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CONTENTS

PAPERS

No. PAGE

2171 DEVELOPMENT OF TRANSPORTATION IN THE UNITED STATES.
By **J. E. Teal** 1

Discussion:
By Fred Lavis 24
William J. Wilgus 25
Joseph B. Eastman 29
J. L. Campbell 30
W. W. Crosby 32
W. B. Irwin 34
F. R. Schanck 36
J. E. Teal 40

2172 STABILITY OF GRANULAR MATERIALS.
By **R. E. Glover and F. E. Cornwell** 46

Discussion:
By R. G. Hennes 64
Stanley U. Benscoter 65
A. Hrennikoff 69
D. P. Krynine 72
R. E. Glover and F. E. Cornwell 78

2173 DRAINAGE OF LEVEED AREAS IN MOUNTAINOUS VALLEYS.
By **Gordon R. Williams** 83

Discussion:
By Merrill Bernard 97
L. K. Sherman 98
C. O. Clark 100
W. W. Horner 103
Walter T. Wilson 108
W. B. Langbein 109
Gordon R. Williams 112

2174 TIMBER FRICTION PILE FOUNDATIONS.
By **Frank M. Masters** 115

Discussion:
By E. H. Connor 141
Glenn B. Woodruff 142
Jacob Feld 143
G. G. Greulich 144
G. S. Paxson 145
Raymond D. Mindlin 147
John W. Solomon 149
A. E. Cummings 162
Frank M. Masters 170

2175 ALLOCATION OF THE TENNESSEE VALLEY AUTHORITY PROJECTS.
By **Theodore B. Parker** 174

Discussion:
By Malcolm Elliott 188
E. L. Chandler 189

No. Page

Eugene L. Grant 202
Sherman M. Woodward 203
Albert R. Arledge 205
C. W. Watchorn and F. A. Allner 210
Theodore B. Parker 215

2176 BENDING MOMENTS IN THE WALLS OF RECTANGULAR TANKS.
By **Dana Young** 217

Discussion:
By I. K. Silverman 231
Dana Young 232

2177 DESIGN OF SIGN LETTER SIZES.
By **Adolphus Mitchell and T. W. Forbes** 233

Discussion:
By Guy Kelcey 243
Charles M. Noble 244
Adolphus Mitchell and T. W. Forbes 245

2178 LATERAL STABILITY OF UNSYMMETRICAL I-BEAMS AND TRUSSES IN BENDING.
By **George Winter** 247

Discussion:
By Robert K. Schrader 261
H. N. Hill 264
George Winter 266

2179 REDUCTION OF MINERAL CONTENT IN WATER WITH ORGANIC ZEOLITES.
By **R. F. Goudey** 269

Discussion:
By James M. Montgomery 281
George L. Davenport 283
Robert Spurr Weston 284
Harvey F. Ludwig and Russell G. Ludwig 286

2180 UNUSUAL EVENTS AND THEIR RELATION TO FEDERAL WATER POLICIES.
By **W. G. Hoyt** 290

Discussion:
By Edward H. Sargent 304
Dana M. Wood 304
James S. Sweet 306
J. L. Campbell 307
David J. Guy 309
Albert S. Fry 312
W. G. Hoyt 314

2181 EFFECT OF VARIATION OF ELASTIC CHARACTERISTICS ON STATIC UNKNOWNS.
By **A. Hrennikoff** 317

Discussion:
By A. H. Finlay 340
Stefan J. Fraenkel 343
Joseph A. Wise 345
A. Hrennikoff 350

2182 BASIC ECONOMIC CONSIDERATIONS AFFECTING SINGLE AND MULTIPLE PURPOSE PROJECTS.
By **Otis M. Page** 353

Discussion:
By G. B. Pillsbury 363

No. Page

2183 ANALYTICAL AND EXPERIMENTAL METHODS IN ENGINEERING SEISMOLOGY.
By **M. A. Biot** 365

Discussion:
By George R. Rich 386
N. J. Hoff 388
Merit P. White 390
N. H. Heck 394
Frank Neumann 395
Jacob Feld 397
George W. Housner 398
Homer M. Hadley 402
Robert E. Glover 405
M. A. Biot 405

2184 HYDRODYNAMICS OF MODEL STORM SEWER INLETS APPLIED TO DESIGN.
By **G. S. Tapley** 409

Discussion:
By Ralph W. Powell 444
G. S. Tapley 445

2185 MECHANICS OF CREEP FOR STRUCTURAL ANALYSIS.
By **Joseph Marin** 453

Discussion:
By A. Hrennikoff 470
C. A. P. Turner 472
Ralph W. Stewart 477
Joseph Marin 479

2186 PROFILE CURVES FOR OPEN-CHANNEL FLOW.
By **Dwight F. Gunder** 481

Discussion:
By Carl Rohwer 489
C. J. Posey 489
J. C. Stevens 492
Boris A. Bakhmeteff and Nicholas V. Feodoroff 492
Hunter Rouse 502
Dwight F. Gunder 504

2187 THE GREASE PROBLEM IN SEWAGE TREATMENT.
By **Almon L. Fales and Samuel A. Greeley** 507

Discussion:
By Arthur D. Weston 525
Harvey F. Ludwig 535
W. S. Mahlie 537
Denis Dickinson 537
Charles R. Velzy 538
F. M. Dawson 539
Almon L. Fales and Samuel A. Greeley 541

2188 DESIGN OF ST. GEORGES TIED ARCH SPAN.
By **J. M. Garrelts** 543

Discussion:
By R. W. Abbett 555
Jacob Karol 555
Alexander Dodge 558
Carlos A. Bejarano 566
A. A. Eremin 568

No. Page

A. M. Freudenthal 569
C. H. Gronquist 570
Harold E. Langley 572
J. M. Garrelts 573

2189 RELATIVE ANGULAR, LINEAR, AND TRAVERSE ACCURACIES IN CITY SURVEYS.
By S. A. Bauer 577

Discussion:
By Cleveland B. Coe 591
W. H. Rayner 592
Philip Kissam 594
George D. Whitmore, C. C. Miner, and W. O. Byrd 598
G. Brooks Earnest 600
S. A. Bauer 601

2190 EARLY CONTRIBUTIONS TO MISSISSIPPI RIVER HYDROLOGY.
By C. S. Jarvis 605

Discussion:
By Robert Follansbee 629
R. W. Davenport 629
Charles Senour 633
John C. Hoyt 637
George O. Guesmer 638
H. T. Cory 638
C. S. Jarvis 639

2191 CLASSIFICATION OF IRRIGABLE LANDS.
By W. W. Johnston 656

Discussion:
By R. Earl Storie 672
Walter W. Weir 672
W. L. Powers 675
W. W. Johnston 675

2192 MOMENT BALANCE: A SELF-CHECKING ANALYSIS OF RIGIDLY JOINTED FRAMES.
By R. J. Cornish 677

Discussion:
By D. D. Matthews 685
William A. Larsen 690
A. A. Eremin 690
L. E. Grinter 691
B. J. Aleck 693
Frederick S. Merritt 694
Ralph W. Stewart 696
G. W. Stokes 698
Bruce Jameyson 700
R. E. Bowles 703
William Morris 704
John Mason 706
R. J. Cornish 710

2193 PERMEABILITY OF MUD MOUNTAIN DAM CORE MATERIAL.
By Allen S. Cary, Boyd H. Walter, and Howard T. Harstad 719

Discussion:
By F. H. Kellogg 729
J. E. Christiansen and Milton Fireman 733
William P. Creager 735
Allen S. Cary, Boyd H. Walter, and Howard T. Harstad 736

No. Page

2194 RESISTANCE OF SOFT FILL TO STATIC WHEEL LOADS.

By **W. Watters Pagon** 738

Discussion:

By Jacob Feld 756

W. Watters Pagon 757

2195 VISCOSITY AND SURFACE TENSION EFFECTS ON V-NOTCH WEIR COEFFICIENTS.

By **Arno T. Lenz** 759

Discussion:

By A. A. Kalinske 783

Glen N. Cox and F. J. Germano 784

Emery H. Willes 785

Ralph W. Powell 792

G. H. Hickox 796

Arno T. Lenz 797

2196 MODEL TESTS ON STRUCTURES FOR HYDROELECTRIC DEVELOPMENTS.

By **L. M. Davis** 803

Discussion:

By Emil P. Schuleen 824

C. M. Allen 828

Adolf A. Meyer 828

L. M. Davis 829

2197 STRESS CONCENTRATIONS IN PLATES LOADED OVER SMALL AREAS.

By **H. M. Westergaard** 831

Discussion:

By George R. Rich 857

Dana Young 858

A. Nádai 860

L. A. MacColl 863

L. W. Teller and Earl C. Sutherland 864

D. L. Holl 868

C. A. P. Turner 881

H. M. Westergaard 884

2198 HYDRAULIC DESIGN OF DROP STRUCTURES FOR GULLY CONTROL.

By **B. T. Morris and D. C. Johnson** 887

Discussion:

By John Hedberg 919

L. Standish Hall 919

J. E. Christiansen 921

Walter T. Wilson 926

N. A. Christensen and Dwight Gunder 927

Boris A. Bakhmeteff and Nicholas V. Feodoroff 930

G. H. Hickox 932

B. T. Morris and D. C. Johnson 935

2199 DISTRIBUTION GRAPHS OF SUSPENDED-MATTER CONCENTRATION.

By **Joe W. Johnson** 941

Discussion:

By H. A. Einstein 957

G. C. Dobson 958

Thomas R. Camp 959

Frank S. Bailey 960

Joe W. Johnson 960

No. Page

2200 EARTH PRESSURE AND SHEARING RESISTANCE OF PLASTIC CLAY: A SYMPOSIUM 965

FOREWORD 966

LINER-PLATE TUNNELS ON THE CHICAGO (ILL.) SUBWAY.

By Karl Terzaghi 970

EARTH-PRESSURE MEASUREMENTS IN OPEN CUTS, CHICAGO (ILL.) SUBWAY.

By Ralph B. Peck 1008

EARTH PRESSURE ON TUNNELS.

By W. S. Housel 1037

Discussion:

By Ralph H. Burke 1059
L. G. Lenhardt 1061
George E. Shafer 1062
M. E. Chamberlain 1064
A. A. Eremin 1066
A. E. Cummings 1067
D. M. Burmister 1072
Gregory P. Tschebotarioff 1075
D. P. Krynine 1078
L. F. Cooling 1086
Karl Terzaghi 1090
Ralph B. Peck 1097
W. S. Housel 1103

2201 STATISTICAL ANALYSIS IN HYDROLOGY.

By L. R. Beard 1110

Discussion:

By L. Standish Hall 1122
H. Alden Foster 1127
Paul V. Hodges 1128
Joe W. Johnson 1130
R. H. Blythe, Jr. 1137
R. W. Davenport 1139
Edward J. Bednarski 1141
Ralph W. Powell 1144
E. J. Gumbel 1144
L. R. Beard 1154

2202 NUMERICAL PROCEDURE FOR COMPUTING DEFLECTIONS, MOMENTS, AND BUCKLING LOADS.

By N. M. Newmark 1161

Discussion:

By Bruce Johnston 1189
M. S. Ketchum, Jr. 1193
John B. Wilbur 1194
Ralph W. Stewart 1198
Stefan J. Fraenkel 1201
Alfred S. Niles 1206
Camillo Weiss 1208
A. A. Eremin 1211
Myron L. Gossard 1212
Robert A. Williamson 1220
I. Oesterblom 1224
C. W. Dunham 1224
N. M. Newmark 1227

No. Page

2203 FUNDAMENTAL ASPECTS OF THE DEPRECIATION PROBLEM: A SYMPOSIUM. 1235

FOREWORD.

By Maurice R. Scharff 1236

RELATIONSHIP TO REGULATED INDUSTRY.

By J. A. Walls 1237

RELATIONSHIP TO COMPETITIVE INDUSTRY.

By Eugene L. Grant 1242

RELATIONSHIP TO PUBLIC WORKS AND GOVERNMENT FINANCE.

By Roy W. Crum and Robley Winfrey 1255

Discussion:

By Edwin F. Wendt and L. T. Fleming 1262
Anson Marston 1264
William G. Atwood 1268
George E. Goldthwaite 1269
Nathan B. Jacobs 1272
H. J. Flagg 1274
Nelson Lee Smith 1276
C. Beverley Benson 1278
Paul T. Norton, Jr. 1282
H. L. Ripley 1284
Carroll A. Farwell 1286
Thomas R. Agg 1287
Conde B. McCullough 1289
W. V. Burnell 1293
Roger L. Morrison 1296
Wallace B. Carr 1297
W. L. Waters 1300
A. G. Mott 1301
David A. Kosh 1303
E. G. Walker 1307
K. Lee Hyder 1311
James T. Ryan 1314
John C. Page 1316
John S. Worley 1321
John F. Miller 1322
Fred Asa Barnes 1323
Robert F. Legget and Erwin E. Hart 1326
Kernan Robson 1329
J. A. Walls 1331
Eugene L. Grant 1335

2204 ENERGY LOSS AT THE BASE OF A FREE OVERFALL.

By Walter L. Moore 1343

Discussion:

By Merit P. White 1361
Boris A. Bakhmeteff and N. V. Feodoroff 1364
Carl E. Kindsvater 1373
J. E. Christiansen 1375
L. Standish Hall 1378
Hunter Rouse 1383
Walter L. Moore 1387

2205 ENTRAINMENT OF AIR IN FLOWING WATER: A SYMPOSIUM 1393

OPEN CHANNEL FLOW AT HIGH VELOCITIES.

By L. Standish Hall 1394

CLOSED CONDUIT FLOW.

By A. A. Kalinske and James M. Robertson 1435

No. PAGE

Discussion:

By WARREN DELAPP 1448
KARL R. KENNISON 1451
ROBERT T. KNAPP 1455
CARL E. KINDSVATER 1459
J. H. DOUMA 1462
JOE W. JOHNSON 1473
J. C. STEVENS 1474
D. C. MCCONAUGHY 1484
T. J. CORWIN, JR. 1493
L. STANDISH HALL 1494
A. A. KALINSKE AND J. M. ROBERTSON 1513

2206 THE SOCIETY TODAY AND TOMORROW: ADDRESS AT THE ANNUAL CONVENTION, LOS ANGELES, CALIFORNIA, JULY 28, 1943 1517

XI

MEMOIRS OF DECEASED MEMBERS

Page

Arthur Newell Talbot, Past-President and Hon. M. Am. Soc. C. E. 1530

Otis Ellis Hovey, Hon. M. Am. Soc. C. E. 1537

Joseph Barlow Lippincott, Hon. M. Am. Soc. C. E. 1543

Frederick Abels Allner, M. Am. Soc. C. E. 1551

Robert Adams Allton, M. Am. Soc. C. E. 1553

Thomas Clark Atwood, M. Am. Soc. C. E. 1556

George Ellsworth Barrows, M. Am. Soc. C. E. 1558

George Henry Blakeley, M. Am. Soc. C. E. 1560

Carolus Morton Broomall, M. Am. Soc. C. E. 1561

Robert Francis Brown, M. Am. Soc. C. E. 1564

Alfred Dickey Butler, M. Am. Soc. C. E. 1565

William Tatem Collings, Jr., M. Am. Soc. C. E. 1568

Edwin Ford Dawson, M. Am. Soc. C. E. 1570

Charles Meirs Denise, M. Am. Soc. C. E. 1571

Manuel Victor Domenech, M. Am. Soc. C. E. 1573

Edward James Dougherty, M. Am. Soc. C. E. 1575

Arthur Sylvester Douglass, M. Am. Soc. C. E. 1576

Walter Jacob Fuston, M. Am. Soc. C. E. 1579

Philip Dakin Gillham, M. Am. Soc. C. E. 1580

Carleton Greene, M. Am. Soc. C. E. 1582

Arthur James Griffin, M. Am. Soc. C. E. 1584

Claude Irving Grimm, M. Am. Soc. C. E. 1585

Alan Gordon Gutteridge, M. Am. Soc. C. E. 1587

Andrew Harkness Harkness, M. Am. Soc. C. E. 1588

Jay Russell Hickox, M. Am. Soc. C. E. 1590

John Mifflin Hood, Jr., M. Am. Soc. C. E. 1592

Jere Chamberlain Hutchins, M. Am. Soc. C. E. 1594

Clark Ellsworth Jacoby, M. Am. Soc. C. E. 1598

David Clayton Johnson, M. Am. Soc. C. E. 1601

Frederick Henry Kester, M. Am. Soc. C. E. 1602

Arthur Kraus, M. Am. Soc. C. E. 1604

Robert Linton, M. Am. Soc. C. E. 1605

Albert Kualii Brickwood Lyman, M. Am. Soc. C. E. 1608

Walter Henry Meier, M. Am. Soc. C. E. 1609

Clarence Eugene Myers, M. Am. Soc. C. E. 1612

John A. Newlin, M. Am. Soc. C. E. 1614

William Warren Orcutt, M. Am. Soc. C. E. 1617

Henry Blanchard Pratt, M. Am. Soc. C. E. 1622

Charles Miller Reppert, M. Am. Soc. C. E. 1623

Barzillai Allen Rich, M. Am. Soc. C. E. 1625

Wallace Carl Riddick, M. Am. Soc. C. E. 1626

Fred Elmer Rightor, M. Am. Soc. C. E. 1629

Nathan Schein, M. Am. Soc. C. E. 1631

Arthur Monroe Shaw, M. Am. Soc. C. E. 1632

Frank Herbert Snow, M. Am. Soc. C. E. 1634

John Rodgers Spelman, M. Am. Soc. C. E. 1638

Herman Stabler, M. Am. Soc. C. E. 1641

Milton Gerry Starrett, M. Am. Soc. C. E. 1645

John Williams Storrs, M. Am. Soc. C. E. 1646

William John Thomas, M. Am. Soc. C. E. 1648

Arthur Clarence Tozzer, M. Am. Soc. C. E. 1649

Alvin Dumond Wilder, M. Am. Soc. C. E. 1651

Edward Gilbert Williams, M. Am. Soc. C. E. 1653

John William Woermann, M. Am. Soc. C. E. 1654

Herbert Scandlin Battie, Assoc. M. Am. Soc. C. E. 1657

Chester Robert Chadwick, Assoc. M. Am. Soc. C. E. 1659

Joseph James Corti, Assoc. M. Am. Soc. C. E. 1660

Marius Schoonmaker Darrow, Assoc. M. Am. Soc. C. E. 1664

Willard Livermore Gorton, Assoc. M. Am. Soc. C. E. 1666

PAGE

Albert Henry Greenwood, Assoc. M. Am. Soc. C. E. 1670
Joseph Churchill Hilton, Assoc. M. Am. Soc. C. E. 1671
William Frederick Jenrick, Assoc. M. Am. Soc. C. E. 1674
Hallett Edward McClintock, Assoc. M. Am. Soc. C. E. 1675
William Griffith Moler, Assoc. M. Am. Soc. C. E. 1678
Henry Aylesbury Stringfellow, Assoc. M. Am. Soc. C. E. 1680
Frank Edgar Walter, Assoc. M. Am. Soc. C. E. 1681

ERRATA

Transactions, Vol. 101 (1936)

Beginning page 901, note modifications in formulas, as follows: Eq. 74, delete coefficients a in two places; Eq. 76, change a^3 to a^2; Eq. 77, change a^3 to a^2 and a^5 to a^4; Eq. 78, change a^6 to a^5; Eq. 79, change a^5 to a^4 and a^7 to a^6; Eq. 80, change a^6 to a^5; and Eq. 81, change a^3 to a^2, a^5 to a^4, and a^7 to a^6.

Transactions, Vol. 107 (1942)

Page 1033, line 1, in the denominator, change "×" to "+."

Page 1085, Table 7, under 1933–1934, change Item 42*a* to read "345,545" instead of "545,545."

Page 1234, last term of Eqs. 14, change "$M_{3,4}$" to "$M_{3,u}$."

PAPERS AND REPORTS OF COMMITTEES PUBLISHED IN PROCEEDINGS, AM. SOC. C. E., OCTOBER, 1942, TO SEPTEMBER, 1943, INCLUSIVE, NOT PUBLISHED IN TRANSACTIONS, VOL. 108

	Number of *Proceedings*.
CONFORMITY BETWEEN MODEL AND PROTOTYPE: A SYMPOSIUM	October, 1942
THE HYDRAULIC JUMP IN SLOPING CHANNELS. By Carl E. Kindsvater	November, 1942
APPLICATION OF SOIL MECHANICS IN DESIGNING BUILDING FOUNDATIONS. By A. Casagrande and R. E. Fadum	November, 1942
FLOW CHARACTERISTICS AT RECTANGULAR OPEN-CHANNEL JUNCTIONS. By Edward H. Taylor	November, 1942
RELATION OF UNDISTURBED SAMPLING TO LABORATORY TESTING. By P. C. Rutledge	November, 1942
CONCRETE RESERVOIRS OF THE VERTICAL-BEAM TYPE. By C. Maxwell Stanley, Jr.	December, 1942
ILLINOIS WATER LITIGATION, 1940–1941. By Langdon Pearse	December, 1942
PHYSICAL PROPERTIES THAT AFFECT THE BEHAVIOR OF STRUCTURAL MEMBERS. By Wilbur M. Wilson	December, 1942
ORGANIZING AND FINANCING SEWAGE TREATMENT PROJECTS. By Samuel A. Greeley	December, 1942
SEISMIC SUBSURFACE EXPLORATION ON THE ST. LAWRENCE RIVER PROJECT. By E. R. Shepard and Reuben M. Haines	December, 1942
PENDLETON LEVEE FAILURE. By Kenneth E. Fields and William L. Wells	December, 1942
AERATION OF SPILLWAYS. By G. H. Hickox	December, 1942
RIVETED AND PIN-CONNECTED JOINTS OF STEEL AND ALUMINUM ALLOYS. By Leon S. Moisseiff, E. C. Hartmann, and R. L. Moore	January, 1943

	Number of *Proceedings*.
DEWATERING, INCINERATION, AND USE OF SEWAGE SLUDGE: A SYMPOSIUM	January, 1943
PRIMARY RÔLE OF METEOROLOGY IN FLOOD FLOW ESTIMATING. By Merrill Bernard	January, 1943
STRAINS, STRESSES, AND SHEAR IN ENGINEERING PROBLEMS. By Silas H. Woodard	February, 1943
DETERMINATION OF KUTTER'S *n* FOR SEWERS PARTLY FILLED. By C. Frank Johnson	February, 1943
CHARACTERISTICS OF HEAVY RAINFALL IN NEW MEXICO AND ARIZONA. By Luna B. Leopold	February, 1943
EFFECT OF TURBULENCE ON SEDIMENTATION. By William E. Dobbins	February, 1943
DEVELOPMENT OF THE CHICAGO TYPE BASCULE BRIDGE. By Donald N. Becker	February, 1943
FLOW AROUND BENDS IN STABLE CHANNELS. By C. A. Mockmore	March, 1943
THE QUEENS MIDTOWN TUNNEL. By Ole Singstad	March, 1943
WATER SUPPLY ENGINEERING: REPORT OF THE COMMITTEE OF THE SANITARY ENGINEERING DIVISION FOR THE THREE YEARS ENDING DECEMBER 31, 1942	March, 1943
A METHOD OF COMPUTING URBAN RUNOFF. By W. I. Hicks	April, 1943
HIGHWAY ENGINEERING EDUCATION: PROGRESS REPORT OF THE COMMITTEE OF THE HIGHWAY DIVISION	April, 1943
SIMPLIFIED ANALYSIS OF SKEWED REINFORCED CONCRETE FRAMES AND ARCHES. By Richard M. Hodges	May, 1943
RÔLE OF THE LAND DURING FLOOD PERIODS. By W. W. Horner	May, 1943
A SYMMETRICALLY LOADED BASE SLAB ON AN ELASTIC FOUNDATION. By Stanley U. Benscoter	May, 1943
SEDIMENTATION IN RESERVOIRS. By Berard J. Witzig	June, 1943
STRENGTH OF SLENDER BEAMS. By George Winter	June, 1943
SLUDGE DRYING DEVELOPMENTS AT CHICAGO, ILL. By Lloyd M. Johnson	September, 1943

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2171

DEVELOPMENT OF TRANSPORTATION IN THE UNITED STATES

By J. E. Teal,[1] M. Am. Soc. C. E.

With Discussion by Messrs. Fred Lavis, William J. Wilgus, Joseph B. Eastman, J. L. Campbell, W. W. Crosby, W. B. Irwin, F. R. Schanck, and J. E. Teal.

Synopsis

The first railroad chartered in the United States was the Baltimore and Ohio, in 1828, and the story of the development of the railroads that followed, which made possible the United States as it exists today, is intensely interesting. Other forms of transportation have come into being, and some have grown to a point where they are having a decided influence on the social and economic environments of every community in the Nation. These agencies of transportation, under present conditions, are all competitive with each other, and certain economic influences have developed that apparently have resulted in unfair competition between them.

With the importance of the issues in mind, the writer has endeavored, in this paper, to sketch the history of transportation during the past one hundred or more years, to outline briefly some of the economic conditions, and to discuss a few of the pertinent factors involved. An effort has been made to correlate facts and data pertaining to the subject, which is now a transportation problem so vital and so far-reaching that a sound national economy may depend on its solution.

Railway Transportation

During the past 113 years, railway development in the United States passed through several distinct stages. The "First Stone" of the Baltimore and Ohio Railroad was laid on July 4, 1828, at which ceremony the venerable Charles Carroll, of Carrollton (Va.), remarked:

> "I consider this among the most important acts of my life, second only to the signing of the Declaration of Independence, even if indeed it be second to that."

Note.—Published in December, 1941, *Proceedings*.

[1] Prin. Roadway Engr., Board of Investigation and Research, Washington, D. C.

From that date to 1850 was a period of exploration, both in the matter of developing road, equipment, and plant facilities, and in the methods of finance and management.

The next forty years was a period of very rapid railway expansion. The railway systems had been pushed west across the Allegheny Mountains to Chicago, Ill., and the Union Pacific and the Central Pacific railroads had been completed by 1869. Railway development was rapid during the period leading to the depression of 1873. The decade 1880 to 1890 witnessed an increase in railway lines aggregating more than 70,000 miles, when a number of transcontinental lines were constructed.

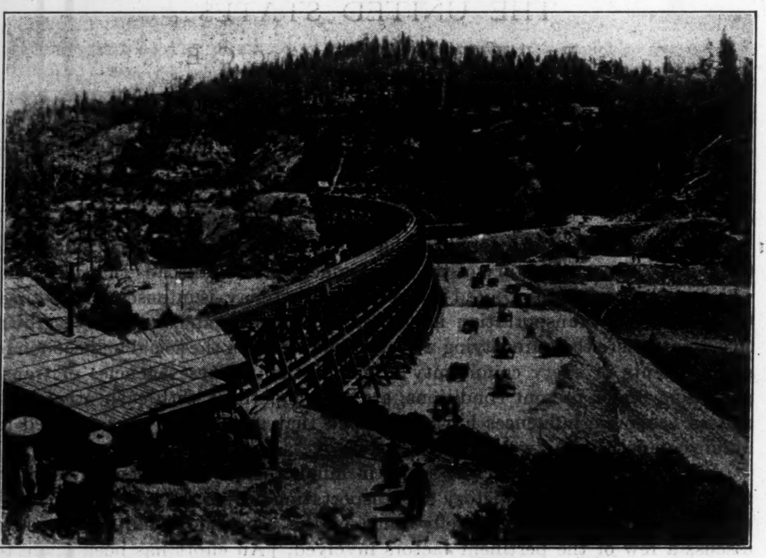

FIG. 1.—RAILROAD DEVELOPMENT WAS RAPID DURING THE PERIOD LEADING TO THE DEPRESSION OF 1873

During this period of rapid railway construction, public authorities and federal, state, and local governments became convinced that the railways were indispensable to the prosperity of the Nation. Railway construction was encouraged by means of federal land grants and financial subsidies until about 1871. Some financial aid was contributed after that year.

From 1890 to the period of federal control (1917 to 1920) might be considered as the finishing period of the railway plant. Railway construction consisted mainly of the building of feeder and cross lines or new branch lines—the rounding out of the railway system as it is known today. Railway mileage was increased from 163,597 in 1890 to the peak of 254,251 in 1916, but this had receded to 235,064 miles in 1939. Track mileage reached a maximum of 429,883 miles in 1930, declining to 408,350 miles in 1939; thousands of miles of branch and feeder lines have become obsolete.

The last period, covering two decades, is of great significance in the railway picture; it begins when highway transportation entered upon a new era of development. As the result of competition growing out of common-carrier service on the highways, the revival of waterways, the expansion of pipe-line transportation, and the coming of the airplane, the railway industry has had to "tighten its belt" in order to meet financial obligations.

Railroad Operating Results Since 1920

The total property investment in Class I railways amounted to $26,527,000,000 in 1930, decreasing to $26,131,000,000 at the end of 1939. Peak revenues reached the total of $6,383,000,000 in 1926, declined to an average of $3,622,000,000 per yr from 1931 to 1937, and were $3,995,000,000 in 1939, a 13-yr decrease of 37%.

Net income, after fixed charges, reached its peak in 1929, with a total of $897,000,000, but only averaged $35,000,000 per annum from 1931 to 1937; in 1938 there was a net deficit of $129,000,000, with a net gain of only $93,000,000 in 1939. Railway taxes increased from $276,000,000 in 1921 to $356,000,000 in 1939, or 29%; and taxes per one dollar of revenue increased from 5.0¢ in 1921 to 8.9¢ in 1939, or 78%. The rate of return earned by the railroads on their investment in transportation property averaged 4.81% in 1929, 1.67% for the years 1931 to 1937, 1.43% in 1938, and 2.26% in 1939, a 10-yr decrease of 53%.

Railroad freight traffic reached 447,000,000,000 ton-miles, its peak, in 1929, averaged 292,000,000,000 ton-miles from 1931 to 1937, and 333,438,000,000 ton-miles in 1939, a 10-yr decrease of 25%. Railroad passenger traffic reached its peak in 1920, when 46,800,000,000 passenger-miles were handled, declined to an average of 19,900,000,000 from 1931 to 1937, and amounted to 22,651,000,000 in 1939, a 19-yr decrease of 52%.

Railroad freight rates declined from 12.75 mills per ton-mile in 1921 to an average of 9.96 mills from 1931 to 1937, and 9.74 mills per ton-mile in 1939, an 18-yr decrease of 23%. Average revenue per passenger-mile declined from a maximum of 3.86¢ in 1921 to 2.033¢ from 1931 to 1937, and 1.839¢ in 1939, an 18-yr decrease of 52%. Competition has been an important factor in the downward trend in railroad freight and passenger rates.

Railroad employment declined from the peak figure of 2,022,000 in 1920 to an average of 1,063,000 from 1931 to 1937, and 988,000 in 1939, a 19-yr decrease of 51%.

Because of the failure of many carriers to meet their interest charges, the credit of the railroad industry in general has been greatly impaired, and must be recouped in the future. The meager financial results of railroad operations for 1938, amounting to 1.43% return on the investment, were secured by heavy reduction in maintenance and other expenditures, some of which must be made up at some future date. Closely related to the subject of maintenance is the decrease in ability of the railroads to contribute to the national economy in the form of purchases of the products of manufacturers and other industry.

The gross capital expenditures made by the railroads for the period 1921 to 1930 were an average of $773,000,000 per annum. For the period 1931 to

1939 this average amounted to only $262,000,000, or a decrease of $511,000,000, or 66%, per annum. Comparing the purchases of fuel, material, and supplies during the same periods, the average annual expenditure was reduced from $1,384,000,000 for the first period to $660,000,000 for the second period, a decrease of $724,000,000, or 52% per annum. These statistics show that under normal conditions the railroads are heavy consumers of the products of other industries and are, therefore, an important factor in the Nation's business. Also, the capability of the railroads to buy equipment and other supplies, and to foster general employment, has been seriously impaired in recent years.

More than 75,000 miles of railroad lines, or about one third of all railroad mileage, are now in the hands of receivers or trustees. This is the largest proportion of rail mileage in bankruptcy ever recorded in the history of the United States.

It is only necessary to note the many improvements that have developed in rail transportation during the two decades 1920–1940, and more particularly, the latter of the two. Every one knows of the modern, streamlined, air-conditioned, high-speed passenger trains, and the faster freight, including the pickup and delivery services.

INLAND WATERWAY TRANSPORTATION

The history of inland waterway transportation, of course, antedates that of railroad transportation. Rivers and canals were among the principal channels of transportation in the United States until the close of the Civil War. In Virginia, the James River Company, organized by George Washington and others, as a stock company, in 1785, had as its object the construction of a canal from the tidal basin of the James River at Richmond, Va., following that river to Clifton Forge, Va., and across the Allegheny Mountains, by highway, to the Kanawha River, a tributary of the Ohio River in West Virginia. The canal was completed for a distance of 200 miles, when financial troubles stopped construction. The property was later taken over by the predecessor of The Chesapeake and Ohio Railway Company. By the close of 1825, the Erie Canal had been completed between the Hudson River and Lake Erie, 351 miles, and in 1828 the Chesapeake and Ohio Canal was under construction from the tidal basin of the Potomac River at Washington, D. C., for a distance of 186 miles, following the Potomac River to Cumberland, Md.

Canal construction spread very rapidly, not only in those states along the eastern seaboard that were in need of lines of communication with the Middle West, but also within the Western States. Following the Civil War, in 1865, the development of railways attracted much of the commerce that formerly moved by water. The railways also created new commerce in many areas not served by rivers, following which traffic on the rivers and canals declined to a condition of little importance.

The first turnpike roads were built in the United States with the view that the full cost of transportation should be borne by the shipper. Likewise, it was the intention to charge tolls for the use of the canals; but returns from this source rarely were sufficient to cover both capital charges and maintenance expenses. The Pennsylvania canals were a complete financial failure. The

New York system of inland waterways, including the Hudson River and the Erie-Champlain canal, made a better showing, but had to be (and still are) supported substantially by general taxation. The total cost of canals to the people of New York from the beginning to 1929 has amounted to more then $346,000,000. From 1930 to 1940, inclusive, the total expense aggregated $109,195,000, or an average of $9,927,000 per annum, divided thus: $31,398,000, or $2,854,000 per annum, for operation and maintenance, and $77,797,000, or $7,073,000 per annum, for carrying charges on construction cost, computed at 4% per annum. This brings the grand total of expenditures, to 1940, to $455,195,000. Annual credit for revenue received by the commissioner of canals and waterways for various services amounted to $607,000 for 1940.

By 1885, the rivers and canals had lost most of the high-grade traffic. However, low-grade traffic, such as sand, gravel, and coal, continued to move in fairly large quantities for a longer period. By the end of the century, with the exception of coal traffic on the Monongahela and Ohio rivers, canal and river transportation was unimportant.

FIG. 2.—THE INLAND WATERWAYS RENAISSANCE RECEIVED A GREAT STIMULUS DURING THE FIRST WORLD WAR

The inland waterways renaissance, which began in the last decade of the nineteenth century, received a great stimulus during the first World War, and since the war has been carried forward with the assistance of fostering legislation and the spur of many government officials, individuals, and organizations. To bring this question down to date, it suffices to state that the report by the chief of engineers of the United States Army for 1939 shows that a total of $3,637,528,923 has been spent or authorized on the construction and maintenance of waterways projects to June 30, 1939, of which approximately $2,655,000,000 has been spent or authorized for river and harbor navigation projects. Of this amount, approximately $1,123,000,000 was on the Mississippi River system, $517,000,000 for other waterways, and the remainder for seacoast and lake harbors and channels and intercoastal canals. The total mileage of inland waterways, according to the Board of Engineers for Rivers and Harbors, as of June 15, 1935, aggregated 26,406, divided as follows:

Inland waterways	Route mileage
Atlantic Coast rivers	5,263
Gulf Coast rivers, including the Mississippi River system	17,381
Pacific Coast rivers	1,796
Intercoastal waterways	1,966
Total	26,406

Of the total mileage, 15,688 miles have a depth of less than 6 ft; 5,394 miles have depths ranging between 6 ft and 9 ft; 4,746 miles have depths ranging from 9 ft to 12 ft; and 1,578 miles have depths greater than 12 ft.

TABLE 1.—WATER-BORNE TONNAGE

Year	SHORT TONS (THOUSANDS)		
	Great Lakes	Other	Total
1920	98,751	125,400	224,151
1925	113,644	204,569	318,213
1930	115,727	226,760	342,487
1935	89,356	225,917	315,273
1936	122,788	276,264	399,052
1937	143,356	313,287	456,643
1938	82,480	277,755	360,235

The total tonnage of water-borne commerce on the Great Lakes, and on rivers, canals, and connecting channels in the United States, after eliminating the known duplications, is shown in Table 1 for the calendar years 1920 through 1938. From this table it will be seen that the tonnage other than on the Great Lakes has increased from 125,400,000 tons in 1920 to 277,755,000 tons in 1938, an increase of 121%.

The total ton-mileage on inland waterways of the United States during 1938, exclusive of the Great Lakes, was 17,742,506,538. The total ton-mileage on the Great Lakes was 49,004,019,901, which, together with that on the rivers, canals, and connecting channels, makes a total ton-mileage of 66,746,526,439. Computed on the basis of 4.46 mills per ton-mile (average for Inland Waterway Corporation 1924 to 1935), the expense to the shippers using rivers, canals, and connecting channels for transporting 17,742 million tons one mile in 1938 would be approximately $79,000,000. The cost of owning and maintaining the waterways, of course, is borne by the federal government.

HIGHWAY TRANSPORTATION

The development of highways in the United States has passed through three periods, each characterized by usage of a distinct character. In the first period, extending to about the middle of the nineteenth century, the initiative in furnishing major highway routes was taken by the federal government. This was gradually relinquished, until about 1856 all federal jurisdiction over the national highway routes had been assumed by the states through which the routes passed. During this time a number of states had found the development of subsidiary highway routes to be burdensome and had placed some of them on a commercial basis by granting charters to individuals or companies, permitting them to build and operate toll roads. The states, however, participated in the cost of these projects and retained the right to regulate them.

Prior to 1892 the development in the field of highway transportation involved no particular complex problems of public policy. Both the building of the roads and the building of the motor vehicles were passing through experimental phases. The use of the public highways for purely commercial purposes had not attained sufficient importance to cause any concern on the part of the established carriers.

The second period in the development of the highway transportation as it exists today embraces the one of rapid railroad building, extending to about 1900. As the railroad mileage expanded, the use and necessity of the highways decreased to a position of minor importance. As transportation over public roads became unnecessary and unprofitable except for short distances, the building and maintenance of these roads were reduced to a matter of local importance only and consequently were left to the small political subdivisions, such as the county, township, and road district. It follows that from the middle to the end of the nineteenth century only a small proportion of highway travel passed beyond the boundaries of local political subdivisions, and the administration and financial responsibility for furnishing highway facilities thus narrowed down to such areas.

The year 1900 marks the beginning of the third period, which is characterized by a very rapid increase in the use of the highways. As the automobile developed, it soon became apparent that transportation on the highways for longer distances was possible, resulting in the creation of traffic that bore little or no relation in point of origin and destination to the political areas through which it passed. Such traffic was of an entirely different character than the type that moved wholly within the local subdivisions. It was also realized that the cost of highways required by these different types of traffic could not with equity be assessed entirely against the local property owners, but that a certain part of the cost should be borne by the taxpayers at large.

From 1900 to 1920 the development of the highway was somewhat similar to the evolution of the automobile—that is, as it developed, it passed through various experimental stages as to types of materials used in its construction. As the volume of traffic grew, the wear and tear on the highways became more apparent, and better and stronger materials were required to maintain them.

During the two decades (1920 to 1940), however, rapid expansion of improved highways occurred, and the perfecting of the motor-vehicle equipment has given rise to a substantial volume of commercial operations on the highways. The competition therein afforded by highway transport has been the cause of grave concern, particularly to the railroads, and it has seriously affected their revenues.

In 1921 the highway mileage of the United States aggregated 2,924,505, of which 202,915 miles, or 6.9%, were under the control of the state authorities. The total mileage increased to 3,009,066 in 1930, of which 324,496 miles, or 10.8%, were under the jurisdiction of state authorities. Only 84,372 miles, or 41.6%, of the state-controlled mileage in 1921 were surfaced, and, of this mileage, only 14,707 miles, or 17.4%, were of the high-type surface.

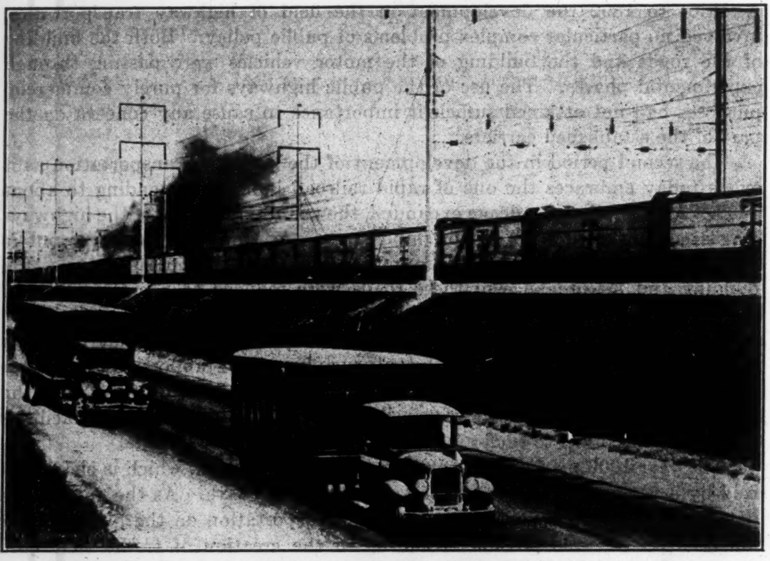

Fig. 3.—The Competition Afforded by Highway Transport Has Been the Cause of Grave Concern to the Railroads

By the close of 1931, significant shifts had occurred in this distribution. The rural highway mileage had expanded only 110,388 miles, or 3.8%, whereas the state highway system had added 126,027 miles, an increase of 62.1%. About 12% of this increase in state mileage was caused by the transfer of local mileage to state systems, rather than by an expansion of the entire road mileage. It is also significant that by the end of 1931 about 74% of the state highway system mileage was surfaced, as contrasted with 42% in 1921. More than 33% of the mileage in 1931 had high-type surface, as compared with 17% in 1921.

During 1931 and subsequent years, various state highway commissions have taken over the supervision of the construction and maintenance of certain county highways which, together with some road mileage previously reported, formed a secondary system of state highways; and at the close of 1938 the existing mileage of public roads and streets under state control aggregated 541,841 miles, distributed as follows:

United States highway system (1938)	Miles
Primary state highway system	326,852
Secondary state highways and county roads under state control	194,428
Total	521,280
Urban expansion of state highways	23,804
Total mileage under state control	541,841

Of this mileage, 424,791 miles, or 78%, have been surfaced. From the foregoing data it may be seen that the state-controlled mileage increased from 202,915 in 1921 to 328,942 miles in 1931, or 62%, and to 541,841 miles in 1938, or 65% over 1931, and 167% over 1921.

Estimates[2] indicate the total cost of the highway system at the close of 1939, including state highways, county roads, and city streets, to be $35,218,000,000, divided as shown in Table 2.

TABLE 2.—HIGHWAY COSTS (MILLION DOLLARS); 1939

Item	Period	State highways	County highways	City streets
1	Prior to 1921	741	4,276	1,921
2	1921 to 1932[a]	4,008	4,720	5,210
3	1933 to 1937[a]	3,933	3,042	3,269
4	1938 to 1939[a]	1,573	1,216	1,308
	Totals	10,255	13,254	11,708
		Grand total		35,217

[a] Inclusive.

The Bureau of Statistics, Interstate Commerce Commission, placed the investment in highways and vehicles used in intercity transportation in the United States at $24,232,000,000, as of 1937. Approximately $12,187,312,000 has been collected from the users of streets and highways through gasoline taxes, the amount for 1939 being $821,656,000 net.

The first automobile appeared in 1892, and from that time the growth of automotive transportation and the development of the extensive highway system in the United States are coincident and interdependent. The record indicates that in 1895 there were four registered motor vehicles. This number increased to 8,000 in 1900, and to 55,000 in 1904. Of these 55,000 vehicles, 410 were classified as trucks and road tractors. From this date, the number of automobiles classified as passenger cars, taxis, buses, trucks, and road tractors increased annually to a total of 9,231,941 in 1920, divided thus: 8,225,859 passenger vehicles and 1,006,082 trucks.

A peak of all motor-vehicle registrations was reached in 1930, when a total of 26,545,281 vehicles was recorded, of which 86.9% were passenger vehicles and 13.1% were trucks. During the next few depression years, the number of registrations decreased, and totaled 23,827,290 in 1933. The number in 1940 was 32,452,861, of which 27,434,979 were privately owned passenger vehicles, 4,590,386 were trucks, and 427,496 were publicly owned vehicles. The percentage of motor trucks had increased from 1.7 in 1904 to 14.2 in 1940.

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[2] "What Is Public Aid to Transportation?" produced and distributed by the Assn. of Am. Railroads. Estimates for periods up to 1937, inclusive, by C. B. Breed, Clifford Older, and W. S. Downs, Cons. Engrs. Estimates for 1938 and 1939 based on previous 5-yr period.

Passenger-car traffic is fairly homogeneous as to both character of traffic and type of equipment; no such uniformity, however, exists in truck operations. Widely different types of vehicles are used in the several kinds of truck operations, and variations in ownership and operation exist and various types of services are rendered.

The number of trucks engaged in short commercial or private hauling over the highways is not accurately known. It has been estimated, however, that, of the 4,590,000 privately owned motor and tractor trucks, approximately 1,307,000 are farm owned, 2,626,000 are privately owned and not operated for hire, and of the remaining, approximately 262,000 are common carriers and 395,000 are contract carriers. Approximately 86% of all the trucks are operated by the owners, and about 14% are common and contract carriers, operated for hire, or under hauling agreements.

TABLE 3.—MOTOR-VEHICLE STATISTICS; 1940, 1939, 1938.

No.	Description	1940	1939	1938
	(a) INCOME (THOUSANDS OF DOLLARS)			
1	State gasoline tax collections—net[a]	$870,692	$822,013	$772,060
2	Registration fees for all vehicles	373,771	353,533	340,061
3	Fees for permits, certificates of title, fines and penalties	65,407	58,961	48,764
4	Total, items 1 to 3[b]	$1,309,870	$1,234,507	$1,160,885
	(b) NUMBER OF MOTOR VEHICLES REGISTERED, AND GASOLINE CONSUMED (THOUSANDS)			
	Number of Motor Vehicles Registered:[c]			
5	Private and commercial vehicles	32,026	30,615	29,486
6	Owned by federal government	143	121	110
7	Owned by state, county, and municipal governments	284	274	257
8	Total	32,453	31,010	29,853
9	Gasoline consumed (gallons)	21,913,000	20,638,000	19,507,000
	(c) UNITS			
10	Average gasoline tax (cents) per gallon[d]	3.97	3.98	3.96
11	Average persons per vehicle in the United States[e]	4.1	4.3	4.4
12	Average car registration fee in the United States[f]	$11.67	$11.55	$11.53

[a] Plus inspection fees, dealers' license fees, fines and penalties, and aviation gasoline.
[b] Motor-vehicle users in all states paid some or all of the taxes and fees included under items 1 to 3.
[c] Includes trucks: 4,590,386 in 1940, 4,320,829 in 1939, and 4,224,031 in 1938 (see Table 4).
[d] Item 1 divided by item 9. Rates for individual states range from 2¢ to 7¢ per gal.
[e] Population divided by item 5. 10.4 persons in 1921.
[f] Item 2 divided by item 5.

Truck operations have been adapted to a wide range of transportation tasks, which vary from the handling of light farm produce and transportation of household goods, merchandise, and machinery, to the heavy services of logging camps and construction work. The distances over which these operations occur vary widely. Common carriers and contract trucking operations are scheduled in runs of from 50 to 100 miles to more than 1,500 miles.

Small-scale operations and individual managements were characteristic features of commercial trucking operations until about 1930. Since that time there has been a tendency to consolidate trucking operations throughout the country.

Under provision of Act of Congress dated August 9, 1935, the common-carrier truck operators and contract operators handling interstate commerce are subject to the jurisdiction of the Interstate Commerce Commission, and for the period closing with March, 1940, the Commission received reports of revenues, expenses, and statistics representing 1,106 motor carriers of property.

Table 3 shows motor-vehicle statistics for 1938, 1939, and 1940, compiled annually by Federal Works Agency, Public Roads Administration. Registered motor trucks by capacity groups as of December 31, 1939, were as shown in Table 4.

TABLE 4.—REGISTERED MOTOR TRUCKS AS OF DECEMBER 31, 1939

Capacity (Tons)		No.	%
From	To[a]		
....	>1.0	1,641,166	38.18
1.0	1.5	84,974	1.98
1.5	2.0	2,247,219	52.27
2.0	2.5	132,935	3.09
2.5	3.5	108,490	2.52
3.5	5.0	44,138	1.03
<5.0	....	40,078	0.93
Total............		4,299,000	100.00

[a] To, but not including.

Information is not readily available covering the financial status of the various trucking concerns, except through the reports to the Interstate Commerce Commission, which show that for 1939 a total of 1,105 carriers were reported as receiving gross operating revenues aggregating $425,373,099, and operating revenues of $20,663,530, divided thus: 82.5% by common carriers and 17.5% by contract carriers.

For the same year (1939), 196 motor carriers of passengers reported revenue aggregating $143,108,224, of which $123,899,080 was for passengers carried between cities and $19,209,144 was received from local or suburban passengers. The net revenue reported amounted to $20,249,003.

OIL AND GASOLINE PIPE LINES

Pipe lines serve the oil industry by: (1) Assembling crude oil at central points for storage or shipment to refineries; (2) shipments of crude oil to refineries or marine shipping terminals by trunk lines; and (3) long-distance shipment of the principal refined products from field or tidewater refineries or tank terminals. Pipe lines are classified as gathering and trunk lines. The gathering lines are usually laid on top of the ground, and range in size to about 4 in. in diameter, whereas the trunk lines are laid underground, below freezing, and are commonly 6 in. to 8 in. or more in diameter.

The pipe-line mileage of companies reporting to the Interstate Commerce Commission is equal to about 42% of the railway main-track mileage, and amounted to 98,661 miles in 1939, divided thus: 59,108 miles in trunk lines and 39,573 miles in gathering lines.

The development of pipe lines marched side by side with the development of the oil industry. The first successful line, 4 miles long, began operation in the oil fields of western Pennsylvania, in December, 1865, and, 10 years later, a 4-in. line, 60 miles long, was laid to Pittsburgh, Pa. In 1892, there were about 3,000 miles of pipe lines in service, mostly 6 in. in diameter. These connected the oil fields of western Pennsylvania with the refineries in the Eastern States. About 96% of the oil produced prior to 1900 was from fields east of the Mississippi River. During the next 30 years this relationship changed and, by 1930, 66% of the oil came from west of the Mississippi River. In that year the states of California, Arkansas, Oklahoma, and Texas, together, produced more than 86% of all the crude oil in the United States.

Between 1892 and 1914 there was great expansion in pipe-line construction, and at the end of that period five major companies operating in the mid-continent field had a total of 6,059 miles of trunk lines and 3,956 miles of gathering lines. This expansion extended into California, and five companies operating there had lines aggregating 2,439 miles in 1919.

Since 1920 the development has been part of the growth of large oil companies. The numerous oil discoveries in the two decades 1920 to 1940 have resulted in adequate oil reserves and the growth of industry in its many phases; the automotive uses for oil and its products, especially, has increased greatly the demands on the oil industry.

Until 1930, pipe lines were used mostly for transporting crude oil to refineries, and refined oil and gasoline to shipping terminals; since that time they also have been utilized in transporting gasoline from the refineries and shipping terminals to market centers. This practice now exists along the eastern seaboard and from mid-continent fields to Lake Michigan and to Ohio, upper Mississippi, and Missouri river valley points.

The transportation of oil and oil products through pipe lines has become a major industry in itself. In 1921, there was a total of 55,260 miles of gathering and trunk pipe lines operated by companies reporting to the Interstate Commerce Commission. This mileage gradually increased to 98,681 miles in 1939. During the same period, the investment in the industry increased from $365,000,000 to $829,646,000 whereas the operating revenue increased from $115,900,000 to $212,466,000. For 1939, the operating expenses were $97,130,000 and taxes were $31,936,000, leaving a net of $83,401,000. (Fig. 4 shows several hundred of the 20,000 surplus tank cars used in 1941 to alleviate the oil shortage on the East Coast due to the National Defense situation.)

The interstate pipe lines became common carriers and were placed under the jurisdiction of the Interstate Commerce Commission under the Hepburn Act of June 29, 1906, since which time the authority of the Commission has extended over this industry.

The extent to which pipe-line companies compete with the railroads and inland waterway carriers is of considerable interest. Data are not readily available as to the volume of oil products now being shipped by common and contract carriers over the inland waterway routes; however, statistics compiled by the Director of the Bureau of Statistics, Interstate Commerce Commission, and published in February, 1939, show that the railway revenue from transportation of crude petroleum and petroleum products decreased from

$287,027,000 in 1928 to $187,977,000 in 1937, whereas pipe-line revenues increased from $206,438,000 in 1928 to $240,362,000 in 1937.

It is obvious that pipe-line competition has been seriously felt by the railroads. In the past it has not been possible to compare relative cost of pipe-line transportation with railroad transportation. It is the general opinion of railway cost engineers that it is impossible to determine the exact cost of rail transportation for handling oil traffic. However, it appears that, as compared with railroads, the cost of pipe-line operations is relatively low, because of the comparatively small number of employees required. On the other hand, the pipe lines are the only form of transportation other than the railways that do not now receive annual subsidies from the government. The pipe lines never received land grants.

FIG. 4.—THE TRANSPORTATION OF OIL HAS BECOME A MAJOR INDUSTRY

AIR TRANSPORT

Aviation with heavier-than-air machines was born on the sand dunes at Kitty Hawk, N. C., on December 17, 1903, when Wilbur and Orville Wright succeeded in flying a machine equipped with planes and a small gasoline motor for a distance of 260 yd. The Wright Brothers continued their experiments, and on October 5, 1905, at Dayton, Ohio, they succeeded in making their heavier-than-air machine, powered with a gasoline motor, fly a distance of 24 miles, at a speed of 38 miles per hr. During the next 10 years many experiments were conducted in this field, and at the beginnning of the first World War belligerent nations found a practical use for flying. This period (1914 to 1917) provided a powerful stimulus to all branches of flying, and great improvements were made in technical equipment. Intensive development during this period led to constant changes not only in the practice but in the application of flying, so that, when peace came, it was expected that progress would be as rapid as in wartime. The expected progress, however, did not seem to materialize immediately, for at the beginning of 1926 many war experiences had not been fully digested, nor their lessons properly formulated.

The Aeronautics Branch of the U. S. Department of Commerce was established in 1926, and efforts were made to provide airways and aids to air navigation. The first domestic contract air-mail routes were established in that year. Since 1926, this industry has taken the best of skill and methods from older industries, and at the same time it has developed types of materials, devices, and methods best suited to its own efficiency and requirements. Today, less than four decades from the beginning, no industry is more alive to the possibility of technical advance.

The development and production of airplanes have made possible an astonishing advance in the field of transportation, which probably has no parallel in the history of any other common carrier. It was in 1929 that scheduled air transport pioneers first began to be operative. Once each day a small single-engine airplane took off from the Newark airport (which had been recently reclaimed from the New Jersey marshes) and made its way toward the Pacific Coast. It carried an occasional passenger and some mail. Today the national air map reveals a great network of air lines, reaching from the Atlantic to the Pacific, and from the Great Lakes to the Gulf of Mexico. These lines normally fly almost twice as many miles as do the combined air lines of Great Britain, France, Germany, Italy, and the Netherlands; they transport more passengers and more mail, and they surpass the air lines of any other nation in comfort, speed, and safety. Within the same brief period of a little more than one decade, the pioneers of air commerce have linked the United States in scheduled air transportation with Hongkong, China, a distance of 8,746 miles across the Pacific Ocean; with Bermuda, 784 miles from New York in the Atlantic Ocean; and with Central and South America, 7,318 miles from Miami, Fla., to Buenos Aires, Argentina. Since July, 1939, the United States has established transatlantic scheduled air transportation with Great Britain, Ireland, France, and Portugal, 3,437 miles. This service, however, has been interrupted in so far as the belligerent nations are concerned. A measure of the progress of aviation is afforded by reference to Table 5, which contains selected statistical information.[3] It shows the progress of civil aeronautics in the United States by calendar years, beginning with 1926.

TABLE 5.—PROGRESS OF AERONAUTICS IN THE UNITED STATES

Year	Total length of airways (miles)	Total Freightage (Lb)		Total Miles Flown		Passengers carried
		Express	Mail	(Revenue)	Passenger	
1929	36,330	257,443	7,772,014	25,141,499	103,747,249[b]	173,405
1940	94,079	14,188,178	32,700,000[a]	119,517,263	1,265,164,059	3,185,278
Increase (%)	159	5,411	321	375	1,119	1,737

[a] Estimation, based on ton-mileage and tons handled in 1935. [b] Data for 1930.

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[3] "Progress of Civil Aeronautics in the United States," U. S. Dept. of Commerce, Civil Aeronautics Administration, Washington, D. C.

Table 6 shows the financial results[4] for all domestic air carriers for the fiscal year 1939 compared with 1935.

Up-to-date information is not readily available as to the private and public investment in the air transport industry. However, the Bureau of Statistics, Interstate Commerce Commission, reports a total investment for airways and aircraft of $171,000,000 in 1937.

TABLE 6.—OPERATING REVENUES AND EXPENSES OF DOMESTIC AIR CARRIERS

No.	Description	1935	1939	Increase (%)
	Operating Revenues:			
1	Other than United States mail.....	$14,128,249	$30,650,695	117
2	United States mail................	8,814,296	16,669,197	89
3	Total........................	$22,942,545	$47,319,892	106
4	Operating expenses................	$26,264,961	$46,326,581	76
5	Profit (P) or Loss (L)..............	$ 3,322,417 (L)	$ 993,311 (P)	...

The government aids airway carriers by: (1) Establishment and maintenance of air navigation facilities, including construction and maintenance of intermediate landing fields, radio broadcasting stations, radio range beacon stations, radio landing beacons and landing fields, and radio lighting facilities; (2) weather services provided by the U. S. Weather Bureau, U. S. Department of Agriculture; and (3) substantial air-mail contracts. It has been estimated that the sum total of all public aids to this transport agency on a cumulative basis to the present time is more than $400,000,000.

INVESTMENT AND TRAFFIC

The foregoing outlines briefly the history of the transportation agencies serving the people of the United States, and demonstrates that there is a large investment in each. It has been impossible to obtain comparable information on the subject, because of the different methods that have been used by the various agencies in accounting for maintenance and investment. However, the total investment in intercity transportation facilities as compiled by the Bureau of Statistics, Interstate Commerce Commission,[5] as of 1937, was $53,215,000,000 as shown in Table 7. This includes steam railways (including express and sleeping car companies), electric railways, highways and vehicles, waterways and equipment, pipe lines, and airways and aircraft.

TABLE 7.—INVESTMENT IN INTERCITY TRANSPORTATION FACILITIES IN THE UNITED STATES (1937)

Facility	Million dollars	Percentage
Steam railways, express, and sleeping car companies..............	23,706	44.6
Electric railways..................	379	0.7
Highways and vehicles..............	24,232	45.5
Waterways and equipment...........	3,782	7.1
Pipe lines..........................	945	1.8
Airways and aircraft................	171	0.3
Total........................	53,215	100.0

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[4] "First Annual Report," Civil Aeronautics Authority, 1939.

[5] Report of Bureau of Statistics, Interstate Commerce Commission, submitted to House Committee on Interstate and Foreign Commerce, January 24, 1939; hearings on H. R. Doc. No. 2531, Vol. 1, pp. 24–25.

The volume of traffic, measured by revenue ton-miles and passenger-miles, handled by the various agencies is shown in Tables 8(*a*) and 8(*b*) for 1939 as compared with 1926. The data in these tables were compiled from the report of the "Committee of Six" appointed in 1938 by the President of the United States to study the transportation situation, corrected to date.

Study No. 3951, made by the Bureau of Statistics, Interstate Commerce Commission, showing the fluctuation in carload freight traffic compared with production for 1928 to 1938 inclusive, shows that the ratio of the actual railroad tons handled to the potential railroad tons declined progressively each year, with the exception of 1934, when the percentage was one point up, until only 78.7% of the potential tonnage was handled in 1938, compared with 1928.

TABLE 8.—COMPARISON OF TRAFFIC AND REV

No.	Facility	(*a*) REVENUE TON-MILES				
		1926		1939		Change[a]
		Millions	%	Millions	%	
	(1)	(2)	(3)	(4)	(5)	(6)
1	Steam railways	447,444	75.4	335,375	61.7	25.0 D
2	Electric railways	1,313	0.2	725	0.1	44.8 D
3	Intercity motor carriers	23,530	3.9	46,000	8.5	95.5 I
4	Great Lakes	90,037	15.2	76,312	14.0	15.2 D
5	Other inland waterways	9,543	1.6	19,937	3.7	108.9 I
6	Pipe lines	21,700	3.7	65,015	12.0	199.6 I
7	Airways	....	....	11	....	....
8	Total	593,567	100.0	543,375	100.0	8.5 D

[a] D = decrease and I = increase. [b] Intercity buses. [c] Includes Pullman and express companies estimated estimated for 1926.

The revenue of commercial carriers for 1939 compared with 1926 is shown in Table 8(*c*). From these tables, it will be seen that the economic position of the steam railways has been declining, and that there has been a very substantial increase in the business of the inland water, highway, and air carriers.

The changes in the transportation "picture" are very significant. What have been the causes resulting in the shift of passengers and freight from the railways to the highways, waterways, and airways? In the broad sense, they are entirely economic. Passengers and shippers use the transportation agencies offering the most satisfactory service at the least direct expense to them. The expense factor being equal, the service factor will govern. The real question before the people, therefore, will be to determine whether or not the free laws of economics govern, and, if not, to what extent competing carriers should be controlled by public authority. It will then be up to Congress to find a remedy to correct the evils and protect fairly and equitably the interests of all concerned.

Taxes Paid by Transportation Agencies

In 1939 the railways paid 8.9¢ of each dollar of revenue for taxes. This is an increase of 78% over 1921, and the trend is still upward. In addition, the railways bear the expense of owning, maintaining, and protecting their properties, amounting to 24¢, which sum, plus taxes, makes a total of 33¢ per dollar of revenue.

The pipe lines reporting to the Interstate Commerce Commission for 1939 had similar expenses, aggregating 15¢ for taxes and 23¢ for owning and maintaining their properties, making a total of 38¢ per dollar of revenue, which compares with the 33¢ borne by the railways.

ENUE IN THE UNITED STATES, IN 1926 AND 1939

(b) Revenue Passenger-Miles					(c) Revenue of Commercial Carriers				
1926		1939		Change[a]	1926		1939		Change[a]
Millions	%	Millions	%		Millions	%	Millions	%	
(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
35,673	75.2	22,713	56.8	36.3 D	$6,701[e]	83.0	$4,140	67.1	38.2 D
5,537	11.7	956	2.4	82.7 D	....[d]	....	....	....	....
4,375[b]	9.2	14,147	35.4	223.4 I	475[e]	5.9	1,048	17.0	120.6 I
1,849	3.9	1,486	3.7	19.6 D	} 721[e]	8.9	708	11.5	1.8 D
....	....	....	....	....	173	2.2	213	3.5	23.1 I
....	....	678	1.7	 I	....	....	56	0.9	....
47,434	100.0	39,980	100.0	15.7 D	$8,070	100.0	$6,165	100.0	23.6 D

for 1926. [d] Comparable data for electric railways not available. [e] Waterway operation and motor carriers

Other transport agencies have somewhat lower taxes in relation to their revenue, due principally to the fact that they do not own and maintain the right of ways or roadbed over which they operate and no taxes accrue against them on that account.

All water carriers reporting to the Interstate Commerce Commission for 1939 paid 2.7¢, and carriers on the Mississippi River System paid 2.5¢ per dollar of revenue in taxes.

Motor truck and bus companies reporting to the Interstate Commerce Commission for 1939 show a tax payment of 7.6¢ per dollar of revenue. The taxes paid by motor truck and bus lines include gasoline taxes and license fees, a large portion of which, plus other tax funds, is used for constructing, maintaining, and protecting the highways over which the carriers operate. The government owns, maintains, and protects the waterways and highways for the benefit of the users, free of charge to them and at public expense.

No portion of the taxes paid by the railways and the pipe lines is used by the government for constructing and maintaining railways and pipe lines. These taxes are used for the normal functions of government, including federal, state, county, and various municipalities. Table 9 shows a comparison of the revenue received and taxes paid by carriers reporting to the Interstate Commerce Commission for 1939.

REGULATION OF TRANSPORTATION AGENCIES

Transportation problems (largely economic) growing out of the increased activity of government, in creating and improving facilities for transportation by water, highway, and air, are becoming more complex. The growth of these various transportation agencies has been sketched briefly in this paper. On inland waterways, improved and maintained at government expense, are barges owned and operated by the government, by common-carrier operators, by contract carriers, and by private individuals or companies. On the coastal waterways, connected by canals constructed and operated by the government, are carriers whose operations are conducted under the fostering care of the government. On the public highways, built partly by gasoline taxes, are carriers operating as common carriers and under contract, and also vehicles operated solely for the use and convenience of their owners. The commercial airplane relies upon the landing facilities generally provided by public authority, and depends for its air operations upon facilities generally provided and maintained by the government.

TABLE 9.—REVENUE AND TAXES, 1939

No.	Description	Group A			Group B			Mississippi River system
		Railways	Pipe lines	Total	Highways	Waterways[b]	Total	
1	Revenue	$3,995,004,000	$212,466,000	$4,207,470,000	$568,481,000	$110,551,000	$679,032,000	$11,744,000
2	Taxes	355,678,000	31,936,000	387,614,000	43,298,000	2,931,000	46,229,000	289,000
3	Tax ratio[a]	0.089	0.150	0.092	0.076	0.027	0.068	0.025

[a] Taxes in dollars per dollar of revenue (item 2÷item 1).

[b] All groups of waterways; includes Atlantic and Gulf coasts, Great Lakes, Mississippi River and its tributaries, and Pacific Coast carriers.

One characteristic in common is applicable to all of these agencies of transportation—their capital is mobile. If their operations prove unprofitable, they can readily be shifted to other scenes, or they can go out of business. This does not apply to the railroads, with their permanent ways and structures; nor does it apply to the pipe lines.

Until recently the problem has been further complicated by the inconsistencies of government policies. There is the related problem of adjustments in the relationships of government to a variety of interests among the several types of carriers, and the all-exclusive groups of taxpayers and of consumers whose welfare it is the duty of the government to conserve.

It has been said that the railways are the backbone of the national transportation system, but this does not deny that other forms of transportation are essential, and in fact indispensable, to the national economy and defense. Certain fundamental problems might be stated thus: (1) To define the field in which each form of transport is superior to the other for service, efficiency, and economy; (2) to determine how services of each form operating in its proper sphere should be coordinated with the services of the other; and (3) to define the extent to which government should impose uniform regulations on all forms of transport, and special regulations, peculiar to its character, on any one form. An incidental question is whether government should undertake by regulation to confine each form of transport to its proper economic sphere or, by permitting unrestricted competition, allow each to find the field in which it can survive. Each form of transport has occupied as much of the transportation field as it could under the conditions existing today. Their services are already coordinated to a considerable degree. Government, both federal and state, has imposed regulations to a very great extent, but not equally upon the different forms of transport.

The regulation of transportation has existed in one form or another from the early time when ways were constructed by the aids of collective effort. In the United States it was not until February 4, 1887, that agreement could be reached in Congress upon a measure that purported to "regulate commerce." In effect, this first act was little more than a declaration of the intention to enter the field of railroad regulation. It created the Interstate Commerce Commission as an agency of enforcement. Since this humble beginning, the authority and powers of the Interstate Commerce Commission have been expanded gradually, until now they are very broad and affect, in varying degrees, the regulation of all forms of transportation, except airways. It was known as the Interstate Commerce Act, and it has been amended many times. Without commenting upon the changes and amendments that have been made in connection with the legislation regulating commerce, perhaps one of the most important is the Transportation Act, approved September 18, 1940. Certain provisions of this Act bear specifically on the problems affecting all transport agencies, and are of vital importance to the national economy. Quoting Title I, Section 1, under the heading, "National Transportation Policy":

> "It is hereby declared to be the national transportation policy of the Congress to provide for fair and impartial regulation of all modes of transportation subject to the provisions of this Act, so administered as to recognize and preserve the inherent advantages of each; to promote safe, adequate, economical, and efficient service and foster sound economic conditions in transportation and among the several carriers; to encourage the establishment and maintenance of reasonable charges for transportation services, without unjust discriminations, undue preferences or advantages, or unfair or destructive competitive practices; to cooperate with the several States and the duly authorized officials thereof; and to encourage fair wages and equitable working conditions;—all to the end of developing, coordinating, and preserving a national transportation system by water, highway, and rail, as well as other means, adequate to meet the needs of the commerce of the United States, of the Postal Service, and of the national defense. All of the provisions of this Act shall be administered and enforced with a view to carrying out the above declaration of policy."

This is the first time the Congress has recognized the necessity of dealing with the various transportation agencies collectively, instead of separately.

There has been no comprehensive view or proper consideration of the interrelation between the various agencies of transportation that are now competing for the same traffic throughout the United States.

For the first time in American history, subject to the provisions of the Act, the Congress is requesting a fair and impartial regulation of all agencies of transportation. The writer mentions this because airways are definitely omitted from the Act. The airways are kept separate from the others. This is a mistake because it is impossible to deal fairly with a single transportation agency without a broader view.

Another provision, under Title III, Section 301, relates to the establishment of a Board of Investigation and Research, to be composed of three persons, who shall be appointed by the President, for the purpose of investigating and reporting upon the following:

> "(1) The relative economy and fitness of carriers by railroad, motor carriers, and water carriers for transportation service, or any particular classes or descriptions thereof, with the view of determining the service for which each type of carrier is especially fitted or unfitted; the methods by which each type can and should be developed so that there may be provided a national transportation system adequate to meet the needs of the commerce of the United States, of the Postal Service, and of the national defense;
>
> "(2) The extent to which right-of-way or other transportation facilities and special services have been or are provided from public funds for the use, within the territorial limits of the continental United States, of each of the three types of carriers without adequate compensation, direct or indirect, therefor, and the extent to which such carriers have been or are aided by donations of public property, payments from public funds in excess of adequate compensation for services rendered in return therefor, or extensions of Government credit; and
>
> "(3) The extent to which taxes are imposed upon such carriers by the United States, and the several States, and by other agencies of government, including county, municipal, district, and local agencies."

The national transportation policy and the provisions of the Act to make such a study as outlined are definitely related. The policy is to be "so administered as to recognize and preserve the inherent advantages of each." What is meant by "inherent advantages"? Would not the words "relative usefulness" be more appropriate? The ability to serve all of the transportation needs of the Nation in all seasons and under all conditions should be an important consideration. The President's Board should be very helpful in determining this policy.

Pipe-line transportation is the only one that is highly specialized in the United States and, of course, no one claims that it is capable of supplanting other agencies for full community needs. It is hoped that in administering the policy, a "yardstick" will be applied that will recognize these facts.

The national policy is to promote "safe, adequate, economical, and efficient service and foster sound economic conditions in transportation and among the several carriers." The terms "safe, adequate, economical, and efficient" require no comment. However, the meaning of "sound economic conditions" may be vague when applied to the several transportation agencies. Is it the intent of the Congress that all agencies of commercial transportation should be self-supporting, or does "sound economic conditions" imply a continued government aid to some forms of transportation?

It would appear that, under Title III, Section 301, of the Act, the President's Board will be expected to weigh all of the factors affecting the economies of the railway, highway, and waterway carriers; it will consider all costs of transportation—those borne by the shippers and those borne by the taxpayers.

The national transportation policy is "to encourage the establishment and maintenance of reasonable charges for transportation services, without unjust discriminations, undue preferences or advantages, or unfair or destructive competitive practices." If fair and impartial application is made of these principles, many of the important factors that are effecting chaos will be eliminated.

The declaration, "all to the end of developing, coordinating, and preserving a national transportation system by water, highway, and rail, as well as other means, adequate to meet the needs of the commerce of the United States, of the Postal Service, and of the national defense" is of interest. It is the opinion of many that there are too many transportation facilities. The national transportation burden is now at its height, for that reason. The three agencies mentioned are more than adequate to meet the needs of commerce, and the government continues to develop more facilities by water, highway, and air. What about national defense? The railway is indispensable, as has been testified by competent Army officers. If additional facilities are necessary for this purpose, a much greater capacity can be developed for the least expenditure, in rail facilities, as compared with waterway and highway expenditures. The Army engineers in charge of waterway investments, from the beginning, have given little consideration, if any, to existing facilities, when contemplating the extension of waterway facilities. The economic justification rests upon diverting traffic from existing facilities to the waterways. Coordination cannot come out of such a policy. No doubt the President's Board will weigh very carefully all of the many controversial factors affecting this problem.

TRANSPORTATION COSTS

This paper will not be complete without a brief discussion of transportation costs as may be considered by the President's Board of Investigation and Research. Among other things, it will be authorized to ascertain, if possible, "The relative economy and fitness of carriers by railroad, motor carriers, and water carriers for transportation service, or any particular classes or descriptions thereof, with the view of determining the service for which each type of carrier is especially fitted or unfitted." One of the most important factors to be considered by the Board will be the actual cost of producing transportation by these types of carriers. This will be most difficult, if not impossible, to accomplish. Railway cost engineers and the Statistical Division of the Interstate Commerce

Commission have had this question before them for many years, as it applies to the railroads, and in recent years to the highways, and up to now (1941) the results have not been comforting. There may be more than one method of approach to this problem. The one that has been so complex and difficult is to determine what it costs to transport persons and property (classified as to commodities) by the various transport agencies without regard to the amounts paid by the users. This method meets with very serious handicaps, because of the impossibility of determining the costs as applied to individual persons and various kinds of property. This would be particularly ~~true in~~ connection with railway, waterway, highway, and airway transportation. Oil and gasoline constitute the major part of property transported in pipe lines in a fluid form, which, of course, simplifies the problem of cost finding. With the railways, waterways, highways, and airways, all of which transport many classes of commodities, the problem has not been solved. It naturally follows that, if this method of approach is used, it will be necessary to determine costs applicable to passenger, mail, and express; costs of l. c. l. (less than carload-lots) packages of all sizes and weights, and freight of all kinds moving in carloads, for long hauls and for short hauls (all mixed up in long trains and short trains, in large or small motor vehicles, steamboats and barge tows, and air transports) and for varying operating conditions as may obtain throughout the United States.

All transportation agencies are competing for business by means of a pricing system and service offering that have resulted in dividing the potential traffic among the various agencies. Records are available of the revenues and expenses of all common carriers reporting to public authority. The differential between the total revenues and total expenses of the carriers is so narrow that the price paid by the shipper might properly be considered as the cost of transportation.

The amount of government aid, particularly in connection with the highway carriers, is not easy to determine. A number of studies have already been made for the purpose of finding out if the commercial users of the highways are paying their full share. The highway carriers claim that the various tax levies plus gasoline taxes more than compensate for the use of the highways. The railroads do not agree. When the total of public aid contributions is added to the total amount of the carriers' expenses, or the total amount paid the carriers by the shippers, it should not be difficult to determine the relative total economy of the various types of transport agencies.

With accurate data as to the amount of government aid affecting transportation, it would then become a matter of public policy as to whether or not such aid should be continued or withdrawn. If all forms of government aid were withdrawn from commercial transportation agencies, economic laws would soon point out which form of transportation was the most economical and satisfactory to the users. The elimination of government aids, except for experimental, promotional, and defense purposes, would solve the problem now so vitally affecting national transportation. Therefore, it seems logical that the first thing the President's Board should do is to develop, very carefully, the amounts of government aid now reflected in the cost of commercial transportation in all of its forms. Much has been accomplished already in this respect.

Joseph B. Eastman, when federal coordinator, evidently had in mind the importance of the question of subsidy, or government aid, to the various transportation agencies when he undertook to prepare the four large volumes of "Public Aids to Transportation," which were begun, through his Research Section, in 1933–1936, and later completed and issued by the chairman of the Interstate Commerce Commission in 1940. This report contains a great wealth of valuable information, together with comments and conclusions.

In October, 1940, J. J. Pelley, President of the Association of American Railroads, released a reply entitled "What Is Public Aid to Transportation?" (produced and distributed by the Association of American Railroads), which is an analysis of the data contained in Mr. Eastman's report, and points to inconsistencies that render the conclusions incomparable, confusing, and misleading.

Here are the results of two studies that have a direct bearing on the future work of the President's Board of Investigation and Research, and it will be well for all who are interested to study in great detail the information contained in both of these reports.

ACKNOWLEDGMENTS

Credit is due the Association of American Railroads; Brookings Institute ("The American Transportation Problem"); Federal Works Agency, Public Roads Administration; Interstate Commerce Commission; Office of Chief of Engineers, U.S. Army; U. S. Department of Commerce, Civil Aeronautics Administration; and others for information contained in this paper. Fig. 1 was supplied by the Southern Pacific Railway Company, and Fig. 4 by the Pennsylvania Railroad Company.

DISCUSSION

Fred Lavis,[6] **M. Am. Soc. C. E.**—The author merits the thanks of all those interested in problems of transportation for having presented such a comprehensive and statistically well-fortified review of transportation developments during the last century, as well as a broad picture of the internal transportation system of the United States as it now exists.

Too many people have only a narrow and meager comprehension of the entire problem which, as the author states in the "Synopsis," is one "so vital and so far-reaching that a sound national economy may depend on its solution."

The author has painted a fair and unbiased picture of the entire situation. He only alludes very briefly, however, to the vast number and great importance of the many indeterminate factors that prevent any reasonably accurate determination of relative costs of transportation by most of the facilities in use today.

What is the value of the property used by the railways? How can such value be determined? What are the costs of highways—original cost, reconstruction, and maintenance? How are the costs of providing and maintaining highways to be distributed among, through long-distance traffic, local freight traffic, and ordinary automobiles used for business, for pleasure, and for access to property?

What are the effects of curvature, varying rates of gradient, rise and fall, distance, smoothness and width of pavement, separation of lanes of traffic, separation of cross traffic, etc? Should drivers of motor vehicles who benefit by improvements pay a special tax for them?

What part of highway costs may be chargeable to military use?

How do railways or highways or even waterways affect the values of adjacent, or more or less near-by, properties? To whom do accretions of value belong? For instance, on a highway like Route 25 in New Jersey through Jersey City, Newark, and Elizabeth, how much of its cost or value should be allocated to traffic between the remainder of the United States and New York, and how much to the State of New Jersey? How much are the cities through which it passes benefited by having this traffic taken off their streets?

What proportion of the costs of the Westchester parkways should be borne by the county, the state, and the City of New York? Should the traffic which uses these parkways, and thereby saves time, tires, and gasoline, pay a proportionate part of their cost? A large book could be written about these and other questions and difficulties of determining costs of transportation.

The associations of highway builders and users have proved to their own satisfaction that all highway costs are covered by taxes paid by their users. The railways have proved equally conclusively (or with equal ignorance) that this is not so. The truth is that neither knows.

Table 7, as given in the paper, might easily be misinterpreted either because of lack of thought or wilfully. There, it is shown that about 90% of the total investment in intercity transportation facilities in the United States is for railways and highways, being divided about equally between the two. The point to be noted, however, is that the railway plant is purely a transportation machine whereas the highways are in a fairly large degree ways of access to lands and, in not a few instances, adjuncts of recreation.

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[6] Cons. Engr., New York, N. Y.

It should be emphasized also that railways perform a service in moving bulk freight and large numbers of passengers that cannot be moved by any other known means of transportation, and one without which the United States could not exist as a nation even in times of peace. For war, of course, their value is inestimable.

The author points out one step that may be taken—namely, the determination, with some approach to accuracy, of the amount of government aid affecting the provision of transport facilities and costs of transportation. That very likely may be of value.

He suggests (see heading "Transportation Costs") that "If all forms of government aid were withdrawn from commercial transportation agencies, economic laws would soon point out which form of transportation was the most economical and satisfactory to the users." In theory this is right. In practice it is not possible.

Water transportation is mixed up with flood control. Air transport has been a development of incalculable benefit to the United States and to the world, if it is conceded that the world, and transportation in the world, must advance. The construction of highways is strictly a governmental function—national, state, county, and municipal. Highways must be built with public funds.

There remain the railways, ground between the millstones of demands of organized labor for higher wages, regulation of rates by the government and, at least to some extent, competition of subsidized highway and water transportation.

The President's Board may be able to arrive at a fair determination of costs by the various different forms of transportation but this is doubtful.

The war has shown, and will show more conclusively, the absolute need of both railways and highways in the United States. The highways will be taken care of. The users, and that means the public generally, will see to that. The railways are a different problem.

It is the suggestion of the writer that the President's Board concentrate on a problem which can be solved—the maintenance in the United States, not of a governmentally owned railway system but, of a railway system that will provide efficient service at the lowest cost consistent with financial integrity.

WILLIAM J. WILGUS,[7] HON. M. AM. SOC. C. E.—This timely and informative paper on the Nation's transportation problem comes to fill a need that has never before been so pressing. The array of pertinent facts and data is impressive. To them, however, it would seem well to add others essential to the making of a sound diagnosis, as does the physician in searching for all the symptoms of a bodily ailment before announcing its causes and the measures, though painful, then to be taken for its cure. The age-old Delphic admonition, know thyself, is to be heeded in this if genuine success is to attend the effort.

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[7] Weathersfield, Ascutney, Vt.

The physical and financial aspects of the problem are not alone to be considered. There is also a psychological aspect that plays an enormous part in the interplay of forces bearing on the movement of persons and property from place to place. Man, by his very nature, prefers to do things for himself when thereby he may gain convenience, time, pleasure, and independence, as in the use of the motorcar. Then, too, there is his inborn urge to escape from what he deems to be the exactions of corporations dominated by unconscionable "high finance," so recently revealed at Washington.[8] Again there is his feeling that his products often may be moved to markets by highway and waterway at rates he can afford to pay, when they would remain immobile were there no such escape from shipment by rail at higher rates. In this attitude he is ignoring the countervailing fact that he must look to the railroad for never-failing service when its competitors are locked in ice and snow. Then he shrinks both from the subsidizing of tax-burdened railroads, controlled as they are by the "high finance" he fears, and the withdrawal of aids to their rivals with resulting higher rates to be borne by him, even though he recognizes the unfairness of favoring one and not the other. Finally, there is his conviction that, were it not for competition offered him by highway, waterway, airway, and pipe line, he would be powerless to secure by rail the rates and service he now enjoys, or is striving to obtain. The argument that through taxation he is paying, indirectly, additional sums for these advantages is to him an abstraction that "leaves him cold," as does the other argument, well founded though it is, that the increasing diversion of his traffic from the railroads to their subsidized competitors will in the end land the railroads, exhausted, in his lap. These are human attitudes that cannot be "poohpoohed." They have had a tremendous effect on the development of transportation and will so continue. They are among the causes that must be given weight.

From the physical and financial angle the situation is no less difficult. The railroads—backbone of the Nation's transportation—are looked upon statistically as a national system, as they are in a physical sense. In this view the public is prone to be led astray. Though interconnected, with traffic flowing over them uninterruptedly to and from all parts of the United States, they are in truth separately owned and controlled by a multitude of corporations having diverse interests and varying in their financial status from the wofully weak to the excessively strong. Collective statistics for all of them are not applicable individually to their widely differing members. Their lack of managerial unity may be compared with the original thirteen states before they were federalized in 1789. They are without a compelling head to direct their efforts impartially for the common good—all for one and one for all—the strong to assist the weak, each being equally necessary for the public good. In the absence of unity their development as a whole has been checked, while their rivals are advancing. Highways are being extended and improved on a vast scale; waterways, often unworthy products of the malodorous "pork barrel," are "on the march"; and airways are in the picture with promise of great things to come. As to pipe lines, sight should not be lost of those devoted to the movement of natural gas and powdered substances in fluid suspension, as well as oil and gasoline mentioned by the author; nor should the effect of electric transmission lines on the movement of rail-borne coal be ignored.

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[8] Senate Rept. No. 25, 76th Cong., 1st and 3d Sessions, Pts. 1 to 27, inclusive, and Senate Rept. No. 26, 77th Cong., 1st Session, Pts. 1 and 2.

In considering the development of transportation affecting the redistribution and relocation of industry, internal and foreign trade, and the Nation's safety in time of war, all of these interrelated means of movement—railroads, highways, waterways, airways, pipe lines, and transmission lines—should, of course, be given heed. The railroads, in particular, should be given consideration in their straightened circumstances.

A glance at the development of Class I railroads will disclose that their property investment from 1916 to 1940 (each of these years preceding the entry of the United States into war) increased from 17.6 billion dollars to 26.3 billion dollars, or nearly 50%, and the stockholders' equity in the hands of the public, in excess of the funded debt, from, say, 8 billion dollars to 16 billion dollars, or 100%. Corresponding to this, net income available for dividends and surplus decreased from 647 million dollars to 189 million dollars, or 71%, while cash actually disbursed in the form of dividends to stockholders with which to buy their bread and butter decreased from 306 million dollars, or 4.4%, to 159 million dollars, or 2%. On their equity, including the additional 8 billion dollars of "venture capital" which they had "plowed back" and otherwise put into their properties, their dividends dropped from something like 4% to a little more than 1%. Even these meager dividends were not earned in the 9 years from 1931 to 1939, inclusive, during which the deficits thus incurred aggregated something like a billion dollars.

The results of this unhappy picture are reflected in the money market where the railroads' aggregate property investment in excess of 26 billion dollars, represented by a net capitalization outstanding in the hands of the public totaling 17.6 billion dollars, has a quoted value of the order of, say, 10 billion dollars. In the face of this decay in values it is easy to understand why pre-depression gross capital expenditures on Class I railroads, averaging upwards of 800 million dollars per annum, dropped to an annual average of 276 million dollars during the 10-yr period from 1931 to 1940, inclusive. The plant as a whole, therefore, could not hold its own in a sharply competitive field in which a billion dollars of new railroad investment annually, according to competent authority, was deemed necessary for that purpose. Between 1916 and 1940 total railway line mileage has retreated from 254,037 to 233,670. In the service of Class I railways, locomotives have decreased in number from 61,332 to 41,721, although their tractive effort has remained the same; freight cars from 2,253,233 to 1,653,663, and in tons capacity from 92,280,335 to 82,722,361; and passenger cars from 52,179 to 37,817. From 1930 to 1940 maintenance was neglected, perforce, and modernizing was put aside—other than on what may be termed the luxury routes. The much more extensive mileage of lesser main lines and branches patronized by an inarticulate public has borne the brunt of deprivation.

Despite competent management as manifested by a remarkable growth in the efficiency of operation, the development of railroad transportation therefore has not been upward; nor are the results of the present war-time boom to be taken as the mark of a permanent change for the better. In the words of Ernest E. Norris, president of the Southern Railway Company:[9]

> "We can foresee the thin trickle of traffic that will flow from post-war readjustment. We can divine the savage competition that the railroads will face for every ton of that traffic. And, without stretching our imagination very far, we can visualize the devastating effects of these two certainties. We can see revenues declining; expenses frozen; profit a dimming memory; capital stoutly—and rightly—refusing to invest in an industry faced with government-nursed competition."

In this it should not be forgotten that, although the owner has received a constantly lessening return on his vastly enlarged investment, the other interests involved (managements, including the interests that control them; labor; shippers; and the general public) have had increasing rewards. As has been said by an authority in the railroad field:[10]

> "Railway management cannot truly boast it is really doing a good job as long as only about one-sixth as much net income is being earned for railway stockholders as was earned for them in the decade before the present depression."

As the evidence clearly points to the impossibility of railway management doing the "good job" with no "prince" in sight to rescue its "Cinderella," the conclusion is inescapable that the railroads in their entirety cannot finance the necessities essential to their continuance as a quasi-public enterprise in private hands. In the words of the foregoing authority:[11]

> "If the United States should begin to climb out of the slough of depression and unemployment in which it has so long reposed, large additions to railroad plant and equipment would be needed, and 2 per cent isn't enough earnings to attract adequate capital. The railroads need a new program to supplement their 'ready' theme; and one which will hold out some hope to railway owners. Why should these latter wish to avoid government ownership, if private ownership continues indefinitely to give them nothing."

Now war is here, with the Nation's transportation situation in what may prove to be "a tight corner." The failure of the railroads to obtain materials constitutes "the greatest and most imminent danger that threatens the provision of adequate and efficient railroad service during the remainder of the present emergency." The President's Board of Investigation and Research, referred to by the author, has been injected into an interterritorial rate controversy quite foreign to its purpose.[12] An unhealthy increase in passenger and freight rates is imminent. Demands for trained personnel, materials, and equipment on an enormous scale are in sight for service overseas, in respect of which the United States is totally unprepared.

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[9] "Railroad Data" from the Eastern Railroad Presidents Conference, November 21, 1941.

[10] Editorial in the *Railway Age*, July 20, 1940.

[11] "The Week at a Glance," *ibid.*

[12] "The Week at a Glance," *ibid.*, January 10, 1942.

The writer, under the circumstances, sees no reason to change the views he expressed in 1939.[13] They point to a way in which, as he believes, the development of railroad transportation may be checked in its dangerous downward course and moved upward as befits the times. The day has gone by for more investigations, further reports, and lengthy studies when Fate is knocking at the door. Action is what is needed, and that quickly. Much responsibility for this action lies in the hands of the newly appointed Director of Defense Transportation.

Joseph B. Eastman,[14] Esq.—Some of the problems treated by Mr. Teal are not quite as simple as they seem. For example, when it comes to determining the relative costs of the different forms of transportation, an over-all picture is afforded by comparing the rates that the public pays and adding something for what one may think is the public aid received; but, after all, the important point is to determine in what particular places each form of transportation can serve best—what particular kinds of hauling they can do better than some other kind of carrier. Now that involves quite a detailed study. There are not many short-cut methods that will yield that answer. It requires a long and very careful study, and the great difficulty is that the picture is changing all the time. It is not static, and that is one of the great difficulties that the President's new Board of Investigation and Research will encounter. That part of the Transportation Act of 1940 was really taken from a bill drawn by the writer in 1938 for the creation of a national transportation authority (Federal Transportation Authority), and at that time he had in mind a permanent agency that would be studying these questions all of the time—not necessarily making big reports on them, but applying pressure here and there, exerting influence, and offering advice where it is thought it would do the most good.

As a purely temporary organization, directed to make a comprehensive study of this question of relative costs, the President's Board will encounter some difficulties before it completes its work. Public aid to transportation is a problem that could be discussed all day and all night. The writer has produced four volumes on that subject. Railroad administrators probably do not disagree with the writer so much when it comes to the water carriers. Probably they do not disagree with him greatly about the air carriers or the railroads. They do disagree most emphatically with respect to the highway carriers, and there are many questions which are not only questions of fact but questions of public policy.

The question of whether it is possible to reduce the number of trackage miles, for example, between Chicago, Ill., and Cincinnati, Ohio, is a very large one. There is no doubt whatever that if the railroad systems were operated as a unit and the traffic were directed over the best routes between the important points, considerable money could be saved. There are some objections to that possibility also. The four lines of railroad between Cincinnati and Chicago do not follow precisely the same routes. There are many places on each of those lines that are not served by the others, and people at those points, of course, would not appreciate it if their service were abandoned.

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[13] "Nationalization of the Railroads?" by William J. Wilgus, *Civil Engineering*, October, 1939, p. 593.

[14] Director, Office of Defense Transportation, Washington, D. C.

An elaborate study of this problem was made under the so-called "Prince plan," which would combine the railroads into seven systems—two in the East, two in the South, and three in the West—and change the routing of traffic, concentrating it in various ways to obtain large savings. A group of railroad engineers and traffic men investigated that plan for the writer when he was federal coordinator. Whereas authors of the plan put the savings at about $700,000,000, that committee reduced it to something like $250,000,000. There is no doubt whatever that, in connection with any such plan, a large volume of objection would arise from the public in various respects.

For example, such a plan would leave Baltimore, Md., served by one railroad system; New York, on the other hand, would be served by two; and Chicago would be served by at least four, and maybe five or six. That kind of an arrangement would cause considerable objection. People at points located on what would become secondary railway lines under such a plan as that would complain because they were removed from the main arteries of commerce. It is not a problem that can be dealt with simply without considering all the various angles. The writer feels that there are important savings that could be made through consolidation of the railroads, combined with coordination (that is, cooperative effort in places where their interests are in common, such as at terminals), and at the same time preserve and probably improve the service to the public; but as he found when he was federal coordinator, this is a very debatable subject.

J. L. Campbell,[15] M. Am. Soc. C. E.—The paper presented is a valuable historical review of the development of transportation by railways, waterways, highways, airways, and pipe lines in Continental United States.

Between the several kinds of transportation, important issues lie in matters of need and capacity, extensions or enlargements, costs, competition, regulation, and coordination of transportation services. Out of right settlements or adjustments of such issues, unified, adequate, and cooperative transportation would flow at fair costs promoting national prosperity and welfare.

Developments of transportation on the highways and the airways are in a fluid status making probable no early sound and adequate solution of the entire problem which should be assigned to a commission composed of men of best ability and experience for the task. Such solution should be translated into appropriate legislation. Meanwhile, "first aid" is necessary for the preservation of railway transportation, the major part of all transportation. Such aid lies within the discretion of the Interstate Commerce Commission.

In terms of volume, value, and indispensability, railway transportation towers above its competitors. It is the backbone of transportation to which other forms should be adjusted in equitable relations and fair competition so that the backbone shall not be broken or incapacitated in the major service it must render. As between railway transportation on one side and waterway and highway transportation on the other side, regulation and competition have been unfairly adverse to transportation by railway because transportation by waterway and highway has not paid the full cost of its service and has not been subjected to the rigors of railway regulation. Regulation should cover fairly all competitive transportation.

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[15] Chf. Engr., Northwestern Pac. R. R. (*Retired*), Oakland, Calif.

While the trend of railway revenue rates has been generally downward or *in status quo*, the trend of unit costs of labor and materials used by the railways has been upward. Since 1920, there has been growing diversion of traffic from the railways to the highways. This is reason enough for the present financial condition of the railways wherein about one third of railway mileage is now in receivership and the remainder of it is unable to earn a fair return upon investment except in periods of abnormal traffic of great volume. For normal traffic, railway revenue is generally below railway expense. This result is largely due to too much limitation of railway revenue by fiat of law and dicta of the Interstate Commerce Commission. A railway dividend is now almost a curiosity to be placed under a glass case in the Smithsonian Institution.

In volume, excellence, and relatively low cost of transportation, the American railway system is unequaled anywhere. It has avoided complete shipwreck and has continued operation as a private enterprise under rigorous regulation by improvements and efficiencies in its plant and service which have reduced cost of transportation and partly bridged the gap between revenues and expenses. It is probable that a point would be reached beyond which the cost of additional improvement and efficiency would widen rather than close the gap. It is improbable that inadequacy of revenue can be completely and permanently eliminated so long as revenue rates are frozen and unresponsive to economic conditions. Unless the rates are substantially maintained in such relation to cost of service as will assure a fair return to private initiative and enterprise, the inevitable end will be government ownership and operation. Thereby, the record indicates that cost of transportation would increase.

Commercial traffic on the highways being so directly and highly competitive with railway traffic should be closely coordinated with its competitor and be subject to the laws controlling railway transportation to the end that regulation of railways and highways as commercial transportation agencies shall be common and equitable to both. Multiplicity of transportation agencies on the highways retards rather than expedites traffic thereon. In so far as the public interest and convenience are promoted thereby, commercial highway traffic should be conducted by the railways.

The national system of highways came into being by the simple and relatively inexpensive process of covering the surface of existing wagon roads with some sort of pavement, the alinement of the roads having remarkable similarity to the meanderings of the covered wagons of the heroic days of Westward Ho. The result is a system of paved highways which, in alinement, gradients, widths, and paved surfaces, is distressingly and dangerously inadequate for the volume and speed of today's highway traffic. Only a beginning has been made on necessary nation-wide reconstruction of this system to standards indispensable to highway requirements of today and the future. This is a reconstruction job which, when done, will have cost vastly more than the original paved roads.

Transportation on the Great Lakes being comparable to transportation on the oceans is the most economical transportation within Continental United States. This is not true of the rivers and canals whereon ton-mile cost is more than on the railways. Relative to the railways, rivers and canals are few in number, limited in mileage, non-elastic in location, and incapable of serving more than a small part of the total tonnage of the country. The railways have surplus capacity more than sufficient to carry the tonnage carried on the rivers and canals. All transportation requirements could be served without the rivers and canals. That would be impossible without the railways. In general, vast expenditures for navigation on rivers and canals have been made without economic justification. Such expenditures should cease.

A vital function of river channels is to carry away the flood waters of the continent. The drainage capacity of these channels should not be diminished by navigation structures without weighty reasons therefor. This was brought vividly to mind during the great 1937 flood on the Ohio. There is some conflict between flood control and navigation on rivers.

Waterway transportation should be required to raise its service rates sufficiently to cover total cost of plant and service and yield a fair return on investment so that its competition shall be fair to its competitors who must fix their rates as described. This also applies to airways and pipe lines.

The airways are too young for final conclusions thereon.

W. W. Crosby,[16] M. Am. Soc. C. E.—In this paper the author has set forth, most interestingly, many of the facts regarding the appearance and growth of the various facilities. He states that under present conditions all are competitive and that "unfair competition" has resulted. From purely financial viewpoints many observers would have the same view; but that is not the only aspect that should be considered in attempting corrections of the existing situation.

The development of waterways had a basis of regional interest besides the economic one. The railways had a considerable basis of national interest underlying their economic one—regardless of their prostitution by "high finance." Pipe lines were based almost wholly on financial interest, although a secondary result may have been to advance regional interests. Air transport has developed from a national, if not international, interest. The interest underlying highway development has been almost wholly local or parochial.

Considering these underlying motivations, it becomes apparent that, although some of the much-needed coordination could be had beneficially by cooperation (and possibly by some regulatory action), a more fundamental remedy must be sought. Legislation against the developments of applied science is not the American way to protect the Nation; nor is it even in the financial interest. Cleaner, clearer, and broader thinking than has prevailed is needed for the national welfare as well as for National Defense.

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[16] Cons. Engr., Coronado, Calif.

All through the warp and woof of highway development have run such threads as "taking the farmer out of the mud," "all-year roads," "farm-to-market highways," "express ways for faster traffic," and "safer and more commodious highways." A very few suggestions of national interest occur. One was indicated perhaps by Congressional provisions for the national roads mentioned by the author. Possibly a second short thread may be found in the "National Highway" movement and others early in the twentieth century. Most of them broke off rather shortly.

The writer believes that careful analysis proves that even these latter were more regional at the bottom than national. The combining of local and regional highway systems to form "national highways" has been more fortuitous than otherwise, and it occurred as an afterthought rather than as an original concept for national benefit. L. I. Hewes,[17] M. Am. Soc. C. E., has stated that "The history of this strategic net [of national or defense highways] goes back to the Federal Highway Act of 1921"; but the earlier results were simply the linking up, for long-distance facility, of existing roads into a "Pershing Map" in 1922. A revision of the Pershing Map was made in 1939, and was thereafter called the "Highway Map Showing Main Traffic Routes of Military Importance." Another revision was made under date of May 15, 1941.

Then Mr. Hewes states that "These main routes are intended to connect all important foci of actual or potential importance for defense. Clearly, the strategic net is used much more by civil than by military traffic." The foregoing quotations seem to the writer to support his statement preceding them.

With the possible exception of the "Harrisburg-Pittsburgh Turnpike," the writer remembers no highway in the United States that has been originated from a conception of the national, or even of a major regional, interest. Italy and Germany did do this many years ago; but the results so far obtained in the Unites States have come merely from piecing together local roads built for local ends and from narrow conceptions of what would gratify local or small regional desires. The federal influence has not been exerted effectively otherwise, and even as late as 1941 federal agencies seem to have continuously disapproved the broader idea.

A further fact in explanation of the insufficiency of present improved highways in National Defense is that the federal road authorities have clung too long to their announced principle that "the improvement of a highway is solely justified by the savings in transportation costs." Another fact may be that state highway authorities too often have been concerned mainly with the influence of highway expenditures on votes. The writer already has emphasized[18] that this criticism is not directed against engineers as such. At times, however, they are forced to share it.

The author states (see heading "Investment and Traffic") that the causes of "The changes in the transportation 'picture'" have been "in the broad sense, * * * entirely economic"—that is, "satisfactory service at the least direct expense." Does he not go too far? The writer thinks that Mr. Eastman (quoted by the author) has showed[19] a more complete understanding when he stated "Competition between different forms of transportation * * * has become essentially a warfare * * *" that "compels attention to the needs and desires, and even the foibles, of the public which is being served."

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[17] "Better Highways for National Defense," by L. I. Hewes, *Civil Engineering*, January, 1942, p. 10 *et seq.*

[18] *Transactions*, Am. Soc. C. E., Vol. 102 (1937), p. 1115.

As the writer has contended heretofore, too great reliance has been placed in highway matters on mathematical formulas, financial returns (real or figurable), local political influences, and personal ambitions, with not enough reasonable regard for convenience, safety, and national interests. He does not believe that the durable solution of the desired coordination will be found in "transportation costs" alone. If "convenience" is to be considered, it may even be that highway competition with railway facilities will be increased. "Technocracy" has its appeal to the scientific mind; but as such its further reasoning was undermined long ago by the truths of Kipling's "Imperial Rescript."

Coordination is needed today. It is even indispensable. The author mentions this but seems to believe that the "coordination" desired is the subordination of other forms of transportation to the financial benefit of the railways. Neither the history of the railroads nor the developments of transportation science justify such a conclusion, economically or otherwise. The Transportation Act that he quotes seems to the writer to indicate a much broader view; and Mr. Eastman himself seems on the side of the writer.

Recently the writer had the occasion to mention to the National Resources Planning Board an instance of deplorable lack of coordination under his personal observation. The reply, dated September 19, 1941, was that "unfortunately, co-ordinating machinery for an over-all transportation approach (to the local problem) does not now exist in the Federal Government." To the writer's further suggestion that possibly the National Resources Planning Board might obtain the help of the Public Work Reserve Agency described by Melvin E. Scheidt,[20] M. Am. Soc. C. E., in October, 1941, the reply was that the Public Work Reserve Agency was "merely advisory to State and local governments on matters of procedure."

The appointment of a federal coordinator of transportation in the person of Mr. Eastman, Chairman of the Interstate Commerce Commission, lends encouragement to the idea that broad consideration may yet be had of national transportation problems with acceptable coordination of present and future facilities. It is to be hoped that the solutions will not be based entirely on consideration of profits to any of them.

W. B. Irwin,[21] M. Am. Soc. C. E.—A broad and comprehensive presentation of current transportation conditions is contained in this paper.

It has been characteristic of the development of the United States that the more convenient and expeditious mode of transportation supplanted the less efficient. Nothing, however, has been evolved to displace the railways in economy or efficiency for mass transportation of passengers or freight.

The trend of railway transportation is to maintain the highest standard of service warranted by traffic requirements and to abandon obsolete lines as rapidly as the requisite authority can be secured. The former has achieved marked success, as evidenced by the prompt movement of large numbers of troops and their equipment, and the carrying of the heaviest freight traffic in history during 1941. The abandonment of obsolete lines has not met with equal success chiefly because of opposition to discontinuing the operation of unprofitable branches.

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[19] "The Transportation of Tomorrow," by Joseph B. Eastman, *Civil Engineering*, January, 1942, p. 1.

[20] "The Public Work Reserve," by Melvin E. Scheidt, *ibid.*, October, 1941, Fig. 1, p. 577.

[21] Asst. to Vice-Pres., G. N. Ry., St. Paul, Minn.

In discussing "Railway Transportation" the author states "Track mileage reached a maximum of 429,883 miles in 1930, declining to 408,350 miles in 1939; thousands of miles of branch and feeder lines have become obsolete." The reduction of 21,533 miles of railroad from the peak mileage is by no means a complete measure of the mileage that has become obsolete; abandonments had begun before the peak mileage was reached in 1930 and are still continuing, with many miles of obsolete line yet in operation which, for the purpose of this discussion, are considered surplus mileage in the sense of unprofitable operation.

It is of interest to consider the reasons for surplus railroad mileage, some of which are as follows:

(*a*) Competitive building of mileage in an area requiring the service of only one railroad;

(*b*) An overbuilding of railroad mileage in areas where the productive capacity of the region failed to develop sufficient traffic to support the operation of a railroad, or where depletion of the natural resources left sparsely settled districts in the wake of the exhaustion of forest areas and mineral deposits; and

(*c*) Reduced utilization of railway service due to the introduction of other transportation agencies.

The provision of the Transportation Act of 1920, requiring the issuance of a certificate of public convenience and necessity for railway construction, virtually put an end to competitive railroad building in excess of transportation needs, but considerable of such mileage remains in unprofitable operation.

Many miles of branch lines were built in agricultural areas, providing a network of railroads within the practicable limit of team-haul. This resulted in a greater railroad mileage than could be operated profitably from the production of the region. In some areas prolonged periods of drought have so reduced the production of grain and livestock that the railroads serving them must operate at a loss. This is also true of branch lines that are required to continue serving communities engaging in limited agricultural production in regions formerly devoted to lumbering or mining operations which have been exhausted.

In productive regions, where highways permit all-weather motor-truck operation, most of the livestock is trucked to market along with a considerable quantity of other farm products; less-than-carload freight, petroleum products, and some coal are trucked from distributing centers to towns on the branch lines, leaving in general only carload shipments of livestock and grain to be handled by rail to terminal markets, and a limited quantity of inbound carload freight to be transported by rail, resulting in unprofitable operation. Improved highways and the use of the motorbus and private automobile have caused local passenger traffic on the railways to diminish almost to the vanishing point.

Although a great many branch or feeder lines are unable to produce any net revenue when only their own operations are concerned, it is generally held that, if some profit is shown when the contribution of the traffic of these lines to the railway system as a whole is considered, the continued operation of the line in question is in the public interest.

In some cases where light traffic or other secondary lines of competing railways were parallel or served substantially the same areas, it has been possible to reduce the surplus mileage by securing a certificate of public convenience and necessity authorizing the abandonment and removal of one of the lines and permitting its owner to operate its trains on the remaining line under a grant of trackage rights.

F. R. SCHANCK,[22] M. AM. SOC. C. E.—In presenting the subject of transportation in the United States, the author brings before the Society one of those basic enterprises, the discussion of which properly includes the broadest aspects of engineering. The consideration of such matters tends to expand the viewpoint of members beyond the rather narrow confines of purely technical engineering. It helps to remind them that they are primarily citizens of communities and of the Nation rather than simply members of an academic debating club.

As the author indicated, a thorough study of the development of transportation in the United States requires research into history. However, the writer does not believe that merely a reference to the "First Stone" of the Baltimore and Ohio Railroad brings to the subject the full light which history can furnish. Besides history, it is necessary to consider economics, geography, law, and industrial and inventive growth, not to mention finance, and politics. In fact, it appears wise to revive one's recollection of the true function of engineering. The definition runs something like this: The study and the utilization of the forces and the materials of nature for the benefit of humanity. The last five words furnish the reason for the very existence of engineers. The author does not seem to have emphasized that ideal in his brief for railroads.

One very interesting and important type of transportation in the United States is not covered by the author; that is, city "mass transportation." During the life time of many, city street cars were drawn by animals—horses, and mules; then many lines used cable traction. Soon practically all these were supplanted by electricity and now, except in a few of the largest cities, buses are used for nearly all surface transportation. Until very recently many street railway companies were constantly emphasizing the fact that passengers may be carried more cheaply by electric street railways than by any other present method. However, as the president of one of the New York surface car lines has said, when requesting from the Board of Local Transportation permission to substitute buses for electric rail cars, "The people like to ride on rubber."

The fact that mass transportation has actually passed through the initiation, the zenith of prosperous operation, and now the almost complete elimination of a universally used type of plant and equipment within a comparatively short time, gives the opportunity to analyze such a development. An unbiased and critical study by engineers of a system so nearly parallel with steam railways should give accurate and helpful data as to the probable destiny of the latter. One assumption that frequently predicates discussion of transportation development is that the system that costs the least will inevitably dominate. The fact that cost alone may not necessarily satisfy the public is demonstrated by the transitions in mass transportation.

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[22] Cons. Engr., Portland, Ore.

Some years ago, when the term of the franchises for the street-car lines of Portland, Ore., had expired, there was such violent and widespread disagreement expressed by the citizens as to renewing the franchises that the City Commission hesitated to take any action. Finally the mayor appointed a committee of citizens with a request that this body make some suggestion as to whether the old type of franchise be offered the Street Railway Company, whether a radically different set of conditions be demanded, or whether the existing company be considered as having a status no different from any other applicant for a franchise. The writer, being a member of this committee, had access to considerable public sentiment on the matter of mass transportation. Preferences for different methods of solving the mass transportation problem of Portland did not seem to be based primarily on costs to the riders.

At this same time, the writer was chairman of a committee of the City Club of Portland which had been appointed to analyze and digest the various suggestions and programs that were advanced as the solution of the mass transportation problem. This committee spent several months in collecting data from all cities of a size comparable with Portland. The committee's position also furnished the opportunity to secure statistics as to the apparent indication that cost may not be the primary factor in the minds of the public when purchasing transportation.

Several thousand questionnaires were placed in the hands of individuals of many groups which together seemed to give a fair cross section of public opinion. These included all the employees of certain laundries, banks, department stores, service clubs, factories, and office building occupants. Such questions as their ownership of automobiles, as well as their class of work, indicated quite clearly the economic status of the individuals. The majority of every group showed that some other consideration besides the amount of fare would determine their selection of means of transportation, except the employees of an interurban street railway company which has since abandoned passenger service.

As long as the selection of a mode of transportation of either freight or passengers is a matter of personal choice, the evidence just cited seems worthy of consideration by all common carriers. The limitations of the factors described in the Transportation Act creating the Board of Investigation and Research appear too narrow for a wise report on such a basic subject. The social and cultural effects of limited or abundant transportation facilities are of the utmost importance. The restraint or the development which a particular region or community may experience by absence of impartiality in its transportation services or charges as compared to other places and a multitude of complex problems all have roots in transportation.

The author has brought each other type of transportation, whose development he has described, into its relationship to railways. He writes of "normal conditions" in which railroads require large amounts of materials for operations and expansion. This normal condition apparently means when railroads have a substantial monopoly of transportation.

To have an adequate picture of the entire history of rail transportation as it is now known, it is necessary to go back at least 50 years prior to the occasion, noted by the author, of the inauguration of the Baltimore and Ohio Railroad construction. The earlier date covers that time in the British Isles when rails were laid upon which horse-drawn vehicles transported freight and passengers, and especially coal, in some of the mining regions of that country.

Where these "rail ways" were constructed along the routes of public highways they seem to have been given the same status as any other toll road, probably with the idea that those toll payers who had their own vehicles equipped with suitable wheels could have the privilege of transporting freight over the improved roadbed. Some of the rail systems permitted the use of vehicles with ordinary type wheels; others required flanged wheels.

The unique fact that at this same time the newly invented steam engine was adapted for locomotion brought about the revolutionary changes which have resulted in the modern railway systems. When steam locomotives were used as the tractive force, it became obvious that several unrelated users could not satisfactorily operate vehicles over these improved "toll roads." The implications of the origin of railways remain even after the complete evolution of the private property idea which now almost obscures the original status. Both the right of eminent domain which railroads possess and their designation as common carriers retain the germ of the origin of railroads as public highways. Permission to construct rail toll roads and to collect revenue for their use was by grants limited in time. The fee title remained in the Crown or other sovereign. Certain of the earlier charters granted to railway companies in the United States contain language which indicates the origin of this governmental status of the railway system.

In this country especially, the evolution to private ownership has been complete so that for many decades court decisions have indicated clearly that railroads are private property. This status of private property in the hands of railroad owners makes a distinction inevitable in the treatment which the railroads must expect in their dealings with governmental bodies. The author seems not to have taken this into account in his comparison of the status and problems of railroads with water, air, and highway transportation systems. All of these are free, legally, to be used by any one wishing to do so. On the other hand, no one can use the railway except the owners.

No individual or group receives any income from the use of the highways, improved air routes, or waterways. Apparently there is no comparable basis for computing the public funds to be spent on public improvements or the taxes to be levied on private property or any other of the subjects which the author indicates are matters of concern.

Although it may sometimes appear to be unjust, yet it does seem one of the requirements of nature that "iniquity of the fathers" is visited "upon the children unto the third and fourth generation." Going back much less than three generations when the railroads were substantially in complete control of all transportation, the president of one of the largest systems was most widely known by his exclamation, "the Public be damned." Even as recently as 1900 or 1910, such apparent elementary rights, which the shippers now possess as designating a route for freight shipments, were not at all privileges assured to those who paid the bill. As indicated in the case of the preference of the public as between street cars and buses even at higher cost, the convenience of door-to-door freighting service, deliveries in a reasonable and fixed time after shipment, and more frequent services all helped to build up the trucking business at the expense of railways.

In other words, more acceptable, convenient, or cheaper transportation was provided. Just as faster, although not cheaper, shipments by rail in the early part of the nineteenth century put canal boat operators out of business, so now an improvement over railways has eliminated many branch lines and caused loss of traffic, both freight and passenger, to automobiles, airplanes, and pipe lines.

There does not seem to have been any reason why the railroad companies themselves might not have foreseen the advantages that the autotruck provides for certain types of business and have used these to supplement their rail business. Instead of this, during the first few years of autotrucking, railroads combated in every way possible the additional convenience which the trucks provided the public. After it became apparent that the truck transportation was here to stay, not infrequently railroads have expended considerable sums in payment for franchises or other intangible rights to established truck or bus operators.

The question of inequity of taxation of railroads seems to be no more complex than that by which highways are financed. Although highways are used for all types of transportation, for access, and for the general convenience of the adjacent property owners and the public, they are financed almost solely by a tax on gasoline imposed on one class of users. In fact, the construction of highways has often provided great traffic outlets for railroads. In the same manner, much of the expenditures for harbor improvements, which the author classes as competition unfair to railroads, has built up water traffic most all of which is hauled to ports by railroads. The great bulk of the foreign commerce of the United States has, in the past, been lumber, wheat, cotton, meat, and food products which involved long rail hauls. Without the improved harbors many industries and the communities they support, with the railroads benefiting, could not have reached their present magnitude.

Soon after the First World War, a Royal Commission on Transport was appointed by the British Government to study all phases of the subject. The reports of this commission are illuminating even if some conditions in Great Britain are very different from those in the United States. The authority and scope of its investigations apparently were much broader than those of the Board of Investigation and Research which the President has been authorized to appoint. The Royal Commission had in its personnel not only those technically qualified to represent various types of transportation but also those whose viewpoint was more that of the well-informed public. Of course, this later viewpoint is the one which should predominate in dealing with matters as vital to all people as transportation.

J. E. TEAL,[23] M. AM. SOC. C. E.—Since the first presentation of this paper in December, 1941, transportation conditions in the United States have changed materially. This change was first manifest in the summer of 1939, and became more pronounced after September 1, 1939, when the war in Europe was started, which resulted in industries in the United States receiving large orders for materials and supplies. When a major change occurs in industrial activity, all transportation agencies immediately feel the effect. This increased industrial activity continued to December 7, 1941, when the United States declared war.

The discussions have been enlightening, as they have revealed points omitted or overlooked by the writer and have expanded others. Because of the great change in transportation conditions, caused by the war effort, it appears to the writer that the discussion might well be terminated by emphasizing the relative importance of the various transportation agencies in connection with the war economy and let the record speak for itself.

The desire of transportation agencies is to assist in winning the war at an early date. The war economy of the United States depends largely on available means of transport. The ability of the Nation to transport men and materials overseas is of vital concern. However, the overseas shipments are greatly exceeded in volume by those within the United States, because here it is necessary to transport vast numbers of troops and military equipment from one point to another, in the process of training, and also to transport the materials required in the construction of military establishments, in the building of ships, and in the manufacture of ammunition, guns, planes, and tanks. All of this transportation for the war effort is given first consideration, of course, and the remainder of the transportation is used for the benefit of civilian activities. The civilian population now (December, 1942) finds traveling more difficult because of the greatly increased demands on transportation agencies.

At the beginning of 1942 the Nation's transportation plant and facilities consisted of approximately 233,000 miles of railway first main track; 1,960,000 freight-train cars, of which 1,685,000 were railroad owned; 38,000 passenger-train cars; 45,000 locomotives; 1,469,000 miles of improved highway, on which were operated 5,000,000 highway trucks, 140,000 buses, and 29,000,000 automobiles; inland waterways, excluding the Great Lakes, which were handling 4% of the Nation's bulk freight; and 105,000 miles of pipe lines, which were moving 10% or more. Vessels that were operated on the Great Lakes and along the coast provided transportation at low cost for large quantities of coal, oil, and lumber. Airplane transportation was increasing at a rapid rate to handle the ever-increasing passenger, mail, and express load.

As stated by Ralph Budd,[24] Hon. M. Am. Soc. C. E.:

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[23] Prin. Roadway Engr., Board of Investigation and Research, Washington, D. C.

[24] "Transportation on the Home Front," by Ralph Budd, *Civil Engineering*, September, 1942, p. 503.

"These various forms of transportation took on the traffic load incident to war abroad and preparedness at home, as it came along, without any considerable strain simply by utilizing a larger proportion of their carrying capacity than had been needed during the preceding years. Thus for the first time they realized the benefits from the extensive improvements that had been made."

Since the outbreak of World War II, on September 1, 1939, the railroads have handled a continuously increasing traffic load, as shown by Table 10. The progressive effect of the increasing traffic load is shown by the percentage data in Cols. 3, 6, and 9, Table 10. Comparing 1942 with 1939, the cumulative freight traffic increase is shown as 26.2% in carloadings and 88.9% in ton-miles. The greater increase in ton-miles is due to the increased load per car and an increase in the length of haul. The average load per car in 1942 will be more than 4 tons greater than the average load in 1939, whereas the average haul per ton of freight in 1939 was 351 miles, and today is about 420 miles. In the passenger service, including troop movements, the number of revenue passengers carried one mile nearly doubled in 1942 as compared with 1939.

TABLE 10.—INCREASE IN FREIGHT AND PASSENGER TRAFFIC ON UNITED STATES RAILROADS

Year	CAR LOADINGS			REVENUE TON-MILES			REVENUE PASSENGER-MILES		
	Thousands (1)	Index No. (2)	Increase (%)[a] (3)	Millions (4)	Index No. (5)	Increase (%)[a] (6)	Millions (7)	Index No. (8)	Increase (%)[a] (9)
1939	33,911	100.0	....	333,438	100.0	....	22,651	100.0	....
1940	36,358	107.2	7.2	373,253	111.9	11.9	23,762	104.9	4.9
1941	42,285	124.7	16.3	475,072	142.5	27.3	29,350	129.6	23.5
1942[b]	42,800	126.2	1.7	630,000	188.9	31.6	53,000	234.0	53.3
Total[c]	....	....	26.8	....	....	87.4	....	....	98.7

[a] Percentage increase over the preceding year. [b] Estimated. [c] Percentage increase, 1942 over 1939.

TABLE 11.—RAILROAD OWNERSHIP OF FREIGHT CARS AND LOCOMOTIVES

Date; September 1:[a]	FREIGHT CARS				LOCOMOTIVES			
	Total owned	Unserviceable		Serviceable	Total owned	In Bad Order		Serviceable
		No.	%			No.	%	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1939	1,650,423	224,603	13.6	1,425,820	42,848	8,381	19.6	34,467
1940	1,640,924	137,869	8.4	1,503,055	41,608	6,268	15.1	35,340
1941	1,648,502	78,044	4.7	1,570,458	41.384	4,247	10.3	37,137
1942[a]	1,738,553	44,931	2.6	1,693,622	41,444	2,157	5.2	39,287

[a] Except 1942, which is for December 1.

Table 11 shows the progress made by the railroads since September 1, 1939, in freight-car and locomotive ownership. It shows the net results the railroads have obtained from installation of new equipment and the repair of existing equipment. The main achievement has been in the reduction in number and proportion of unserviceable cars and locomotives.

The number of freight cars in bad order on September 1, 1939, was 224,603, or 13.6% of total ownership. This was reduced to 44,931 freight cars on December 1, 1942, or 2.6% of total ownership. Thus, although the total ownership of freight cars increased by only 88,130 cars, the number of serviceable units increased by 267,802 cars.

In the case of locomotives, the ownership on December 1, 1942, was less than on September 1, 1939, by 1,404 locomotives. On the other hand, the number of serviceable engines increased by 4,820 units, those in bad order having been decreased from 8,381 on September 1, 1939, to 2,157 on December 1, 1942.

TABLE 12.—IMPROVEMENT IN THE UTILIZATION OF RAILWAY EQUIPMENT

Year	FREIGHT CARS		FREIGHT LOCOMOTIVES		PASSENGER LOCOMOTIVES	
	Ton-miles[a]	Index No.	Ton-miles[b]	Index No.	Car-miles[c]	Index No.
1939	692	100.0	167,870	100.0	1,408	100.0
1940	721	104.2	178,344	106.2	1,462	103.8
1941	833	120.4	198,997	118.5	1,532	108.8
1942[d]	945	136.6	214,544	127.8	1,615	114.7

[a] Freight-car utilization, in net ton-miles per active freight-car day. [b] Freight-locomotive utilization, in gross ton-miles per active freight-locomotive day. [c] Passenger-locomotive utilization, in passenger-train car-miles per active locomotive day. [d] First six months of 1942.

It is clear, however, that the railroads could not have met the heavy traffic demands of today without improved performance. Table 12 shows the steady improvement in utilization of equipment that has occurred since 1939. With respect to freight-car utilization, the average car in 1942 has been equal to more than one and one-third cars in 1939. Freight-locomotive utilization has increased 27.8%, and passenger-locomotive utilization has increased 14.7%. The railways have been doing a good job.

Table 8 shows that the relative volume of traffic for the year 1939 aggregated 543.4 billion revenue freight ton-miles and 40 billion revenue passenger-miles. It is estimated from data in Table 10 (which were furnished by the Association of American Railroads) that railway traffic will be increased 291.6 billion revenue ton-miles, or 87.4%, and revenue passenger-miles will be increased 22.3 billion miles, or 98.7%, in 1942 compared with 1939. Reliable information is not available (as of December, 1942) as to just what the traffic will be for other carriers; however, information at hand indicates that the traffic of these agencies also will be materially increased over that of 1939. It is obvious that changed conditions due to the war effort have greatly increased the demand for transportation by the highway, waterway, pipe line, and air carriers, as well as by the railroads. All of these agencies are being handicapped by the lack of equipment, and it now seems that the gasoline shortage in the East and the rubber-tire situation may have a retarding effect on the capability of the highway carriers to continue to handle the volume of traffic now prevailing.

As previously stated, the inland waterways have been improved and modernized during the past quarter of a century. The improvements of some of them were projected in the feverish rush of World War I. World War II threatens to start a similar expansion of waterway improvements. As illustrations, construction of the Florida Ship Canal, the Lake Erie-Ohio River Canal, and the St. Lawrence Deep Seaway—each possessing strong political and commercial supporters—is being advocated earnestly by their backers, either as absolutely necessary to the war effort or as desirable "make work projects" as soon as the war is over. Therefore, it seems to the writer that an examination of previous waterway developments is appropriate at this time, and in particular the Ohio River waterway system, which is probably the most complete of any improved waterway system in the United States.

This was a subject given consideration by Committee XVI—Economics of Railway Location and Operation—of the American Railway Engineering Association (A.R.E.A.) in 1940; the following conclusions[25] were formulated at that time:

> "1—The Ohio River navigation system consists of 3,612 route miles (981 on the Ohio river between Pittsburgh and Cairo, and 2,621 on tributary streams), and represents an investment on the part of the taxpayers of the United States of some $234,000,000, and a total expenditure (as of January 30, 1938) including maintenance and a relatively small amount of unexpended appropriations, of $362,000,000. Based on the operation during 1937, a total of about 4,205,000,000 ton-miles of freight was transported on these rivers, a large percentage of which would have moved on the railroads had government improved waterways not been available.
>
> "The annual cost of transportation on these rivers to the nation is estimated at approximately $34,000,000, of which about one-half is paid or absorbed by the taxpayers and the other half by the users of the waterways.
>
> "There is no evidence of record that the users of the waterways pass on to the public in general the savings that accrue to them as compared with the rail freight rate. Waterway transportation may be cheap to the favored shipper but is costly to the taxpayers, who foot the bill for improving, operating and maintaining the locks, dams and channels."

A more recent study of this subject has been reported by C. Emery Troxel,[26] who develops some very interesting cost figures (excluding the cost paid by the shipper) representing the cost per ton-mile of all traffic on this waterway system which is borne by the government or taxpayers.

The facts presented raise questions as to the effectiveness of waterways (compared to railways) as a means of maximizing wartime transportation, and of the economic justification of most, if not all, of such enterprises at any time. Concerning these questions Professor Troxel states:

> "The presence of the Ohio Waterways thereby may relieve the pressure on railroads and improve the war effort of the country. Yet this gain would be a sort of windfall for the nation, for these waterways were developed more for peacetime than for wartime use. Such additional transport capacity, granted that it may prove convenient, might have been provided more economically by other means. * * * It is customary for

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[25] *Proceedings*, A.R.E.A., Vol. 41, 1940, p. 149.

[26] "How 'Cheap' Is River Transport?" by C. Emery Troxel, *Railway Age*, September 12, 1942, p. 408.

> waterway protagonists to acclaim the railroad-rate reductions which follow the opening of a waterway. Though they correctly note the transportation expense reduction for some commodities and companies, they fail to recognize the possibility of contrary consequences elsewhere. Perhaps the rate reductions to meet water competition have the effect of forcing rate increases for commodities or areas where there is less competition. Or, in the event of a continuance of deficit operations, railroads may be forced to abandon parts—the weaker branches—of their systems. Communities near the waterway or elsewhere—for toll-free water transportation may provoke abandonments at inland points—may be compelled to make more expensive (for some goods) use of trucks. These effects may appear slowly, and yet they may be inevitable."

Under date of September 18, 1942, the Board of Investigation and Research transmitted its annual report to the President and the Congress of the United States, in accordance with Section 305 of Part 1, Title III, Transportation Act of 1940. In this first report the Board outlines the scope of its investigations, which include the flow of traffic, the cost and efficiency of transportation services, amount of public aids received and taxes paid by each type of carrier, labor relationships, shipper-carrier relationships, regulatory policy and practices, and interterritorial freight rates. The following statement[27] is of particular significance:

> "The coming of the war has imposed extraordinary burdens upon the national transportation system. Within a very brief period carriers which for years had suffered from insufficient traffic were obliged to expand their services to the limits of their capacity, and danger signs warning of critical shortage of transportation began to appear. This sudden shift has not, however, altered the fundamental problems of transportation, nor has it changed the essential characteristics of the various types of carriers; rather, it has focused attention upon certain aspects of these problems, demonstrating the need for adequate capacity and flexibility in transportation and making more certain than ever the prospect of far-reaching readjustments in our transportation facilities.
>
> "These emergency conditions have not diminished the significance of the problems which the Board is directed to investigate. This was recognized by the President when, by proclamation of June 26, 1942, he extended the term of the Board to its statutory limit, September 18, 1944. The proclamation declares that—an efficient transportation system is essential to the nation in peace and war, and the national interest requires the development of informed policies by which such a system may be promoted, strengthened and maintained."

The Board of Investigation and Research reports progress in its studies of relative economy and fitness, public aids, taxation, and interterritorial freight rates. Under the caption "Further Reports" appears the following:

> "The Board is proceeding as rapidly as possible with the research contemplated by the act. The entry of this country into the war has complicated the problems which the Board is directed to study, but has not changed their essential character. The scarcity of trained personnel and the burdens imposed upon the carriers by wartime demands have in some ways increased the difficulty of the Board's investigations. Yet the war has emphasized the importance of the work and has facilitated it in

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[27] Annual Report of Board of Investigation and Research, September 22, 1942, p. 3.

some important respects. It has made available recent data concerning the capacity of our transportation facilities. It has already brought forth new operating methods and has tested expedients which otherwise could be dealt with only in theory. These experiences are of great value in considering methods of developing a national transportation system adequate for the present and future needs of commerce and national defense.

"The Board's studies of the interterritorial freight rate structure will be filed in January 1943; its report on public aids to carriers will be filed early in 1943; its report on carrier taxation will be completed by the end of the current fiscal year. Its investigations of relative economy and fitness of carriers are of such magnitude and importance as to require a longer period for completion. These further reports will contain the full findings and recommendations of the Board."

The finding and recommendations of the Board will be of national interest and may have an important bearing on the future policy affecting the national transportation economy.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2172

STABILITY OF GRANULAR MATERIALS

By R. E. Glover,[1] Esq., and F. E. Cornwell,[2] Esq.

With Discussion by Messrs. R. G. Hennes, Stanley U. Benscoter, A. Hrennikoff, D. P. Krynine, and R. E. Glover and F. E. Cornwell.

Synopsis

Results of studies made for the purpose of developing a theory of stability for granular materials, based directly on fundamental laws of behavior, are presented in this paper. Some elementary linear solutions are obtained which are applied to the problem of estimating the stability of earth dams.

Introduction

The primary purpose of soil mechanics is to develop methods of correlating laboratory tests with the performance of structures in the field. The correlation problems for granular materials are similar, in many respects, to those for elastic materials. Methods for calculating the performance of elastic structures have received much attention and the analytical tools for making the calculations have been grouped into a "theory of elasticity." Much of the theory of elasticity is built upon the postulates of equilibrium and of geometrical continuity subject to the relation between stress and strain imposed by Hooke's law. It is important to note that simplicity of the fundamental postulates does not insure a simple solution of the correlation problem since the elastic equations, in many cases, are difficult to solve.

The equations of equilibrium are as important for granular materials as for elastic materials but it can be seen readily that the continuity condition may not be appropriate for granular materials since these can often be sheared extensively without losing their properties.

In the elastic case, it has been proved[3] that a solution to the problem of stress distribution is unique provided it meets the foregoing equilibrium and continuity requirements and, in addition, satisfies the condition peculiar to the problem. These latter conditions are usually imposed at boundaries where conditions are known and, therefore, they are called "boundary conditions."

Note.—Published in November, 1941, *Proceedings*.

[1] Senior Engr., Bureau of Reclamation, Denver, Colo.

[2] Asst. Engr., Bureau of Reclamation, Denver, Colo.

[3] "A Treatise on the Mathematical Theory of Elasticity," by Augustus E. H. Love, University Press, Cambridge, England, 4th Ed., 1927, paragraph 118.

In the analysis to be described, equilibrium equations expressed in terms of stresses have been retained. The continuity condition is also retained where the stress system is such as will prevent relative movement between grains. Where slippage between grains is possible, the continuity condition has been replaced by a condition between the maximum and minimum principal stresses. This relation is the one commonly referred to as Coulomb's law, which states that resistance to shearing is compounded of an intrinsic cohesion, and a frictional resistance proportional to the normal stress acting on the plane of shear.

The trained engineer will find little that is new in this description and the methods to be described are not much more than a mathematical mobilization of relations with which the engineer has long been familiar. The inducement for the expenditure of effort, on analyses of the type described herein, comes from the possibility that the use of simplifying assumptions may be avoided and the reliability of the results thereby enhanced.

In developing a mathematical approach to the solution of any physical problem, one is faced with the necessity of reducing the problem to an exact status by means of specific definitions. It is usually necessary to replace the actual material, which is almost never continuous, by an ideal continuous material. Of course, it is desirable that, in so far as possible, the properties of the two materials be identical. In the present case, certain consequences of the granular structure of the actual material are imposed upon the ideal. Other limitations arise from purely mathematical difficulties.

In the theory of elasticity, stress is defined in terms of the resultant force transmitted across a small plane area within the body. In the present case it is expedient to modify this concept because of the presence within the voids of water under pressure. The term "stress," as used herein, is defined as the force transmitted by the contact pressures between grains over a small area divided by the area, and therefore excludes the percolation pressures.

To form the concept of stress in the granular material, pass a plane through it and find the surface that passes through no particle but which lies as close to the plane as possible. The particles on one side of the surface touch those on the other at several points, and at each of these points a force is transmitted from particle to particle. Now consider a small part of the area of the surface. The average stress on the small area is defined as the resultant of the forces transmitted from particle to particle across it, divided by the area of its projection on the plane. Conceptions of average normal and shear stresses can be formed in the same way. Since it is apparent that the diameter of the small area previously mentioned must be large compared to the diameter of a grain, in order to include enough contact points to yield a valid average, a stress computation made for a point, as designated by coordinates, will represent an average stress in the immediate vicinity of the point rather than a stress at a point.

The stress convention is shown in Fig. 1. Forces of gravity, flotation, and fluid friction, as well as an additional body force which is indicated as due to earthquake, are included. When it is desired to obtain the total force transmitted across a section, pore-pressure forces must be added.

The solution requires that every small element of volume must be in equilibrium under the stress resultants and forces that act upon it. In addition to equations of equilibrium, it is required that the stresses satisfy the equations of elasticity in those regions where the relation between stresses is such as will not permit grains to change their relative positions, and in the regions where this condition is not met, it is assumed that the relation between the principal stresses is such as to make slippage imminent on the planes of least resistance.

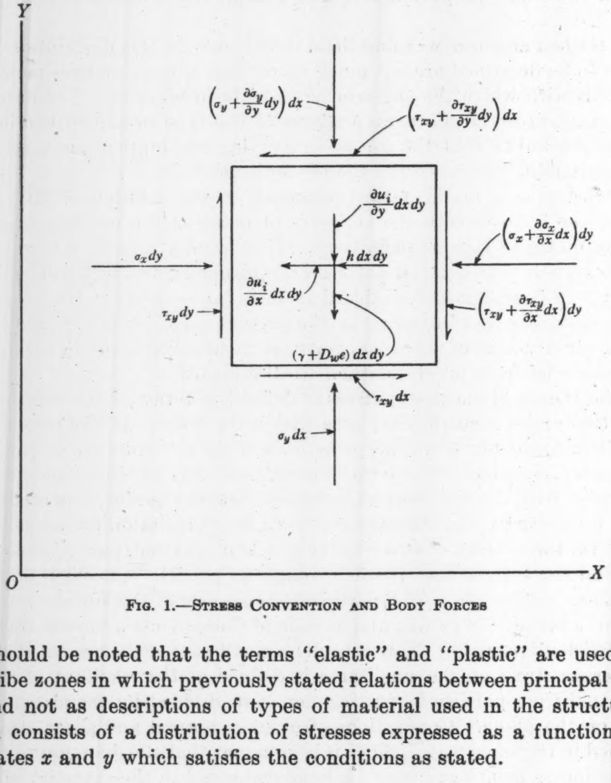

Y
$\left(\sigma_y + \frac{\partial \sigma_y}{\partial y} dy\right) dx$
$\left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} dy\right) dx$
$\frac{\partial u_i}{\partial y} dx\, dy$
$\left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) dy$
$\sigma_x dy$
$h\, dx\, dy$
$\frac{\partial u_i}{\partial x} dx\, dy$
$\tau_{xy} dy$
$\left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx\right) dy$
$(\gamma + D_w e)\, dx\, dy$
$\tau_{xy} dx$
$\sigma_y dx$
O
X

FIG. 1.—STRESS CONVENTION AND BODY FORCES

It should be noted that the terms "elastic" and "plastic" are used herein to describe zones in which previously stated relations between principal stresses exist and not as descriptions of types of material used in the structure. A solution consists of a distribution of stresses expressed as a function of the coordinates x and y which satisfies the conditions as stated.

NOTATION

The letter symbols used in this paper are defined where they first appear, and they are assembled for convenience of reference in Appendix I.

STRESS RELATIONS IN THE ELASTIC AND PLASTIC ZONES

Let σ be the normal stress on any plane, positive when compressive, and τ the shear stress. According to the theory of failure, which is accepted herein, it is true, for every plane at every point, that

$$|\tau| \leqq C + f\,\sigma \quad \text{.........................} \quad (1a)$$

in which C and f are positive constants, known as the cohesion and the coefficient of friction, respectively, whose values depend upon the material; and it is also true that relative sliding of the material on the two sides of the plane is incipient when

$$|\tau| = C + f\,\sigma \quad \text{.........................} \quad (1b)$$

but impossible when

$$|\tau| < C + f\,\sigma \quad \text{.........................} \quad (1c)$$

Relations 1 may be written in terms of the function

$$R = C + f\,\sigma - |\tau| \quad \text{...................} \quad (2)$$

as follows: $R \geqq 0$, $R = 0$, and $R > 0$. In Eq. 2 R is a function of the position of the point and the orientation of the plane. Let R_0 be the minimum value of R at a given point. Then Eq. 1*a*, which is satisfied everywhere, is equivalent to $R_0 \geqq 0$.

Plastic Region.—At any point where $R_0 = 0$ there must be one or more planes; it is found that there are two, for which $R = R_0 = 0$, and along which, therefore, slipping is incipient. At such a point the material will be said to be in the "plastic state," and any region at every point of which the material is in the plastic state will be called a "plastic region."

Elastic Region.—At any point where $R_0 > 0$ there is no plane along which slipping is possible, since $R \geqq R_0 > 0$; and at such a point the material is assumed to obey Hooke's law and will be said to be in the "elastic state." Any region at every point of which the material is in the elastic state will be called an "elastic region."

As a consequence of Hooke's law, certain conditions of compatibility are satisfied at all points of an elastic region. For a two-dimensional body these conditions (which are discussed at length in works on the theory of elasticity) are reduced to the single equation:

$$\frac{\partial^2 \sigma_y}{\partial x^2} + \frac{\partial^2 \sigma_x}{\partial y^2} - 2\,\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = 0 \quad \text{...} \quad (3)$$

The Functions R and R_0.—Functions R and R_0 may be expressed in terms of the principal stresses or in terms of the normal and shear stresses. Let σ_1 and σ_2 be the principal stresses, positive when compressive, and σ_1 algebraically the greater; and let ϕ be the angle measured counterclockwise from the line of action of σ_1 to the plane of σ and τ. Equilibrium of the triangular element shown in Fig. 2 requires that

$$\sigma = \sigma_1 \sin^2 \phi + \sigma_2 \cos^2 \phi \quad \text{...................} \quad (4a)$$

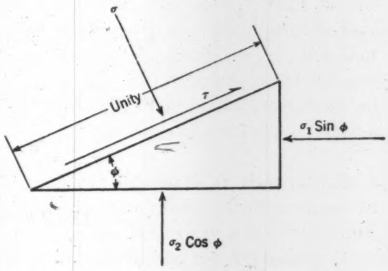


FIG. 2

and

$$\tau = (\sigma_1 - \sigma_2) \sin \phi \cos \phi \ldots\ldots (4b)$$

or

$$\sigma = \frac{\sigma_1 + \sigma_2}{2} - \frac{\sigma_1 - \sigma_2}{2} \cos 2\phi \ldots\ldots (5a)$$

and

$$\tau = \frac{\sigma_1 - \sigma_2}{2} \sin 2\phi \ldots\ldots (5b)$$

After substituting Eqs. 5 in Eq. 2 and simplifying,

$$R = C + f\frac{\sigma_1 + \sigma_2}{2} - \frac{\sigma_1 - \sigma_2}{2}(f \cos 2\phi + |\sin 2\phi|) \ldots\ldots (6a)$$

Then remembering that R_0 is the minimum of R at a given point, and that at a given point σ_1 and σ_2 are constants, one finds

$$R_0 = C + f\frac{\sigma_1 + \sigma_2}{2} - \sqrt{1 + f^2}\,\frac{\sigma_1 - \sigma_2}{2} \ldots\ldots (6b)$$

This minimum is attained when

$$\phi = \pm \operatorname{arc\,tan}\left(\sqrt{1 + f^2} - f\right) \ldots\ldots (7)$$

The two planes thus defined form the acute angle, arc tan $\frac{1}{f}$, which is bisected by the line of action of σ_1, the greater of the principal stresses. These are the planes along which slip occurs, if it does occur.

Other useful relationships, the derivations of which are omitted, are:

$$\frac{\sigma_1 + \sigma_2}{2} = \frac{\sigma_x + \sigma_y}{2} \ldots\ldots (8a)$$

$$\frac{\sigma_1 - \sigma_2}{2} = \tfrac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau^2_{xy}} \ldots\ldots (8b)$$

$$\sigma_x = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta \ldots\ldots (8c)$$

$$\sigma_y = \frac{\sigma_1 + \sigma_2}{2} - \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta \ldots\ldots (8d)$$

$$\tau_{xy} = \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta \ldots\ldots (8e)$$

$$\tan 2\theta = \frac{2\,\tau_{xy}}{\sigma_x - \sigma_y} \ldots\ldots (8f)$$

$$\sin 2\theta = \frac{2\,\tau_{xy}}{\sigma_1 - \sigma_2} \ldots\ldots (8g)$$

and

$$\cos 2\theta = \frac{\sigma_x - \sigma_y}{\sigma_1 - \sigma_2} \ldots\ldots (8h)$$

in which θ is the angle measured counterclockwise from the x-axis to the line of action of σ_1. It frequently happens that confusion arises in the computation of θ; and it should be remembered, therefore, that: $\sigma_1 - \sigma_2$ is never negative; θ is referred always to σ_1, the greater—never to σ_2, the lesser; and θ is defined not by the value of $\tan 2\theta$ (Eq. 8*f*), but by the value of $\tan 2\theta$ and the algebraic sign of either $\sin 2\theta$ or $\cos 2\theta$. If Eqs. 8*a* and 8*b* are substituted in Eq. 6*b*, the following expression for R_0 is obtained:

$$R_0 = C + f\frac{\sigma_x + \sigma_y}{2} - \frac{\sqrt{1+f^2}}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau^2_{xy}} \;\ldots\ldots\ldots (9)$$

COHESIONLESS MATERIAL

The types of solutions available at present are applicable only to a cohesionless material; that is, one for which $C = 0$. For such a material the law $R_0 \geqq 0$ becomes, in terms of the principal stresses, from Eq. 6*b*,

$$f\frac{\sigma_1 + \sigma_2}{2} \geqq \sqrt{1+f^2}\,\frac{\sigma_1 - \sigma_2}{2} \;\ldots\ldots\ldots\ldots\ldots\ldots (10a)$$

or, in terms of σ_x, σ_y, and τ_{xy}, from Eq. 9,

$$f\frac{\sigma_x + \sigma_y}{2} \geqq \frac{\sqrt{1+f^2}}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau^2_{xy}} \;\ldots\ldots\ldots\ldots (10b)$$

By letting $B = \dfrac{\sqrt{1+f^2}}{f}$ Eqs. 10 reduce to

$$\sigma_2 \geqq \frac{B-1}{B+1}\,\sigma_1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11a)$$

and

$$\sigma_x + \sigma_y \geqq B\,\sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau^2_{xy}} \;\ldots\ldots\ldots\ldots\ldots (11b)$$

The equalities represented by Eqs. 11 are two ways of stating the plasticity condition, and the inequalities represent a relation that must be satisfied in elastic regions. These expressions imply, and it can be proved, that tensile stresses are impossible, that inside an elastic region neither of the direct stresses can vanish; also that inside or on the boundary of a plastic region, if either direct stress is zero, then all the stresses are zero. It follows that all stresses vanish at a free boundary, and it should be noted that any boundary subjected to water load alone is free, provided the water pressure is continuous across the boundary.

Boundary Between Elastic and Plastic Region.—It is clear that, across a boundary between an elastic and plastic region, the direct stress normal to the boundary and the shear stress tangential to the boundary are continuous, but it is not clear whether the direct stress tangential to the boundary is or is not continuous.

It appears that the theory must permit such a boundary to move as the loading on the outer boundaries of the body is varied. If the direct stress tangential to the boundary is not continuous, then, as the boundary moves, there must be sudden changes of stress at points on the boundary, and sudden movements of the particles, involving infinite accelerations. Such consequences are to be avoided, and it is postulated, therefore, that all stresses are continuous across the boundary. Of course, this postulate holds only in case the properties of the material are continuous across the boundary.

APPLICATION TO THE PROBLEM OF EARTH-DAM STABILITY

The structure to be analyzed is idealized by dividing it arbitrarily into a series of elastic and plastic regions (see Fig. 3). Odd Roman numerals (I, III, V, etc.) are used to designate regions assumed to be plastic and even ones (II, IV, VI, etc.) denote regions assumed elastic. In every case it is assumed that the inner boundaries, those joining two regions, are straight lines. The use, in the equations of inner boundaries, of arbitrary constants α and β with subscripts associating them with the elastic region involved is also illustrated. The corresponding terms for the outer boundaries (that is, numbers n_1 and n_2) used in the equations of the downstream and upstream slopes are constant, of course, for any one dam and are obtainable directly from known data.

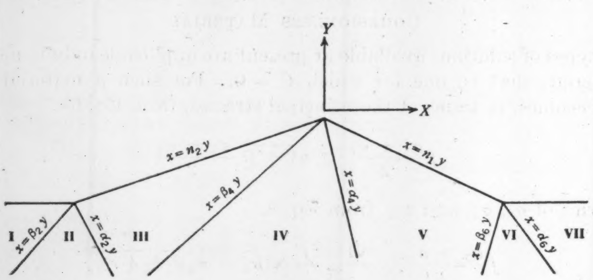


FIG. 3

Note that the problem, as illustrated, is cut into three parts in which are grouped regions I, II, III, regions III, IV, V, and regions V, VI, VII. The separate parts are referred to as applying, respectively, to the upstream toe, the dam, and the downstream toe. For each part a new system of axes, with its origin at the point common to all three regions and with the x and y axes parallel to and with the sense of those for regions III, IV, V, is used. Additional parts are needed when the slope of either fill is not uniform throughout. In such a case the actual slope is approximated by two or more sections of constant slope and then a new group is introduced at each point where there is a change.

EQUATIONS AND LINEAR SOLUTIONS FOR ELASTIC REGIONS

The forces acting on an element of the material are shown in Fig. 1. If the accelerations of the water are negligible, the equations of equilibrium are seen to be

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial u}{\partial x} + h = 0 \ldots\ldots\ldots\ldots\ldots\ldots (12a)$$

and

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial u}{\partial y} + \gamma + D_w\, e = 0 \ldots\ldots\ldots\ldots\ldots (12b)$$

The three partial differential equations (Eqs. 3 and 12) must be satisfied by any solution for an elastic region. Consider a definite region "i," and make

$$C_{2i} = \gamma + D_w e + \frac{\partial u_i}{\partial y} \qquad (13a)$$

and

$$C_{2i} C_{1i} = h + \frac{\partial u_i}{\partial x} \qquad (13b)$$

Then the general linear solution of Eqs. 3 and 12 is

$$\sigma_x = C_{2i} (a_{1i} x + a_{2i} y + a_{3i}) \qquad (14a)$$

$$\sigma_y = C_{2i} (b_{1i} x + b_{2i} y + b_{3i}) \qquad (14b)$$

and

$$\tau_{xy} = C_{2i} [- (1 + b_{2i}) x - (a_{1i} + C_{1i}) y + d_{3i}] \qquad (14c)$$

in which a_{1i}, a_{2i}, a_{3i}, b_{1i}, b_{2i}, b_{3i}, and d_{3i} are arbitrary constants.

In general, however, one does take the vertex or common point of all the regions involved as the origin of the coordinate system. With this in mind it is evident that all stresses must vanish at $x = 0$, $y = 0$. Then the solution in the elastic region "i" reduces to

$$\sigma_x = C_{2i} (a_{1i} x + a_{2i} y) \qquad (15a)$$

$$\sigma_y = C_{2i} (b_{1i} x + b_{2i} y) \qquad (15b)$$

and

$$\tau_{xy} = C_{2i} [- (1 + b_{2i}) x - (a_{1i} + C_{1i}) y] \qquad (15c)$$

In these expressions, the four arbitrary constants a_{1i}, a_{2i}, b_{1i}, and b_{2i} will be determined, along with α_i and β_i, by the boundary conditions along $x = \alpha_i y$ and $x = \beta_i y$.

The inequality $R_0 > 0$ expressed in Eq. 11b as

$$\sigma_x + \sigma_y > B \sqrt{(\sigma_x - \sigma_y)^2 + 4 \tau^2_{xy}} \qquad (16a)$$

must be satisfied in order to comply with the definition of an elastic region.

EQUATIONS AND LINEAR SOLUTIONS FOR PLASTIC REGIONS

In any plastic region "i" a solution must satisfy two partial differential equations and one algebraic equation; namely, Eqs. 12 and the equation arising from the law $R_0 = 0$. This latter equation, called the plasticity equation, is from Eq. 11b,

$$\sigma_x + \sigma_y = B \sqrt{(\sigma_x - \sigma_y)^2 + 4 \tau^2_{xy}} \qquad (16b)$$

The general linear solution of Eqs. 12 and 16b, in which C_{1i} and C_{2i} are defined by Eqs. 13, is

$$\sigma_x = - C_{2i} (C_{1i} + n - n^2 b_{1i}) (x - n y + \epsilon') \qquad (17a)$$

$$\sigma_y = C_{2i} b_{1i} (x - n y + \epsilon') \qquad (17b)$$

and

$$\tau_{xy} = - C_{2i} (1 - n b_{1i}) (x - n y + \epsilon') \qquad (17c)$$

in which b_{1i} is determined by

$$(1 + n^2)^2 b^2_{1i} - 2\left[(n + C_{1i})(n^2 - 1 - 2f^2) + 4n(1 + f^2)\right] b_{1i} + (n + C_{1i})^2 + 4(1 + f^2) = 0 \qquad (18)$$

except that the foregoing solution does not include the case corresponding to $n \rightarrow \infty$, which is

$$\sigma_x = C_{2i}\, a_{2i}\, (y + \epsilon') \qquad (19a)$$

$$\sigma_y = -\, C_{2i}\, (y + \epsilon') \qquad (19b)$$

and

$$\tau_{xy} = -\, C_{2i}\, C_{1i}\, (y + \epsilon') \qquad (19c)$$

in which

$$a_{2i} = -\left[\frac{B^2 + 1}{B^2 - 1} \pm \frac{2B}{B^2 - 1}\sqrt{1 - C^2_{1i}(B^2 - 1)}\right] \qquad (20)$$

Note that in each type of solution for a plastic region there are two cases, coming (for example) from the two solutions of the quadratic equation for b_{1i}. These are distinguished, whenever the two are different, by denoting that in which σ_x is the larger of the two values of σ_x as the passive case, the other as the active case.

As for elastic regions, the most important case is that in which the stresses are assumed to vanish at $x = 0$, $y = 0$. For such conditions,

$$\sigma_x = -\, C_{2i}\, (C_{1i} + n - n^2 b_{1i})\, (x - n\, y) \qquad (21a)$$

$$\sigma_y = C_{2i}\, b_{1i}\, (x - n\, y) \qquad (21b)$$

and

$$\tau_{xy} = -\, C_{2i}\, (1 - n\, b_{1i})\, (x - n\, y) \qquad (21c)$$

in which b_{1i} is determined by Eq. 18. Note that along $x = n\, y$ these expressions yield $\sigma_x = \sigma_y = \tau_{xy} = 0$. Another useful case where $C_{1i} = 0$ is that in which all stresses are zero along $y = 0$. Then for the active case,

$$\sigma_x = -\, C_{2i}\, \frac{B - 1}{B + 1}\, y \qquad (22a)$$

$$\sigma_y = -\, C_{2i}\, y \qquad (22b)$$

and

$$\tau_{xy} = 0 \qquad (22c)$$

and for the passive case (in addition to Eqs. 22*b* and 22*c*)

$$\sigma_x = -\, C_{2i}\, \frac{B + 1}{B - 1}\, y \qquad (23)$$

The condition that all three stresses vanish along the outer boundary is sufficient to exhaust the arbitrary constants in a plastic region. Hence the solution is then known completely in an "outer" plastic region—that is, in a plastic region part of whose boundary is part of the boundary of the structure.

Slip Lines.—The direction of the slip lines may be determined by computing the two angles θ and ϕ by formulas already presented. However, this computation may be shortened as follows: Let ψ equal the angle between the y-axis and a slip line, counted positive clockwise. Then the directions of the two families of such lines are,

$$\tan \psi_1 = \lambda_1 = -\frac{2 f \tau_{xy} + \sigma_y - \sigma_x}{2 \tau_{xy} - 2 f \sigma_y} \qquad (24a)$$

and

$$\tan \psi_2 = \lambda_2 = -\frac{-2 f \tau_{xy} + \sigma_y - \sigma_x}{2 \tau_{xy} + 2 f \sigma_y} \qquad (24b)$$

Note that if the stresses are linear, as they are in the present solutions, then λ_1 and λ_2 are constants and the slip lines are two families of parallel straight lines. It is convenient to use the exact check formula,

$$\left| \frac{\lambda_1 - \lambda_2}{1 + \lambda_1 \lambda_2} \right| = \frac{1}{f} \qquad (25)$$

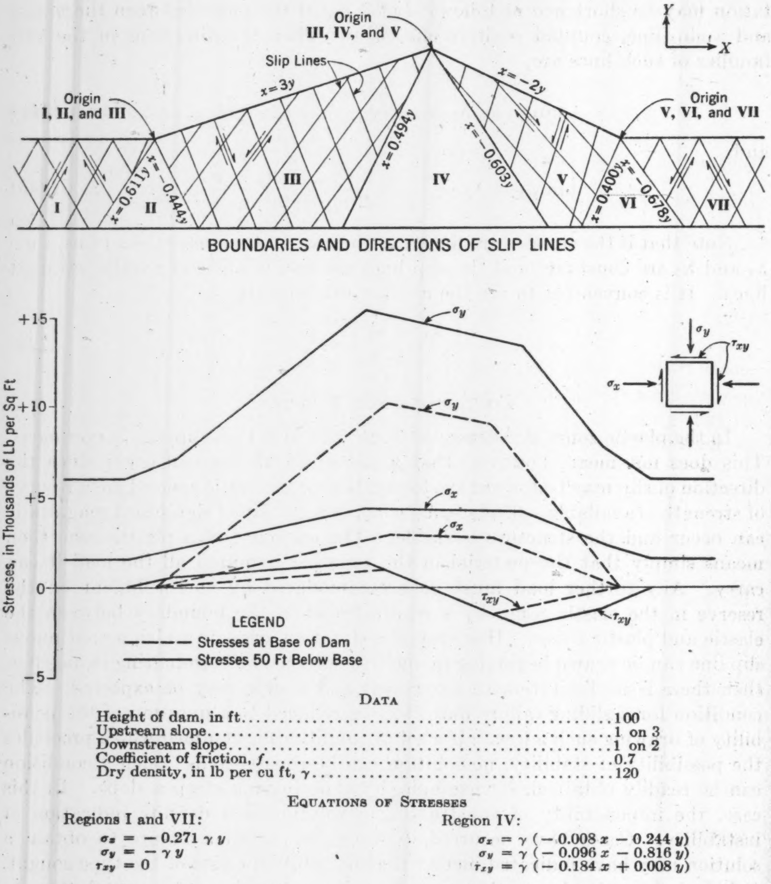
CONDITIONS FOR FAILURE

In the plastic zones the stress distribution is such that slippage is imminent. This does not mean, however, that a failure by sliding can occur since the direction of slip may be toward the foundation or an elastic zone where a reserve of strength is available. Under such condition, no slip of significant magnitude can occur and the structure is stable. The existence of a plastic zone then means simply that the material in the zone has acquired all the load it can carry. Any further load must be accommodated by encroachment on the reserve in the elastic zones by a readjustment of the boundary between the elastic and plastic zones. However, if a situation arises in which a continuous slip line can be drawn beginning in one free surface and terminating in another, then there is no limitation to movement and a slide may be expected. The condition for a sliding failure may then be reduced to a question of the possibility of drawing such a line. If a set of conditions is assumed which precludes the possibility of stability, no solution can be constructed. Such a condition can be readily obtained, for example, by choosing too steep a slope. In this case, the impossibility of constructing a solution is a definite indication of instability. Cases have occurred, however, in which a failure to obtain a solution may have indicated merely that no solution exists of the type sought. It is hoped that such questions may be answered as soon as more solutions to the fundamental equations are obtained.

The type of analysis described does not lead to an evaluation of a factor of safety but to a determination of stability or instability. An idea of the reserve strength available may be obtained, however, by investigating the reserve R in the elastic zones and by studying the solution to determine whether small or large changes of loading would be necessary to produce the conditions for a slide.

APPLICATIONS

Example 1. Solution of a Typical Uniform Material Dam of 1 on 3 Upstream and 1 on 2 Downstream Slopes, with Reservoir Empty.—Fig. 4 shows the analysis



DATA

Height of dam, in ft	100
Upstream slope	1 on 3
Downstream slope	1 on 2
Coefficient of friction, f	0.7
Dry density, in lb per cu ft, γ	120

EQUATIONS OF STRESSES

Regions I and VII:

$$\sigma_x = -0.271\,\gamma\,y \qquad \sigma_y = -\gamma\,y \qquad \tau_{xy} = 0$$

Region II:

$$\sigma_x = \gamma\,(0.066\,x - 0.311\,y) \qquad \sigma_y = \gamma\,(0.176\,x - 1.108\,y) \qquad \tau_{xy} = \gamma\,(0.108\,x - 0.066\,y)$$

Region III:

$$\sigma_x = 0.099\,\gamma\,(x - 3\,y) \qquad \sigma_y = 0.344\,\gamma\,(x - 3\,y) \qquad \tau_{xy} = 0.033\,\gamma\,(x - 3\,y)$$

Region IV:

$$\sigma_x = \gamma\,(-0.008\,x - 0.244\,y) \qquad \sigma_y = \gamma\,(-0.096\,x - 0.816\,y) \qquad \tau_{xy} = \gamma\,(-0.184\,x + 0.008\,y)$$

Region V:

$$\sigma_x = -0.171\,\gamma\,(x + 2\,y) \qquad \sigma_y = -0.543\,\gamma\,(x + 2\,y) \qquad \tau_{xy} = 0.085\,\gamma\,(x + 2\,y)$$

Region VI:

$$\sigma_x = \gamma\,(-0.129\,x - 0.359\,y) \qquad \sigma_y = \gamma\,(-0.281\,x - 1.190\,y) \qquad \tau_{xy} = \gamma\,(0.190\,x + 0.129\,y)$$

FIG. 4.—TYPICAL ANALYSIS FOR EMPTY RESERVOIR

as completed and also lists data used in the computations. An outline of the solution follows.

Upstream Toe.—For region I, $C_{11} = 0$ and $C_{21} = \gamma$. Stresses were computed by Eqs. 22.

For region III, $C_{13} = 0$, $C_{23} = \gamma$, $n = 3$. Stresses were computed by Eqs. 21, using the value of b_{13} from Eq. 18 for the active case.

For region II, $C_{12} = 0$, $C_{22} = \gamma$. Stresses for this region are given by Eqs. 15. In order to evaluate the four arbitrary constants in these equations and the two arbitrary constants in the equations of the boundaries, the direct and shear stresses were equated across the boundaries—that is, across $x = \alpha_2 y$ and $x = \beta_2 y$. This led to the six simultaneous equations,

$$\left.\begin{aligned} a_{12}\beta_2 + a_{22} &= -0.27108 \\ b_{12}\beta_2 + b_{22} &= -1 \\ (1 + b_{22})\beta_2 + a_{12} &= 0 \\ a_{12}\alpha_2 + a_{22} &= 0.09886\,(\alpha_2 - 3) \\ b_{12}\alpha_2 + b_{22} &= 0.34432\,(\alpha_2 - 3) \\ \text{and} \qquad (1 + b_{22})\alpha_2 + a_{12} &= -0.03295\,(\alpha_2 - 3) \end{aligned}\right\} \ldots\ldots\ldots\ldots(26)$$

Eqs. 26 were solved and the numerical values placed in Eqs. 15. The resulting expressions for σ_x, σ_y, and τ_{xy} are given in Fig. 4. For a check of the inequality which must be satisfied in any elastic region (see Eq. 16*a*), the point $x = 0$, $y = -1$ was used, from which $\sigma_x = 0.311\,\gamma$, $\sigma_y = 1.108\,\gamma$, $\tau_{xy} = 0.066\,\gamma$, and $B = 1.7438$; then (see Eq. 16*a*) $1.409\,\gamma < \sigma_x + \sigma_y$ (as required).

Dam.—For region III the values determined in the computations for the upstream toe were applicable. This is true because of the fact that a shift in the origin of coordinates, to any point along the boundary where all stresses are zero, produces no change in the expressions for stresses. Stresses for region V were computed by the use of the same formulas and data as for region III except that $n = -2$. The stresses in region IV and expressions for the boundaries were established by equating direct and shear stresses across $x = \alpha_4 y$ and $x = \beta_4 y$. The inequality was checked as at the upstream toe.

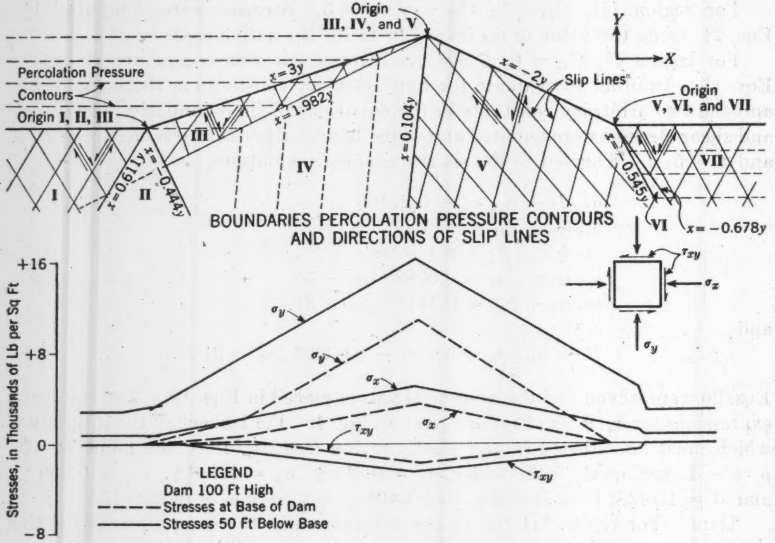
Downstream Toe.—Stress values for region V were obtainable directly from the solution for the dam. For region VII the stresses were the same as for region I at the upstream toe. The work of obtaining the stresses in region VI and of fixing the boundaries between regions was done by equating stresses as before, only this time the boundaries $x = \alpha_6 y$ and $x = \beta_6 y$ were used.

Directions of the slip lines as shown in the various plastic regions, Fig. 4, were obtained by substituting stresses for the region in question in Eqs. 24.

Example 2. Solution of a Typical Zoned Dam of 1 on 3 Upstream and 1 on 2 Downstream Slopes, with Reservoir Full.—Fig. 5 shows the results of the analysis and data used in the computations. An outline of the procedure follows.

Upstream Toe.—In this area the percolation pressure contours are taken as horizontal and as having a hydrostatic spacing vertically. The percolation function may then be written as $u = -D_w\,(y - H)$. This gives $\frac{\partial u}{\partial y} = -D_w$ and for a value of $\gamma + D_w\,e$ of 120 lb per cu ft; then, $C_2 = 57.6$; and $C_1 = 0$. The remainder of the solution is similar to that in Example 1.

Dam.—Stresses for region III may be taken from the upstream toe solution. Region V, which is taken as dry, was computed as in Example 1. In region IV


Origin
III, IV, and V
Y
X
Percolation Pressure
Contours
Origin I, II, III
x=3y
x=1.982y
x=−2y
Slip Lines
Origin
V, VI, and VII
x=0.104y
x=0.611y
x=−0.44y
x=0.545y
I
II
III
IV
V
VI
VII
x=−0.678y
BOUNDARIES PERCOLATION PRESSURE CONTOURS
AND DIRECTIONS OF SLIP LINES
+16
+8
0
−8
Stresses, in Thousands of Lb per Sq Ft
σy
σy
σx
σx
τxy
τxy
τxy
σx
σy
LEGEND
Dam 100 Ft High
Stresses at Base of Dam
Stresses 50 Ft Below Base

DATA

Height of dam, in ft	100
Dry density, in lb per cu ft	95
Wet density, in lb per cu ft	120
Coefficient of friction, *f*	0.7

EQUATIONS OF STRESSES

Regions I and VII:
$\sigma_x = -15.614\, y$
$\sigma_y = -57.6\, y$
$\tau_{xy} = 0$

Region II:
$\sigma_x = 3.788\, x - 17.930\, y$
$\sigma_y = 10.140\, x - 63.797\, y$
$\tau_{xy} = 6.197\, x - 3.788\, y$

Region III:
$\sigma_x = 5.695\, (x - 3\, y)$
$\sigma_y = 19.833\, (x - 3\, y)$
$\tau_{xy} = 1.898\, (x - 3\, y)$

Region IV:
$\sigma_x = 15.092\, x - 35.711\, y$
$\sigma_y = 46.984\, x - 113.32\, y$
$\tau_{xy} = -10.117\, x + 18.123\, y$

Region V:
$\sigma_x = -16.234\, (x + 2\, y)$
$\sigma_y = -51.558\, (x + 2\, y)$
$\tau_{xy} = 8.117\, (x + 2\, y)$

Region VI:
$\sigma_x = -60.195\, x - 56.423\, y$
$\sigma_y = -130.968\, x - 146.390\, y$
$\tau_{xy} = 88.790\, x + 60.195\, y$

FIG. 5.—TYPICAL ZONED DAM; ANALYSIS FOR FULL RESERVOIR

the percolation pressure contours were assumed parallel to the boundary $x = \alpha_4\, y$, so the percolation function was written as

$$u_4 = -D_w \frac{x - \alpha_4\, y}{\beta_4 - \alpha_4} \qquad (27)$$

Then

$$\frac{\partial u_4}{\partial y} = \frac{D_w\, \alpha_4}{\beta_4 - \alpha_4} \qquad (28a)$$

and

$$\frac{\partial u_4}{\partial x} = -\frac{D_w}{\beta_4 - \alpha_4} \qquad (28b)$$

When $\gamma + D_w\, e = 120$ lb per cu ft,

$$C_{24} = \frac{120\,\beta_4 - 57.6\,\alpha_4}{\beta_4 - \alpha_4} \qquad (29a)$$

and

$$C_{14} = \frac{-\,62.4}{120\,\beta_4 - 57.6\,\alpha_4} \qquad (29b)$$

The remainder of the solution is similar to that given in Example 1.

Downstream Toe.—Horizontal percolation pressure contours, spaced hydrostatic vertically, were taken in regions VI and VII. Then for these regions: $u = -\,D_w\, y$; $C_2 = 57.6$; and $C_1 = 0$.

Stresses in region V were computed for the dam assuming that it is dry. These same stresses were used when equating stresses across $x = \beta_6\, y$. This involves no inconsistency because the stresses computed for region VI are contact stresses and do not include the percolation pressure. The upward body forces due to flotation below river-bed level in zone V, however, are neglected. The details of the remainder of the procedure have been given previously.

By letting the percolation function in region VII be variable, its limiting value for stability may be determined by joining regions V and VII directly. This may be accomplished by equating normal and shear stresses across a common boundary and solving for the value of the pressure in the percolation function. This computation has been made for various conditions and the results are given in Table 3, Appendix II.

CONCLUSION

All present available solutions, of the fundamental equations that have been established in this paper, are linear in form. This definitely limits the stress systems in the various regions and makes mandatory a system of linear boundaries between regions. More general solutions are needed to avoid many of the present limitations. The present linear solutions should be considered as representing a first step in the development of a general method of approach.

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APPENDIX I

NOTATION

The following symbols, used in this paper, conform essentially with "Soil Mechanics Nomenclature" approved by the Society in 1941 and published as Manual of Engineering Practice No. 22:

a = arbitrary constant (a_{1i}, a_{2i}, a_{3i}, etc.);

$B = \dfrac{\sqrt{1+f^2}}{f}$;

b = arbitrary constant (b_{1i}, b_{2i}, b_{3i}, etc.);

C = cohesion; also arbitrary constant (C_{1i}, C_{2i}, C_{3i}, etc.);

D_w = density of water;

d = arbitrary constant (d_{1i}, d_{2i}, d_{3i}, etc.);

e = ratio of voids to total volume;

f = coefficient of internal friction;

H = depth of water in reservoir;

h = horizontal earthquake acceleration force expressed as a force per unit of volume;

n_1, n_2, n_3, etc. = arbitrary constants in the equations of the outer boundaries;

R = a function representing the reserve strength to resist shear failure;

R_0 = the minimum of R;

u = percolation pressure;

x and y = rectangular coordinates;

α = arbitrary constant in the equations of the inner boundaries;

β = arbitrary constant (see α);

γ = dry weight per unit of volume;

ϵ' = an arbitrary constant;

θ = angle of the x-axis to the line of action of σ_1, counted positive counterclockwise;

λ = tangent of the angle ψ;

σ = normal component of stress, positive when compressive;

σ_x and σ_y = normal stresses on planes normal to the x and y axes respectively;

τ = shear stress;

τ_{xy} = shear stress in direction of x on plane normal to y;

ϕ = angle from the line of action of σ_1 to the plane under consideration, counted positive counterclockwise;

ψ = angle from the y-axis to the slip lines, counted positive clockwise.

APPENDIX II

TABLE 1.—LIMITING SLOPES FOR VARIOUS EARTHQUAKE INTENSITIES

Earthquake acceleration—fraction of gravity	Wet density, lb per cu ft	COEFFICIENT OF FRICTION						
		0.50	0.55	0.60	0.65	0.70	0.75	0.80
0.00 *g*	115	4.63	4.25	3.94	3.69	3.47	3.28	3.11
	120	4.41	4.06	3.76	3.51	3.30	3.12	2.97
	127.5	4.15	3.81	3.54	3.30	3.11	2.94	2.79
	135	3.94	3.62	3.36	3.13	2.95	2.79	2.65
	142.5	3.77	3.46	3.21	3.00	2.82	2.66	2.53
0.05 *g*	115	5.99	5.37	4.89	4.49	4.17	3.90	3.67
	120	5.64	5.07	4.61	4.24	3.94	3.69	3.47
	127.5	5.22	4.70	4.29	3.95	3.67	3.44	3.24
	135	4.90	4.41	4.03	3.72	3.46	3.24	3.05
	142.5	4.64	4.19	3.82	3.53	3.29	3.08	2.90
0.10 *g*	115	8.41	7.23	6.37	5.71	5.19	4.77	4.43
	120	7.74	6.69	5.92	5.32	4.85	4.47	4.16
	127.5	6.99	6.08	5.40	4.87	4.46	4.11	3.83
	135	6.43	5.62	5.01	4.53	4.15	3.84	3.58
	142.5	6.00	5.26	4.70	4.26	3.91	3.62	3.38
0.15 *g*	115	13.91	10.93	9.05	7.76	6.82	6.11	5.55
	120	12.18	9.75	8.18	7.07	6.25	5.63	5.13
	127.5	10.43	8.52	7.23	6.31	5.62	5.09	4.66
	135	9.25	7.65	6.56	5.76	5.16	4.68	4.30
	142.5	8.40	7.02	6.05	5.34	4.80	4.37	4.03
0.20 *g*	115	38.58	21.79	15.33	11.92	9.81	8.38	7.35
	120	27.73	17.60	13.00	10.38	8.69	7.51	6.64
	127.5	20.11	13.97	10.79	8.84	7.53	6.59	5.88
	135	16.18	11.81	9.37	7.81	6.73	5.94	5.34
	142.5	13.77	10.38	8.39	7.08	6.15	5.46	4.93
0.25 *g*	115	....	....	47.35	24.87	17.06	13.10	10.70
	120	....	81.50	30.53	19.04	13.97	11.11	9.29
	127.5	....	37.14	20.66	14.47	11.22	9.23	7.89
	135	59.76	25.07	16.06	11.93	9.56	8.03	6.95
	142.5	36.42	19.44	13.40	10.32	8.44	7.19	6.29

Table 1 can be used to investigate the stability of slopes in the following cases:

(*a*) Downstream slopes of dams below the point where the phreatic line emerges;

(*b*) Upstream slopes following a rapid drawdown; and

(*c*) Slopes of cuts or fills when saturated.

TABLE 2.—LIMITING SLOPES FOR VARIOUS VALUES OF C_1

f	VALUES OF C_1:										
	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.50	2.00	2.28	2.63	3.07	3.67	4.50	5.75	7.83	12.00	24.5	....
0.55	1.82	2.06	2.34	2.71	3.17	3.79	4.66	5.96	8.13	12.48	25.50
0.60	1.67	1.87	2.12	2.42	2.80	3.29	3.93	4.84	6.20	8.47	13.00
0.65	1.54	1.72	1.94	2.20	2.51	2.91	3.41	4.09	5.04	6.46	8.83
0.70	1.43	1.59	1.78	2.01	2.28	2.61	3.03	3.56	4.27	5.26	6.75
0.75	1.33	1.48	1.65	1.85	2.09	2.38	2.72	3.16	3.71	4.46	5.50
0.80	1.25	1.39	1.54	1.72	1.93	2.18	2.48	2.84	3.30	3.89	4.67

The type of failure contemplated is shown in Fig. 6(*a*) to consist of a sliding of material down the slope. The fill material is assumed to be cohesionless. The percolation pressures are assumed to have a hydrostatic distribution in the vertical direction. The numbers in Table 1 represent the steepest slopes that can be expected to remain stable under assumed conditions.

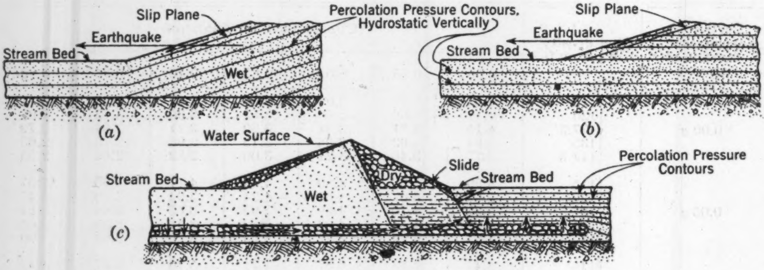
Slip Plane
Earthquake
Stream Bed
Wet
Percolation Pressure Contours, Hydrostatic Vertically
(a)
Slip Plane
Earthquake
Stream Bed
(b)
Water Surface
Stream Bed
Wet
Dry
Slide
Stream Bed
Percolation Pressure Contours
(c)

FIG. 6

Table 2 is useful for investigating either the stability of upstream slopes of earth dams under "reservoir full" conditions or the stability of dry slopes.

TABLE 3

Density, in lb per cu ft		Coefficient of Friction				Coefficient of Friction			
dry	wet	0.50	0.60	0.70	0.80	0.50	0.60	0.70	0.80
		(*a*) Slope on Downstream Face, 1 on 2				(*c*) Slope on Downstream Face, 1 on 4			
90	115	97.78	104.89	108.02	110.03	101.55	105.61	108.32	110.16
100	120	100.87	108.76	112.25	114.47	105.02	109.56	112.57	114.62
110	127.5	106.45	115.14	118.97	121.42	111.06	116.02	119.33	121.59
120	135	112.04	121.51	125.70	128.37	117.07	122.47	126.09	128.55
130	142.5	117.63	127.89	132.42	135.32	123.07	128.93	132.85	135.51
		(*b*) Slope on Downstream Face, 1 on 3				(*d*) Horizontal			
90	115	101.22	105.47	108.25	110.13	115.00	115.00	115.00	115.00
100	120	104.69	109.41	112.51	114.59	120.00	120.00	120.00	120.00
110	127.5	110.66	115.85	119.26	121.55	127.50	127.50	127.50	127.50
120	135	116.63	122.30	126.01	128.51	135.00	135.00	135.00	135.00
130	142.5	122.60	128.74	132.76	135.47	142.50	142.50	142.50	142.50

The type of failure contemplated is shown in Fig. 6(*b*) to consist of a sliding of material down the slope. The fill material is assumed cohesionless. The percolation pressures for saturated slopes are assumed to have a hydrostatic distribution in the vertical direction. The numbers in Table 2 represent the steepest slopes that can be expected to remain stable under assumed conditions.

The quantity C_1 is defined for wet material as

$$\frac{\text{Fraction of gravity} \times \text{wet density}}{\text{Wet density} - \text{density of water}},$$

and for dry material as fraction of gravity.

Table 3 is useful for evaluating the effects of excessive uplift or percolation pressure gradients at the downstream toe of earth dams. It is also useful for estimating the intensity of the upward percolation pressure gradients necessary to produce piping.

The computations indicate that a slide should occur at the downstream toe before actual piping is observed. However only a small difference is indicated between the gradients necessary to cause sliding and piping, and, if the conditions were local, it would be reasonable to expect the piping phenomena to appear first.

The individual entries in Table 3 are the upward percolation pressure gradients in pounds per cubic foot necessary to produce failure, as shown in Fig. 6(*c*). The gradients necessary to produce piping are indicated in that part of the table marked "Horizontal."

DISCUSSION

R. G. Hennes,[4] Assoc. M. Am. Soc. C. E.—In this paper the authors have advanced the development of soil mechanics and have, perhaps, somewhat shifted the course of that development. All tools available to the engineer for the design or analysis of earthwork and foundations are notably of limited application. Each new tool supplements rather than supplants the older. In the opinion of the writer stability analysis by summation of forces along a curved potential slip surface will remain the more convenient and more reliable method for estimating the stability of earth slopes. Quite possibly, the accuracy of the conventional procedure can be improved by determining the distribution of normal pressure along the slip curve by the proposed method of analysis, instead of by assuming a vertical pressure at each point proportional to the height of the overlying column of earth.

In studying the stability of the foundation of embankments the new method offers opportunity for improvement over solutions that depend upon formulas based on the assumption of a semi-infinite solid, by providing a better approximation of stress distribution over the base of the embankment.

The authors state (see "Introduction") "that the terms 'elastic' and 'plastic' are used herein to describe zones * * * and not as descriptions of types of material used in the structure." There is danger that the casual reader might construe this statement to mean that the validity of the analysis is independent of the elastic properties of the material, whereas actually such is not the case. The authors have not freed the soil analyst from the necessity of pretending that sand and a gravel follow Hooke's law. This they recognize elsewhere in the paper. The assumption that stress is proportional to strain, of course, is common to all the usually accepted equations for stress distribution; but the error thus introduced is less consequential than in the present case when, as in consolidation studies, the object of the analysis is the computation of relatively small pressure changes that occur at considerable depth due to a surface loading. In the latter case only a small part of the stress-strain curve is involved, and this short arc might reasonably be replaced by a straight line. On the other hand, by its very nature a stability analysis of an earth dam presupposes the possibility of stress intensities ranging in value from zero at a free boundary to maximum resistance in plastic zones. The curvature of the stress-strain graph for such a stress range in an embankment cannot be ignored without adversely affecting the validity of the analysis.

It is to be hoped that further development of the authors' attack on the problem of stability may introduce a substitute for Eq. 3. It appears to the writer that triaxial compression tests provide grounds for believing such a development at least a possibility. Data from a limited number of triaxial tests on sand performed at the University of Washington at Seattle, Wash., indicate that, for a given lateral pressure, the major part of the axial stress-strain curve can be fairly well described by the equation

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[4] Associate Prof., Civ. Eng., Univ. of Washington, Seattle, Wash.

$$\epsilon = \zeta^{\frac{\sigma - n}{m}} \qquad (30)$$

in which ϵ represents strain, m and n are constants for the given test, and ζ is the base of natural logarithms. The next step would be to formulate an expression for strain involving both axial and lateral pressures.

Eq. 3 can be obtained by substituting for strain in terms of stress in the equation:

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} - \frac{\partial^2 \epsilon_{xy}}{\partial x \, \partial y} = 0 \qquad (31)$$

If more thorough investigation should confirm the writer's hypothesis that there is available a better approximation of the facts than Hooke's law, then substitution of the former in Eq. 31 would provide a basis for improving upon Eq. 3, and would possibly lead to more realistic solutions than Eqs. 15.

It is true that such a development, if successful, would increase somewhat the amount of laboratory work prerequisite for mathematical analysis. This in itself would be a healthy situation. The general adoption of more rigorous mathematical methods has a tendency to obscure the dependence of all analysis on the accuracy of the basic physical measurements. Enforced reliance on the laboratory serves to modify overconfidence engendered by mathematical precision, while at the same time it helps to justify use of these same theoretical refinements.

STANLEY U. BENSCOTER,[5] JUN. AM. SOC. C. E.—The stress analysis of a cohesionless embankment, which the authors have presented, is a definite step forward in the knowledge of the behavior of granular materials. Many contributions, both theoretical and experimental, in this field of stress analysis are needed to create a reasonable conception of the state of stress at all points within a granular mass. This field has been neglected too long, most probably because of the supposedly insurmountable mathematical difficulties involved in dealing with a granular mass. If physicists and aeronautical engineers have been able to treat, mathematically, the behavior of groups of particles having variable velocities in thermodynamics and aerodynamics, soil physicists certainly should be able to treat, mathematically, groups of particles having variable masses. It is almost certain that statistical methods and other branches of advanced mathematics will find application eventually in the stress analysis of soils.

The authors have obtained a solution for stresses in an embankment which is possible of existence so far as can be determined from the geometry of a Mohr diagram and the assumption of a failure law representable as a straight line passing through the origin of the Mohr axes. The division of the embankment and foundation into elastic and plastic regions offers an interesting approach to the problem. The reported behavior of the elastic regions is reasonable since they decrease with increasing height of the embankment. The concept of the reserve strength R is valuable and should find much application in the future.

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[5] Structural Design Section, Bridge Branch, U. S. Engrs., Fort Belvoir, Va.

The geometry of a Mohr diagram yields a quadratic algebraic equation (Eq. 11*b*) relating stresses and the internal friction coefficient in a plastic region. The authors have shown that linear expressions for stresses can be found which satisfy this equation and the two equations of equilibrium of an element. As the authors state, future investigations must be made to find the possibilities of expressing stresses in a plastic region by higher degree polynomials or transcendental functions. This suggests that several solutions may be found for stresses that are possible of existence; and there will probably be a need for one or more additional laws to govern the solution before the correct one (or the most probable one) can be chosen.

If the stresses are expressed in terms of an Airy function, thus assuring equilibrium of elements, the quadratic algebraic equation becomes a quadratic differential equation. Prof. Theodor von Kármán, M. Am. Soc. C. E., drew attention to this equation as one among many non-linear differential equations with which the engineer is struggling.[6] It seems strange that government engineering agencies interested in the behavior of soil are not financing a program of mathematical research in the behavior of granular masses.

Another manner of dealing with the non-linear equation governing stresses in a granular mass is the use of the principal stress lines as a coordinate system. In this orthogonal curvilinear system the shearing stress vanishes and the non-linear equation becomes linear. The resulting equation is merely a proportionality relation between the principal stresses as given by Eq. 11*a*. A simple example will illustrate: Consider the two dimensional polar coordinate system to form a set of principal stress lines in a granular mass and the body forces to be negligible. The shearing stress $\tau_{r\theta}$ is zero at all points, and the Mohr diagram reveals that

$$\sigma_r = K\,\sigma_\theta \quad \text{(32)}$$

in which K is Rankine's coefficient, either active or passive. The formulas for σ_r and σ_θ, in terms of an Airy function, are

$$\sigma_r = \frac{\partial^2 F}{\partial r^2} \quad \text{(33a)}$$

and

$$\sigma_\theta = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} \quad \text{(33b)}$$

Assuming symmetry about the origin, try the following simple stress function,

$$F = A\,r^n \quad \text{(34)}$$

Differentiating to obtain σ_r and σ_θ and substituting in Eq. 32:

$$\sigma_r = p\left(\frac{r}{a}\right)^{K-1} \quad \text{(35a)}$$

and

$$\sigma_\theta = \frac{p}{K}\left(\frac{r}{a}\right)^{K-1} \quad \text{(35b)}$$

In Eqs. 35, p is the value of σ_r at $r = a$. These formulas (represented graphically in Fig. 7) indicate the distributions that are possible in a solid or hollow cylinder of dry sand acted upon by a uniform external pressure. The inside pressure of the hollow cylinder may vary between minimum and maximum values. If the inside pressure has an intermediate value, the internal stress distribution is not known. This may be a statistical problem rather than one in elasticity. For the solid cylinder, Curve II must be rejected since it becomes infinite at the origin. Curve I gives a possible stress distribution; yet it is certainly questionable.

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[6] "The Engineer Grapples with Nonlinear Problems," by T. von Kármán, *Bulletin*, Am. Mathematical Soc., Vol. 46, No. 8, August, 1940, pp. 615–683.

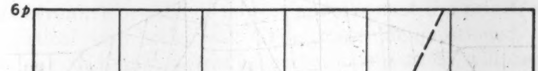

FIG. 7.—VALUES OF σ_r

The state of stress at a point in a body, in the two-dimensional case, may be completely defined by giving the stress circle on a Mohr diagram and the directions of principal stresses. The directions of principal stresses at all points within the body may be represented by drawing the principal stress lines. Thus it is apparent that investigators should strive to find laws governing the shape and location of principal stress lines. Today they know only that these lines form two orthogonal curvilinear systems which are continuous even through boundaries between elastic and plastic regions. They are not necessarily representations of harmonic functions, or solutions of La Place's equation, as one might be tempted to hope. In this connection it is interesting to note that, in 1941, M. A. Sadowsky[7] showed that in perfectly plastic behavior (no internal friction) principal stress lines form an equi-areal pattern. In elastic bodies these lines must be parallel and perpendicular to unloaded surfaces. In a granular mass this may not be true.

The principal stress lines for Example 1 (Fig. 1) may readily be drawn as shown in Fig. 8. These lines must be questioned, with a view toward encouraging further study, because they are not parallel and perpendicular to the embankment surface. If the soil contains the slightest amount of cohesion, this orthogonal boundary condition must be satisfied since a stress circle can then be drawn on a Mohr diagram passing through the origin and tangent to the limiting line. It is possible that for a granular cohesionless mass the failure law should be represented by a stair-stepped line passing through the origin, but with extremely small steps. If such were true, a tiny circle could then be drawn tangent to the limiting line and passing through the origin of the Mohr axes. This would demand the aforementioned orthogonal boundary condition.

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[7] "Equiareal Pattern of Stress Trajectories in Plane Plastic Strain," by M. A. Sadowsky, *Journal of Applied Mechanics*, Vol. 8, No. 2, June, 1941, pp. A-74–A-76.

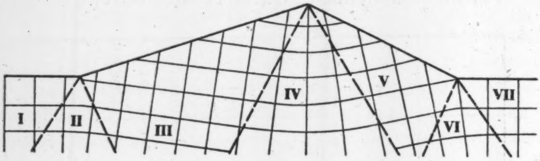

Fig. 8

In an attempt to sketch a set of principal stress lines orthogonal to the boundary one may try a set of circular and radial type lines. If an attempt is made to picture the changing shape of these lines as the height of the embankment is increased from zero to a maximum, this set of lines is rejected for a set of the type sketched in Fig. 9. These lines correspond to a non-linear solution. A few interesting observations can be made from this set of principal stress lines, assuming that the embankment has the maximum possible height. At point A, Fig. 9, the principal stress σ_1 will be the minimum principal stress, whereas at point B the stress σ_1 is the maximum principal stress. Thus some point between A and B, on any path that may be chosen, is an isotropic point or one at which $\sigma_1 = \sigma_2$. At this point, Mohr's circle degenerates to a point and there are no shearing stresses in any direction. Similarly, at point C, σ_2 is the minimum principal stress and becomes the maximum at point D. The trajectory of isotropic points might appear as curve EFG. This curve would probably appear first around the fillets and expand from there to its final position as the height of the embankment is increased. Instinctively one expects the plastic state to develop around the fillets from stress concen-

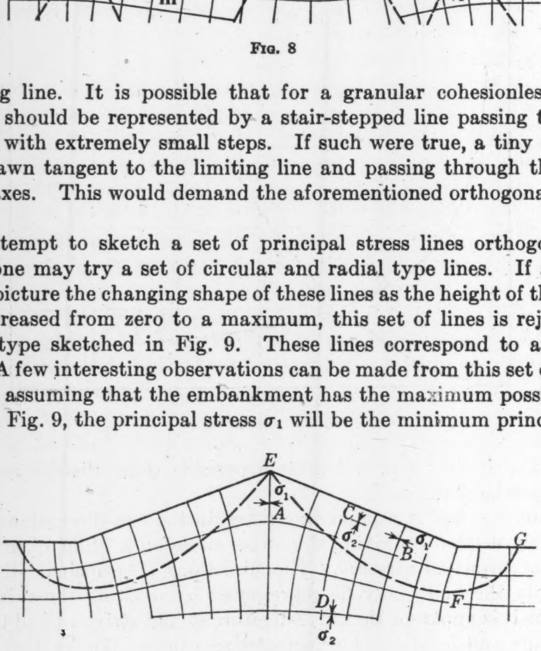

Fig. 9

tration as it occurs in elastic bodies. Heavy thrusts at the toe and heel, parallel to the surface, are in agreement with M. Frontard's solution.[8] The trajectory of isotropic points should lie within a band of elastic-state stress although the stress in such a band may be a matter of statistical mechanics rather than elasticity.

If the author's solution were applied to a symmetrical embankment constructed to maximum possible height, it appears that the principal stress lines would be vertical and horizontal. This must be questioned. This writer believes that at maximum height of embankment there will be a maximum amount of curvature and rotation of principal stress lines, which is another way to express "arching" action. The authors find Rankine's state of stress in regions I and VII of Example 1, whereas the truth would be a natural state with "at rest" pressures. This is a minor criticism. It is not apparent how the elastic and plastic regions extend downward into the foundation below the region shown in Fig. 4. The linear solution presented seems to be reasonable at an external corner, or apex of an embankment, but must be questioned when applied to a reentrant corner or fillet since it does not appear to predict the expected stress concentration.

The comments presented in this discussion are not intended to be critical of the solution presented by the authors, but rather they are meant to assist in stimulating increased interest in dealing with soil-stress problems by mathematical theory. Attention might be called to the fact that, at present, soil analysts are unable to detect experimentally the state of stress in a granular mass at various points. Hence there is no manner of proving by the "scientific method" that a given solution is true. So long as investigators are content to struggle with metallic pressure cells they will make little progress. There is great need for creative thought in both the experimental and theoretical solution of problems in the stress analysis of soils.

A. Hrennikoff,[9] Assoc. M. Am. Soc. C. E.—The novel approach to the problem of stability analysis of cohesionless granular materials, presented in this paper, is interesting.

The granular mass is visualized by the authors as divided into alternate elastic and plastic zones. The state of stress in the former obeys Hooke's law and is governed by the equations of equilibrium (Eqs. 12) and by the equation of compatibility or continuity, which for elastic materials takes the form of Eq. 3. The state of stress in the plastic regions is characterized by impending sliding at every point in the region, and is subject to the plasticity equation (Eq. 16) in addition to the equations of equilibrium (Eqs. 12), whereas Eq. 3 does not apply since the continuity has been broken. In these equations, allowance is made for the pressure of percolating water and for the constant horizontal earthquake acceleration. Expressions obtained for the stresses are linear functions of the coordinates.

The foregoing presents an outline of the proposed method, and now some of the limitations of the assumptions involved in it will be discussed.

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[8] "Calculs de Stabilité des Barrages en Terre," by M. Frontard, *Transactions*, Second Congress on Large Dams, Vol. IV, pp. 243–293.

[9] Asst. Prof., Civ. Eng., Dept. of Civ. Eng., Univ. of British Columbia, Vancouver, B. C., Canada.

1. *Uniqueness of Solution.*—A disconcerting feature of the problem of stress analysis in a mass of soil consisting of a combination of elastic and plastic regions is the multitude of the states of stress that may be found for the same exterior boundary conditions after some more or less artificial conditioning of the soil mass. This important feature distinguishes sharply the problem discussed here from the problems of the theory of elasticity, in which the solutions are unique.

To illustrate this point, consider a dam similar to the one shown in Fig. 4 and assume, for the sake of simplicity, that it is symmetrical about the vertical plane. Imagine, now, two adjacent thin, vertical, smooth walls cutting through the dam along the axis of symmetry and dividing it into two equal parts. Holding the right wall for the time being in place, move the left wall horizontally to the left. A passive plastic state of stress forms now in the soil in the vicinity of the wall, and this plastic region extends gradually to the left, as the stresses increase with further advance of the wall. Next, leave the left side of the dam at the location it has reached and perform a similar operation on the right half by moving the right wall an equal amount. When the soil is so conditioned, eliminate the central gap between the two halves of the dam by moving one of the parts, together with its foundation, toward the other, and after that withdraw the walls. The two latter operations need not change the state of stress previously created in the two halves of the dam. Likewise wall movements may be imagined in the directions away from the soil creating an active plastic region in the middle of the dam. An infinite number of similar but unsymmetrical movements may also be conceived leading to a wide variety of the resultant states of stress, every one of which, no matter how artificial may be its origin, satisfies the same differential equations of equilibrium, compatibility, and plasticity, and the same exterior boundary conditions, as the solutions found by the proposed method.

Most of these theoretically conceivable states of stress bear no resemblance to the natural state resulting from the actual conditions of construction of the embankment, application of loads, and the character of possible failure.

It seems likely that the natural state of stress in the dam, as distinct from these artificial states, is also not unique in view of the influence of such rather indefinite factors as settlement, bulking, and shrinkage caused by temperature and moisture changes; but the range of such fluctuations in the natural state should not be wide.

This discussion leads to the conclusion that a solution of the differential equations cannot be accepted even as an approximate solution of the embankment problem, not to mention an exact solution, without at least demonstrating first by some appropriate qualitative analysis that such solution is reasonable under the given conditions. This qualitative investigation of the problem with the view to placing the elastic and plastic regions into locations where they are likely to develop, and of assigning to the latter the active or passive states as may be demanded by the circumstances, is completely overlooked in the method proposed. A stereotyped assumption of three regions, two plastic on the outside and one elastic between them, is used instead at every point of break in slope, and apparently under all circumstances—an assumption very convenient from the mathematical point of view, but totally unjustified physically.

Referring to the specific examples presented in Figs. 4 and 5, the presence of plastic states in regions I and VII above and below the dam seems particularly objectionable, resembling too much the generally discarded Rankine's theory of earth pressure. Also one does not see any reason for the existence of plastic states in zones III and V of Fig. 4. A dam with slopes as low as those shown in this example, and subjected to no loads other than its own weight, should, it would seem, be in an elastic state throughout.

If the stress pattern found by the proposed analysis may thus be deemed to "fall short of the mark," so the possible contention that the method answers the question of stability or instability of the embankment must likewise be considered untenable in view of the arbitrariness of location of the elastic and plastic regions.

2. Linear Variation of Percolation Pressure.—Such linear variation is assumed throughout the granular mass, as is implied in the form of the expressions for the integration constants, Eqs. 13*a* and 13*b*, making the flow lines of the percolating water straight and parallel in each region. This simple but unrealistic assumption misrepresents the effect of water and greatly reduces danger to stability arising from convergence of the flow lines and from the related phenomena of boiling and piping at the toe of the dam.

3. Hooke's Law.—Some error results from the application of this assumption to the elastic regions for the following reasons:

(*a*) The stress-strain relationship for the soil is not quite linear;

(*b*) The curve of unloading does not completely coincide with the curve of loading;

(*c*) There is some uncertainty about the legitimacy of applying the compatibility equation to soil, since this condition implies preservation of elastic continuity of deformations throughout all the past history of the soil mass from the unloaded condition to its present state—that is, through the period of construction, settlement, and the following volume fluctuations caused by temperature and moisture changes.

This uncertainty is not removed by the fact that, in the method proposed, the compatibility condition does not contribute to the results, being satisfied automatically by any linear solution for stresses. It must be admitted, however, that compliance of the soil with Hooke's law has been postulated by many investigators.

4. Elastic and Plastic Regions.—When loading decreases or reverses, the regional boundaries between the elastic and plastic zones undergo some shifting. A region that was formerly plastic but is no longer so after the load change is neither elastic nor plastic in the sense used by the authors, since the limiting sliding condition exists no more, and at the same time the compatibility equation does not hold in view of the continuity having been broken during the plastic stage. This consideration seems to invalidate the applicability of the method to the conditions that involve a reversal or a substantial change in the character of the loading. Such conditions are present in an earthquake or in the case of a sudden drawdown in a reservoir retained by the earth dam—cases which the authors apparently list within the scope of their theory.

5. Continuity of Stresses.—The assumption of continuity across the elastic-plastic boundaries of the direct (normal) stresses parallel to the boundaries has not been substantiated and appears improbable.

Some minor inconsistencies and obscurities in presentation may also be mentioned.

In Example 2 it is difficult to interpret physical conditions corresponding to the solution given in Fig. 5. Judging by the pressure contours there seems to be a lack of equilibrium in the water conditions at the boundary of zones V and VI. Again, the part played by region V with regard to percolating water is puzzling. This region seems to be composed of a very pervious material through which the water coming from region IV falls as through a chute. This naturally raises the questions: Where does this water go, and how can this situation be reconciled with the presence of pressure head in regions VI and VII?

Referring again to Fig. 5, it appears that the condition of equality of the boundary stresses at the border of zones II and IV is not satisfied. The same difficulty arises also in other places.

As the authors correctly point out, the state of stress in any elastic region must satisfy the inequality (see Eq. 16*a*) $R_0 > 0$. It is insufficient, however, to check this inequality at an arbitrary point, as the authors do, without proving that the function R_0 retains the same sign throughout the entire region.

It is not clear whether the data presented in Appendix II are based on the theory proposed; and, in case they are, how they are derived from it.

The foregoing reasons lead the writer to the belief that the proposed method of analysis of stresses in granular materials is unsatisfactory in view of its highly speculative nature and arbitrariness. However, this conclusion should not minimize the credit due the authors for the novelty of approach and mathematical ingenuity in attacking a baffling engineering problem.

D. P. KRYNINE,[10] M. Am. Soc. C. E.—The method of analysis described in this interesting paper consists in subdividing an earth dam into several regions by straight lines passing through the three vertexes of the triangle representing an idealized dam. These regions in alternate order are assumed to be "plastic" and "elastic," the difference between them being that the "plastic" regions are assumed to be on the point of losing "stability" by shear, whereas there is no shear danger in the "elastic" regions. As the authors themselves state, these terms do not define the type of the material. In fact, the latter, in general, is neither plastic nor elastic. It is regrettable that this inaccurate, though common, terminology has been adopted.

If an earth mass fails, it is generally due to a deficiency in lateral support. A vertical pressure requires a corresponding amount of horizontal pressure for stability. The "plastic" regions are brought to the state of incipient motion by so changing both the vertical and the horizontal pressures as to just satisfy Mohr's condition of failure.

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[10] Research Associate in Soil Mechanics, Dept. of Civ. Eng., Yale Univ., New Haven, Conn.

Existing Methods of Analysis of Earth Structures.—The existing methods of analyzing earth structures may be subdivided into two main classes: (*A*) Methods by which stability is checked without computing stresses at all individual points of the earth mass; and (*B*) methods by which such stresses are computed.

Methods of Class (*A*).—A geological section of the locality should be traced, and the cross section of the earth structure located on it. For this purpose the general dimensions of the earth structure are first chosen mostly from experience.

From the study of the geological section and the cross section of the structure the designing engineer tries to answer the question: "What might happen to my structure?" As Hardy Cross, M. Am. Soc. C. E., frequently says in his lectures, this question should be answered by any designer analyzing his structure—in any branch of structural design. The most common case of possible failure of an earth structure is sliding along a more or less regular surface (failure line) which in cross section is generally assumed to be an arc of a circle (see Fig. 10). The analysis should be made for all possible cases of separation of part I of the structure (hereafter termed "wedge") from the remaining part II. Conditions of equilibrium for the case of incipient motion of wedge I are established, and the safety factor found.

This approach is deficient mostly because of unwarranted assumptions that must be made as to the distribution of the shearing stress and the shearing resistance along the assumed failure line AB. The methods of this class are very valuable, however, from several other points of view. Since wedge I is considered as a solid non-deformable body, no knowledge of stresses within it is needed. Furthermore, in establishing the shape of a failure line, the designer never loses sight of the actual physical conditions of both the structure and its foundation. An appreciable percentage of earth-structure failures are foundation failures, and one of the principal aims of the designer is to avoid them. Among the methods of class (*A*) the so-called "Swedish method" and the "ϕ-circle method"[11, 12] should be mentioned.

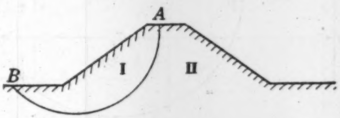

FIG. 10.—SEPARATION OF PART I OF A STRUCTURE FROM PART II ALONG THE FAILURE LINE AB

Methods of Class (*B*).—The writer limits himself to mentioning two methods of this class, both emanating from the Bureau of Reclamation. In one of them[13] the vertical normal stress, σ_y, equals the weight of the material above the given point, as in the Swedish method, and to take care of the horizontal normal stress a "compaction factor" is introduced. There are correction stresses designed to make the stresses satisfy the boundary conditions. Shearing stresses are checked against the shearing strength of the material, taking into account both friction and cohesion. In this way the factor of safety at

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[11] "Contributions to Soil Mechanics 1925–1940," by Donald W. Taylor, published by Boston Soc. of Civ. Engrs., 1940, pp. 337–386; also, "Stability of Earth Slopes," by the same author, *Journal*, Boston Soc. of Civ. Engrs., Vol. XXIV, 1937, p. 197.

[12] "Analysis and Control of Landslides," by R. G. Hennes, *Bulletin*, Univ. of Washington, Eng. Experiment Station, 1936.

[13] "Rational Design of Earth Dams," by J. H. A. Brahtz, *Transactions*, 2d Cong. of Large Dams, Washington, D. C., 1936, Vol. IV, pp. 543–576.

different points of the dam is computed, and necessary corrections are introduced into the preliminary design of the structure.

The peculiarities of the method described in the paper as compared to the method by J. H. A. Brahtz, just outlined, are the following: (*a*) Cohesion is not taken into consideration; and (*b*) there is no evaluation of a factor of safety, but simply a determination of what the authors call "reserve of strength" in the "elastic" regions. Since the "plastic" regions used in this method are those in which a failure is likely to occur, the method in question approaches the methods of class (*A*) more closely than does that of Brahtz.

Incipient Motion in "Plastic" Regions and Mohr's Circle.—If the stresses σ_x, σ_y, and $\tau_{xy} = \tau_{yx}$ at a point are in equilibrium, the ends of the vectors representing these values (σ_x = BC, σ_y = DE, $\tau_{xy} = \tau_{yx}$ = F′ E = CF, Fig. 11(*a*)) are located at a circle termed, as is well known, Mohr's circle. Vectors BC and DE are plotted horizontally from a vertical line BAD, A being a point located in the horizontal line passing through the center of the circle, O. Trace a line AG, its slope being equal to the coefficient of friction of the given earth material, *f*. In other words, make the angle GAO equal to the angle of friction. The point in question belongs to a "plastic" region if Mohr's circle touches line AG as in Fig. 11. If Mohr's circle is located below line AG and does not touch it, the point in question belongs to an "elastic" region.

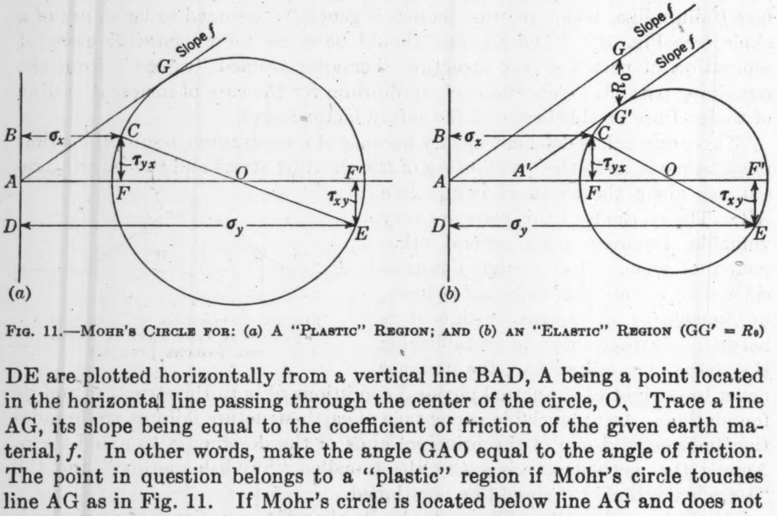

FIG. 11.—MOHR'S CIRCLE FOR: (*a*) A "PLASTIC" REGION; AND (*b*) AN "ELASTIC" REGION (GG′ = R_0)

Mohr's circle also may be constructed readily for the case of a cohesive material ($C > 0$). All formulas of the paper from Eq. 4*a* to Eq. 11*b*, inclusive, may be directly, or practically directly, seen on Mohr's circle. No doubt these equations are correct, but most of them as represented in an analytical form are not necessary for the purposes of the paper. In particular, R (Eq. 6*a*) is nothing else but the vertical distance between different points of Mohr's circle and line AG. Its minimum value, function R_0 (Eq. 6*b*), is the vertical distance between parallel lines AG and A′G′, the latter being tangent to the circle (Fig. 11(*b*)).

The values of the vertical pressure, σ_y, and the horizontal pressure, σ_x, to be used in the construction of Mohr's circle, should be corrected for percolation and earthquake. The vertical and horizontal components of the force of percolation, per unit of volume of the structure, are denoted by $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial x}$ in the paper. The vertical component $\frac{\partial u}{\partial y}$ ("uplift") acts against gravity. The symbol C_2 as used in the paper is the unit weight of dry earth material, plus the weight of moisture in pores, if any, minus uplift if any (Eq. 13*a*).

The horizontal component of the percolation force, $\frac{\partial u}{\partial x}$, and the horizontal force of earthquake, *h*, per unit of volume of the dam, decrease the lateral support. The algebraic sum of these two quantities, designated by $C_1 C_2$ in the paper, is given per unit of volume of the dam. It must be recomputed per square unit of the cross section of the dam as shown hereafter.

Differential Equations Are Not Necessary in "Plastic" Regions.—The use of differential equations to compute stresses in "plastic" regions may be avoided by using the following simple procedure. Refer to Fig. 12(*a*) in which *x* and *y* correspond to the point where stresses are to be computed. Cut from the dam a vertical prism ABCD having DC = 1 sq unit for base, this "unit" being very small. The given point is supposed to be at the middle of this base. The latter carries the vertical volume forces of the prism ABCD, which amount to

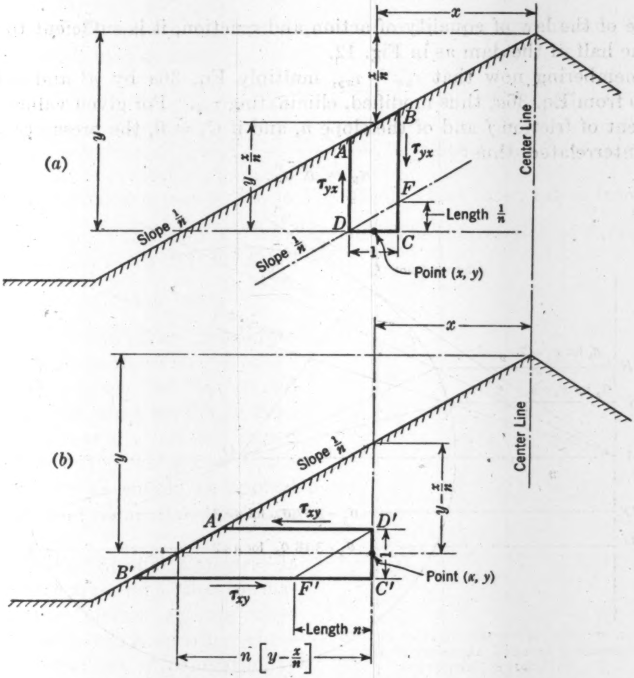

FIG. 12.—NORMAL STRESSES AND SHEARS IN A "PLASTIC" REGION

$C_2\left(y - \frac{x}{n}\right)$, this being the absolute value of those forces. In addition, there is an excess of shearing stress, τ_{yx}, acting along area FC $= \frac{1}{n}$ sq units, since the shearing stresses along areas AD and BF are balanced. Hence, the vertical pressure, σ_y, at point (x,y), is:

$$\sigma_y = -C_2\left(y - \frac{x}{n}\right) + \frac{1}{n}\tau_{yx} = \frac{C_2}{n}(x - n\,y) + \frac{\tau_{yx}}{n} \ldots\ldots (36a)$$

In a similar manner, Fig. 12(*b*) represents a horizontal prism A′B′C′D′ cut from the dam, the given point (x,y) being at the middle of its base D′C′ = 1 sq unit. Fig. 12(*b*) is a complete analogy of Fig. 12(*a*), and the horizontal pressure at point (x,y) is:

$$\sigma_x = C_1\,C_2\,n\left(y - \frac{x}{n}\right) + n\,\tau_{xy} = -\,C_1\,C_2\,(x - n\,y) + n\,\tau_{xy} \ldots (36b)$$

Because of the law of equality of action and reaction, it is sufficient to study only one half of the dam as in Fig. 12.

Remembering now that $\tau_{yx} = \tau_{xy}$, multiply Eq. 36*a* by n^2 and subtract Eq. 36*b* from Eq. 36*a*, thus modified, eliminating τ_{xy}. For given values of the coefficient of friction f and of the slope n, and if $C_1 = 0$, the pressures σ_y and σ_x are interrelated, thus:

$$\sigma_y = K\,\sigma_x \ldots\ldots\ldots\ldots (37)$$

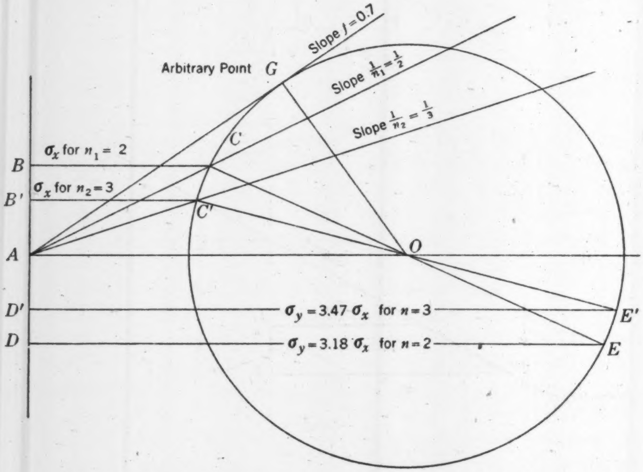
Slope $f = 0.7$
Arbitrary Point G
Slope $\frac{1}{n_1} = \frac{1}{2}$
Slope $\frac{1}{n_2} = \frac{1}{3}$
C
σ_x for $n_1 = 2$
B
σ_x for $n_2 = 3$
B'
C'
A
O
D'
$\sigma_y = 3.47\,\sigma_x$ for $n = 3$
E'
D
$\sigma_y = 3.18\,\sigma_x$ for $n = 2$
E

FIG. 13.—RATIO BETWEEN THE VERTICAL AND THE HORIZONTAL NORMAL STRESSES IN A "PLASTIC" REGION (CASE $C_1 = 0$; FIGS. 4 AND 5)

in which K is a constant to be measured on Mohr's circle (Fig. 13). First draw a horizontal line AO and a line AG with a slope $f = 0.7$ (as in the examples of the paper). A perpendicular line GO at an arbitrary point G of line AG determines the position of the center, O, of Mohr's circle. The two lines AC $\left(\text{slope } \frac{1}{n_1}\right)$ and AC′ $\left(\text{slope } \frac{1}{n_2}\right)$ determine vectors BC and B′C′ corresponding to the horizontal pressures σ_x. In their turn, vectors DE and D′E′ correspond to the vertical pressures, σ_y.

If C_1 is not equal to zero, the dam may be visualized as having a steeper slope $\frac{1}{n_0}$ and being at the "plastic" state under the action of the vertical body forces only. Let:

$$\sigma_x = n_0\,\tau_{xy} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38)$$

from which, using Eq. 36*b*:

$$\tau_{xy} = \frac{C_1\,C_2}{n - n_0}\,(x - n\,y) \ldots\ldots\ldots\ldots\ldots\ldots (39a)$$

and

$$\sigma_x = \frac{n_0\,C_1\,C_2}{n - n_0}\,(x - n\,y) \ldots\ldots\ldots\ldots\ldots\ldots (39b)$$

Introducing the value of τ_{xy} from Eq. 39*a* into Eq. 36*a* and dividing Eq. 36*a*, thus modified, by Eq. 39*b*:

$$\frac{\sigma_y}{\sigma_x} = \frac{n - n_0 + C_1}{n\,n_0\,C_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (40)$$

Compute the values: $\frac{\text{DE}}{\text{BC}}$ and $\frac{n - n_0 + C_1}{n\,n_0\,C_1}$, in which n and C_1 are given, and $n_0 = \frac{\text{BC}}{\text{CF}}$, for various points C as in Fig. 11(*a*). Plot these values from points C outward, along the radii of Mohr's circle. The intersection of curves I and II thus obtained (Fig. 14) furnishes the true value of the ratio $\frac{\sigma_y}{\sigma_x}$. A simple trial-and-error procedure would also serve the same purpose.

It is easy to see that Eqs. 36 and 37 furnish the same results as Eqs. 21. In the case of a "plastic" region such as I or VII, in which $n = \infty$, the Rankine formula should be applied (Eqs. 22), and no special derivation is needed.

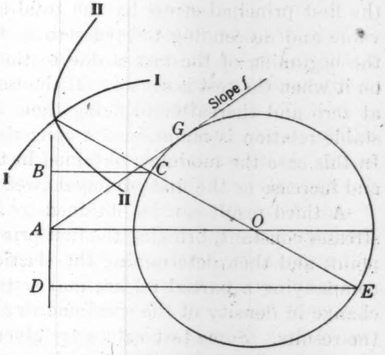

FIG. 14.—RATIO BETWEEN THE VERTICAL AND THE HORIZONTAL NORMAL STRESSES IN A "PLASTIC" REGION ($C_1 \gtrless 0$)

Stresses in "Elastic" Regions.—It is postulated in the paper that the material in the "elastic" regions obeys Hooke's law, which is not true. Afterward the condition of compatibility is used under the form of Eq. 3, which presumes that the material obeys Hooke's law. Using this condition, it is found that the stress distribution in "elastic" regions is rectilinear. This result is all that is needed for finding stresses in the "elastic" regions. As soon as the stresses at the boundaries of the plastic regions at a certain elevation are computed, the stresses in the "elastic" regions at the same elevation may be found simply by tracing straight lines as in Figs. 4 and 5, since equality (not "continuity" as the authors state) of stress across a boundary is postulated. Under the circumstances, it would be much simpler to postulate, directly, the rectilinear stress distribution in the "elastic" regions.

Slip Lines.—Instead of using Eqs. 24 and 25, the slip lines in a boundaryless "plastic" region may be located simply from Mohr's circle (Fig. 11). The major principal stress, σ_1, makes half the angle F′OE with the vertical. The two systems of slip lines make the angle $45° - \frac{1}{2}$ arc tan f with the direction of the major principal stress, σ_1, on both sides of it.

Title of the Paper.—The writer believes that the title of the paper is inappropriate. It is very questionable whether this method can be applied in the case of embankments made of a material different from the foundation, especially on a transverse slippery slope. Such is the case of many highway and railway embankments, dikes, and levees.

Conclusion.—In the opinion of the writer, the method described in this paper is interesting and original. Unfortunately, the method in question, although simple in itself, has been developed and presented in such a manner that quite a screen of mathematical symbols obscures its physical significance.

R. E. GLOVER,[14] ESQ., AND F. E. CORNWELL,[15] ESQ.—Professor Hennes objects to the use of Hooke's law for the material in the elastic zones, and quotes test results in support of his objection. The type of stress-strain relation expressed in Eq. 30 is obtained in a triaxial test when the second and third principal stresses are held constant and the first principal stress is raised progressively until failure occurs. The elastic modulus obtained by dividing the first principal stress by the total strain is variable, beginning at a finite value and descending toward zero as the test proceeds. The finite value at the beginning of the test is due to the initial pressure which must be placed on it when the test is set up. If the test is performed by starting all pressures at zero and thereafter bringing them all up together, so that a definite and stable relation is maintained among them, a different result will be obtained. In this case the modulus, obtained in the manner described, will start at zero and increase as the first principal stress increases.

A third result can be obtained by holding the second and third principal stresses constant, bringing the first principal stress about halfway to the failure point, and then determining the elastic modulus by observing the strains accompanying a partial release and restoration of the first principal stress. A change in density of the specimen due to a different packing will also modify the results. Some test values are given in Table 4.

In view of these differences it would appear that a closer approximation to the actual stress-strain relation than the simple one used by the authors could be chosen only after a careful study of the material, the proposed method of placing it in the dam, and the loads to be placed on it thereafter.

In connection with Figs. 8 and 9, Mr. Benscoter presents a discussion that appears reasonable for a granular mass with cohesion. The horizontal direct stress at a point in Fig. 8 corresponding to point A, Fig. 9, although not a principal stress, is only slightly greater than a fourth of the vertical stress. Similarly, for point B, slightly below the surface, the direct stress parallel to the downstream slope is almost twice that perpendicular to the slope, although these stresses are not the principal ones. As he states, Fig. 9 corresponds to non-linear solutions, whereas Fig. 8 was for linear ones.

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[14] Senior Engr., Bureau of Reclamation, Denver, Colo.

[15] Asst. Engr., Bureau of Reclamation, Denver, Colo.

TABLE 4.—THE ELASTIC MODULUS, COMPUTED FROM TESTS OF A FINE NATURAL SAND
(Grain Size, 0.04 to 0.60 mm)

CASE 1.—COMPUTED FROM VOLUME CHANGES PRODUCED BY THE APPLICATION OF HYDROSTATIC PRESSURE[a]			CASE 2.—COMPUTED FROM A TRIAXIAL TEST; INITIAL HYDROSTATIC PRESSURE, 100 LB PER SQ IN.[b]		
Test pressure (lb per sq in.)	Volume strain (in. per in.)	Elastic modulus (lb per sq ft)	Deviator stress (lb per sq in.)	Axial strain (in. per in.)	Elastic modulus (lb per sq ft)
10	0.02388	181,000	118	0.0105	1,618,000
25	0.02453	440,000	184	0.0215	1,232,000
50	0.02541	850,000	223	0.0323	994,000
100	0.02641	1,636,000	249	0.0435	824,000
....	....	....	267	0.0542	709,000

[a] Computed by dividing the stress by the total linear strain. The linear strain is computed by dividing the volume strain by 3. [b] Computed by dividing the deviator stress by the total axial strain after the start of the test.

Good agreement was found between predicted failure at the reentrant corner common to regions V, VI, and VII and actual performance of several models, when instability was caused by raising percolation gradients in region VII. These tests are discussed further herein under remarks on Professor Hrennikoff's discussion.

The writers agree with Mr. Benscoter that the application of mathematical theory to soil analysis problems is behind similar practice in other fields, and they look forward to further development. They also agree that new methods for measuring pressures in embankments are needed urgently.

Professor Hrennikoff comments at some length on the uniqueness of the solutions and concludes that the proposed solutions are unacceptable unless some additional proof of their reasonableness can be supplied. To support his conclusion he describes procedures by which he attempts to show that a wide variety of stress states could be installed in the dam by some artificial conditioning of the soil mass. The writers object to his demonstration on the following grounds: The passive, plastic-state, stress distribution requires the presence of shearing stresses on horizontal and vertical planes, and therefore it would be impossible to set up such a state by using a smooth wall in the manner he describes. This statement is true also for the active, plastic-state, stress distribution and all the possible intervening elastic states so long as the boundary slope is not horizontal. A similar difficulty will be encountered if the wall is to be made rough. In this case, any of the foregoing stress states could be set up because the shearing stresses required on the vertical planes could now be maintained; but it would not be possible to eliminate the central gap without altering the shearing stresses because a removal of the walls would leave a discontinuity in the shear stresses along the plane of symmetry. This discontinuity represents a major violation of the requirement for equilibrium. If an elastic zone is introduced, these difficulties can be removed and the shear stress can be made to pass through zero at the plane of symmetry, as it should.

These difficulties with Professor Hrennikoff's demonstration are considered to be sufficiently serious to warrant a reconsideration of the adverse criticisms in his section 1. In the latter part of this section he objects to the use of plastic states in zones III and V of Fig. 4 and expresses a preference for an elastic state throughout for slopes as low as those shown. In reply to this objection it can be stated that a solution satisfying the elastic equations, and free from the aforementioned difficulties, will require the existence of stresses along the boundaries and these cannot be sustained by a cohesionless material. The active state was chosen for regions I and VII for the examples in Figs. 4 and 5 because it was considered to be the state which probably would exist in a granular mass that had been deposited by running water, as would be the case with many stream beds.

Having disposed of these details, a consideration of the question of uniqueness may be resumed. Professor Hrennikoff is correct when he states that the solutions are not unique. The writers would like to urge, however, that the difficulty is not due to the form of their proposed solutions but is inherent in the nature of the problem. This may be illustrated by the retaining wall problem. Here the active and passive states represent limiting states beyond which stress distributions of the linear type cannot exist. Between these limits, however, there are a multitude of elastic states possible, and it should be noted that all of them, including the active and passive cases, are solutions of the differential equations (Eqs. 3 and 12) which meet the boundary conditions. This does not include all the possibilities, since additional solutions of non-linear type probably can exist also. It is evident, then, that Rankine's active state solution for the case of no surcharge, for example, is not unique in the sense that this distribution must exist to the exclusion of all others. It remains a physical possibility, however, in spite of the fact that tests and experience show that greater horizontal pressures can exist. In practice, other factors besides the satisfaction of the differential equation and the boundary conditions must be taken into account when a decision must be made as to the design pressure of a retaining wall. These factors will include a consideration of the manner of placing the backfill and the conditions of service after completion.

It is necessary, also, to exercise an intelligent choice when selecting the stress states for the plastic zones in a dam. An active state for the downstream slope might be reasonable for a dam that is to be constructed by rolled-fill methods, whereas a hydraulic fill dam might require a different choice. These remarks will apply to the comments in Professor Hrennikoff's sections 1 and 4. The questions raised in section 3, regarding the use of Hooke's law, have already been dealt with in connection with Professor Hennes' discussion.

In regard to the question raised by Professor Hrennikoff in section 5, concerning continuity of stress, it may be stated that the choice here was a matter of judgment. In the case of a dam subjected to a changing load, where the boundaries between elastic and plastic zones must move, it was considered that a discontinuity of the normal stress parallel to the boundary would lead to difficulties in regard to compatibility of strains on the elastic side of the boundary in the case where this zone is widening. In other cases, a choice of equality for shear stresses along the boundary and for direct stresses normal to it seems to be indicated. In Fig. 5, if the percolation pressure gradients in zone VII are raised a certain amount, the elastic zone VI will collapse to a simple boundary, subjected to such a stress condition. In Example 2, the percolation pressure portion is incomplete, as was stated in the original description. This is due to the limitations imposed by the available solutions. In this case, the choice of a percolation pressure was made so that the actual body forces would be represented as well as possible. To determine how seriously the validity of the solutions would be affected by these limitations, some tests were conducted on model dams. These were built in a special tank which was about 20 ft long, 4 ft deep, and 3 ft wide. The model dams were approximately 2 ft high and rested on a foundation about 2 ft deep. The case shown in Fig. 6(*c*) was tested on several occasions with a dam whose upstream and downstream slopes were 1 on 3 and 1 on 2, respectively. The computations indicated that a sliding failure should occur at the downstream toe, with an upward percolation pressure gradient in zone VII, which would be slightly less than that required to cause piping. The exact figures for one of the models tested were 113.25 and 123.10 lb per sq ft per ft to cause sliding and piping, respectively. The sliding failure was observed to take place at the expected gradient and before any material had been removed by piping. The estimated gradient to cause piping was later confirmed very closely. It was concluded from this test that the neglected portions of the percolation pressure pattern did not influence the results of the computations greatly.

It is only necessary to test one point inside an elastic region if the equality is satisfied at the two boundaries. The reason for this may be stated as follows: The equation for R_0 is second degree in x for any constant value of y (say y_1) when linear stresses and a uniform coefficient of friction are assumed. A plot of this equation will show intercepts on the x-axis at both $x_1 = \beta_i\, y_1$ and $x_2 = \alpha_i\, y_1$ when plastic regions bound the elastic zone along $x = \beta_i\, y$ and $x = \alpha_i\, y$. Therefore, to check this inequality, the condition $R > 0$ at only one point along the line between (x_1,y_1) and (x_2,y_1) is sufficient. The same reasoning would apply for any other constant value of y, such as y_2, y_3, etc.; therefore, to check the inequality at only one point within the entire region is sufficient.

Tables 1 and 2 of Appendix II were worked out by restricting solutions of Eq. 18 to real values so that stresses as given by Eqs. 21 would be real. The discriminant was set equal to zero, and values of n were determined for the various weights, coefficients of friction, etc., shown in the tables. The two systems of percolation pressure contours used are shown in Figs. 6(*a*) and 6(*b*). Table 3 was computed by determining the percolation pressure gradient in zone VII which would cause this zone to assume the passive state. For somewhat lesser gradients, zone VI is absent and zone VII is in the elastic state, as explained previously.

Professor Krynine presents a summary of present methods of analysis of earth structures, which he follows with a brief geometrical interpretation of conditions in the plastic regions. This latter interpretation, together with his explanations of some of the mathematics, should prove helpful to any one going through the method for the first time. His section showing that differential equations are not necessary in plastic regions is an interesting presentation but would not shorten the computations for such regions. It was postulated that the material in the elastic regions obeys Hooke's law, but this assumption need not be made if a more reasonable behavior can be stated. Whether Hooke's law is postulated, or rectilinear stress in the elastic region is assumed, makes no difference, of course, when the designer is limiting himself to linear solutions, but as stated in the conclusion to the paper a general method of approach was being developed. The procedure as used in the paper would hold even if neither Hooke's law nor rectilinear stress were assumed.

In closing, the writers wish to take this opportunity to express their sincere thanks to those who have contributed to the discussions.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852

TRANSACTIONS

Paper No. 2173

DRAINAGE OF LEVEED AREAS IN MOUNTAINOUS VALLEYS

BY GORDON R. WILLIAMS,[1] ASSOC. M. AM. SOC. C. E.

WITH DISCUSSION BY MESSRS. MERRILL BERNARD, L. K. SHERMAN, C. O. CLARK, W. W. HORNER, WALTER T. WILSON, W. B. LANGBEIN, AND GORDON R. WILLIAMS.

SYNOPSIS

Various methods of disposing of the drainage from streams tributary to leveed areas are outlined in this paper. Brief consideration is given to the characteristics of these methods in fulfilling the design criterion that local drainage must be disposed of without causing damage appreciably greater than if the streams could flow unobstructed to the main river at low stage.

The details of a method of analyzing local hydrology and developing capacities of drainage structures under various conditions are presented. Graphs show volumes and rates of rainfall and runoff used in the design storms and floods, and relations between selected capacities and available storage for numerous designs for drainage structures.

INTRODUCTION

The drainage of leveed areas on the flood plains of deep valleys results in more problems, and fortunately more possible solutions to those problems, than does the drainage of other types of leveed areas. For example, in providing for local drainage of areas behind levees in the lower Mississippi River Valley, in general there is no alternative but to install pumping stations, as there is no opportunity for providing head or storage for other types of works. Also, the drainage of an urban area in relatively low country already provided with a storm-water drainage system can be disposed of only by a pumping station directly connected to the outfalls.

The problems of levee protection and subsequent local drainage in a mountainous valley may differ from those in level or rolling country in the following respects:

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[1] Hydr. Engr., U. S. Engr. Dept., Wilkes-Barre, Pa.

(1) Levee protection is usually justified only for urban areas, in which case right-of-way costs will be high;

(2) The narrowness of the flood plain and the concentration of buildings limit the space for, and type of, structures to be built;

(3) The topography increases the drainage area tributary to the leveed area; and

(4) The topography requires that drainage be provided for streams whose hydrologic characteristics may differ from those of the protected areas.

Conditions (1) to (4) have been encountered in connection with flood-protection projects on the North and West branches of the Susquehanna River in Pennsylvania at the communities of Wilkes-Barre, Hanover Township, Kingston, Edwardsville, Plymouth, and Williamsport. The hydrologic procedures to be presented in this paper were developed particularly for the design of the protective works at Hanover Township, Plymouth, and Williamsport.

Local Drainage Requirements

Authorized projects of the U. S. Engineer Department for the protection of communities specify the river against which protection is to be provided. The degree of protection afforded against the specified river is based on studies of past floods and economic considerations. The adopted project usually provides for a substantial freeboard against the confined flow of the greatest known flood. A corresponding degree of protection is not required against flows from tributaries that pass through leveed areas, but it is required that the protective works on the main river shall not aggravate conditions caused by floods on the tributary streams. If a community desires flood protection from a tributary stream, or any other local improvements not a part of an authorized project, it must obtain authorization from Congress, which in turn may request the U. S. Engineer Department to make additional investigations for such improvements. Flood protection from small tributaries is very rarely justified.

The problem of preventing conditions within the leveed area from being no worse than if the tributary stream could flow unobstructed to the river is a difficult one and requires the consideration of a variety of situations. It is obvious from a hydraulic standpoint that a perfect solution is not possible, but from a damage standpoint a satisfactory solution can be obtained. In other words, it is possible to dispose of interior runoff at a rate that will prevent local damage from being appreciably greater than with no levee and the river at low stage. Benefits resulting from the fact that the main river has been prevented from entering the leveed area do not affect the economics of the local drainage problem because the elimination of main river damage has already been considered in determining the justification for the entire project.

There are two conditions for which the effect of floods on local tributaries must be considered:

(*a*) When the main river is at low stage and normal drainage to the river can take place; and

(*b*) When the river is at or above flood stage and special drainage works must be provided.

On the North Branch of the Susquehanna River conditions (*a*) and (*b*) are largely confined to well-defined seasons and therefore require a detailed study of probable hydrologic conditions that are characteristics of those seasons.

METHODS OF DISPOSING OF LOCAL DRAINAGE

There are six basic methods of disposing of local drainage from tributaries passing through leveed areas. These methods are: (1) Levees or walls along the tributary stream, (2) culvert with floodgate, (3) pressure conduit, (4) diversion channel, (5) storage reservoir, and (6) pumping station. Any of these methods may be used alone, but two or more are customarily used in combination. The selection of any method or combination of methods is based on studies of comparative costs and consideration of special local conditions.

(*1*) *Levees or Walls Along the Tributary Stream.*—From the standpoint of simplicity this method is to be preferred. However, it may be relatively costly for small tributaries or those with a drainage area of less than about 50 sq miles. The levees or walls must be extended upstream along the banks of the tributary to high ground and must be at an elevation sufficient to provide freeboard under all reasonable combinations of coincident discharges at the outlet of the tributary. If the tributary flows through an urban area, right-of-way costs for levees or walls may be prohibitive. Some tributaries along the Susquehanna River enter the flood plain at right angles to the course of the main river and then flow parallel to the river, sometimes for several miles, before finally emptying into the river. Under such conditions levees along the banks of a tributary become a major project in themselves. Levees or walls may be constructed part way up a tributary to connect with other works, such as a pressure conduit or pumping station. Levees have been provided in the Susquehanna River projects only on Lycoming Creek at Williamsport, which has a drainage area of 269 sq miles, but they were considered for other tributaries much smaller in size.

(*2*) *Culvert with Floodgates.*—Relief culverts with floodgates are used for floods occurring when the river is at low stage. Floodgates will pass local drainage at any stage in the river, provided there is sufficient differential head resulting from ponding at the intake. The selected size of culvert depends upon the design inflow and the allowable ponding elevation at the entrance. The latter elevation is dependent on the damage that will result from ponding. Obviously it is impossible to eliminate all ponding at the levee and still obtain an effective head on the culverts. Where conditions require excessively large batteries of culverts, there may be justification for providing smaller openings in combination with a fuse plug in the levee that can be blown out if conditions warrant. Relief culverts and floodgates have been constructed or planned at every point of natural outlet at levees in the Susquehanna River projects.

(*3*) *Pressure Conduit.*—A pressure conduit may be used if there is sufficient head. The total energy head or difference in elevation between stage in the river and the energy elevation at the entrance must be sufficient to overcome losses in the conduit and to produce velocity of flow under various combinations of river stage and conduit discharge. To obtain the head it is usually necessary to place the entrance some distance from the main river, resulting in a reduction in the area diverted and an increase in the length of the conduit. Pressure conduits are most practicable where the flood plain is narrow so that the construction and right-of-way costs are at a minimum. Sometimes the conduit can be laid in the original stream channel, but if that channel has been used as an outlet for storm-water drains, a separate intercepting sewer must be built or connections with gates or valves provided in the conduit. Pressure conduits also require supplementary works consisting of a pumping station and a relief culvert. The pumping station is required to dispose of runoff from the undiverted area and runoff from the diverted area in excess of the capacity of the conduit. The latter runoff will usually occur during low stages in the river and will pass through the relief culverts. The only pressure conduit under construction in the Susquehanna River flood-control projects is on Toby Creek at Kingston, but others have been considered.

(*4*) *Diversion Channel.*—Diversion of tributary streams to a point outside of leveed areas is usually practicable only when a stream is close to the limits of the protected area. In such cases changing the natural course of a tributary is often the most economical procedure. However, if the fall from the point of diversion to the river is great, and if there are highways and railroad tracks to be crossed, the costs of drop structures and bridges may indicate that some other method is more economical. Diversion channels have been planned for Coal Creek at Plymouth (diverted area, 1.6 sq miles) and Millers Run at Williamsport (diverted area, 7.8 sq miles).

(*5*) *Storage Reservoirs.*—Under certain combinations of favorable conditions it may be economical to construct storage reservoirs to impound the runoff of tributaries during floods in the main river. Reservoirs may be located at levees or at higher elevations farther upstream. In order to satisfy the criterion adopted for the design of local drainage structures, the reservoir need not provide flood control during low stages in the main river and hence has only to provide storage for the limited period in which the relief culverts cannot operate. Such reservoirs may be effective with only 2 or 3 in. of storage instead of the 6 in. usually considered necessary for flood control. Reservoirs with dams must be constructed back in the hills where dam sites are available, in which case there usually will be a considerable uncontrolled area, the runoff from which must be pumped. In addition, the usual relief culverts must be provided for floods not controlled by the reservoir. Reservoirs with dams have been considered but never found to have been justified in the solution of levee drainage problems in the Susquehanna River Basin; but reservoirs adjacent to levees have played an important part in the solution of such problems.

(*6*) *Pumping Station.*—If the hydrographs for design floods are routed through existing ponding areas and constrictions in the channel and valley of a tributary stream, it often will be found that, with only a relatively small rate of outflow, water-surface elevations can be kept close to conditions prior to the construction of the levee. Hence, a pumping station is used more than any other method to satisfy the drainage criterion. Such pumping installations usually take full advantage of the modifying effect of available storage adjacent to the levee. Such storage is usually in low marginal areas where development has been limited and where damage will be slight. The chance that such areas may be developed and filled after the levee is built must be considered and requires an appraisal of possible future development. Relief culverts must be provided in connection with pumping stations to dispose of runoff that may occur when the river is at low stage. A total of eleven pumping stations is planned, for unsewered or partly sewered areas on Susquehanna River levee projects in Pennsylvania. An additional eleven pumping stations are planned for completely sewered areas.

LOCAL HYDROLOGY

The determination of appropriate rates and volumes of runoff to be used in the determination of the capacities of drainage structures is exceedingly difficult, especially when there are no available records in the immediate vicinity of a project. It requires a consideration of the seasons in which different conditions may occur, an analysis of rainfall and runoff characteristics of these seasons, and a determination of the runoff characteristics of each sub-area to be drained. In usual storm-water drain design it is considered necessary only to determine peak rates of runoff, and the effect of storage is neglected. In the design of drainage structures for natural areas behind levees, it is necessary to route complete hydrographs through existing and proposed ponding areas in order to determine the modified peak rates for which to provide. The hydrologic problem is to determine, as well as to route, hydrographs that have a reasonable probability of occurrence under different conditions.

Basic Data.—It is desirable to have actual records or estimates of stage in the main river, of runoff from the tributaries, and of local rainfall and snowfall. If the main stream is important as is the Susquehanna River, long records of stage are usually available or can be determined from near-by places by means of gage-relation curves. Continuous records of stage on the North Branch at Wilkes-Barre began in 1896 and on the West Branch at Williamsport in 1895.

The tributary creeks, on the other hand, are small and have been considered too unimportant to be gaged. On some there are a few flood marks that make it possible to estimate the maximum floods known to the residents. For the most part there are no records on which to base times of concentration or unit-hydrograph characteristics.

Rainfall records at nonrecording gages are available at many places in the Susquehanna River Basin, and some of the records have been kept for 45 years or more. Unfortunately, there are only two recording gages in the basin in Pennsylvania upon which to determine the frequency of rainfall intensities of short durations. These gages are located at Harrisburg and Scranton. All these records have been analyzed and form the initial basis for determining the capacity of drainage structures.

In order to obtain more reliable data on the runoff characteristics of the small drainage areas adjacent to active or proposed flood-control projects, the

U. S. Engineer Department has established, in cooperation with the U. S. Geological Survey, a number of gaging stations on small areas. The smallest areas on which gaging stations were established are listed in Table 1.

TABLE 1.—GAGING STATIONS ON SMALL AREAS IN THE SUSQUEHANNA RIVER BASIN IN PENNSYLVANIA

No.	Stream	Place	Area[a]
1	Paxton Brook	Harrisburg	11.2
2	Solomon Creek	Wilkes-Barre	15.4
3	Hageman Run	South Williamsport	6.3
4	Grafius Run	Williamsport	3.1

[a] Drainage area in square miles.

A recording rain gage was also established on the headwaters of Solomon Creek. The new system of recording rain gages established by the Federal State Flood Forecasting Service of Pennsylvania is proving of increasing value in connection with newly established stream gaging stations. These streamflow and rainfall gages, even when in operation for only one flood season, often give information that results in the saving of large sums of money in the design of drainage structures.

Flood Season.—The flood season on the Susquehanna River was determined from an analysis of stage records at Wilkes-Barre. The number of flood peaks in the record above damage stage are as follows:

Month	No.	Month	No.
October	3	April	6
November	3	May	1
December	1	June	0
January	3	July	2
February	4	August	0
March	13	September	0

It is evident that 83% of the peak damage stages have occurred in the season from November to April, inclusive, and that March and April are the most severe flood months. The purpose of defining the flood season is to determine when the drainage structures will be used and what rainfall and runoff conditions can be expected during the period of use. Runoff from snow cover can be expected throughout the selected season. If the season had been extended farther to include October and May, the absence of snow runoff and the increased losses from infiltration would counteract the higher rainfall rates to be expected during those months.

Rainfall in Flood Season.—It should be emphasized that the drainage areas under consideration are so small that point rainfall data are sufficient for winter storms and lead to a desirable factor of safety for summer storms, and that area-depth relations need not be introduced to add to the complexity of the problem. A study was made first of 1-day and 2-day peak rainfalls in the flood season for eighteen stations in northeastern Pennsylvania. The excessive rainfalls were tabulated by months and then combined into a seasonal record. The frequency of the various rainfall depths was computed and plotted and mean curves drawn. A summary of the results is shown in Table 2.

Considering that the rainfall depths were obtained from calendar-day records of rainfall, and that the actual durations of the rainfalls probably varied over a wide range, there is a remarkable correlation between the results. This is due in part to the fact that winter storms cover wide areas and cause a relatively uniform distribution of precipitation. Such a correlation, which has been anticipated by other investigators,[2] should not be misconstrued as indicating a high degree of accuracy in the frequency-depth relations. Some other period of record might show a similar correlation but not the same frequency-depth relations.

TABLE 2.—SUMMARY OF RAINFALL RECORDS FOR THE SEASON, NOVEMBER TO APRIL, INCLUSIVE

Station (Pennsylvania)	Elevation, in ft above mean sea level	Length of record, in years	Rainfall, in Inches[a]					
			5 Years		10 Years		15 Years	
			1-day	2-day	1-day	2-day	1-day	2-day
(a) SUSQUEHANNA RIVER BASIN								
Wilkes-Barre	544	44	1.8	2.3	2.0	2.7	2.1	2.9
Scranton	746	39	2.0	2.3	2.2	2.6	2.4	2.8
Forest City	....	18	1.9	2.4	2.2	2.7	2.3	2.9
Towanda	754	44	1.9	2.3	2.2	2.6	2.4	2.8
Montrose	1,656	36	1.7	2.0	2.0	2.3	2.1	2.5
Morris Run	....	14	2.2	2.9	2.7	3.2	2.9	3.5
Lawrenceville	1,000	43	1.9	2.3	2.2	2.7	2.4	2.9
Wellsboro	1,319	43	2.1	2.6	2.4	3.0	2.6	3.2
Ansonia	1,136	26	1.6	2.3	1.8	2.7	1.9	3.0
Williamsport	542	43	2.2	2.7	2.4	3.1	2.6	3.3
Lock Haven	554	43	2.0	2.4	2.2	2.8	2.3	3.0
Muncy Valley	864	28	2.2	2.7	2.5	3.1	2.7	3.4
Catawissa	520	28	2.0	2.6	2.3	2.9	2.5	3.2
(b) DELAWARE RIVER BASIN								
Pleasant Mount[b]	....	15	2.2	2.5	2.4	2.8	2.6	2.9
Freeland[b]	1,900	25	2.2	2.9	2.5	3.3	2.7	3.5
Gouldsboro	1,890	25	2.2	2.9	2.5	3.1	2.6	3.3
Mount Pocono	1,740	24	2.6	3.3	2.9	3.9	3.1	4.2
Stroudsburg	500	23	2.2	2.8	2.5	3.1	2.6	3.4

[a] Rainfall to be equaled or exceeded once in the period indicated. [b] On divide between Susquehanna River and Delaware River basins.

Because the critical rainfalls on small areas are those of high intensity and short duration, records of calendar-day rainfall have little application except where extensive flood storage is planned. It is believed, however, that these rainfalls can be used as an index of the possible regional variations in rainfalls of shorter durations. Furthermore, the data in Table 2 indicate that, for the winter season, the increase in depth of rainfall with decrease in frequency is relatively small. For example, the average increase in the expected depth of rainfall between the 5-yr and 10-yr frequencies or between the 10-yr and 15-yr frequencies is only about 10%. This characteristic of the winter storms is significant in the design of drainage structures and indicates that the degree of protection against interior flooding can be increased often without a large increase in expenditure.

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[2] "The Reliability of Rainfall Intensity-Frequency Determinations," by C. W. Thornthwaite, *Transactions*, Am. Geophysical Union, Pt. II, 1937, pp. 476–484.

In order to obtain data on the winter rainfalls of short duration, the original charts for the recording gages at Scranton and Harrisburg were studied. All peak rainfall depths for durations of from 5 min to 12 hr were determined. The rainfall depths for each duration were arranged in order of magnitude, and the frequency with which each value was equaled or exceeded was computed and mean curves drawn. Frequency curves for the Scranton record, which differ but little from those obtained from the Harrisburg record, are shown in Fig. 1. Total depths of rainfall, instead of hourly rates, were used because they can be applied directly to unit hydrographs and because depth-frequency curves when plotted on semilogarithmic paper indicate straight-line relations.

The computed frequencies for the rainfalls of all durations, including the 1-day and 2-day rainfalls, were determined by the formula

$$f = \frac{2\,N}{2\,m - 1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which N = length of record in years; m = serial number of each item, when items are arranged in descending order of magnitude; and f = frequency, in years, with which any item will be equaled or exceeded. Eq. 1, which is based on the theory of least squares, has been universally accepted in estimating flood-peak frequencies but has been rarely used in determining rainfall frequencies. No doubt the reason is that rainfall records furnish more items for analysis, and investigators have considered that theoretical adjustments to available records were not necessary. The formula is just as applicable to one type of frequency analysis as it is to another, including the construction of duration curves.[3]

Design Storm for Flood Season.—In selecting a design storm there are two possible courses of procedure. One is to use the actual rainfalls from some severe storm of record in the flood season, and the other is to derive a synthetic storm from rainfalls of equal frequency. The first method may result in either under-design or over-design of drainage structures unless the characteristics of the design storm are compared carefully with many other storms, particularly those that have occurred during high stages in the main river. The second method, which was chosen for use in the Susquehanna Basin, results in a storm that is rarer than the indicated frequencies of the component rainfalls but eliminates to a large extent the uncertainties resulting from the use of a particular storm. This procedure has been suggested by other engineers.[4,5]

An examination of the rainfall depths in Table 2 indicated that the studies of the recording gage record at Scranton could be applied to areas in the vicinity of Wilkes-Barre. In applying the same studies to areas in the vicinity of Williamsport, the rainfalls of short duration were increased by the ratio between the Williamsport and the Scranton daily rainfalls of the same frequency.

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[3] "Duration Curves," by H. Alden Foster, *Transactions*, Am. Soc. C. E., Vol. 99 (1934), pp. 1213–1267.

[4] Discussion by Merrill M. Bernard of "Relation Between Rainfall and Run-Off from Small Urban Area," by W. W. Horner and F. L. Flynt, *ibid.*, Vol. 101 (1936), p. 189.

[5] "Storage Basins as a Supplement to Storm Sewer Capacity," by John A. Rousculp, *Civil Engineering*, November, 1940, pp. 715–718.

The design storms were derived by combining rainfalls of equal frequency

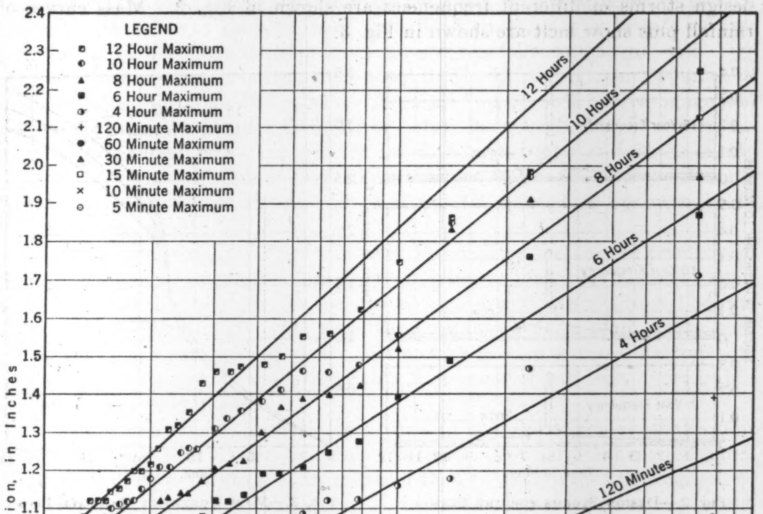

FIG. 1.—FREQUENCY ANALYSIS OF PRECIPITATION AT SCRANTON, PA., FOR THE SEASON NOVEMBER TO APRIL, INCLUSIVE

into a histogram which was nearly symmetrical and which had the maximum intensity at the midpoint. As 10-min unit hydrographs were used for areas less than about 8 sq miles, rainfalls in the design storms were computed for

10-min intervals. In northern Pennsylvania it was assumed that there would be 100% runoff in the winter season and that, in addition, there would be runoff from melting snow at the rate of 1 in. in 24 hr. The histograms for the design storms of different frequencies are shown in Fig. 2. Mass curves of rainfall plus snow melt are shown in Fig. 3.

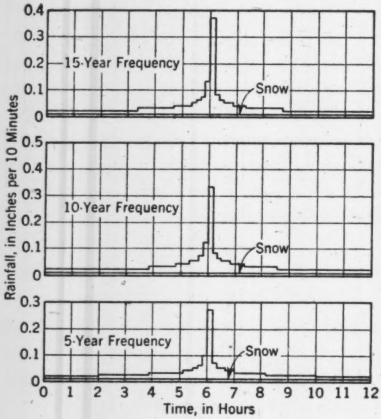

FIG. 2.—DESIGN STORMS FOR THE SEASON NOVEMBER TO APRIL, INCLUSIVE

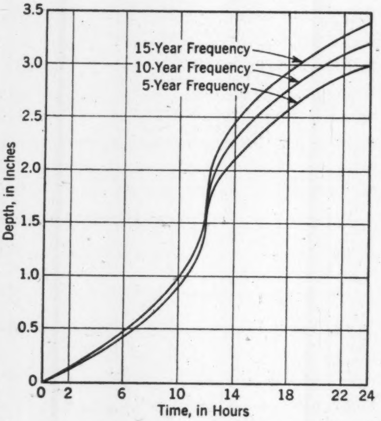

FIG. 3.—MASS CURVES OF RAINFALL PLUS SNOW MELT FOR DESIGN STORMS

At this point it is well to raise the question as to whether the design storm should be based on all rainfalls in the flood season or on rainfalls that have occurred coincidental with flood stage in the main river. Theoretically the latter rainfalls should be used, but practically they do not form a reliable basis for design in the locality under consideration. Using available records of daily rainfall it was found, for example, that the 10-yr, 1-day rainfall for the entire flood season was equaled or exceeded about once in 35 or 40 years in coincidence with high stage in the river. A similar study was conducted taking the hourly rainfalls at Scranton that occurred during high stage at Wilkes-Barre. The results were not conclusive and no reliable frequency curve could be drawn. It was noted, however, that many intense hourly rainfalls did coincide with high stage in the river. In order to determine frequencies of rainfalls coincident with high river stages, which would be as reliable as the frequencies of rainfall throughout the entire flood season, it would be necessary to have a record of coincident events three or four times as long as the available rainfall record. It should also be emphasized that winter floods on the river and on the small tributaries do not have independent meteorological causes. The winter storm is widespread and is usually accompanied by warm temperatures that cause melting of snow throughout the storm area. Peaks of floods on the tributary streams will rarely coincide with peaks on the main stream, but tributary peaks will often coincide with high stages on the main stream. Furthermore, the phenomenon of two or more storms in succession is a common occurrence in the Susquehanna Basin, as in March, 1936. In the flood of March-April, 1940, there were three distinct flood rises above the damage stage at Wilkes-Barre in a period of 12 days.

On rivers with drainage areas in excess of about 10,000 sq miles, or smaller rivers that take many days to reach their flood crests, it may be safe to base the design of drainage structures on known rainfalls coincident with high stages. Such studies have been conducted by the U. S. Engineer Department for cities along the Ohio River, where there are long records of river stage and coincident rainfall. On the Ohio River below Pittsburgh, Pa., it can be assumed safely that there is no relation between local runoff and floods on the main river. The duration of high stages is much longer than on the Susquehanna River at Wilkes-Barre, and therefore there is more opportunity to record coincident rainfall.

Design Hydrograph for Flood Season.—Flood flows that would result from the design storms were determined by applying rainfall plus assumed snow melt to the unit hydrographs for the various local areas. As mentioned before, the entire storm rainfalls were considered to represent rainfall excess, and infiltration was assumed to be zero. The unit hydrographs were obtained from actual records of runoff where available, but most of the unit hydrographs were derived synthetically.

Unit Hydrograph.—The basic procedure in determining a synthetic unit hydrograph was first to derive a unit hydrograph for a rainfall period equal to the time of concentration of an area and then to modify that hydrograph so that it was applicable to the rainfall period used in the design storm, which was 10 min for most areas. The basic theory involved has received consideration elsewhere.[6]

Determination of the time of concentration was a problem in itself. A procedure developed by Z. P. Kirpich,[7] Jun. Am. Soc. C. E., was used. Briefly, this method is to compute, for areas with known times of concentration, correlation factors based on length of stream channel, slope of channel, and average slope of basin. These factors are plotted against known times of concentration, and the resulting curve is used to determine times for other areas where only the correlation factor is known. The results from the data plotted by Mr. Kirpich were extrapolated to give a basis for determinations applicable to the larger and more mountainous areas in the Susquehanna Basin. The extrapolation was checked closely from field data obtained from one mountainous area of about 15 sq miles. It is realized that no results of great precision are obtained by such a procedure. However, it should be more reliable than mere guessing at velocities of overland and stream-channel flow. Furthermore, when more field data become available, it is believed that extension of the same procedure will yield results of greater precision.

The peak discharge of the unit hydrograph for a rainfall period equal to the time of concentration was determined by use of the principle that, if the rainfall excess continues for a period equal to the time of concentration, the rate of runoff will equal the rate of rainfall excess. The familiar rational formula

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[6] "Analysis of Run-Off Characteristics," by Otto H. Meyer, *Transactions*, Am. Soc. C. E., Vol. 105 (1940), pp. 83–141.

[7] *Civil Engineering*, June, 1940, p. 362.

$$Q = C\,i\,A \quad \text{.......................} \quad (2)$$

is based on this theory. If C, the runoff coefficient, is considered equal to unity, the formula can be considered to represent rainfall excess. As the unit hydrograph represents 1 in. of runoff from a uniform rainfall, the rate of rainfall excess equals

$$i = \frac{1}{T_c} \quad \text{.......................} \quad (3)$$

in which T_c equals the time of concentration in hours. As T_c for small areas is usually expressed in minutes, i equals $\frac{60.}{T_c}$ The area, A, is expressed in acres, and the peak rate of discharge, Q_p, for the unit hydrograph is expressed by the formula

$$Q_p = \frac{60\,A}{T_c} \quad \text{.......................} \quad (4)$$

The writer believes that this theoretical rate of discharge is never reached in the time T_c for natural areas having appreciable channel and depression storage. However, such an assumption is conservative and must be used until future studies indicate how the storage effect can be evaluated.

To estimate the shape of rising and falling limbs of the synthetic unit hydrograph, contours representing equal times of travel to the outlet were drawn. From these contours a time-area graph was constructed, and the ordinates to the unit hydrograph were made proportional to this graph. Of course, the use of time contours and time-area graphs is not new, but the relation between those principles and that of the unit graph appears to have been first demonstrated by Merrill Bernard,[8] M. Am. Soc. C. E., and later developed by others,[6] who introduced the unit hydrograph for the time of concentration. In order to conform more nearly to natural unit hydrographs, the falling limbs were arbitrarily flattened so that the time from the peak to zero flow was equal to $\frac{3}{2}\,T_c$. Typical unit hydrographs for an area with a time concentration of 33 min are shown in Fig. 4.

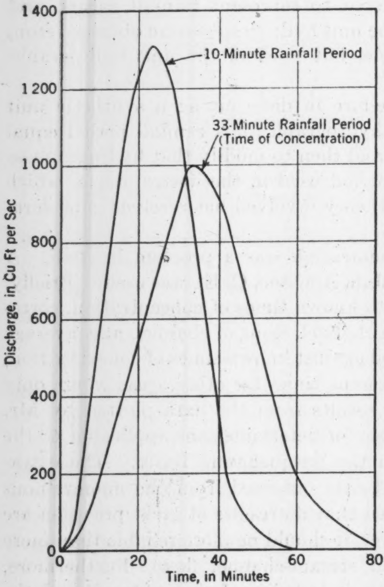

FIG. 4.—TYPICAL UNIT HYDROGRAPHS; STREAM: FOX HOLLOW RUN, WILLIAMSPORT, PA.; DRAINAGE AREA: 548 ACRES

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[8] "An Approach to Determinate Stream Flow," by Merrill Bernard, *Transactions*, Am. Soc. C. E., Vol. 100 (1935), p. 349.

Maximum Probable Storm and Flood.—In order to provide adequate relief openings when the main river is at low stage, a maximum probable storm and flood were computed for each area. The maximum probable storm would be of the summer cloudburst type. The rainfalls for such a storm were determined from an envelope of maximum known rainfalls for northeastern United States, which has been published previously.[9] Using this curve a synthetic storm was constructed in the same manner as the winter design storm. The maximum probable flood hydrograph was derived by deducting an estimated infiltration loss from the rainfalls and then applying the net depths to the unit hydrographs. Studies of storms on larger areas where stream-flow records were available indicated that an infiltration rate of 0.3 in. per hr was conservative for summer storms. This rate assumes that there has been antecedent rainfall, and much higher rates would prevail if the rain fell on dry ground.

No claim is made that the hydrologic analyses presented herein represent perfection in such studies, but rather that they represent a workable and reasonable solution to a difficult engineering problem. Further refinements probably could not be justified from the standpoint of the reliability of the available basic data and from the probable savings in the design of the structures.

DETERMINATION OF CAPACITY OF DRAINAGE WORKS

The hydrographs for the winter design flood and the maximum probable flood, determined as described in the previous section, were routed through all areas of appreciable existing and proposed storage. Most of the effective storage is found to be in the flood-plain areas. In the case of a proposed diversion channel or pressure conduit there is usually little or no storage above the entrance to modify the inflow hydrograph for the diverted area and to reduce the required capacity. On the other hand, structures such as pumping stations and relief culverts, which are located at levees, take full advantage of all modifying storage in the flood plain, which reduces the required capacity without violating the design criterion. Of course, the area drained is greatest at the levee, but the increase in the volumes of runoff is usually offset by the increase in ponding facilities.

It was considered reasonable for the drainage works to have a capacity sufficient to dispose of runoff from the 10-yr winter flood without causing damage in the flood plain appreciably greater than if water could flow freely to the river at low stage.

In order to determine the incremental increase in damage that would result from various ponding elevations at the levee, stage-damage curves were plotted for each area affected. Data for these curves were obtained from detailed surveys for the entire flood plain. The latter surveys had been made previously in order to determine the justification for each local protection project as a whole.

Because of the characteristics of the winter storms, already discussed in this paper, the selection of the rainfall frequency as between 5, 10, or 15 years usually does not have a very material effect on the size of drainage structures. The greater the flood-plain storage, the less the variations in inflow rates affect the capacity of structures. Fig 5 shows the relations between pumping capacity and storage for a number of designs based on 10-yr winter rainfalls. Local conditions affect the flood routing and in turn affect the selected pumping capacities, thus causing a rather wide dispersion of the plotted points. The pumping capacities required in terms of inches per day give rather large values, but it must be remembered that these capacities represent peak rates that are needed for only a few hours at a time.

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[9] *Transactions*, Am. Soc. C. E., Vol. 106 (1941), Fig. 21, p. 1504.

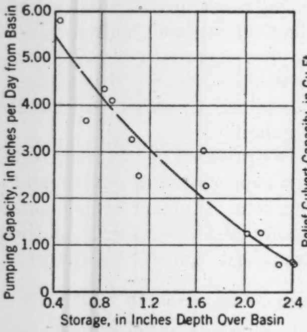

FIG. 5.—RELATION BETWEEN CAPACITY AND FLOOD-PLAIN STORAGE FOR VARIOUS PUMPING STATION DESIGNS

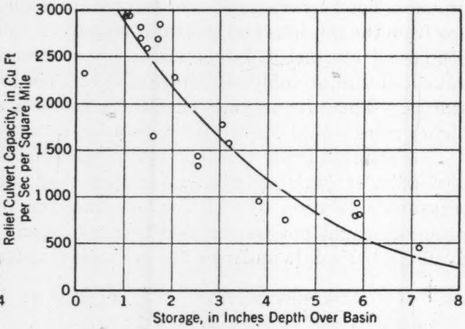

FIG. 6.—RELATION BETWEEN CAPACITY AND FLOOD-PLAIN STORAGE FOR VARIOUS RELIEF-CULVERT DESIGNS

The relations between relief-culvert capacity and flood-plain storage for a number of designs are shown in Fig. 6. In this case the inflow is that from the maximum probable flood. These relations are also greatly affected by channel and storage conditions in each individual basin.

It may appear to some that the basis of design for drainage structures presented herein is rather conservative, but it should be for the following reasons:

(*a*) Only a part of the present flood-plain storage is within the right of ways of the projects and hence is subject to reduction by filling in for future development;

(*b*) Future development may increase incremental damage resulting from ponding; and

(*c*) Future development may cause an increase in the volume and concentration of runoff.

ACKNOWLEDGMENTS

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DISCUSSION

Merrill Bernard,[10] M. Am. Soc. C. E.—Although Mr. Williams has confined his discussion to projects created by the construction of levee systems of considerable magnitude and importance, similar situations of lesser extent are to be found in nearly all of the great metropolitan areas and in many smaller cities and towns. His paper is a contribution to the solution of these problems and should be an encouragement to those who would treat them rationally rather than empirically.

In planning for the disposal of tributary flow within leveed areas, the ideal situation to be met is that of a synchronization of flow which, by virtue of a consistently shorter concentration period, would insure the discharge of the lateral flow under positive head through automatic floodgates for all conditions of outside stage. These circumstances may be encountered on small watersheds as headwaters are approached, but are never sufficiently realized to preclude the necessity of resorting to other means of control discussed by the author. In addition to the six methods mentioned by him, opportunity might occasionally present itself to consider the use of a secondary levee provided with gates and pumps. Under reasonably favorable channel and surface slope conditions, liberal gate dimensions to insure maximum duration of flow under natural head, and minimum supplemental pumping, the effect would be to enhance the utilization of areas dedicated to temporary storage and to reduce the depth and duration of flooding on developed, contiguous, low-lying areas. The storage basin itself could be closed in without reducing its capacity by substituting depth created by the secondary levee for area covered by the unconfined basin. The sequence of operation would be: (1) Gravity flow through upper and lower gates, (2) lower gates closed with the reversal of head, (3) lower pumping started, (4) upper gates closed with reversal of head, (5) upper pumping started, (6) upper pumping stopped, and (7) lower pumping stopped.

The effect, on cost, of dividing the pumping installation into two units, the additional cost of constructing the secondary levee, and the greater operating cost, if any, would have to be weighed against the benefits of added protection in each particular case.

The author has capitalized reasonably on the fact that flooding in eastern Pennsylvania is confined to a rather definite "flood season." The 12-hr precipitation record at Harrisburg, Pa., demonstrates the differences between annual rainfall and that within the adopted flood season:

F	All storms or maximum yearly	All storms or maximum seasonal
2	2.0	1.5
5	2.7	1.8
10	3.5	2.0
40	5.0	2.4

The column headings indicate a confirmation of the author's opinion that the selection of the maximum yearly or seasonal values, when treated through

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[10] Hydrologic Director, U. S. Weather Bureau, Washington, D. C.

Eq. 1, will produce the same frequency-depth relations as if all rainfall amounts within the record were considered, any differences being confined to frequencies of less than 2 years. Also, the differences in the values of the second and third columns represent reduction in design values gained by restricting the frequency series to the flood season. This in effect is offsetting the higher intensities of summer-type rainfall with the greater reductions for infiltration prevailing during the off-flood season. The frequency of the greater amounts occasioned by the inclusion of all storms within the record is reduced by increasing the value of *m* in the frequency equation.

The value of 1 in. in 24 hr for the contribution of melting snow to runoff seems somewhat low. There is little secure knowledge of this phenomenon but, on watersheds sufficiently small to allow snow-melt runoff from the entire area within a melting period, greater amounts could be expected. The average temperature for the locality that can occur with snow cover and coincidental rainfall could be as high as 55° F for a 24-hr period. The general opinion is that from 0.05 to 0.15 in. of snow-melt runoff per degree-day can be expected, depending upon the state of the snow mantle. Under these conditions runoff depth from 1.1 to 3.5 in. is possible. This does not include the contribution of free water present in a mantle of snow of low quality.

L. K. SHERMAN,[11] M. AM. SOC. C. E.—An interesting and logical analysis of the problem of local or interior drainage of leveed areas in mountainous valleys is presented in this paper. The author calls attention to the difference between conditions in the lower Mississippi River and the Susquehanna River valleys. Be that as it may, the conditions on the upper Mississippi and the Illinois rivers differ from those on the Susquehanna River in degree rather than in principles involved. Many of the levee districts are adjacent to steep bluffs extending to a height of from 200 to 600 ft. The runoff from the lateral streams is swift and flashy.

In either location, the first principle of design is the elimination, in so far as possible, of all of the hill water entering the leveed area. In the case of bottom lands to be used as agricultural drainage and levee districts, the upstream and downstream limits are generally confined to the distance between two lateral streams. Major lateral streams, through a town or between two adjacent drainage districts, must be carried direct to the river in a wide floodway and with levees of ample height at their junction with the bluffs or hills. Many levee districts were drowned out during the floods of the Illinois and Mississippi rivers in 1926–1927 due to overtopping of the lateral levees when there was ample freeboard along the main river. The Illinois and Mississippi rivers probably have a longer, higher stage period than do the rivers in Pennsylvania. The opportunity for the occurrence of peak lateral flow with fairly high river stages is not so great in Pennsylvania. Nevertheless, the additional insurance cost is so relatively small that the design of lateral levees should be predicated upon the synchronization of peak flow in both lateral and river channels. The writer would take exception to the author's remark under "Local Drainage Requirements": "A corresponding degree of protection is not required against flows from tributaries that pass through leveed areas."

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[11] Cons. Engr., Chicago, Ill.

The author states (see heading "Methods of Disposing of Local Drainage: (1) Levees or Walls Along the Tributary Stream"): "Some tributaries along the Susquehanna * * * then flow parallel to the river, sometimes for several miles, before finally emptying into the river." This is also a common phenomenon on the Mississippi River. The flood profile of the lateral, at the upper end of the chute, is generally at a higher elevation than the main river across the dividing strip of land. In the case of the Fabius River, in Missouri, advantage of this situation was taken to dig a pilot channel cutoff across the strip between the Fabius and the Mississippi rivers. In less than two years, the floods of the Fabius had enlarged the cutoff to full size and had completely filled up the old stream channel, below the diversion, to a depth of 3 or 4 ft above ordinary low water. A serious menace and expense in levee maintenance were overcome by this simple expedient.

Without resort to excessive factors of safety, the author has presented a logical analysis of the rainfall-runoff problem for small tributary streams. Neglect to realize the importance of this problem, undoubtedly, is the reason for the numerous levee failures heretofore mentioned.

Fig. 4 may be a little puzzling to those who have become accustomed to using only the percentage distribution graph. The two hydrographs shown are true unit hydrographs. Each one represents the volume of 1 in. of runoff in specified times.

There appears to be little agreement on the definition of time of concentration. The writer defines it as the time period from the beginning of a uniform rate of rainfall on a basin until the runoff rate becomes a constant.

On this basis the writer tested the 33-min hydrograph in Fig. 4. Six average ordinates were scaled from the 10-min unit hydrograph. The summation process was applied to a series of 10-min net rains of 0.303 in. or a rate of 1.818 in. per hr. The average rate of runoff between 30 and 40 min was 940 cu ft per sec; checking a peak somewhat less than 1,000 cu ft per sec, the average between 40 and 50 min was 993 cu ft per sec and at 50 min the discharge was 1,000 cu ft per sec and continued thereafter at that rate as long as the uniform rain persisted.

This curve here developed by the writer is the "cumulative curve" introduced by Russell Morgan and D. W. Hullinghorst.[12]

A unit hydrograph, if available, reflects storage effect, and independent storage determination is not necessary; $T c$ is not an exact constant for a given basin. It varies with the condition of initial soil moisture and time of transit. The latter is governed by stage or rate of discharge. For all practical purposes, the author's derivation of $T c = 33$ min checks with the writer's definition of time of concentration.

The use of zero infiltration capacity is sound for most levee districts when the river stands against the levee. Many Illinois and upper Mississippi districts, with sandy substrata, show an actual inflow seepage of 0.25 in. or more per day. These same districts in July and August will have a large infiltration capacity.

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[12] See "The Flood Hydrograph," by Howard M. Turner and Allen J. Burdoin, *Journal*, Boston Soc. of Civ. Engrs., July, 1941, discussion p. 272.

It is to be expected that the requisite pumping capacity for interior drainage should be greater for leveed municipalities than for leveed agricultural districts. Fig. 5 indicates an astounding difference. The experience of years has shown that a pumping capacity of 0.27 in. in 24 hr is adequate for all districts except those affected by continuous backwater from the river. In this case 0.50 in. in 24 hr is required.[13] A part of this difference is probably due to the availability of the interior ditches and low lying waste lands for storage, and a part is due to the fact that a material infiltration capacity exists during the season of possible crop damage.

C. O. CLARK,[14] JUN. AM. SOC. C. E.—The drainage of leveed areas in mountainous valleys is soundly analyzed by the author. The approach to the drainage problem by way of rainfall frequency, runoff, and the unit hydrograph has several important advantages over empirical design. First, it is well balanced, in that it is neither grossly inadequate, as when inapplicable empirical relations are applied, nor wastefully over-adequate, as when provision is made for some recent occurrence that is really phenomenal and very unlikely to recur. Second, the solution is approached from the over-adequate side, since cost forces a reduction of adopted size below the first calculation, with the result that, in the usually broad range of acceptable solutions, the more adequate are adopted. Third, the extreme cost of some of the first-calculated solutions leads to an ingenuity of solution not inspired by the approach from the inadequate side. Fourth, utilizing rainfall frequency data, the engineer usually comes to his answer with the deliberate knowledge that the provided capacity may be overtaxed, but that the payment of the damage will be cheaper than its prevention.

The possibilities for utilization of rainfall records coincident with flood stages present a promising opportunity for reduction of cost of pumping installations. The writer, in 1937, made for the U. S. Engineer Department the first studies along this line for the cities of Louisville, Ky., Evansville, Ind., and Cairo, Ill., along the Ohio River, where the procedure is unquestionably sound. The procedure can be utilized for drainage along streams which are so small that crests are reached in as little as 36 hr.

There are many circumstances in which the frequency of rainfalls that have occurred coincidently with flood stage in the main river can be used. Whereas rainfall during damaging flood stages may be too infrequent to study, rainfall during moderate and low stages is common enough to indicate the frequency with which it can be expected at high stages.

The writer believes that most of the floodtime rainfall may be treated as a phenomenon independent of the flood-producing rainfall, and he found this to be true in the following example. This depends, of course, upon the difference in time between flood peaks on the main stream and on the tributary. If the time difference is less than 24 to 36 hr, partial dependence is likely, since both floods might result from the same storm. In this case, however, drainage should be estimated for the flood-producing storm, rather than some other synthetic storm, as both together are unlikely.

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[13] "Cost of Pumping for Drainage," by J. G. Sutton, *Technical Bulletin No. 327*, U. S. Dept. of Agriculture, October, 1932.

[14] Engr., U. S. Engr. Office, Winchester, Va.

If the time difference is more than 36 hr, independence of storm causes is essentially attained. Although general conditions are more favorable for rainfall than the average, and rainfall therefore might be more probable than average, the normal movement of storms in the Eastern United States is such that, 2 days after a severe storm, a region is quite likely to be in the high-pressure area, and to be relatively fair, with a reduced probability of rainfall.

However, it is not necessary to assume that the causes are strictly dependent or independent. The degree of dependency can be established. Thus, from the correlation of rainfall and low-river stages, it can be established that rainfall during flood stages is one, two, or several times as probable as average. This conclusion can be extended to higher stages. The probability of heavy rainfall is directly proportional to the total length of experience, almost any rainfall being probable in a long enough period of exposure. If, during a particular period of time, such as floodtime, rainfall were twice as probable as normal, the same rainfall could be expected in that period as in a normal period twice as long as that under consideration.

The duration of main-stream floods is very short (only a few days), so it may take a good many years to build up a year of exposure to floodtime rainfall. If pumping is necessary at the annual flood stage, and the average duration of floods above annual stage is 2 days each, the exposure of an area to floodtime rainfall during 50 years is only 100 days. If rainfall is twice as likely to occur, or two times as probable as normal, the exposure is equivalent to 200 days of normal exposure, and the rainfall to be expected is that which normally occurs once in 200 days.

To establish the relative probability of rainfall during floodtime, it is only necessary to establish a sufficiently low limit of river stage to include the desired number of rainfall occurrences. The probable normal experience for the floodtime period is computed for the same period of time as the aggregate period during which the river was above the critical stage, and is compared with the actual floodtime experience as plotted or computed in the manner suggested by the author. If the computed and observed frequencies do not agree, they will be found to be approximately parallel to each other, when plotted with a logarithmic frequency scale, and one can be expressed as a percentage of the other. Thus, if twice as many occurrences are reported as computed, it is safe to conclude that the probability of rainfall during floodtime is twice the normal. Because of the larger number of hourly amounts recorded, the study of hourly records is easier than the study of daily records. It is not essential that the rainfall and river stages be observed at the same place. Since the river stage is used only to identify those periods in which floods are likely as compared with the average, in the author's example, recording rainfall records at Scranton and Harrisburg could be utilized with stage records at the locality under consideration—Williamsport.

Data studied for Richmond, Va., on the James River will illustrate the point in question. Rainfall during floodtime at Richmond is rare. Almost 2 days elapse between the rainfall that produces the James River floods and the arrival of the peak at Richmond. By that time, local streams have fallen to low levels, and they rise again only if additional rain falls. By the criterion of the writer, in a preceding paragraph, floodtime rainfall should be independent and normal.

A study of the hourly and daily rainfall records of the Richmond Weather Bureau for the period 1905–1939 showed that rainfall during floodtime was no more, and no less, probable than at any other period. In 34 years the James River is normally above a stage of 8.0 ft for 4,080 hrs, 170 days, or 0.46 year. A stage of 8.0 ft was reported during each of 193 days in the period 1905–1939. Normally, rainfall of 0.01 in. or more is reported for 590 hr per yr, or 275 times in 170 days.

TABLE 3.—NORMAL RAINFALL IN INCHES, RICHMOND, VA.

Maximum probable rainfall	LENGTH OF PERIOD, IN DAYS (NOT NECESSARILY CONSECUTIVE)							
	1	2	5	10	20	50	100	200
Hourly...	0.03	0.07	0.14	0.23	0.35	0.56	0.76	0.98
Daily....	0	Trace	0.10	0.38	0.72	1.15	1.55	2.00

Table 3 shows the normal rainfall experience of Richmond. The quantities listed define the hourly and daily rainfall frequency curves from 1 to 200 days. (Note: The fact that hourly rainfall is greater than daily rainfall for short periods of time is correct, and merely means that, although rainfall of 0.01 in. or more occurs in only one of every 3 or 4 days, it usually rains for several hours when it does rain.)

The records of hourly and daily precipitation were compiled for all days, 1905–1939, inclusive, upon which a stage of 8.0 ft was reported, a total of 193 days. The precipitation for the entire day was included, regardless of whether the river remained above 8.0 ft. This inclusion is partly offset by the omission of records of days when the river has exceeded the stage for a few hours without having been reported.

Table 4(*a*) compares the normal experience of any 170 days at Richmond, selected at random, with the record of all days, 1905–1939, upon which the James River was reported above 8.0 ft. Table 4(*b*) shows a similar comparison for the experience above a stage of 20.0 ft, which is an average of 11 days in 34 years.

Tables 4(*a*) and 4(*b*) show essential agreement between the floodtime experience and that of a normal period of equal length. If the number of floodtime occurrences were 2.0 or 3.0 times the number of normal occurrences, the conclusion would be that floodtime rainfall was 2.0 or 3.0 times as probable as normal. In this case, there is no essential deviation, and the writer draws the conclusion that the probability is the same, and that the estimate of rainfall at higher stages on a similar basis is warranted.

In this connection, it is well to note that it is not necessary to have a knowledge of the true duration of floods above given stages; but it is adequate to use a flood-stage frequency curve with any assumed number of days' duration per rise. If the duration is assumed too long, the probability ratio will be correspondingly smaller. Likewise, a seasonal or monthly rainfall frequency curve could be used, instead of an annual curve as used by the writer, with a corresponding compensative change in the indicated relative probability of rainfall during floodtime and in the normal period for which a rainfall curve is available.

TABLE 4.—COMPARISON OF FLOODTIME AND NORMAL RAINFALL, RICHMOND, VA.; NO. OF OCCURRENCES

Excess (in.)	(a) RIVER ABOVE 8.0 FT				(b) RIVER ABOVE 20.0 FT			
	Hourly Rainfall		Daily Rainfall		Hourly Rainfall		Daily Rainfall	
	Floodtime[a] 1905 to 1939	Normal for 170 days	Floodtime[a] 1905 to 1939	Normal for 170 days	Floodtime 1905 to 1939	Normal for 11 days	Floodtime 1905 to 1939	Normal for 11 days
Trace	....	....	54	73	....	....	5	5
0.01	354	275	36	54	16	18	4	4
0.03	....	....	....	....	8	10	....	....
0.05	....	....	....	....	5	7	3	3
0.10	71	55	....	....	1	3	2	2
0.20	22	21	29	26	0	1	....	....
0.30	7	11	....	....	....	....	2	1
0.50	1	4	12	13	....	....	0	1
0.87	1	1	....	....	....	....	....	....
1.00	....	....	6	5	....	....	....	....
1.50	....	....	2	2	....	....	....	....
2.00	....	....	0	1	....	....	....	....

[a] During any part of a day on which a stage of 8.0 ft or higher was reported.

The writer believes that the theory of this method of analysis can be extended easily to runoff instead of rainfall. A pumping plant to be used above the 5-yr river stage, where average duration of floods is 2 days, is exposed to conditions under which it must operate for only 20 days in 50 years. One year of normal or near normal runoff would contain 18 such occurrences. Observations of runoff certainly should be made before any expensive final drainage plan is adopted.

W. W. HORNER,[15] **M. AM. SOC. C. E.**—Mr. Williams' paper deals with a class of engineering problems which appears to be occurring more and more frequently in engineering practice, and about which relatively little has been published. The title is somewhat too narrow, since the principles involved seem to apply clearly to the drainage of any relatively narrow levee-protected area where there is considerable lateral inflow from a tributary stream. For example, the writer has encountered exactly the same conditions in connection with the "industrial property" area along the Trinity River at Dallas, Tex. At this point the lateral inflow was almost entirely from storm-sewered urban areas. He also has found such conditions in his work for the East Side Levee and Sanitary District at East St. Louis, Ill., where the lateral inflow is from small streams of 1 to 20 sq miles flowing out of the higher-level Illinois bluff drainage areas. Possibly by a little stretch of the imagination, these bluff lands might be considered as mountains. In the first instance, the only possible solution was through the use of pressure conduits; and in the second, cutoff channels, detention storage, and pumping are involved.

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[15] Cons. Engr. (Horner & Shifrin), St. Louis, Mo.

The procedure suggested by Mr. Williams is interesting, and in part new. It appears to justify some comments and a number of questions which the writer hopes that the author may be able to answer in his closing discussion. The two conditions referred to as (*a*) and (*b*) under "Local Drainage Requirements" are fundamental to all such problems. In the experience of the writer, item (*b*) is always the critical one and must be considered first, and item (*a*) should be taken up after the primary solution of item (*b*) has been determined.

Under the head of "Local Hydrology" Mr. Williams discusses the requirement for the consideration of complete hydrographs. This usually has not been undertaken in local problems of this kind, but it is fundamental to any reasonable solution. The one exception to this requirement is that of pressure conduits draining completely sewered urban areas where no storage is provided and the conduit capacity must be equivalent to the peak rate of flow from the storm sewers. For this situation, it is necessary to study the coincident probability of discharges of various frequencies with concordant stages in the main river.

In the case of the Dallas project (1931), a preliminary report had indicated pressure conduits of a capacity to carry the peak rate from the determined rainfall frequency of the urban storm system, in this instance a 5-yr frequency; but the preliminary design of the pressure conduits provided this capacity against the maximum recorded river stage in the Trinity River. This situation was quite critical to the economics of design, as the lower-lying portions of the areas served by the storm sewers were but little above the maximum recorded flood stage.

TABLE 5.—RESULTS OF LEVEES AND FLOODWAY CLEARANCE ON STAGE DURATIONS, TRINITY RIVER AT DALLAS, TEX.

Description	FLOOD FLOW (CU FT PER SEC):			
	75,000	63,000	49,000	23,000
Record Data (27 Years):				
Stage (ft)	42[a]	41	39	35
Days of above stage in 27 years	6	10	38	68
Rate of same excess with reservoirs	69,000	32,000	15,000	12,500
Stage, with cleared floodway between levees	39.5	35	33	30 ±

[a] Second high flood in 27 years; highest stage 52.6 ft.

This case was complicated by the then recent completion of reservoirs above Dallas, and by the changed régime involved in the construction of the levees and cleared floodway, and required a complex analysis of the effect of these structures on stage durations. The result of these studies is summarized in Table 5.

The required hydraulic grade to serve critical improvements in the vicinity of the head of the pressure conduit was at stage 54. The design stage at the outlet was fixed at 35, giving a drop of 19 ft in the 4,700 ft of pressure conduit.

For this situation, an additional check also was made on the actual occurrence of rainfall equal to or exceeding the 1-yr frequency, with concurrent river stages. It was found that, although many of the precipitation periods were related to the same storm which produced the river rises, these precipitation periods never occurred at the passing of the peak of the flood, and were generally related to flood stages well down on the rising side of the hydrograph. It was found that, within the duration of stages in excess of 41 (or 35 with improvements), precipitation had occurred on only 1 day and in an amount of only 0.75 in.

Mr. Williams' experiences with basic data deficiencies closely parallel those of the writer, except that the writer generally has found more adequate recording rain-gage information than seemed to be available in this instance. The installation of stream gaging stations on small areas, such as that in Mr. Williams' project, seems to the writer to be a fundamental requirement for this class of work, the value of which has not been fully appreciated heretofore. Hydrologists are too prone to think that stream-flow records are of no value until a record of a number of years is available. Actually, a good record of two or three major rises of the hydrograph for a small watershed makes unnecessary reams of calculations in the field of synthetic hydrograph preparation, most of which do not lead to very convincing results.

The author's division of both rainfall and flood stages into two major seasons would seem to be adequate for the Susquehanna River. The writer used a similar division in the Dallas studies, determining rainfall depth-frequency relations for each of two 6-month periods; but because of climatic differences, the periods were not the same as those used by Mr. Williams. A study of rainfall frequency records for the winter seasons and the use of the results as indexes appear to be valid procedures for the project described. It is to be noted that the author does not propose a similar procedure for the summer season, and the writer agrees that the indexes for that season would not be sufficiently significant.

The use of point rainfall data for the complete drainage basin seems satisfactory for the winter season, but the writer would not utilize it for summer storms where the area is appreciably in excess of 1 sq mile.

Under the head of "Local Hydrology: Design Storm for Flood Season," the author struggles with a methodology that has been a cause of concern to all hydrologists. In the writer's experience, there is no fully satisfactory answer to this requirement. The best and probably most valid answer involves not the choice of some severe storm, but the actual analysis of runoff for a large number of "severe" storms throughout the period of record, and the subjection of the results of a flood-frequency analysis. Something on this order was utilized in the writer's recent study of flood flow on the Trinity River for the U. S. Department of Agriculture. It involves a great amount of work, however—an amount that would be in excess of what normally would be undertaken for projects such as those described by the author, which involve a considerable number of small streams. The second method suggested, and the one which the author utilized in setting up a synthetic storm pattern from rainfalls of equal frequency, would seem to be definitely and probably considerably on the conservative side. The writer would hesitate to adopt it without further tests of the results. It unquestionably provides synthetic storm patterns of a much rarer and infrequent type than the chosen design frequency. A third method that sometimes has been utilized involves the setting up of a pattern for the design frequency precipitation in accordance with average occurrence. This involves a considerable analysis to determine average storm patterns, but extensive work in this field already has been done. The method was utilized by the Department of Agriculture in the Little Tallahatchie flood-control studies in Mississippi, and there are also available independent studies reported by D. I. Blumenstock.[16] The St. Louis regional rainfall records have been subjected to such studies under the supervision of the writer. Important conclusions which may be drawn from these studies are that:

(1) Such a design storm will never contain short-period intensities as great as the short-period intensities of equal frequency; and

(2) High rates will not occur at the midpoint in the storms, but more nearly at the first one-third point.

The effect of these differences is to give a somewhat flatter and somewhat more advanced histogram of rainfall than those shown by the author.

It is interesting that the author does not consider infiltration rates for the winter storms, possibly due to the fact that so little data have as yet been published for these conditions. The writer is inclined, however, to question the propriety of using the runoff as 100% of the precipitation, plus snow melt, in the design storm. It would seem that, except under very rare conditions, some infiltration and interception losses must be inevitable. The use of such a value in the maximum storm might be justified if the maximum storm were to occur in winter. The application of an infiltration "rate" of 0.3 in. per hr for summer storms, presumably of the short thunderstorm type, would seem to the writer to be overconservative. However, he is of the opinion that proper rate cannot be determined from stream flows on larger areas unless larger areas can be found that have approximately the same coverage of soil and vegetation. The only reasonable approach to the determination of a proper infiltration rate would be on the basis of studies of the identical soil and cover involved in the particular drainage basin. Otherwise, considerable error in either direction may be involved.

The writer is interested in the approach to the development of a synthetic unit hydrograph. In the Trinity River investigation, the writer, in collaboration with Waldo E. Smith, M. Am. Soc. C. E., analyzed a large number of hydrographs at the Rockwall gaging station (821 sq miles). For this station relations were developed as follows:

$$\frac{Q_p}{Q} = 0.025 + 0.07\,(Q - 0.7) \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

for values of Q, or mass surface runoff in excess of 0.7 in., and for Q_p, the peak-hour rate in inches per hour; also, from the recession curves,

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[16] *Technical Bulletin No. 698*, U. S. Dept. of Agriculture, Washington, D. C.

$$q = K_s S \qquad (6)$$

in which q is in cubic feet per square mile and S is hour-second-feet per square mile. For Rockwall, the value of K_s was 0.046.

A study of smaller basins in the same physiographic province indicated that their relationship could be made a function of the basin area in square miles, as follows:

$$K_s = \frac{7.4}{A^{0.25}} \qquad (7)$$

for watersheds in excess of 20 sq miles. For 1 in. of runoff,

$$Q_p = \frac{1.11}{A^{0.6}} \qquad (8)$$

The base of the hydrograph in hours is

$$B = 2.5\,A^{0.6} \qquad (9)$$

and the lag time in hours from center of mass of excess rainfall to center of mass of the hydrograph is

$$L = 0.88\,A^{0.6} \qquad (10)$$

From these relationships it was possible to construct distribution graphs for basins of various areas. To these graphs excess rainfall was applied in time increments varying from 1 hr for basins of 23 sq miles or less to 6 hr for basins in excess of 200 sq miles.

These last four relationships (Eqs. 7 to 10) were most useful and were extremely helpful in working out the hydrograph form.

The use of the "time of concentration" as discussed by the author has not been found to be very satisfactory for natural drainage basins of this type, and the writer would prefer not to use a methodology that involved the so-called rational formula. He would like to call attention to the fact that the "C" in the rational formula involves both losses and delay factors. In the latter category it includes the effect of channel storage. If by some means a proper value of "C" has been determined, then it would seem that something like flood routing already has been done, and the hydrograph would include channel-storage effects above whatever point it was applied. The writer is in agreement with the statement made immediately below Eq. 4 of the paper, but would go further and state that, in most instances, such an assumption is not only conservative but excessively conservative. In general, the writer is of the opinion that much better synthetic hydrographs can be developed from a study of even two or three actual hydrographs than through any approach involving time of concentration and equal time contours. In this connection he would call attention again to the importance of doing what was done under the described projects—namely, the installation of stream gaging stations at the point where the lateral streams enter the flood plain.

WALTER T. WILSON,[17] ASSOC. M. AM. SOC. C. E.—The writer likes Mr. Williams' analytical approach, that is, studying floods by examining their causes. The logic and merit of studying floods by relating stream flow to the antecedent and causative rainfall cannot be denied. Without wishing to detract from Mr. Williams' excellent paper, the writer, feels that the study of successive causes and effects leading to floods should be carried at least one step nearer the first cause. The Hydrometeorological Section of the Weather Bureau has been studying the causes of major storms.

The expression "maximum probable" flood seems unsatisfactory because the term "maximum" implies an upper limit, whereas the term "probable" involves a wide range of meaning and begs a specific definition. A flood or a storm must have a finite physical upper limit. The writer, with an increasing number of hydrologists and meteorologists, is perfectly willing to "stick his neck out," and study this thing of which all hydrologists are thinking—the "maximum possible" storm or flood.

Storm magnitude is determined and limited by the nature and magnitude of its causes. These include the available quantity of moisture in the air, the rate at which moisture laden air flows into the flood area, and the rate at which the moisture can be condensed and precipitated. These factors have finite limits, which involve an admittedly difficult but not hopelessly impossible study.

By way of adverse criticism, the writer feels that the subject of snow is passed by with an inadequacy both in its treatment and melting rate. Mr. Williams' estimated design rate of an inch of snow melt per day has an obscure basis and seems too low. This is in contrast not only to reasonable degree-day factors but also in contrast to actual records. Reference is made to the 1936 flood in New England. The total runoff for the second storm period (March 16 to 19) from the basin of the East Branch of the Pemigewasset River, above Lincoln, N. H. (104 sq miles), has been estimated at 12 in. to 15 in. The nature of the topography and lack of good precipitation data require estimation of the precipitation during this period, which by preparation of isohyetal maps and careful consideration of meteorological factors indicates a value of 4-in. to 5-in. average depth. The difference between the 12 in. to 15 in. of runoff and the 4 in. to 5 in. of rain probably includes some ground-water contribution, but leaves about 8 in. of snow. The writer believes that at least 4 in. of snow melted and ran off in less than one day. By assuming a reasonable velocity of concentration and a reasonable ratio of stream length to basin area, Eq. 4 "boils down" to a Myer's rating of about 30% to 50%.

Under the heading "Local Drainage Requirements," the author states that flood protection from small tributaries is rarely justified. The writer realizes the danger of separating one sentence from the remainder of the text, and may not understand the meaning in this case. However, Mr. Williams has exposed himself to criticism by proponents of upstream engineering and flood control in source regions.

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[17] Asst. Hydrologic Engr., Weather Bureau, Washington, D. C

W. B. Langbein,[18] **Assoc. M. Am. Soc. C. E.**—Any treatment of the flood problem even though directed to one specific design is likely to involve the application of a large range of hydrologic principles. The designer of drainage structures or leveed areas finds it necessary not only to familiarize himself with, but also to evaluate, rainfall and snow-melt characteristics, the absorptive qualities of the ground and the laws of open-channel flow, besides the hydraulic and mechanical properties of the drainage structures themselves. There is special need, therefore, for methods of solving the many hydrologic questions that arise in the design of hydraulic structures and are common to most flood and even drought problems. Methods of treatment in one of them merit consideration in another, and Mr. Williams' paper, although perhaps limited in title, contributes much to this end.

The writer was most interested in Mr. Williams' construction of flood discharge hydrographs from known or assumed amounts of effective rainfall and snow melt. This feature of the rainfall-runoff relation has been a favorite subject for some years; and with increasing experience, fairly satisfactory results have been attained, mainly through the unit hydrograph, a hydrologic tool which reaches its tenth anniversary this year (1942). However, there is noted in Mr. Williams' treatment a co-mingling of the so-called rational method of earlier origin with a synthetic unit-graph method. At the possible risk of confusing the subject, the writer offers the following method for solving Mr. Williams' problem, which breaks away entirely from the so-called rational procedure, substitutes some of the newer unit-graph technique and flood-routing principles, and yet is believed to be as direct and simple as the method used by Mr. Williams.

Fig. 7 shows standard dimensionless unit hydrographs based on a dimensionless S-hydrograph previously derived by the writer.[19] The scale is expressed in terms of the lag between center of mass of effective rainfall and flood runoff, a constant or near constant for any drainage basin. The lag has three other corollary meanings: (1) It is equal to the quotient between an increment in storage in the basin and the corresponding increment in discharge; thus it represents the storage element, and is equivalent to the flood-routing coefficient K;[20] (2) it is equal to a weighted mean time of wave travel from the several component parts of the basin; and (3) it is equal to the storage coefficient as derived from recession limbs of flood hydrographs. It is computed by either comparing times of center of mass of effective rainfall and center of mass of direct runoff, or by analysis of the recession curve.[21] The lag thus defined and computed is descriptive of the storage and concentration characteristics of a basin and may be correlated with basin topographic characteristics in ways dependent on local conditions. Generalizations cannot be given; in some regions, good correlations can be obtained simply with length of a basin; in others, the slope or altitude needs to be introduced. F. F. Snyder,[22] Jun. Am. Soc. C. E., has published data applicable to Pennsylvania, and G. T. McCarthy,[20] Assoc. M. Am. Soc. C. E., has derived a method applicable to the Connecticut River Basin. With such correlations a unit hydrograph for application to ungaged areas within a limited region can be approximated. The procedure then is as follows: Suppose a basin of 3 sq miles has a lag of 1.67 hr and it is desired to compute the peak discharge using rainfall of critical duration conforming to the design storms given in Fig. 2. For a 10-yr storm the amounts of the rainfall plus snow melt in specified periods of time are given in Col. 3 of Table 6. Col. 2 of this table contains the time scale expressed non-dimensionally in terms of the basin lag (100 min in this example). The peak of unit graphs of various durations of inflow (in terms of lag) is given by the envelope curve of Fig. 7. The corresponding values are entered in Col. 4, Table 6. It will be noted that for lengthening storm durations the volume of supply increases, whereas the peak of a unit hydrograph decreases. What is the critical duration of the given storm-rainfall-characteristics of this basin? The answer

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[18] Associate Hydr. Engr., U. S. Geological Survey, Washington, D. C.

[19] "Channel Storage and Unit Hydrograph Studies," by W. B. Langbein, *Transactions*, Am. Geophysical Union, 1940, p. 626.

[20] "The Unit Hydrograph and Flood Routing," by G. T. McCarthy, unpublished ms. presented at Conference of North Atlantic Div., Corps of Engrs., U. S. Army, June 24, 1938.

[21] "Channel Storage and Unit Hydrograph Studies," by W. B. Langbein, *Transactions*, Am. Geophysical Union, 1940, p. 625.

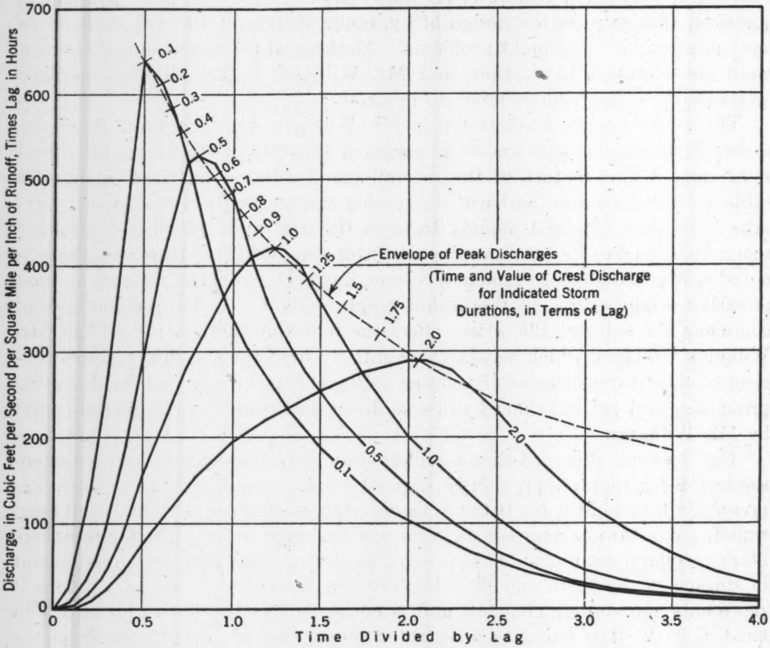

FIG. 7.—DIMENSIONLESS DIAGRAM SHOWING FAMILY OF STANDARD UNIT HYDROGRAPHS FOR SEVERAL SELECTED DURATIONS OF RUNOFF GENERATION

is found in Col. 6, which is the product of Cols. 3 and 4. A storm of 120-min duration gives the maximum peak of this basin. The formula is thus:

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[22] "Synthetic Unit-Graphs," by F. F. Snyder, *Transactions*, Am. Geophysical Union, 19th Annual Meeting, Pt. 1, 1938, pp. 447–454.

$$Q_p = S \frac{\text{Vol}}{\text{Lag}} \qquad \text{(11}a\text{)}$$

or

$$Q = S \frac{\text{Vol}}{\text{Lag}} A \qquad \text{(11}b\text{)}$$

in which Q_p is the discharge in cubic feet per second per square mile, Q is the discharge in cubic feet per second, S is the ordinate of Fig. 7, and A is drainage area in square miles. A unit hydrograph for the critical duration can then be read from the family of graphs given in Fig. 7.

TABLE 6.—COMPUTATION OF FLOOD-PEAK DISCHARGES FOR STORM PERIODS OF SELECTED DURATIONS

TIME		Volume of supply (in.) ×	Peak of graph (Fig. 7) (csm-hr per in.)[a] ÷	Lag (hr) =	Peak discharge[a] Q_p
Minutes (1)	Lag (%) (2)	(3)	(4)	(5)	(6)
0	0	0	...	...	...
30	0.30	0.55	585	1.67	193
60	0.6	0.75	500	1.67	224
90	0.9	0.92	440	1.67	242
120	1.2	1.05	390	1.67	245 = Max.
150	1.5	1.14	350	1.67	239
180	1.8	1.22	315	1.67	230
210	2.1	1.29	280	1.67	216
240	2.4	1.35	250	1.67	202

[a] Cubic feet per second per square mile.

The writer proposes the lag time as a measure of the time-discharge characteristics of drainage basins in lieu of the so-called time of concentration. The time of concentration is a concept that has not been, and seemingly cannot be, defined accurately. For example, the usual definition is the "time required for rainfall excess to flow from the most remote part of the basin to the gaging station." Mr. Williams adapts another to his problem—the time required for runoff to equal the rate of rainfall excess. Still another is the time between beginning of the flood rise and its peak. It has never been demonstrated that these are equivalent. Incidentally, C. E. Ramser, M. Am. Soc. C. E., used the latter meaning in determining time of concentration which Mr. Kirpich used in his article, to which Mr. Williams referred.[7] In discussing Mr. Kirpich's article,[23] the writer stated that the correlation presented was equivalent to a velocity of 4 ft per sec and the inclusion of the slope factor was not well supported by the available evidence. The lag time, however, can be computed in more than one way, each of which is theoretically equivalent subject only to errors or limitations of data.

Eq. 11*b* is similar in form to Eq. 4 if the latter is multiplied by the volume of runoff, except that lag is used to evaluate the storage effect and no assumption need be made as to the critical length of the storm. It is proposed for use instead of the "rational" formula $Q = C\,i\,A$. Insufficient data are given by Mr. Williams to compare the results obtained by the two methods, except to the following extent:

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[23] *Civil Engineering*, August, 1940, p. 533.

Fox Hollow Run, according to Fig. 4, has a lag (center of mass of supply to center of mass of hydrograph) of 20 min.

In dimensionless terms the peaks of the hydrographs shown are as follows:

(1) "Ten-minute rainfall period" = 0.50 lag; $\dfrac{1{,}300\ \dfrac{\text{ft}^3}{\text{sec-in.}}}{\dfrac{548}{640}\ \text{miles}^2} \times 0.33\ \text{hr}\ (= \text{lag})$ = 500 (no units). The envelope curve in Fig. 7 for inflow period of 0.5 lag gives 530.

(2) "Thirty-three-minute rainfall period" = 1.67 lag; $\dfrac{1{,}000\ \dfrac{\text{ft}^3}{\text{sec-in.}}}{\dfrac{548}{640}\ \text{miles}^2} \times 0.33$ hr (= lag) = 388. The envelope curve in Fig. 7 for inflow period of 1.67 lag gives 330.

GORDON R. WILLIAMS,[24] ASSOC. M. AM. SOC. C. E.—The volume of discussion pertaining to this paper was gratifying to the writer, particularly since the subject was not one to arouse interest during wartime. Many interesting angles were stressed, and considerable space could be devoted to developing further the points in question. There was general agreement that the title of the paper was too restricted and that the principles presented have wide application to drainage problems, both urban and rural, in many parts of the United States.

In discussing the methods of disposing of local drainage, Mr. Bernard proposed an additional method, or rather a combination of methods, in which a secondary levee and pumping station would be provided upstream from the main river levee and pumping station. Such a method might be worthy of consideration under special conditions, but the use of two sets of pumping installations and gates rarely would be justified from an economic standpoint.

The importance of having adequate levee heights under method (1) was demonstrated by Mr. Sherman with an example. The writer has utilized the following method for developing adequate height for tributary levees:

(*a*) From known records of discharges, develop a relation between coincident flood discharges on the main river and on the tributary;

(*b*) Compute flow lines up the tributary, using a range of coincident discharges; and

(*c*) Draw an envelop of the flow lines.

The envelop curve plus a normal freeboard for wave action will constitute the levee profile. The height of the downstream end will be governed by the magnitude of the design discharge in the main river, but the height of the upstream end usually will be governed by the magnitude of the design discharge in the tributary. In this connection, Mr. Sherman took exception to the statement that "A corresponding degree of protection is not required against flows from tributaries that pass through leveed areas." This statement was intended to apply particularly to the capacity of structures such as pressure conduits and pumping stations. The writer agrees with Mr. Sherman that in the application of method (1) the statement should not be taken too literally and that adequate upstream height is well worth the relatively small additional cost. On other types of drainage structures, it is obviously not sound engineering practice to provide discharge capacity for a flood having a probable frequency of occurrence of once in a hundred years or less.

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[24] Hydr. Engr., U. S. Engr. Dept., Wilkes-Barre, Pa.

Mr. Sherman commented on the high pumping capacities in inches per day, shown in Fig. 5, as compared with pumping capacities found necessary for leveed areas on the Illinois and upper Mississippi rivers. The latter areas are flat, provide much more storage per square mile of tributary area, and have a resulting delayed runoff. The pumping stations along the Susquehanna River must dispose of a hydrograph resulting from quick runoff from mountain slopes. Records of flood runoff coincident with high stages in the Susquehanna River already have been obtained on one area of 18 sq miles and indicate that a pumping capacity of 0.8 in. per day in connection with more than 2.0 in. of flood-plain storage will be required to meet conditions which have already been experienced and which were not unusual. Such a capacity is in close agreement with the mean curve in Fig. 5.

Several of the discussers questioned the assumed rate of snow melt of 1 in. in 24 hr in connection with the design storm. Messrs. Bernard and Wilson considered the rate too low, whereas Mr. Horner considered that the use of 100% runoff plus snow melt is too conservative. The writer admits that there were few data upon which to base the assumption, but is interested to note the wide divergence of opinion among recognized specialists in hydrology. The selected rate of snow melt was intended to have the same probable frequency of occurrence as the design storm rainfalls—namely, 5 to 15 years. Actually, the rate of snow melt, which is only 0.04 in. per hr, has little effect on the peaks of the hydrographs and is merely inserted for its influence on the 1-day and 2-day storage requirements.

Mr. Wilson took exception to the statement that "Flood protection from small tributaries is very rarely justified," and stated that the writer "has exposed himself to criticism by proponents of upstream engineering and flood control in source regions." In making the statement the writer has reference to the economic justification for the erection of expensive structures, such as large storage dams, when the benefits are to accrue only to the tributaries themselves. The burden of proof of the efficacy as well as the justification of other methods, such as terraces, check dams, and crop control, is squarely up to the proponents of such methods. The determination of the annual charges for such work will not be difficult, but the computation of the reduction in flood stages at points downstream for floods of different magnitudes and frequencies will test the ingenuity of agricultural engineers and agronomists.

The use of the rational formula (Eq. 2) to determine the peak of the time of concentration unit hydrograph received considerable comment. Mr. Langbein and Mr. Sherman were of the opinion that the writer was attempting to establish a new definition for time of concentration, but that was not the case. The confusion seems to result from the fact that the writer was concerned only with rainfall that would run off—that is, rainfall excess. In accordance with the rational theory, if a uniform rainfall falls on saturated soil and continues to fall until all parts of the area are contributing, the theoretical rate of runoff will equal the assumed uniform rainfall rate and will continue at that rate as long as the rainfall continues. Mr. Horner pointed out that the *C* in the rational formula, as generally used, involves both losses and delay factors. By using a time of concentration derived from actual stream-flow records, the writer has introduced, in part at least, some of the delay factors. Some of the conservatism derived from the use of the rational theory is nullified in the application of the final flood hydrographs, which are greatly modified by routing through existing storage and constrictions. It is agreed that the method of deriving the unit hydrograph is conservative, and should not be resorted to if actual records are available.

Messrs. Horner and Langbein presented methods of deriving synthetic unit hydrographs by correlation of lag time with the characteristics of recorded hydrographs. The writer believes that the principles involved are perhaps superior to those presented in his paper, but are difficult to apply for the following reasons:

(*a*) Records of complete hydrographs from areas of less than 10 sq miles are difficult, if not impossible, to obtain for a wide variety of drainage area characteristics; and

(*b*) The extrapolation and application to very small areas of relations obtained from hydrographs for relatively large areas will lead to erroneous results and to serious under-design.

Mr. Clark considered, in detail, studies for determining rainfalls expected to occur coincident with different river stages. The writer agrees that this is the ideal solution for drainage areas tributary to large rivers or at any locality where there are sufficient records. The results of a study of this kind have been presented by Frank A. Marston,[25] M. Am. Soc. C. E. Rainfalls coincident with different river stages were considered in Mr. Horner's discussion, where he outlined the problem of the design of a pressure culvert at Dallas, Tex.

The expression "maximum probable" flood is questioned by Mr. Wilson, and he prefers the term "maximum possible" flood. The writer agrees that his expression involves two words which are somewhat contradictory, but what he attempted to determine was an extraordinary flood which by nature could not be defined within close limits. If the "maximum possible" flood could be determined, it would have no application to the structures under consideration, and would have little application to engineering design except in the determination of reserve spillway capacity for very large dams. The advisability of using even the "maximum probable" flood in determining the capacity of relief openings is questionable, except where great damage will result from backwater. As stated in the description of method (2), it will be found in many cases to be good engineering and sound economic practice to plan to blow a hole in earth levees in the case of a very rare occurrence rather than spend large sums on oversized gates and culverts.

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[25] *Civil Engineering*, July, 1939, pp. 409–411.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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Paper No. 2174

TIMBER FRICTION PILE FOUNDATIONS

By Frank M. Masters,[1] M. Am. Soc. C. E.

With Discussion by Messrs. E. H. Connor, Glenn B. Woodruff, Jacob Feld, G. G. Greulich, G. S. Paxson, Raymond D. Mindlin, John W. Solomon, A. E. Cummings, and Frank M. Masters.

Synopsis

The theory of timber friction pile behavior presented in this paper was developed in connection with pile tests on the Morganza Floodway in Louisiana. It may afford a means of determining, with reasonable accuracy, the safe loads to impose on groups of piles in any arrangement, and of any length, from data secured by the loading to failure of one or more individual friction piles at the site.

Introduction

Many large areas of land in the United States on which structures imposing heavy foundation loads are constructed are formed of silty and clayey materials, being delta deposits of the larger streams. Sandy deposits or other such firm materials suitable for the support of heavy concentrated foundation loads are frequently found at such great depths below the surface of these deposits that it is not economically possible to construct foundations that bear directly on the firm materials, or that are supported on point bearing piles driven into these firm materials. Moreover, timber piles, which are relatively inexpensive and of remarkable durability when embedded entirely below the ground water, are becoming increasingly scarce in the longer lengths required to reach such firm bearing strata. Consequently, many existing structures are supported on piles embedded in the overlying silt and clay materials. It is becoming more and more important, therefore, to find some reasonably accurate method of determining the safe carrying capacity of single piles and groups of piles that do not reach point bearing in a firm stratum, but depend instead for their support entirely upon the adhesion of the surrounding clay or silt to the embedded pile surfaces.

Such piles, popularly called "friction" piles, are the subject of analysis in this paper. The adhesion that supports them transmits their load to the soil

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Note.—Published in November, 1941, *Proceedings*.

[1] Cons. Engr. (Modjeski & Masters), Harrisburg, Pa.

by vertical shearing stresses, which are greatest in the soil at the periphery of the single pile or of the pile group.

The tip of a "friction" pile delivers practically no load in bearing, so that the load-carrying capacity of a single pile is dependent upon the length of the pile, the diameter and taper of the pile, and the shearing strength of the soil. The last may increase somewhat from the surface downward due to the additional internal friction generated in the soil on account of horizontal earth pressures from the weight of the overlying materials. However, due to the fact that the butt diameter of timber piles is larger than the tip diameter, the load delivered to the soil per unit of length of a timber pile will usually be greatest at the butt and smallest at the tip.

Load tests have shown that pile-driving formulas, in which the weight of the ram and the blows per foot of penetration are used to determine the carrying capacity of the pile, do not apply to friction piles, probably because the characteristics of clay or silt during driving are different from what they are when the pile is at rest in the clay or silt. Static load tests which simply measure the load that creates a predetermined amount of settlement within a fixed amount of time, also do not properly measure the ultimate carrying capacity of friction piles, which is measured rather by the shearing value of the soils in which the piles are embedded than by the compressibility of the underlying materials.

It has long been known that an average individual pile in a group arrangement of friction piles does not have the same carrying capacity as an isolated single pile and that the arrangement and spacing, center to center, of the piles in a group affect the load capacity of the piles. Because of the lack of tests to failure of group arrangements of friction piles, compared with individual friction piles driven in the same character of soil, there has been little or no comparable information available concerning the supporting capacity of single piles and piles in group arrangements with varying spacing, from which a theory might be developed for the economic design of friction pile footings.

THE MORGANZA PILE TESTS

As a part of the Mississippi River flood-control program, the U. S. Engineer Corps is required to construct railway and highway structures across the Morganza Floodway in Louisiana. In order to determine the most economical design of these structures, which are being erected by the railway lines, the Engineer Corps arranged with the railway companies to make tests of various types of piles on which to support the proposed trestles. The results of these tests have been made available by the Corps of Engineers for use in this paper. The locations of these tests, along the lines of the proposed railway structures, with relation to borings and subsurface investigations, are shown in Fig. 1. It excludes those tests carried to point bearing and shows only the tests of timber and concrete friction piles that were loaded to failure. The soil above the "sand line" in Fig. 1 is of approximately uniform quality along the Floodway, consisting of successive strata of loam, silt and clay, and mixtures of those materials. At the Port Allen (La.) Branch Crossing (not shown in Fig. 1), pile C22, Station 2348 + 70, has 77.5 ft of penetration, with its tip just reaching the sand line. The driving records show that until a pile had reached the sand stratum, where it came up suddenly, the blows per foot increased only gradually

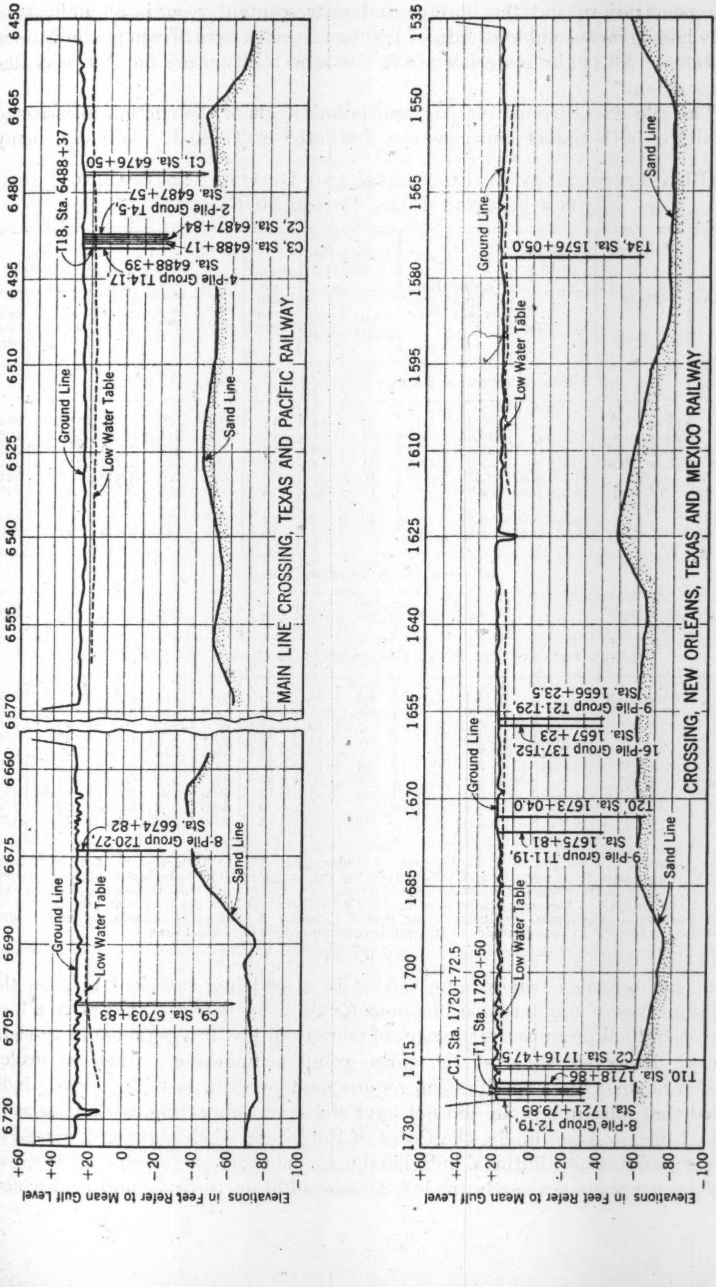

FIG. 1.—PROFILES OF RAILWAY CROSSINGS OF THE MORGANZA FLOODWAY SHOWING LOCATIONS OF FRICTION PILES TESTED TO FAILURE

with penetration, and the borings and water-content records all indicate a fairly homogeneous material into which the test piles were driven at the various locations. All of the materials above the sand stratum are of silty, clay-like composition.

The pile sizes, driving records, and failure loads of the friction piles tested to failure, both timber and concrete, are given in Table 1. Actually, many tests, not recorded herein, were carried to a load just sufficient to meet the requirements of standard specifications for timber pile tests—that is, to a load that would not produce a settlement of more than ¼ in. in a period of forty-eight hours. However, some of the early group arrangements when so loaded settled beyond these specification requirements, and in so failing clearly indicated that piles in a group did not have the same supporting capacity as individual piles. Accordingly, the Corps of Engineers, when requested, agreed to load to failure one additional individual pile, and two 9-pile groups, all of which had been previously partly loaded, and in addition to drive and load a new

TABLE 1.—SUMMARY OF PILE SIZES, AND DRIVING AND LOADING DATA, FOR FRICTION PILES TESTED TO FAILURE

Pile[a]	Type[b]	DIAMETER, IN IN.[c]		Weight,[d] in kips per pile	Hammer No.[e]	BLOWS PER FOOT		Formula safe load (tons)[g]	Penetration (ft)	Average contact area (sq ft)	Average settlement (ft)[h]	AT FAILURE	
		Butt	Tip			Half length[f]	Last foot					Load[i]	Skin friction[j]
(a) TEXAS AND PACIFIC RAILWAY; MAIN LINE CROSSING (T. & P.)													
C1	27-in. HOC	...	...	28.8	OR	10	83	48	52.2	389	2.021	117.5	604
C2	24-in. SOC	...	...	29.8	OR	11	22	46	50.5	335	1.826	117.5 –	702
C3	30-in. HOC	...	...	35.6	OR	21	29	59	50.5	418	2.513	117.5	562
C9	30-in. HOC	...	...	63.2	OR	34	314	219	92.0	762	0.032	210	551
T4–5	UYP	16.56	8.87	2.25	2	16	27	13	50.0	166	0.009	60	721
T14–17	YP and cypress	16.80	8.35	2.1	2	15	27	14	50.0	165	Tipped	50	610
T18	YP	16.50	8.50	2.1	2	13	25	12	50.5	165	1.510	70	847
T20–27	YP	15.91	8.10	2.1	2	9	20	10	50.0	158	0.049	40	510
(b) PORT ALLEN BRANCH CROSSING													
C22	30-in. HOC	...	...	52.7	OR	28	452	238	77.5	609	0.351	210	688
(c) NEW ORLEANS, TEXAS AND MEXICO RAILWAY CROSSING (N.O.T. & M.)													
C1	30-in. HOC	...	...	35.4	0	22	58	79	51	415	0.360	87.5	420
C2	30-in. HOC	...	...	62.6	0	42	493	196	81.5	669	1.329	127.5	380
T1	UYP	17.50	8.50	2.2	1	4	16	18	52.0	195	Failed	66.5	685
T2–9	UYP	16.91	10.20	2.3	1	6	11	13	50.4	179	0.042	42.5	476
T10	UYP	17.75	11.50	3.8	1	6	13	15	59.5	214	Failed	96.5	900
T11–19	UYP	18.08	10.67	3.2	1	6	14	16	59.7	219	0.231	56.3	520
T20	UYP	17.50	8.00	3.2	1	3	30	30	85.0	295	0.023	100	680
T21–29	UYP	18.08	10.50	3.0	1	5	11	13	59.9	224	0.032	55.33	495
T34	CYP	16.75	8.50	3.1	1	3	8	9	83.0	283	Failed	110	777
T37–52	UYP	16.86	8.31	2.35	1	4	11	13	60.0	203	0.236	42.9	423

[a] C = "concrete"; T = "timber"; T37–52, etc., = timber piles T37 to T52; that is, a 16-pile group.
[b] YP = yellow pine; U = untreated; CYP = creosoted yellow pine; HOC = hollow octagonal concrete pile; and SOC = solid octagonal concrete pile.
[c] Average for the group.
[d] In thousands of pounds per pile.
[e] Vulcan type of hammer; OR = manufacturer's number.
[f] Blows per foot with pile half driven.
[g] Safe load, in tons per pile, by "Engineering News" formula.
[h] Average permanent settlement, after unloading, in feet.
[i] In tons per pile.
[j] In pounds per square foot of contact area.

16-pile group so as to obtain information that would show the carrying capacity of single piles of different lengths as well as groups of piles in different arrangements and spacings. The behavior of these three groups was similar; each

TABLE 2.—TEST RECORDS FOR INDIVIDUAL TIMBER FRICTION PILES, LOADED TO FAILURE; N.O.T. & M. CROSSING[a]

(a) PILE T10; 59.5 FT LONG

Total load, in tons (1)	Time (hr:min) (2)	Settlement, in ft (3)
0	0	0
16.5	1:00	0.002
	5:45	0.002
31.5	6:00	0.004
	10:00	0.005
	14:00	0.006
	21:00	0.006
41.5	21:30	0.007
	24:00	0.009
	29:00	0.009
51.5	32:00	0.013
	35:00	0.013
	38:00	0.014
	41:00	0.015
	56:30	0.015
	61:00	0.014
	69:00	0.014
	71:45	0.015
	75:20	0.015
46.5	75:20	0.013
41.5	75:20	0.012
36.5	75:20	0.012
31.5	75:20	0.011
26.5	75:20	0.010
21.5	75:20	0.010
16.5	75:30	0.009
21.5	75:50	0.010
26.5	75:50	0.011
31.5	75:50	0.011
36.5	75:50	0.011
41.5	75:50	0.012
46.5	75:50	0.012
51.5	75:50	0.013
56.5	76:00	0.013
	79:00	0.013
	82:30	0.015
	91:00	0.015
	94:45	0.013
61.5	94:55	0.013
	97:00	0.016
	99:30	0.020
66.5	100:15	0.020
	109:00	0.020
	113:00	0.021
	119:45	0.021
71.5	120:00	0.022
	122:00	0.022
	124:00	0.023
	125:50	0.023
76.5	126:00	0.023
	130:00	0.024
	134:00	0.025
	142:00	0.025

(a) PILE T10 (continued)

Total load, in tons (1)	Time (hr:min) (2)	Settlement, in ft (3)
81.5	142:50	0.027
	144:00	0.028
86.5	144:30	0.028
	147:00	0.029
	149:00	0.030
91.5	149.00	0.031
96.5	149:10	0.031
	149:30	0.034
	149:53	0.040
	149:55[c]	....
....	....	....
....	....	....

(b) PILE T18; 50.5 FT LONG

Total load, in tons (1)	Time (hr:min) (2)	Settlement, in ft (3)
0	0	0
10	0	0.004
	4:00	0.004
15	4:00	0.006
	8:00	0.006
20	8:00	0.008
	10:00	0.008
	12:00	0.009
25	12:00	0.009
	14:00	0.009
	16:00	0.010
30	16:00	0.011
	18:00	0.012
	20:00	0.012
35	20:00	0.013
	22:00	0.013
	24:00	0.015
40	24:00	0.017
	26:00	0.017
	28:00	0.018
45	28:00	0.018
	32:00	0.018
50	32:00	0.021
	36:00	0.021
55	36:00	0.026
	38:00	0.026
	40:00	0.027
60	40:00	0.028
	42:00	0.029
	44:00	0.031
65	44:00	0.032
	46:00	0.037
	48:00	0.038
70[b]	48:00	0.039
	49:00	1.510

(c) PILE T20; 85 FT LONG[d]

Total load, in tons (1)	Time (hr:min) (2)	Settlement, in ft (3)
0	0	0
7.5	0	0
15.0	0:25	0.003
25.0	0:35	0.003
	4:00	0.004
	8:00	0.004
	12:00	0.005
	16:00	0.005
	18:25	0.006
30.0	18:30	0.008
40.0	18:35	0.009
50.0	18:40	0.013
	22:00	0.015
	32:01	0.015
	36:00	0.016
	41:50	0.016
60.0	42:00	0.016
70.0	42:10	0.019
75.0	42:15	0.020
	44:01	0.024
	48:30	0.025
	52:00	0.025
	56:01	0.026
	65:25	0.026
80.0	65:35	0.027
90.0	65:45	0.028
100.0	66:00	0.028
	69:30	0.032
	73:00	0.033
	77:00	0.034
	83:00	0.034
	86:00	0.035
	104:01	0.035
	108:00	0.036
	116:01	0.036
	142:00	0.042
	161:00	0.044
	185:00	0.045
80.0	185:25	0.043
70.0	185:28	0.042
60.0	185:30	0.040
50.0	185:34	0.038
40.0	185:40	0.035
30.0	188:10	0.029
25.0	188:13	0.028
15.0	188:18	0.027
7.5	188:20	0.027
0	188:25	0.026
....	198:00	0.023
....	....	....
....	....	....
....	....	....
....	....	....

(d) PILE T1; 52 FT LONG

Total load, in tons (1)	Time (hr:min) (2)	Settlement, in ft (3)
0	0	0
11	0	0.003
	4:00	0.003
16.5	4:00	0.005
	6:00	0.006
	8:00	0.006
21.5	8:00	0.008
	10:00	0.010
	12:00	0.010
26.5	12:00	0.011
	16:00	0.011
31.5	16:00	0.012
	18:00	0.013
	20:00	0.014
36.5	20:00	0.017
	22:00	0.018
	24:00	0.018
41.5	24:00	0.020
	28:00	0.020
46.5	28:00	0.024
	30:00	0.025
	32:00	0.025
51.5	32:00	0.038[e]
	34:00	0.039[e]
	36:00	0.041[e]
56.5	36:00	0.038[e]
	38:00	0.040[e]
	40:00	0.040[e]
61.5	40:00	0.029
	42:00	0.030
	44:00	0.030
66.5	44:00	0.051
	46:00	0.058
	48:00	....[c]

(e) RETEST, PILE T34; 83 FT LONG

Total load, in tons (1)	Time (hr:min) (2)	Settlement, in ft (3)
0	0	0
30	0	....
55	....	....
80	21:45	0.047
	24:45	0.051
	29:25	0.051
90	29:25	0.056
	38:25	0.059
100	43:55	0.063
	53:35	0.063
110	53:35	....[f]
	57:25	0.199
	70:25	0.239[g]

[a] Except pile T18 (Table 2(*b*)), which is T. & P. Crossing. Several intermediate readings where there were no time-settlement changes have been omitted for brevity. [b] This pile settled under 70 tons until the bracing system rested on the ground. [c] Failure. [d] Pile T20 did not plunge sharply; but because it continued to settle under the 100-ton load, at least partial failure occurred. [e] Readings incorrect; pile not plumb. [f] Rapid settlement. [g] Further settlement was prevented by the shoring under the loading blocks.

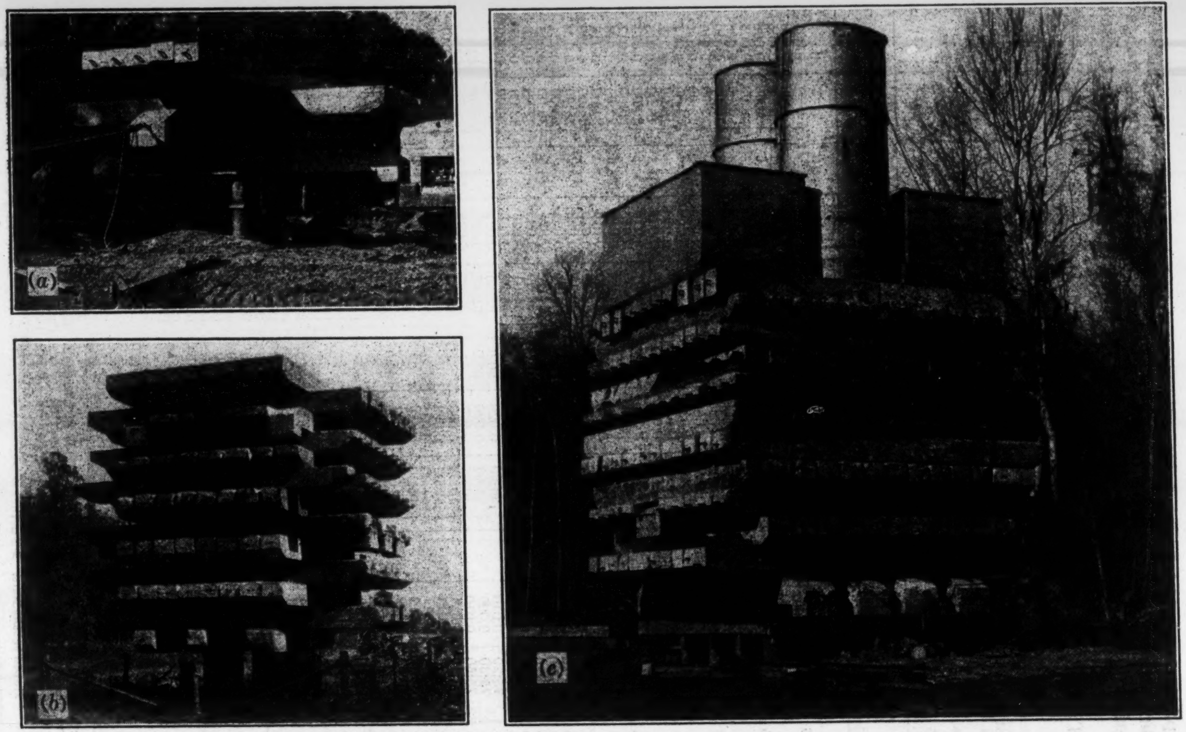

FIG. 2.—PILE TESTS TO FAILURE: (*a*) Pile T1, N.O.T. & M., After Failing Under a 66.5-Ton Load; (*b*) 9-Pile Group at Station 1656 + 23.5 (see Fig. 1), Under a Total Load of 498 Tons; (*c*) 16-Pile Group at Station 1657 + 23 (Fig. 1), Under a Total Load of 775 Tons

failed by leaning toward a corner or a side generally having the smallest diameter piles, and in failing carried down the earth surface with it in a dished depression extending outside the group and having a maximum value near the group center.

TABLE 3.—TEST RECORDS FOR GROUPS OF TIMBER FRICTION PILES LOADED TO FAILURE

N.O.T. & M. CROSSING[a]

(a) PILES T21–T29; NINE, 60-FT PILES

Total load per pile (tons) (1)	Time: Days (2a)	Time: Hr:min (2b)	Settlement, in ft (3)
0	0	0	0
5	0	2:30	0.002
10	0	3:15	0.003
	0	8:00	0.004
	0	12:00	0.004
	0	16:00	0.005
	1	1:30	0.005
15	1	1:30	0.007
	1	5:00	0.008
	1	19:00	0.009
	1	23:00	0.009
20.3[b]	1	23:30	0.011
20.3[b]	243	0	0.011
25.3	243	0	0.012
30.3	243	5:00	0.012
	243	22:00	0.016
	244	4:00	0.016
	245	4:45	0.0162
35.3	245	4:45	0.0166
	245	19:00	0.0176
40.3	246	2:30	0.0186
	247	3:00	0.0205
	247	22:00	0.0219
	248	5:00	0.0219
45.3	248	5:00	0.023
	248	21:00	0.0242
	249	22:00	0.0266
	250	23:00	0.0286
50.3	251	5:00	0.0303
	251	19:00	0.0329
	252	0	0.0349
	252	23:00	0.0361
	254	3:00	0.0376
	254	23:00	0.0376
55.3	255	0:30	0.0416
	255	3:30	0.045
	255	22:00	0.0565[c]

(b) PILES T2–T9; EIGHT, 50.5-FT PILES

Total load per pile (tons) (1)	Time: Days (2a)	Time: Hr:min (2b)	Time, in hr (2)	Settlement, in ft (3)
0	0	0		0
10.3	0	3:00		0.006
10.3			7	0.008
			10	0.009
			14	0.0095
			17	0.010
15.3			18	0.012
			21	0.012
			24	0.014
			26	0.014
			29	0.015
20.3			30	0.017
			34	0.018
20.3			38	0.0185
			41	0.019
25.3			42	0.023
			46	0.025
			53	0.025
30.3			55	0.028
			58	0.030
			61	0.031
			64	0.032
35.3			66	0.035
			69	0.038
			72	0.038
			75	0.039
			77	0.039
37.2			78	0.041
37.2			81	0.043
			84	0.044
			87	0.0445
			90	0.045
39.75			94	0.047
42.5			97	0.048
			98	0.049[d, f]

T. & P. CROSSING

(c) PILES T4–T5; TWO, 50-FT PILES

Total load per pile (tons) (1)	Time, in hr (2)	Settlement, in ft (3)
0	0	0
15	0	0.001
	2	0.005
	4	0.005
20	4	0.006
	6	0.006
	8	0.0065
25	8	0.009
	12	0.009
30	12	0.012
	16	0.012
35	16	0.014
	18	0.014
	20	0.015
40	20	0.015
	24	0.015
45	24	0.021
	28	0.021
50	28	0.021
	30	0.021
	32	0.022
55	32	0.024
	34	0.024
	36	0.026
60	36	0.030
	38	0.033
	40	0.034
	42	0.034
	44	0.036[d]

(d) PILES T14–T17; FOUR, 50-FT PILES

Total load per pile (tons) (1)	Time, in hr (2)	Settlement, in ft (3)
0	0	0
20	2	0.010
	6	0.010
	8	0.011
25	8	0.013
	10	0.014
	12	0.015
	14	0.016
30	14	0.019
	16	0.021
	18	0.022
	20	0.024
35	20	0.025
	22	0.026
	24	0.027
	26	0.028
40	26	0.029
	28	0.029
	30	0.030
	32	0.0305
45	32	0.034
	34	0.035
	36	0.0355
	38	0.036
50	38	....[e]

(e) PILES T20–T27; EIGHT, 50-FT PILES

Total load per pile (tons) (1)	Time, in hr (2)	Settlement, in ft (3)
0	0	0
22.5	7:30	0.012
	10:30	0.015
	13:30	0.0175
	16:30	0.0175
	19:30	0.018
26.25	19:30	0.019
	22:30	0.019
	25:30	0.0205
	28:30	0.021
	31:30	0.021
30.0	31:30	0.023
	37:30	0.023
	40:30	0.025
	43:30	0.025
33.75	43:30	0.027
	46:30	0.027
	49:30	0.032
	55:30	0.032
37.5	55:30	0.034
	58:30	0.034
	61:30	0.038
	64:30	0.041
	67:30	0.041
40	67:30	0.0435
	70:30	0.044
	73:30	0.0455
	79:30	0.0455
	82:30	0.047
	85:30	0.0485
	88:30	0.049
	91:00	0.0495

[a] Data for groups T37–T52 and retest group T11–T19, plotted in Figs. 2 and 3, are not included in Table 3. [b] After 243 days, no appreciable increase in settlement was observed. [c] The group was leaning, settling 0.082 ft toward a corner having the piles with the smallest diameters. [d] Load was now removed because of the steepness of the settlement curve. [e] Group failed by tipping and breaking piles. [f] After unloading, the group had an average permanent settlement of 0.042 ft.

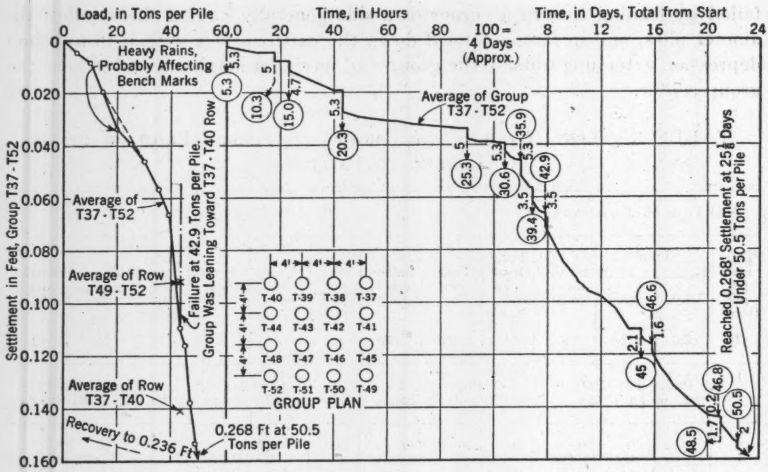

FIG. 3.—SIXTEEN-PILE GROUP T37–T52; 60-FT PILES; N.O.T. & M. CROSSING

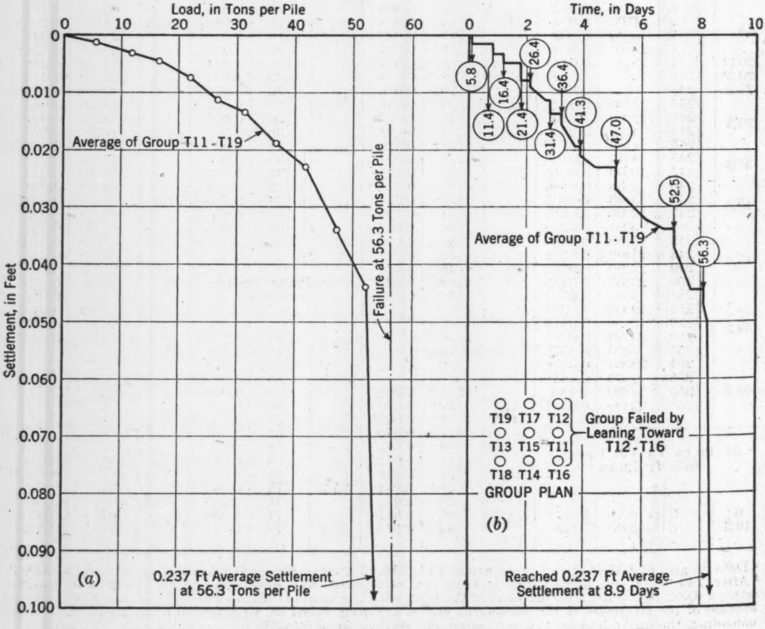

FIG. 4.—RETEST—NINE-PILE GROUP T11–T19; 60-FT PILES; N.O.T. & M. CROSSING

As illustrated in Fig. 2, the test loads were applied in the form of long, narrow, concrete blocks completely spanning the group and laid crosswise in successive layers. They were applied in increments until failure had been reached as indicated by "plunging" or by a continuation of settlement with time. These concrete blocks were supplemented in the case of the 16-pile group by additional loads in the form of water tanks. Settlements were measured accurately from bench marks in the ground surface that were far enough from the groups not to be influenced by the loads.

The results of all tests, to failure, of timber piles plotted both with relation to the load and settlement, as well as to the time and settlement, are given in Tables 2 and 3 and in Figs. 3 and 4. In these figures, the numbers in circles are the total loads per pile, in tons. The numbers on the adjoining arrows are load increments. These tests all demonstrate that larger groups fail more slowly than smaller groups and that individual piles fail by plunging. Table 4 gives the skin friction values at failure for each of the timber friction piles loaded to failure, and for each of the concrete friction piles loaded to failure.

TABLE 4.—SKIN FRICTIONS, INDIVIDUAL PILES

(a) TAPERED TIMBER PILES				(b) CONCRETE PILES, PARALLEL SIDES		
Crossing	Pile No.	Length (ft)	Skin friction[a]	Crossing	Pile No.	Skin friction[a]
N.O.T. & M.	T1	52	685	N.O.T. & M.	C1	420
N.O.T. & M.	T10	59.5	900	N.O.T. & M.	C2	380
N.O.T. & M.	T20	85	680	T. & P.	C1	604
N.O.T. & M.	T34	83	777	T. & P.	C2	702
T. & P.	T18	50.5	847	T. & P.	C3	562
Average	..	....	778	T. & P.	C9	551
				Port Allen	C22	688
				Average	..	558

[a] In pounds per square foot of embedded area based on the average diameter.

The parallel-sided concrete piles, when loaded to failure, did not develop the same supporting capacity per unit of embedded area as did tapered timber piles, although one would assume that the frictional resistance of the surface of a concrete pile should be at least equal to the frictional resistance of the smooth surface of a timber pile. This may be due either to the wedging action of the tapered timber pile or to some very favorable stress distribution along its sides due to the taper. The average skin friction of all tapered timber piles loaded to failure in this series of tests was 778 lb per sq ft of embedded area, whereas the average for the parallel-sided concrete piles was only 558 lb per sq ft.

NOTATION

The following notation is used in this paper:

L = length of pile as driven;

L_1 = depth from pile tip to point in ground below tip;

P = applied load; P' = point load, or load delivered by circumferential band of height dz;

q_h = unit horizontal pressure in radial direction;

q_v = unit vertical pressure;

R = diagonal distance from point load to point in soil at which pressure or shear is desired;

R_c = ratio of load delivered per linear foot of pile at top of pile to load delivered per linear foot at bottom of pile;
r = radial distance from pile center to point in ground;
s_v = unit vertical shear perpendicular to r;
s_x = unit horizontal shear parallel to x;
Z = depth from ground surface to point in ground above pile tip;
Z_1 = depth from ground surface to point in ground below pile tip.

THEORY OF FRICTION PILE BEHAVIOR

Various theories and formulas for the distribution of the load delivered by a pile to the soil were considered in this study. An attempt was made to apply some simple rule for the distribution of pile loads like the straight-line distribution at 60° to the horizontal that is common in some specifications; but this series of tests when plotted would not support any straight-line treatment, and pointed to the necessity for a more complex treatment; nor did the test results appear to check the formulas proposed by R. D. Mindlin,[2] Assoc. M. Am. Soc. C. E., which prescribe tension in the soil for loads applied beneath the ground surface. Only the classic Boussinesq formulas for the distribution of pressure through homogeneous material beneath a point load, when expanded for application to a line load, were found to check with the test results. If further tests are made, they may reveal that some simple and yet reasonably approximate

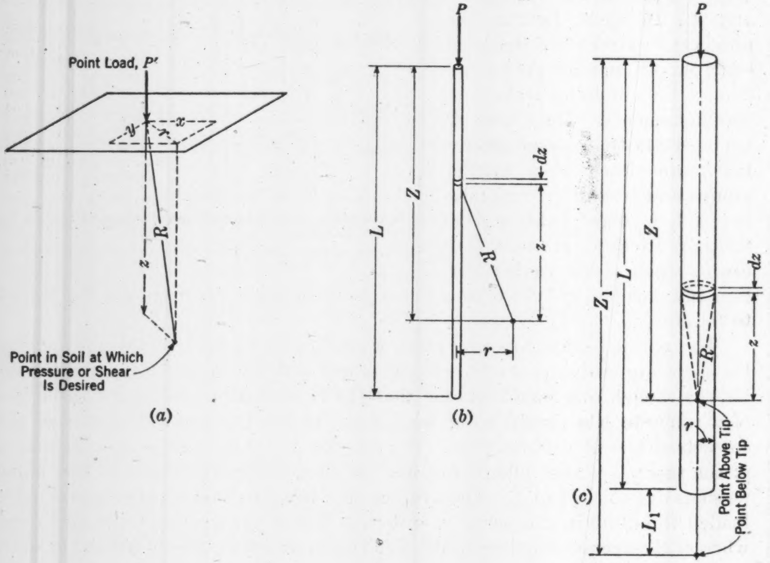

FIG. 5.

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[2] "Force at a Point in the Interior of a Semi-Infinite Solid," by Raymond D. Mindlin, *Physics*, May, 1936.

theory may be applicable. The Boussinesq point-load formulas, which have been used frequently to determine pressures beneath buried foundations, and which were made the basis of the following analysis, are as follows (see Fig. 5(*a*)):

The unit vertical pressure (parallel to the z-axis) is

$$q_v = \frac{3\,P'\,z^3}{2\,\pi\,R^5} \qquad \text{(1a)}$$

The unit horizontal pressure, parallel to r (if Poisson's ratio for the soil is 0.5), is

$$q_h = \frac{3\,P'\,r^2\,z}{2\,\pi\,R^5} \qquad \text{(1b)}$$

The unit vertical shear, perpendicular to r, is

$$s_v = \frac{3\,P'\,r\,z^2}{2\,\pi\,R^5} \qquad \text{(1c)}$$

The unit horizontal shear, parallel to x, is

$$s_x = \frac{3\,P'\,x\,z^2}{2\,\pi\,R^5} \qquad \text{(1d)}$$

The unit horizontal shear, parallel to r, is numerically equal to that in Eq. 1*c*.

A friction pile delivers to the soil a vertical line of load through vertical shearing stresses, developed at the surface of the pile, to result in bearing pressures beneath the pile and shearing stresses around it. It was, therefore, necessary to expand Boussinesq's point-load formulas for pressures and shears by integration to apply to such a vertical line of load. These integrations follow:

Vertical Shear at Radius "r" from Pile.—If the load delivered to the soil per linear foot of pile varies from some value at the tip to R_c times that value at the butt, the load delivered by any circumferential band of height dz (see Fig. 5(*b*)) may be regarded as a point load having the value:

$$P' = \left[1 + \frac{z + L - Z}{L}\,(R_c - 1)\right]\frac{2\,P\,dz}{L\,(R_c + 1)}$$

$$= \left[\frac{(L\,R_c - Z\,R_c + Z) + z\,(R_c - 1)}{L}\right]\frac{2\,P\,dz}{L\,(R_c + 1)} \qquad \text{(2a)}$$

Based on the Boussinesq formula,

$$s_v = \frac{3\,P'\,r\,z^2}{2\,\pi\,(r^2 + z^2)^{2.5}} \qquad \text{(2b)}$$

the vertical shear at a point above the tip at depth Z is:

$$s_v = \int_0^z \frac{3\,r\,z^2}{2\,\pi\,(r^2 + Z^2)^{2.5}}\left[\frac{(L\,R_c - Z\,R_c + Z) + z\,(R_c - 1)}{L}\right]$$

$$\times \frac{2\,P\,dz}{L\,(R_c + 1)} = \frac{P\,r}{\pi\,L^2\,(R_c + 1)}\left[\frac{(L\,R_c - Z\,R_c + Z)Z^3}{r^2\,(r^2 + Z^2)^{1.5}}\right.$$

$$\left. + (R_c - 1)\left(\frac{2}{r} - \frac{Z^2}{(r^2 + Z^2)^{1.5}} - \frac{2}{(r^2 + Z^2)^{0.5}}\right)\right] \qquad \text{(2c)}$$

Vertical Pressure at Radius "r," or on Axis of Pile of Radius "r."—If the load delivered to the soil per linear foot of pile varies from some value at the tip to R_c times that value at the butt, the load delivered by any circumferential band of height dz (see Fig. 5(c)):

$$P' = \left[1 + \frac{z + L - Z}{L}(R_c - 1)\right]\frac{2\,P\,dz}{L\,(R_c + 1)}$$

$$= \left[\frac{(L\,R_c - Z\,R_c + Z) + z\,(R_c - 1)}{L}\right]\frac{2\,P\,dz}{L\,(R_c + 1)} \quad \ldots\ldots\ldots (3a)$$

Based on the Boussinesq formula,

$$q_v = \frac{3\,P'\,z^3}{2\,\pi\,(r^2 + z^2)^{2.5}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3b)$$

the vertical pressure at a point above the tip at depth Z is

$$q_v = \int_0^z \frac{3}{2\,\pi} \times \frac{z^3}{(z^2 + r^2)^{2.5}} \times \frac{(L\,R_c - Z\,R_c + Z) + z\,(R_c - 1)}{L}$$

$$\times \frac{2\,P\,dz}{L\,(R_c + 1)} = \frac{3\,P}{\pi\,L^2\,(R_c + 1)}\left\{-\frac{(L\,R_c - Z\,R_c + Z)\,(3\,Z^2 + 2\,r^2)}{3\,(r^2 + Z^2)^{1.5}}\right.$$

$$+ \frac{2\,(L\,R_c - Z\,R_c + Z)}{3\,r} + (R_c - 1)\left[-\frac{Z^3}{3\,(Z^2 + r^2)^{1.5}}\right.$$

$$\left.\left. - \frac{Z}{(Z^2 + r^2)^{0.5}} + \log_e\left(\frac{Z + (Z^2 + r^2)^{0.5}}{r}\right)\right]\right\} \quad \ldots\ldots\ldots (3c)$$

The Vertical Pressure at Any Point Below the Tip at Depth Z_1.—The load delivered by any circumferential band of height dz:

$$P' = \left[1 + \frac{(z - L_1)}{L}(R_c - 1)\right]\frac{2\,P\,dz}{L\,(R_c + 1)}$$

$$= \left[\frac{(Z_1 - R_c\,L_1) + z\,(R_c - 1)}{L}\right] \times \frac{2\,P\,dz}{L\,(R_c + 1)} \quad \ldots\ldots\ldots (4a)$$

Applying the limits of Z_1 and L_1, the vertical pressure at a point below the tip is expressed by

$$q_v = \frac{3\,P}{\pi\,L^2\,(R_c + 1)}\left\{\frac{-(Z_1 - R_c\,L_1)\,(3\,Z^{2}_{1} + 2\,r^2)}{3\,(r^2 + Z^{2}_{1})^{1.5}} + (R_c - 1)\right.$$

$$\times \left[\frac{-Z^{3}_{1}}{3\,(Z^{2}_{1} + r^2)^{1.5}} - \frac{Z_1}{(Z^{2}_{1} + r^2)^{0.5}} + \log_e\left(\frac{Z_1 + (Z^{2}_{1} + r^2)^{0.5}}{r}\right)\right]$$

$$+ \frac{(Z_1 - R_c\,L_1)\,(3\,L^{2}_{1} + 2\,r^2)}{3\,(r^2 + L^{2}_{1})^{1.5}} - (R_c - 1)\left[\frac{-L^{3}_{1}}{3\,(L^{2}_{1} + r^2)^{1.5}}\right.$$

$$\left.\left. - \frac{L_1}{(L^{2}_{1} + r^2)^{0.5}} + \log_e\left(\frac{L_1 + (L^{2}_{1} + r^2)^{0.5}}{r}\right)\right]\right\} \quad \ldots\ldots\ldots (4b)$$

For pressures on the pile axis, use a value of r equal to the radius of the pile at, or immediately above, the point in the ground. Precisely the same formulas determine the vertical pressures around a pile, assuming it to be a vertical line of load, and setting r = radial distance from the pile axis.

The ratio "R_e" introduced in these integrations produces flexibility in the theory. As applied to timber piles, it takes account of the natural pile taper as well as the internal friction of the soil. When an untreated timber friction pile is withdrawn from soil after having been loaded, it is generally coated with a thin film of the material, indicating excellent adhesion between the pile and the soil, and showing that failure occurred by shearing the soil. If the

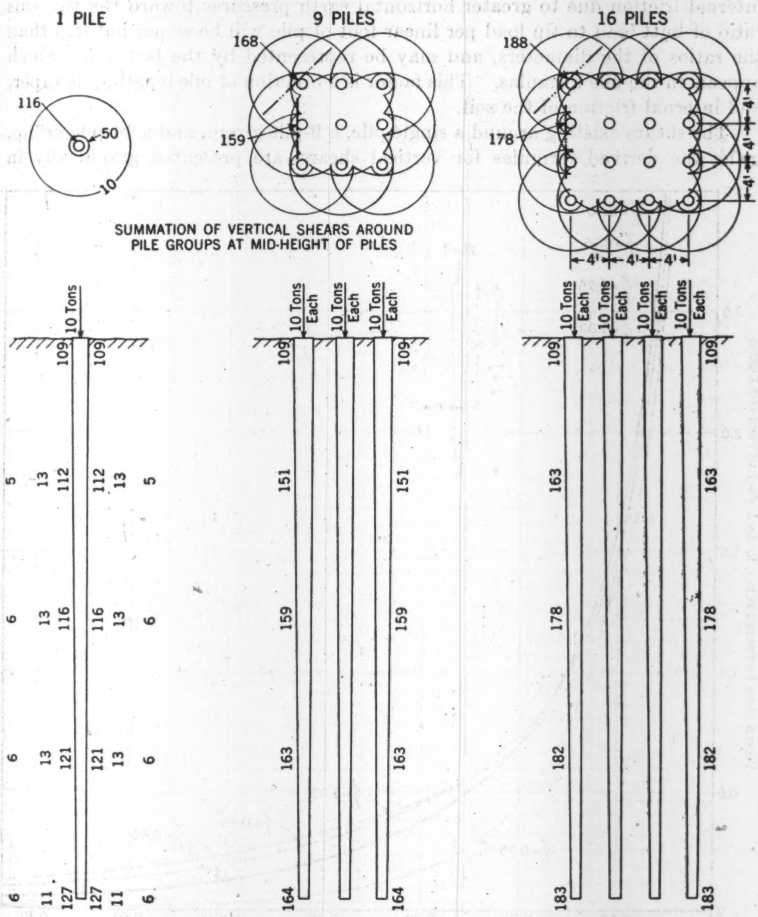

1 PILE
9 PILES
16 PILES
116
50
10
168
159
188
178
4'
4'
4'
4'
4'
4'
SUMMATION OF VERTICAL SHEARS AROUND PILE GROUPS AT MID-HEIGHT OF PILES
10 Tons
10 Tons Each
10 Tons Each
10 Tons Each
10 Tons Each
10 Tons Each
10 Tons Each
10 Tons Each
109
109
5
13
112
112
13
5
6
13
116
116
13
6
6
13
121
121
13
6
6
11
127
127
11
6
109
109
151
151
159
159
163
163
164
164
109
109
163
163
178
178
182
182
183
183

Fig. 6.—Vertical Shears (in Pounds per Square Foot) Under Loads of 10 Tons per Pile (L = 60 Ft; Diameters at Butt and Tip, 14 In. and 8 In.; 4-Ft Spacing; and R_e = 1.5)

shearing strength of the soil were constant from butt to tip, the load delivered by shear to the soil per linear foot of pile would vary with the diameter and would be greater at the butt than at the tip. Since the shearing strength of the soil is slightly increased from butt to tip, however, as a result of a build-up of internal friction due to greater horizontal earth pressures toward the tip, this ratio of butt load to tip load per linear foot of pile will be somewhat less than the ratios of the diameters, and may be represented by the factor R_c, which appears in the pile formulas. This factor is a function of pile length, pile taper, and internal friction of the soil.

The shears existing around a single pile, a 9-pile group, and a 16-pile group, using the derived formulas for vertical shears, are presented graphically in Fig. 6. The shears are based on the assumption of $R_c = 1.5$, which value substantially checks the test results for timber piles. The computations were facilitated by the use of the curves of Fig. 7, in which the general shear formula (Eq. 2c) is plotted for various depths around a pile of unit length and under a

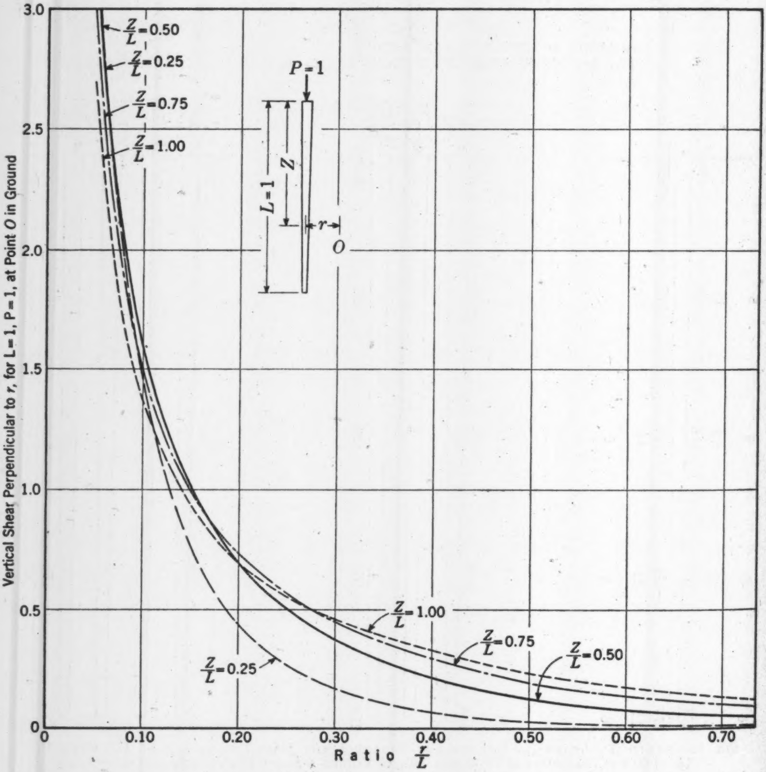


FIG. 7.—UNIT VERTICAL SHEARS AROUND A SINGLE FRICTION PILE ($L = 1$; $P = 1$; $R_c = 1.5$)

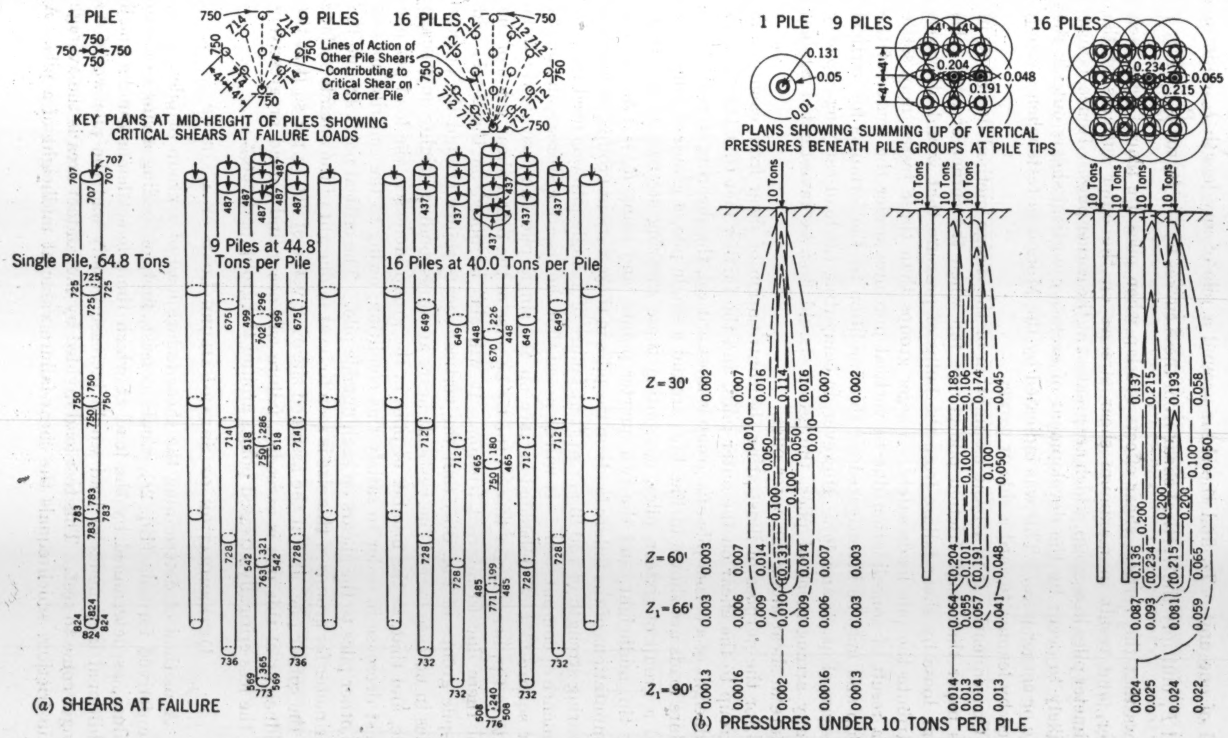

1 PILE
9 PILES
16 PILES
750
714
712
Lines of Action of Other Pile Shears Contributing to Critical Shear on a Corner Pile
KEY PLANS AT MID-HEIGHT OF PILES SHOWING CRITICAL SHEARS AT FAILURE LOADS
Single Pile, 64.8 Tons
9 Piles at 44.8 Tons per Pile
16 Piles at 40.0 Tons per Pile
707
725
783
824
487
675
499
296
702
518
286
728
542
321
763
736
569
365
773
437
649
448
226
670
465
180
485
199
771
732
508
240
776
(a) SHEARS AT FAILURE
1 PILE
9 PILES
16 PILES
0.131
0.05
0.01
4'
0.204
0.048
0.191
0.234
0.065
0.215
PLANS SHOWING SUMMING UP OF VERTICAL PRESSURES BENEATH PILE GROUPS AT PILE TIPS
10 Tons
Z = 30'
Z = 60'
Z₁ = 66'
Z₁ = 90'
0.002
0.007
0.016
0.114
0.010
0.050
0.100
0.003
0.014
0.131
0.006
0.009
0.0013
0.0016
0.189
0.106
0.174
0.045
0.101
0.055
0.064
0.057
0.041
0.014
0.013
0.137
0.193
0.058
0.200
0.136
0.087
0.093
0.081
0.059
0.024
0.025
0.022
(b) PRESSURES UNDER 10 TONS PER PILE

FIG. 8.—VERTICAL SHEARS AND PRESSURES AROUND 60-FT PILES (Diameters at Butt and Tip, 14 In. and 8 In.; 4-Ft Spacing; and $R_s = 1.5$)

load of one unit. To find the shear around a pile of any length L, under a load P, multiply the vertical shear from these curves by the factor $\frac{P}{L^2}$. Fig. 6 shows that the vertical shear effects overlap when piles are placed near each other, and result in maximum shear stresses on the outer surfaces of the perimeter piles in a group, which are greatest at the corner piles. Hence, failure is likely to occur by the development of excessive vertical shears outside the edge or corner piles. This was indicated by the Morganza tests when a group failed by leaning toward a side or corner.

Consequently, it is a basic assumption of the design method described in this paper that small friction pile groups fail when the corner or edge piles break loose by shear failure from the bulbs of pressure that would otherwise distribute the pile loads safely. Larger groups than those tested may settle as a result of consolidation due to vertical pressures under the center of the group and may be investigated for this condition by the formulas for vertical pressure (Eqs. 3*c* and 4*b*). However, in computing the load-carrying capacity of any arrangement of piles, the group should first be investigated against failure in shear along the outer piles.

For the purpose of illustrating the contribution of shear from each pile in a group to the shear on the outer piles, and the further fact that the shear at midheight of a corner pile of a group is most critical, the shearing stresses under failure loads are shown in Fig. 8(*a*) around a single pile, a group of nine piles, and a group of sixteen piles, computing these shearing stresses at the butt, the tip, midheight, and the two quarter points, and using $R_c = 1.5$. In the computations of the failure loads on the piles in these various groups, an average shearing strength of 750 lb per sq ft at failure of a single pile was used for convenience of computation. It will be noted that the shearing stresses, in pounds per square foot of embedded surface on the single pile, increase slowly from butt to tip, being larger toward the tip because of the better shear value of the soil there due to internal friction. It will also be noted on the 9-pile and 16-pile groups of Fig. 8(*a*) that the vertical shears shown around the outside piles in no case exceed the vertical shears at corresponding heights on a single pile, and that, as the number of piles in the group increases, the load per pile must decrease in order to satisfy this condition, owing to the contributions of all other piles to the shears on each outside pile. The critical point around the perimeter for groups of this size is seen to be at midheight of the outside surface of the corner pile, and, if the assumed shearing value of 750 lb per sq ft at the midheight of this pile is exceeded, failure will occur and spread progressively to the less critically stressed points around the group perimeter.

DETERMINATION OF SHEAR-FAILURE LOAD OF A GROUP

A method of determining the shear-failure load of a group of piles, using the derived formula (Eq. 2*c*), would consist, first, of loading a single pile to failure, as determined by the load at which the pile settlement under small additional increments of load would increase rapidly and the pile would no longer come to rest. This test load divided by the contact area is the average skin friction, which equals the shear-failure value at midheight of a pile. A group is assumed to fail by shear at this test-determined value per square foot along a plane intersecting the outer surface of a corner pile, and drawn approximately normal to the line connecting this corner pile with the center of gravity of the group. A load P is assumed on each pile of the group arrangement to be investigated. The contribution of each pile in the group to the midheight shear of the corner or critical pile is found by using the general shear formula (Eq. 2c) and multiplying the contribution of each of the other piles by the sine of the angle of action with the diagonal plane of failure. When these contributions from all piles in terms of a load P per pile are added to the usual midheight shear on the surface of the corner pile (which is equal to P divided by its penetrated contact area), a decimal number is obtained, multiplied by P. The shear-failure value per square foot is divided by this decimal number, and the quotient is the failure load P per pile.

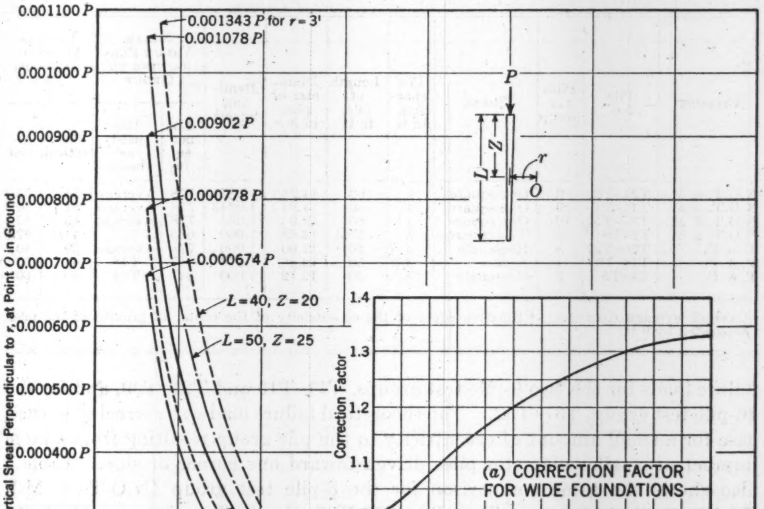


FIG. 9.—VERTICAL SHEARS AROUND FRICTION PILES, AT MIDHEIGHT (R_c = 1.5)

To facilitate the computation, curves have been drawn in Fig. 9, plotting the shear formula for R_c = 1.5 for various actual pile lengths from 40 ft to 80 ft, giving the shear at midheight of a pile at any radius from the pile. From these curves it is possible to read directly the fractional part of P contributed as shear by any pile in the group to the shear on the critical pile at the corner, on a plane normal to the line connecting the two piles.

In order to demonstrate the accuracy of the use of these derived formulas, Table 5 has been compiled to show the comparison of theoretical and actual failure loads for the two 9-pile test groups, T11–T19 and T21–T29, and for the 16-pile test group, T37–T52. The theoretical failure load was corrected in each case for a small amount of eccentricity in the pile group resulting from a large number of smaller diameter piles driven toward one corner or side. Table 5 also shows the same comparison for the 8-pile test group (N.O.T. & M.), T2–T9; 8-pile test group (T. & P.), T20–T27; the 4-pile test group (T. & P.) T14–T17; and 2-pile test group (T. & P.), T4 and T5.

TABLE 5.—COMPARISON OF THEORETICAL AND ACTUAL FAILURE LOADS PER PILE, MORGANZA TIMBER TEST PILE GROUPS

Crossing	Pile No.	Piles per group	Shape	Pile spacing, in ft	Length of pile, in ft	Diameter of pile, in in.[a]	Bending effect[b]	Shear Value Pertaining to Group		Failure Loads, in Tons per Pile	
								Lb per sq ft	Pile used as basis	Theoretical	Actual
N.O.T. & M.	T21–T29	9	One square	4	60	14.265	1.022	778	Average	54	55.3
N.O.T. & M.	T11–T19	9	One square	4	60	14.31	1.0243	778	Average	54	56.3
N.O.T. & M.	T37–T52	16	One square	4	60	12.93	1.065	778	Average	43	42.9
N.O.T. & M.	T2–T9	8	Two squares[c]	3	50.5	13.55	1.000	685	T1	41.5	42.5
T. & P.	T20–T27	8	Rectangle	3	50	12.00	1.000	778	Average	39	40
T. & P.	T14–T17	4	Square	3	50	12.56	1.076	847	T18	47	50
T. & P.	T4–T5	2	Rectangle	3	50	12.72	1.000	847	T18	60	60

[a] Over-all average diameter. [b] Relative effect of the eccentricity of the center of gravity of the piles.
[c] Separated by 9 ft.

Unfortunately individual piles were not driven and tested to failure adjacent to all of the groups because time and available funds were both limited, and, in computing the carrying capacity of those groups where no single pile test was made nearby, the average shear value of 778 lb per sq ft of all single piles tested to failure was used. On the other hand, where single piles had been driven and tested to failure near some of the smaller groups, the actual shear values of these single piles were used in computing the theoretical failure load. It will be noted that theoretical failure loads computed on this basis checked very closely with the actual failure loads.

MODIFICATIONS OF METHOD FOR LARGE GROUPS

Since each pile distributes its load outward and downward, it is evident that, as the group size increases, the point of critical shear on the outer piles will tend to move downward. From computations of various sizes and arrangements of groups, it has been found that, where the side of a square group is more than one third of the length of the piles, the critical shear on the corner piles occurs lower than the midheight of the pile. Thus, the failure load determined from curves plotted for midheight computations (Fig. 9(*b*)) should be divided by the factors in Fig. 9(*a*).

COMPUTATION OF SAFE LOADS

This method primarily establishes the failure load per pile of a group in shear. In the actual design of structures supported on friction piles, the working load chosen per pile should provide ample safety factors (1) against possible shear failure, and (2) against excessive settlements from vertical pressures, giving due regard to the relative proportions of dead, live, and dynamic loads. The dead load is the constant load that may produce excessive settlement over long periods of time without necessarily resulting in a shear or total failure. It is probable that a working load, or dead plus live load, may be chosen some-

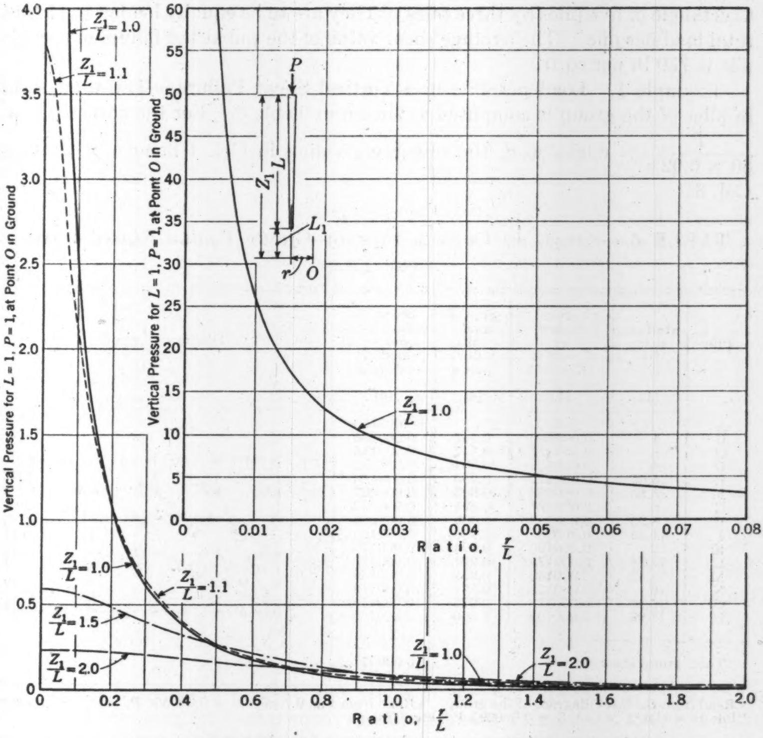


FIG. 10.—UNIT VERTICAL PRESSURES BENEATH A SINGLE FRICTION PILE ($L = 1$; $P = 1$; $R_s = 1.5$)

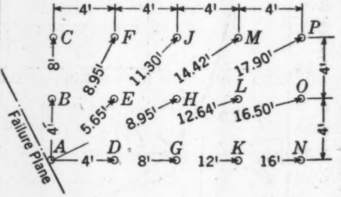
what closer to the shear-failure load in large groups than in small groups, because of the relatively sluggish failure of larger groups as illustrated by the behavior of the 16-pile group under test.

Illustrative Problem.—The following example illustrates the manner in which this pile theory is applied to an actual problem of computing the failure loads and pressures of a 15-pile group. The calculations of vertical pressures were facilitated by the use of the curves of Fig. 10, in which the vertical pressure formulas have been plotted for a pile length $L = 1$ and load $P = 1$.

In this problem, fifteen 60-ft piles, with a specified minimum tip diameter of 8 in. and a minimum butt diameter of 14 in., are laid out at 4-ft centers in a rectangle of five piles by three piles. They are to be equally loaded to 15 tons total load per pile. The average shear value of the soil at the failure of a single pile is 750 lb per sq ft.

Example 1. Load per Pile in a Vertical Shear Failure.—The total shear in piles of the group is computed as shown in Table 6. For the corner pile A, $\frac{P}{60 \times 0.92\,\pi} = 0.005780\,P$, the remaining values in Col. 4 being Col. 2 times Col. 3.

TABLE 6.—SHEAR ON CORNER PILE OF GROUP UNDER EQUAL LOADS PER PILE

Pile	Radius, r, in ft[a] (1)	Vertical shears[b] at $Z = 30$ ft ($R_e = 1.5$) (2)	Sine of angle with failure plane (3)	Shear on failure plane under equal loads P per pile[c] (4)	Diagram of group
A	....	....	....	0.005780	 4' 4' 4' 4' C F J M P 8' 8.95' 11.30' 14.42' 17.90' 4' L O B E H 5.65' 8.95' 12.64' 16.50' 4' 4' Failure Plane A D G K N 4' 8' 12' 16'
B	4	0.000660	0.447	0.000295	
C	8	0.000327	0.447	0.000146	
D	4	0.000660	0.894	0.000590	
E	5.65	0.000472	0.950	0.000449	
F	8.95	0.000283	0.802	0.000227	
G	8	0.000327	0.894	0.000292	
H	8.95	0.000283	1.000	0.000283	
J	11.30	0.000205	0.950	0.000195	
K	12	0.000190	0.894	0.000170	
L	12.64	0.000175	0.991	0.000173	
M	14.42	0.000140	0.993	0.000139	
N	16	0.000125	0.894	0.000112	
O	16.50	0.000118	0.977	0.000115	
P	17.90	0.000105	1.000	0.000105	
Total group shear =				0.009071 P	

[a] Read from the inset diagram of the group. [b] Read from Fig. 9, "pile B = 0.000660 P," etc. [c] Compute: "Pile B = Col. 2 × Col. 3 = 0.000295 P," etc. (Col. 4).

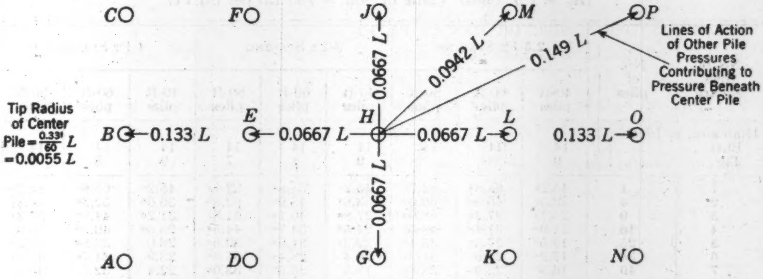
From the total group shear in Table 6, the failure load P per pile equals

$$\frac{750 \text{ lb per sq ft (failure shear)}}{0.009071} = 82{,}700 \text{ lb} = 41.4 \text{ tons}.$$

(In the inset diagram of Table 6 the failure plane is drawn normal to the line connecting the corner pile with the center of the group—a 45° plane gives almost the same results.)

Example 2. Vertical Pressures Beneath the Group Under 15 Tons per Pile.—In Table 7, the inset diagram shows the group dimensions reduced to proportional parts of L ($= 60$ ft), with radial distances from the center pile to each other pile. The table shows the method and results of the computation of vertical pressures.

TABLE 7.—VERTICAL PRESSURES BENEATH THE GROUP, UNDER 15 TONS PER PILE


CO
FO
J
M
P
Lines of Action of Other Pile Pressures Contributing to Pressure Beneath Center Pile
0.0667 L
0.0942 L
0.149 L
Tip Radius of Center Pile = $\frac{0.33'}{60}L$ = 0.0055 L
B
0.133 L
E
0.0667 L
H
0.0667 L
L
0.133 L
O
0.0667 L
AO
DO
G
KO
NO

Depth Below Tip of Pile H		Values Read from Fig. 10					Total Pressure	
In ft	Ratio Z_1/L	Pile H (Tip $r = 0.0055\ L$)	Four piles E, G, J, L ($r = 0.0667\ L$)	Four piles D, F, K, M ($r = 0.0942\ L$)	Four piles A, C, P, N ($r = 0.149\ L$)	Two piles B, O ($r = 0.133\ L$)	In units	In tons per sq ft[a]
0	1	47.10	15.50	10.72	6.16	3.56	83.04	0.347
6	1.1	3.76	10.40	8.88	6.00	3.36	32.40	0.135
30	1.5	0.59	2.32	2.27	2.12	1.08	8.38	0.035
60	2	0.23	0.92	0.92	0.88	0.46	3.41	0.014

[a] The actual pressures in last column are equal to units in preceding column multiplied by $15/60^2$.

Two important conclusions are to be noted in the solutions of Examples 1 and 2: First, that the failure load per pile, of the 15-pile group, is slightly greater than that of the approximately equivalent 16-pile square group of Fig. 8(a); and second, that the vertical pressures under the center pile of the 15-pile group are slightly less than those beneath the interior piles of the 16-pile square group of Fig. 8(b) if it is loaded to 15 tons per pile.

These conclusions suggest that tables of shear-failure load per pile, and of pressures beneath groups for square groups of many dimensions, may be a useful guide to design. Tables 8 and 9 have been prepared with that purpose in mind. Table 8 has been prepared for a shear-failure value of 750 lb per sq ft from which results may be proportioned for other values; and Table 9 has been prepared for 10 tons per pile, from which results may be proportioned for other loads. The tables should be used subject to the following rules:

(1) A rectangular or circular group will fail in shear at a larger load per pile than a square group containing the same number of piles.

(2) A rectangular group will have smaller vertical pressures and a circular group will have larger vertical pressures than a square group containing the same number of piles.

TABLE 8.—SHEAR FAILURE LOADS PER PILE, IN TONS, FOR SQUARE GROUPS OF PILES OF VARIOUS LENGTHS AND SPACINGS

(R_c = 1.5; Shear Value of Soil = 750 Lb per Sq Ft)

No. of rows square	No. of piles	2.5-Ft Spacing			3-Ft Spacing			4-Ft Spacing		
		40-ft piles	60-ft piles	80-ft piles	40-ft piles	60-ft piles	80-ft piles	40-ft piles	60-ft piles	80-ft piles
Diameter, in Inches:										
Butt		14	14	14	14	14	14	14	14	14
Tip		9	8	7	9	8	7	9	8	7
1	1	45.2[a]	64.8[a]	82.7[a]	45.2[a]	64.8[a]	82.7[a]	45.2[a]	64.8[a]	82.7[a]
2	4	32.0[a]	46.5[a]	60.0[a]	33.6[a]	48.9[a]	62.8[a]	36.0[a]	52.3[a]	66.8[a]
3	9	25.7[a]	37.2[a]	48.0[a]	27.8[a]	40.1[a]	51.8[a]	31.2[a]	44.8[a]	57.3[a]
4	16	21.9[a]	31.4[a]	40.6[a]	24.5[a]	34.7[a]	44.5[a]	28.5[a]	40.0[a]	50.9[a]
5	25	19.5[a]	27.5[a]	35.4[a]	22.3[a]	31.0[a]	39.6[a]	26.0	36.8[a]	46.2[a]
6	36	17.9[a]	24.9[a]	31.7[a]	20.4	28.4[a]	35.8[a]	23.9	34.7[a]	43.0[a]
7	49	16.5	22.9[a]	28.8[a]	18.6	26.5[a]	33.0[a]	22.4	32.1	40.4[a]
8	64	15.3	21.4[a]	26.7[a]	17.5	24.9	30.9[a]	21.3	30.1	38.1
9	81	14.4	20.2[a]	24.9[a]	16.4	23.1	29.1[a]	20.2	28.4	35.8
10	100	13.5	18.9	23.5[a]	15.9	21.7	27.4	19.6	27.2	33.9
11	121	12.9	17.8	22.3[a]	15.1	20.7	26.1	19.0	25.9	32.4
12	144	12.3	17.0	21.1	14.5	19.9	24.7	18.5	25.0	31.1
13	169	11.9	16.2	20.1	14.2	19.1	23.6	18.2	24.3	30.0
14	196	11.7	15.5	19.1	13.8	18.4	22.5	17.8	23.6	28.9
15	225	11.5	15.1	18.1	13.5	17.9	21.7	17.7	22.9	28.3
16	256	11.2	14.6	17.5	13.3	17.4	21.2	17.5	22.5	27.5
17	289	10.9	14.2	17.0	13.1	16.9	20.3	17.4	22.2	27.0
18	324	10.7	13.7	16.3	13.0	16.6	19.8	17.3	21.9	26.3
19	361	10.5	13.5	15.8	12.9	16.2	19.4	17.2	21.5	25.9
20	400	10.4	13.2	15.3	12.7	15.9	18.7	17.2	21.3	25.5
21	441	10.3	13.0	14.9	12.7	15.6	18.4	17.1	21.2	25.1
22	484	10.3	12.6	14.6	12.6	15.5	18.0	17.0	21.0	24.8

[a] Governed by shear summation at midheight of corner pile; all others governed by shear summation at some point below midheight of corner pile.

PRACTICAL APPLICATION OF THIS METHOD TO DESIGN

On the basis of the preceding data, the following would be the design procedure:

(1) Make sufficient borings at the site to establish the approximate uniformity of the soils over the entire area being considered, and to establish further the fact that a firm stratum for the support of point bearing piles is not economically accessible.

(2) Drive a test pile of about the length contemplated to be used in the design, and if the number of hammer blows per foot increases only gradually with increased penetration, the pile will be a friction pile, distributing load approximately in the manner suggested by this pile theory.[a] More than one test pile may be used to sample the area thoroughly.

(3) Load the test pile, by increments, to failure, which will be indicated by a sharp increase in settlement that does not cease with time, and which should result in at least several inches of settlement. From this failure load, the shear-failure value, in pounds per square foot, may be computed by dividing by the penetrated contact area.

(4) With this shear value in mind determine, by an examination of Table 8, approximately what the failure load per pile will be for the approximate number of piles required, and select a working load per pile that provides an ample safety factor consistent with the group size and the loading characteristics. For an unusual group, the determination of failure load may be made more precisely by the method given in Example 1.

TABLE 9.—VERTICAL PRESSURES, IN TONS PER SQUARE FOOT, BENEATH CENTER PILE OF VARIOUS SQUARE PILE GROUPS LOADED TO 10 TONS PER PILE

(R_c = 1.5; 40-Ft Piles, 14-In. Butt, 9-In. Tip; 60-Ft Piles, 14-In. Butt, 8-In. Tip)

Depth, in ft, below tip	One pile	3 rows = 9 piles	4 rows = 16 piles	5 rows = 25 piles	7 rows = 49 piles	9 rows = 81 piles	11 rows = 121 piles	13 rows = 169 piles	15 rows = 225 piles	17 rows = 289 piles
					(a) 40-FT PILES, 3-FT SPACING					
0	0.172	0.316	0.376	0.448	0.562	0.663	0.752	0.826	0.890	0.940
4	0.023	0.139	0.197	0.266	0.386	0.490	0.585	0.660	0.728	0.789
20	0.004	0.032	0.054	0.082	0.146	0.218	0.295	0.374	0.448	0.520
40	0.001	0.013	0.023	0.035	0.066	0.106	0.154	0.202	0.251	0.298
					(b) 40-FT PILES, 4-FT SPACING					
0	0.172	0.278	0.318	0.367	0.443	0.505	0.555	0.593	0.625	0.649
4	0.023	0.117	0.157	0.208	0.287	0.352	0.407	0.453	0.490	0.523
20	0.004	0.031	0.051	0.076	0.131	0.187	0.245	0.299	0.348	0.392
40	0.001	0.013	0.022	0.034	0.064	0.099	0.137	0.173	0.211	0.251
					(c) 60-FT PILES, 3-FT SPACING					
0	0.131	0.228	0.271	0.324	0.413	0.492	0.568	0.635	0.698	0.755
6	0.010	0.074	0.113	0.160	0.245	0.326	0.405	0.478	0.543	0.600
30	0.002	0.014	0.025	0.039	0.073	0.112	0.157	0.207	0.259	0.310
60	0.000	0.006	0.010	0.016	0.030	0.049	0.073	0.099	0.129	0.161
					(d) 60-FT PILES, 4-FT SPACING					
0	0.131	0.204	0.234	0.271	0.331	0.384	0.433	0.475	0.510	0.541
6	0.010	0.064	0.093	0.129	0.190	0.245	0.295	0.340	0.379	0.413
30	0.002	0.014	0.025	0.039	0.070	0.104	0.142	0.182	0.221	0.258
60	0.000	0.006	0.010	0.016	0.030	0.048	0.070	0.094	0.119	0.144

(5) Find the approximate vertical pressures beneath the center of the group, under the working load chosen per pile, by examination of Table 9 (some interpolation may be necessary). If the pressures seem dangerous, or if the group is unusual, a more precise calculation may be made by the method given in Example 2.

(6) If consolidometer tests have been made on samples of the soil beneath and around the piles, it will be possible to compute, from the vertical pressures determined by use of the formulas and curves, a theoretical maximum settlement of the pile tops along the vertical center line of the footing under the working load. This theoretical settlement cannot be realized fully, especially in piles supporting a stiff pier base where the stiffness of the footing will restrict

settlements because of the lower pressures existing under the edges of the footing. If the area under a large building is thought of as one large footing in sections, however, the indicated center settlement may be realized.

MODIFICATION OF PRESSURE DISTRIBUTION DUE TO VARIATION IN MATERIALS

The materials encountered on the Morganza Floodway are believed to distribute loads in a manner representative of most true friction pile foundations. If the materials were to grow progressively denser with increased penetration, the distribution of pile load to the soil would become greater toward the tip, of course, and the effect of this could be estimated by reducing the value of R_c in all the formulas from the value of 1.5 used in this paper. Ratio $R_c = 1$ would represent a pile with uniform load per foot, and $R_c = 0$ would represent a distribution of load increasing from zero at the butt to a maximum value at the tip.

As the materials would become denser and denser toward the tip, the reduction in pile strength due to grouping would become progressively less pronounced until, with pile tips bearing on rock, the permissible load per pile in a group would be equal to the permissible load on a single pile.

TABLE 10.—MINDLIN-RUDERMAN STRESS COEFFICIENTS FOR PILE FOUNDATIONS

(Poisson's Ratio Assumed 0.5; All Values Negative (or Compression) Except As Noted by + in Table 10)

Ratio Z/L	Ratio r/L									
	0	0.1	0.2	0.3	0.5	0.7	1	1.5	2	3
(a) FRICTION-LOAD STRESS DISTRIBUTION $p_Z = P_{\text{Friction}} \times K_f/L^2$										
0.25	0.698	0.568	0.355	0.209	0.079	0.033	0.009	0.001	0.000	0.000
0.50	0.554	0.522	0.442	0.345	0.184	0.091	0.031	0.005	0.001	0.000
1.00	....	1.550	0.750	0.478	0.249	0.145	0.067	0.024	0.006	0.001
1.25	0.902	0.790	0.593	0.432	0.246	0.152	0.078	0.027	0.010	0.001
1.5	0.419	0.404	0.364	0.312	0.216	0.147	0.083	0.033	0.013	0.003
2.00	0.180	0.178	0.172	0.163	0.139	0.112	0.077	0.039	0.019	0.005
3.00	0.068	0.068	0.067	0.066	0.062	0.057	0.048	0.034	0.022	0.009
(b) POINT-LOAD STRESS DISTRIBUTION $p_Z = P_{\text{Point}} \times K_p/L^2$										
0.25	+0.174	+0.166	+0.142	+0.090	+0.021	0.009	0.012	0.004	0.001	0.000
0.50	+0.755	+0.678	+0.476	+0.268	+0.034	0.032	0.024	0.062	0.029	0.000
1.00	......	0.120	0.119	0.110	0.096	0.078	0.051	0.021	0.008	0.001
1.25	3.89	2.59	1.221	0.494	0.145	0.082	0.051	0.023	0.011	0.001
1.5	1.04	0.948	0.736	0.513	0.237	0.118	0.061	0.026	0.012	0.002
2.00	0.289	0.281	0.263	0.239	0.183	0.130	0.076	0.035	0.012	0.004
3.00	0.086	0.086	0.084	0.082	0.076	0.068	0.055	0.036	0.026	0.008

MODIFICATION OF RESULTS BECAUSE OF OTHER THAN BOUSSINESQ DISTRIBUTION

The Boussinesq distribution of stress is always downward from a point of load, and has been used with success in determining pressures beneath many buried foundations. Professor Mindlin's formulas[2] indicate a stress distribution upward by tension as well as downward by compression from a point load buried in the ground. D. M. Burmister, Assoc. M. Am. Soc. C. E., has presented the Mindlin formulas in usable form for pile foundations,[3] as reproduced in Table 10.

From Table 10(*a*), showing stress distribution around a pure friction pile with uniform load per linear foot, it has been possible to compute a curve of vertical shears around a pile at midheight. In Fig. 11 this curve is shown in comparison with the corresponding curve from the general shear formula (Eq. 2*c*) derived in this paper, using $R_c = 1.5$, and with a curve from the same formula using $R_c = 1$ for the condition of uniform load per foot of pile.

The comparative tendency of the Mindlin distribution is to reduce pressures beneath the pile and increase the shears around the pile, thus throwing more load into distant piles of a group and making group action more dangerous than the Boussinesq distribution would indicate.

Since the curve for $R_c = 1.5$ under Boussinesq distribution checks the pile tests more closely than does the curve for Mindlin distribution, it would seem that timber friction piles distribute load more nearly in accordance with the Boussinesq formulas.

IMPLICATIONS OF THIS THEORY OF FRICTION PILE BEHAVIOR

If this theory is correct in stating that shear failure occurs through the building up of critical shear stresses on the corner piles of square or rectangular groups, it would follow logically that a more efficient grouping arrangement against shear failure could be obtained by eliminating corner piles through the use of octagonal footings containing the same number of piles.

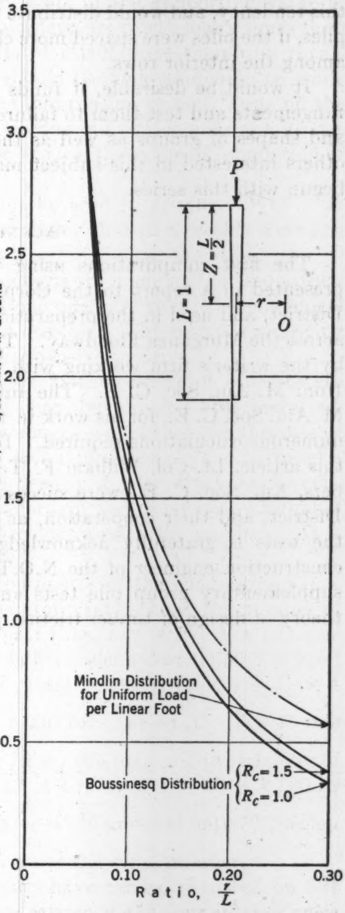


FIG. 11.—COMPARISON OF UNIT VERTICAL SHEARS AROUND A SINGLE FRICTION PILE BY BOUSSINESQ AND MINDLIN DISTRIBUTIONS (AT MIDHEIGHT OF PILE)

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[3] "Proceedings of the Purdue Conference on Soil Mechanics and Its Applications," Purdue Univ., Lafayette, Ind., September 2 to 6, 1940, p. 339 (citing also "Stress Distribution Around a Loaded Pile," by J. Ruderman, Thesis No. 500 submitted to the Dept. of Civ. Eng., Columbia Univ., New York, N. Y., in 1939, in partial fulfilment of the requirements for the degree of Master of Science).

Where a large group of friction piles supports a stiff pier base, some redistribution of pile loads toward the edges from those calculated will take place in order that the settlement of the edge piles due to vertical pressures may become equal to the settlement of the center piles. It would be in accord with this tendency, and would distribute the pier load more equally to the individual piles, if the piles were spaced more closely in the exterior rows of the group than among the interior rows.

It would be desirable, if funds were available, to make various group arrangements and test them to failure so as to have additional test data on sizes and shapes of groups as well as the most efficient spacing. It is hoped that others interested in this subject may be able to continue the tests that were begun with this series.

ACKNOWLEDGMENTS

The first computations using the formulas derived in this paper were presented in a report to the Corps of Engineers, U. S. Army, New Orleans District, and used in the preparation of designs for proposed railway crossings across the Morganza Floodway. The report was presented in December, 1940, by the writer's firm working with that of J. F. Coleman, Past-President and Hon. M. Am. Soc. C. E. The author is indebted to Harry J. Engel, Assoc. M. Am. Soc. C. E., for his work in the derivation of the equations used and the numerous calculations required. During the progress of the pile tests used in this article, Lt.-Col. William F. Tompkins and Lt.-Col. Clark Kittrell, Members, Am. Soc. C. E., were successive district engineers for the New Orleans District, and their cooperation, as well as that of their assistants, in making the tests is gratefully acknowledged. Roscoe Owen, M. Am. Soc. C. E., construction engineer of the N.O.T. & M. Railway Company, supervised the supplementary group pile tests and has made valuable comments upon the theory of design of timber friction pile foundations presented in this paper.

DISCUSSION

E. H. CONNOR,[4] M. AM. SOC. C. E.—The tests of the supporting value of piles in various groups, as reported by Mr. Masters, are worth many times their cost, and the formulas deduced and results of calculations therefrom reveal many interesting facts. Many more tests should be made and could be made with a saving of millions of dollars from the information gained: Tests on the spacing of piles in groups, on the result of leaving out center piles of a group, on the supporting values in sand in the river bed, on the value of foundation of piles driven on a batter, etc.

Some engineers specify that a jet cannot be used in driving. Why? It would save money in many cases. Do they know that in a week's time one pile driven in sand or clay would support a lesser load than the other.

It is evident that many pile-foundation designs are made on the assumption that 60 piles will carry 60 times as much as one pile, and that it is economical to crowd these piles into as small an area as possible. The result is a spacing so close that the piles are very difficult and expensive to drive.

Of course, the volume and the cost of the footing course of concrete are reduced. The strength of the foundation may be reduced still more. A proper spacing of piles to secure the same strength would increase the cost of concrete but reduce the cost of piles. The economical design then becomes a question of unit costs of concrete and piles, length of piles, thickness of concrete base or sealing course. The cost of cofferdam and excavation also must be considered; but the proper design now becomes a scientific problem. It is still not an exact science.

The writer often has thought that the perimeter of the entire group in a certain location determined the power of the group rather than the number of piles, if the piles were not spaced farther apart than, say, 4 ft. For instance (see Table 1(*c*)), a group of nine piles T11–19 failed under a load of $(56.3 \times 9 =)$ 507 tons. Assuming that the perimeter of group length is $(2 \times 4 + 1) \times 4 \times 59.7 = 2{,}149$ sq ft, the shear is $\frac{507}{2{,}149} = 0.236$ tons per sq ft. The 16-pile group T37 to 52 failed under load of $(42.9 \times 16 =)$ 686 tons. The perimeter of the group multiplied by the penetration is $(3 \times 4 + 1) \times 4 \times 60 = 3{,}120$ sq ft, and the shear is $\frac{686}{3{,}120} = 0.220$ tons per sq ft—a difference of only 7%, which amount could easily exist in two similar groups of the same number.

How much would the group of sixteen piles have carried if spaced on 3-ft centers? How much would this group have carried if the four interior piles had been omitted?

Another feature has come to the attention of many. By standing unloaded a few days after driving, a pile will carry an increased load. Four piles were driven in San Francisco Bay to support a derrick. They were 90 ft long and penetrated about 25 ft of water and 55 ft of clay. They drove easily and did not show greatly increased resistance to driving as the penetration increased.

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[4] Vice-Pres. and Cons. Engr., Mo. Val. Bridge & Iron Co., Leavenworth, Kans.

It was suggested that the length of the piles be increased to 120 ft. After waiting two days, a single-acting steam hammer (5,000-lb ram, 36-in. stroke) could not move the 90-ft piles. These lengths were used for a year without settlement.

In 1900, the writer's firm built a creosoted pile trestle across Biloxi Bay. The piles drove easily in the mud—too easily. They would not support much more than a heavy hammer. The contractor was alarmed. The specifications required that the causeway carry a load of 100 lb per sq ft. It was completed and several panels loaded to 100 lb without any settlement and the causeway served the public nearly thirty years.

In 1912 the Kansas City Terminal Railway Company made tests of concrete piles driven in dry loam, testing them immediately after driving. Some were tapered; some had parallel sides. The tapered piles carry considerably more than those with parallel sides but they are difficult to keep plumb and they yield more readily to side movement from obstructions struck while driving.

From the contractor's point of view, one other point in spacing piles should be given more consideration; that is, the distance from the outside edge of the concrete base to the center of the outside row of piles. This is usually shown on the plans as 18 in. In a deep excavation or cofferdam, wales at least 12 in. wide are needed. A 5,000-lb, single-acting steam hammer (36-in. stroke) has an over-all clearance of 40 in. requiring 36 in., with 4-in. clearance, from the center of the outside pile to the inside of sheet piles unless a follower is used.

Glenn B. Woodruff,[5] M. Am. Soc. C. E.—Many engineers have been aware of the general principles demonstrated by this paper. The thanks of the profession are due the author for presenting the results of these tests in a manner that gives a mathematical statement of these principles.

Although Table 5 shows a remarkable agreement between theoretical and actual results, the author probably will agree that delta deposits are generally so heterogeneous that extreme precision in predicting results is unwarranted. It appears permissible therefore to simplify several of the equations by eliminating terms that will have minor influence on the results.

The load tests were made shortly after the piles were driven. Possibly, over a long period of years, the distribution of the loads from the piles to the surrounding soil will change materially. In most silty soils, the process of soil consolidation under its own weight is still continuing. If the strata near the surface are settling, these strata, rather than taking load from the piles, will be supported partly by the piles and will add to the pile load. It will be difficult to separate the results of this action from the settlements resulting from the consolidation of the strata below the pile tips, but it is to be hoped that the owners of the structures to be built will arrange for such a record of settlements under piers of varying size that certain conclusions can be drawn.

Tables 8 and 9 show clearly the advantages of wide pile spacing. This wide spacing has the disadvantage that larger, and consequently heavier, footings are required to distribute the superstructure loads to the piles. In some cases, it becomes advisable to use battered piles to combine the advantages of a small pile cap and a wide distribution of the load. In this way, each of the piles in a comparatively large cluster may be made to approach the load-carrying capacity of a single pile.

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[5] Chf. Engr., Utah-Pomeroy-Morrison, Provo, Utah.

JACOB FELD,[6] M. AM. SOC. C. E.—A complete solution of the subject of friction piles has been presented in this paper, and the author is to be complimented for his derivation of empirical tables for practical use in design.

The average frictional resistance along the surface of a single pile at failure for timber piles (Table 4) is given as 778 lb per sq ft, as against 558 for parallel-sided concrete piles. The discrepancy is merely the result of an assumption. Shear failure in the case of either wood or concrete piles is not along the pile surface but along a soil surface, formed by the adhesion of some soil to the pile. In the case of tapered wood piles, the proper surface area seems to be governed by the maximum pile diameter, not by the average. Certainly at failure, the soil directly below the tapered surface is compressed sufficiently to act as part of the pile. Table 11 shows the data of Table 4(*a*) with such adjustment; the average soil friction for the wood piles is 600 lb per sq ft of cylinder surface, with the butt diameter.

TABLE 11.—SKIN FRICTIONS, INDIVIDUAL, TAPERED, TIMBER PILES (SEE TABLE 4(*a*))

Pile No.	Length (ft)	Load at failure (tons)	DIAMETERS (IN.)		SKIN FRICTION, IN LB PER SQ FT	
			Tip	Butt	Average diameter	Maximum diameter
T1	52	66.5	8.5	17.5	685	560
T10	59.5	96.5	11.5	17.75	900	700
T20	85	100	8.0	17.5	680	500
T34	83	110	8.5	16.75	777	600
T18	50.5	70	8.5	16.5	847	640
Average					778	600

Since the minimum value occurs with the minimum tip and the maximum value occurs with the maximum tip, there is certainly some variation with tip size. The number of test values is not sufficient to draw more definite conclusions.

A group of friction piles in itself is a means of transferring loads to lower soil layers, very much in the manner of a spread footing. The similarity of the two rules in Example 2 with the conclusions drawn from tests with spread footings, therefore, is not surprising. Only by exposure of loaded surface to soil not carrying its full capacity can loads be carried safely.

The writer has used a very rough rule for the ratio of load-carrying capacity of a group of piles to that of a single pile. Since it seems to have some relation to the result presented in this paper, the rule is given herein. The value of a pile in a group is the value of a single pile less one sixteenth for each adjacent diagonal or row pile (see Table 12). In view of the unknown relative effects of tip area and length upon the unit bearing value of a pile, the suggested reduction in pile values of a group is an easy approximation. A pile fully enclosed in a group, as the center pile in a square array of nine piles, has a value of 50% of a single pile. A minimum pile spacing distance is assumed.

This method, applied to the author's examples in Table 6 and Fig. 8, gives values of 10.25 and 10.75 equivalent single pile units for the 15-pile and 16-pile groups. This would indicate that in the 15-pile group an individual pile would fail at about 60 tons, if the average failure load equals 41.4 tons. This value is consistent with the individual failure tests. Fair agreement is also found with the data in Table 5.

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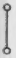
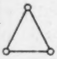
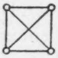
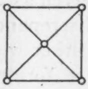
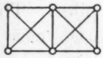
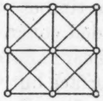
[6] Cons. Engr., New York, N. Y.

TABLE 12.—EQUIVALENT PILE VALUES OF CLUSTERS

(Total Value of Cluster Given in Parentheses, in Terms of a Single Pile Value)

					
2 Piles @ $\frac{15}{16}$ (1.87)	3 Piles @ $\frac{15}{16}$ (2.81)	4 Piles @ $\frac{13}{16}$ (3.25)	5 Piles 4 @ $\frac{13}{16}$ 1 @ $\frac{12}{16}$ (4.00)	6 Piles 4 @ $\frac{13}{16}$ 2 @ $\frac{11}{16}$ (4.62)	9 Piles[a] 4 @ $\frac{13}{16}$ 4 @ $\frac{11}{16}$ 1 @ $\frac{8}{16}$ (6.50)

NOTE: – Similarly for Other Clusters.

[a] Omitting Center Pile of the 9-Pile Cluster Does Not Affect Bearing Value of the Cluster.

G. G. GREULICH,[7] ASSOC. M. AM. SOC. C. E.—Careful and adequate driving and load tests made preliminary to the final preparation of a design are described in this paper. It is apparent that, after reviewing all available recorded data, previously expounded theories and formulas were expanded and modified in an intelligent manner so as to be theoretically compatible with the test results. The closeness of the approximations is shown by Table 5. However, the close agreement shown in the table may be due, in a large measure, to the exceptionally uniform soil conditions prevailing in this portion of the Lower Mississippi Valley. In areas where more heterogeneous soils may be encountered, it is doubted whether such close agreement would be found.

The theoretical failure loads given in Table 5 are remarkably close to the actual failure loads and demonstrate that, on occasion, the law of averages will take care of widely varying factors. Reference to Table 1 shows that there is very little relationship between the hammer blows for the last foot of penetration and the actual skin friction values developed at failure. For example, in the T. & P. tests, pile T18, requiring 25 blows for the last foot of 50.5 ft of penetration, shows a skin friction value of 847 lb per sq ft, as compared with 610 lb per sq ft for T14–T17, where the number of blows for the last foot of approximately the same length of penetration (50 ft) was 27, or two more than for the much higher load developed by the first pile. Reference to the N. O. T. & M. tests, pile T34, requiring 8 blows for the last foot of 83 ft of penetration, shows a skin friction value of 777 lb per sq ft, as compared with 680 lb per sq ft for T20, where the number of blows for the last foot of 85 ft of penetration was 30.

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[7] Sales Representative, Specialty Sales Div., Carnegie-Illinois Steel Corp., Pittsburgh, Pa.

In spite of the wide variation of skin friction values for individual piles, the average value from Table 4 apparently gave exceptionally accurate results when used by Mr. Masters in predicting theoretical group behavior. This is especially notable in view of the normal and often wide variation found in the shape of timber piles and their behavior in driving when driven in isolated groups. The writer wonders if this method will prove as accurate when applied to groups at other locations. Only additional similar test data will develop this point. In the meantime, it has been demonstrated again that, having access to carefully recorded test data, it is possible to develop rules or formulas that do give reasonable results for any areas of ground which are comprehensively covered by test data.

In the writer's opinion, Mr. Masters has done a great service to the profession by the incorporation of Table 8. Perusal of this table should give any practicing engineer a vivid picture of the rapid reduction in load capacities of piles when used closely spaced in large groups. This table throws light on a remark made some years back by an elderly engineer whose name the writer cannot recall. In his practice, he had built a number of swing bridges with the main center pier founded on piles. In spite of the fact that the timber piles he ordinarily used tested satisfactorily in clusters of four for safe loads of 16 to 18 tons per pile, he found that, in order to keep settlements within workable limits for his movable structures, he could not load the piles to more than 7.5 to 8 tons.

Referring to Table 8, it will be noted that, for 2.5-ft spacing of 40-ft piles, the failure load for clusters of four is computed as 32 tons per pile. On such a pile, a 16-ton to 18-ton load might ordinarily be considered a safe working load. Going on down the tabulation to 16.5-ton load per pile, which would represent a safe load of 7.5 to 8 tons, it is noted that a group of 49 piles would have this theoretical capacity per pile:

> Multiplying 49 piles by 6.25 sq ft of contributory area, on 2.5-ft spacing, the resulting gross area is approximately 306 sq ft.

This area could well have been the area of the main piers of some of the early, light, swing bridges of this type which were approximately 20 ft in diameter. As stated before, it is believed that this paper gives a vivid explanation of the elderly engineer's remarks, and emphasizes the necessity for drastic reduction in assumed design loads for individual friction piles when closely spaced in large clusters.

G. S. PAXSON,[8] ASSOC. M. AM. SOC. C. E.—There is probably no part of structural design less understood than foundations, especially foundations supported by friction piles. For many years, it has been known that the value of a group of piles was not the sum of the values of the individual piles taken separately. The method of design proposed by Mr. Masters provides a quantitative solution of the problem. The remarkable correspondence between the theoretical and actual loads at failure in the Morganza Floodway tests gives great support to the proposed method as a practical design tool.

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[8] Bridge Engr., State Highway Comm., Salem, Ore.

The writer was particularly interested in the approach that the paper provides to the question of piles versus spread footings on cohesive soils. There are many contradictory statements in current engineering literature on this subject, but, so far as the writer knows, no quantitative comparison has been given.

In Fig. 12, showing the approximate vertical pressure, q_v, contours are given for a foundation supported on piles and for the same foundation resting directly on the soil. The pressure contours on the left-hand side of the figure show the pressures in pounds per square foot for the foundation resting directly on the soil, whereas those on the right-hand side show the pressures under a foundation supported by piles. The foundation chosen for the example was a square, 12 ft on a side. For the pile foundation, 16 piles were used. The piles were symmetrically placed, spaced at 3 ft on centers with 1.5 ft allowed between the pile centers of the outside rows and the edges of the foundation. A load of 10 tons per pile and a pile penetration of 40 ft below the bottom of the foundation were assumed. The piles were assumed to have a 14-in. butt and an 8-in. tip. In the formulas for pressure, a value of $R_c = 1.5$ was used. The spread foundation was the same size as the pile foundation and carried the same total load, 160 tons.

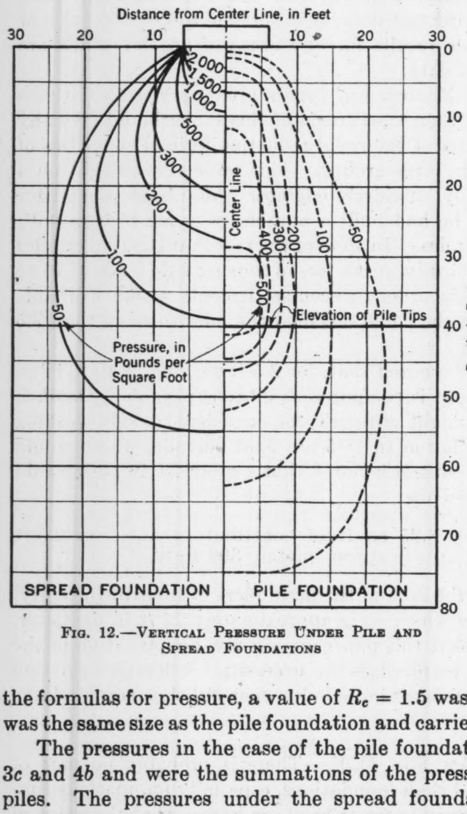

Fig. 12.—Vertical Pressure Under Pile and Spread Foundations

The pressures in the case of the pile foundation were computed from Eqs. 3*c* and 4*b* and were the summations of the pressures contributed by all of the piles. The pressures under the spread foundation were computed by the Boussinesq formula applied to subdivisions of the foundation area. The pressure contours are those in a vertical plane passing through the center of the foundation and perpendicular to the sides. It should be noted that this plane does not pass through any pile and does not show the local pressure concentrations close to the peripheries of the piles.

It is interesting to note the differences in pressure distribution. Under the assumed loading, the greatest pressure under the pile foundation does not exceed 600 lb per sq ft and is confined to a zone around the lower 10 ft of the pile lengths, whereas, under the spread foundation, the pressures reach the unit load on the slab (2,220 lb per sq ft) with the higher pressures near the ground surface. In the usual case of a fairly uniform soil, it is of advantage to transfer the loads to considerable depths below the surface as soil samples generally will show a decreasing voids ratio with increasing depth. There are other cases, of course, in which the surface stratum is better than the underlying material. Under such a condition, a pile foundation might show more total settlement than a spread foundation of the same size.

The consolidation of soil held between piles that are spaced reasonably close together is undoubtedly a complicated procedure and probably not susceptible of analysis. It is possible that the entire block settles as a unit and that the pressure at the pile tips is the pressure of the foundation slab, less the pressure transferred to the surrounding soil by the outside rows of piles. Since, in general, the soil consolidation is proportional to the logarithm of the applied pressure, it would seem that piles in the ordinary soil formation would decrease the total settlement of a foundation considerably.

RAYMOND D. MINDLIN,[9] ASSOC. M. AM. SOC. C. E.—An unfortunate error made by the author invalidates most of the conclusions drawn from the mathematical portion of his investigation. A solution of the equations of elasticity, known as the Boussinesq solution, is used in this paper, in an attempt to establish a rational estimate of the stress distribution in the soil around a friction pile. The applicability of the classical theory of elasticity to stresses in soils is debatable, but may be allowed as a first approximation on the basis that no better theory exists. However, once this barrier is passed and the investigation enters the realm of elasticity theory, it is not permissible to violate the mathematical and physical laws which immediately become effective.

It has been observed before[10] that, when the Boussinesq solution is integrated along a line parallel to the direction of the applied force, as the author has done, additional distributed loading is automatically introduced on the surface. The resulting stresses in the interior of the body are then those due not only to the line load penetrating into the body, but also to the distributed surface load.

The author implies that his formulas (Eq. 2*c*, for example) represent stresses at depth Z due to the distribution of delivered load P' (Eq. 2*a*) extending from the butt to the tip of a pile of length L. He should state, at this point, that the form of P' is either an assumption, or is based on experimental or theoretical results which he should supply. (Actually, the determination of P' is the most difficult part of the problem.) Furthermore, Eq. 2*c* gives, in fact, the distribution of s_v at (and only at), a depth Z due to (1) that part of P' (and only that part) between the surface and depth Z, plus (2) a surface distribution of s_v and q_v.

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[9] Asst. Prof., Civ. Eng., Columbia Univ., New York, N. Y.

[10] "On the Distribution of Stress Around a Pile," by R. D. Mindlin, discussion of papers, Section E—Stress Distribution in Soils, *Proceedings*, International Conference on Soil Mechanics and Foundation Eng., Cambridge, Mass., Vol. III, 1937.

The fact that the surface distribution exists in the author's solution is masked by the fact that, in Eq. 2c, the symbol Z represents both the coordinate of the plane on which s_v is to be calculated and the depth to which the line of load extends. If Z is allowed to approach zero in Eqs. 2c and 3c, s_v and q_v also approach zero; but this is not because no surface loading s_v (or q_v) is implied by the author's integration. Because of the dual meaning of Z, one cannot move the plane on which the stress is to be examined without changing the length of the line of load. Thus, if it is desired to use Eq. 2c or Eq. 3c to investigate the stress at the surface, one will, at the same time, automatically reduce the length of the line of load to zero (that is, the entire solid is unloaded), resulting in no stresses in the solid at all, and, of course, no stresses at the surface. The expression for s_v the author would have found, if he had prosecuted his incorrect theory rigorously, is (confining attention to the case $R_c = 1$, for simplicity):

$$s_v = \frac{P\,r}{2\,\pi\,L}\left\{\frac{Z^3}{r^2\,[r^2 + Z^2]^{1.5}} - \frac{(Z - Z')^3}{r^2\,[r^2 + (Z - Z')^2]^{1.5}}\right. \cdots\cdots\cdots (5)$$

in which Z is the distance below the surface of any point, and Z' is the distance, below the surface, to which the line of load extends. It is then seen that if no distinction is made between Z and Z', the author's result is obtained. In Eq. 5, however, it is seen that the stress does not vanish at the surface ($Z = 0$). Hence the stress distribution found by applying the Boussinesq solution is one due to a line distribution of load, plus a surface distribution which does not actually exist. The author's formulas represent only a part of the results to be expected from integration of the Boussinesq solution, and, since the Boussinesq solution does not apply to his case, even the part that he gives is incorrect.

Furthermore, in Eq. 5, Z' should be taken equal to L (the length of the pile); otherwise, if it is taken equal to Z, it is assumed incorrectly (and the author has implicitly done this) that the portion of the pile extending below the plane on which the stress is calculated does not affect the stress on that plane.

The author refers to a solution of the elasticity equations by the writer[2] and states that it does not check the test results quoted in the author's paper. This is not surprising on several counts, but chiefly because the solution and the tests do not deal with the same problem. The writer's solution is for the stresses due to a force applied at a point in the interior of a semi-infinite elastic solid, just as the Boussinesq solution is for a force at a point on the surface of the semi-infinite solid. Neither solution deals directly with stresses due to line distributions of forces, although both may be extended to such cases, by integration, with the important reservation that the Boussinesq solution is restricted to loads distributed on the surface of the solid.

The author refers to a thesis by J. Ruderman as mentioned in a paper by D. M. Burmister.[3] Mr. Ruderman's work contains the correct integration for a uniform line distribution of load in the interior of the semi-infinite elastic solid. This thesis gives the results the author should have obtained for the case $R_c = 1$. Table 10(*a*) of the paper contains only a small portion of Mr. Ruderman's results and it is then stated that "From Table 10(*a*) * * * it has been possible to compute a curve of vertical shears around a pile at midheight. In Fig. 11 this curve is shown * * *." On the contrary, it is not possible to perform such a computation from the data given in Table 10(*a*), with the result that the curve in Fig. 11 designated as "Mindlin Distribution for Uniform Load per Linear Foot" is incorrect.

The author uses the principle of superposition to find the distribution of stress for a group of piles. It is important to emphasize that the distribution, along a pile, of load transfer from pile to soil, is influenced by the presence of neighboring piles, so that superposition is not strictly permissible in this case.

The real problem, for either a single pile or a group of piles, is to find the distribution, along the pile, of the load transfer from pile to soil. The remainder is comparatively simple, as long as elasticity is assumed. A study such as Mr. Ruderman's, even though an assumption as to this distribution is made at the start, is useful because information can be obtained from it that may be helpful in a more comprehensive attack on the problem. Mr. Masters, however, has produced a set of formulas whose only connection with the problem he has assigned himself is that they accidentally conform in part with his experimental data.

JOHN W. SOLOMON,[11] ESQ.—There are many interesting facts in this paper that certainly will be useful to the practicing foundation engineer. It is always worth while to inquire into the relationship between the computed failure load (bearing power) of a group of friction piles and the actual load required to produce failure. By referring to experiences with friction piles of reinforced concrete the writer proposes to demonstrate that the differences between failure loads of parallel sided and tapered piles, reported by Mr. Masters, can be explained easily. In addition, the differences in the solution proposed by Professor Mindlin[2] and the so-called Boussinesq formula can also be explained.

In the work that the writer has in mind numerous reinforced concrete friction piles were molded in the soil (cast in place) by first sinking a casing, then setting steel bars, and finally pouring concrete. The concrete was compacted by compressed air, and the casings finally removed. Such friction piles were constructed in the dried bed of a former river, and along the banks of a flowing river. The pile lengths varied from 40 ft to 50 ft, and the diameters from 16 in. to 17 in. The soil was a mixture of very fine sand, silt, and clay, with sometimes a pocket or stratum of medium sand.

Tests to failure, at the electric power station of Comines, in the northern part of France, gave skin friction determinations of 350 to 420 lb per sq ft. In the vicinity of the Alps, piles were driven for the towers of a high-tension line from Bissorte to Montmelian. Where this line crossed the River Arc, the skin friction was found to be only 320 lb per sq ft, the piles being 50 ft long, and the soil a very fine material composed of incongruent elements (mud). Some piles were tested by applying a horizontal pull at a point two thirds the height of the tower superstructure. This cantilever pull imposed compression

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[11] New York, N. Y.; formerly Engr. in Chg. of Roumanian Pavilion, New York World's Fair, 1939 and 1940.

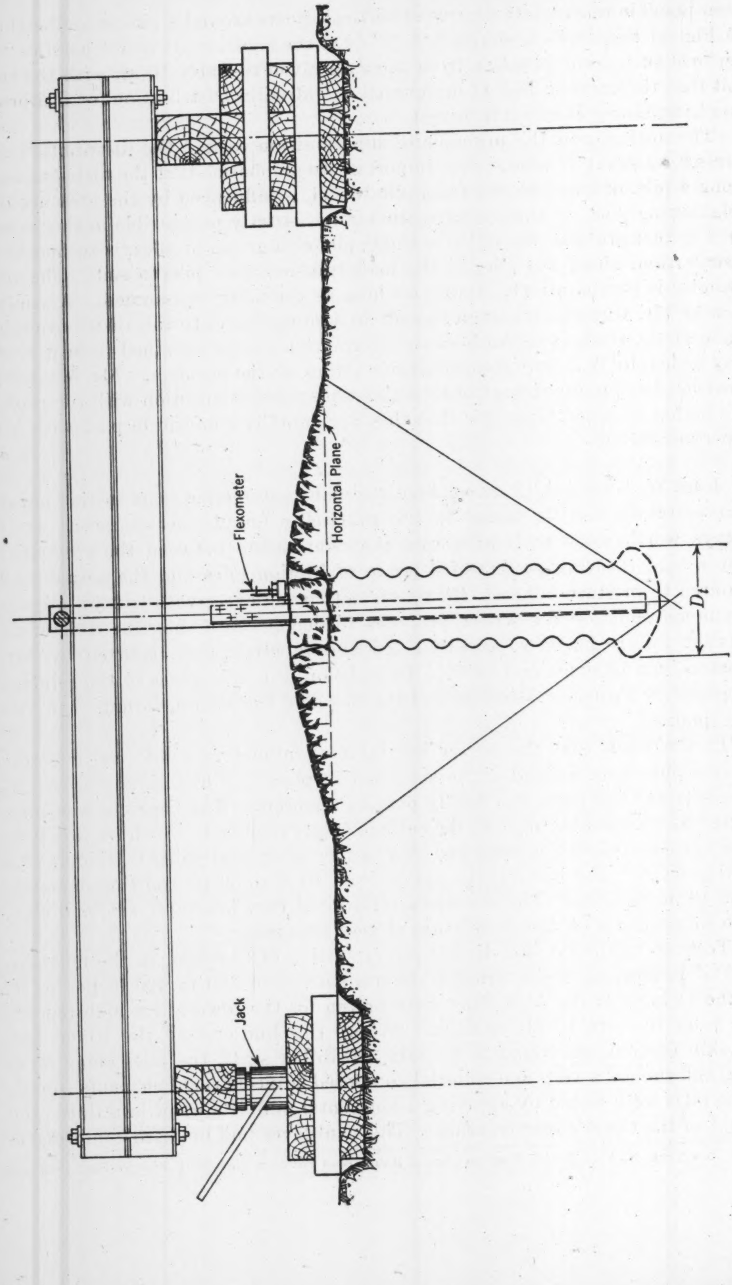
Flexometer
Horizontal Plane
D_t
Jack

FIG. 13.—LOADING SETUP FOR PULLING TEST. FOR A PERFECT PULLING TEST USE JACK ON BOTH ENDS

on two piles, and tension on two piles. Deducting the point loads it was found that the piles in tension failed at smaller loads than those in compression. Similar tests with piles varying in length from 8 ft to 12 ft, of which 6 ft to 8 ft was embedded in pebbles and stones up to 6 in. diameter, developed a tensile resistance of from 60 to 65 tons. The compression of the concrete piles against the soil created some kind of a cementation of the soil. Conditions at the ground surface indicated that the area influenced by the test was a segment of an almost perfect sphere, with its center at the enlarged tip of the pile (see Fig. 13). For all practical purposes the simple theory of a tension cone was considered sufficiently accurate. For two piles adjacent to each other, the writer obtained the formula

$$K_R = \frac{l^3}{2\,(2\,L \tan \phi + D_t)^3} + \tfrac{1}{2} \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

in which K_R is a reduction coefficient; l is the distance between the centers of the two piles; L is the length of the pile; ϕ is the angle of the cone of failure; and D_t is the diameter of the enlarged tip of the pile.

In 1935 individual piles and groups of piles were placed for a large storehouse at Grande Couronne, near Rouen, on the River Seine. These piles were 70 ft long, driven into a solid chalk stratum. The soil above the chalk was a 25-ft to 30-ft stratum of fill and mud, overlying a stratum of sand and pebbles 6 ft or 8 ft deep, with alternating layers of fine sand and clay, 8 ft to 10 ft deep, to the level of the chalk. Twelve single piles were driven to a stand test load of 140 tons each (including the weight of the pile), and one group of nine piles, capped by a reinforced concrete slab, was tested at 1,350 tons. In all cases the load was applied by filling a box with iron pyrites. For the individual piles, in addition, a hydraulic jack was applied, and the settlement of the head of the pile was registered by two flexometers (see Fig. 14). The load was increased gently, at regular intervals, by increments of 5 tons. Precautions were taken so that the points of contact at the bottom of the loaded box were some distance from the piles, and the same precautions were made in fixing the support for the flexometers.

In the case of the pile group, the box of the pyrites was centered on the reinforced concrete caps. A load was applied by adding a constant quantity of pyrites to the box until the total of 1,350 tons had been reached. In this case, five registering flexometers, one at each corner, and one at the center, were used to record the settlement of the group. Every test proved that the load was transmitted to the soil from the pile, beginning quite close to the surface, and extending down to the tips. This was interpreted to mean that, for relatively small loads, only the friction surface of the upper part of the piles was working. The flexometer indicated an elastic contraction of the pile which was much less than could be expected under the assumption that the load, however great or small, was transmitted by the full length of the pile, regardless of the distribution formula adopted. At Grande Couronne the tip of the 70-ft pile began to "work" only after the superposed load had exceeded 50 tons. In any foundation project involving the use of piles, the test diagram of each pile and a record of the nature of the soil for the full depth of the pile are essential.

In the case of friction piles, an exact picture of the distribution of the load resistance at points adjacent to the entire depth of the pile require loading diagrams with load increments increased slowly. Cylindrical piles, driven in almost homogeneous soils, have a constant load distribution, beginning at a certain depth below the surface. Diagrams of the tests, such as these, permit the investigator to study the building up of stresses in the soil, as a pile is loaded, and thus to determine the interaction between the piles of a given group. Especially it is possible to appraise the final distribution under the failure load.

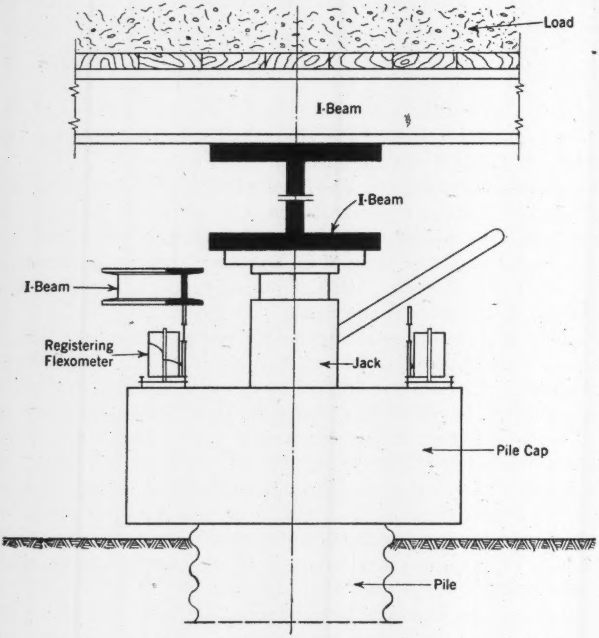

FIG. 14

The contraction of the pile itself has been neglected by Mr. Masters. Fig. 6, for example, showing shears for piles loaded to 10 tons, is not consistent with actual facts. Under only 10 tons per pile, no load would be transmitted at the tip. Eq. 2*c* was based on a positive value of the point load at the tip, greater than zero. There would be no load delivered by the tip for the piles in Fig. 6 when the applied load is 10 tons or less. The same applies to the 60-ft piles in Table 9.

Consider a friction pile in place, ready for its working load (see Fig. 15). If a load P_1 is applied, the upper length of the pile, L_1, shortens by an amount Δ_1, and the surrounding soil, for the same distance, resists that shortening.

The resulting force system represents the state of equilibrium between the work of the elastic deformation of the pile and the work of the elastic and plastic deformation of the soil. The load distribution for this depth begins at 0 at the surface, goes quickly to a value of p_1 per square foot, and continues at this almost constant value to the lower end of depth L_1, when it decreases again to practically 0.

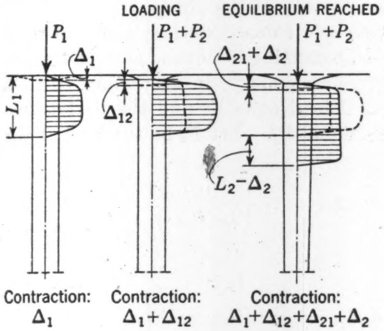

FIG. 15

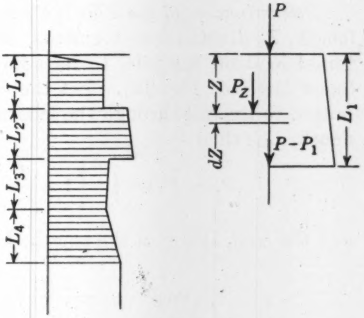

FIG. 16

The load is now increased from P_1 to a total of $P_1 + P_2$. Immediately the equilibrium between the pile and the soil is disturbed, the length L_1 is shortened to a total of $\Delta_1 + \Delta_{12}$, corresponding to the new superposed load. The hydrodynamic stresses in the soil, likewise, including the water stresses within the voids of the soil, are reduced due to the escape of water, in a period varying from a few hours or days, so that the equilibrium is further disturbed because of this condition. To reestablish equilibrium for the new load, it is necessary to add the work done by the soil surrounding the length L_2 of the pile. The final settlement for the total load corresponds to a shortening of length L_2 by Δ_2, plus a supplementary shortening of L_1 by Δ_{21}, due to the diminishing load distribution in length L_1. The load distribution, through $L_1 + L_2$, is balanced according to the same law as that applying to L_1; that is, it changes to a value greater than p_1. This sequence continues for successive increments of load increases, until $P_t = \Sigma P_i$ has made use of the entire resisting power of the pile to the tip, where $\Sigma L_i = L$, the total length of the pile. Only then will the contact between the tip of the pile and the soil have any effect, and only then will the pile settle under its applied working load.

If a load, P_c, traverses the pile for a depth, L_c, without encountering any resistance from the soil surrounding the pile in that distance, P_c can be called a free, or almost free, circulating load. This is the condition that occurs in the case of inconsistent soil (mud, etc.).

It should be borne in mind that the settlement of a pile head may be due, in addition to the shortening of the pile, to the consolidation of the soil about the pile, as well as the settlement of the pile through the soil, after the applied load has passed its critical value. The complexity of the load distribution during a test period depends upon the nature of the soil, and the rapidity with which test loads are applied or released. By using greater loads, P_t, a dynamic coefficient becomes involved, which, in turn, imposes a load distribution of its own, down along the pile.

After the load has begun to exert its influence at the tip of the pile, the load distribution is further readjusted, and then the pile continues to settle until the critical load is reached. If the load is transmitted to the pile by a jack, the pile descends in jerks, with every stroke of the pump.

Determination of the Elastic Contraction of a Pile.—Consider a pile of total length, L, divided into segments, each transmitting a certain part of the external load on the pile, to its adjacent soil segments, in accordance with a linear law (see Fig. 16). Let the external load equal P and the part transmitted to the soil through the length, L_1, equal P_1. Let P_z be the load at the depth, L_{1z}; then

$$P_z = P - \int_0^Z \left[1 + \frac{L_1 - Z}{L_1}\,(R_{c1} - 1)\right] \frac{2\,P_1\,dZ}{L_1(R_{c1} + 1)} \quad\ldots\ldots\ldots (7a)$$

and the contraction of the length, L_1, of the pile is

$$\Delta_1 = \int_0^{L_1} \frac{P - \dfrac{2\,P_1}{L_1(R_{c1} + 1)}\left[Z\,R_{c1} - \dfrac{Z^2}{2\,L_1}\,(R_{c1} - 1)\right]}{A_z\,E}\,dZ \quad\ldots\ldots\ldots (7b)$$

in which E = the modulus of elasticity of the pile, and the cross-section area at the point Z is

$$A_z = \pi\left[\frac{D_b}{2} - \frac{Z\,(D_b - D_t)}{2\,L_1}\right]^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

where D_b and D_t = the pile diameters at the butt and tip, respectively. The total contraction of the pile is given by a summation of the contractions, Δ_1; thus:

$$\Delta_{\text{total}} = \sum_{i=1}^{i=n} \Delta_i = \sum_{i=1}^{i=n} \int_0^{L_i}$$

$$\times \frac{P - \sum\limits_{i=1}^{i=i} P_{i-1} - \dfrac{2\,P_i}{L_i(R_{ci} + 1)}\left[Z_i\,R_{ci} - \dfrac{Z^2{}_i}{2\,L_i}\,(R_{ci} - 1)\right]}{A_{zi}\,E}\,dZ_i \quad\ldots (9)$$

In Eq. 9, P_0 equals 0, Z_i equals 0 for the butt end of L_i, and Z_i equals L_i for the tip end of L_i. If the soil surrounding the pile has one characteristic, L_i can be taken, for practical purposes, equal to 0; that is, when $i > 1$.

For a cylindrical pile with parallel sides, the total contraction is

$$\Delta = \sum_{i=1}^{i=n} \left[\frac{\left(P - \sum\limits_{i=1}^{i=i} P_{i-1}\right) L_i}{E\,A} - \frac{2\,P_i\,(R_{ci} - 1)\,L_i}{3\,(R_{ci} + 1)\,E\,A}\right] \quad\ldots\ldots (10)$$

and when n equals 1:

$$\Delta = \frac{P\,L}{E\,A}\left[\frac{2}{3} - \frac{R_c}{3\,(R_c + 1)}\right] \quad\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

Typical solutions of Eq. 11 yield:

R_c	Δ
Free circulating load	$\dfrac{P\,L}{E\,A}$
0	$\dfrac{2\,P\,L}{3\,E\,A}$
1	$\dfrac{P\,L}{2\,E\,A}$
1.5	$\dfrac{7\,P\,L}{15\,E\,A}$
∞	$\dfrac{P\,L}{3\,E\,A}$

In the case of a tapered pile, the total contraction is

$$\Delta_{\text{total}} = \sum_{i=1}^{i=n} \left\{ \frac{\left(P - \sum\limits_{i=1}^{i=i} P_{i-1}\right) L_i}{\pi\,E\,\dfrac{D_{ti}\,D_{bi}}{4}} - \frac{2\,P_i\,R_{ci}\,L_i \left(\dfrac{D_{bi}}{D_{ti}} - 1 - \log_e \dfrac{D_{bi}}{D_{ti}}\right)}{\pi\,E\,(R_{ci}+1)\left(\dfrac{D_{bi}-D_{ti}}{2}\right)^2} \right.$$

$$\left. - \frac{P_i\,(R_{ci}-1)\,L_i}{\pi\,E\,(R_{ci}+1)\left(\dfrac{D_{bi}-D_{ti}}{2}\right)^3} \left[\frac{D_{ti}}{2} - \frac{\left(\dfrac{D_{bi}}{2}\right)^2}{\dfrac{D_{ti}}{2}} + 2\,\frac{D_{bi}}{2} \log_e \frac{D_{bi}}{D_{ti}} \right] \right\} \ldots (12)$$

and for $n = 1$, and $D_t = \dfrac{D_b}{m}$, the contraction becomes

$$\Delta = \frac{A(m) + B(m)\,R_c}{1 + R_c} \times \frac{P\,L}{E\,A_b} \ldots\ldots\ldots\ldots\ldots (13)$$

in which:

$$A(m) = \frac{m\,(-\,3\,m^2 + 4\,m - 1 + 2\,m^2 \log_e m)}{(m-1)^3} \ldots\ldots\ldots (14a)$$

and

$$B(m) = \frac{m\,(m^2 - 1 - 2\,m \log_e m)}{(m-1)^3} \ldots\ldots\ldots\ldots\ldots (14b)$$

The solution of these equations for representative values of m is presented in Table 13. For a given length, L_c, the load P_c is a traversing or a free, circulating load, as stated previously, and the contraction is expressed by

$$\Delta_c = \frac{m_c\,P_c\,L_c}{E\,A_{bc}} \ldots\ldots (15)$$

TABLE 13.—VALUES OF $A(m)$ AND $B(m)$

m	$A(m)$	$B(m)$
1.000	0.6666	0.3333
1.100	0.7157	0.349
1.200	0.7628	0.364
1.300	0.810	0.377
1.400	0.853	0.391
1.500	0.896	0.403
1.600	0.936	0.415
1.700	0.975	0.426
1.800	1.016	0.436
1.900	1.054	0.447
2.000	1.0896	0.456

Analysis of a Loading Test Considering the Elastic Behavior of the Pile.—The part of a test diagram, which is usually neglected—the beginning of the test—is the most valuable for determining the load distribution in the soil adjacent to a pile. From the five tests presented in Table 2, test (*d*) is useless for this analysis because part of the readings are incorrect, the pile not being plumb. Test (*e*) is also useless for this analysis because the first reading was taken at 80 tons. It should be emphasized that the accuracy of bench-mark readings, using a standard construction level, is generally not sufficient to determine the elastic behavior of a pile. This error may be as high as ±0.001 ft. Pile T20, in Table 2, affords a good illustration of the load distribution during the loading period.

In analyzing Pile T20, the author has assumed that the load is transmitted to the soil through the entire length of the pile, following a linear law. Let $D_b = 17.50$ in., $D_t = 8.00$ in., and $m = 2.19$; then by Eq. 13, the elastic contraction is

$$\Delta = \frac{1.158 + 0.471\, R_c}{1 + R_c} \times \frac{P\, L}{E\, A_b} \quad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

in which $E = 1{,}400{,}000$ lb per sq in. (if E is greater, R_c is smaller); $A_b = 240.5$ sq in., and $L = 85$ ft. The values of R_c computed from Eq. 16 for test readings of Δ and P, are given in Table 14, the last column of which indicates values of R_c, for $E = 1{,}600{,}000$ lb per sq in. The negative value for R_c in Table 14, under the 15-ton load, indicates that no load distribution can be found which will transmit the load to the soil through the full length of the pile. For a load of $P = 25$ tons, a value of R_c equal to 170.0 indicates that the load transmitted to the soil is 1,170 lb per ft at the butt of the pile, and 6.89 lb per ft at the tip; under a load of $P = 25$ tons, there is practically no load transmission at the lower part of the pile, inasmuch as Δ equals 0.006 ft, which corresponds to an ultimate condition of rest. Therefore, high values of R_c, and even the value $R_c = 8.55$, for the case of $E = 1{,}600{,}000$, are impossible, even as a first approximation in the case of actual soil conditions. On the basis of Table 14, and of the intermediate reading given in Table 2, it can be stated as a practical matter that, for $E = 1{,}400{,}000$ lb per sq in., the full length of the pile does not transmit load to the soil until the load reaches 70 tons; for $E = 1{,}600{,}000$ lb per sq in., the full length of the pile does not act until $P = 50$ tons.

TABLE 14.—SOLUTIONS OF EQ. 16 TO DETERMINE R_c FOR PILE T20

P (tons)	Settlement of pile head (ft)	VALUES OF R_c	
		E = 1,400,000	E = 1,600,000
15	0.003	−10.15	−39.10
25	0.006[a]	+170.00	+8.55
30	0.008	11.05	4.20
50	0.013[b]	14.61	4.82
	0.015	4.58	2.30
	0.016[a]	3.21	1.70[d]
75	0.020[b]	11.05	4.20[c]
	0.024	3.21[c]	1.70[c]
	0.026[a]	2.20[d]	1.19[d]
100	0.028[b]	7.20[c]	3.21[c]
	0.030	4.58[c]	2.30[c]
	0.032	3.21[c]	1.70[c]
	0.045[a]	0.628[d]	0.254[d]

[a] Final. [b] Initial. [c] Includes the buckling effect; that is, the permanent settlement and deformation, if any, caused by the eccentricity of the load and the stiffness of the pile tip embedded in the solid sand layer. The 25-ton increment, however, did not reach the tip of the pile and the equilibrium of the pile-soil system is not yet re-established. [d] Includes the buckling effect.

The settlement readings reported by Mr. Masters were stopped too soon after unloading. Readings should have been continued for a period equal to the loading time (188 hr) or at least the period required for the load of 100 tons

—that is, 120 hr. Had this been done, the final settlement would have been 0.004 ft to 0.008 ft less, corresponding to a maximum elastic contraction of Δ varying from $(0.045 - 0.019 =)\ 0.026$ to $(0.045 - 0.015 =)\ 0.030$, for a 100-ton load. It seems to the writer that a value of $E = 1{,}600{,}000$ lb per sq in. corresponds more nearly to the actual elasticity modulus, unless the residual settlement of the pile head was actually less than could be estimated from the paper.

The load distribution for pile T20, with P less than 70 tons (or less than 50 tons), begins with a zero distribution at the ground surface, increases rapidly to a value of p_1 per square foot, which corresponds to a value of R_c very much less than 1. Then it increases gradually to the lower part of the effective length of the pile, where the value of R_{c2} varies from about m to 2, and finally decreases to zero. Static equilibrium is considered to be established, or nearly so. Eq. 16 for this condition was solved by determining the working length, L_w, corresponding to the load P, the results being tabulated in Table 15. For this analysis no information was available except the few readings listed in Table 2. With these data it was necessary to make a few assumptions in order to limit the range of possible R_c-values. For example, it was assumed that when the loading tests were applied, the dynamic stresses due to driving the pile were no longer present. Also, final equilibrium is assumed to have been attained after a few hours of practically no settlement.

TABLE 15.—SOLUTION OF EQ. 16, FOR VARIOUS EXTERNAL LOADS

Load P (tons)	Modulus of elasticity E (lb per sq in.)	Effective length L_w (ft)[b]	m	$A(m)$	$B(m)$	Final settlement[c] (ft)	R_c
15	1,400,000	50	1.47	0.885	0.399	0.003	0.774
	1,400,000	50	1.47	0.885	0.399	0.0027[d]	1.35
	1,600,000	55	1.54	0.910	0.408	0.003	0.725
	1,600,000	55	1.54	0.910	0.408	0.0027[d]	1.28
25	1,400,000	58	1.59	0.933	0.414	0.006	0.830
	1,400,000	58	1.59	0.933	0.414	0.0057	1.09
	1,600,000	65	1.71	0.980	0.427	0.006	0.950
	1,600,000	65	1.71	0.980	0.427	0.0057	1.23
50	1,400,000	72	1.85	1.036	0.442	0.016	0.935
	1,600,000	81 to 83[a]	2.10	1.125	0.464	0.016	1.30–1.45
65 to 70	1,400,000	81 to 83[a]	2.10	1.125	0.464	0.019–0.023	4.60–2.10

[a] 81 ft to 83 ft corresponds to the level of the sand stratum. [b] L_w is the approximate effective length of the pile distributing load P to the soil through friction. [c] Indicating a state of equilibrium. [d] Comparing the settlements for 15 and 25 tons it is evident that this value (0.0027 ft) should be even smaller.

To check the load distribution it is advisable to make an unloading test at about 30% to 50% of the failure load of the pile, just before the effective length of the pile coincides with the actual length. This was done partly for pile T10. The settlement of the pile caps should be recorded carefully from the very beginning, and the applied load should be increased slowly, and at regular intervals. In Table 2(*b*), pile T18 indicates that the entire length of the pile is effective in friction, under a load of 10 tons. It appears that the soil becomes denser below 40 ft. Pile T10, Table 2(*a*), indicates the following characteristics for $E = 1{,}400{,}000$ lb per sq in.:

Load P (tons)	Length L_w (ft)
16.5	35
31.5	45
41.5	55
45	59.5

For concrete piles the settlement can be determined at any point, although usually at midheight will be sufficient. For this purpose a steel pipe 1½ to 2 in. in diameter is embedded in the axial center of the pile to the depth at which settlement is to be observed. A free, movable rod is inserted within the pipe to the point of observation, and it is the top end of this rod that indicates the settlement. This suggests a very simple method for determining more exactly the load distribution in the soil during the progress of the test. More particularly is this true for piles driven in soils of complex and different characteristics. Only by this simple test can some of the problems in soil mechanics ever be solved.

Under the same load, P, the load distribution varies for piles of different total lengths. On a project dealing with timber friction piles, the test specimens should be in lengths varying by 10 ft—40, 50, 60, 70 ft, etc. For concrete cylindrical friction piles—that is, with parallel sides—it is sufficient for specimens to vary in 20-ft lengths—40, 60, 80 ft, etc. A qualitative description of the soil gained by studying borings, which indicates the depth of the different layers of soil, should always be available.

Table 3, Groups of Piles.—Consider a pile with a total length, L_1. The elastic contraction due to an external load, P_1 is Δ_1 and in this case, R_c is equal to R_{c1}, which, as the load approaches the failure load, is proportional to $m = \frac{D_b}{D_t}$. For a group of piles under the same circumstances, tests would show an elastic contraction, Δ_2, per pile, of $\frac{P_{1f}}{P_{gf}}\Delta_1$; the value of the external load, P_1, remaining the same. This is greater than Δ_1 and corresponds to a much smaller value of R_{c2}. In the foregoing P_{1f} is the failure load for individual piles, and P_{gf} is the failure load per pile in the group. This relationship indicates that in group piles the load distribution increases toward the tip of the pile, as may be expected. Test T10 can be compared with tests T21 to T29. The elastic contraction of the piles at the time of failure for the group is only a little smaller than that for an individual pile at failure; but even more important, if P_1 is small, and corresponds to $L_w \ll L$, then the tests show that the settlement in the group Δ_2 is from 50% to 200% larger than the expected ratio, $\frac{P_{1f}}{P_{gf}}\Delta_1$, if L_w would have equaled L.

In the two tests, P_{1f} equals 96.5 tons, and P_{gf} equals 55.3 tons. A comparison of predicted and actual measured contraction is given in Table 16.

In Table 3(*a*), group T21–T29, the effective length, L_w, is equal to the total length, L, for all loads beginning at 28 tons.

Table 4, Skin Friction of Individual Piles.—The differences in the average skin friction between concrete piles with parallel sides and tapered timber piles is less than revealed by Table 4. By adding the weight of the concrete piles (28 kips to 63 kips) the average skin friction of the latter increases to 645 lb per sq ft.

TABLE 16.—COMPARISON OF PREDICTED AND MEASURED CONTRACTIONS

Load P_1 (tons)	Contraction Δ_1 (ft)	CONTRACTION Δ_2 (FT)	
		Predicted	Measured
16.5	0.002	0.0035	0.0095[a]
31.5	0.006	0.0105	0.0163[b]

[a] Increase of 170% or more. [b] Increase of 55% or more; includes non-elastic settlement.

For the tapered piles the average skin friction at the time of failure is 853 lb per sq ft (tests T1, in which the pile was not plumbed, and T20, which was not tested to failure, were discarded from this average). The point resistance of timber piles and of concrete piles with larger tip surface was neglected. The difference between 645 and 853 corresponds to the effect of wedging action on the tapered piles. The swelling of wooden piles in the ground might also have been a factor in producing an increase of the critical load compared to that of concrete piles.

Friction Pile Theory.—The author's interpretation of test results is somewhat open to question. This entire problem is concerned with the distribution of the stresses in the soil as produced by a variably located point load in the soil. As early as 1848 Lord Kelvin[12] offered a solution of the stresses due to a point load in an indefinitely extended solid. Later Boussinesq gave his well-known solution of stresses in a semi-infinite body due to forces in the boundary.[13]

More recently, in 1936, Professor Mindlin, using the Galerkin vector, offered a solution for the distribution of stresses in a semi-infinite body, with a point load at any place in the body.[2] The latter solution was the connection between Lord Kelvin's and Boussinesq's problems. Each of these three theories is rigorously valid only for the theoretical, elastic and isotropic bodies (those that follow Hooke's law and which permit the superposition of small strains and stresses). It has been found that the Boussinesq theoretical distribution of stresses does not check for all soils by any means, and especially for cohesionless, or nearly cohesionless, bodies. In 1933 Froehlich introduced a coefficient ν (which he calls a concentration factor) varying from 3 to 6. In the Boussinesq solution ν equals 3 corresponds with a Poisson ratio of 0.5. Froehlich still adheres to the validity of the linear law.[14]

As the name indicates, the coefficient ν helps take into consideration the fact that the modulus of elasticity of certain soils, at various depths, is not constant. It also helps to concentrate stresses more around the point loads, corresponding to the difference between the hypothetical bodies used in the

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[12] *Cambridge and Dublin Mathematical Journal*, Vol. 3, 2d Series, 1848, p. 88.

[13] *Application des potentiels*, Paris, 1885.

[14] "Druckverteilung im Baugrunde," Dr.-Ing. O. K. Froehlich.

Lord Kelvin, Boussinesq, and Mindlin stress distribution, and the real soil. Consider a point at *O* in the soil, and a pile driven not far from it (see Fig. 17). It is clear that the influence of the upper part of the pile, not far from the ground surface, would follow theoretically the Boussinesq law. At distances farther from the ground surface the distribution would approach more nearly that of Lord Kelvin's theory. Since the Kelvin distribution has an effect about half as great as the Boussinesq distribution in the horizontal plane containing point *O*, it is clear that by using a coefficient, R_m, the stress value at the point *O* can be determined with as good an approximation as may be desired. For example, if it is assumed that the pile distributes the load to the soil at a constant rate per linear foot, and if the Boussinesq formula is applied to the calculation of the stresses, giving each elementary load a coefficient dictated by a certain rule, then by integration, the stresses can be approximated as accurately as may be desired to the actual stresses. This is true, even if the influence of the point loads below the plane of point *O* is neglected. If the load distribution is not constant, a variable coefficient, R_m, and a load distribution, R_c, can be introduced. It is in this sense that the coefficient, R_c, used by Mr. Masters, is to be interpreted as corresponding to the coefficient, R_m, of the present discussion, and not as a load distribution along the depth of the piles.

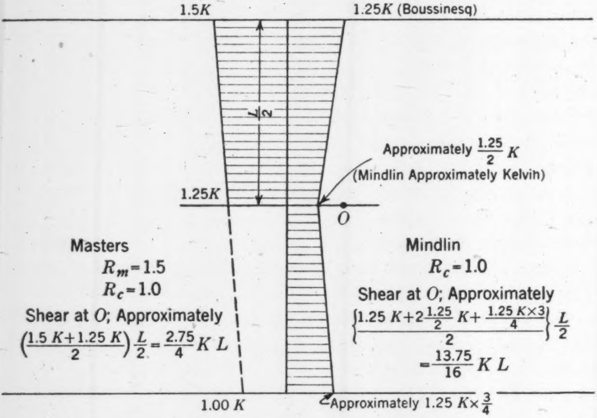
1.5K
1.25K (Boussinesq)
$\frac{L}{2}$
Approximately $\frac{1.25}{2}K$
(Mindlin Approximately Kelvin)
1.25K
O
Masters
$R_m = 1.5$
$R_c = 1.0$
Shear at O; Approximately
$\left(\frac{1.5K + 1.25K}{2}\right)\frac{L}{2} = \frac{2.75}{4}KL$
Mindlin
$R_c = 1.0$
Shear at O; Approximately
$\left\{\frac{1.25K + 2\frac{1.25}{2}K + \frac{1.25K \times 3}{4}}{2}\right\}\frac{L}{2}$
$= \frac{13.75}{16}KL$
1.00 K
Approximately $1.25 K \times \frac{3}{4}$

FIG. 17.—VERTICAL SHEAR AT *O*, MIDHEIGHT (SCHEMATIC)
Mindlin Values Approximately 25% Larger for $R_c = 1$

Consequently, if point *O* is selected at the midheight horizontal plane of the piles, a value of R_m can be obtained which will yield the stresses in that plane. Another value of R_m would give stresses at another point *O* at a different depth. This is why the Mindlin stress distribution yields tensile stresses independent of the cohesion factor of the soil, and also why the shearing stresses computed by Professor Mindlin's method are higher than those in actual soil. In the future it may be possible to take these two factors into consideration by introducing the cohesion coefficient and the internal friction angle of the soil.

This will be a difficult task, however. With the foregoing explanation in mind, the coefficient R_c in the paper should actually be called R_m to differentiate it from the actual R_c, which is the ratio of the load delivered per linear foot of pile at the top of the pile to the load delivered per linear foot at the bottom of the pile. When this has been done, vertical shears at midheight of friction piles (corresponding to an average of vertical shears for the full length of piles) and safe loads may be computed in practice as indicated in the paper.

Approximate Formula for Vertical Shear in a Pile Group.—Table 6 for the determination of the interaction of piles in a group can also be used by considering the vertical shears as inversely proportional to the center-to-center distances between the piles, in which case the error would not exceed 3%. In the case of pile A—

$$s_{vA} = \frac{P}{2\,\pi\,r_1\,L} \quad \text{(17)}$$

equals 0.005780 at the midheight of the pile. An even better approximation can be obtained by assuming that the influence of a distant pile A_n on pile A_1 is inversely proportional to $\frac{r_{1n}\,(L^2 + 4\,r^2_{1n})^{1.5}}{r_1\,(L^2 + 4\,r^2_1)^{1.5}}$, in which r_{1n} equals the distance, $A_1\,A_n$, r_1 = radius of pile A_1 at midheight. This value is then multiplied by the sine of the angle of the line of action. The foregoing proportion is obtained from Eq. 2*c* by neglecting the term with the coefficient $R_c - 1$, and letting Z equal $\frac{L}{2}$; then,

$$s_v = \frac{P\,L^2}{2\,\pi\,r\,(L^2 + 4\,r^2)^{1.5}} \quad \text{(18)}$$

In Eq. 18, R_c from the first part of Eq. 2*c* disappears, as was to be expected for Z equals $\frac{L}{2}$ with a linear load distribution. By using Eq. 18 instead of Eq. 2*c* the error is less than 1%, and for the practicing engineer it is more convenient. The vertical pressure beneath the group of piles can be computed as indicated in the paper. In this case again, Eq. 4*b* can be simplified for the plane through the tip of the pile, as follows:

$$\phi_v = \frac{2\,P}{\pi\,L\,r\,(R_m + 1)} \quad \text{(19)}$$

This gives a very good approximation (R_m from Eq. 19 has the same value as R_c from Eq. 4*b*.

In conclusion, it may be stated that by using a coefficient R_m, a better approximation of the stress distribution around the pile corresponding to the true distribution in the soil can be obtained. A better insight into the true behavior of soil under the forces exerted by a friction pile can be obtained only by actual tests. The theoretical approach of the load distribution for a friction pile, as developed in this discussion, was completed by the writer during the aforementioned tests of Grande Couronne in 1935. Then it was found that only through the elastic behavior of the pile, and by the measurements of displacements of different points alongside the pile, could the real distribution of the load be obtained, and some other problems of the soil mechanics approached.

A. E. CUMMINGS,[15] M. AM. SOC. C. E.—The question of the relative carrying capacities of single piles and pile groups has been a controversial subject among foundation engineers for many years. The experiments described in this paper might have served to add materially to the available knowledge on this problem. Unfortunately, it appears as though too much effort was devoted to the development of an unsound mathematical theory and too little effort to the development of a carefully thought out test program that would include field measurements of all of the quantities that are important in problems of this kind.

Referring to the mathematical theory, the author's effort to use the Boussinesq point-load equations simply adds another chapter to the long story of the incorrect application of these equations to all kinds of foundation problems. Several years ago, the writer published[16] the original derivation of the Boussinesq equations together with a statement of the fundamental assumptions on which the derivation is based. Whenever these equations are applied to an actual problem, it is necessary to keep these basic assumptions in mind and to make sure that the conditions of the problem do not conflict with the assumptions under which the equations were derived.

For example, one of Boussinesq's fundamental assumptions is that the load is applied at the plane boundary of the semi-infinite, elastic, isotropic, solid and not at some point below the surface of the solid. The author has used the Boussinesq equations for loads that are applied below the surface of the ground although it has already been explained by Raymond D. Mindlin,[17] Assoc. M. Am. Soc. C. E., why the Boussinesq equations cannot be used in this manner to determine the distribution of stresses around a pile. Furthermore, the author does not appear to have given any consideration to some of the other fundamental assumptions of the mathematical theory of elasticity on which the Boussinesq equations are based. In particular, the paper says nothing about strains or displacements, although these are just as important as the stresses in problems of this kind. Also, the author's extension of his analysis from the single pile to the pile group presupposes the validity of the principle of superposition. It is of considerable interest to examine the author's proposed theoretical solution from the standpoint of displacements and the principle of superposition.

As is well known, the entire mathematical theory of elasticity is based on Hooke's law concerning the proportionality of stress and strain. Whenever a force is applied to an elastic body, some kind of displacement occurs as a result of that force. The problem under consideration is no exception to this rule and the stresses determined by the author would necessarily be accompanied by displacements in the soil surrounding the pile. Since the author has assumed the soil to be incompressible (Poisson's ratio = 0.5), the horizontal tangential stress vanishes everywhere and the two normal stresses given by Eqs. 1*a* and 1*b* will determine the horizontal radial displacements by means of the equation

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[15] Research Engr., Raymond Concrete Pile Co., New York, N. Y.

[16] *Transactions*, Am. Soc. C. E., Vol. 103 (1938), p. 324.

[17] "On the Distribution of Stress Around a Pile," by R. D. Mindlin, *Proceedings*, International Conference on Soil Mechanics and Foundation Eng., Cambridge, Mass., Vol. III, 1936, p. 71.

$$u = -\frac{r}{2E}(q_v + q_h) \qquad \text{.......................(20)}$$

in which u is the radial displacement, and E is Young's modulus of the soil. The vertical displacements, w, may be obtained in a similar manner from the author's stress equations.

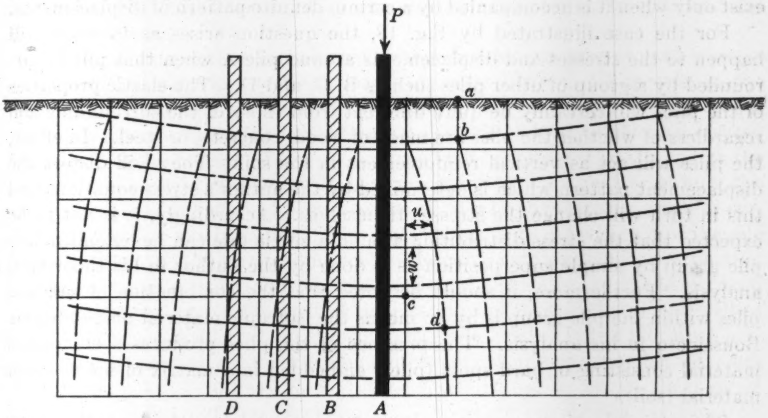

Fig. 18

Fig. 18 represents any meridional plane in the soil through the vertical axis of the pile A. The solid lines represent an imaginary rectangular network existing in the ground after the pile is driven and before any load is applied to the pile head. When the load, P, is placed on pile A, stresses are set up in the soil around the pile, and these stresses are accompanied necessarily by certain vertical and radial displacements in the soil. The dotted lines in Fig. 18 are a qualitative representation of the distortions that would occur in the rectangular network due to the displacements determined by the author's theoretical stresses. The actual amounts of the displacements would depend on the magnitude of the load, P, and on the numerical value of Young's modulus, E, of the soil. All points on the ground surface would be displaced vertically downward and there would be no radial displacements at the ground surface because of the assumption of incompressibility (Poisson's ratio = 0.5). For example, a soil particle originally at a would move to b. Within the soil, all vertical cylindrical surfaces concentric with the vertical axis of the pile would be bulged radially outward and all horizontal planes would be bent downward. A soil particle originally at point c would move to point d.

This system of displacements is a necessary consequence of the author's stress system. If the stresses in the soil were those determined by the author, the displacements would have to be of the type shown in Fig. 18. It is conceivable that such a system of stresses and displacements could exist in the soil in the vicinity of a single pile. However, if the soil around the pile were subjected to restraints that would affect the displacement pattern, these restraints would also affect the stress pattern. This follows inevitably from the fact that the entire mathematical theory of elasticity is based on a definite set of stress-strain relationships. Boussinesq's stress equations were derived with the aid of the basic differential equations which relate the stresses to the space derivatives of the displacements. Accordingly, the Boussinesq stress pattern can exist only when it is accompanied by a certain definite pattern of displacements.

For the case illustrated by Fig. 18, the question arises as to what will happen to the stresses and displacements around pile A when that pile is surrounded by a group of other piles such as B, C, and D. The elastic properties of the piles will certainly be quite different from those of the surrounding soil regardless of whether the piles are made of wood, concrete, or steel. In effect, the piles will act as vertical reinforcement in the soil. They will distort the displacement pattern which is determined by the author's stress equations and this in turn will change the stresses themselves. Accordingly, it is not to be expected that the stress distribution around a single pile can be extended to a pile group by simple superposition as is done by the author in his theoretical analysis. Furthermore, it should be noted that the combination of soil and piles within the pile group is by no means the isotropic material postulated by Boussinesq in his analysis. The presence of the piles produces a compound material consisting of hard spots (piles) embedded in a matrix of much softer material (soil).

The principle of superposition is sometimes used to determine the stress distribution in foundation problems. For example, a large mat foundation is often divided into small segments and the stress distribution for each segment is computed as if the segment were a point load.[18] The stresses produced by each segment are then added by simple superposition to obtain the stress distribution for the entire mat. This application of the principle of superposition to a mat foundation is something quite different from the application of that principle to a pile group. In the case of the pile group, each additional pile affects the elastic properties of the underground. Even though the soil around a single pile might be considered as having the properties of an elastic isotropic material, the introduction of additional piles destroys the isotropic properties and materially changes the elastic properties. Accordingly, the author's method of applying the principle of superposition to a pile group is an incorrect procedure.

Another factor in this problem to which the author has paid no attention is the contraction of the pile itself. After the pile is in place in the ground and before any load has been applied to the pile head, there is no shearing stress in the soil such as that given by Eq. 2*c*. When a load is applied to the pile head, there is a downward movement of the upper end of the pile due to the elastic shortening of the pile. As the pile starts to move with respect to the surrounding soil, the movement mobilizes the shearing resistance of the soil. Since the movement begins at the top of the pile, the mobilization of shearing resistance in the soil also begins at the top of the pile near the ground surface. When a small load is applied to a long friction pile, all of the reaction for the load is developed in the soil around the upper part of the pile. The load produces no stress in the lower part of the pile and consequently no elastic contraction and no mobilization of shearing resistance in the soil surrounding the lower part of the pile. As the load is increased the stress in the pile approaches the pile point but it does not actually reach the point until the load attains a certain value which is determined by the elastic properties of the pile and the surrounding soil.

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[18] *Proceedings*, Am. Soc. C. E., May, 1933, p. 784.

In Fig. 6, the author has computed the shearing stresses around a single pile 60 ft long carrying a 10-ton load. A value of $R_c = 1.5$ is used in the calculation. Actually the value of R_c would be very much greater than 1.5 for the conditions shown in the figure because little or no load would be delivered to the soil at the bottom of the pile. If no load is delivered to the soil at the bottom of the pile, the value of R_c becomes infinite. As a matter of fact, the determination of the numerical value of R_c should be the starting point for any theory of friction pile behavior. The real problem is to determine how the pile delivers its load to the surrounding soil. The author states that R_c is a function of pile length, pile taper, and internal friction of the soil. He has overlooked the fact that it is also a function of the elastic properties of the pile as well as a function of the load that is applied to the pile head. If the pile were perfectly rigid so that no elastic shortening of the pile would occur under any load, it might be possible to have $R_c = 1.5$ for all loads and to have a shear-stress distribution similar to that shown around the single pile in Fig. 6. Since all actual piles are elastic and will therefore shorten under compressive stresses, this elastic shortening will influence the transfer of load from the pile to the soil. The value of R_c, therefore, will depend on the elastic properties of the pile and on the amount of the applied load as well as on the three factors mentioned by the author.

The usual procedure in the case of a pile load test is to place the load on top of the pile and then to measure the downward movement of the head of the pile for each load increment. It is not possible to determine the elastic shortening of the pile from such measurements because nothing is known about the movement of the pile point. Neither can such measurements furnish any information as to the manner in which the pile transfers the load to the surrounding soil. On the other hand, if the actual elastic shortening of the pile were determined by suitable measurements, it would be possible to determine from these measurements approximately how the pile was transferring its load to the soil. For example, it is a simple matter to compute what the elastic shortening of the pile would be if it were a free-standing strut with no part of the load supported by friction on the sides of the strut. When any part of the load is supported by friction along the sides of the pile, the elastic shortening is always less than it would be for the same pile carrying the same load as a free-standing strut. When all of the load is carried by side friction as in the author's problem, it is possible to compute the elastic shortening of the pile for any assumed distribution of friction along the sides of the pile. This can be done even if the friction distribution is assumed to be discontinuous due to soil stratification.

Conversely, if the elastic shortening of the pile is carefully measured in the field, it is possible to deduce the friction distribution from the field measurements.

Experiments of this kind have been made in The Netherlands by C. Franx,[19] G. C. Boonstra,[20] and others. The piles tested in The Netherlands were precast concrete piles. When they were cast, a piece of 1.5-in. pipe extending the full length of the pile was placed in the vertical axis. A 1-in. rod was inserted in this pipe and fastened at the pile point. The test load was applied to the driven pile in such a manner that it was possible to measure the deflection of the pile head and also the movements of the 1-in. rod which gave the deflections of the pile point. The difference between the two readings for any given load represented the elastic shortening of the pile at that load. In Fig. 19, curves a and b are reproductions of a pair of load-settlement curves measured by Boonstra. The solid line shows the settlement of the pile head and the broken line that of the pile point. The specimen was a reinforced pre-cast concrete pile of octagonal cross section, 16 in. in diameter and 70 ft long. It was driven through 50 ft of soft clay and peat, 15 ft of fine sand, and then into sand and gravel for a depth of about 6 ft. The condition was similar to that of pile T20 of the author's tests.

As can be seen from Fig. 19, there was practically no movement of the pile point until the load amounted to 15 or 20 tons although the head of the pile began to move as soon as the loading was started. The vertical distances between the two curves, a and b, represent the elastic shortening of the pile, but in order to show this relationship more clearly, curve c, Fig. 19, was plotted with loads as abscissas and elastic shortening as ordinates. It is easily seen from this figure that the relationship between load and elastic shortening is not linear. If the distribution of load transfer from pile to soil were constant throughout the entire load range (R_c = constant), and if the pile material were perfectly elastic, curve c, Fig. 19, would be a straight line. As to the elastic properties of the pile, it could be expected to behave under load like a reinforced concrete column. The stress-strain curve for such a column might be slightly curved but the curve would be concave toward the strain axis, whereas most of curve c, Fig. 19, is concave toward the stress axis. Accordingly, the only conclusion that can be drawn from curve c is that the distribution of load transfer from pile to soil varies throughout the load range. In other words, R_c is not a constant. As the load is increased, the transfer of load from pile to soil distributes itself in various ways depending on the elastic properties of the pile and the soil and on the amount of the load. The sudden break in curve c at about 150 tons indicates that the upper part of the pile probably broke loose from the surrounding soil entirely so that practically all of the applied load was resisted by the lower end of the pile which was in the sand and gravel.

Consideration of this type of test data raises a serious question as to the validity of the figures presented in Table 8. All of the pile loads given in this table are computed with $R_c = 1.5$ on the assumption that the distribution of load transfer from pile to soil is constant over a wide range of loads and pile lengths. The author's computation implies that R_c is an independent variable to which numerical values can be assigned for the purpose of computing pile loads. Actually, the situation is just the reverse. The amount of the load influences the manner in which the pile transfers load to the soil so that the numerical value of R_c depends on the load.

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[19] "Proefbelasting van Heipalen te Rotterdam," by C. Franx, *Polytechnisch Weekblad*, 1934, Nos. 23, 24, and 25.

[20] "Eenige Beschouwingen over den Puntweerstand van Palen," by G. C. Boonstra, *De Ingenieur*, February 23, 1940, Vol. 55, No. 8, p. B-33.

This aspect of the problem is something that could easily have been investigated in the Morganza tests. It would have been possible to set three or four rods in a hollow concrete pile, such as pile C9, so that settlement observations could have been recorded not only at the pile head but also at the pile point and at several places along the length of the pile. If the strains in the pile itself had been measured, it would be possible to determine how the pile was transferring load to the soil at each load increment.

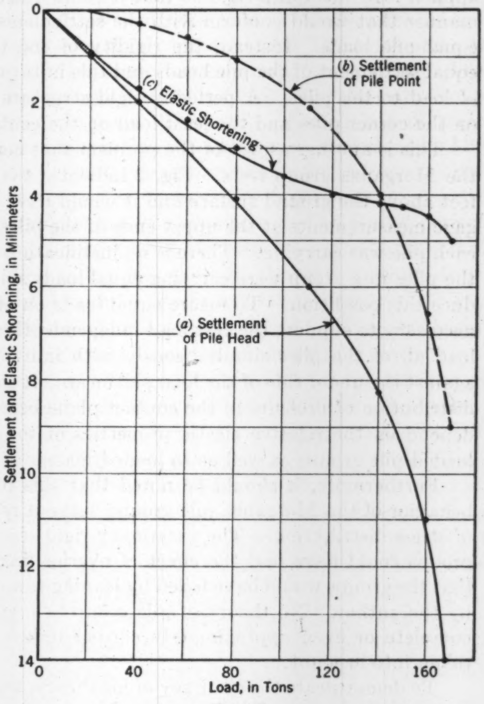

FIG. 19

Because he has completely neglected the entire question of strains and displacements, the author has developed a mathematical analysis that is theoretically unsound and of no practical value. The analysis is based on the mathematical theory of elasticity but nowhere in the paper is there any mention of the elastic properties of the piles themselves. The author has treated the piles as though they did not exist except as imaginary vertical line loads in the soil. Unfortunately, this method has led to a theoretical stress distribution which, in the writer's opinion, must also be classified as imaginary. The piles are actual physical bodies having certain elastic properties, and neither the single pile nor the pile group can be analyzed without taking these elastic properties into account.

Throughout his theoretical analysis of pile groups the author has assumed that each pile in the group would be subjected to the same load as all of the others. He has made this same assumption in connection with the groups that were actually tested. This question of load distribution over the pile heads is an important factor in pile group behavior. The author has referred to this subject in his conclusions by remarking that the piles under the outer edges of a stiff pier base would be subjected to greater loads than those under the center. This remark is perfectly true but the question arises as to what was happening in this respect during the Morganza group tests.

Fig. 2(*c*) shows the load arrangement that was used for the 16-pile group. If each pile in the group were subjected to the same load, the center piles could be expected to settle the most, the edge piles would settle somewhat less, and the corner piles would settle least of all. The photograph indicates that the applied load was quite rigid so that it could scarcely be expected to warp in a manner that would conform with the settlements that would be produced by equal pile loads. Instead, the rigidity of the test apparatus would require equal settlement of the pile heads and this in turn would affect the distribution of load to the piles. A perfectly rigid structure would put the greatest load on the corner piles and the least load on the center piles.

This is another aspect of the problem that could have been investigated in the Morganza group tests. Fig. 2 indicates that the pile heads were several feet above the ground surface and it would have been possible to make strain-gage measurements at the upper ends of the piles to determine how much load each pile was carrying. There is no justification for the assumption that all of the piles in a group were carrying equal loads since no effort was made to produce this condition. To insure equal loads on all piles in a group it would be necessary to support the test load independently of the pile group and then to load all of the piles simultaneously with individual hydraulic jacks reacting against the under side of the load platform. It should be kept in mind that the distribution of pressure in the contact plane between any two solid bodies will depend on the relative elastic properties of the two bodies. This applies to loaded pile groups as well as to loaded masses of earth or other materials.

Furthermore, it should be noted that this phenomenon could explain the behavior of the Morganza pile groups without reference to the author's theory of stress distribution. The practically rigid structures built on top of the pile groups could have had the effect of overloading the corner and edge piles so that the groups would have failed by leaning toward a corner or side as described by the author. No theory of pile group behavior can be considered as being complete or even approximately correct unless this phase of the problem is taken into account.

To demonstrate the accuracy of his theory, the author has prepared Table 5, which indicates a remarkably close agreement between theoretical and actual failure loads. However, the writer finds it difficult to follow the line of reasoning used by the author in his selection of shear values on which the comparisons are based. An average value is used for four of the group tests, and the other three are compared with values determined for certain individual piles. Referring to Fig. 1, these two profiles represent cross sections of the floodway 4 or 5 miles wide. The profiles themselves are some miles apart along the length of the floodway. The single piles, as well as the pile groups, are therefore scattered over distances that are measured in miles, and the first question to be decided is whether the comparisons should be based on averages or whether each group test should be compared with the results of the nearest single pile.

It is of considerable interest to examine Table 5 with these ideas in mind. To check the T2–T9 group, the author used pile T1 which was 130 ft from the group. However, pile T10 was only 294 ft from the group. Accordingly, if the proper way to handle this problem is by means of averages, it would seem reasonable to average T1 and T10 in order to check this group. Such a procedure would not check the theory so closely because the average friction developed by T1 and T10 was 792 lb per sq ft, whereas the author used 685 lb per sq ft from T1 alone. On the other hand, if the proper way to handle this problem is by comparing each group with the nearest single pile, groups T11–T19, T21–T29 and T37–T52 should be compared with T20 alone and not with an average that includes piles that are miles away from these groups. The distance from group T11–T19 to group T21–T29 is 1,958 ft and the third group as well as pile T20 are between the two. However, a comparison of these groups with pile T20 would not check the theory so well because pile T20 developed a friction value of only 680 lb per sq ft, whereas the author checked these groups by using the general average value of 778 lb per sq ft.

The data given in Table 5 indicate that the tests provided an almost perfect check of the author's theory. The agreement between theoretical and actual loads is really too perfect in view of the fact that the theory is unsound from the point of view of both mathematics and theoretical mechanics. In the writer's opinion, the author's theory is analogous to the old Ptolemaic theory of a geocentric solar system. For 1,400 years the Ptolemaic system was accepted generally because it appeared to be able to account for the motions of the planets. After Galileo invented the astronomical telescope, astronomers were able to stop speculating and start measuring. The very first measurements caused the abandonment of Ptolemy's geocentric system in favor of the modern heliocentric system. The fact that Galileo got into trouble for measuring things and upsetting an ancient theory should not discourage modern engineers from making all the measurements that can possibly be made in a given experiment. A theory based on insufficient data can appear to be practically perfect and yet be completely wrong.

The writer joins the author in the hope that engineers interested in pile foundations will continue to make tests to determine how such foundations behave under load. However, the writer considers it necessary to supplement that hope with several suggestions. To develop an adequate theory for the behavior of a single friction pile, it will be necessary to determine, by means of suitable field measurements, how the pile transfers its load to the surrounding soil. In order to develop an adequate theory for the behavior of a group of friction piles, field measurements must be made to determine the distribution of load to the pile heads in addition to the measurements that are required to determine the distribution of load transfer from pile to soil. Field measurements that include only a record of the settlement and rebound of the pile head are inadequate for the development of a theory of friction pile behavior.

Frank M. Masters,[21] **M. Am. Soc. C. E.**—It was the author's idea to present, in this paper, test data which seemed to demonstrate some facts about group action in timber friction piles that previously had been largely a matter of conjecture. The tentative theory for the calculation of failure loads of groups proposed in the paper appeared to check the test results and therefore seemed to have some merit. Such a theory can be of value only if it meets all conditions and if it is improved by criticism. The writer, therefore, sincerely appreciates the resulting comment and criticism.

Mr. Connor offers a number of practical suggestions on pile spacing that are too often neglected by the designer. He also gives data on heavy loads actually carried by friction piles driven into soft material which, during driving, did not appear to be able to support such loads. Mr. Connor further presents an interesting method of comparing the strengths of different groups of piles on the basis of comparative perimeter areas of soil around the groups. This method evidently gives a close approximation as between groups; but it is not possible to use this method to compare single piles with groups.

Mr. Woodruff raises several points of interest. The writer agrees that all predictions of pile strength using this theory will not have the accuracy of agreement between theory and tests apparently indicated in Table 5; nor does it seem that such accuracy is altogether necessary in the design of pile foundations. Further, it is quite possible, as Mr. Woodruff suggests, that soil consolidation around the piles over a long period of time may affect their settlement somewhat; but it does not seem likely that such a long-time consolidation should affect the shear strength of the soil adversely and lead to pile failure. It is anticipated that settlement readings will be made on the Morganza Floodway structures over a period of time. It is to be noted in connection with long-time loadings that one of the pile groups on the N. O. T. & M. Crossing, group T21–T29, was loaded over a period of 243 days with a constant load of 20.3 tons per pile without apparent additional settlement after the first day.

Mr. Feld suggests that a tapered timber pile actually may carry with it, in failing, a complete cylinder of soil having a diameter equal to the butt diameter of the timber pile. He points out that under this assumption the shearing value of the soil at failure would be 600 lb per sq ft and would be more nearly in correspondence with the value of 558 lb per sq ft indicated in Table 4 for parallel-sided concrete piles. He also describes an approximate rule for determining group strengths by subtracting progressive one sixteenths from the value of all piles after the first, as every new pile is added to the group. This apparently gives satisfactory results for the examples cited, but does not take account of pile spacing in relation to length or diameter, which the writer believes has an important bearing on the strength of piles in a group.

Mr. Greulich states that the law of averages probably contributed to the agreement between theory and tests as recorded in Table 5, and with this the writer agrees. Mr. Greulich cites an interesting example of group action in piles under the large foundations of pivot piers of swing bridges, where an earlier designer found he must restrict working loads to as low as 7½ to 8 tons per pile.

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[21] Cons. Engr. (Modjeski & Masters), Harrisburg, Pa.

Mr. Paxson gives a graphic comparison of pressures beneath pile foundations as compared with pressures beneath spread foundations, showing the considerably greater pressure under the spread foundation, and giving emphasis as well to the fact that this greater pressure is at a point near the surface where soil conditions are generally poorer than near the elevation of pile tips.

Professor Mindlin states that a solution of the equations of elasticity, known as the Boussinesq solution, was used in the writer's paper, and that the integrated Boussinesq formula is not applicable to pile foundations. It is not to be inferred that, because the writer used the Boussinesq solution, he accepts the concept of elasticity in soils. The Boussinesq solution was used because it appeared to fit satisfactorily the results of tests to failure better than did any other formula. Nowhere in the writer's paper was there any mention of the word "elasticity"; this was a deliberate omission because it is not believed that soils are perfectly elastic. This is especially true in connection with the tests given in the paper which were carried to failure, well beyond any possible range of elasticity in the soil.

Professor Mindlin states that it is not possible to compute a curve of Mindlin distribution for a uniform load per linear foot from the data given in Table 10. The method used in the preparation of Fig. 11 was as follows: Part (*a*) of Table 10 gives the pressures at various radii from the pile for various depths. These data permit a computation of the total vertical force on a circle of 0.1 radius, 0.2 radius, 0.3 radius, 0.5 radius, etc., at such depths. For a pile transmitting a uniform load per linear foot to the soil, the total load distributed from the top to 0.25 L depth is $\frac{1}{4}P$, and to 0.50 L depth is $\frac{1}{2}P$. If, from these distributed loads, there are subtracted the total vertical forces acting on the circles of various radii already cited, the difference in each case is the total shear carried by the cylindrical surface of the cylinder of earth from the top of pile to the depth 0.25 L or 0.50 L. For each radius a curve of total shear may be drawn, starting from 0 at the top to a calculated value at 0.25 L, to a larger calculated value at 0.50 L. The slope of this curve at any point is the shear transmitted by a circumferential band one unit high. If this slope is divided by $2\pi r$, it will give the shearing intensity per square unit. The curve of Mindlin distribution in Fig. 11 was based on the slope of the various curves at midheight of the pile. Any error that exists as a result of the graphical method used is believed to be slight.

Professor Mindlin questions the principle of superposition applied to pile groups. Of course, there are many unusual stresses created between piles in a group due both to the existence of the other piles and to soil stresses created during driving. Nevertheless, there seems to be no reasonable way of determining the build-up of stresses under pile loads unless loads from various piles are superimposed upon one another.

Professor Mindlin states that the real problem is to determine the distribution along the pile of the load transfer from pile to soil. There are certain practical ways of arriving at this solution. When a timber pile is started in the ground and the hammer is placed upon it, it will often settle into the soil a distance of several feet. In the case of the Morganza piles, this distance of embedment into the soil under the weight of the hammer used was such as to suggest that the shearing strength of the soil near the surface was reasonably close to the average shearing strength at failure of the pile. As was emphasized in the paper, when some allowance is made for the comparatively larger diameter of the butt as compared with the tip of a pile, and when there is reason for the belief that the shearing strength of soil increases slightly with depth, it is reasonable to arrive at such a value of R_c as was used in the paper. Incidentally, such practical data as these, secured during driving operations, may often give a clue as to the variations in material from ground to tip of pile. In a recent study undertaken in the writer's office, dealing with a pile foundation at another locality, it was discovered that this sinking of the pile under the weight of a 9,700-lb hammer amounted to 15 ft, indicating a shearing strength of soil at the surface of the ground of only 400 lb per sq ft, whereas the entire pile failed under an average skin friction of 975 lb per sq ft. This indicated a considerable variation in strength of material and suggested a method of arriving at a value of R_c for the pile group in question.

Mr. Solomon demonstrates mathematically the effect of the elasticity of the pile upon R_c and proves that for small pile loads the pile tip actually may deliver no load to the soil. Since the shearing stresses computed in the paper were primarily related to failure loads on the piles, the value of R_c used must be considered as applying particularly to such failure loads. Mr. Solomon also suggests that some of the difference in skin friction value between concrete and timber piles may be due to the effect of weight of the concrete piles as compared with the weight of timber piles. All of the larger concrete piles were hollow and had a weight no greater than the weight of the earth which they displaced. If a cylinder of earth is visualized as standing in soil, there will be no vertical shear along its vertical surfaces except that created by an applied load on top of the cylinder. For that reason, the weights of the piles themselves were uniformly and properly neglected in computing skin frictions under loads.

Mr. Cummings declares that the Boussinesq solution is not applicable to piles in an elastic soil, and shows that a combination of piles and soil is not isotropic. He restates the effect of contraction of the pile upon the distribution of load to the soil, and questions the equality of loads on the various piles of a group in the Morganza tests. Mr. Cummings evidently feels that not enough measurements were made and that the Morganza tests were not up to the experimental standard established by Galileo. In that connection, although the writer has heard many suppositions as to the effect of grouping upon pile strength, he knows of no previous tests of the scope of the Morganza tests to establish by measurement what this grouping effect is at failure. The writer also wishes to restate a fact which appeared in the paper—that these tests were originally intended to satisfy the requirements of current specifications, and that certain failures of small groups suggested that the tests be extended to larger groups. The writer is indebted to the Corps of Engineers, U. S. Army, for their cooperation in making possible these supplementary failure tests on large groups. The theory was then developed, using that method of distribution which would check the tests to failure and, by such a practical check, the method served to cancel the numerous variables which are cited by Mr. Cummings.

Mr. Cummings also objects to the use of average value of shear strength over the several crossings in cases where single pile tests were not close at hand. This was done because it seemed the most reasonable approach, since funds had not been available to make a single pile test close to each group test. However, after the paper was published, the N. O. T. & M. Railway, at its own expense, made a failure test on a single 50-ft pile immediately adjacent to the 16-pile group, T37–T52, and obtained a failure shear value of 735 lb per sq ft at this location. The 16-pile group used 60-ft piles, which might be expected, due to their increased depth, to develop somewhere between this value of 735 and the average value of 778 lb per sq ft used in the study.

In conclusion, the writer hopes that opportunity may arise to study the effect of grouping by all of the methods proposed, both in the discussions and in the paper, and urges that every effort should be made to relate all theories developed to actual load tests in the field.

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TRANSACTIONS

Paper No. 2175

ALLOCATION OF THE TENNESSEE VALLEY AUTHORITY PROJECTS

By Theodore B. Parker,[1] M. Am. Soc. C. E.

With Discussion by Messrs. Malcolm Elliott, E. L. Chandler, Eugene L. Grant, Sherman M. Woodward, Albert R. Arledge, C. W. Watchorn and F. A. Allner, and Theodore B. Parker.

Synopsis

The various theories applicable to the allocation of costs of a multiple-purpose water control project, and the procedure followed by the Tennessee Valley Authority in allocating the cost of its first three dams to navigation, flood control, and power, are presented in this paper. It is shown that a large difference in the allocation of the common costs of a project affects to a much lesser degree the proportions of the total cost charged to the individual purposes. Consequently, the final allocations are approximately correct, within reasonable limits, regardless of the exact method adopted.

Requirements of the Act

The allocation problem of the Tennessee Valley Authority (TVA) was necessitated by Sections 9*a* and 14 (see Reference 1 in the Appendix) of the Tennessee Valley Authority Act as amended. It is to be noted at the outset that Section 9*a* specifically provides for the generation of power only in so far as it is consistent with the primary purposes of navigation and flood control and that the proceeds of the sale of power are "to assist in liquidating the cost or aid in the maintenance of the projects of the Authority."

The location and the minimum pool elevations of the Authority's dams on the main Tennessee River (Fig. 1) were determined by navigation requirements, whereas the surcharge pool elevations were determined by the maximum amount of flood control storage space consistent with a minimum damage to the cities and agricultural areas. Dams on the tributary rivers were chosen principally with regard to flood control purposes. All of these dams operating together as a unified power system, in so far as possible after satisfying the primary purposes of navigation and flood control, produce a large amount of power.

Note.—Published in December, 1941, *Proceedings*.

[1] Chf. Engr., TVA, Knoxville, Tenn.

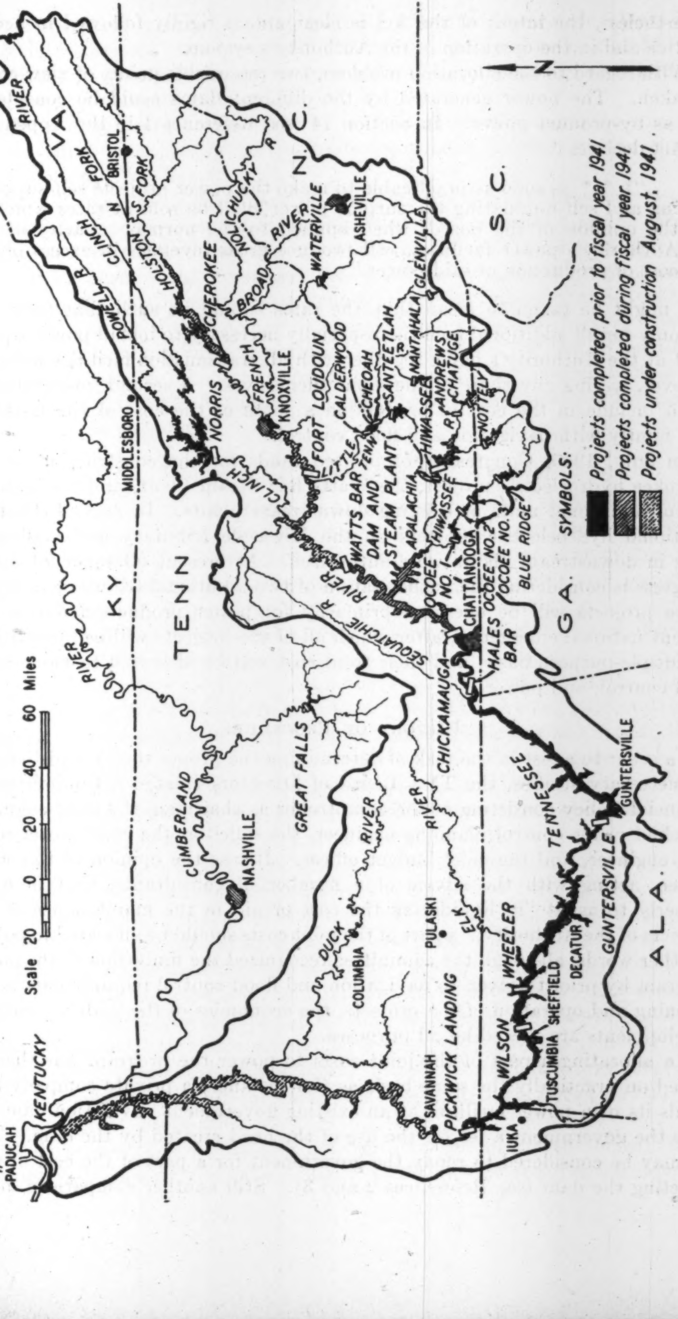
Scale
20 0 20 40 60 Miles
KY
TENN
VA
N.C.
S.C.
GA.
ALA.
N
SYMBOLS:
Projects completed prior to fiscal year 1941
Projects completed during fiscal year 1941
Projects under construction August, 1941
PADUCAH
KENTUCKY
CUMBERLAND
NASHVILLE
RIVER
GREAT FALLS
MIDDLESBORO
BRISTOL
KNOXVILLE
CHATTANOOGA
ASHEVILLE
SAVANNAH
PICKWICK LANDING
IUKA
TUSCUMBIA
SHEFFIELD
WILSON
WHEELER
DECATUR
GUNTERSVILLE
GUNTERSVILLE
TENNESSEE
CHICKAMAUGA
HALES BAR
OCOEE NO.2
OCOEE NO.1
OCOEE NO.3
BLUE RIDGE
APALACHIA
HIWASSEE
HIWASSEE R.
CHATUGE
NOTTELY
ANDREWS
NANTAHALA
GLENVILLE
SANTEETLAH
CHEOAH
CALDERWOOD
FORT LOUDOUN
LITTLE TENN.
WATTS BAR DAM AND STEAM PLANT
SEQUATCHIE R.
RIVER
CLINCH
NORRIS
FRENCH BROAD RIVER
WATERVILLE
NOLICHUCKY R.
CHEROKEE
HOLSTON
S. FORK
N. FORK
CLINCH
POWELL R.
RIVER
DUCK
RIVER
COLUMBIA
PULASKI
ELK RIVER

FIG. 1.—TENNESSEE RIVER VALLEY

Nevertheless, the intent of the Act is clear and is rigidly followed in actual practice and in the operation of the Authority's system.

With regard to the allocation problem, two reasonable points of view could be taken. The power generated by the different dams could be considered only as by-product power. In Section 14 (see Reference 1 in the Appendix) the Act declares

> "* * * as soon as practicable to make the power projects self-supporting and self-liquidating the surplus power shall be sold at rates which in the opinion of the board, when applied to the normal capacity of the Authority's power facilities, will produce gross revenues in excess of the cost of production of said power."

This might be taken to mean that the rates should be such that they will pay only for all additional facilities especially necessary to install power equipment at the Authority's dams together with all transmission facilities without, however, paying any share of the dams themselves. A second interpretation would include in the cost of production a share of the cost of the facilities used jointly with navigation and flood control.

In July, 1940, Congress appropriated funds for the construction of the Cherokee hydroelectric project, the Watts Bar Steam Plant, and the installation of additional units at existing downstream plants. In July, 1941, four additional hydroelectric projects on the Hiwassee tributary and additional units in downstream plants were authorized. At present (October 20, 1941) Congress is considering the authorization of two additional tributary projects. These projects will be operated primarily for power production during the present national emergency, after which all of the projects will be operated on a multiple-purpose basis, and their total cost will be allocated to navigation, flood control, and power.

History of Procedure

In order to assist in the task of determining the proper theory and to make the necessary studies, the TVA Board of Directors created a Committee on Financial Policy consisting of the comptroller as chairman, the chief engineer, the chief water control planning engineer, the solicitor, the chief power planning engineer, and the chief budget officer. It was the opinion of this committee, acting with the advice of a number of consultants, that in order properly to assist "in liquidating the cost or aid in the maintenance of the projects of the Authority," a part of the joint costs should be allocated to power. In other words, although the committee recognized the limitation of the power program by priority given to navigation and flood control requirements in the planning and operation of the projects, the economies of the multiple-purpose developments are shared by all purposes.

In allocating a part of the joint costs to power the program was thereby placed on practically the same basis as (for instance) a private company that builds its own power facilities at an existing government navigation dam and pays the government a fee for the use of the head created by the dam. This fee may be considered to repay the government for a part of the cost of constructing the dam (see References 2 and 3). Still another comparison might be made with a private company building a dam and power development on a navigable stream, the government building a navigation lock at the same location and granting the company a reduction or waiver of the license fee which would normally be required.

Such agreements between the U. S. Government and private companies are in effect examples of cost allocations, similar in character to the cost allocations at TVA dams.

At all times, the TVA Board of Directors was kept informed of the committee's work, and after the first "Report of the Investment in the Wheeler, Norris and Wilson Projects and the Allocation of the Investment Among the Several Purposes of the Tennessee Valley Authority Act," the committee submitted to the Board each year a report on the allocation of the Authority's investment in the system completed during the previous fiscal year. The first report, which was published as House Document No. 709, 75th Congress, 3d Session, included a full description of the committee's considerations leading to the allocation of the Authority's investment in the multiple-purpose projects to the various purposes.

THEORIES OF JOINT-COST ALLOCATIONS

The various theories of joint-cost allocations described in House Document No. 709 are summarized herein. It should be observed that allocations of costs are concerned principally with the proper apportionment of the cost of the structures and facilities which are used jointly for the various purposes of the development. Expenditures made solely for any single purpose are to be charged directly to that purpose. The following methods of allocation were considered by the committee.

Vendibility Theory.—Under competitive open market conditions, products jointly produced will generally be sold at prices totaling the joint cost of production plus a reasonable profit. Apportionment of costs will depend largely upon the demand for the products separately. This theory breaks down because there is no open market in which the services produced by the Authority under conditions of joint supply càn be sold. Navigation and flood control, under present conditions, are not vendible commodities.

Benefit Theory.—This theory advocates apportionment of joint costs on the basis of estimated benefits derived from the completed development.

The difficulties peculiar to this theory are principally those of evaluation of the benefits. Beneficial effects of any one project are multitudinous. Some of these are gains to individual property owners, reduction in costs to consumers, profits to merchants and industrialists, advantages of a social and governmental character; some are specific, others general in nature; some are easily evaluated, others are intangible; some are local, others remote from the Valley; some are immediately effective, others will benefit future generations. House Document No. 709 presents this theory in some detail and furnishes much pertinent information.

It is necessary to distinguish between two versions of the benefit theory. According to one version, the benefits would include the over-all advantage to the entire country from the projects constructed by the Authority. According to the other version, the benefits would be measured by the alternative cost of obtaining equivalent results by single-use projects, provided the advantages to the country can be shown to be sufficient to justify the expenditure necessary to construct such a single-use project. This second interpretation is probably the most workable of any of the methods that have been considered and is discussed in House Document No. 709 under the name of "alternative-justifiable-expenditure theory." A similar method was suggested for allocating joint costs between navigation and power for the St. Lawrence River project on the international section of that stream, and has also been used as a basis for allocating joint costs of water utilities between commercial service and fire-protection service.

Use of Facilities Theory.—This theory would distribute joint costs upon the basis of comparative use of the joint facilities. To each single function would be allocated such share of the joint cost as is measured by the extent of its use.

In the steam railroad field, for example, the apportionment of common cost between freight, mail, express, and passenger business may be made upon some comparable use units like car-mile, passenger-mile, or ton-mile measure.

When applied to the TVA problem, the theory becomes very involved; an acre-foot of reservoir capacity or an acre-foot of water released has been suggested as the common unit; but when considering power, head is an essential feature, whereas for flood control, head is not an essential factor. It is possible, however, that the theory might be usable after sufficient records have been collected over a considerable period of time regarding the actual use which has been made of the facilities. For the immediate future, the theory appears not to be workable.

Equal Apportionment.—This may be considered as a common-sense rule of equity to be used when it is felt that no truly scientific basis of apportionment can be found. However, such a rule does not seem practicable where the respective uses for each function are not equal.

Special Costs.—This theory suggests the apportionment of the costs to the several uses in direct proportion to the special expenditures made for each use. The procedure appears irrelevant where the object of the allocation is to impute responsibility for joint costs.

Alternative Justifiable Expenditure.—This theory proposes to divide the joint costs in proportion to the alternative justifiable expenditure less the direct cost for each function. By constructing projects which serve multiple uses, savings may be achieved in expenditures over those necessary for single-use projects.

According to the theory (see Reference 4 in the Appendix), the direct cost for any one purpose corresponds to the investment that could have been eliminated from the total project cost if that purpose had not been included in the development of the project. The alternative justifiable expenditure for any one purpose is the lowest cost of realizing an equal benefit to that obtained in the multiple-use project by a development undertaken solely for that purpose provided such expenditure is justified by the benefits obtainable.

Table 1 illustrates the application of this theory to the following simple hypothetical case—the assumed allocation of $1,000,000:

Assumed Direct Costs:		
Navigation	$ 150,000	
Flood control	100,000	
Power	250,000	
Total		$ 500,000
Common costs		500,000
Total project cost		$1,000,000
Assumed Alternative Costs:		
Navigation	$ 400,000	
Flood control	300,000	
Power	600,000	
Total	$1,300,000	

A conclusion as to the various allocation theories considered is given in the Appendix (see Reference 5).

TABLE 1.—ASSUMED ALLOCATION OF $1,000,000

Facility (1)	Alternative costs (2)	Direct costs (3)	Remaining costs; Col. 2 minus Col. 3 (4)	Percentage of total (5)	Allocation of common costs (6)	Direct costs added; Col. 3 plus Col. 6 (7)	Percentage of total (8)
Navigation	$400,000	$150,000	$250,000	31.3	$156,000	$306,000	30.6
Flood control	300,000	100,000	200,000	25.0	125,000	225,000	22.5
Power	600,000	250,000	350,000	43.7	219,000	469,000	46.9
Total	$1,300,000	$500,000	$800,000	100.0	$500,000	$1,000,000	100.0

In Table 2, Items 1 to 21 comprise a summary of the trial computations based on the alternative-justifiable-expenditure method for the successive steps in the TVA program of construction up to the close of fiscal year 1940 and including the Wilson, Wheeler, Norris, Pickwick Landing, Chickamauga, Guntersville, and Hiwassee developments. In Table 2, Items 22 to 34 are a summary of the final allocations for this same program as recommended by the Financial Policy Committee. It will be seen that the latter vary from the trial computations in Table 2, Items 1 to 21, by relatively small sums. This variation represents the factor of judgment that entered into the committee's consideration of the problem in making their final determination, as is explained subsequently. (Costs used in the three-plant allocation exclude $4,774,905 reserved for Wilson Dam depreciation from date of completion to date of acquisition by the Authority. In all subsequent allocations, total costs have been used—reconstruction cost new at Wilson—and depreciation reserves at all projects have been set up in a special account, independent of allocations.)

TABLE 2.—System Allocations

No.	Facility	Three-Plant System, Including Norris, Wheeler, and Wilson Dams		Four-Plant System, Including Pickwick Landing Dam		Five-Plant System, Including Guntersville Dam		Six-Plant System, Including Chickamauga Dam		Seven-Plant System, Including Hiwassee Dam	
		Dollars	%	Dollars	%	Dollars	%	Dollars	%	Dollars	%
1	Total investment allocated	98,900,576	....	127,300,982	....	158,584,469	....	193,297,469	....	210,279,469	....
	Alternative Costs:										
2	Navigation	48,334,000	....	64,006,000	....	85,510,000	....	105,160,000	....	106,160,000	....
3	Flood control	33,210,000	....	38,447,000	....	44,435,000	....	52,310,000	....	63,260,000	....
4	Power	79,230,000	....	100,689,000	....	115,236,000	....	138,894,000	....	150,055,000	....
5	Total	160,774,000	....	203,142,000	....	245,181,000	....	296,364,000	....	319,475,000	....
	Direct Costs:										
6	Navigation	4,408,807	....	9,709,037	....	12,500,024	....	17,177,896	....	17,177,896	....
7	Flood control	2,600,000	....	3,621,000	....	3,621,000	....	4,661,000	....	4,871,000	....
8	Power	26,059,335	....	36,020,676	....	44,218,468	....	53,114,788	....	56,368,996	....
9	Total	33,068,142	....	49,350,713	....	60,339,492	....	74,953,684	....	78,417,892	....
	Remaining Alternative Costs:										
10	Navigation	43,925,193	34.4	54,296,963	35.3	73,009,976	39.5	87,982,104	39.7	88,982,104	36.9
11	Flood control	30,610,000	24.0	34,826,000	22.6	40,814,000	22.1	47,649,000	21.5	58,389,000	24.2
12	Power	53,170,665	41.6	64,668,324	42.1	71,017,532	38.4	85,779,212	38.8	93,686,004	38.9
13	Total	127,705,858	100.0	153,791,287	100.0	184,841,508	100.0	221,410,316	100.0	241,057,108	100.0
	Allocation of Common Costs:										
14	Navigation	22,643,459	34.4	27,516,445	35.3	38,806,766	39.5	46,982,483	39.7	48,656,922	36.9
15	Flood control	15,779,470	24.0	17,616,761	22.6	21,712,140	22.1	25,443,914	21.5	31,910,502	24.2
16	Power	27,409,505	41.6	32,817,063	42.1	37,726,071	38.4	45,917,388	38.8	51,294,153	38.9
17	Total	65,832,434	100.0	77,950,269	100.0	98,244,977	100.0	118,343,785	100.0	131,861,577	100.0
	Allocation of Total Costs:										
18	Navigation	27,052,266	27.3	37,225,482	29.2	51,306,790	32.3	64,160,379	33.2	65,834,818	31.3
19	Flood control	18,379,470	18.6	21,237,761	16.7	25,333,140	16.0	30,104,914	15.6	36,781,502	17.5
20	Power	53,468,840	54.1	68,837,739	54.1	81,944,539	51.7	99,032,176	51.2	107,663,149	51.2
21	Total	98,900,576	100.0	127,300,982	100.0	158,584,469	100.0	193,297,469	100.0	210,279,469	100.0
22	Total investment allocated	94,125,671	....	127,300,982	....	158,584,469	....	193,297,469	....	210,279,469	....
	Direct Costs:										
23	Navigation	4,075,988	....	9,709,037	....	12,500,024	....	17,177,896	....	17,177,896	....
24	Flood control	2,600,000	....	3,621,000	....	3,621,000	....	4,661,000	....	4,871,000	....
25	Power	23,967,177	....	36,020,676	....	44,218,468	....	53,114,788	....	56,368,996	....
26	Total	30,643,165	....	49,350,713	....	60,339,492	....	74,953,684	....	78,417,892	....
	Allocation of Common Costs:										
27	Navigation	22,218,877	35.0	28,062,097	36.0	38,315,541	39.0	46,154,076	39.0	47,470,168	36.0
28	Flood control	15,870,627	25.0	18,708,064	24.0	20,631,445	21.0	24,852,195	21.0	31,646,778	24.0
29	Power	25,393,002	40.0	31,180,108	40.0	39,297,991	40.0	47,337,514	40.0	52,744,631	40.0
30	Total	63,482,506	100.0	77,950,269	100.0	98,244,977	100.0	118,343,785	100.0	131,861,577	100.0
	Allocation of Total Costs:										
31	Navigation	26,294,865	27.9	37,771,134	29.7	50,815,565	32.0	63,331,972	32.7	64,648,064	30.7
32	Flood control	18,470,627	19.6	22,329,064	17.5	24,252,445	15.3	29,513,195	15.3	36,517,778	17.4
33	Power	49,360,179	52.5	67,200,784	52.8	83,516,459	52.7	100,452,302	52.0	109,113,627	51.9
34	Total	94,125,671	100.0	127,300,982	100.0	158,584,469	100.0	193,297,469	100.0	210,279,469	100.0

COMMENT

The allocation of the costs of the multiple-purpose projects is required by the TVA Act, which states that the cost of such projects shall be charged to flood control, navigation, fertilizer, national defense, and the development of power. It also requires that the findings of the allocation shall be used thereafter for the purpose of keeping the book value of these properties. Although not explicitly stated, it may be assumed that the reason Congress required such allocation was to determine equitable rates for the sale of power. This assumption is further borne out by the language of the same section of the Act concerning the keeping of accounts, which states that

> "* * * for the purpose of accumulating data useful to the Congress in the formulation of legislative policy in matters relating to the generation, transmission, and distribution of electric energy * * * the Board shall keep complete accounts of its cost of generation, transmission, and distribution of electric energy."

Although the allocations made thus far have not as yet been used in the actual setting of power rates, tentative allocations were used by the engineers of the Joint Investigating Committee, appointed by Congress, in their study of the power rates actually in effect. They found that these rates would produce revenues sufficient to retire more than all of the cost of generation, transmission, and distribution of power, including interest. In fact, without such an allocation of costs, it would have been nearly impossible for this study to have been made.[2]

The various theories of joint-cost allocation considered by the TVA Financial Policy Committee have been previously described herein. Arguments for and against such theories are of academic interest, since the real difficulties arise in their application.

Such difficulties constitute the real objection to the direct use of the benefit theory, so-called. This method appeals to engineers as being reasonable and equitable. It has long been used in assessing costs of irrigation, flood control, and land drainage, where the costs of the projects themselves are paid for by taxation on the real property served or protected.

A little reflection will convince any experienced engineer that the value of the benefits from the various components of the Authority's program cannot be estimated with sufficient accuracy to serve as the basis for accounting procedures. These benefits are very real and very considerable, but by their very nature include a wide variety of both tangible and intangible items that cannot be evaluated with any pretense of accuracy.

There is no real justification for basing the allocation of costs for accounting purposes upon any one mathematical formula. The principal method adopted by the Authority serves more as an aid to judgment than as a formula for the accurate determination of definite values.

By this method the common costs are allocated in proportion to the remaining costs resulting from the subtraction of direct costs from the total estimated alternative costs. This amounts to assuming that the savings from multiple-purpose development are shared in by the various purposes in this same proportion. It is also equivalent to allocating the common cost in proportion to the cost of constructing separately those facilities which in multiple-purpose development are provided by the jointly-used portions of the project.

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[2] "Engineers' Report of the Joint Committee Investigating the Tennessee Valley Authority," S. Doc. No. 56, Pt. III, 76 Cong., 1st Session.

The exactness of such a method from the mathematical point of view depends upon the accuracy with which it is possible to estimate the cost of hypothetical "alternative" single-purpose developments. These estimates from the accounting standpoint have practically no accuracy at all, since they must depend upon broad assumptions regarding the design of such developments.

Nevertheless, these computations are a most valuable aid to the judgment in arriving at final allocations of cost, since they show the trends of different items, and the effect of the different and variable factors.

In the end, the actual allocation of costs must be an arbitrary and empirical process, and the writer can see no reason why this should not be recognized frankly. Since no one method has received any official or judicial approval, it is entirely reasonable to use any or all methods when these are appropriate and equitable.

Limits of Variation

It is pertinent to examine, at this point, the limits of variation in the final allocations as affected by the application of the theory. In the cost allocations the direct costs of the various purposes can be readily ascertained and are not subject to variations due to possible inaccuracies of estimates. The allocations based on the alternative-justifiable-expenditure theory, however, are affected by possible inaccuracies in the estimates of alternative costs. These possible inaccuracies have been held to a minimum in so far as possible in the allocations made thus far, since the costs which were used for alternatives were based largely on the costs of the multiple-purpose projects with comparatively minor alterations. Moreover, a comparatively large variation in the alternative costs would result in a minor variation in the final allocations as is demonstrated by Table 3, which shows the effect of variations made in the seven-plant allocation based on the alternative-justifiable-expenditure theory.

TABLE 3.—Relative Effect of Changes in Alternative Costs (Percentages)

Facility	Reduction in cost of one-purpose system[a]	Corresponding Change in the Allocation of Costs to Power	
		Common costs	Total costs
Navigation	10	+1.8	+1.1
Flood control	10	+1.1	+0.6
Power	10	−4.1	−2.6

[a] Other systems remain the same.

Viewing this problem in its broadest sense, it is evident that no allocation can be determined with mathematical precision. On the other hand, there are reasonable limits to the variations in allocations of common cost.

Where three major purposes are involved, it would be an obvious absurdity to charge more than half of the common cost to power alone, especially where power production is designated by congressional legislation as incidental to the other two purposes. On the other hand, where power is the only "paying partner," it is difficult to justify charging to power less than one third of the common costs in any event.

The TVA allocation of common costs to power has been set at 40% for the three-plant to seven-plant systems, resulting in an average allocation to power of the total costs of about 52%. Were the allocation of common costs increased from 40% to 50%, this would increase the total to only 58% whereas a decrease in allocation of the common costs from 40% to 33⅓% would reduce the total to only 48%.

The extreme reasonable range of the total allocation to power, regardless of the method adopted, therefore, is from 48% to 58%. Considering the difficulty of the problem and the wide divergence of opinion expressed in this connection, this is distinctly reassuring.

OTHER SIMILAR ALLOCATIONS

Bonneville Project.—Reference 6, in the Appendix, contains material quoted from the Federal Power Commission's release No. 380 with reference to the allocation of costs for the Bonneville project.

Boulder Power Plant.—The Swing-Johnson Bill, which provided for construction of Boulder project, required that before construction of the dam was started contracts should be engaged for sale of power at a price sufficient to repay the entire cost of the construction of the dam, together with interest at 4%, in a 50-yr period. The actual contracts as entered into with the various lessees provided that each shall pay a certain price for use of falling water at a dam, and that the cost of the power installation shall also be paid by these lessees at a rate sufficient to repay the cost of the power plant in 25 years, together with interest. These contracts are now in force and apparently the cost of the dam will be repaid as provided in the Act.

However, the City of Los Angeles, Calif., has endeavored to obtain a modification of the prices paid for power whereby a substantial allocation would be made to flood control, the cost of power equipment would be repaid in 50 years instead of 25, and the rate of interest on the investment would be reduced from 4% to 3%. This bill was passed by Congress in July, 1940.

CONCLUSIONS

The TVA allocation of costs may be summarized as follows:

1. The allocation was made as directed by Act of Congress;
2. Allocation is not based exactly on any one mathematical calculation or formula, the final sums being fixed by judgment and not by computation;
3. Items used solely for a particular purpose are charged to that purpose alone; it is therefore only the remaining "common costs" which require allocation;
4. A considerable variation in the allocation of common costs affects, to a much lesser degree, the proportions of the total cost charged to the three purposes; and
5. Because only the allocation of common costs is at all open to question, the possible variation in the totals due to different methods of allocation is

comparatively small. It can be shown, therefore, that the final allocations are approximately correct, within reasonable limits, regardless of the method adopted.

Most of the material used in the preparation of this paper, including estimates and similar data, was contributed by W. L. Voorduin, Assoc. M. Am. Soc. C. E., of the TVA engineering staff.

APPENDIX

DOCUMENT CITATIONS AND REFERENCES

Ref. 1.—Tennessee Valley Authority Act:

"Sec. 9*a*. The board is hereby directed in the operation of any dam or reservoir in its possession and control to regulate the stream flow primarily for the purposes of promoting navigation and controlling floods. So far as may be consistent with such purposes, the board is authorized to provide and operate facilities for the generation of electric energy at any such dam for the use of the Corporation and for the use of the United States or any agency thereof; and the board is further authorized, whenever an opportunity is afforded, to provide and operate facilities for the generation of electric energy in order to avoid the waste of water power, to transmit and market such power as in this act provided, and thereby, so far as may be practicable, to assist in liquidating the cost or aid in the maintenance of the projects of the Authority.

"Sec. 14. The board shall make a thorough investigation as to the present value of Dam Numbered 2, and the steam plants at nitrate plant numbered 1, and nitrate plant numbered 2, and as to the cost of Cove Creek Dam, for the purpose of ascertaining how much of the value or the cost of said properties shall be allocated and charged up to (1) flood control, (2) navigation, (3) fertilizer, (4) national defense, and (5) the development of power. The findings thus made by the board, when approved by the President of the United States, shall be final, and such findings shall thereafter be used in all allocations of value for the purpose of keeping the book value of said properties. In like manner, the cost and book value of any dams, steam plants, or other similar improvements hereafter constructed and turned over to said board for the purpose of control and management shall be ascertained and allocated.

"The board shall, on or before January 1, 1937, file with Congress a statement of its allocation of the value of such properties turned over to said Board and which have been completed prior to the end of the preceding fiscal year, and shall thereafter in its annual report to Congress file a statement of its allocation of the value of such properties as have been completed during the preceding fiscal year.

"For the purpose of accumulating data useful to the Congress in the formulation of legislative policy in matters relating to the generation, transmission, and distribution of electric energy and the production of chemicals necessary to national defense and useful in agriculture, and to the Federal Power Commission and other Federal and State agencies, and to the public, the Board shall keep complete accounts of its costs of generation, transmission, and distribution of electric energy and shall keep a complete account of the total cost of generating and transmission facilities constructed or otherwise acquired by the Corporation, and of producing such chemicals, and a description of the major components of such costs according to such uniform system of accounting for public utilities as the Federal Power Commission has, and if it have none, then it is hereby empowered and directed to prescribe such uniform system of accounting, together with records of such other physical data and operating statistics of the Authority as may be helpful in determining the actual cost and value of services, and the practices, methods, facilities, equipment, appliances, and standards and sizes, types, location, and geographical and economic integration of plants and systems best suited to promote the public interest, efficiency, and the wider and more economical use of electric energy. Such data shall be reported to the Congress by the board from time to time with appropriate analyses and recommendations, and, so far as practicable, shall be made available to the Federal Power Commission and other Federal and State agencies which may be concerned with the administration of legislation relating to the generation, transmission, or distribution of electric energy and chemicals useful to agriculture. It is hereby declared to be the policy of this Act, that, in order, as soon as practicable, to make the power projects self-supporting and self-liquidating, the surplus power shall be sold at rates which, in the opinion of the board, when applied to the normal capacity of the Authority's power facilities, will produce gross revenues in excess of the cost of production of said power and in addition to the statement of the cost of power at each power station as required by section 9 (*a*) of the 'Tennessee Valley Act of 1933,' the board shall file with each annual report a statement of the total cost of all power generated by it at all power stations during each year, the average cost of such power per kilowatt-hour, the rates at which sold, and to whom sold, and copies of all contracts for the sale of power."

Ref. 2.—"Rules of Practice and Regulations" of the Federal Power Commission (dated June 1, 1938), Section 11.22:

"Reasonable annual charges for recompensing the United States for the use of Government dams or other structures owned by the United States and for the use, occupancy, and enjoyment of the lands of the United States adjoining and pertaining thereto will be based upon the estimated value for power purposes of the properties and privileges for which a license is issued * * *."

Some examples of power facilities built by private companies subject to this provision are:

Company	River	Development
Henry Ford and Son, Inc.	Hudson	Green Island
Louisville Gas and Electric Company	Ohio	Dam No. 41
Ford Motor Company	Mississippi	Twin Cities
Kanawha Valley Power Company	Kanawha	Marnet, London, and Dam No. 1
Kentucky Utilities Company	Kentucky	Lock No. 7

Ref. 3.—The Third Annual Report of the Federal Power Commission (for the fiscal year ended June 30, 1923) states regarding Project No. 362 (Twin Cities Development), Section 7, p. 256:

"To recompense the United States for the use of said dam and appurtenant structures and lands adjoining and pertaining thereto, the license to pay as a reasonable annual charge the sum of $95,440 said sum being interest * * * and depreciation * * * upon $1,193,000 an amount deemed to be so much of the cost of said dam, structures, and lands as would be justified if built or purchased solely for purposes of power development."

Ref. 4.—H. Doc. No. 709, pp. 19–20:

"The concept of alternative cost is premised on the theory that the expenditure in single-use projects would be justified by the benefits obtainable; hence, the alternative cost may be taken as a measure of the investment which the individual purposes would be justified in expending in a joint venture. Where such a cost would not have been justified, it is necessary to substitute a lower estimate based on evaluation of the benefits.

"Through the medium of alternative costs, a relative measure of value applicable to all purposes is utilized. In applying this method it is necessary to secure estimates of the lowest alternative cost by means of which substantially the same quantity and quality of service for each separate function can be obtained. The fundamental assumptions which underlie the cost estimates for single-use structures must be as reasonable and practical as they can be made and must be based upon experience and after adequate investigation."

Ref. 5.—H. Doc. No. 709, p. 20:

"After a review and extensive exploration of all the relevant methods, the committee has reached the conclusion that no one method furnishes the best general basis upon which to proceed to a final allocation that would reflect the various shares of the total joint costs to be imposed on the several functions of the Wilson, Norris, and Wheeler projects. The committee recognizes that each of the relevant methods has a measure of validity under some circumstances but that there are objections to or difficulties in the application of all the methods to the Authority's projects. Much time and effort were devoted to testing out the various hypotheses involved on the existing and proposed projects of the Authority. The allocations finally recommended are admittedly based on an exercise of judgment after considering all facts; nevertheless the committee is unanimous in believing that they furnish a fair basis for determining the relative shares of the various functions in the joint costs."

With respect to the three-plant allocation, the committee further decided that:

"The three completed projects represent a great national-defense asset, but because they are not in operation for that purpose during peacetime, it appears to be impracticable to attempt to allocate any portion of the investment to their wartime use, and the committee recommends that no part of the investment be assigned to that function. It is further recommended that no part of the joint investment be allocated to fertilizer, since fertilizer operations are being currently charged for services rendered by other departments of the Authority."

Ref. 6.—Release No. 380, Federal Power Commission:

"The initial power development at Bonneville, which will have installed electric generating capacity of 86,400 kilowatts or approximately one-fifth of the probable ultimate installation, is slated for completion by June 30, 1938, at a total estimated cost, including interest during construction, of $9,180,500 for facilities solely for power purposes. The Commission has allocated this entire sum to initial power development and, in addition, has allocated $2,501,900 of the total estimated cost to June 30, 1938, of facilities having joint value for the purposes of navigation and power development.

"The $2,501,900 figure was obtained through the determination that, ultimately, power development may fairly bear 32.5 percent of the cost of facilities having joint value for the production of electric energy and other purposes. The total estimated cost of such facilities to June 30, 1938, including fishways, is $38,490,700 and the present allocation to initial power development is one-fifth of 32.5 percent of this sum.

"The total cost of the Bonneville project when completed is estimated at $74,144,600. In setting this figure, the Commission has estimated the cost of ultimate power development with ten generating units having an aggregate capacity of 504,000 kilowatts, together with the cost of all at-site appurtenant facilities for power use only, to be $29,448,000; and the cost of facilities having joint value for navigation and power purposes, including fishways, to be $39,179,000. It is estimated that the cost of facilities for navigation purposes only will be $5,517,600 as of June 30, 1938, and no additional capital costs for facilities solely for navigation purposes are contemplated.

"On the basis of the Commission's present determination that ultimately power development may fairly bear 32.5 percent of the cost of facilities having joint value for the production of electric energy and other purposes, $42,181,000 or approximately 57 percent, of the total ultimate cost of the Bonneville project would be allocated to power development."

DISCUSSION

Malcolm Elliott,[3] M. Am. Soc. C. E.—The author has made a convincing presentation of the desirability of using the alternative-justifiable-expenditure theory as a basis for the allocation of costs of multiple-purpose projects.

It is assumed that this theory of cost allocation is applicable to preliminary studies of a project, made for the purpose of determining whether a project or any feature of it is economically warranted, as well as to the allocation of the known total cost after the project is completed. As far as preliminary estimates of costs, budgeting appropriated funds, and final accounting of expended funds are concerned, it appears that the alternative-justifiable-expenditure theory forms as good a guide as any that can be devised for determining what proportion of the total estimated or expended funds should be allocated to the various purposes. However, there is another use that must be made of the cost allocations of the TVA projects in which it is questionable whether the alternative-justifiable-expenditure theory is properly applicable.

The author has quoted from the TVA Act (see heading "Comment"):

> "* * * for the purpose of accumulating data useful to the Congress in the formulation of legislative policy in matters relating to the generation, transmission, and distribution of electric energy * * * the Board shall keep complete accounts of its cost of generation, transmission, and distribution of electric energy."

This language, the debates in Congress, statements in committee hearings, and arguments in lawsuits involving the TVA leave little doubt that the intent of Congress was in part to create a "yardstick" by which rates charged by private power companies could be measured. It goes without saying that the "yardstick" itself should be fairly measured; otherwise the measurements made with it will be unfair either to the public or to the private companies.

An important element in the cost of production of power is the cost of financing, including interest, the sale of securities, and amortization to extinguish the investment at the end of the useful life of the project. Naturally, these costs will vary with the amount of the investment. The cost of a dam built for power alone must be carried entirely by power revenues, whereas the cost of a dam built for power, navigation, and flood control is divided among these three purposes. Hence, the capital expenditures for power are likely to be much less in the case of a government-built, multiple-purpose project than in the case of a privately built, single-purpose power plant. In the example used in Table 1, the first cost of a single-purpose power plant is $600,000, whereas a plant of equal capacity constructed as a part of a multiple-purpose project is but $469,000, or about 78% of a single-purpose plant.

The private owner, of course, would base his rates on a first cost of $600,000; but the government, if it employs the alternative-justifiable-expenditure theory advocated by the author, will measure the private company rates with a "yardstick" based on a first cost of $469,000, and therefore might conclude that as much of the private rates as depends on financing and interest charges should be reduced by 22%. The foregoing is merely an assumed numerical example; but, on examining the cost figures as computed for actual projects (Table 2), even greater disparities are found between the cost of single-purpose (alternative) power plants and power features included in multiple-purpose plants. For example, considering the three-plant system in Table 2, the alternative power cost is $79,230,000, whereas the final allocation of the multiple-purpose cost to power is but $49,360,179—roughly 62½% of the alternative cost.

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[3] Col., Corps of Engrs., U. S. Army; Div. Engr., Upper Mississippi Val. Div., St. Louis, Mo.

Many people in the United States are favorable to production of power by the government on the theory that power is such a vital public necessity that its production should be at the lowest possible cost and not burdened with the earning of profits for private investors. If such theory were the established policy of the government, as determined by Congress, the government properly would take all possible advantage of its unique opportunity of combining power with public uses, such as navigation and flood control, in a way to reduce the first cost of power installations even though private companies were "run out of business." In the TVA Act, however, Congress has not adopted such a policy. The evident purpose of the Act is to make use of the power produced in connection with navigation and flood control works, to market the power in order to help defray the cost of such works, and to accumulate data useful to the Congress in the formulation of legislative policy relating to production and marketing of power.

It is the view of the writer that the allocation of costs based on the alternative-justifiable-expenditure theory, as described by the author, will fall short of furnishing Congress the data it will need for the formulation of policies unless it is clearly explained that the first costs of power installations thus computed are not for comparison with the necessary and reasonable costs that must be incurred by private companies for single-purpose power plants.

E. L. CHANDLER,[4] M. AM. SOC. C. E.—Proper allocation of the costs of multiple-purpose projects is a complicated problem, and the author has made a useful contribution to available data on the subject.

The alternative-justifiable-expenditure theory of allocation (see heading "Theories of Joint-Cost Allocations: Alternate Justifiable Expenditure) results in a solution that is applicable to most allocation problems. It appears to warrant more wholehearted approval than that accorded to it by the TVA Financial Policy Committee. In the introduction to its "Notes on Allocation," a supplement to the TVA formal report on cost allocations (H. Doc. No. 709, 75th Cong.), the Committee states that it "found most merit in the 'alternative-justifiable-expenditure' theory" and, in the last sentence of the paragraph on "Conclusion as to Allocation Theories," states that the "allocations finally recommended are admittedly based on an exercise of judgment after considering all facts."

A summary of allocations to date is given in Table 2. By comparing the percentages shown in items Nos. 18, 19, and 20 with corresponding values in items Nos. 31, 32, and 33, it will be noted that the exercise of judgment resulted in a change of 1.6% of the allocation to power in the three-plant system, as compared with the value determined by calculations based on the justifiable-expenditure formula. Changes in the allocations to navigation and flood control were 0.6% and 1.0%, respectively. The changes happen to have decreased step by step until, for the seven-plant system, there is an increase of only 0.7% of the allocation to power. It is scarcely reasonable to suppose that the accuracy of any allocation can be guaranteed within closer limits than these. Perhaps it is not exceeding the bounds of caution to say that the justifiable-expenditure method is the adopted method. There is merit in standardization if conditions warrant it.

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[4] With Charles T. Main, Inc., Wilmington, Del.

Added confidence in the merit of this method is found in the fact that "it held its own," from the early days of allocation study, against all of the proposed methods, some of which were ardently supported by their sponsors. It was first used under the title of "benefit theory," and tentative allocations for the first four projects were prepared in 1936 and 1937. The term "benefit theory" was distasteful to some who were concerned with development of the allocations, chiefly because they did not find it possible to arrive at measures of benefits that seemed satisfactory to them. Later, with some changes in terminology and many improvements in details of the procedure used in deriving the estimated justifiable expenditures, but with no change in the application of the formula, the method emerged under the name "alternative-justifiable-expenditure" theory. The writer considers this to be very satisfactory nomenclature.

The maximum total justifiable expenditure for any one of the objectives served by a multiple-purpose project would be measured by the total benefit (with respect to the purpose under consideration) that is derived from the development of the project. Assuming that the value of benefits for any given objective exceeds the cost of the least expensive single-purpose project that might be devised for accomplishing equivalent results, the expenditure necessary for such a single-purpose project may be considered as a lower limit of justifiable expenditure. If the expenditure necessary for such a single-purpose project would not be justified by results to be accomplished, then a lower value for justifiable expenditure should be substituted. It should be some measure of the benefits to result from the expenditure.

The most suitable values to be used as "justifiable expenditures" in any given allocation problem may be determined in any one of several ways, as circumstances dictate. Perhaps a suitable measure of total benefits accruing to each of the purposes may be feasible. It may be found that values based on use of facilities afford a suitable basis, etc. In any event, the values used for all of the objectives of the problem at hand should be maintained on as nearly an equivalent basis as possible. In the TVA allocation, estimates for the cost of alternative single-purpose projects, deemed capable of providing services equivalent to that furnished by the multiple-purpose project, have been used for the purpose.

It is obvious that the accuracy of final allocations under this method depends largely on the soundness of estimates for the alternative justifiable expenditures. Assuming the adequacy of those estimates, the steps taken in developing the TVA allocation give results within what are, probably, suitable limits of accuracy, although they are not absolutely correct. During the allocation studies, methods of derivation were developed that lead to mathematically accurate results for all steps after original estimates have been established.

Method of Determining Directly Segregable Portions of Multiple-Purpose Project Costs.—For any multiple-purpose project, probably certain parts of the expenditures will have been made solely in the interests of specific objectives such as flood control, navigation, or power generation. It also will be true that certain parts of the structures serve more than one purpose. The first step in the problem of allocation is a determination of the directly segregable portions of the cost. A method for determining those segregable amounts has been developed geometrically and arithmetically as indicated in Fig. 2. This diagram and the following computations are intended to establish principles that may be applied to the solution of any given allocation problem:

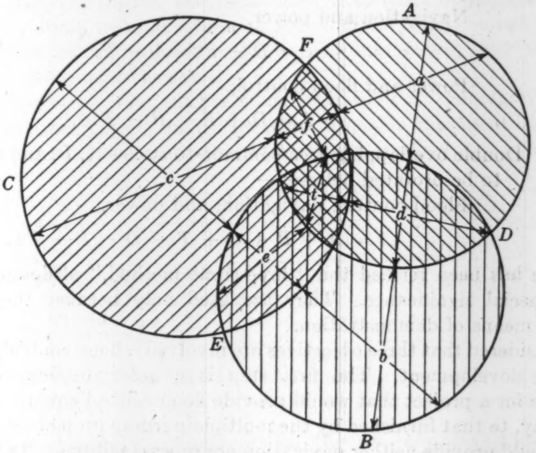

FIG. 2.—SEGREGATION OF DIRECTLY ALLOCABLE PARTS OF THE COST OF A MULTIPLE-PURPOSE PROJECT

Area (see Fig. 2)	Description
	Complete circle; total project cost if built for—
A	Flood control only
B	Navigation only
C	Power only
	Combined total project cost if built for—
D	Flood control and navigation (overlapping combination of areas *A* and *B*)
E	Navigation and power (overlapping combination of areas *B* and *C*)
F	Power and flood control (overlapping combination of areas *C* and *A*)
T	Flood control, navigation, and power (overlapping combination of areas *A*, *B*, and *C*; not indicated in Fig. 2)
	Part of circle not overlapped in combining the three circles is the cost directly allocated to—
a	Flood control,

$$a = T - E \qquad \text{(1a)}$$

b Navigation,

$$b = T - F \tag{1b}$$

c Power,

$$c = T - D \tag{1c}$$

Single overlapping area common to two circles is the cost to be prorated between—

d Flood control and navigation

$$d = E + F - C - T \tag{2a}$$

e Navigation and power,

$$e = D + F - A - T \tag{2b}$$

f Power and flood control,

$$f = D + E - B - T \tag{2c}$$

Double overlapping area common to circles *A*, *B*, and *C* is the cost to be prorated between—

t Flood control, navigation, and power,

$$t = A + B + C + T - D - E - F \tag{3}$$

This device has been termed the "three-circle method," although the circles have no special significance. They are used only because they furnish a convenient means of demonstration.

It is considered that three objectives are involved—flood control, navigation, and power development. The first step is a determination of justifiable expenditure for a project that would provide flood control equally satisfactory, in every way, to that furnished by the multiple-purpose project (area *A*, Fig. 2) but that would provide neither navigation nor power facilities. In like manner, the justifiable expenditure must be determined for a structure that would provide only for navigation equal to that furnished by the multiple-purpose project (area *B*, Fig. 2), and also for a single-purpose project (area *C*, Fig. 2) that would furnish the equivalent of power facilities included in the multiple-purpose development.

It is assumed that, if a dual-purpose project were built to provide for flood control and navigation, the resulting expenditure would be less than the combined sum represented by the total of areas *A* and *B*. Therefore, to represent such a dual-purpose project graphically, the two circles are shown overlapping in the diagram, and the combination is designated as area *D*, which is less than the total area *A* plus the total area *B* by the amount of overlapping (in this instance, indicated by the area *d* plus *t*). In the same way, if a dual-purpose project were to be built to serve the purposes of navigation and power, with no provision for flood control, the resulting expenditure is represented on the diagram by that part labeled *E*, which includes overlapping areas *B* and *C*. The net resulting sum in this case is less than the combined total of areas *B* and *C* by the amount of overlapping between the two areas, which is graphically represented by area *e* plus area *t*, Fig. 2. In the same way, a dual-purpose project built for purposes of flood control and power only is represented by the overlapping of area *A* and area *C*, the area being less than the combined total of *A* plus *C* by the amount of overlapping indicated by area *f* plus *t*. The three purposes are served at a cost materially less than would be the case if each of the three were to be gained by a single structure or if any combination of dual-purpose and single-purpose projects were developed.

Eqs. 1 to 3 will give the proper part of the total cost to be allocated directly to each of the three objectives; the part that is common solely to each combination of two objectives; and that part common to all three of the objectives. These values are fundamental, and allocation of the cost of the tri-purpose project is based on them.

In the case of a project involving more than three objectives, the method outlined would still be applicable. A fourth circle would be introduced and a larger number of simultaneous equations would result, but the principles involved would remain the same.

The discussion thus far has been purely theoretical. Segregation of the cost of a hypothetical project in accordance with the aforementioned theory is indicated arithmetically and diagrammatically in Fig. 3. The incremental costs are those expenditures made solely in the interest of single purposes. The first step in such a segregation is a determination of the justifiable expenditures for single-purpose projects that would have to be built to satisfy each one of the three objectives, and also for those projects that would serve each of the three dual combinations to an extent equal to the service rendered by the actual tri-purpose project. The following tabulation indicates the assumed cost of the project that would have resulted in each case:

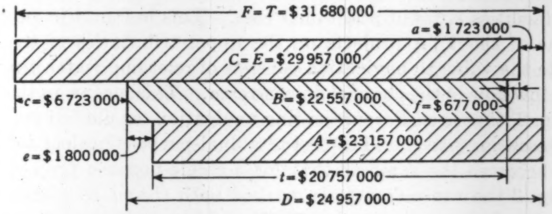

FIG. 3

Area (see Fig. 2)	Objective	Cost
A	Flood control only................	$23,157,000
B	Navigation only................	22,557,000
C	Power only......................	29,957,000
D	Flood control and navigation......	24,957,000
E	Navigation and power...........	29,957,000
F	Flood control and power..........	31,680,000
T	Total..........................	31,680,000

For the sake of clarity, in Fig. 2 the areas are shown overlapping by only small segments. As a matter of fact, in the case of most projects the cost common to two or more objectives is likely to be a large proportion of the total cost, and such a condition has been assumed for the hypothetical example. Then, it is difficult to indicate the true conditions to scale by the use of circles, and a different type of diagram has been devised in Fig. 3.

A study of this diagram will show that the cost of a project that would give flood-protection facilities equivalent to those assumed would be $23,157,000. Such a project would not be built in the interests of navigation or power development. This is indicated in Fig. 3 by the lower rectangle, length *A*. A project that would serve purposes of navigation only, without regard to flood control or to power, would cost $22,557,000 (length *B*, Fig. 3). The cost of a project built for purposes of power generation only would be $29,957,000 (length $C = E$, Fig. 3). The three rectangles have been arranged in a horizontal position such that the various other elements of cost desired are shown diagrammatically. For instance, if a dual-purpose project were built that would furnish flood-control and navigation facilities equivalent to those furnished by the tri-purpose project, but which would provide nothing specifically for power development, the cost would be $24,957,000. This is represented by length *D*.

The hypothetical tri-purpose project is such that, if a single-purpose project were built for the primary purpose of providing an equivalent of the existing power generation facilities, it would be sufficient also to provide the existing navigation facilities without additional cost. This is found to be true in the case of some of the TVA projects on the upper tributaries of the Tennessee River, where the value to navigation consists of releasing water during periods of normally low flow in the channel downstream. This means, in the case of the circles in Fig. 2, that area *B*, representing navigation, would fall entirely within area *C*, representing power. The cost of a dual-purpose project for navigation and power becomes the same as that for a single-purpose project for power. Thus, in Fig. 3 the top rectangle is marked both *C* and *E*. It also has been assumed that a dual-purpose project that would provide flood-control facilities and power-generation facilities equivalent to those now existing would be the same as the existing structure. Therefore, the combined dual cost represented by length *F* (Fig. 3) would be the same as the total cost of the existing project, *T* (= $31,680,000). The assumptions have been made with the purpose of setting up a rather complicated example.

Costs represented by lower-case letters and the overlapping of circles in the theoretical discussion have been determined as follows:

The cost directly allocable to flood control, represented by length *a*, is $1,723,000. In Fig. 3 it is represented by that part of the lower rectangle which extends beyond the right-hand limit of the upper rectangles. A project that would provide suitable power-generation facilities would automatically provide the necessary navigation facilities. Therefore, there is no part of the cost of the project to be allocated directly to navigation and, consequently, the value of *b* becomes zero. That part of the cost properly allocable solely to power generation, represented by length *c*, is $6,723,000. This is the part of the top rectangle which extends to the left beyond the limit of the lower rectangles. There is no value indicated for length *d* which, in the theoretical discussion, is that part of the cost common to flood control and navigation only. Considering only a dual-purpose project for purposes of flood control and navigation, the common element length *d* becomes $20,757,000. However, a dual-purpose project involving navigation and power would cost the same as a single-purpose project for power only. Thus, in Fig. 3, rectangle *B*, representing navigation, is completely overlapped by rectangle *C*, representing power, and it is evident that any part of the cost common to lengths *A* and *B* also must be common to length *C*. Therefore, the sum of $20,757,000, which is common to lengths *A* and *B*, actually becomes length *t* in this study (it is common to all three of the objectives), and *d* becomes zero. This means that no part of the cost is common to costs *A* and *B* which is not also common to cost *C*. That part of the cost common to navigation and power only (length *e*, Fig. 3) is $1,800,000. That part of the cost common to flood control and power only (length *f*, Fig. 3) is $677,000. The sum of $20,757,000, common to all three of the objectives, is shown as length *t* in Fig. 3.

The foregoing procedure appears to be mathematically correct. The fact that some of the values may prove to be zero has no bearing at all on the soundness of the method. The matter of resolving the total cost of a multiple-purpose project into the incremental costs directly allocable to the several objectives and into costs that are common is relatively simple if the foregoing procedure is followed. Dependability of the results obtained depends wholly on the accuracy with which the costs represented by the several capital letters are determined.

To segregate the costs for a tri-purpose project, it will be noted that estimates of cost for seven different projects may be required, although, as in the foregoing hypothetical case, the number may be reduced if two functions are served by a project built for any one of the objectives. In such a case a single-purpose project automatically assumes the character of a dual-purpose project.

During discussions of the allocation procedures, the estimating required in order to apply the method seemed to be a formidable objection to its use in the minds of some non-engineering members of the TVA staff. It does not appear too serious from an engineering viewpoint. Granting that any estimate is likely to be somewhat in error, carefully prepared estimates, formulated by experienced engineers, should fall within suitable limits of accuracy to satisfy the needs at hand. This is especially true when consideration is given to the relatively minor effect on ultimate allocations that may result from substantial variations in original estimates. This has been demonstrated by Mr. Parker in Table 3. The writer investigated these relations during the course of allocation studies and found that, on a basis substantially the same as the foregoing, with variations as large as 20% in the estimates of justifiable expenditures, the greatest change in allocation was approximately 4%, and the average was somewhat less than 2%. Careful estimates should not be more than 20% out of balance, and there seems little likelihood that allocations can be determined with any certainty of accuracy within 2%.

Method of Allocating Common Costs Under Alternative-Justifiable-Expenditure Theory.—The non-segregable costs common to more than one objective are determined under the "three-circle method" of segregation. In the cases studied, only tri-purpose projects are considered, although the same method of segregation and allocation may be applied to any multiple-purpose project.

In the "three-circle method" of segregation an "incremental cost" was determined for each of the three purposes, that being the expenditure incurred in each case over and above what would be expended if a dual-purpose project were built and the purpose under consideration were non-existent in the project. The costs common to each possible combination of two purposes were established on a basis of hypothetical dual-purpose projects, and the cost common to all three purposes was established.

After determining the "justifiable expenditure" for each objective, the segregated "incremental cost" is deducted, and the result becomes a sum that may be termed "justifiable additional investment." Having already charged to the objectives those parts of the multiple-purpose project costs that are directly segregable, it is logical to allocate the remaining common costs on the basis of "justifiable additional investments."

As described under the three-circle method of segregation, there are four items of common cost to be allocated. For exact allocation, each common cost should not be distributed directly on the basis of the "justifiable additional investments" with which it is concerned. Rather, each should be distributed in proportion to the remainders that would be left after the other three items of common cost have been allocated, and such allocations have been deducted from the corresponding "justifiable additional investments." The resulting amounts may be termed "remaining justifiable additional investments." The development of mathematical equations by which common costs may be directly allocated on this basis follows.

The following symbols are adopted to represent the quantities involved:

Symbol	Description
	Justifiable additional investment for—
M	Flood control
N	Navigation
O	Power
	Cost common to (see Eqs. 2 and 3)—
d	Flood control and navigation only
e	Navigation and power only
f	Power and flood control only
t	Flood control, navigation, and power
	Proportion of d allocated to—
d_m	Flood control
d_n	Navigation
	Proportion of e allocated to—
e_n	Navigation
e_o	Power
	Proportion of f allocated to—
f_m	Flood control
f_o	Power
	Proportion of t allocated to—
t_m	Flood control
t_n	Navigation
t_o	Power

Allocation of t.—The steps in the allocation of cost t are shown in Table 4(*a*). The total common cost t is to the part of t to be allocated to flood control

(t_m) as the total "remaining justifiable additional investment" is to the remaining justifiable additional investment for the part allocated to flood control.

TABLE 4.—Allocation Formulas; Justifiable Additional Investment

Justifiable additional investment	(a) Allocation of t		(b) Allocation of d		(c) Allocation of e		(d) Allocation of f	
	Previously allocated[a]	Breakdown	Previously allocated[a]	Breakdown	Previously allocated[a]	Breakdown	Previously allocated[a]	Breakdown
M	$d_m + f_m$	t_m	$f_m + t_m$	d_m	....	....	$d_m + t_m$	f_m
N	$d_n + e_n$	t_n	$e_n + t_n$	d_n	$d_n + t_n$	e_n	....	....
O	$e_o + f_o$	t_o	....	....	$f_o + t_o$	e_o	$e_o + t_o$	f_o
$M+N+O$	$d + e + f$	t	$e_n + f_m + t_m + t_n$	d	$d_n + f_o + t_n + t_o$	e	$d_m + e_o + t_m + t_o$	f

[a] Items considered as if previously allocated.

Expressing this proportion algebraically and transposing t to solve for t_m:

$$t_m = t \frac{M - (d_m + f_m)}{(M + N + O) - (d + e + f)} \quad \text{(4a)}$$

Similarly, the total common cost t is to the part of t allocated to navigation (t_n) as the total remaining justifiable additional investment is to the remaining justifiable additional investment for the part allocated to navigation; thus:

$$t_n = t \frac{N - (d_n + e_n)}{(M + N + O) - (d + e + f)} \quad \text{(4b)}$$

Finally, the total common cost t is to the part of t allocated to power (t_o) as the total remaining justifiable additional investment is to the remaining justifiable additional investment for the part allocated to power:

$$t_o = t \frac{O - (e_o + f_o)}{(M + N + O) - (d + e + f)} \quad \text{(4c)}$$

Allocation of d.—Since it is common to M and N only (see Table 4(*b*)), d is broken down into d_m and d_n. This eliminates line O, and the comparable equations are—

For flood control:

$$d_m = d \frac{M - (f_m + t_m)}{(M + N) - (e_n + f_m + t_m + t_n)} \quad \text{(5a)}$$

and, for navigation:

$$d_n = d \frac{N - (e_n + t_n)}{(M + N) - (e_n + f_m + t_m + t_n)} \quad \text{(5b)}$$

Allocation of e.—In this case, e is common to N and O only (see Table 4(*c*)) and therefore is broken down into e_n and e_o. Line M is thus eliminated and—

For navigation:

$$e_n = e \frac{N - (d_n + t_n)}{(N + O) - (d_n + f_o + t_n + t_o)} \quad \text{(6a)}$$

and, for power:

$$e_o = e \frac{O - (f_o + t_o)}{(N + O) - (d_n + f_o + t_n + t_o)} \tag{6b}$$

Allocation of f.—Being common to M and O only (see Table 4(d)), f is divided into f_m and f_o. This eliminates N, the justifiable additional investment for navigation. The remaining formulas are—

For flood control:

$$f_m = f \frac{M - (d_m + t_m)}{(M + O) - (d_m + e_o + t_m + t_o)} \tag{7a}$$

and, for power:

$$f_o = f \frac{O - (e_o + t_o)}{(M + O) - (d_m + e_o + t_m + t_o)} \tag{7b}$$

Major Equations.—First eliminate Eqs. 4 by substitution. For simplicity these may be written:

$$t_m = \frac{t}{K} (M - d_m - f_m) \tag{8a}$$

$$t_n = \frac{t}{K} (N - d_n - e_n) \tag{8b}$$

and

$$t_o = \frac{t}{K} (O - e_o - f_o) \tag{8c}$$

in which

$$K = (M + N + O) - (d + e + f) \tag{9}$$

In Eq. 5a (for example), substitute the values of t_m and t_n defined by Eqs. 8, and simplify the resulting form algebraically to

$$d_m = d \left[\frac{L\,(M - f_m) + t\,d_m}{L\,(M + N - e_n - f_m) + t\,d} \right] \tag{10}$$

in which $L = K - t$. Eq. 10 may be simplified further to the general formula:

$$d_m\,(M + N - e_n - f_m) = d\,(M - f_m) \tag{11}$$

Similarly, Eqs. 8 substituted in Eq. 6a yield (simplified algebraically)—

$$e_n = e \left[\frac{L\,(N - d_n) + t\,e_n}{L\,(N + O - d_n - f_o) + t\,e} \right] \tag{12}$$

which may be simplified to the general formula

$$e_n\,(N + O - d_n - f_o) = e\,(N - d_n) \tag{13}$$

In Eq. 7b, substitute t_o and t_m from Eqs. 4 to yield

$$f_o = f \left[\frac{L\,(O - e_o) + t\,f_o}{L\,(M + O - d_m - e_o) + t\,f} \right] \tag{14}$$

or

$$f_o\,(M + O - d_m - e_o) = f\,(O - e_o) \tag{15}$$

Eqs. 11, 13, and 15 can be written in the following form:

$$(M + N - f + f_o - e_n)\, d_m = d\,(M - f + f_o) \ldots\ldots\ldots (16a)$$

$$(N + O - d + d_m - f_o)\, e_n = e\,(N - d + d_m) \ldots\ldots\ldots (16b)$$

and

$$(O + M - e + e_n - d_m)\, f_o = f\,(O - e + e_n) \ldots\ldots\ldots (16c)$$

Solving Eq. 16*c* for f_o and substituting this value in Eqs. 16*a* and 16*b*, successively:

$$\begin{aligned} d_m\,[(O + M - e + e_n - d_m)\,(M + N - e_n - f) + f\,(O - e + e_n)] \\ = d\,[(O + M - e + e_n - d_m)\,(M - f) + f\,(O - e + e_n)] \ldots (17a) \end{aligned}$$

and

$$\begin{aligned} e_n\,[(O + M - e + e_n - d_m)\,(N + O - d + d_m) - f\,(O - e + e_n)] \\ = e\,[(N - d + d_m)\,(O + M - e + e_n - d_m)] \ldots\ldots\ldots (17b) \end{aligned}$$

The foregoing derivations have been given in considerably more detail than is necessary to obtain the results but the full procedure is given for the sake of clarity. Beyond Eqs. 17, further modification is inadvisable. By substituting known quantities in these formulas, solutions of d_m and e_n may be obtained. It is then a simple matter to derive allocations of the other common costs by substituting those values in preceding equations. Frequently, some one or more of the hypothetical dual-purpose projects contains no cost that is common only to the two objectives used in combination (that is, d, e, or $f = 0$). Under such circumstances, Eqs. 17 become much simplified. If desired, answers may be obtained by a graphic solution of the equations. In some instances the graphic method will be found to be preferable.

"Cut-and-Try" Solution.—The foregoing equations give results that are mathematically accurate. However, in most instances it is possible to arrive at results that are within satisfactory limits of accuracy by using a cut-and-try method of solution, which involves considerably less work without departing from the fundamental principles of the equations.

As a first step in the cut-and-try method, d, e, and f are broken down in direct proportion to the "justifiable additional investments," M, N, and O. This gives a set of tentative allocations for d, e, and f. Based on these values, a set of tentative "remaining justifiable additional investments" is obtained. These form a basis for a tentative allocation of t. Then the latter allocated values (t_m, t_n, and t_o) exchange places with the tentative breakdown of one of the first three common costs (say f), and a second set of "remaining justifiable additional investments" is obtained. Based on these, f is reallocated.

Following a similar procedure, the latest allocated values (f_o and f_m) now exchange places with the tentative values for e, and a new allocation for e is determined. Extending the process another step, d is reallocated. This completes a cycle that furnishes a new breakdown for each of the common costs. By the nature of the process these values are closer to a true allocation of the common costs than those first used.

It will be noted that these tentative allocations have been based on "remaining justifiable additional investments," but that the latter are not those which would result from deducting the true allocations from the original "justifiable additional investments."

TABLE 5.—CUT-AND-TRY SOLUTION

(Units Are Dollars)

JUSTIFIABLE ADDITIONAL INVESTMENT		APPROXIMATE BREAKDOWN OF MULTIPLE-PURPOSE COSTS				Remaining justifiable investment	ALLOCATION OF MULTIPLE-PURPOSE COMMON COST			
Symbol	Cost (1)	*d* (2)	*e* (3)	*f* (4)	*t* (5)	(6)	*t* (7)	*f* (8)	*e* (9)	*d* (10)
(*a*) FIRST CYCLE										
Allocation of *t*:										
M	500	269	. . .	133	. . . .	98	56	. . . .	. . . .	. . . .
N	800	431	222	. . .	. . . .	147	84	. . . .	. . . .	. . . .
O	1,000	. . .	278	267	. . . .	455	260	. . . .	. . . .	. . . .
M+*N*+*O*	2,300	700	500	400	. . . .	700	400	. . . .	. . . .	. . . .
Allocation of *f*:										
M	500	269	. . .	. . . .	56	175	. . . .	109.89	. . . .	. . . .
N	800	431	222	. . . .	84	63	. . . .	. . . .	. . . .	. . . .
O	1,000	. . .	278	. . . .	260	462	. . . .	290.11	. . . .	. . . .
M+*N*+*O*	2,300	700	500	. . . .	400[a]	700	. . . .	400	. . . .	. . . .
Allocation of *e*:										
M	500	269	. . . .	109.89	56	65.11	. . . .	. . . .	. . . .	. . . .
N	800	431	. . . .	. . . .	84	285.00	. . . .	. . . .	193.91	. . . .
O	1,000	. . .	. . . .	290.11	260	449.89	. . . .	. . . .	306.09	. . . .
M+*N*+*O*	2,300	700	. . . .	400[b]	400[a]	800	. . . .	. . . .	500	. . . .
Allocation of *d*:										
M	500	. . . .	. . . .	109.89	56	334.11	. . . .	. . . .	. . . .	273.16
N	800	. . . .	193.91	. . . .	84	522.09	. . . .	. . . .	. . . .	426.84
O	1,000	. . . .	306.09	290.11	260	143.80	. . . .	. . . .	. . . .	. . . .
M+*N*+*O*	2,300	. . . .	500[c]	400[b]	400[a]	1,000	. . . .	. . . .	. . . .	700
(*b*) SECOND CYCLE										
Allocation of *t*:										
M	500	273.16	. . . .	109.89	. . . .	116.95	66.83	. . . .	. . . .	. . . .
N	800	426.84	193.91	. . . .	. . . .	179.25	102.43	. . . .	. . . .	. . . .
O	1,000	. . . .	306.09	290.11	. . . .	403.80	230.74	. . . .	. . . .	. . . .
M+*N*+*O*	2,300	700	500	400	. . . .	700	400	. . . .	. . . .	. . . .
Allocation of *f*:										
M	500	273.16	. . . .	. . . .	66.83	160.01	. . . .	102.70	. . . .	. . . .
N	800	426.84	193.91	. . . .	102.43	76.82	. . . .	. . . .	. . . .	. . . .
O	1,000	. . . .	306.09	. . . .	230.74	463.17	. . . .	297.30	. . . .	. . . .
M+*N*+*O*	2,300	700	500	. . . .	400[a]	700	. . . .	400	. . . .	. . . .
Allocation of *e*:										
M	500	273.16	. . . .	102.70	66.83	57.31	. . . .	. . . .	. . . .	. . . .
N	800	426.84	. . . .	. . . .	102.43	270.73	. . . .	. . . .	182.26	. . . .
O	1,000	. . . .	. . . .	297.30	230.74	471.96	. . . .	. . . .	317.74	. . . .
M+*N*+*O*	2,300	700	. . . .	400[b]	400[a]	800	. . . .	. . . .	500	
Allocation of *d*:										
M	500	. . . .	. . . .	102.70	66.83	330.47	. . . .	. . . .	. . . .	273.51
N	800	. . . .	182.26	. . . .	102.43	515.31	. . . .	. . . .	. . . .	426.49
O	1,000	. . . .	317.74	297.30	230.74	154.22	. . . .	. . . .	. . . .	. . . .
M+*N*+*O*	2,300	. . . .	500[c]	400[b]	400[a]	1,000	. . . .	. . . .	. . . .	700

[a] *t*(=$400) in Col. 5 taken from preceding determination of *t* in Col. 7. [b] *f*(=$400) in Col. 4 taken from preceding determination of *f* in Col. 8. [c] *e*(=$500) in Col. 3, taken from preceding determination of *e*(=$500) in Col. 9.

TABLE 5.—(*Continued*)

Justifiable Additional Investment		Approximate Breakdown of Multiple-Purpose Costs				Remaining justifiable investment	Allocation of Multiple-Purpose Common Cost			
Symbol	Cost (1)	d (2)	e (3)	f (4)	t (5)	(6)	t (7)	f (8)	e (9)	d (10)
(*c*) Ninth Cycle (Following Successive Intermediate Cycles)										
Allocation of t:										
M	500	272.28	....	101.02	....	126.70	72.40	....	....	....
N	800	427.72	173.48	....	....	198.80	113.60	....	....	....
O	1,000	....	326.52	298.98	....	374.50	214.00	....	....	....
$M+N+O$	2,300	700	500	400	....	700	400	....	....	....
Allocation of f:										
M	500	272.28	....	....	72.40	155.32	....	101.05	....	....
N	800	427.72	173.48	....	113.60	85.20	....	....	....	....
O	1,000	....	326.52	....	214.00	459.48	....	298.95	....	....
$M+N+O$	2,300	700	500	....	400[a]	700	....	400	....	....
Allocation of e:										
M	500	272 28	....	101.05	72.40	54.27	....	....	....	....
N	800	427.72	....	....	113.60	258.68	....	....	173.44	....
O	1,000	....	....	298.95	214.00	487.05	....	....	326.56	....
$M+N+O$	2,300	700	....	400[b]	400[a]	800	....	....	500	....
Allocation of d:										
M	500	....	....	101.05	72.40	326.55	....	....	....	272.28
N	800	....	173.44	....	113.60	512.96	....	....	....	427.72
O	1,000	....	326.56	298.95	214.00	160.49	....	....	....	....
$M+N+O$	2,300	....	500[c]	400[b]	400[a]	1,000	....	....	....	700

[a] t(=$400) in Col. 5 taken from preceding determination of t in Col. 7. [b] f(=$400) in Col. 4 taken from preceding determination of f in Col. 8. [c] e(=$500) in Col. 3, taken from preceding determination of e(=$500) in Col. 9.

Based on these allocations, other cycles are computed. With each cycle the adjustment in allocations becomes less and less as the true values are approached. The process is continued until the adjustments fall within desired limits of accuracy.

A hypothetical example is presented in Table 5, in which each of the aforementioned steps are shown. Nine cycles were used in order to study the rate at which the adjustment in the allocation values diminished. Computations for cycles intervening between the second and ninth are omitted since they add little to clarifying the procedure. Completion of the fourth cycle brought the allocation into such close agreement with the true values that further work was not justified with the small gain in accuracy. Hence, four cycles are deemed sufficient for a practical application of the method. In practice it is possible to omit the values of t and solve for d, e, and f; but, as in the development of the fundamental equations, all of the steps are included for the sake of clarity.

This cut-and-try method was found to be very satisfactory in the solution of the problem. A check may be made on the calculations at any time, and the work stopped after the degree of accuracy justified has been obtained. The variation of results obtained by each cycle indicates the rate at which the allocation values are approaching correctness. By studying these reductions it is readily possible to determine when the results have reached a point beyond which the refinement gained makes further calculations unnecessary.

The foregoing demonstration develops a mathematically correct method of allocating the joint or common costs in any multiple-purpose project, although in most instances it may be found that the increase in accuracy which is gained by the refinements of allocating the costs common only to two objectives (d, e, and f in the equations) is not of sufficient consequence to justify the work required. Under such circumstances it should be sufficient to allocate all of the joint costs ($d + e + f + t$) in one operation in proportion to the "justifiable additional expenditures" and to omit the other steps.

Acknowledgment.—The fundamentals of the foregoing mathematical derivations were proposed by Sherman M. Woodward, M. Am. Soc. C. E., chief water control planning engineer, under whose direction the writer acted as chief estimator in the TVA organization.

EUGENE L. GRANT,[5] M. AM. SOC. C. E.—Once a multiple-purpose project is completed, it may be operated in various ways. The demands of flood control for empty reservoirs and of power for full reservoirs are particularly in conflict. If a project really is operated with flood control as a primary objective and power a secondary one, the firm power generated may be only a small fraction of that obtainable with operation for power as a primary or coördinate objective. This may be true even though the method of operation may not have much effect on the average annual kilowatt-hours of energy generated.

The alternative-justifiable-expenditure method of cost allocation must necessarily be based on some assumed scheme of reservoir operation. This, in turn, determines the alternative plant and the alternative cost. Thus different prospective plans of operation will involve some differences in the costs allocated among the various purposes. However, it seems probable that—in some multiple-purpose projects, at least—the differences in benefits to the several purposes of the project with various plans of operation will be much greater than the differences in costs allocated to these purposes. This may be particularly evident where the value of firm power from one plan of operation is compared with the value of surplus power from some other plan. It would be interesting if, in his closure, Mr. Parker would comment on this point in relation to the TVA cost allocation study. Were cost allocations made for more than one plan of operation? If so, were allocated costs compared with benefits, and particularly power costs with power values, under the various plans?

Whatever the prospective plan of operation adopted in the design of a multiple-purpose project, it seems likely that circumstances will operate to cause departures from this plan. For instance, under war-time conditions of power shortage in the Tennessee Valley, it would seem reasonable to give major consideration to power in operating TVA reservoirs. In more normal times, the pressures on the operators of government multiple-purpose projects are likely to be strongest from those beneficiaries of the projects who make no special contribution to their costs—that is, from flood control and navigation interests. In this connection the propensity of congress to designate multiple-purpose projects as primarily for flood control and navigation regardless of their real justification may prove to be unfortunate. The Central Valley project in California, conceived as an irrigation project to be financed in large part from sale of power, but designated by congress as primarily a navigation and flood control project, is a case in point.

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[5] Prof. of Economics of Eng., Stanford Univ., Dept. of Civ. Eng., Stanford University, Calif.

Sherman M. Woodward,[6] **Hon. M. Am. Soc. C. E.**—For many people the best approach to the allocation problem is from the point of view of the most reasonable distribution of the joint savings or profits in a multiple-purpose project. The data given in Table 1 are most convenient to illustrate this approach. Here a triple-purpose project for navigation, flood control, and power is assumed to cost only $1,000,000, whereas the justifiable expenditure for the three purposes, planned separately (see Col. 2, Table 1), totals $1,300,000. Thus, there is a savings or profit of $300,000 due to combining the three facilities in a joint undertaking rather than building them independently of each other. The amount saved is then allocated to each purpose so as to reduce the amount of the investment charged to each, as illustrated in Table 6. The final allocated cost values shown in Table 6 are the same as in Table 1.

TABLE 6.—Assumed Allocation of $1,000,000

Facility (1)	Justifiable investment (2)	Direct costs (3)	Remaining justifiable investment[a] (4)	Percentage of total (5)	Allocation of joint savings (6)	Allocated cost[b] (7)	Percentage of total (8)
Navigation.........	$400,000	$150,000	$250,000	31.3	$94,000	$306,000	30.6
Flood control.......	300,000	100,000	200,000	25.0	75,000	225,000	22.5
Power.............	600,000	250,000	350,000	43.7	131,000	469,000	46.9
Total..........	$1,300,000	$500,000	$800,000	100.0	$300,000	$1,000,000	100.0

[a] Col. 2 minus Col. 3. [b] Col. 2 minus Col. 6.

The allocation of joint costs seems to be a matter of accounting rather than of economics. It resembles somewhat the making of appraisals—a common procedure in connection with condemnation cases, bankruptcy cases, inheritance cases, insurance settlements, and damage cases, as well as in commercial transactions of all kinds. It requires knowledge, experience, and judgment in making the voluminous computations.

In the method of allocation used in Table 2, if one takes the difference between the columns for the seven-plant system and the six-plant system in an attempt to find the effect of the Hiwassee project, Items 2 and 10 seem to show that the justifiable investment in Hiwassee for the benefit of navigation is $1,000,000, whereas Items 14 and 18 seem to show that the actual computed allocation of the Hiwassee investment to navigation is $1,674,439. Of course, it was not intended that Table 2 should be used in this way, but if any one thinks that this is an important question, it is a simple matter to modify slightly the method of making the computation and remove the apparent inconsistency, as is illustrated in Table 7.

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[6] Chf. Water Control Planning Engr., TVA, Knoxville, Tenn.

TABLE 7.—SYSTEM ALLOCATIONS (ALTERNATE METHOD)
(Amounts in Dollars Are Given to the Nearest Hundred Thousand)

No.	Facility	SIX-PLANT SYSTEM, INCLUDING CHICKAMAUGA DAM		SEVEN-PLANT SYSTEM, INCLUDING HIWASSEE DAM		COL. 5 MINUS COL. 3		EFFECT OF HIWASSEE DAM	
		Dollars	%	Dollars	%	Dollars	%	Dollars	%
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1	Total investment allocated...	1,933	....	2,103	....	170	....	...	....
	Justifiable Investment:								
2	Navigation..............	1,052	....	1,062	....	10	....	...	....
3	Flood control............	523	....	633	....	110	....	...	....
4	Power..................	1,389	....	1,500	....	111	....	...	....
5	Total..............	2,964	....	3,195	....	231	....	...	....
	Direct Costs:								
6	Navigation..............	172	....	172	....	...	....	...	....
7	Flood control............	47	....	49	....	2	....	...	....
8	Power..................	531	....	563	....	32	....	...	....
9	Total..............	750	....	784	....	34	....	...	....
	Remaining Justifiable Investment:								
10	Navigation..............	880	39.7	890	36.9	10	5.1	...	....
11	Flood control............	476	21.5	584	24.2	108	54.8	...	....
12	Power..................	858	38.8	937	38.9	79	40.1	...	....
13	Total..............	2,214	100.0	2,411	100.0	197	100.0	...	....
	Allocation of Common Costs:								
14	Navigation..............	470	39.7	487	36.9	17	12.5	7	5.1
15	Flood control............	254	21.5	319	24.2	65	47.8	75	54.8
16	Power..................	459	38.8	513	38.9	54	39.7	54	40.1
17	Total..............	1,183	100.0	1,319	100.0	136	100.0	136	100.0
	Allocation of Total Costs:								
18	Navigation..............	642	33.2	659	31.3	17	10.0	7	4.1
19	Flood control............	301	15.6	368	17.5	67	39.4	77	45.3
20	Power..................	990	51.2	1,076	51.2	86	50.6	86	50.6
21	Total..............	1,933	100.0	2,103	100.0	170	100.0	170	100.0

This table includes the first 21 items for part of the columns of Table 2. To simplify the presentation, all of the amounts stated in dollars have been changed to the nearest hundred thousand and then the last five ciphers are omitted. Col. 7, obtained by subtracting Col. 3 from Col. 5, shows that the total investment in Hiwassee to be allocated is $17,000,000; the total justifiable investment is $23,100,000; and the total direct costs amount to $3,400,000. This leaves $13,600,000 of common costs to be allocated, whereas the total remaining justifiable investment is $19,700,000. Col. 9 shows that if the common costs in Hiwassee are allocated in proportion to the components of the remaining justifiable investment in the Hiwassee project, finally the total investment in the Hiwassee project would be allocated approximately as follows: $700,000 to navigation, $7,700,000 to flood control, and $8,600,000 to power. Compared with the results in Col. 7, the figures in Col. 9 show about $1,000,000 less charged to navigation, $1,000,000 additional charged to flood control, and no change in the charge to power.

Albert R. Arledge,[7] **M. Am. Soc. C. E.**—Mr. Parker has made an interesting and useful presentation of the theories of allocation for general application as well as specifically for the TVA. The writer believes that many engineers have been puzzled as to how the various costs were determined for each element in a multiple-purpose project. This paper, the writer feels, has been helpful in clearing away the mist that formerly hung over such projects.

Under the heading "Other Similar Allocations," the author includes Boulder Power Plant. The writer is interested in presenting a fuller and clearer picture of this project in regard to allocations. A thorough treatment of this subject would be too long and involved. Consequently, only a few of the selected high lights that bear on the financing will be treated.

The Boulder Canyon project and other major multiple-purpose projects are fairly similar in the purposes for which they were constructed—that is, to improve navigation, control floods, store water for irrigation and domestic supply, generate electrical energy, etc.—and in allocations made for the different functions. They are dissimilar when it comes to the financing or repayment to the government of the cost of the various projects. In different localities different functions receive emphasis. For example, the flood control and water conservation features were emphasized in the publicity advocating the passage of the Boulder Canyon Project Act, but power was selected to make the project "a self-supporting and financially solvent undertaking."

The principal reason for the lack of uniformity in public policy regarding these large projects is probably the fact that the Boulder Canyon Project Act was passed in December, 1928, before the Depression and the great wave of unemployment swept the United States, whereas the other projects were authorized subsequently to this crisis in a constructive attempt to reduce unemployment and improve labor conditions. In any event, between the time of enactment of the Boulder Canyon Project Act and the setting up of other multiple-purpose projects, the Congress changed its public policy from that of requiring self-liquidation to that of granting financial aid by allocating federal funds to functions other than power. These funds ranged from 67.5% to 43% of the ultimate cost of the Bonneville project and from 60% to 48% of the ultimate cost in the case of the TVA. Power is not obligated to repay such allocations. Power at Boulder Dam is obligated to repay all costs of the project, including the allocation to flood control.

Attention is directed to the fact that the Congress of the United States set public policy by the terms of the Boulder Canyon Project Act,[8] approved December 21, 1928, when it authorized the Secretary of the Interior, in Section 1, in part, as follows:

> "* * * for the purpose of controlling the floods, improving navigation and regulating the flow of the Colorado River, providing for storage and for the delivery of stored waters thereof for the reclamation of public lands and other beneficial uses exclusively within the United States, and for the generation of electrical energy as a means of making the project herein authorized a self-supporting and financially solvent undertaking, * * * subject to the terms of the Colorado River Compact herein men-

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[7] Prin. Civ. Engr., Los Angeles Bureau of Power & Light, Los Angeles, Calif.

[8] "Boulder Canyon Project Act," *Public No. 642*, H. R. 5773, 70th Cong., 2d Session.

> tioned * * * to construct, operate, and maintain a dam and incidental works in the main stream of the Colorado River at Black Canyon or Boulder Canyon adequate to create a storage reservoir of a capacity of not less than twenty million acre-feet of water * * *: Provided * * * also to construct and equip, operate, and maintain at or near said dam, or cause to be constructed, a complete plant and incidental structures suitable for the fullest economic development of electrical energy from the water discharged from said reservoir * * *."

Although the foregoing section authorized the Secretary of the Interior to construct the Boulder Canyon project, it was necessary to meet the further provision of financing before work could proceed legally. This provision follows:

> "Sec. 4(b). Before any money is appropriated for the construction of said dam or power plant, or any construction work done or contracted for, the Secretary of the Interior shall make provision for revenues by contract, in accordance with the provisions of this Act, adequate in his judgment to insure payment of all expenses of operation and maintenance of said works incurred by the United States and the repayment, within fifty years from the date of the completion of said works, of all amounts advanced to the fund under subdivision (b) of Section 2 for such works, together with interest thereon made reimbursable under this Act." [8]

The "Project Act" makes an allocation for flood control purposes. C. C. Elder,[9] M. Am. Soc. C. E., has given the background of this project in greater detail than can be given herein. To quote Mr. Elder on this subject:

> "All advances to the Colorado River Dam fund from the U. S. Treasury were required to be repaid with four per cent interest, but a definite allocation of $25,000,000 was made to flood control to be repaid out of 62.5 per cent of excess revenues, if and when available. If this repayment were not completed within the general 50-year amortization period, ending May 31, 1987, repayment was to be continued out of the same proportion of the project's net revenues. Thus deferment was permissible for the flood control allocation's repayment, if this became necessary, but provision was made for full repayment finally, with four per cent compound interest. The amount of this flood control allocation appears to have been fixed arbitrarily, so far as is now evident, but is approximately the sum that might have been needed for a program of flood control by means of a low dam for this purpose solely, and a system of raised and strengthened levees. This solution would have given immediate, but evidently only temporary, relief from flood damages, due to relatively rapid filling with silt of a small reservoir. But the federal obligation to furnish some degree of flood control on the lower Colorado River was recognized, and this is probably the only case to date where repayment, even though possibly on a deferred basis, of a government flood control structure, or its allocated share of the construction cost, has ever been required or promised.
>
> "In addition to the flood control allocation, other excess revenues of the Colorado River Dam fund were allotted as follows:
>
> > " 'If during the period of amortization the Secretary of the Interior shall receive revenues in excess of the amount necessary to meet the periodical payments to the United States as provided in the contracts executed under this act, then, immediately after the settlement of such periodical payments, he shall pay to the State of Arizona 18¾%

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[9] "Boulder Canyon Project, Original and Readjusted Repayment Contracts," by C. C. Elder—presented before the Engineering Economics Division, Am. Soc. C. E., 89th Annual Meeting, New York, N. Y., January 22, 1942.

> of such excess revenues and to the State of Nevada 18¾% of such excess revenues.
>
> "'After the repayments to the United States of all money advanced with interest, charges shall be on such basis and the revenues derived therefrom shall be kept in a separate fund to be expended within the Colorado River Basin as may hereafter be prescribed by the Congress.'
>
> "Records of Congressional Committee hearings indicate that the intent of Congress was that Arizona and Nevada should receive some payments in lieu of taxes which would have been levied if the project had been privately developed. The amount is stated as having been arrived at on the basis of a fairly parallel case, the leasing of public oil lands, where the rule had become established that the State involved should receive 37 per cent of the net revenue in lieu of taxes. The disposition of the balance of excess revenues, after completion of repayment of U. S. Treasury advances to the fund, was such a remote problem as not to excite immediate interest, so that it was merely left to future action of Congress, though allotted for Colorado River Basin developments if and when available."

In compliance with the foregoing authorization and provisions, the Secretary of the Interior, after investigation and study of regional market conditions, set a schedule of rates in Article (16) of the contract, dated April 26, 1930, a part of which follows:

> "(16) In consideration of this lease, the lessees severally agree:
>
> "(1) To pay the United States for the use of falling water for the generation of energy for their own use, respectively, by the equipment leased hereunder (except as otherwise provided in Article Seventeen (17) hereof), as follows:
>
> "(*a*) One and sixty-three hundredths mills ($0.00163) per kilowatt-hour (delivered at transmission voltage) for firm energy;
>
> "(*b*) One-half mill ($0.0005) per kilowatt-hour (delivered at transmission voltage) for secondary energy; * * *.
>
> "At the end of fifteen (15) years from the date of execution of this contract and every ten (10) years thereafter, the above rates of payment for firm and secondary energy shall be readjusted upon demand of any party hereto, either upward or downward as to price, as the Secretary (of Interior) may find to be justified by competitive conditions at distributing points or competitive centers."[10]

Based on the rates established in the aforementioned contracts, prepared and executed under the original act, the two lessees, the Los Angeles Municipal Bureau of Power and Light and the Southern California Edison Company, Ltd., assumed the financial responsibility for the return of the full costs of the dam, power house, and incidental structures in 50 years at 4% interest, and for the return of the cost of machinery and equipment in ten annual installments, as set up in Article (9)(a) of the contract which, in part, follows:

> "(9)(a) Compensation for the use, for the periods of lease thereof, of machinery and equipment furnished and installed by the United States, for each lessee respectively, for the generation of electrical energy, equal to the cost thereof, including interest charges at the rate of four per centum (4%) per annum, compounded annually from the date of advances to the

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[10] Contract for Generation and Transmission of Power, Boulder Canyon Project—Bureau of Reclamation, U. S. Dept. of the Interior; the City of Los Angeles, Calif.; and Southern California Edison Company, Ltd., April 26, 1930.

> Colorado River Dam Fund for the purchase of such equipment and machinery to June 1st of the year next preceding the year when the initial installment becomes due under this article, shall be paid to the United States by the lessees, severally, in ten (10) equal annual installments, so as to amortize the total cost (including interest as fixed above), and interest thereafter upon such total cost at the rate of four per centum (4%) per annum * * *."[10]

The power obligation of these two lessees totaled 64% of the firm power (this included the 18% allotment to each of the states of Arizona and Nevada, divided equally between the two lessees). The states assumed no financial responsibility in regard to the capital outlay. At that time, the Metropolitan Water District of Southern California, whose allotment was 36% of the Boulder Dam firm energy, was not considered a going concern, as the bonds for the construction of its Colorado River Aqueduct had not yet been voted.

From the preceding sketchy review of the requirements and provisions of the Project Act and the contracts, and the additional fact that the Metropolitan Water District long since had voted its bonds and had become a going concern, it would seem apparent that a revision of the contracts was necessary to maintain justice and equity between the government and the lessees.

The Boulder Canyon Project Adjustment Act[11] permits the contracts for energy to be adjusted to rates more comparable with the liberal public policy adopted by the Congress for other multiple-purpose projects throughout the United States. It recognized 100% power allotment for amortization purposes instead of 64% of the allotment as set in the original contracts. The "Adjustment Act" also provides more certain payments to the states of Arizona and Nevada and to the Colorado River Development Fund.

The Secretary of the Interior was interested, from an administration standpoint, in changing the lessee from a "Contract for Lease of Power Privilege" to that of an "Agent of the United States for the Operation of the Power Plant." The Adjustment Act provided the mechanics under which the desirable changes suggested could be made. The adjusted rates for Boulder Dam energy will more nearly meet "competitive conditions at distributing points or competitive centers," will still comply with the rigorous requirements of the original Act for a self-supporting project, and will reduce the initial rates roughly 30%.

Under "Limits of Variation" the author has well stated:

> "Viewing this problem in its broadest sense, it is evident that no allocation can be determined with mathematical precision. On the other hand, there are reasonable limits to the variations in allocations of common cost.
>
> "Where three major purposes are involved, it would be an obvious absurdity to charge more than half of the common cost to power alone, especially where power production is designated by congressional legislation as incidental to the other two purposes. On the other hand, where power is the only 'paying partner,' it is difficult to justify charging to power less than one third of the common costs in any event."

The writer feels that the author, in these two paragraphs, has expressed a policy in regard to allocations which could well be adopted by the Congress as a guide to simplicity and uniformity in public policy for the benefit of all multiple-purpose projects that it initiates.

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[11] "The Boulder Canyon Project Adjustment Act," approved July 19, 1940, *Public No.* 756, H. R. 9877, 76th Cong., 3d Session.

In summarizing, the Adjustment Act has effected changes, as follows:

1. Lessee to operating agent;
2. 4% interest to 3% interest;
3. (*a*) The rate for firm energy from 1.63 mills per kw-hr to 1.163 mills per kw-hr; and (*b*) the rate for secondary energy from 0.50 mill per kw-hr to 0.34 mill per kw-hr;
4. Payment for machinery and equipment from ten annual installments to 50 years;
5. The postponement of the payment of the flood control allocation without interest until the end of the 50-yr amortization period, May 31, 1987;
6. The payments of 18¾% of

> "* * * revenues in excess of the amount necessary to meet the periodical payments to the United States as provided in the contract, or contracts, executed under this Act * * *"[8]

if any, to each of the states of Arizona and Nevada, to the payment

> "* * * to each of said States of the sum of $300,000 for each year of operation, beginning with the year of operation ending May 31, 1938, and continuing annually thereafter until and including the year of operation ending May 31, 1987 * * *"[11]; and

7. The payment for the Colorado River Basin development changed from the stipulation that

> "* * * after the repayment to the United States of all money advanced with interest, charges shall be on such basis and the revenues derived therefrom shall be kept in a separate fund to be expended within the Colorado River Basin as may hereafter be prescribed by the Congress,"[8]

to a more assured and earlier payment, as defined by:

> "* * * the sum of $500,000 for the year of operation ending May 31, 1938, and the like sum of $500,000 for each year of operation thereafter, until and including the year of operation ending May 31, 1987 * * *."[11]

Items 6 and 7 are subject to a further provision as set forth in Section 3 of the Adjustment Act, as follows:

> "If, by reason of any act of God, or of the public enemy, or any major catastrophe, or any other unforeseen and unavoidable cause, the revenues, for any year of operation, after making provision for costs of operation, maintenance, and the amount to be set aside for said year for replacements, should be insufficient to make the payments to the States of Arizona and Nevada and the transfers to the Colorado River Development Fund herein provided for, such payments and transfers shall be proportionately reduced, as the Secretary may find to be necessary by reason thereof."[11]

An unusual feature of the Adjustment Act is its retroactive character, requiring these changes to become effective June 1, 1937.

C. W. WATCHORN,[12] ESQ., AND F. A. ALLNER,[13] M. AM. SOC. C. E.—The problem of allocating the cost of an enterprise to the various services it renders or to the commodities it produces is difficult, even when the services rendered or the commodities produced are all of the same general kind. The problem of allocation is still more complicated for multiple-purpose enterprises or projects of which the services or products are of very different kinds.

Strictly speaking, the so-called theories of joint-cost allocations, with the possible exception of the use-of-facilities theory, are never actually what their names indicate; rather, they are mathematical methods that arbitrarily assign responsibility for the joint cost to the various purposes of a project. These cost-allocating methods are just as applicable to a single-purpose project with respect to the various users of its services as to a multiple-purpose project with respect to its various purposes. It is not difficult to imagine situations where, for a single-purpose project, the application of these cost-allocation theories would lead to inequitable and discriminative results. There are many cases in which the cost of obtaining a given service in one way has no conceivable relation to the cost of obtaining it in another way; for example, the customers of a water utility may be able to supply their own requirements privately at varying unit costs, depending on their respective circumstances. There is no basis for assuming that the cost to the utility to supply these various customers would bear any relation to the costs to such users to supply themselves. Since the application of the alternative cost or of any other method may lead to arbitrary and discriminative results when applied to such a relatively simple case, it follows that the same result may be obtained when these same methods are applied to the more complicated problem of allocating cost for multiple-purpose projects. As stated by the author, however, the more generally accepted methods for allocating the cost of a single-purpose project scarcely would be adaptable to a multiple-purpose project because of the absence of common characteristics of the services of the various purposes of such projects.

Even for electric utilities, there is no one generally accepted method for allocating cost to the various classes of customers. The reason that cost allocations are made in such cases is to determine the reasonableness of the charges made or to obtain a basis for determining what charges should be made. Regardless of the allocation of cost and the resulting charges for services, the total annual revenue to be received by the utility would depend only on the total annual cost of rendering service, provided, of course, that such cost is less than the cost of an alternative supply.

The justification of a multiple-purpose project, from an engineering point of view, depends only on the total annual cost of the services from the project being less than the minimum total annual justifiable cost (that is, cost which is less than the annual benefits) of obtaining the same services otherwise. The total annual revenue, as for a single-purpose project, should depend only on the total annual cost of rendering the services, provided it is not greater than the minimum annual cost of obtaining the same services otherwise. Except for this limitation, the total annual revenue should be independent of both the cost of obtaining the same services otherwise and of any cost allocation that may be made.

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[12] Efficiency Engr., Pennsylvania Water & Power Co., Baltimore, Md.

[13] The late Pres., Safe Harbor Water Power Corp., Baltimore, Md. Mr. Allner died on July 18, 1942.

The only reason for making a cost allocation is to obtain a basis for the rates and charges for various services that will result in a total annual revenue as nearly equal to the total annual cost as possible. If it is the public policy not to collect revenue for various services of government-owned, multiple-purpose projects and to make only nominal charges for others, the resulting deficiencies will have to be subsidized from general taxes. The amount of such subsidies should be determined directly as a matter of public policy and not indirectly (as is most often done), by means of a cost allocation.

The questions as to whether or not a project should be subsidized and the extent of the subsidy are political questions rather than problems of engineering economics. The application of engineering economics to a project as a whole ends when it has been determined whether or not the project is economically justified; and the political decisions with respect to such projects, rather than the engineering and economic facts, should be based on the declared public policy. The answer to the question as to what extent, if any, it is the public policy to subsidize such projects depends in turn on the public's knowing the true annual cost, including the controversial question as to the "cost of money."

The money invested belongs to the public regardless of whether the project is "privately" or "publicly" owned; in the one case the money invested belongs to specific individuals or groups and in the other to the public generally. The economics of a project do not change merely because it is sponsored by the government. An important difference between the financing of a project by the government or by a private company is that the government, particularly the federal government, has much greater resources. An apparent difference arises from the fact that, because of tax regulations and the safety to investors in government securities provided by its taxing powers, the government is usually able to borrow at a lower rate than a private enterprise. This does not necessarily mean, as is often assumed, that the cost of money to the government for any particular purpose is less than that to a responsible private company for the same kind of undertaking. This is true even though the government, if it so desires, may finance a project entirely with borrowed money, whereas private companies must usually provide substantial portions of equity money in addition to that borrowed.

It is frequently forgotten that the government's "equity" is provided by the taxpayers who must bear the burden of all losses suffered on account of investments in such projects. It is difficult to understand logic that assumes the public places less value on its money and is willing for the money to earn a lower return simply because it was invested through the government instead of a private company. The answer is that the money used in the same project is worth the same, regardless of who made the investment, assuming equally capable management in each case, since the risks would then be the same. Further, it is not believed that the interest rate paid by the government alone tells the entire story as to the cost of the public's money used by the government for such purposes. The present situation as to taxes makes this point more clear. The unevaluated portion of the cost of the public's money used by the government is the necessary sacrifice of the public to make money available to the government for its various activities. This fact has always existed and is only made more clear by present conditions. For example, if one assumes no public debt (as a result of a pay-as-you-go program), the government would pay no interest on the money expended by it for any purpose. Should one then assume zero cost for the money spent by the government for such projects? The fact of the matter is that there is no magic in government ownership, and the cost of money to it is the average return demanded by the public itself for investments in reasonably safe private undertakings. Such a basis for determination serves as an indirect, but nevertheless reliable, method of evaluating previously neglected elements of cost of money to the government. The economic justification of the project and its true annual total cost should both be computed on the foregoing basis for determining the cost of money. The total annual revenue should be equal to this total annual cost to make the project self-supporting. However, it may be the public policy that no revenue be received from some of the purposes of the project, with only nominal revenue from others, and normal revenue from the remaining purposes.

To illustrate, assume a multiple-purpose project built for flood control, navigation, power and energy production, and recreation; and assume that it is the public policy that no direct charges be made for flood-protection benefits, that only nominal charges be made for use of the recreation facilities, and that normal charges be made for navigation and power and energy. The total annual cost of the project is $3,700,000, of which $300,000 is the annual cost incurred specifically for flood control, $400,000 specifically for navigation, $1,500,000 specifically for power and energy production, $100,000 specifically for recreation, and the remainder of $1,400,000 for joint facilities. The nominal charges made for the use of the recreation facilities result in an annual revenue of $20,000. The problem is to determine the rates and charges that should be made for the use of the navigation facilities and for power and energy. It is believed, as stated previously, that there is no correct engineering solution to this problem, in the scientific sense, and that the determination of the cost allocation to the various purposes is necessarily arbitrary, regardless of how it may be derived.

The total annual revenue desired is likewise a matter of public policy, which one may assume would be either (1) as large as possible, (2) approximately equal to the total annual cost of the project even though flood control is to be wholly, and recreation largely, subsidized, (3) equal to some specified amount, (4) less than the total annual cost of the project by some specified amount, or (5) equal to some specified percentage of the total annual cost of the project.

It is believed that the underlying principle of the vendibility theory of cost allocation (namely, that services and commodities jointly produced are mutual by-products of each other and that the actual total annual revenue should be as nearly as possible equal to the desired total annual revenue) is the most logical basis for arriving at an equitable, but admittedly arbitrary, basis for allocating the cost of a government-owned, multiple-purpose project to its various purposes. The total revenue received on account of any of the purposes of the project is immaterial, so long as it is not greater than that required to defray the annual cost for the most economical means of obtaining the same services otherwise.

The charges for each of the services should be equal to or less than the smaller of (1) the economic rate (that is, the rate that would produce the maximum net income), and (2) the rate required to be paid for the most economical available means of obtaining the same services otherwise.

If the total annual revenue resulting from the predetermined nominal rates for certain services and those determined in the foregoing manner for the other services (which may be called either separately or in total the unadjusted revenue) is greater than the total desired for the project (as determined by public policy) the rates should be further reduced so that the desired over-all results will be obtained. Such an excess of revenue may result from various combinations of different amounts of earnings on account of the various purposes of the project. Whatever the combination is that causes this excess, it is suggested that the reduction be based on comparisons of the unadjusted annual revenue with the annual cost of the specific facilities for each of the purposes of the project, and of the total unadjusted annual revenue with the total revenue desired.

The simplest case results when the excess, if any, of unadjusted total annual revenue over the desired total annual revenue is derived from only one purpose of the project. This will be the case when there is only one purpose for which the unadjusted annual revenue exceeds the annual cost of specific facilities and this excess is greater than the amount the unadjusted total annual revenue exceeds the desired total annual revenue. It would appear illogical, under such circumstances, to further reduce the annual revenues for those purposes for which the unadjusted annual revenues fail to cover the annual cost of specific facilities; and consequently, such reduction should all be applied to the one purpose for which the unadjusted annual revenue exceeds the annual cost of specific facilities. However, in the event (even though there is only one purpose for which the unadjusted annual revenue exceeds the annual cost of specific facilities for that purpose) that the excess of the unadjusted total annual revenue over the desired total annual revenue is greater than such excess for that one purpose, the situation becomes rather complex and more complicated treatment of the problem, as discussed later, is necessary.

If there are several purposes of the project for which the unadjusted annual revenues exceed the annual cost of specific facilities, there are, as brought out in the preceding special case, two possible conditions, one of which is relatively simple and the other more complex. The simple condition results when the excess of the unadjusted total annual revenue over the total annual revenue desired is less than the total of the excesses of the unadjusted annual revenues over the annual cost of specific facilities for those purposes of the project for which there are such excesses. The reduction to be made for the simple case should be on a proportional basis to those purposes of the project having such excess revenues.

To illustrate, assume for the example given previously that the unadjusted annual revenue is $1,500,000 for navigation and $2,800,000 for power and energy, which for navigation is $1,100,000 and for power and energy $1,300,000 in excess of the annual cost of the specific facilities for those purposes. These unadjusted annual revenues, together with those from recreation, give \$4,320,000 as the total annual revenue from the project. If the desired annual revenue is \$3,000,000, the total reduction for navigation and power and energy would be \$1,320,000, whereas the sum of the excess revenues for these purposes is \$2,400,000. The reduction to be applied to navigation then would be $\frac{11}{24} \times \$1{,}320{,}000$, or \$605,000, and to power and energy $\frac{13}{24} \times \$1{,}320{,}000$, or \$715,000. The resulting adjusted revenue for navigation would be \$895,000, and for power and energy \$2,085,000.

The more complex situations result when the desired total annual revenue of the project is so low, in comparison with the unadjusted total annual revenue, that the necessary reduction is greater than the total excess of unadjusted revenue over the annual cost of specific facilities for those purposes of the project for which such excess revenues exist. (The case in which there was no purpose of the project for which the unadjusted annual revenue exceeded the annual cost of specific facilities would also be included here.) In order to make the desired reduction in total revenue for such cases, not only revenues from otherwise "profitable" purposes of the project, if any, must be reduced below the annual costs of specific facilities for these purposes, but revenues from the "unprofitable" purposes of the project may also be further reduced.

The solution suggested herein involves the finding of a ratio, less than unity, which when applied to the specific cost of facilities for any purpose will give the adjusted revenue for that purpose, provided the revenue is reduced thereby (that is, the revenue for any purpose is to be reduced or must remain unchanged). The ratio to be determined must also satisfy the condition that adjusted revenues (including those that remain unchanged) must total to the desired revenue from the project.

To illustrate, assume the same data for the project as previously used, except that the total desired annual revenue is only \$1,000,000 (that is, the subsidy has been increased \$2,000,000 as compared with the previous example). The total annual cost of specific facilities, excluding those for flood control, is \$2,000,000. This total multiplied by 50%, the ratio of the desired annual revenue to the total annual cost of specific facilities, would give the desired \$1,000,000 total revenue. It is immediately apparent, however, that the 50% ratio applied to the cost of facilities for recreation (\$100,000) would give a revenue in excess of the \$20,000 predetermined for this purpose. Consequently, the computation of the ratio must be adjusted by excluding the annual cost of recreation facilities and recreation revenue. The total annual cost of specific facilities for navigation and power and energy is \$1,900,000, and the desired revenue, after deducting \$20,000 for recreation, \$980,000. The ratio of desired revenue to cost is then 0.516. The adjusted revenue for navigation would then be \$400,000 × 0.516, or \$206,400, and for power and energy \$1,500,000 × 0.516, or \$773,600. The total adjusted revenue, including \$20,000 from recreation, is the desired \$1,000,000.

These two methods of distributing the excess of the unadjusted total annual revenue over the total annual revenue desired among the revenues from the various purposes of a multiple-purpose project have a common point at which their results necessarily agree. At this point the desired reduction in total revenue is just equal to the sum of the excesses of unadjusted annual revenues over the annual cost of specific facilities for those purposes for which excesses exist.

This suggested basis for determining the charges to be made for the various services rendered by multiple-purpose projects is not, in a strict sense, a basis for allocating cost. It is merely a basis for the determination of the charges to be made for these various services in accordance with a declared public policy, not only as to what purposes of the project are to be subsidized, but also as to what extent the project as a whole is to be subsidized. The procedure used by the author (assuming that the rates for the various services are based on the cost allocation) results in a subsidy for the project as a whole which varies as the cost allocated to the various purposes varies. The subsidy increases as the cost allocated to the no-revenue and small-revenue producing purposes of the project increases. On the other hand, the application of the foregoing suggested basis for determining the charges to be made for the various services of the project assures the return that is desired from the project as a whole in accordance with public policy, and the public policy as to what purposes shall be subsidized as no-revenue and small-revenue producing. At the same time it makes available the various services, for which normal charges are to be made, at the same or less cost than the minimum for which they could be obtained otherwise. Thus, the beneficiaries of such services have no basis for complaint, since they are not hurt, whereas the interests of the general public are protected to a definite predetermined extent.

THEODORE B. PARKER,[14] M. AM. SOC. C. E.—The discussions have elicited much additional information which undoubtedly will be of value to those interested in the subject. It is particularly gratifying to note that the discussers seem to be in substantial agreement with the methods used by the Tennessee Valley Authority.

Colonel Elliott agrees generally with the methods used in determining the costs to be allocated to the various purposes of a multiple-purpose project, but questions the use of a cost allocation, in particular with reference to the so-called power "yardstick." If it were conceded, for the purpose of the argument, that the costs of private and public enterprises were different, or, for that matter, that the costs of construction or types of structures required in different parts of the country were different, it would seem to the writer that adjustments for these differences could be made readily. Without a cost allocation to which such adjustments could be applied, there would be no base from which to begin in making any comparison. It should also be kept in mind that the capital cost of the generating stations is only a comparatively small part of the cost of power and that transmission, distribution, and marketing costs are not subject to allocation. It follows that even if the cost of generation for a public enterprise were 70% of the cost of generation of a private enterprise, the actual costs of power at market would be more nearly equal.

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[14] Chf. Engr., TVA, Knoxville, Tenn.

Mr. Chandler presents, in some detail, interesting variations of the alternative-justifiable-expenditure method which will greatly assist the uninitiated in a better understanding of the problem. However, it appears to the writer that, in the use of the so-called "three-circle" method, simplicity of the alternative-justifiable-expenditure method is sacrificed without much, if any, gain in accuracy since a considerably larger number of estimates of cost are required.

Professor Grant raises an interesting and pertinent point by requesting more information on the effect of different plans of operation on the cost allocations. In accordance with the TVA Act, flood control and navigation are the primary purposes, and power is the by-product of the development of the Tennessee River and its tributaries. The Act is clear and specific, the various developments are planned and operated accordingly, and in normal times, therefore, there cannot be any choice as to plan of operation; the cost allocations are made to conform with this plan. However, the Authority has in operation (among others) the Cherokee Dam on the Holston River and the Nottely and Chatuge developments in the upper Hiwassee drainage basin, which, by special appropriation of the Congress, were authorized to alleviate a threatened power shortage in the region. During the hearings before the Congressional committees on the bill authorizing the construction of these projects, it was fully explained that these dams were to be operated temporarily for the primary purpose of maximum power production, and that at the conclusion of the war the operation of the dams would revert to the normal multiple-purpose operating plan.

To conform to the realities of this situation, the writer is recommending that the Authority submit two cost allocations for approval by the President of the United States—one to apply for the duration of the war and charging the entire cost of the additional developments to power, the other to go into effect after the Congress has declared the war emergency to be ended and giving effect to the planned multiple-purpose operation of the projects.

It is interesting to note that the methods used are sufficiently flexible to allow for this difference in operation as well as sufficiently accurate so that the difference in operation does not disappear in the process of allocation.

Mr. Woodward's discussion is a worth-while addition to the conception of allocations, and offers an alternative allocation principally involving the cost of Hiwassee Dam and affecting only the allocated costs of navigation and flood control.

Mr. Arledge summarizes clearly and concisely the various steps taken by the Congress with regard to the financing of the Boulder Canyon project. This discussion contains much valuable information which the individual engineer could only find after considerable research, if at all. The remarks of Mr. Watchorn and the late Mr. Allner do not appear to be particularly pertinent to the solution of the problem of cost allocation of the TVA projects as directed by the Congress. However, their discussion presents a thoughtful opinion on the general subject of allocations.

In conclusion, the writer wishes to express his sincere appreciation to all the discussers.

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Paper No. 2176

BENDING MOMENTS IN THE WALLS OF RECTANGULAR TANKS

BY DANA YOUNG,[1] ASSOC. M. AM. SOC. C. E.

WITH DISCUSSION BY MESSRS. I. K. SILVERMAN, AND DANA YOUNG.

SYNOPSIS

An attempt is made herein to obtain a theoretical solution for the deflection and moments in the vertical side walls of a rectangular tank which is open at the top. More precisely, a formal analysis is made of a rectangular plate that is entirely free or unsupported at the top, perfectly fixed along the sides and bottom and subjected to a transverse hydrostatic loading. In certain cases, the walls of a rectangular tank, as shown in Fig. 1, may be considered to act as plate with these idealized edge conditions, and it is only in such cases that this analysis applies. For a tank that is square in plan and with a heavy base, perfected fixity at the sides and bottom would be closely approached. If one horizontal dimension is much greater than the other, the assumed edge conditions would not prevail. This analysis may be said to furnish the solution for an idealized case and the designer must use judgment in applying the results to a particular problem.

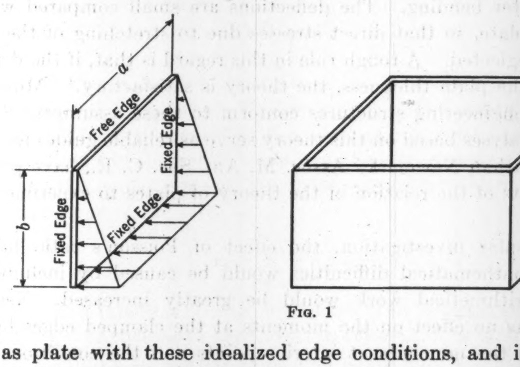

FIG. 1

NOTE.—Published in November, 1941, *Proceedings*.

[1] Prof., Applied Mechanics, The Univ. of Texas, Austin, Tex.

There are other possible applications besides that of a tank wall. The vertical slab in a counterfort retaining wall may be considered as such a plate providing the bottom edge is designed for fixity. The flat slab of a buttress dam is in the same category.

The analysis given herein is based on the so-called ordinary theory of bending of thin plates. According to this theory, the load on the plate and the shape of the deflected surface of the plate are related by a partial differential equation. An analysis of the problem involves finding a solution of the differential equation which also satisfies the specified edge conditions for the plate. After this deflection function is found, the bending moments may be determined from the second derivatives of the function. The derivation of the differential equation and the expressions for moments, shears, and reactions are given by various writers such as S. Timoshenko[2] and A. Nadai.[3]

INTRODUCTION

The theory of the bending of plates used in this paper is based upon several assumptions that must be considered in interpreting the results of the analysis. The principal assumptions may be noted briefly. The plate is assumed to be homogeneous, isotropic, and of constant thickness. The thickness of the plate is small compared with its lateral dimensions, so that the effect of shearing stresses on the deflection may be neglected. A line that is normal to the plane of the plate before bending remains straight and normal to the deflected middle surface after bending. The deflections are small compared with the thickness of the plate, so that direct stresses due to stretching of the middle surface may be neglected. A rough rule in this regard is that, if the deflection is less than half the plate thickness, the theory is satisfactory.[4] Most plates or slabs used in engineering structures conform to these assumptions closely enough so that analyses based on this theory serve as reliable guides for design. A. Nadai[3] and Nathan Newmark,[5] Assoc. M. Am. Soc. C. E., have presented detailed statements of the relation of the theory of plates to experiments and design.

In this particular investigation, the effect of Poisson's ratio has been neglected. No mathematical difficulties would be caused by including this item, but the arithmetical work would be greatly increased. Neglecting Poisson's ratio has no effect on the moments at the clamped edges but does affect, somewhat, the moments at interior points and the conditions at the free edge. Corrections to the interior moments for any given value of Poisson's ratio can be made.[2,3]

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[2] "Theory of Plates and Shells," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1940.

[3] "Die elastischen Platten," by A. Nadai, Julius Springer, Berlin, 1925.

[4] "Theory of Plates and Shells," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1940, p. 51.

[5] "What Do We Know About Concrete Slabs?" by N. M. Newmark, *Civil Engineering*, September, 1940, pp. 559–562.

NOTATION

The letter symbols in this paper are defined where they first appear and are assembled for reference in the Appendix.

GENERAL PROCEDURE

This problem requires the solution of the differential equation

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \qquad (1)$$

with the boundary conditions (assuming Poisson's ratio as zero):

At $x = 0.5\,a$,

$$w = 0 \quad \text{and} \quad \frac{\partial w}{\partial x} = 0 \qquad (2a)$$

At $y = b$,

$$w = 0 \quad \text{and} \quad \frac{\partial w}{\partial y} = 0 \qquad (2b)$$

At $y = 0$,

$$\frac{\partial^2 w}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^3 w}{\partial y^3} + 2\frac{\partial^3 w}{\partial x^2\,\partial y} = 0 \qquad (2c)$$

In Eqs. 1 and 2: w is the deflection of the plate at any point; q is the intensity of a distributed lateral load; x and y are the rectangular coordinates in the plane of the middle surface of the plate; a and b are the dimensions of the plate in the x and y directions, respectively; and D, the flexural rigidity of the plate, is equal to

$$D = \frac{E\,h^3}{12\,(1 - \mu^2)} \qquad (3)$$

in which: E is the modulus of elasticity of the plate material; h is the thickness of the plate; and μ represents Poisson's ratio. The difficulty in solving this problem arises from the fact that there is no known procedure for solving the differential equation subject to these boundary conditions. The most general solution for rectangular plates is the one due to Levy[6] which uses for the deflection an expression of the form

$$w = \sum_{m=1}^{\infty} X_m \sin\frac{m\,\pi\,y}{b} \qquad (4)$$

in which X_m is a function of x alone; and m is an integer (which has the value 1, 2, 3 ··· to ∞ in successive terms of the series). The solution requires that the plate be simply supported along two edges and any boundary conditions may be prescribed at the other two edges. Therefore, it cannot be applied directly to the problem at hand. However, it is possible (in a manner somewhat analogous to that proposed by S. Timoshenko[7] for plates clamped at all four edges) to solve the problem by a proper combination of several loadings, each of which is of the form of Eq. 4.

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[6] "Die elastischen Platten," by A. Nadai, Julius Springer, Berlin, 1925, p. 120.

[7] "Bending of Rectangular Plates with Clamped Edges," by S. Timoshenko, *Proceedings*, Fifth International Cong. for Applied Mechanics, Cambridge, Mass., 1938, p. 40.

For this purpose, the four cases shown in Fig. 2 are taken, each of which is simply supported along one pair of edges. Fig. 2(*a*) carries an hydrostatic loading, is simply supported at $x = \pm 0.5\,a$, fixed at $y = b$, and has a free edge at $y = 0$. Fig. 2(*b*) has no lateral load but is bent by arbitrary edge moments

$$(M_x)_{x=\pm\frac{a}{2}} = \sum_{m=1}^{\infty} F_m \sin\frac{m\,\pi\,y}{b} \ldots\ldots\ldots\ldots\ldots\ldots (5a)$$

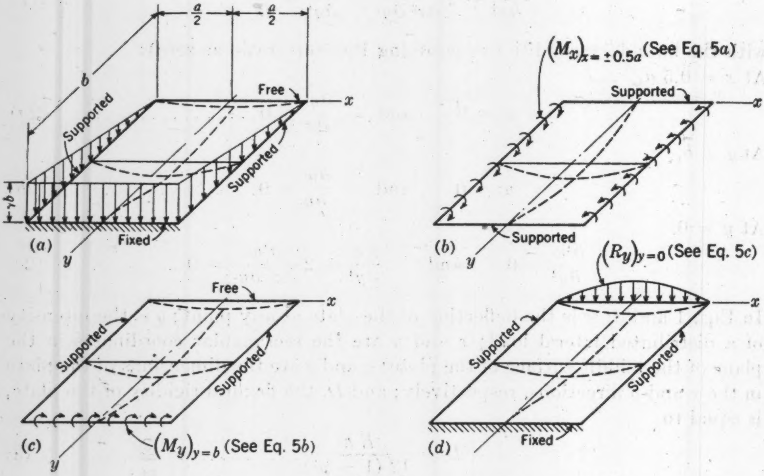
$\frac{a}{2}$
$\frac{a}{2}$
b
Free
x
Supported
Supported
γb
Fixed
(*a*)
y
$(M_x)_{x=\pm 0.5a}$ (See Eq. 5*a*)
Supported
x
Supported
(*b*)
y
$(R_y)_{y=0}$ (See Eq. 5*c*)
Free
x
Supported
Supported
(*c*)
$(M_y)_{y=b}$ (See Eq. 5*b*)
y
x
Supported
Supported
Fixed
(*d*)

FIG. 2

the distribution of which along the edges is taken in the form of a trigonometric series with undetermined coefficients F_m. Fig. 2(*c*) is bent by edge moments

$$(M_y)_{y=b} = \sum_{k=1,3,5,\cdots} E_k \cos\frac{k\,\pi\,x}{a} \ldots\ldots\ldots\ldots\ldots (5b)$$

and Fig. 2(*d*) is bent by arbitrary edge reactions at $y = 0$ of the form

$$(R_y)_{y=0} = \sum_{k=1,3,5\cdots} R_k \cos\frac{k\,\pi\,x}{a} \ldots\ldots\ldots\ldots\ldots\ldots (5c)$$

In Eqs. 5, M_x and M_y are the bending moments per unit length of plate on sections perpendicular to the x and y axes, respectively; R_x and R_y are the reactions per unit length of plate along edges perpendicular to the x and y axes, respectively; m and k are integers, with values as indicated on the summation signs; and F_m, E_k, R_k, etc., are constants. The boundary conditions at the edges of these plates are as shown in Fig. 2. The deflection surface for each of these plates may be obtained by Levy's method and then these four solutions combined in such a manner that Eq. 1 and the boundary conditions, Eqs. 2, are satisfied. This leads to a set of simultaneous algebraic equations from which the values of the coefficients F_m, E_k, and R_k may be calculated.

The method of obtaining the deflection equation for the four cases will be illustrated by considering Fig. 2(*d*). Results for the other cases will then be given without proof since the same procedure applies.

DEFLECTION EQUATION FOR FIG. 2(*d*)

Denote the deflection in this case by w_4. Since there is no distributed lateral load on the plate, the differential equation (Eq. 1) reduces to

$$\frac{\partial^4 w_4}{\partial x^4} + 2\frac{\partial^4 w_4}{\partial x^2\,\partial y^2} + \frac{\partial^4 w_4}{\partial y^4} = 0 \qquad \text{(6)}$$

The boundary conditions are:

At $x = \pm 0.5\,a$,

$$w_4 = 0 \quad \text{and} \quad \frac{\partial^2 w_4}{\partial x^2} = 0 \qquad \text{(7a)}$$

At $y = b$,

$$w_4 = 0 \quad \text{and} \quad \frac{\partial w_4}{\partial y} = 0 \qquad \text{(7b)}$$

At $y = 0$,

$$\frac{\partial^2 w_4}{\partial x^2} = 0 \quad \text{and} \quad -D\left(\frac{\partial^3 w_4}{\partial y^3} + 2\frac{\partial^3 w_4}{\partial x^2\,\partial y}\right) = \sum_{k=1,3,5\cdots} R_k \cos\frac{k\,\pi\,x}{a} \qquad \text{(7c)}$$

Take the deflection function in the form

$$w_4 = \sum_{k=1,3,5\cdots} Y_k \cos\frac{k\,\pi\,x}{a} \qquad \text{(8)}$$

in which Y_k is a function of y alone. Substituting Eq. 8 in Eq. 6, the function Y_k is found to be

$$\begin{aligned} Y_k = {} & G_k \cosh\frac{k\,\pi\,y}{a} + G'_k\,\frac{k\,\pi\,y}{a}\cosh\frac{k\,\pi\,y}{a} \\ & + G''_k \sinh\frac{k\,\pi\,y}{a} + G'''_k\,\frac{k\,\pi\,y}{a}\sinh\frac{k\,\pi\,y}{a} \end{aligned} \qquad \text{(9)}$$

From the boundary conditions Eqs. 7*b* and 7*c*, the four constants in Eq. 9 are found to be

$$\left.\begin{aligned} G_k &= \frac{2\,a^3\,R_k}{k^3\,\pi^3\,D\,K_k}\left[\tanh\alpha_k - \alpha_k\,(1 - \tanh^2\alpha_k)\right] \\ G'_k &= \frac{a^3\,R_k}{k^3\,\pi^3\,D\,K_k}\,(2 - \tanh^2\alpha_k) \\ G''_k &= G'_k + \frac{a^3\,R_k}{k^3\,\pi^3\,D} \\ G'''_k &= -\,0.5\,G_k \end{aligned}\right\} \qquad \text{(10)}$$

in which

$$K_k = \tanh^2\alpha_k - \alpha^2_k\,(1 - \tanh^2\alpha_k) - 4 \qquad \text{(11)}$$

α_k being equal to $\dfrac{k\,\pi\,b}{a}$.

Deflection Equation for Figs. 2(*a*), 2(*b*), and 2(*c*)

The deflection equations for the other cases are derived in a similar manner; thus, for Fig. 2(*a*):

$$w_1 = \frac{\gamma a^4}{\pi^5 D} \sum_{k=1,3,5\cdots} \frac{(-1)^{\frac{k-1}{2}}}{k^5} \left(4\,y + A_k \cosh\frac{k\,\pi\,y}{a} + A'_k \frac{k\,\pi\,y}{a} \cosh\frac{k\,\pi\,y}{a} + A''_k \sinh\frac{k\,\pi\,y}{a} + A'''_k \frac{k\,\pi\,y}{a} \sinh\frac{k\,\pi\,y}{a}\right) \cos\frac{k\,\pi\,x}{a} \qquad (12)$$

in which: $A_k = \frac{a\,\delta_k}{k\,\pi\,K_k}$; $A'_k = \frac{a\,\omega_k}{k\,\pi\,K_k}$; $A''_k = A'_k - \frac{8\,a}{k\,\pi}$; $A'''_k = -\,0.5\,A_k$; and

$$\delta_k = 16\,\alpha_k\,(1 - \tanh^2\alpha_k) - 16\tanh\alpha_k + \frac{8}{\cosh\alpha_k}(\alpha_k - \tanh\alpha_k + \alpha^2_k \tanh\alpha_k) \qquad (13a)$$

$$\omega_k = 8\tanh^2\alpha_k - 16 + \frac{1}{\cosh\alpha_k}(8 + 4\,\alpha^2_k - 8\,\alpha_k\tanh\alpha_k) \qquad (13b)$$

and K_k is defined by Eq. 11.

Similarly, the deflection equation for Fig. 2(*b*) is:

$$w_2 = \frac{b^2}{2\,\pi^2 D} \sum_{m=1}^{\infty} \frac{F_m}{m^2\cosh\beta_m} \times \left(\beta_m \tanh\beta_m \cosh\frac{m\,\pi\,x}{b} - \frac{m\,\pi\,x}{b}\sinh\frac{m\,\pi\,x}{b}\right) \sin\frac{m\,\pi\,y}{b} \qquad (14a)$$

and the deflection equation for Fig. 2(*c*) is:

$$w_3 = \frac{a^2}{\pi^2 D} \sum_{k=1,3,5\cdots} \frac{E_k}{k^2\cosh\alpha_k} \left(B_k \cosh\frac{k\,\pi\,y}{a} + B'_k \frac{k\,\pi\,y}{a}\cosh\frac{k\,\pi\,y}{a} + B''_k \sinh\frac{k\,\pi\,y}{a} + B'''_k \frac{k\,\pi\,y}{a}\sinh\frac{k\,\pi\,y}{a}\right) \cos\frac{k\,\pi\,x}{a} \qquad (14b)$$

In Eqs. 14: $\beta_m = \frac{m\,\pi\,a}{2\,b}$; $\alpha_k = \frac{k\,\pi\,b}{a}$; and—

$$\left.\begin{aligned} B_k &= \frac{\alpha_k + \tanh\alpha_k}{3\tanh\alpha_k + \alpha_k\,(1 - \tanh^2\alpha_k)} \\ B'_k &= \frac{0.5\,\alpha_k\tanh\alpha_k - 1}{3\tanh\alpha_k + \alpha_k\,(1 - \tanh^2\alpha_k)} \\ B''_k &= B'_k \\ B'''_k &= -\,0.5\,B_k \end{aligned}\right\} \qquad (15)$$

EQUATIONS FOR DETERMINING F_m, E_k, AND R_k

The solution for the given problems is to be obtained by superposition of the solutions for the four cases in Fig. 2, by taking

$$w = w_1 + w_2 + w_3 + w_4 \quad \text{(16)}$$

Coefficients F_m, E_k, and R_k are to be so determined that the boundary conditions in Eq. 2 will be satisfied. Thus, for the condition that the slope is zero at $y = b$,

$$\left(\frac{\partial w_2}{\partial y}\right)_{y=b} + \left(\frac{\partial w_3}{\partial y}\right)_{y=b} = 0 \quad \text{(17)}$$

Differentiating, Eq. 14*a* gives

$$\left(\frac{\partial w_2}{\partial y}\right)_{y=b} = \frac{b}{2\,\pi\,D} \sum_{m=1}^{\infty} \frac{(-1)^m F_m}{m \cosh \beta_m} \times \left(\beta_m \tanh \beta_m \cosh \frac{m\,\pi\,x}{b} - \frac{m\,\pi\,x}{b} \sinh \frac{m\,\pi\,x}{b}\right) \quad \text{(18a)}$$

and from Eq. 14*b*

$$\left(\frac{\partial w_3}{\partial y}\right)_{y=b} = \frac{a}{2\,\pi\,D} \sum_{k=1,3,5\cdots} \frac{E_k}{k} \frac{(1 + \alpha^2_k)(\tanh^2 \alpha_k - 1) - 3}{3 \tanh \alpha_k + \alpha_k (1 - \tanh^2 \alpha_k)} \cos \frac{k\,\pi\,x}{a} \quad \text{(18b)}$$

Before substituting the two slope expressions in Eq. 17, it is expedient to expand the expression inside the brackets in Eq. 18*a* in a Fourier cosine series. Doing this, and then substituting in Eq. 17 the following equation is obtained connecting the coefficients E_k and F_m:

$$E_k \frac{(1 + \alpha^2_k)(1 - \tanh^2 \alpha_k) + 3}{3 \tanh \alpha_k + \alpha_k (1 - \tanh^2 \alpha_k)} = \frac{8\,(-1)^{\frac{k-1}{2}}\,k^2\,b^3}{\pi\,a^3} \sum_{m=1}^{\infty} \frac{(-1)^m\,m\,F_m}{\left(m^2 + \dfrac{k^2\,b^2}{a^2}\right)^2} \quad \text{(19)}$$

There will be one equation of the form of Eq. 19 for each value of $k = 1, 3, 5, \cdots$, and each of these will have an infinite number of terms on the right-hand side.

In a similar manner, from the condition that the reaction

$$R_y = -D\left(\frac{\partial^3 w}{\partial y^3} + 2\,\frac{\partial^3 w}{\partial x^2\,\partial y}\right) \quad \text{(20a)}$$

at $y = 0$ must be zero, the following system of equations is obtained:

$$R_k = -\frac{4\,(-1)^{\frac{k-1}{2}}\,k\,b}{a^2} \sum_{m=1}^{\infty} m\,F_m \frac{m^2 + \dfrac{2\,k^2\,b^2}{a^2}}{\left(m^2 + \dfrac{k^2\,b^2}{a^2}\right)^2} \quad \text{(20b)}$$

From the condition that the slope at $x = 0.5\,a$ must be zero:

$$
\begin{aligned}
&F_m\left(\tanh\beta_m + \frac{\beta_m}{\cosh^2\beta_m}\right) + \frac{4\,m^2 a^3}{\pi\,b^3}\sum_{k=1,3,5\cdots}\frac{k\,(-1)^{\frac{k-1}{2}}\,E_k}{\left(k^2 + \frac{m^2 a^2}{b^2}\right)^2} \\
&\times\left\{\frac{(\alpha_k + \tanh\alpha_k)\left(2 + \frac{m^2 a^2}{k^2 b^2}\right)}{[3\tanh\alpha_k + \alpha_k\,(1 - \tanh^2\alpha_k)]\cosh\alpha_k} - (-1)^m\right\} \\
&+ \frac{8\,m^2 a^4}{\pi^2 b^3}\sum_{k=1,3,5\cdots}\frac{(-1)^{\frac{k-1}{2}}\,R_k}{K_k\left(k^2 + \frac{m^2 a^2}{b^2}\right)^2}\Bigg\{[\tanh\alpha_k - \alpha_k\,(1 - \tanh^2\alpha_k)] \\
&\times\left(2 + \frac{m^2 a^2}{k^2 b^2}\right) + (-1)^m\frac{\alpha_k + \tanh\alpha_k}{\cosh\alpha_k}\Bigg\} = -\frac{4\,\gamma\,a^4}{\pi^4 b}\sum_{k=1,3,5\cdots}\frac{1}{k^3} \\
&\times\Bigg\{-\frac{(-1)^m\,4\,k\,\frac{b}{a}}{k^2 + \frac{m^2 a^2}{b^2}} + \frac{\frac{m^2 a^2}{\pi\,b^2}}{\left(k^2 + \frac{m^2 a^2}{b^2}\right)^2 K_k}\Bigg[\delta_k\left(2 + \frac{m^2 a^2}{k^2 b^2}\right) \\
&\qquad + (-1)^m\cosh\alpha_k\,(-\,\delta_k + 2\,\omega_k\tanh\alpha_k)\Bigg]\Bigg\} \qquad\qquad (21)
\end{aligned}
$$

Eqs. 19, 20*b*, and 21 provide the necessary relations for determining F_m, E_k, and R_k. Each one defines an infinite system of equations with an infinite number of terms. To solve these numerically, it is necessary to assume that the series converge and to neglect all terms higher than a given order. The result is a finite system of simultaneous equations that may be solved to obtain values for the coefficients. This completes the solution of the differential Eq. 1 for the specified boundary conditions. The moments at any point may then be determined from the known relations of the plate theory. For example, the moment per unit length of plate on sections perpendicular to the x-axis is given by

$$
M_x = -\,D\frac{\partial^2 w}{\partial x^2} = -\,D\left(\frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_3}{\partial x^2} + \frac{\partial^2 w_4}{\partial x^2}\right) \qquad (22)
$$

assuming that Poisson's ratio is zero.

NUMERICAL EXAMPLE

For an example, assume a tank with square sides for which $a = b$; and in each of Eqs. 19, 20*b*, and 21 take only the first four terms of each series. Then in Eq. 19, using $m = 1, 2, 3$, and 4, and taking $k = 1, 3, 5$, and 7, successively, the numerical values of the constants may be computed to obtain the four equations:

$$
\left.
\begin{aligned}
1.0228\,E_1 &= -\,0.6366\,F_1 + 0.2037\,F_2 - 0.0764\,F_3 + 0.0352\,F_4 \\
1.0000\,E_3 &= 0.2292\,F_1 - 0.2712\,F_2 + 0.2122\,F_3 - 0.1467\,F_4 \\
1.0000\,E_5 &= -\,0.0942\,F_1 + 0.1514\,F_2 - 0.1652\,F_3 + 0.1515\,F_4 \\
1.0000\,E_7 &= 0.0499\,F_1 - 0.0888\,F_2 + 0.1113\,F_3 - 0.1181\,F_4
\end{aligned}
\right\} \qquad (23a)
$$

Calculation of the constants in Eq. 20*b* gives the system:

$$\left.\begin{aligned}
a R_1 &= -\,3.0000\,F_1 - 1.9200\,F_2 - 1.3200\,F_3 - 0.9965\,F_4 \\
a R_3 &= 2.2800\,F_1 + 3.1243\,F_2 + 3.0000\,F_3 + 2.6112\,F_4 \\
a R_5 &= -\,1.5089\,F_1 - 2.5683\,F_2 - 3.0623\,F_3 - 3.1410\,F_4 \\
a R_7 &= 1.1088\,F_1 + 2.0335\,F_2 + 2.6718\,F_3 + 3.0220\,F_4
\end{aligned}\right\} \ldots (23b)$$

Finally, from Eq. 21, compute the following set of numerical equations:

$$\left.\begin{aligned}
&1.1666\,F_1 + 0.4314\,E_1 - 0.0382\,E_3 + 0.0094\,E_5 \\
&\qquad - 0.0036\,E_7 - 0.1685\,a R_1 + 0.0057\,a R_3 \\
&\qquad\qquad - 0.0008\,a R_5 + 0.0002\,a R_7 = -\,0.04694\,\gamma a^3 \\
&1.0197\,F_2 - 0.0589\,E_1 + 0.0903\,E_3 - 0.0303\,E_5 \\
&\qquad + 0.0127\,E_7 - 0.2608\,a R_1 + 0.0156\,a R_3 \\
&\qquad\qquad - 0.0028\,a R_5 + 0.0008\,a R_7 = -\,0.03474\,\gamma a^3 \\
&1.0014\,F_3 + 0.2640\,E_1 - 0.1063\,E_3 + 0.0496\,E_5 \\
&\qquad - 0.0238\,E_7 - 0.2450\,a R_1 + 0.0225\,a R_3 \\
&\qquad\qquad - 0.0050\,a R_5 + 0.0016\,a R_7 = -\,0.02586\,\gamma a^3 \\
&1.0001\,F_4 + 0.0799\,E_1 + 0.0976\,E_3 - 0.0606\,E_5 \\
&\qquad + 0.0338\,E_7 - 0.2603\,a R_1 + 0.0261\,a R_3 \\
&\qquad\qquad - 0.0068\,a R_5 + 0.0024\,a R_7 = -\,0.03485\,\gamma a^3
\end{aligned}\right\} \ldots (23c)$$

There are now 12 unknowns in 12 simultaneous equations. Solving these equations yields the following values for the coefficients:

$$\begin{aligned}
F_1 &= -\,0.0314\,\gamma a^3 & E_1 &= 0.0190\,\gamma a^3 & R_1 &= 0.1076\,\gamma a^2 \\
F_2 &= -\,0.0030\,\gamma a^3 & E_3 &= -\,0.0063\,\gamma a^3 & R_3 &= -\,0.0999\,\gamma a^2 \\
F_3 &= -\,0.0026\,\gamma a^3 & E_5 &= 0.0023\,\gamma a^3 & R_5 &= 0.0765\,\gamma a^2 \\
F_4 &= -\,0.0043\,\gamma a^3 & E_7 &= -\,0.0011\,\gamma a^3 & R_7 &= -\,0.0608\,\gamma a^2
\end{aligned}$$

The bending moments and the deflection at any point may now be computed, the results being shown in Figs. 3 and 4.

The numerical values given are approximate due to the fact that only four terms were taken in each series. However, each of the functions used satisfies the differential equation and gives zero deflection along the sides and bottom of the plate. The inaccuracy arises from the fact that the slope at these edges will not quite be zero, and the top edge will not quite be free, unless an infinite number of terms is used. An insight into the error involved may be

TABLE 1.—Numerical Values[a] for Edge Slopes

Location		Edge Slopes for Plates Shown in:				Resultant slope
x	y	Fig. 2(*a*)	Fig. 2(*b*)	Fig. 2(*c*)	Fig. 2(*d*)	
0	b	0	0.00281	−0.00280	0	0.00001
0.5 a	0	−0.00783	0	−0.00072	0.00797	−0.00058
0.5 a	0.25 b	−0.00773	0.00446	−0.00089	0.00428	0.00012
0.5 a	0.50 b	−0.00663	0.00569	−0.00113	0.00202	0.00005
0.5 a	0.75 b	−0.00336	0.00401	−0.00122	0.00060	0.00003

[a] Numerical values in this table are coefficients that should be multiplied by $\frac{\gamma a^4}{D}$ to obtain slope.

obtained by computing the actual slope along the edges for the problem as solved. Results of such a calculation are given in Table 1. From this it is seen that the actual edge slopes remaining are only a few per cent of the corresponding slopes for Fig. 2(a) which has simply supported edges. So it may

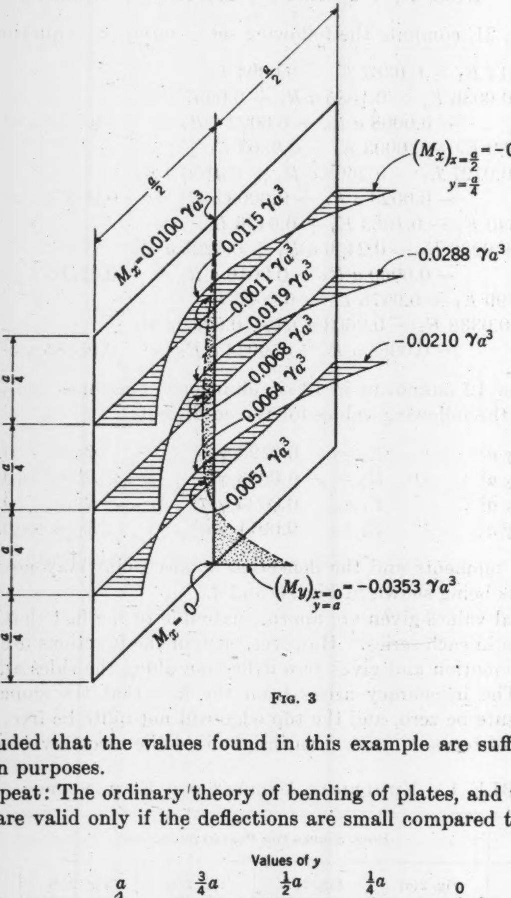
$\frac{a}{2}$
$\frac{a}{2}$
$(M_x)_{x=\frac{a}{2},\ y=\frac{a}{4}} = -0.0270\ \gamma a^3$
$M_x = 0.0100\ \gamma a^3$
$0.0115\ \gamma a^3$
$-0.0288\ \gamma a^3$
$0.0011\ \gamma a^3$
$0.0119\ \gamma a^3$
$-0.0210\ \gamma a^3$
$0.0068\ \gamma a^3$
$0.0064\ \gamma a^3$
$0.0057\ \gamma a^3$
$\frac{a}{4}$
$\frac{a}{4}$
$\frac{a}{4}$
$\frac{a}{4}$
$(M_y)_{x=0,\ y=a} = -0.0353\ \gamma a^3$
$M_x = 0$

FIG. 3

be concluded that the values found in this example are sufficiently accurate for design purposes.

To repeat: The ordinary theory of bending of plates, and the results found from it, are valid only if the deflections are small compared to the wall thick-

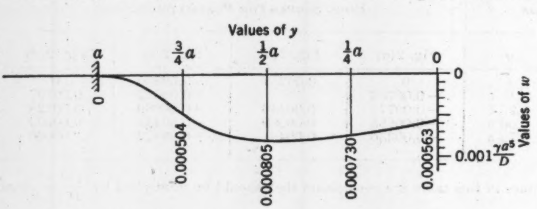

FIG. 4.—DEFLECTION ALONG VERTICAL CENTER LINE

ness. This condition may be readily investigated for any particular design. For example, take a steel water tank with sides 100 in. square. The maximum moment per linear inch is $M = -0.0353\,\gamma\,a^3 = -0.0353 \times 100^3 \times \frac{62.5}{1{,}728}$ $= -1{,}280$ in-lb per in. Assuming an allowable bending stress of 18,000 lb per sq in., $\frac{18{,}000\,h^2}{6} = 1{,}280$; from which $h = 0.654$ in., say $\frac{11}{16}$ in. The maximum deflection is given by $w = 0.000805\,\frac{\gamma\,a^5}{D} = 0.33$ in. using $E = 30{,}000{,}000$ lb per sq in. and $\mu = 0.3$. This gives a ratio of deflection to wall thickness of $\frac{0.33}{0.6875} = 0.48$ which is just about the maximum permissible.

OTHER SOLUTIONS

This same problem has been solved by other investigators using different methods. P. T. Cheng[8] approached the problem by choosing deflection functions containing arbitrary constants which he determined numerically by satisfying the boundary conditions at a finite number of points along the edges. His final values for the moments and deflections check, very closely, the values given herein. Calculations by the method of finite differences have been made by A. Smotrow[9] for the same problem, using a network of 36 squares. The moment values thus obtained are appreciably smaller than those found in this paper. For example, the maximum moment at the base is given by Smotrow as $0.030\,\gamma\,a^3$ as compared to $0.0353\,\gamma\,a^3$ given in Fig. 3. This is to be expected, and it is believed that a closer network of points would have to be taken in the finite difference method to obtain the accuracy of the superposition method used herein. Smotrow also completed the calculations for plates with $\frac{b}{a}$-ratios of 1.5 and 0.67.

MOMENTS FOR PLATES OF OTHER PROPORTIONS

Calculations have been given, in this paper, only for a square plate in which the ratio $\frac{b}{a}$, of the height to the width, is unity. Having the moment values for this case as a standard of comparison, it is possible to approximate moments for plates with other $\frac{b}{a}$-ratios by the consideration of known solutions of plates with the same loading but having other edge conditions.

Consider first the moment M_y at the center of the base of Figs. 5(*a*), 5(*b*), and 5(*c*), all of which carry the same hydrostatic load. Fig. 5(*a*) is the one treated in this paper. Fig. 5(*b*) is similar except that the top edge is simply supported, whereas Fig. 5(*c*) has simply supported side edges and is the same as Fig. 2(*a*). It is clear that the base moment for Fig. 5(*a*) will be greater than for Fig. 5(*b*) and less than for Fig. 5(*c*). For values of $\frac{b}{a}$ greater than unity, the moment for Fig. 5(*a*) will approach that of Fig. 5(*b*), and for values of $\frac{b}{a}$ less than unity, it will approach that of Fig. 5(*c*). Moments for Fig. 5(*b*) with various $\frac{b}{a}$-ratios have been published,[10] and values for Fig. 5(*c*) may be readily calculated from Eq. 12. In Fig. 6 estimated values for Fig. 5(*a*) are indicated by the broken curve drawn through the control point at $\frac{b}{a} = 1$ and approaching the other two curves for other values of $\frac{b}{a}$.

---

[8] "The Analysis of Rectangular Plates Under Various Loading and Boundary Conditions," by P. T. Cheng, unpublished research work at Univ. of Minnesota, Minneapolis, Minn., 1940.

[9] "Berechnung von Platten," by A. Smotrow (in Russian), *Staats-Verlag für Literatur des Bauwesens*, Moskau und Leningrad, 1936.

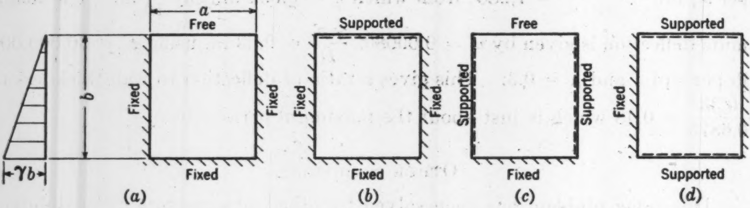
a
Free
b
Fixed
Fixed
Fixed
γb
(a)
Supported
Fixed
Fixed
Fixed
(b)
Free
Supported
Supported
Fixed
(c)
Supported
Fixed
Fixed
Supported
(d)

Fig. 5

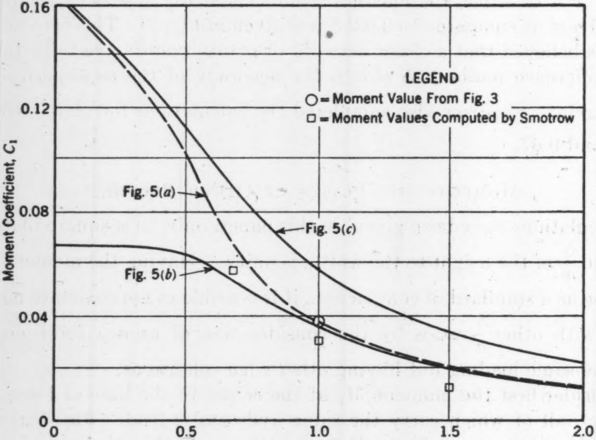

Fig. 6.—Moment at Center of Base: $M_y = -C_1 \gamma b^3$

For the edge moment at the midpoint of each side, it seems reasonable to assume that the value for Fig. 5(*a*) will lie between that for Fig. 5(*b*) and that for Fig. 5(*d*). Values for the case represented by Fig. 5(*d*) are given by Professor Timoshenko.[11] Curves for Figs. 5(*b*) and 5(*d*) are plotted in Fig. 7 from data in the references cited and the assumed curve for Fig. 5(*a*) is indicated by the broken line.

---

[10] "Analysis of Clamped Rectangular Plates," by D. Young, *Journal of Applied Mechanics*, December, 1940, p. A-139.

It is believed that these moment values are adequate for design purposes. For plates with $\frac{b}{a}$-ratios not too different from unity $\left(\text{say from } \frac{2}{3} \text{ to } \frac{3}{2}\right)$ the distribution of moments along the base may be taken as parabolic and along the sides may be conservatively considered as constant over the middle half and then decreasing uniformly to one half its maximum value at either end.

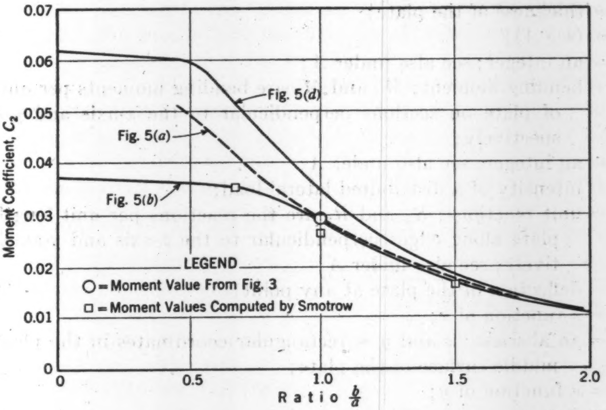

FIG. 7.—MOMENT AT CENTER OF SIDE: $M_x = -C_2 \gamma b^3$

APPENDIX

NOTATION

The following letter symbols, adopted for use in this paper, conform essentially to Standard Letter Symbols for Mechanics, Structural Engineering and Testing Materials, prepared by a Committee of the American Standards Association with Society representation and approved by the Association in 1932.[12] A few obvious conflicts—such as D (standard symbol for diameter) = flexural rigidity and w (standard symbol for uniformly distributed load) = deflection of plate—have been introduced to adapt the notation to common terms found in current literature on flat plate analysis.

A = a constant; with subscripts m or k, the symbols A, B, E, F, G, K, R, δ, and ω are each constants defined where they are used;

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[11] "Theory of Plates and Shells," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., p. 210.

[12] ASA—Z10a—1932.

a = length of the plate in the x-direction;
B = (see A);
b = length of the plate in the y-direction;
D = flexural rigidity of the plate (see Eq. 3);
E = modulus of elasticity of the plate material; see also under A;
F = (see A);
G = (see A);
h = thickness of the plate;
K = (see A);
k = an integer; see also under A;
M = bending moment; M_x and M_y are bending moments per unit length of plate on sections perpendicular to the x-axis and y-axis, respectively;
m = an integer; see also under A;
q = intensity of a distributed lateral load;
R = unit reactions; R_x and R_y are the reactions per unit length of the plate along edges perpendicular to the x-axis and y-axis, respectively; see also under A;
w = deflection of the plate at any point;
X = a function of x;
x = an abscissa; x and y = rectangular coordinates in the plane of the middle surface of the plate;
Y = a function of y;
y = (see x);
$\alpha_k = \frac{k \pi b}{a}$;
$\beta_m = \frac{m \pi a}{2 b}$;
γ = specific weight of water; weight per unit volume of water;
δ = (see A);
μ = Poisson's ratio;
ω = (see A).

DISCUSSION

I. K. Silverman,[13] Assoc. M. Am. Soc. C. E.—Even with such a tractable ratio of $\frac{b}{a}$ equal to unity, the amount of calculation involved in the solution offered by the author is very great; and it is probably for this reason that Professor Young resorted to the "roughing in" of the curves denoted by 5(*a*) in Figs. 6 and 7. For determining the character of these curves, a single control point and two limiting curves were available.

Because of the amount of computation involved, any approximate solution that will give sufficiently accurate values with less computation is to be welcomed. Such an approximate solution is available[14] and the relative accuracy of the two can be compared. Unfortunately the Marcus approximate method was applied to the ratio $\frac{b}{a} = 0.5$ for case (*a*), Fig. 5, and the results for this case must be compared with values from Curves 5(*a*) of Figs. 6 and 7 which are approximate. Moreover, the Marcus method was applied to case (*b*), Fig. 5, and Table 2 shows the values obtained by Marcus for cases (*a*) and (*b*) compared with those taken from the curves of Figs. 6 and 7.

TABLE 2.—Comparison of C for $b/a = 0.5$

Method	C_1		C_2	
	Fig. 5(*a*)	Fig. 5(*b*)	Fig. 5(*a*)	Fig. 5(*b*)
Young....	0.102	0.064	0.049	0.036
Marcus...	0.082	0.063	0.042	0.039

It is seen that the results of the author's method and the Marcus approximate solution agree fairly well for case (*b*), but there is quite a discrepancy in the values given for case (*a*), Fig. 5. Since the Young values are taken from a curve that is "roughed in," no definite statement can be made concerning the accuracy of the Marcus method until a mathematically exact solution is made of curves 5(*a*), Figs. 6 and 7, using the Young method. However, the comparison of the values for case (*b*) would seem to indicate that the Marcus method can be relied upon. Another distinguishing feature is that the Marcus solution for $b/a = 0.5$ shows that the maximum value of M_x occurs at $x \doteq a/2, y = b/4$, and its value is $0.050\,\gamma\, b^3$

As the author states, the solution given in the paper can be applied to counterforted walls subjected to hydrostatic pressure by assuming constant thickness and fixity at the base and counterfort. An interesting problem is the analysis of the part of the counterforted wall, ABCD (Fig. 8(*a*)). Not only are the questions of moments and shears of interest, but the following question may be posed: What must be the ratio of b/a so that the deflection of line AD shall be sensibly the same as that of a cantilever of the same height?

The writer has applied the Marcus method to such a plate ABCD of Fig. 8(*a*) with b/a = unity and obtained results as shown in Fig. 8(*b*). For ratios $b/a < 1$ the Marcus method is not satisfactory when applied to plates ABCD of Fig. 8(*a*), and Marcus states that the computations become tedious for $b/a < 0.5$ when applied to walls of cases (*a*) and (*b*), Fig. 5.

---

[13] Portland, Me.

[14] "Beitrag zur Untersuching von Behältern mit ebeben Wandungen," by H. Marcus, *Der Bauingenieur*, February 7, 1936, p. 40; also "La Résistance des Réservoirs à Parois Planes," by the same author, *Annales des Ponts et Chaussées*, April, 1937, p. 540.

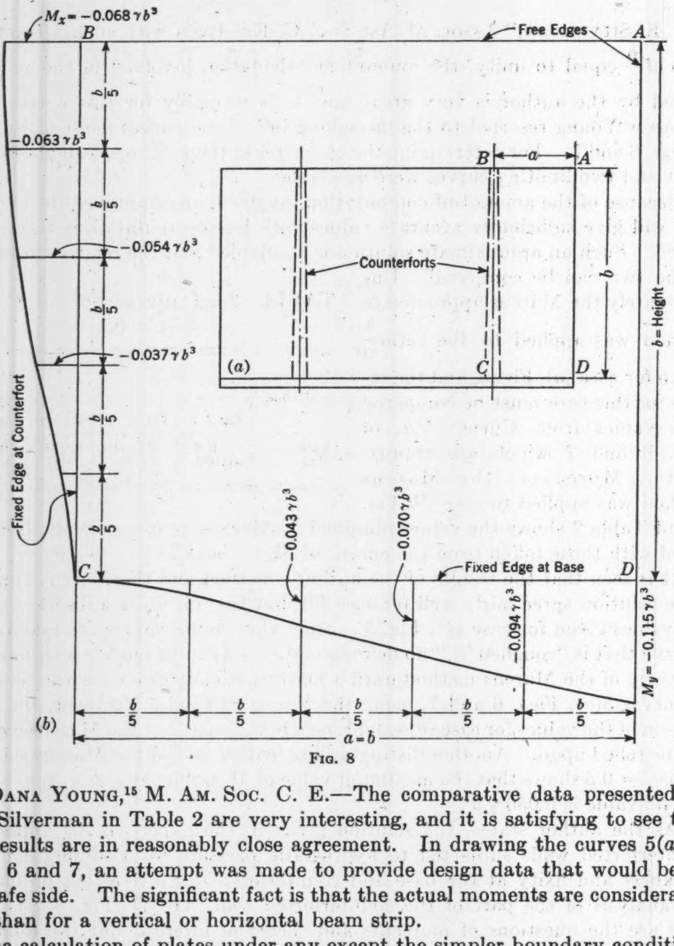
$M_x = -0.068\,\gamma b^3$
B
Free Edges
A
$-0.063\,\gamma b^3$
$-0.054\,\gamma b^3$
$-0.037\,\gamma b^3$
$\frac{b}{5}$
$\frac{b}{5}$
$\frac{b}{5}$
$\frac{b}{5}$
$\frac{b}{5}$
Fixed Edge at Counterfort
C
B
a
A
Counterforts
b
(a)
C
D
b = Height
$-0.043\,\gamma b^3$
$-0.070\,\gamma b^3$
Fixed Edge at Base
D
$-0.094\,\gamma b^3$
$M_y = -0.115\,\gamma b^3$
(b)
$\frac{b}{5}$
$\frac{b}{5}$
$\frac{b}{5}$
$\frac{b}{5}$
$\frac{b}{5}$
a = b

FIG. 8

DANA YOUNG,[15] M. AM. SOC. C. E.—The comparative data presented by Mr. Silverman in Table 2 are very interesting, and it is satisfying to see that the results are in reasonably close agreement. In drawing the curves 5(*a*) of Figs. 6 and 7, an attempt was made to provide design data that would be on the safe side. The significant fact is that the actual moments are considerably less than for a vertical or horizontal beam strip.

The calculation of plates under any except the simpler boundary conditions is at best a tedious task, and it is not expected that the detailed calculations should be made by every designer. Fortunately, it is not necessary to do so, any more than it is necessary for a designer to integrate the equation of the elastic curve every time he wishes to know the moments in a clamped beam. A table of moments for all useful cases is what is needed; the results given represent a start in this direction.

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[15] Prof., Applied Mechanics, The Univ. of Texas, Austin, Tex.

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DESIGN OF SIGN LETTER SIZES

BY ADOLPHUS MITCHELL,[1] ASSOC. M. AM. SOC. C. E., AND
T. W. FORBES,[2] ESQ.

WITH DISCUSSION BY MESSRS. GUY KELCEY, CHARLES M. NOBLE, AND
ADOLPHUS MITCHELL AND T. W. FORBES.

SYNOPSIS

There has been a growing interest and attention among traffic engineers, and others, in improving the design of highway warning and destination signs. The effectiveness and usefulness of the highway are increased by proper signs just as the proper use of a mechanical article of merchandise is insured by clear and adequate instructions on the box.

There are two characteristics of properly designed highway signs: (1) They must be readable and they must give the motorist sufficient warning to permit making the maneuver comfortably; and (2) they must be designed so as to gain the motorist's attention. To gain attention (characteristic (2)) is a complex problem which deserves separate treatment and is beyond the scope of this paper. Since certain standard color combinations are now agreed upon by state and national highway agencies, these standard color combinations will be assumed in the paper.

FUNDAMENTAL FACTORS IN LETTER SIZE

On the basis of measurements that have been made of the various factors resulting in the readability, or legibility, of a sign, it is possible to derive methods and formulas for practical determination of the minimum letter size necessary to make them effective under given traffic conditions on the highway.

It is obvious that the size of the letters must depend not only on traffic characteristics (since drivers need more warning at high speeds), but also upon vision because the effectiveness of sign copy depends upon what the driver can see. Knowledge is now available from studies of these various factors to set up a method of the size of letters on highway signs. To compute the size of a letter for a highway sign it is necessary to consider design speed, the location of the

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[1] Senior Traffic Engr., State Highway and Public Works Comm., Raleigh, N. C.

[2] Bureau for Street Traffic Research, Yale Univ., New Haven, Conn.

sign, warning time, type of letter (letter proportion), and legibility distance. The following sub-factors influence the determination of warning time: Time for the minimum glance, safety factor in case the sign is not seen at the first possible moment, perception-reaction time, and stopping or deceleration time as necessitated by circumstances.

Design Speed.—Design speed may be defined as that speed at, or below which, a certain proportion of the vehicles operate at the location in question. The proportion (or percentile speed) used by different engineers has ranged from 80% to 90%,[3] and D. W. Loutzenheiser, Jun. Am. Soc. C. E., in 1940, reported[4] an analysis of speed data in an attempt to derive a similar definition for the design application. This report indicates that at least a 90 percentile speed is desirable. Since the higher percentile value results in a higher speed and a longer warning distance, it represents the conservative choice; and the 90 percentile speed of vehicles operating on the highway at the point to be signed is to be recommended for the design of sign-letter size.

Obviously, it is not practicable to make a "spot speed study" at all points along the highway, and since most highways in open country show similar speed characteristics throughout certain sections, it is sufficient to use one speed determination for such sections and another for special conditions, such as suburban and congested urban conditions. At special locations, such as curves, the techniques now in use for speed zoning would be continued. Thus the spot speed check might be replaced by the use of a speed determination from the ball bank indicator, calibrated for such special application.

Sign Location.—The position of the sign on the highway will influence the necessary letter size longitudinally and laterally. In the first place, the sign placed ahead of the hazard will allow the use of a smaller letter size, whereas one at or behind the hazard will require a larger letter size if the driver is to see it at a given distance or warning time away. In the second place, the lateral placement of the sign will influence the point at which the driver's gaze must leave the sign and return to the road. Longitudinal placement is illustrated diagrammatically in Fig. 1. If the driver of vehicle 1 is to read the sign at the position indicated, and if *O* represents the point of hazard or the point at which the maneuver must begin, a sign placed at a distance *A* ahead of this point will require a smaller letter size than a sign placed at a position *A′* behind this point.

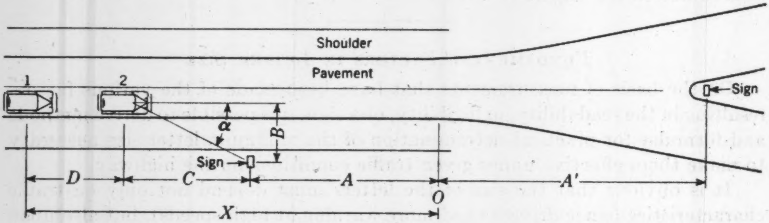

FIG. 1

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[3] "Report of Committee on Safe Approach Speeds," by H. H. Hammond, *Proceedings*, Highway Research Bd., 1940, Vol. 20, p. 653.

[4] "Percentile Speeds on Existing Highways," by D. W. Loutzenheiser, *ibid.*, 1940.

Therefore the distance A may be subtracted or the distance A' added to the necessary warning distance in deriving the necessary legibility distance.

Whether the signs should be placed at position A' or at position A involves the question of attention values, which is beyond the scope of this paper. The lateral position of the sign affects the "cutoff" point of the driver's vision as illustrated in Fig. 1 by the position of vehicle 2. At this point, the line of sight, as the driver is looking at the sign, will form angle α with the center line of the lane of travel as shown. If the maximum allowable angle α is known, the position of vehicle 2 beyond which the driver's line of sight must leave the sign and return to the road can be determined; thus:

$$C = B \cot \alpha \quad \text{.........................(1)}$$

in which B is the distance in feet from the sign to the center line of the traveled lane. The clearest seeing occurs[5] in a small zone in the center of the visual field which is delimited by an optical angle of about 5°. In the area just outside of this 5°, the degree of clear vision reduces rapidly until at about 10° (5° to either side of the line of sight) there is a sharp break in the acuity curve indicating that in a position more lateral than this, the vision experienced in the periphery of the eye is of the fuzzier type. The driver ordinarily glances about the landscape with short quick eye movements, resembling flashes of a highly focused spot light or search light.[6] A practical value for the angle α may be obtained on the basis of allowing two minimum 5° eye movements to the side of the center line of the traveled lane. This will allow some part of the traveled lane to remain in the 10° cone of fairly clear vision up to about 50 ft ahead of the car.

Then, assuming $\alpha = 10°$, Eq. 1 becomes

$$C = B \cot 10° = 5.7\,B \quad \text{....................(2)}$$

From this relationship, if B is 17.5 ft (one half of the 10-ft lane plus a 10-ft shoulder, plus 2.5 ft from the shoulder to the center of the sign) the distance C will be approximately 100 ft. This gives a rule-of-thumb idea of the magnitudes involved. The hypothetical position of vehicle 2 should be in the lane most unfavorable for vision of the sign in the location in question.

Warning Time.—The importance of thinking in terms of time in order to allow the driver sufficient warning has already been mentioned. Since the process of reading, as well as the perception and reaction of the driver, may be assigned fairly definite time values, and since the use of time values automatically corrects for the greater distance at higher speeds, they are fundamental. If sufficient warning time is given the driver and if he is duly impressed with the nature of the hazard to which he is coming, he will be able to operate comfortably and without any surprise element. To be adequate, warning time must allow for: (1) Minimum time for reading, (2) a safety factor in case the sign is not seen at the first possible instant, (3) perception-reaction time, and (4) stopping or deceleration time necessary for the maneuver involved. Ordinarily, the first three of these time values (those in connection with reading and perception reaction) will pass before any slowing occurs. Therefore, they may be treated as pure time values and converted into distance values by multiplying by the design speed.

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[5] "Textbook of Physiology," by W. H. Howell, W. B. Saunders Co., Philadelphia, Pa., 1940.

[6] *Transactions*, Am. Soc. C. E., Vol. 106 (1940), p. 456.

Minimum Reading Time and Safety Factor.—The shortest possible glance from the road to read the sign and back to the road consumes from 0.6 to 1.0 sec. During this glance the maximum amount of copy which can be read by the ordinary person is from three to four familiar words.[7] This time value results from the fact that it takes approximately 0.2 sec for the eye to stop, focus and read, and another 0.2 sec for the eye to start and move through one of the 5° or 10° jumps. To be conservative, therefore, 1.0 sec is adopted as the time necessary for a single minimum glance from road to sign and back to the road, allowing the shortest possible glance at both the road and the sign.

If more than three familiar words are included in the copy, it has been shown that the time for reading the sign may be increased to as much as from 3 to 11 sec,[8] and such signs, therefore, are impracticable on high-speed highways. Where they are unavoidable the reading time should be increased by one second for each additional three or four familiar words, thus making allowance for the driver to glance back to the road between glances at the sign.

It is still necessary to add a time interval as a safety factor in case the sign is not seen at once. The smallest possible safety factor is one 1-sec glance; that is, the minimum reading time is 2 t_g, in which t_g is the time required for a single glance or 1.0 sec and when signs contain the minimum number of words. When the sign contains more than three words:

$$t_g = \frac{N}{3} + 1.0 \qquad (3)$$

in which N equals the number of familiar words on the sign. If the attention or target value has been properly designed into the sign, the motorist will have seen it before he is able to read it, and Eq. 3, with 2 sec as an absolute minimum, will guarantee him time to read the sign twice unless something distracts him or obstructs his vision.

Perception-Reaction Time.—It takes time for the human nervous and muscular system to react, just as it takes time to complete a call through the telephone exchange. This time value increases as the complexity of the process increases, so that there is a range of time values that may be classified under three headings in order of magnitude—that is, simple brake-reaction time such as that in an emergency situation (from 0.5 to 0.7 sec),[9] perception-reaction time in ordinary traffic situations of medium complexity, and finally, judgment perception-reaction time (2.8 to 3.5 sec).[10]

The perception-reaction time needed to observe signs is intermediate between the simple reaction time and the complex skilled judgment that involves the longest time interval. Interpolated between the values of these two, it will not be far in error to assume a perception-reaction time of 1.5 sec for sign design.

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[7] "A Method for Analysis of the Effectiveness of Highway Signs," by T. W. Forbes, *Journal of Applied Psychology*, Vol. 23, 1939, p. 669.

[8] "Improvement in Highway Safety," by A. R. Lauer, *Proceedings*, Highway Research Bd., Vol. 12, 1932, p. 389.

[9] "Driver Test Results," by H. R. DeSilva and T. W. Forbes, Harvard Traffic Bureau and WPA of Massachusetts, Boston, 1937.

[10] "Methods of Measuring Judgment and Perception Time in Passing on the Highway," by T. W. Forbes, *Proceedings*, Highway Research Bd., Vol. 19, 1939, p. 218.

Deceleration Time.—Consider a point at which some vehicles are decelerating (case 1) while others are continuing with practically no change of speed. This case is best illustrated at grade-separated intersections where some drivers are decelerating and turning off while others are going on through the intersection. Obviously, those drivers who wish to turn off must have the time and distance to decelerate comfortably; and at the same time those who proceed at initial velocity must have sufficient warning to choose the correct lane and to avoid incoming and outgoing vehicles. Somewhat similar conditions hold for intersections at grade on low-volume high-speed highways.

A report by John Beakey,[11] in which the speed checks were made on vehicles approaching a stop sign at a rural intersection, indicates that the general driving public utilized approximately 10 sec on the average for decelerating from 55 miles to 20 miles per hr. If the deceleration times for lower initial speeds as reported in this study are examined, it will be found that deceleration to 20 miles per hr in each case approximated 8 sec. These measurements would seem to justify the use of a value of 8 to 10 sec for comfortable deceleration time on most highways. Since this paper is dealing with the minimum values for necessary warnings, the lower value of 8 sec will be adopted. Deceleration to 20 miles per hr was chosen as being applicable in the majority of highway intersection applications. Time values for higher final speeds are given elsewhere.[11]

Regardless of the speed at which they are operating or the point at which they begin their deceleration, various motorists whose vehicles were checked give themselves about the same time to decelerate. The motorists traveling at 30 miles per hr or less could have decelerated in a shorter time if they had desired. It may be assumed, therefore, that 8 sec represents a comfortable time to the motorist and that where a vehicle is proceeding straight through at ordinary speed, a similar time interval should be allowed for weaving and dodging incoming or outgoing cars. Since the through speed will be V_i (which is assumed as 55 miles per hr), it will always be a higher velocity than the average velocity $\frac{V_i + V_f}{2}$. Therefore, $t = 8$ sec and V_i for determining warning distances will allow sufficient time and distance for both the high speed and the decelerating vehicles.

When all cars must come to a stop (case 2), the stopping distance is the controlling factor, but it must be borne in mind that the value of deceleration used must be such as to allow the motorist 8 sec or more in order to satisfy the time requirement for comfortable operation as derived for the case of simple deceleration without stopping. Transforming Eq. 3,

$$t = \frac{1.47\,(V_i - V_f)}{a} = 8 \text{ sec} \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

in which V_i is in miles per hour, V_f equals zero, and a is deceleration in feet per second per second.

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[11] "Acceleration and Deceleration Characteristics of Private Passenger Vehicles," by John Beakey, *Proceedings*, Highway Research Bd., Vol. 18, 1938, Fig. 9, p. 81.

MANEUVERING OR DECELERATION DISTANCE

Case 1.—For this case, utilizing the foregoing values for simple deceleration time, the maneuvering distance D is as follows:

$$D = 1.47\ V_i t = 8 \times 1.47\ V_i \qquad (5)$$

in which V_i is the design speed (90 percentile speed of approach and of through vehicles) in miles per hour.

Case 2.—If the distance necessary to stop comfortably results in a corresponding stopping time greater than that shown (case 2), the stopping distance may be used for computing the total warning distance. Deceleration and stopping distance is given by the following familiar formula:[12]

$$s = \frac{(1.47)^2\,(V^2_i - V^2_f)}{2\,a} \qquad (6)$$

in which V_i is initial velocity in miles per hour, V_f is the final velocity at deceleration (which becomes zero for a stopping distance), and a is the deceleration in feet per second per second. The stopping distance thus becomes:

$$s = \frac{1.08\ V^2_i}{a} \qquad (7)$$

To determine comfortable stopping distances for sign-design purposes, it is necessary to choose a value of deceleration a that will fulfil several conditions. It must result in comfortable stopping, must allow the motorist sufficient warning time, and must fulfil the requirements of safe stopping under adverse surface conditions, such as ice, snow, and wet pavements. In other words, the distance allowed the motorist should be adequate under all conditions. Mr. Beakey's study shows a time of 12 sec for a stop from 55 miles per hr, which corresponds to an average deceleration value a of 6.8 ft per sec². However, the actual instantaneous deceleration from which this average value was computed ran as high as approximately 12 ft per sec² as the vehicle approached the actual stopping point. A report by E. E. Wilson[13] in 1940 indicates 8.6 ft per sec² to represent a comfortable (average) deceleration as reported by drivers and passengers in stops from 50, 60, and 70 miles per hr. Thus, from 6 to 8 ft per sec² as an average deceleration apparently represents the comfortable deceleration range chosen by motorists. Maximum instantaneous values will be much higher, of course, but this does not concern the present application.

It is recommended that an average deceleration value of 4 ft per sec² be used for designing sign letters. This will maintain the maneuvering time in the 8-sec range indicated by Mr. Beakey's study for the lower speeds. For instance, in computing stopping distance from 30 miles per hr, 8 ft per sec² gives $t = \frac{1.47 \times 30}{8} = 5.5$ sec, which is an inadequate time value. The deceleration of 4 ft per sec² raises this value to 11 sec, which is adequate. Furthermore, since this value corresponds to a coefficient of friction of about $f = 0.12$, it should be adequate for icy conditions.[13] Therefore, the sign design value recommended is $a = 4$ ft per sec^2.

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[12] "Skidding Characteristics of Automobile Tires," by R. A. Moyer, *Bulletin No. 120*, Iowa Eng. Experiment Station, 1934.

[13] "Deceleration Distances for High Speed Vehicles," by E. E. Wilson, *Proceedings*, Highway Research Bd., Vol. 20, 1940, p. 393.

WARNING DISTANCE

Having now determined all the components of warning time and warning distance and explained the basis upon which they have been derived, it is possible to assemble the factors to yield the total warning time and warning distance, in Figs. 1 and 2.

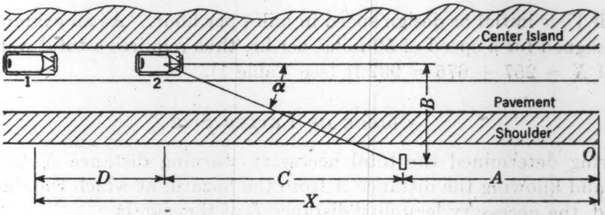

FIG. 2

Case 1.—For case 1 the total warning time T is given by

$$T = 2\,t_g + t_p + t_d \qquad (8)$$

in which t_g is the time for a single glance at a sign, t_p is perception-reaction time, and t_d is deceleration time. Assuming phrases of not more than three familiar words and substituting the value derived previously, this becomes $T = 2.0 + 1.5 + 8.0 = 11.5$ sec. When the sign contains a greater number of words, the first term must be increased accordingly.

The total warning distance then becomes

$$X = 1.47\,V_i\,T = 1.47 \times 11.5\,V_i \qquad (9)$$

in which V_i is the sign-design speed of the highway, and T is the total warning time. For convenience, it is suggested that the time value be rounded off to 11 sec, giving $X = 16.2 \times V_i$ in miles per hr, or $X = 11 \times V_i$ in ft per sec. This will yield a warning distance in which the sign placement and design should allow the sign to be read at 810 ft for a design speed of 50 miles per hr. Other values are given in Table 1.

TABLE 1.—TYPICAL WARNING DISTANCES, X, IN FEET[a]

Deceleration distances required to:	VELOCITY, V_i, IN MILES PER HOUR			
	30	40	50	60
Enter an intersection at 20 miles per hr	486	648	810	972
Stop at an intersection	393	638	932	1,281

[a] Level grades have been assumed throughout.

Case 2.—When all cars must stop at or before the hazard, the total warning distance may be computed as follows:

$$D = 1.47\, V_i\, (2\, t_g + t_p) + s \quad \ldots\ldots (10)$$

in which s is the stopping distance defined by Eq. 7. Substituting the values derived in the foregoing sections:

$$X = 1.47\, V_i \times (2.0 + 1.5) + \frac{1.08\, V^2_i}{a} = 1.47\, V_i \times 3.5 + \frac{1.08\, V^2_i}{4} \ldots (11)$$

in which V_i is initial or design speed in miles per hour. As an example, a stop sign, designed for a speed of 50 miles per hr, thus requires a total warning distance of $X = 257 + 675 = 932$ ft (see Table 1).

CHOICE OF LETTER SIZE

Having determined the total necessary warning distance X (see Figs. 1 and 2) and knowing the distance A from the hazard, at which the sign is to be installed, the necessary legibility distance L of the sign is

$$L = X - A = C + D \quad \ldots\ldots (12)$$

Legibility distances for letters of different widths[14] have been determined in an extended series of tests using from 100 to 250 different people as observers.[15] The findings for daylight conditions may be briefly summarized as follows:

U. S. standard series	Width	Letter legibility, l, in feet per inch of letter height
B	Narrow	33
C	Medium	42.5
D	Wide	50

Dividing the necessary legibility distance (L) by the appropriate letter legibility (l) gives the letter size indicated; thus:

$$\frac{L}{l} = H \quad \ldots\ldots (13)$$

in which H is the letter height of that particular design letter needed.

Again illustrating, if $X = 932$ ft, $A = 400$ ft, and a wide letter is to be used: $H = \frac{932 - 400}{50} = \frac{532}{50} = 10.6$ in., and a 12-in. letter is indicated. Since it is quite possible to locate a sign so far in advance of the hazard that this computation would call for an absurdly small letter height, the position chosen for locating the sign must be checked. The distance D must be great enough to allow at least minimum reading time (see Eq. 3). Therefore, from Fig. 2,

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[14] "Manual on Uniform Traffic Control Devices for Streets and Highways," A.A.S.H.O. and National Conference on Street and Highway Safety, Washington, D. C., 1937.

[15] "Legibility Distances of Highway Destination Signs in Relation to Letter Height, Letter Width, and Reflectorization," by T. W. Forbes and R. S. Holmes, *Proceedings*, Highway Research Bd., 1939, Vol. 19, p. 321.

$$D = X - A - C = X - A - \text{cotan } 10° \ldots\ldots\ldots\ldots (14)$$

For a sign with three words or less, $D \overset{=}{>} 1.47\ V_i \times 2\ t_g$.

Case 1.—When some cars proceed, some decelerate, but none must stop, the design speed is determined first by computing X from Eq. 9, or reading it from Table 1. If the sign is to be 400 ft ahead of the hazard ($A = 400$) the necessary legibility distance $L = X - 400$ ft. Next obtain the legibility per inch of letter height (l) from the text (see heading "Choice of Letter Size").[15] The letter height $H = \frac{L}{l} = \frac{X - 400}{l}$. To check for sufficient reading time, obtain C by Eq. 2. Then the following condition must hold (see Eq. 13): $D = L - C \overset{=}{>} 1.47\ V_i \times 2\ t_g$.

For 3-word or 4-word signs, this becomes $1.47\ V_i \times 2$. For more words, t_g should be increased as shown in Eq. 3.

Case 2.—Where all vehicles must stop, the design speed is determined as before; the total warning distance X is computed from Eq. 11 or Table 1; and A or A' is subtracted or added, respectively. Again suppose it is desired to place the sign 400 ft ahead of the hazard; then $L = X - 400$ and the letter height $H = \frac{X - 400}{l}$, in which l is legibility distance of the letters to be used.[15] Finally, the designer should check for sufficient reading time as in case 1. In both cases, if the reading time is too short, determine A, or the sign location, by reversing the last few steps. Obtain D from Eq. 14, C from Eq. 2, and $A = X - (C + D)$. The sign must be mounted at the position indicated. Finally, compute L and H for the new sign position by the appropriate method for either case 1 or case 2.

Practical Considerations

Obviously certain practical decisions as to where the sign is to be placed, whether both pre-warning and a sign at the hazard or intersection are necessary, and similar questions must be answered before the necessary letter height can be determined. Furthermore, in choosing which of the standard alphabet series to use for the sign copy the designer must take into consideration such factors as the length of place names involved and the general shape and limiting dimensions allowable for his sign. In certain cases where practical requirements make it necessary to include more than three words in a given sign, it has been found desirable to use two different letter sizes in the copy. A larger size has been used for certain important words intended to be read at the greatest distance and a smaller size for the less important legend intended to be read at a closer position and with a second glance of the driver's eye.[16]

Summary

A method has been described for determining the necessary letter size for effective highway sign design in terms of warning time, warning distance, and a

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[16] Sign Specifications of the Pennsylvania Turnpike Commission, Missouri State Highway Dept., and others.

90 percentile design speed. The increased warning distance necessary at higher speeds has been provided through the use of driver warning time as the basic consideration. Appropriate braking distances have been introduced where necessary.

Table 1 (or Eqs. 9 and 11) furnishes values for convenient computation of the letter height needed, and a simplified outline of procedure is given in the paper.

By use of such a method, it is possible to fit the sign to the highway and the driver and to obtain consistently effective signs for widely varying conditions of velocity and sign location with respect to the hazard or the maneuver point.

DISCUSSION

Guy Kelcey,[17] **M. Am. Soc. C. E.**—In their presentation of this too little explored subject, Messrs. Mitchell and Forbes have made a definite contribution in the direction of more efficient and, therefore, of safer roadway usage. Although the subject presented may be of quite limited interest to many engineers, it is of great importance to those who are responsible for best use of roadways for purposes of transportation.

A traffic sign has two values. A driver's attention is attracted to a sign, before he is within its legibility range, by its target value. At closer approach he is able to read its message or to perceive its symbol. If sign copy is too small to be useful to high-speed traffic, except at very close range, special attention must be given to the target value of such signs by the use of conspicuous background colors and shapes, such as octagon, round, square, or diamond, so that the target characteristic of the sign will tell drivers from a distance what general type of road condition is ahead. When, as Messrs. Mitchell and Forbes suggest, sign copy is enlarged so that its legibility has adequate distance value, in terms of the present-day speed characteristics of the location, target value, beyond that provided by the copy or symbol, becomes of lesser or of no importance.

This leads to an interesting conclusion. In the past, traffic sign design has tended to ignore legibility value and copy was condensed into standard sized target space. Pursuit of the principles presented by Messrs. Mitchell and Forbes would reverse this procedure. It would result in designing to legibility value and then adapting the size and shape of the sign background to the resulting copy dimensions. Since the copy or symbol would be large enough to provide any target value needed, no great attention need be paid to providing target value in the sign background. If so, the most conspicuous colors could be used in the copy or symbol and even greater legibility values achieved.

The practice of placing traffic signs in advance of the point to which their message applies is regarded by some as a needless carry-over from conditions of many years ago. Painted signs were originally placed in advance of hazards largely because they could not be read at night until a driver was close; he then needed some distance—generally 300 ft—in which to act. This practice of advance location of traffic signs lost its major purpose with the introduction of reflectors and other means to illuminate copy for legibility from a distance after dark.

Just as a lantern is best placed directly on a pile of brick in a street and not 300 ft in advance, there is much to be said in favor of placing a luminous or illuminated warning or information sign at, or close to, the point to which its message applies. Thus, the message is not only projected to drivers at a distance, but the drivers are brought all the way up to the point to which the message applies.

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[17] Atlanta, Ga.

It is generally overlooked, too, that the location of signs well in advance of a hazard often introduces a danger. Like a structural worker walking high steel, a good driver looks up and not down. His gaze is normally focused several hundred feet or more ahead. When a traffic sign, with copy legibility of from 200 to 300 ft, comes into view, a driver reads its message from or near that distance and the focus of his attention sweeps on. If this distance is added to the 300 ft that the sign is placed in advance of the danger, it will be noted that it is possible for a driver, whose attention is focused on the road in front, to travel the last 500 ft or more with no further reminders of the condition ahead unless the danger itself is conspicuous. At night a driver can and often does forget the message on signs placed in advance of danger points, and being unable to see the hazard ahead, is taken by surprise.

Messrs. Mitchell and Forbes present a method by which sign copy size requirements may be determined accurately for each sign location, thus to provide adequate legibility values to satisfy determined conditions. In practice it should be possible to establish a series of copy sizes in 3-in. or in 6-in. steps and to use the nearest suitable size. Tailor-made traffic signs, with copy exactly proportioned to variable legibility distance needs, could result in a multiplicity of copy dimensions that would increase the cost of manufacture and of inventory and complicate installation and maintenance.

Free space available adjacent to the traveled part of roadways and streets is often so limited that signs, larger than those now used, could not always be accommodated if installed in accordance with present standards. If legibility values greater than those provided by currently standard sign copy are to be employed, it will be recognized that much larger traffic signs would be used. If so, in many cases, they must either be suspended over the roadway or bracketed over the shoulder or the sidewalk. Indeed this has already been done at scattered locations in some jurisdictions and results are reported to be excellent.

Charles M. Noble,[18] M. Am. Soc. C. E.—Marking the second step in the development of highway sign technique, it is hoped that this paper will stimulate interest in continuing the advance in research in order that complete practical scientific data will be available to the highway and traffic engineer.

The use of adequate signs is becoming increasingly important due to the advancing development of modern highway design that utilizes the principle of free flowing traffic. This principle, involving as it does the elimination of cross traffic and left turns by the use of grade separations with right turns only, results in a type of highway facility that is confusing to the uninitiated motorist in that he is required to turn right when he wishes to go left. Turning right in order to "go left" is contrary to the instincts of the average motorist. Consequently, it is of the utmost importance that directions (the sign) be of such design, size, and clarity that the driver will have ample time to observe, read, react, and maneuver properly. The modern "cloverleaf" is designed on the assumption that the motorist will make the proper turns, and he must do so if he is to avoid losing valuable time and "getting lost." Adequate signing is fundamental, therefore, if the economic value of the expensive grade-separated type of highway is to be fully realized; and the utmost care and thought should be given to the design and placement of highway direction signs.

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[18] Arlington, Va.

The authors have presented, step by step, a logical reasoning in an analysis for determinating the correct letter sizes to be used in designing a sign for particular conditions of design speed and site location. The principle of breaking the elements of reading, perception, and reaction into units of time which in turn can be converted into distance and then related to the distance at which various sized letters can be read by the motorist, is fundamentally sound. (Incidentally the table in the paper which presents the visibility distances of letters is based on the eighty percentile group of motorists.)

A study of the material presented reveals that letters of a much larger size than customarily utilized should be used if the highway sign is to be fully effective. These large letters when assembled together into a sign message will result in a sign panel of such dimensions that it will stagger the designer when compared with ordinary standards. In this connection it is interesting to note that some of the direction signs on the Pennsylvania Turnpike are 14 ft wide by 9 ft high and these signs do not give the impression of being oversize for the conditions. The practice of utilizing letters of different size on the same sign, as outlined by the authors, was adopted successfully on the Turnpike, and it is believed that this is a practical method of keeping the size of the sign panel within reasonable bounds.

The analysis by the authors indicates that a smaller size of direction sign may be utilized if the sign is placed in advance of the turnout rather than at the point of turnout. For example, at a speed of 50 miles per hr an advance sign must be seen for a distance of 530 ft (six-lane highway width) requiring 12-in. series D letters, whereas a sign placed at the point of turnout should be seen a distance of 870 ft requiring 18-in. series D letters.

These required distances lead to a complication when signing urban highways in congested metropolitan areas where turnouts are frequent and often very close together with insufficient distance available to satisfy the requirements of speed and vision. Such conditions will tax the ingenuity of the sign designer.

The authors have covered the subject admirably, having placed a new tool in the hands of the engineer which will add to the effectiveness of the highway as well as to the safety and comfort of the traveling public, and have therefore performed a valuable service.

ADOLPHUS MITCHELL,[19] ASSOC. M. AM. SOC. C. E., AND T. W. FORBES,[20] ESQ.—It was hoped that more discussions would be written; but, since the two published were by men who are leaders in this field, it may be said that the lack of quantity was made up for by the quality.

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[19] Senior Traffic Engr., State Highway and Public Works Comm., Raleigh, N. C.

[20] Stevens Inst. of Technology, Hoboken, N. J.; formerly Bureau for Street Traffic Research, Yale Univ., New Haven, Conn.

In closing, it appears to be in order to restate that it was not intended, in the paper, to discuss the attention-getting properties of signs. This element in sign work usually will determine the location of the signs. After the position of each sign has been determined, the sign letter sizes can be designed in accordance with the principles stated in the paper.

The writers wish to thank Messrs. Kelcey and Noble for their fine discussions.

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LATERAL STABILITY OF UNSYMMETRICAL I-BEAMS AND TRUSSES IN BENDING

BY GEORGE WINTER,[1] ESQ.

WITH DISCUSSION BY MESSRS. ROBERT K. SCHRADER, H. N. HILL, AND GEORGE WINTER.

SYNOPSIS

An analysis is given herein of the lateral stability of unsymmetrical I-beams, both free and horizontally restrained along the bottom edge, and of beams of rectangular cross section horizontally restrained along the bottom edge. The cases of horizontally restrained beams apply to sheet stiffeners in bending. It is explained that the torsional rigidity of trusses and of thin-wall I-beams is smaller than would be expected from the use of the usual formulas for torsional stiffness. This is due to the fact that horizontal bending of such elements is accompanied by distortion of the cross section. For this reason, it is suggested that satisfactory, although slightly conservative, estimates of the critical loads of such beams may be obtained by neglecting the torsional rigidity entirely.

THE PROBLEM

The solutions of the problem of lateral instability of beams known in the literature do not cover the entire range of cases arising in practice. A few of these are:

(1) The general case of an I-beam non-symmetrical about its horizontal axis;

(2) The case of beams of various cross sections horizontally restrained along the tension edge (this situation is realized for sheets stiffened by projecting ribs and subject to bending in such a way that the sheet is in tension); and

(3) The lateral stability of trusses and of I-beams made of thin sheet metal.

These problems are analyzed herein for the case of pure bending and for the case of a simple beam with a concentrated load at the center of the span, applied at any arbitrary level.

NOTE.—Published in December, 1941, *Proceedings*.

[1] Asst. Prof. of Civ. Eng., Cornell Univ., Ithaca, N. Y.

Notation.—The letter symbols in this paper are defined where they first appear, and are assembled for reference in Appendix I.

METHOD

The work-energy method is used throughout this investigation. The horizontal displacement of the beam due to lateral instability is resolved into two components (Fig. 1): A horizontal bending displacement y of the flexural center, and a rotation ϕ about a horizontal axis through the flexural center.

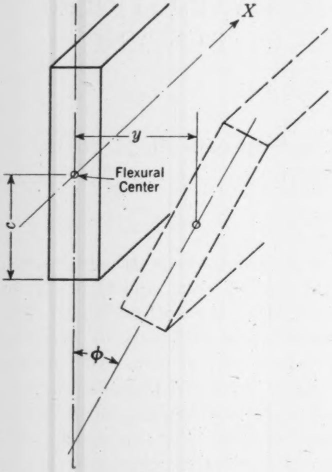

FIG. 1

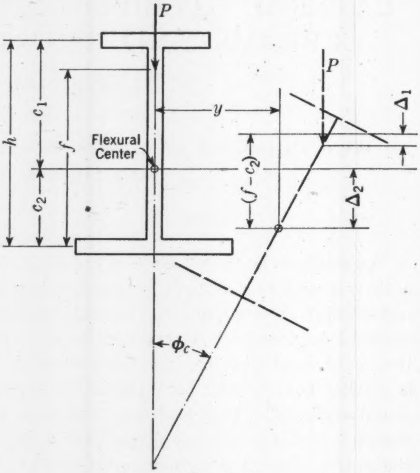

FIG. 2

The angle ϕ is an unknown function of the longitudinal ordinate x (origin at one of the supports). The beam is assumed to be supported in such a way that rotation of the end sections about the longitudinal axis of the beam is prevented. Consequently, the condition that must be satisfied by ϕ at the ends is: $\phi = 0$ at $x = 0$ and $x = L$.

Empirically it is known that, after loss of stability, the horizontal deformation of the beam has the form of a single half wave. This general character is represented and the boundary conditions are satisfied by taking

$$\phi = a \sin \frac{\pi x}{L} \qquad \text{(1)}$$

in which a = a coefficient modifying the total angular deformation ϕ (see Eq. 1); and L = span length.

The assumption of this approximate form of $\phi(x)$ involves a certain error, the magnitude of which could be estimated by expanding $\phi(x)$ in a Fourier series with unknown coefficients instead of taking only the first term of such a series. Without making use of this elaborate procedure it will be shown by comparison that the error involved is negligible for all practical purposes.

VALUES OF INTEGRALS FREQUENTLY ARISING IN THE ANALYSIS

To simplify the mathematical presentation, the solutions of definite integrals which will frequently appear in the following pages are given as follows:

$$\left.\begin{aligned} \int_0^L \phi^2\,dx &= \frac{a^2\,L}{2} \\ \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx &= a^2\,\frac{\pi^2}{2\,L} \\ \int_0^L \left(\frac{d^2\phi}{dx^2}\right)^2 dx &= a^2\,\frac{\pi^4}{2\,L^3} \\ \int_0^{L/2} \phi\,\frac{d^2\phi}{dx^2}\,x\,dx &= -\frac{a^2}{4}\left(1+\frac{\pi^2}{4}\right) \\ \int_0^{L/2} \phi^2\,x^2\,dx &= \frac{a^2\,L^3}{8\,\pi^2}\left(1+\frac{\pi^2}{6}\right) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (2)$$

I-BEAMS ASYMMETRICAL ABOUT THE HORIZONTAL AXIS

(*a*) *Pure Bending.*—For such beams the location of the flexural center is given by:[2]

$$c_1 = h\,\frac{I_2}{I} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3a)$$

and

$$c_2 = h\,\frac{I_1}{I} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3b)$$

in which (see Fig. 2) c_1 and c_2 are the distances to the flexural center from the centers of the compression and tension flanges of the beam, respectively; $h = c_1 + c_2$ = the total effective depth; and I, I_1, and I_2, respectively, are the inertia moments of the total beam, the compression flange, and the tension flange with respect to the vertical axis of the web. In computing I the moment of the web about its vertical axis is neglected.

The strain energy of the beam due to lateral bending and rotation about its flexural center consists of three parts (Fig. 2): (*a*) The energy of bending of the compression flange corresponding to a horizontal deflection $y + c_1\,\phi$; (*b*) the energy of bending of the tension flange corresponding to a horizontal deflection $y - c_2\,\phi$; and (*c*) the torsional energy corresponding to a rotation of the entire beam through the angle ϕ. Consequently, the strain energy is

$$W_e = \tfrac{1}{2}\left\{E\,I_1 \int_0^L \left[\frac{d^2(y + c_1\,\phi)}{dx^2}\right]^2 dx \right.$$
$$\left. + E\,I_2 \int_0^L \left[\frac{d^2(y - c_2\,\phi)}{dx^2}\right]^2 dx + G\,K \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx\right\} \ldots\ldots (4)$$

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[2] "Formulas for Stress and Strain," by R. J. Roark, McGraw-Hill Book Co., Inc., New York, N. Y., 1938, p. 118.

or, by substitution of c_1 and c_2 from Eq. 3,

$$W_\epsilon = \frac{E}{2}\left[I \int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx + \frac{I_1 I_2}{I} h^2 \int_0^L \left(\frac{d^2\phi}{dx^2}\right)^2 dx + C \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx \right] \quad (5)$$

In Eq. 4, K = the torsional constant for the given cross section, G = the shear modulus—

$$G = \frac{E}{2\,(1+\mu)} \quad (6)$$

and μ = Poisson's ratio. In Eq. 5, C, the reduced torsional constant, equals

$$C = \frac{K}{2\,(1+\mu)} \quad (7)$$

If M is the vertical bending moment, its horizontal component acting on the beam in the displaced position is $M_y = M\,\phi$ and

$$\frac{d^2y}{dx^2} = -\frac{M\,\phi}{E\,I} \quad (8)$$

Substituting this expression for $\frac{d^2y}{dx^2}$ and also the appropriate values for the integrals, one arrives at

$$W_\epsilon = E\,\frac{a^2}{2}\left(\frac{M^2}{E^2\,I}\,\frac{L}{2} + \frac{I_1 I_2}{I}\,h^2\,\frac{\pi^4}{2\,L^3} + C\,\frac{\pi^2}{2\,L}\right) \quad (9)$$

The only external forces acting on the beam are the moments M applied at the ends (Fig. 3). If β designates the vertical angle of rotation of each of the end sections due to the horizontal deformation of the beam, the work of the external forces is

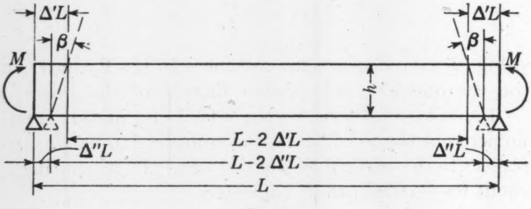

FIG. 3

$$W = 2\,\beta\,M \quad (10)$$

The magnitude of β is obtained by determining half of the difference in length of the chords of the horizontal curves formed by the axes of the top and bottom flanges in the laterally bent state ($\Delta'L - \Delta''L$ in Fig. 3). Thus,

$$\beta = \frac{1}{2\,h}\left\{ \int_0^{L/2} \left[\frac{d(y + c_1\,\phi)}{dx}\right]^2 dx - \int_0^{L/2} \left[\frac{d(y - c_2\,\phi)}{dx}\right]^2 dx \right\}; \text{ or}$$

$$\beta = \tfrac{1}{4}\left[2 \int_0^L \frac{dy}{dx}\,\frac{d\phi}{dx}\,dx + h\,\frac{I_2 - I_1}{I} \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx \right\} \quad (11)$$

From Eqs. 1 and 8, however,

$$\frac{dy}{dx} = \frac{a\,M}{E\,I}\left(\frac{L}{\pi}\cos\frac{\pi\,x}{L}\right)\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

Since $W = 2\,\beta\,M$ (Eq. 10) one obtains, by substitution of the appropriate values for the integrals,

$$W = a^2\frac{M}{2\,I}\left[h\,(I_2 - I_1)\,\frac{\pi^2}{2\,L} + \frac{M\,L}{E}\right]\ldots\ldots\ldots\ldots(13)$$

The critical moment is now determined from the work-energy condition, $W_\epsilon = W$, which results in

$$M_{cr} = E\,h\,\frac{\pi^2}{2\,L^2}\left[(I_1 - I_2) + \sqrt{I^2 + 4\,C\,I\,\frac{L^2}{\pi^2\,h^2}}\right]\ldots\ldots(14a)$$

For the case of a symmetrical I-beam with $I_1 = I_2 = I_0$, Eq. 14*a* reduces to

$$M_{cr} = E\,\frac{\pi^2}{L^2}\sqrt{I^2_0\,h^2 + 2\,C\,I_0\,\frac{L^2}{\pi^2}}\ldots\ldots\ldots\ldots\ldots(14b)$$

The exact solution for this latter special case has been developed by S. Timoshenko. If Professor Timoshenko's value of $\frac{a^2}{L^2}$ is substituted[3] in his comparable equation[4] for M_{cr}, the result is identical with Eq. 14*b*. In other words, for the case of pure bending, Eq. 1 happens to be the exact expression for $\phi(x)$.

(*b*) *Center Load.*—Eq. 5 for the strain energy remains the same as in the previous case. Between $x = 0$ and $x = \frac{L}{2}$,

$$\frac{d^2y}{dx^2} = -\,\frac{M\,\phi}{E\,I} = -\,\frac{P\,x\,\phi}{2\,E\,I}\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

in which P = a vertical concentrated load applied at an arbitrary distance f from the bottom flange. Upon substitution of this expression the first integral in Eq. 5 becomes

$$\int_0^L\left(\frac{d^2y}{dx^2}\right)^2 dx = \frac{P^2}{2\,E^2\,I^2}\int_0^{L/2}\phi^2\,x^2\,dx\ldots\ldots\ldots\ldots(16)$$

If the corresponding values of the integrals are introduced, the strain energy in this case is

$$W_\epsilon = a^2\left[\frac{P^2}{E\,I}\,\frac{L^3}{32\,\pi^2}\left(1 + \frac{\pi^2}{6}\right) + h^2\,\frac{I_1\,I_2}{I}\,\frac{\pi^4}{4\,L^3}\,E + \frac{C\,E}{4}\,\frac{\pi^2}{L}\right]\ldots(17)$$

The vertical displacement of the force P, applied at an arbitrary distance f from the bottom flange, may be resolved into two components, Δ_1 and Δ_2.

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[3] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., Eq. 157, p. 258.

[4] *Ibid.*, Eq. 159, p. 258.

The component Δ_1 (Fig. 2) represents the relative displacement of the point of load application with respect to the flexural center. It is easily seen geometrically that the amount of this displacement is

$$\Delta_1 = (f - c_2)(1 - \cos \phi_c) \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

in which ϕ_c is the angle ϕ at the center of the span. Developing $\cos \phi$ in a power series in ϕ and, for a small value of ϕ, retaining only the first two terms:

$$\Delta_1 = a^2 \frac{f - c_2}{2} \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

The second component Δ_2 represents the relative displacement of the flexural center with respect to the supports caused by the horizontal curvature of the longitudinal axis of the beams through the flexural center. Thus

$$\Delta_2 = -\int_0^{L/2} \frac{d^2y}{dx^2}\, \phi\, x\, dx = \frac{P}{2\,E\,I} \int_0^{L/2} \phi^2 x^2\, dx \ldots\ldots\ldots (20)$$

Consequently, the work of the force P is $W = P(\Delta_1 + \Delta_2)$ or,

$$W = \left[\frac{P^2}{E\,I}\,\frac{L^3}{16\,\pi^2}\left(1 + \frac{\pi^2}{6}\right) + \frac{P}{2\,I}(f\,I - h\,I_1)\right] a^2 \ldots\ldots\ldots (21)$$

The critical load is again obtained by making $W_s = W$, which yields

$$P_{cr} = \frac{48\,\pi^2 E}{L^3(6 + \pi^2)}\left[(h\,I_1 - f\,I) + \right.$$

$$\left. + \sqrt{(h\,I_1 - f\,I)^2 + \frac{L^2(6 + \pi^2)}{48}\left(C\,I + \frac{h^2 I_1 I_2 \pi^2}{L^2}\right)}\,\right] \ldots\ldots (22a)$$

For the special case of the symmetrical **I**-beam ($I_1 = I_2 = I_0$) with P applied at the centroid $\left(f = \dfrac{h}{2}\right)$ this expression simplifies to

$$P_{cr} = \frac{12\,\pi^2}{L^2} E \sqrt{\frac{I_0}{3(6 + \pi^2)}\left(2\,C + \frac{h^2 \pi^2}{L^2} I_0\right)} \ldots\ldots\ldots (22b)$$

The exact solution for this special case in explicit form is not known to the writer. Using the energy method, Professor Timoshenko derived an integral equation[5] for this case. In solving it he uses only the first term of a Fourier expansion[6] which is identical with Eq. 1. It can be shown easily that, in this case, Professor Timoshenko's solution is identical with Eq. 22*b*. According to Professor Timoshenko, the error involved in taking this approximate expression for $\phi(x)$ amounts to only 0.5% in this special case. Without further detailed investigation it is reasonable to state that, in the general case of an unsymmetrical I-beam, loaded at any arbitrary level f, the error involved is of the same order of magnitude, and is therefore negligible for all practical purposes.

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[5] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., Eq. e, p. 266.

[6] *Ibid.*, Eq. f, p. 266.

RECTANGULAR BEAMS HORIZONTALLY RESTRAINED ALONG THE TENSION EDGE

Imagine a simply supported rectangular beam restrained along its bottom (tension) edge in such a way that horizontal displacements are prevented, whereas vertical deflections remain entirely unaffected. This case is very nearly realized if, for some practical reason, horizontal bracing is applied only along the bottom edge. It is granted that such an arrangement in practice will be used only in exceptional cases. However, this problem is not only of general interest, but it leads to the analysis of the more important question of stability of sheet stiffeners of **I**-section and **T**-section, which will be treated subsequently.

(*a*) *Pure Bending.*—The horizontal displacement of the beam (Fig. 4) is again resolved into the horizontal deflection y of its flexural center and into a rotation about the same point. The general expression for the strain energy is then

$$W_\epsilon = \frac{E}{2}\left[I \int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx + C \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx \right] \quad (23)$$

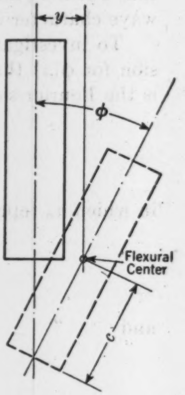

FIG. 4

As seen from Fig. 4 for small angles ϕ

$$y = c\,\phi \quad (24)$$

Substituting Eq. 24 in Eq. 23 yields

$$W_\epsilon = \frac{E}{2}\left[I\,c^2 \int_0^L \left(\frac{d^2\phi}{dx^2}\right)^2 dx + C \int_0^L \left(\frac{d\phi}{dx}\right)^2 dx \right] \quad (25)$$

On the basis of the same considerations as applied before to **I**-beams, the angle of rotation β of the end section is

$$\beta = \frac{1}{2\,c}\int_0^{L/2} \left(\frac{dy}{dx}\right)^2 dx = \frac{c}{2}\int_0^{L/2} \left(\frac{d\phi}{dx}\right)^2 dx \quad (26)$$

and the work of the external moments M becomes

$$W = 2\,\beta\,M = \frac{M\,c}{2}\int_0^L \left(\frac{d\phi}{dx}\right)^2 dx \quad (27)$$

It should be noted that in this case the moments M are not the only external forces acting on the beam. In fact, along the bottom edge horizontal forces are exerted by the restraint, which are of such magnitude that they prevent horizontal displacement of this edge. However, since the points of application of these forces are not subject to any displacement in the direction of the forces, they do no work. For this reason, Eq. 27 gives the total amount of work of the external forces.

It is well established, empirically and analytically, that laterally unrestrained beams buckle in a single half wave. For this reason it was legitimate to use Eq. 1 for an approximate representation of ϕ for free beams. However, in the case of beams restrained laterally along the bottom edge, it is possible that buckling may occur in more than one half wave. Buckling of such multiwave character is known to occur in thin plates, struts on elastic supports, etc.

To investigate this possibility, one must assume the most general expression for $\phi(x)$ that will satisfy the end conditions. Such a general expression is the Fourier series

$$\phi = \sum_{n=1,2,\cdots}^{\infty} a_n \sin \frac{n\,\pi\,x}{L} \tag{28}$$

in which a_n represents undetermined coefficients. Then

$$\int_0^L \left(\frac{d\phi}{dx} \right)^2 dx = \frac{\pi^2}{2\,L} \sum_{n=1}^{\infty} n^2\, a_n^2 \tag{29}$$

and

$$\int_0^L \left(\frac{d^2\phi}{dx^2} \right)^2 dx = \frac{\pi^4}{2\,L^3} \sum_{n=1}^{\infty} n^4\, a_n^2 \tag{30}$$

Substituting these values in Eqs. 25 and 27 and equating $W_e = W$, one arrives at

$$M_{cr} = E\, \frac{I\,c\,\dfrac{\pi^2}{L^2} \displaystyle\sum_{n=1}^{\infty} n^4\, a_n^2 + \dfrac{C}{c} \displaystyle\sum_{n=1}^{\infty} n^2\, a_n^2}{\displaystyle\sum_{n=1}^{\infty} n^2\, a_n^2} \tag{31}$$

All possible values of a_n^2 are fundamentally positive quantities. Consequently, the smallest value of M_{cr} corresponds to $n = 1$. It is thus shown that a laterally restrained beam buckles in a single half wave; in addition, in the particular case of pure bending, Eq. 1 is seen to be the exact expression for the shape of the beam.

Therefore, with $n = 1$,

$$M_{cr} = E \left(c\,\frac{\pi^2}{L^2}\,I + \frac{C}{c} \right) \tag{32a}$$

Since, for a rectangular beam, $c = \dfrac{h}{2}$,

$$M_{cr} = E \left(h\,\frac{\pi^2}{2\,L^2}\,I + \frac{2\,c}{h} \right) \tag{32b}$$

(*b*) *Center Load.*—The expression for the strain energy is the same as in the previous case (Eq. 25).

The vertical displacement of the load P is obtained by the same reasoning as in the case of I-beams. Thus (see Eq. 19):

$$\Delta_1 = a^2 \frac{f - c}{2} \qquad (19)$$

$$\Delta_2 = - c \int_0^{L/2} \phi \frac{d^2\phi}{dx^2} x \, dx \qquad (33)$$

and

$$W = P \left(a^2 \frac{f - c}{2} - c \int_0^{L/2} \phi \frac{d^2\phi}{dx^2} x \, dx \right) \qquad (34)$$

From $W_\epsilon = W$, the following expression for the critical load is obtained:

$$P_{cr} = E \frac{\pi^2}{L} \frac{I c^2 \dfrac{\pi^2}{L^2} + C}{2f + c\left(\dfrac{\pi^2}{4} - 1\right)} \qquad (35a)$$

or, since for a rectangle, $c = \dfrac{h}{2}$,

$$P_{cr} = E \frac{\pi^2}{L} \frac{\left(\dfrac{\pi h}{2L}\right)^2 I + C}{2f + 0.5\, L\left(\dfrac{\pi^2}{4} - 1\right)} \qquad (35b)$$

It may be noted that both Eq. 32*a* and Eq. 35*a* also apply to cross sections other than rectangles, provided that the nature of the section is such that its total resistance to torsion is proportional to K only. This is generally true for cross sections that do not have transversely projecting parts (flanges, etc.).

I-Beams Horizontally Restrained Along Tension Flange

This case is realized for sheets or plates stiffened by projecting ribs of I-beam or T-beam shape and subject to bending of such nature that the sheet or plate is in tension. The sheet is then to be regarded as an integral part of the stiffener—that is, it is to be included in the determination of the moment of inertia I_2 of the tension flange. Whether or not it is also to be included in the determination of the torsional constant K depends upon the type of joint between stiffener and sheet. For joints acting rather in a hinge-like manner (that is, allowing rotation of the stiffeners with respect to the sheet), K should be taken for the stiffener only. The same is true for rather thin sheets (as compared with the wall thickness of the stiffener), since in this case the sheet is likely to bend slightly out of its plane in the immediate vicinity of the joint rather than to twist monolithically with the stiffener.

(*a*) *Pure Bending.*—It is seen from Fig. 5 that, in this case, for small angles ϕ,

$$y = c_2\,\phi \qquad (36)$$

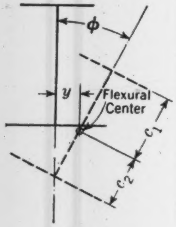

FIG. 5

Substituting Eq. 36 in Eq. 5 and remembering that

$$c_2 = h\,\frac{I_1}{I} \qquad (37)$$

the strain energy (after introducing the corresponding values for the integrals) becomes

$$W_\epsilon = E\,\frac{\pi^2 h^2}{4\,L}\left(I_1\,\frac{\pi^2}{L^2} + \frac{C}{h^2}\right)a^2 \qquad (38)$$

On the other hand, substituting Eq. 36 in Eq. 11 for β, one arrives at

$$W = \frac{h\,M}{4}\,\frac{\pi^2}{L}\,a^2 \qquad (39)$$

Again, making $W_\epsilon = W$, it follows that the critical moment

$$M_{cr} = E\,h\left(I_1\,\frac{\pi^2}{L^2} + \frac{C}{h^2}\right) \qquad (40)$$

Eq. 40 is seen to be independent of I_2, because the tension flange is not subject to any horizontal displacement and the position of the external forces (that is, M) is in no way connected with the position of the flexural center. For this reason Eq. 40 applies to symmetrical as well as to unsymmetrical **I**-beams.

(*b*) *Center Load.*—Eq. 38 for the strain energy remains the same as in the previous case. The vertical displacement of the load, with $y = c_2\,\phi$, becomes

$$\Delta = \Delta_1 + \Delta_2 = a^2\,\frac{f - c_2}{2} - c_2\int_0^{L/2} \phi\,\frac{d^2\phi}{dx^2}\,x\,dx \qquad (41)$$

and the work of the external force P

$$W = \frac{P}{2\,I}\left[f\,I + \frac{h\,I_1}{2}\left(\frac{\pi^2}{4} - 1\right)\right]a^2 \qquad (42)$$

From $W_\epsilon = W$

$$P_{cr} = E\,I\,\frac{\pi^2 h^2}{2\,L}\,\frac{I_1\,\frac{\pi^2}{L^2} + \frac{C}{h^2}}{f\,I + \frac{h\,I_1}{2}\left(\frac{\pi^2}{4} - 1\right)} \qquad (43a)$$

and for the special case of $I_1 = I_2 = I_0, f = \frac{h}{2}$,

$$P_{cr} = 2\,E\,h\,\frac{\pi^2}{L}\,\frac{I_0\,\frac{\pi^2}{L^2} + \frac{C}{h^2}}{\frac{\pi^2}{4} + 1} \qquad (43b)$$

TRUSSES AND I-BEAMS WITH THIN WEBS

These elements require special consideration for the following reason: In general the critical load depends upon the horizontal flexural rigidity and upon the torsional rigidity of the beam. For ordinary beams with thick webs the torsional constant may be determined from well-known formulas.[7] These constants have been derived on the assumption that the shape of the cross section does not change during torsion, or, in other words, that all component parts of the cross section are rotated through the same angle. This is not the case for latticed trusses, however (see Fig. 6(*a*)), and for beams with thin webs (see Fig. 6(*b*)). In such elements the lateral rigidity of the web, in many cases, is so slight that it bends out of its plane when the beam becomes unstable. For this reason the angle of rotation α_1 of the chords or flanges is seen to be less than the angle of rotation α_2 of the line AB in Fig. 6(*c*). Consequently the actual torsional rigidity of such elements will be less than that obtained from a formal application of the usual formulas for the torsional constant *K*. The determination of the actual torsional resistance of such members would require highly involved computations in each particular case and thus is beyond the scope of this paper.

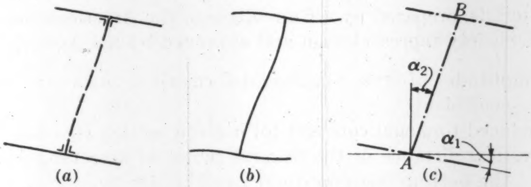

FIG. 6

However, for beams of this type, if *K'* is computed on the basis of the usual formulas (that is, without considering the distortion of the cross section) it will be found that the contribution to the critical load of the lateral flexural rigidity *E I* is usually far in excess of the contribution of the torsional rigidity *G K* to the critical load. Since the actual torsional constant *K* for beams of this type usually is considerably less than *K'* derived by formula, it seems advisable to determine the critical load of such beams by disregarding *K* entirely (that is, by assuming $K = 0$). The value thus obtained will be on the conservative side. On the other hand, if the effect of distorting the cross section were disregarded (that is, if *K'*, determined by formula, were used instead of the unknown actual *K*), the computed critical loads would be decidedly in excess of the actual loads. It is believed, therefore, that the first of these procedures (that is, assuming $K = 0$ for beams of this type) is by far preferable in current design practice.

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[7] "Formulas for Stress and Strain," by R. J. Roark, McGraw-Hill Book Co., Inc., New York, N. Y., 1938, p. 154.

CONCLUSION

In order to facilitate the use of the results of this investigation for beams with very small torsional constants, the formulas presented in this paper are assembled for reference in Appendix II.

APPENDIX I

NOTATION

The following letter symbols, adopted for use in this paper, conform essentially to Standard Letter Symbols for Mechanics, Structural Engineering and Testing Materials, prepared by a Committee of the American Standards Association with Society representation and approved by the Association in 1932.[8]

a = amplitude of the angular deformation: a_n = an undetermined coefficient;
C = reduced torsional constant for a given section (see Eq. 7);
c = vertical distance of the flexural center of a rectangular beam from the bottom (tension edge) (see Fig. 1):
    c_1 = distance from the center of the compression flange;
    c_2 = distance from the center of the tension flange (Fig. 2);
E = Young's modulus;
f = vertical distance from the bottom flange to the point of application of P;
G = shear modulus (Eq. 6);
h = effective height of section, between centers of flanges $(c_1 + c_2)$;
I = total moment of inertia $(I_1 + I_2)$ of a cross section with respect to the vertical axis of symmetry, particularly for I-beams (the moment of inertia of the web about its vertical axis is ignored):
    I_1 = inertia moment of the compression flange about the same axis;
    I_2 = inertia moment of the tension flange about the same axis;
K = torsional constant for a given cross section: K' = a computed constant as compared with (K) an actual constant;
L = span length, between supports:
    $\Delta' L$ = longitudinal displacement of the top (tension) flange, at the support;
    $\Delta'' L$ = longitudinal displacement of the bottom (compression) flange, at the support;
M = vertical bending moment:
    M_{cr} = critical moment;
    M_y = lateral component of M;
P = a concentrated external load (Eq. 5);
W = work of external forces: W_e = strain energy;

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[8] ASA—Z10a—1932.

x = a longitudinal ordinate (see Fig. 1);
y = deflection, horizontal bending displacement (see Fig. 1);
α = angle of rotation:
  α_1 = rotation of the chords or flanges of an I-beam;
  α_2 = rotation of the web (Fig. 6);
β = vertical angle of rotation of the end section (Fig. 3);
Δ = total linear deformation (see also ΔL):
  Δ_1 = relative displacement of the point of application of the load P, with respect to the flexural center;
  Δ_2 = relative displacement of the flexural center with respect to the support, caused by the horizontal curvature of the longitudinal axis of the beams through the flexural center;
μ = Poisson's ratio;
ϕ = total angular deformation (Fig. 1): ϕ_c = angle ϕ at the center of the span.

APPENDIX II

Design Equations

The design equations discussed in some detail in the paper are Eqs. 14, 22, 32, 35, 40, and 43. For beams with very small torsional rigidity, in particular for open web trusses and thin wall beams, assuming in this case $K = 0$, one obtains the following simplified expressions—

Unsymmetrical I-beams, pure bending, unrestrained:

$$M_{cr} = E\,h\,I_1 \frac{\pi^2}{L^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (44a)$$

Symmetrical I-beam, pure bending, unrestrained:

$$M_{cr} = E\,h\,I_0 \frac{\pi^2}{L^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (44b)$$

Unsymmetrical I-beam, center load, unrestrained:

$$P_{cr} = \frac{48\,\pi^2 E}{L^3\,(6 + \pi^2)} \left[(h\,I_1 - f\,I) + \sqrt{(h\,I_1 - f\,I)^2 + \frac{6 + \pi^2}{48}\,h^2\,\pi^2\,I_1\,I_2} \right] \ldots (45a)$$

Symmetrical I-beam, center load, $f = \frac{h}{2}$, unrestrained:

$$P_{cr} = E\,h\,I_0 \frac{\pi^3}{L^3} \sqrt{\frac{48}{6 + \pi^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (45b)$$

Unsymmetrical I-beam, pure bending, restrained:

$$M_{cr} = E\,h\,I_1 \frac{\pi^2}{L^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (46a)$$

Symmetrical I-beam, pure bending, restrained:

$$M_{cr} = E\,h\,I_0\,\frac{\pi^2}{L^2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots (46b)$$

Unsymmetrical I-beam, center load, restrained:

$$P_{cr} = E\,I\,\frac{\pi^3}{2\,L^3}\,h^2\,\frac{I_1\,\pi}{f\,I + \dfrac{h\,I_1}{2}\left(\dfrac{\pi^2}{4} - 1\right)}\ldots\ldots\ldots\ldots (47a)$$

Symmetrical I-beam, center load, $f = \dfrac{h}{2}$, restrained:

$$P_{cr} = E\,I\,\frac{\pi^3}{L^3}\,h\,\frac{2\,\pi}{\dfrac{\pi^2}{4} + 1}\ldots\ldots\ldots\ldots\ldots\ldots (47b)$$

The formal requirements for the use of these formulas are: For Eqs. 44 and 45*b*, $C \ll \dfrac{I}{4}\left(\dfrac{\pi\,h}{L}\right)^2$; for Eq. 45*a*, $C \ll \dfrac{I_1\,I_2}{I}\left(\dfrac{\pi\,h}{L}\right)^2$; and for Eqs. 46 and 47, $C \ll I_1\left(\dfrac{\pi\,h}{L}\right)^2$ ($\ll$ denotes "much less than").

As stated previously, in beams of this type, the actual value of K is generally unknown and it is only evident that $K < K'$. For this reason these formal requirements should be used with critical care as guides rather than as rigorously applicable criteria.

DISCUSSION

Robert K. Schrader,[9] **Jun. Am. Soc. C. E.**—Two types of loading commonly encountered in practice—(*a*) uniformly distributed load, and (*b*) two equal loads symmetrically located on the beam—lend themselves readily to analysis by the method presented in this paper. In his treatment of this problem, herein, the writer leaves the directions of the coordinate axes unchanged, but the origin is transferred to the center of the undistorted beam.

(*a*) *Uniformly Distributed Load.*—With the coordinate system thus transferred, the author's approximate expression for the angular deformation, Eq. 1, now becomes:

$$\phi = a \cos \frac{\pi x}{L} \qquad (48)$$

Eq. 5, for the strain energy W_e, remains unchanged except that the limits of integration now extend from $-\frac{L}{2}$ to $+\frac{L}{2}$. For uniformly distributed load the vertical bending moment M to be substituted in Eq. 8 is

$$M = \frac{q}{2}\left(\frac{L^2}{4} - x^2\right) \qquad (49a)$$

Consequently,

$$\left(\frac{d^2y}{dx^2}\right)^2 = \frac{q^2}{4\,E^2\,I^2}\left(\frac{L^4}{16} - \frac{L^2\,x^2}{2} + x^4\right)\phi^2 \qquad (49b)$$

Substitution of Eqs. 1 and 49*b* in the equation for the strain energy yields, for this case,

$$W_e = \frac{a^2}{4\,I}\left[\frac{q^2\,L^5\,(\pi^4 + 45)}{120\,\pi^4\,E} + \frac{\pi^4\,h^2\,I_1\,I_2\,E}{L^3} + \frac{\pi^2\,C\,E\,I}{L}\right] \qquad (50)$$

The loss in potential energy, or the work done by the load, is equal to the magnitude of the load multiplied by the distance the load falls as the beam deflects laterally. This distance is again divided into two components, Δ_1 and Δ_2 (see Fig. 2). It is seen that for any section

$$\Delta_1 = (f - c_2)(1 - \cos\phi) \qquad (51a)$$

Substituting c_2 from Eq. 3*b*, expanding $\cos\phi$ in a power series in ϕ, and neglecting all terms beyond the second in this expansion, one obtains

$$\Delta_1 = \frac{(f\,I - h\,I_1)}{2\,I}\,\phi^2 \qquad (51b)$$

From Fig. 7 it is seen that

$$\Delta_2 = \Delta'_2 - \Delta''_2 \qquad (52)$$

in which Δ'_2 is the vertical displacement of the flexural center at midspan due to lateral displacement, and Δ''_2 is the difference between this displacement and that of a section at a distance $\bar{x}$ from the origin. From the geometry of the problem the magnitudes of these two components are given by the following expressions:

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[9] 2d Lt., Air Corps, U. S. Army, Oak Park, Ill.

$$\Delta'_2 = \int_0^{L/2} \phi \frac{d^2y}{dx^2}\left(\frac{L}{2} - x\right) dx \ldots\ldots\ldots\ldots\ldots (53a)$$

and

$$\Delta''_2 = \int_0^{\bar{x}} \phi \frac{d^2y}{dx^2}(\bar{x} - x)\, dx \ldots\ldots\ldots\ldots\ldots (53b)$$

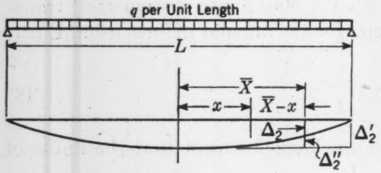

FIG. 7

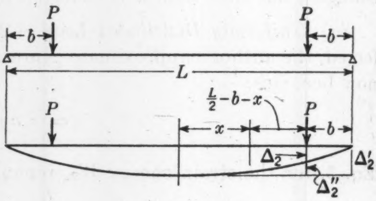

FIG. 8

The total work done by the uniformly distributed load, q, is

$$W = \int_{-L/2}^{L/2} q\,(\Delta_1 + \Delta_2)\, dx \ldots\ldots\ldots\ldots\ldots (54a)$$

By substitution of the expressions for Δ_1 and Δ_2 given in Eqs. 51 and 52 and by evaluation of the integrals, the total work becomes

$$W = \frac{a^2}{4\,I}\left[q\,L\,(f\,I - h\,I_1) + \frac{q^2\,L^5\,(\pi^4 + 45)}{60\,\pi^4\,E}\right] \ldots\ldots\ldots (54b)$$

The critical load is obtained by equating W_e from Eq. 50 to W from Eq. 54b, which results in

$$q_{cr} = \frac{60\,\pi^4\,E}{L^4\,(\pi^4 + 45)}\left[(h\,I_1 - f\,I) + \sqrt{(h\,I_1 - f\,I)^2 + \frac{\pi^4 + 45}{30}\left(h^2\,I_1\,I_2 + \frac{C\,I\,L^2}{\pi^2}\right)}\right] \ldots\ldots\ldots (55)$$

For the special case of a symmetrical I-beam loaded along its gravity axis, the results from Eq. 55 coincide with those obtained by Professor Timoshenko for this case.[10]

(b) *Two Equal Loads, Symmetrically Placed.*—Considering the right half of the beam shown in Fig. 8, it is seen that, from $x = 0$ to $x = (L/2 - b)$, the bending moment is

$$M = P\,b \ldots\ldots\ldots\ldots\ldots\ldots (56a)$$

and therefore, for this part of the beam,

$$\left(\frac{d^2y_1}{dx^2}\right)^2 = \frac{P^2\,b^2}{E^2\,I^2}\,\phi^2 \ldots\ldots\ldots\ldots\ldots\ldots (56b)$$

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[10] "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., Table 23, p. 268.

From $x = (L/2 - b)$ to $x = L/2$ the moment is

$$M = P\left(\frac{L}{2} - x\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (57a)$$

and therefore,

$$\left(\frac{d^2 y_2}{dx^2}\right)^2 = \frac{P^2}{E^2 I^2}\left(\frac{L^2}{4} - L\,x + x^2\right)\phi^2 \ldots\ldots\ldots\ldots (57b)$$

The strain energy of one half of the beam is then obtained from

$$W_e = \frac{E}{2}\left[I\int_0^{(L/2-b)}\left(\frac{d^2 y_1}{dx^2}\right)^2 dx + I\int_{(L/2-b)}^{L/2}\left(\frac{d^2 y_2}{dx^2}\right)^2 dx\right.$$
$$\left. + h^2\frac{I_1\,I_2}{I}\int_0^{L/2}\left(\frac{d^2\phi}{dx^2}\right)^2 dx + C\int_0^{L/2}\left(\frac{d\phi}{dx}\right)^2 dx\right] \ldots\ldots (58a)$$

Substitution of Eqs. 56*b* and 57*b* and evaluation of the integrals yield

$$W_e = \frac{a^2}{8\,I}\left[\frac{P^2}{E}\left(b^2 L - \frac{4\,b^3}{3} - \frac{b\,L^2}{\pi^2}\cos\frac{2\,\pi\,b}{L} + \frac{L^3}{2\,\pi^3}\sin\frac{2\,\pi\,b}{L}\right)\right.$$
$$\left. + \frac{\pi^4\,h^2\,I_1\,I_2}{L^3} + \frac{\pi^2\,C\,E\,I}{L}\right] \ldots\ldots\ldots\ldots\ldots\ldots (58b)$$

To find the work done by each of the two forces P, the distance the load falls will be divided into two components Δ_1 and Δ_2 as before, and again

$$\Delta_1 = \frac{(f\,I - h\,I_1)}{2\,I}\,\phi^2_b = \frac{(f\,I - h\,I_1)}{2\,I}\,a^2\sin^2\frac{\pi\,b}{L} \ldots\ldots\ldots (59)$$

in which ϕ_b is the angular displacement of the load. From Fig. 8 it is seen that, in analogy with Eq. 52, $\Delta_2 = \Delta'_2 - \Delta''_2$, as before, in which Δ'_2 is again the vertical displacement of the flexural center at midspan, and Δ''_2 is now the difference between this displacement and that at the loaded section. Consequently,

$$\Delta'_2 = \int_0^{(L/2-b)}\phi\,\frac{d^2 y_1}{dx^2}\left(\frac{L}{2} - x\right)dx + \int_{(L/2-b)}^{L/2}\phi\,\frac{d^2 y_2}{dx^2}\left(\frac{L}{2} - x\right)dx \ldots (60a)$$

and

$$\Delta''_2 = \int_0^{(L/2-b)}\phi\,\frac{d^2 y_1}{dx^2}\left(\frac{L}{2} - b - x\right)dx \ldots\ldots\ldots\ldots (60b)$$

The work done by P is then

$$W = P\,(\Delta_1 + \Delta_2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (61a)$$

By substitution of Eqs. 59 and 60, and by evaluation of the integrals, the work becomes

$$W = \frac{a^4}{8\,I}\left[4\,P\,(f\,I - h\,I_1)\sin^2\frac{\pi\,b}{L}\right.$$
$$\left. + \frac{P^2}{E}\left(2\,b^2 L - \frac{8\,b^3}{3} - \frac{2\,b\,L^2}{\pi^2}\cos\frac{2\,\pi\,b}{L} + \frac{L^3}{\pi^3}\sin\frac{2\,\pi\,b}{L}\right)\right] \ldots (61b)$$

The critical load is obtained by equating W_e from Eq. 58*b* to W from Eq. 61*b*, which results in

$$P_{cr} = A \frac{E}{L^3}\left[(h\,I_1 - f\,I) + \sqrt{(h\,I_1 - f\,I)^2 + B\,h^2\,I_1\,I_2 + D\,C\,I\,L^2}\right] \;..(62)$$

In Eq. 62, A, B, and D are numerical coefficients the values of which, for different ratios of b/L, are given in Table 1.

TABLE 1.—NUMERICAL COEFFICIENTS FOR THE SOLUTION OF EQ. 62

Coefficients	Ratios b/L									
	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
A	19.59	19.20	18.64	18.00	17.30	16.62	16.00	15.51	15.09	14.93
B	101.6	26.57	12.68	7.84	5.63	4.53	3.83	3.48	3.31	3.27
D	10.3	2.69	1.28	0.794	0.570	0.448	0.369	0.353	0.335	0.331

It may be ascertained easily that, for $b/L = 0.5$, the value $2\,P_{cr}$ of this analysis is identical with that for a concentrated center load obtained by the author. Also, for $b/L = 0.05$, almost the entire length of the beam is subject to pure bending. The critical moment so computed differs from that obtained by the author for pure bending of the entire beam by only 0.1%.

Acknowledgment.—This discussion is based on a thesis prepared under the supervision of the author and submitted to Cornell University, at Ithaca, N. Y., in 1941 in partial fulfilment of the requirements for the degree of Master of Civil Engineering.

H. N. HILL,[11] ASSOC. M. AM. SOC. C. E.—In 1937 the writer had occasion to solve the problem of the lateral buckling of an unsymmetrical I-beam subjected to pure bending, and derived an expression which differs somewhat from that determined by the author. The writer's solution can be expressed in the notation of the paper, if an additional term is introduced—"e," which is defined as the distance from the shear (or flexural) center to the centroid of the section. The centroid will lie between the shear center and the narrow flange. The term "e" will have a positive sign if the wide flange is in compression, and a negative sign if the narrow flange is in compression.

Using the disposition of axes as shown in Fig. 2 of the paper, the differential equations of equilibrium for lateral bending and twisting can be written: $E\,I\,\frac{d^2y}{dx^2} = -\,M\,\phi$ (see Eq. 8), and

$$G\,K\,\frac{d\phi}{dx} - E\,(I_1\,c^2_1 + I_2\,c^2_2)\,\frac{d^3\phi}{dx^3} = M\,\frac{d}{dx}\,(y - 2\,e\,\phi) \;.......(63)$$

In Eq. 63 the expression for the twisting moment is only approximately correct. It is applicable for unsymmetrical I-sections of the kind dealt with in this paper, and in which the two flange areas do not differ greatly. The unsymmetrical

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[11] Research Structural Engr., Aluminum Co. of America, New Kensington, Pa.

I-beams commonly encountered in structural fields are of this nature. A more exact expression for the twisting moment may be obtained by summing up the twisting moments produced by the longitudinal stress on each unit differential area of the cross section. The expression may be derived as follows, considering the z-axis as coinciding with the center line of the web, with the origin at the shear center, and assuming that the flanges are narrow compared to the depth of the beam:

$$M_T = \int_A \frac{M}{I_Y} (z - e)\, dA \left(\frac{dy}{dx} - z \frac{d\phi}{dx} \right) z \ldots\ldots\ldots\ldots (64)$$

in which I_Y = moment of inertia about the principal axis normal to the web. Expanding:

$$M_T = \frac{M}{I_Y} \left[\frac{dy}{dx} \left(\int_A z^2\, dA - e \int_A z\, dA \right) - \frac{d\phi}{dx} \left(\int_A z^3\, dA - e \int_A z^2\, dA \right) \right] \ldots (65)$$

in which

$$\int_A z\, dA = e\, A \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (66a)$$

$$\int_A z^2\, dA = I_Y + e^2 A \ldots\ldots\ldots\ldots\ldots\ldots\ldots (66b)$$

and

$$\int_A z^3\, dA = Z_Y + 3\, e\, I_Y + e^3 A \ldots\ldots\ldots\ldots\ldots (66c)$$

In Eq. 66c, Z_Y is the third moment of the area about the principal axis normal to the web. Therefore,

$$M_T = M \frac{d}{dx} \left[y - \left(2\, e + \frac{Z_Y}{I_Y} \right) \phi \right] \ldots\ldots\ldots\ldots\ldots (67)$$

For an I-beam in which the two flange areas are not greatly different, it is evident that the centroid will not vary greatly from a point at half the depth, and the term $\frac{Z_Y}{I_Y}$ will be very small. The areas of the two flanges of an unsymmetrical I-beam may be the same and yet their lateral stiffnesses may differ greatly.

Differentiating Eq. 63 and substituting for $\frac{d^2y}{dx^2}$ its value from Eq. 8 results in the differential equation

$$\frac{d^4\phi}{dx^4} - \frac{G\, K + 2\, M\, e}{E\, (I_1\, c_1^2 + I_2\, c_2^2)} \frac{d^2\phi}{dx^2} - \frac{M^2}{E^2\, I\, (I_1\, c_1^2 + I_2\, c_2^2)} \phi = 0 \ldots\ldots (68)$$

The solution of this fourth-order linear differential equation, with constants determined from the boundary conditions ($\phi = 0$, $\frac{d^2\phi}{dx^2} = 0$ when $x = 0$ and $x = L$), results in the trigonometric equation

$$\sin L \sqrt{-\frac{G\, K + 2\, M\, e}{2\, E\, (I_1 c_1^2 + I_2 c_2^2)} + \sqrt{\frac{(G\, K + 2\, M\, e)^2}{4\, E^2\, (I_1 c_1^2 + I_2 c_2^2)^2} + \frac{M^2}{E^2\, I\, (I_1 c_1^2 + I_2 c_2^2)}}} = 0 \ldots (69)$$

The smallest non-trivial root of Eq. 69, solved for the bending moment M, gives, for the critical bending moment,

$$M_{cr} = \frac{\pi^2 E\, I}{L^2} \left(\sqrt{\frac{L^2 G\, K}{\pi^2 E\, I} + \frac{I_1 I_2}{I^2} h^2 + e^2} + e \right) \qquad (70)$$

Expressed in the form of Eq. 14*a*, Eq. 70 would read

$$M_{cr} = E\, h \frac{\pi^2}{2\, L^2} \left(I \frac{2\, e}{h} + \sqrt{4\, I_1 I_2 + I^2 \frac{4\, e^2}{h^2} + 4\, C\, I \frac{L^2}{\pi^2 h^2}} \right) \qquad (71)$$

For the case of a symmetrical I-beam ($I_1 = I_2 = I_0$, and $e = 0$), Eq. 71 reduces to Eq. 14*b*.

The difference between Eqs. 71 and 14*a* is largely in the inclusion of the term e in the former. In the author's solution, no account is taken of the location of the centroid of the cross section. The reason for this discrepancy between the two solutions apparently lies in the method used by the author for evaluating the work of the external forces (Eqs. 10, 11, and 13). Eq. 10 is correct only if the end sections, which rotate through an angle β, remain plane. Under the combined lateral bending and twisting, the end sections do not remain plane. That this is so can be seen from Eq. 11. The change in chord length of a longitudinal fiber varies as the square of the lateral deflection of that fiber at the middle of the span. Since this deflection varies linearly with the depth of the beam, the change in chord length cannot vary linearly with the depth.

To determine the work of the external moment, it is probably necessary to sum up the work done by the forces acting on each unit differential area of the cross section. This would involve the stress distribution on the section and consequently the location of the centroid.

GEORGE WINTER,[12] ESQ.—In connection with the general tendency toward higher working stresses and more slender members in engineering structures, problems of the stability of such members are becoming increasingly important and are given wide attention in engineering research. Since the first publication of the writer's paper in December, 1941, three articles on related topics have been published. Prof. J. N. Goodier[13,14] has given a comprehensive and, in many respects, fundamentally new treatment of stability problems involving the simultaneous occurrence of compression, bending, and twist. Mr. Hill[15] has investigated the buckling of unsymmetrical I-beams in pure bending—that is, the same problem which forms part of the writer's paper. Whereas the writer derived his equations by means of energy principles, Mr. Hill used differential equations for this purpose. His discussion of the writer's paper represents an abstract of his aforementioned article.[15]

For the case of pure bending, Mr. Hill derives Eq. 71 which, in form, differs from Eq. 14*a* for the same case. As Mr. Hill rightly points out, his equations are rigorously exact only if the centroid of the section is midway between the flanges. The same holds true for the writer's equation. This does not mean that these equations apply to symmetrical beams only. Although the areas of the flanges must be equal in this case, the $\frac{b}{t}$-ratios for the two flanges may be different (b = width, t = thickness). In many instances beams are so designed in structural practice, especially to avoid local crippling of the compression flange due to large $\frac{b}{t}$-ratios.

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[12] Asst. Prof. of Civ. Eng., Cornell Univ., Ithaca, N. Y.

[13] "The Buckling of Compressed Bars by Torsion and Flexure," by J. N. Goodier, *Bulletin No. 27*, Cornell Univ. Eng. Experiment Station, Ithaca, N. Y., December, 1941.

[14] "Flexural-Torsional Buckling of Bars of Open Section," by J. N. Goodier, *Bulletin No. 28*, *ibid.*, January, 1942.

[15] "The Lateral Instability of Unsymmetrical I-Beams," by H. N. Hill, *Journal of Aeronautical Sciences*, Vol. 9, No. 5, March, 1942.

The writer is gratified to find that, for the case of equal flange areas (the only one for which both equations are strictly exact), Mr. Hill's and his own expressions are identical. To demonstrate this fact let Fig. 9 represent an I-beam with a centroid C, a flexural center F, and a midpoint M. Then the distance e, introduced by Mr. Hill, is the distance from C to F. For equal flange areas the points C and M coincide. In this case, with the notation of the paper and Mr. Hill's rule of signs,

$$e = -\left(\frac{h}{2} - c\right) = \frac{h}{2}\,\frac{I_1 - I_2}{I} \ldots\ldots (72)$$

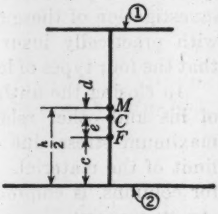

FIG. 9

Substitution of this expression in the second term of Eq. 71 gives:

$$I\,\frac{2\,e}{h} = I_1 - I_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (73)$$

which is identical with the second term of Eq. 14*a*. Likewise, substituting the value of e in the fourth term of Eq. 71, and combining it with the third term, results in:

$$4\,I_1\,I_2 + I^2\,\frac{4\,e^2}{h^2} = I^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (74)$$

which, in turn, is identical with the corresponding term in Eq. 14*a*. For this case, then, Mr. Hill's and the writer's expressions are the same.

If C and M do not coincide, neither Mr. Hill's nor the writer's equations are strictly exact. In this case the difference between the two approaches is that Mr. Hill implicitly takes account of the distance d from the centroid C to the midpoint M. However, he neglects the quantity $\frac{Z_Y}{I_Y}$ (see Eq. 67). The magnitude of d is found from

$$d = \frac{h}{2}\,\frac{A_1 - A_2}{A_1 + A_2 + A_w} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (75)$$

in which A_1, A_2, and A_w are, respectively, the areas of the two flanges and of the web. It is easily seen that the quantity $\frac{Z_Y}{I_Y}$ is of the same order of magnitude as d. Therefore, little is gained in accuracy by taking account of d but neglecting $\frac{Z_Y}{I_Y}$. Except for the rare case of radically different values of A_1 and A_2, the quantity d is seen to be rather small as compared with $\frac{h}{2}$ or c. Therefore, the writer's simple Eq. 14*a* represents a close approximation even for the case in which the flange areas are different.

The practical advantage of the writer's work-energy method is that it provides a rather simple way of analyzing the stability of beams under transverse loading rather than only in pure bending. The writer's equations for pure bending and for center load are supplemented in Lieutenant Schrader's contribution by those for uniformly distributed loading and for two equal loads symmetrically located on the beam, derived by the writer's method. An investigation of these cases by means of differential equations would have met with practically insurmountable mathematical complexities. It is believed that the four types of loading analyzed cover the most important practical cases.

In closing the author wishes to draw attention to the fact that the results of his and other related investigations are directly applicable only if the maximum stress due to the critical load is smaller than the proportional limit of the material. This limitation, the same as that of the Euler formula for columns, is common to all calculations concerning the elastic stability of structures.

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TRANSACTIONS

Paper No. 2179

REDUCTION OF MINERAL CONTENT IN WATER WITH ORGANIC ZEOLITES

BY R. F. GOUDEY,[1] M. AM. SOC. C. E.

WITH DISCUSSION BY MESSRS. JAMES M. MONTGOMERY, GEORGE L. DAVENPORT, JR., ROBERT SPURR WESTON, AND HARVEY F. LUDWIG AND RUSSELL G. LUDWIG.

SYNOPSIS

In general, the idea of deliberately reducing the mineral content in water has been limited to the field of boiler-feed water for steam generation purposes. Some incidental reduction in total solids in water has resulted when "temporary" hard waters have been softened with lime for domestic and industrial uses. Modern organic zeolites can now be used to secure an almost complete removal of solids even to the point of competing with distilled water. Already a number of high-pressure steam plants, ice plants, and a single instance of a large munitions plant, are using organic zeolites to produce water of a low mineral content for boiler-feed or rinsing purposes. The use of organic zeolites, while still expensive, has passed the academic stage. It has been recommended (1)[2] that the latest types of organic zeolites, called resinous exchangers, could be used most satisfactorily for processing and precipitate washing at drug and chemical plants.

It is possible to remove one or more of the positive and negative major ions from water including calcium, magnesium, sodium, alkalinity, sulfate, chloride, and nitrates. The full significance of this fact should be realized. The writer suggests that these newer methods of water treatment be considered in the following general instances:

(1) To reduce any major ion in water that exceeds United States Treasury Department Standards, such as sulfates, 250 ppm; chlorides, 250 ppm; and total solids, 1,000 ppm, or the present proposed limit of 500 ppm;

(2) To remove or alter objectionable ions in water for irrigation use with particular attention to lowering the sodium ratio and reducing total solids to within safe limits, even though calcium and magnesium salts can be kept high; and

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[1] San. Engr., Bureau of Water Works and Supply, Los Angeles, Calif.

[2] Numerals in parentheses, thus: (1), refer to corresponding items in the Bibliography (Appendix).

(3) To extend the applications in which undesirable waters can be modified for safe industrial use, particularly where relatively high costs can be tolerated.

These suggestions do not recommend the production of distilled water for all uses; but, by blending portions of water variously treated and, in some instances, the addition of compounds, there can be a selective modification of various ions in water to produce a final water satisfactory for the purposes intended. Such a development opens up many practical applications of far-reaching economic importance.

DEVELOPMENT OF ZEOLITES

Although the treatment of water with zeolites consists, for the most part, in exchanging calcium and magnesium for sodium, and regenerating the zeolites when exhausted with salt which results in a softer water, there is an actual increase in total solids because the equivalent sodium ions are heavier than the replaced ions. Nevertheless, the treatment by zeolites has been an important stepping stone to the development of more efficient exchange materials which can now be used to remove any of the major ions. The history in the development of zeolites, as applied to water treatment, started with the use of green inorganic sands (glauconite) as early as 1910. This material had an exchange ratio of sodium, for calcium and magnesium, based on about 2,800 grains of hardness (expressed as calcium carbonate) for a cubic foot of material. These natural zeolites were greatly improved during the following decade by processing which doubled the exchange capacity and increased the filtering rate. Common commercial products that were installed at Neville Island, Pennsylvania, in 1926 (2), and at the plant of the City of South Orange, N. J., in 1928 (3), marked the beginning of the use of zeolites for municipal softening.

The first artificial inorganic zeolites were produced as solutions of sodium silicate in 1906, and in the following year the reversible properties of such solutions were discovered. Artificial zeolites are manufactured either from sodium aluminate and sodium silicate to give a ratio of one part sodium oxide to 5.5 or 6.0 parts of silica or from aluminum sulfate and alkalized sodium silicate to give twice the aforementioned ratio. These materials are controlled to form a stiff solid gel or a gelatinous precipitate in liquor. The material is dried either naturally or artificially until it cracks into grains when sprinkled with water. Further control in processing determines the grain size, which is one of the factors in determining the capacity reaction and the loss of head through the filters. Modern inorganic zeolites, therefore, are referred to as "gel-type" zeolites. Their capacity varies from 9,000 to 12,500 grains of hardness removal as calcium carbonate. Most manufacturers guarantee 8,000 grains of hardness removed for each cubic foot of material, although this is not a fair basis for a guarantee as the capacity up to 12,500 grains of hardness removed is dependent on the amount of salt used for regeneration. The rate of filtration for natural zeolites is 5 to 8 gal per sq ft per min with a 40-min period of regeneration. Artificial zeolites require less rinsing than natural zeolites. Downward filtration may be used with all inorganic zeolites.

Salt consumption for regeneration theoretically should average 0.167 lb of salt for each 1,000 grains of hardness removed. The reaction, however, requires the presence of a salt excess and the actual regeneration dosage under the most advantageous conditions, such as municipal plants, varies from 0.33 to 0.4 lb per 1,000 grains of hardness removed. The amount of salt for regeneration at industrial plants is usually 0.4 to 0.5 lb per 1,000 grains of hardness removed. Even higher salt consumption is allowed for individual domestic units—namely 0.5 to 0.7 lb per 1,000 grains of hardness. The lowest figure for salt consumption requires either reuse of portions of the brine, which is a patented process, regeneration of the zeolite at about 85% capacity, or less, or utilization of longer periods of regeneration.

A large proportion of material used in industry includes solar dried commercial products manufactured at Yuma, Ariz., and Chicago, Ill. Processed material, not solar dried, is manufactured in New York, N. Y., and Chicago. More than forty municipal plants have been constructed since 1935 using zeolites for municipal water softening (4).

A process of adding lime has been developed to remove temporary hardness and to treat a part of the lime-softened water with sodium zeolite to zero hardness for the removal of permanent hardness followed by mixing the raw, lime-treated water and the zeolite-treated water to obtain any final residual hardness desired. Such plants have been located at Findlay, Ohio, and the Metropolitan Water District of Southern California (5).

Lime is the most efficient agent for softening calcium bicarbonate waters because the reaction drops out the lime added and the temporary hardness, except for the solubility value of calcium carbonate, which is usually 40 to 60 ppm. The use of excess lime may remove half this residual. Only zeolites can reduce the calcium and magnesium ions to practically zero. Magnesium bicarbonate hardness requires twice as much lime as calcium, and noncarbonate hardness requires the addition of both lime and soda ash. The sodium zeolites remove these substances with about twice the salt required by the stoichemical reaction, and the advantage of lime and soda ash over zeolites becomes a matter of studying costs of chemicals involved. Usually zeolite softening is cheaper than lime and soda ash when the ratio of noncarbonate hardness is 50% or more of the total hardness (4).

The disadvantages of the sodium zeolite softening are: First, it increases the sodium ratio; second, the sodium zeolite must be used at pH-values between 7.0 and 7.8 as there is disintegration at both higher and lower pH-values; and third, water having passed through a municipal zeolite plant containing a residual hardness and a high pH-value such as 9.0 will cause disintegration of zeolites at local industrial plants which must further treat such waters for complete removal of hardness. Sodium zeolite waters in boiler-feed installations are often treated with sulfuric acid and aerated to remove bicarbonate, but the total solids are slightly increased.

Organic Zeolites.—Following the work of B. A. Adams and E. L. Holmes (6) in 1935, significant progress has been made in the development of organic zeolites (1). Cation exchangers, or organic zeolites, can be operated either on a sodium or hydrogen cycle depending on whether they are regenerated by salt or mineral acid. Most commercial plants are using 1% sulfuric acid for regeneration purposes. The first four compounds were formerly called carbonaceous zeolites and the last compound is referred to as synthetic resin. Resins are carbonaceous compounds and so the word "organic zeolites" has been selected to cover both phases of development.

The second group of materials which have been recently developed includes the anion exchangers for negative ion removal.

In 1940, the writer presented (7) a short bibliography of the development of carbonaceous zeolites. A more complete bibliography is to be found in a later work by R. J. Myers, J. W. Eastes, and F. J. Myers (1). Artificial organic zeolites have now been developed that are highly condensed, very hard, resistant, and extremely insoluble, and which have properly sized grains with large inner surfaces, high reactivity, and extreme mechanical strength. These zeolites, which are basically either tannin formaldehyde, or m-phenylenediamine resins, have been developed from a great number of compounds. Organic zeolites are somewhat lighter than inorganic zeolites and swell more without rupturing the internal lattice, even to the extent of increasing their size 25% before regeneration. After regeneration the organic zeolites return to normal size. This normal swelling and contracting is probably due to the difference in the size of molecules being exchanged. One third more freeboard for backwashing is required because the material is lighter.

Organic zeolites are installed in filters made acid or alkali resistant with an upward flow rate of 10 gal per sq ft per min with a backwash of 1 to 2 gal per sq ft per min and a rinse of 2 to 3 gal per sq ft per min. The capacity of hydrogen zeolite is proportional to the amount of acid or salt used for regeneration and is proportionately higher for increased capacities. Ordinarily, 0.3 lb of sulfuric acid per 1,000 grains of hardness or its equivalent removed is considered the most efficient method of operation. This represents a removal rate of about 8,000 grains of hardness per cubic foot of material.

In sodium silicate zeolites, the salt ratio is twice the theoretical amount required under the most favorable conditions of operation. Until recently the hydrogen zeolites required 0.292 lb of sulfuric acid per 1,000 grains of hardness removed (7) at a removal rate of 8,550 grains of hardness extracted for each cubic foot of material. This amount of acid is 2.08 times that which is theoretically required. No experiments have yet been published to establish the minimum acid requirements. The writer has used rates as low as 0.225 lb of sulfuric acid per 1,000 grains of hardness removed when operating on calcium only. This is 1.6 times the theoretical amount. That some excess acid must be added is obvious; yet the report of the increased chemical efficiency of resin exchangers used on a sodium cycle as compared with the sodium silicate zeolites indicates that considerably less acid may be required when resin exchangers are operated on a hydrogen cycle and particularly where sodium is high. For purposes of this paper and since the sulfuric acid requirement for treatment is the principal factor, an arbitrary figure of 0.2 lb of sulfuric acid per 1,000 grains of ions removed and expressed as calcium carbonate will be assumed.

Water that has passed through an organic cation exchanger becomes acid and the bicarbonates are converted to carbonic acid gas. The latter can be removed by aeration; hence the cation exchangers remove not only calcium, magnesium, and sodium, but also carbonates and bicarbonates.

The anion or negative ion exchangers have been developed somewhat more slowly. The first exchangers "leaked" and apparently the lattice structure became easily ruptured. Likewise, there was a tendency for the material to go into solution. The present anion exchangers, however, are extremely efficient. They are acid and alkali resistant, have a capacity of 29,800 grains per cubic foot of hardness removed, rinse with about one half the water used formerly, and give twice the filter runs. Regeneration with 1% sodium hydroxide or soda ash is used.

Organic zeolites exchange positive ions with hydrogen when mineral acids are used for regeneration. The exchange is probably on the phenolic hydroxyl group. The equations for initial and final stages are shown in Table 1.

TABLE 1.—TYPICAL EXCHANGE REACTIONS

No.	(a) CATION REACTIONS			(b). ANION REACTION		
	Hydrogen cycle	Sodium cycle	Magnesium cycle	Hydroxyl cycle	Carbon dioxide cycle	Chloride cycle
	H_2Z	Na_2Z	Mg Z	$Z(OH)_2$	Z CO_3	Z Cl_2
	+	+	+	+	+	+
1 2 3	$\left.\begin{matrix}Ca\\Mg\\Na_2\end{matrix}\right\}\left\{\begin{matrix}CO_3\\(H\,CO_3)_2\\SO_4\\Cl_2\end{matrix}\right.$	$\left.\begin{matrix}Ca\\Mg\end{matrix}\right\}\left\{\begin{matrix}CO_3\\(H\,CO_3)_2\\SO_4\\Cl_2\end{matrix}\right.$	$Ca\left\{\begin{matrix}CO_3\\(H\,CO_3)_2\\SO_4\\Cl_2\end{matrix}\right.$	H_2SO_3 or 2 H Cl or H_2SO_4	2 H Cl + H_2SO_4 ↓	H_2SO_4 ↓ Z Cl_2
	↓	↓	↓			
	$\left.\begin{matrix}Ca\\Mg\\Na_2\end{matrix}\right\}$ Z	$\left.\begin{matrix}Ca\\Mg\end{matrix}\right\}$ Z	Ca Z	↓	$Z\left\{\begin{matrix}Cl_2\\SO_4\end{matrix}\right.$	2 H Cl
	+	+	+	$Z\left\{\begin{matrix}CO_3\\Cl\\SO_4\end{matrix}\right.$	+ H_2CO_3	
1	H_2CO_3	$Na_2\left\{\begin{matrix}CO_3\\(H\,CO_3)_2\end{matrix}\right.$	Mg $(H\,CO_3)_2$			
2	H_2SO_4	Na_2SO_4	Mg SO_4	+		
3	2 H Cl	2 Na Cl	Mg Cl_2	2 H_2O		

The removal of negative ions may be due strictly to adsorption or exchange, but for all practical purposes the hydroxide ion behaves parallel with the hydrogen ion and water treated by both methods. The writer found that chlorine replaced sulfates after the hydroxide is exhausted (7). In an experiment of passing 90 gal of Los Angeles (Calif.) aqueduct water through 277 g of a cation exchanger (7) the hydrogen zeolite became successively a sodium and a magnesium zeolite as the capacity to remove sodium and magnesium was reached. This means that hydrogen zeolites may be used to remove calcium only or calcium and magnesium together. Also, 100 gal of cation-treated water without aeration was passed through 333 g of an anion exchanger, and alkalinity, chlorides, and sulfates were removed in the same order as sodium, magnesium, and calcium. The alkalinity and chlorides exchanged successively with chlorides, finally being replaced with sulfates. This indicates that as ions sulfates can be removed more economically than chlorides. The equations in Table 1(*b*) indicate the initial and final stages of the negative ion reactions.

Regeneration of cation and anion exchangers follows the reactions:
Cation regeneration—

$$\left.\begin{matrix}\mathrm{Ca}\\ \mathrm{Mg}\\ \mathrm{Na_2}\end{matrix}\right\}\mathrm{Z} + \mathrm{H_2SO_4} = \mathrm{H_2Z} + \left.\begin{matrix}\mathrm{Ca}\\ \mathrm{Mg}\\ \mathrm{Na_2}\end{matrix}\right\}\mathrm{SO_4} \ldots\ldots\ldots\ldots\ldots (1)$$

anion regeneration—

$$\mathrm{Zx}\left\{\begin{matrix}\mathrm{Cl}\\ \mathrm{SO_4}\\ \mathrm{H\,CO_3}\end{matrix}\right. + 2\,\mathrm{Na\,OH} = \mathrm{Zx\,OH} + \mathrm{Na_2}\left\{\begin{matrix}\mathrm{Cl_2}\\ \mathrm{SO_4}\\ \mathrm{(H\,CO_3)_2}\end{matrix}\right. \ldots\ldots\ldots (2)$$

CARBONACEOUS ZEOLITES VERSUS SODIUM ZEOLITES

There is a growing tendency to use carbonaceous zeolites in place of sodium zeolites whether artificial or natural, for the reason that the sodium zeolites require narrow operating limits such as a pH-value from 7.0 to 7.8 beyond which zone there is softening of the structure with subsequent loss of silica. There is very little addition of silica to treated water within safe operating limits. At most small plants, however, the pH-variation is not subject to rigid control and there is, as stated, a tendency to favor a zeolite that is acid and alkali resistant. There are a number of installations using carbonaceous zeolites on a sodium cycle. One of the more recent developments has been the combined use of two carbonaceous zeolite filters (see Fig. 1), one operating with 50% of water on a sodium cycle and the other operating on the other 50% of the water supply on a hydrogen cycle with subsequent mixing of the two waters. There are a number of installations where carbonaceous zeolite is used on a sodium cycle followed by acid treatment to remove alkalinity and thereafter to degasify. This treatment, however, increases the total solids rather than decreases them, although it will produce a water low in hardness and alkalinity.

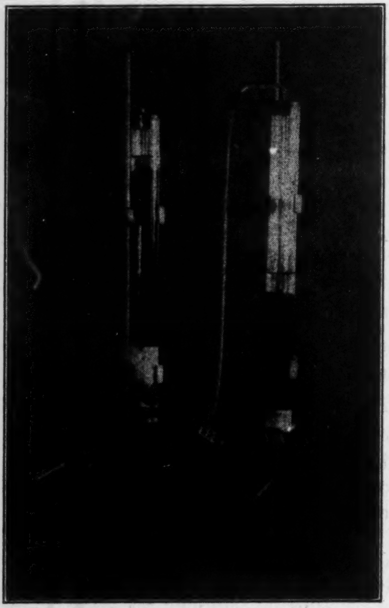

FIG. 1.—LABORATORY CARBONACEOUS ZEOLITE FILTERS

REDUCTION OF SOLIDS BY EVAPORATORS

In the high-pressure steam plant boiler-feed water field, evaporators have been designed to reduce total solids. With appropriate facilities (including separators so that water can be produced having 4 ppm, or less) total solids and the remaining ions are approximately proportional to the ratio of ions originally present. The first cost is relatively high—namely, about $100,000 per million gal capacity. Operation cost is that of distillation, which is about $2,000 per million gal. Plants that have heat exchange facilities allow practically nothing for the cost of operating evaporators. This process is too limited for general application; yet, where boiler make-up is high, evaporators would be extravagant and hydrogen zeolites offer the most economical treatment.

COST

Layouts for filters using sodium zeolites have been fairly well established. Municipal zeolite units run from $25,000 to $35,000 per million gal capacity, although the 70-million gal unit of the Metropolitan Water District of Southern California cost $5,000 per million gal. Treatment plants using the hydrogen cycle require greater tank depth but considerably less area and the equipment must be made acid proof. However, there should not be much difference in the final first cost of plants for equivalent treatment capacity. For the removal of alkalinity by aeration, additional equipment is required. Anion exchangers operate at more than twice the rate as cation exchangers so that the cost of filter assemblies is less than that of the corresponding cation exchanger for the same total flow.

The filtering material of exchangers is naturally a major item of expense and the cost of filtering material in any particular case is based on the quantities used. Freight for material shipped cross-country is a large part of the cost where large quantities cannot be manufactured near the points of use. Roughly, the current prices for sodium zeolite in large quantities average $4.50 per cu ft, organic zeolites on the hydrogen cycle at $9 per cu ft, and organic zeolites on the hydroxyl cycle at $36 per cu ft. These prices, while somewhat disconcerting at first glance, are certain to drop downward as use and production increase. From the standpoint of ease in manufacture, there is no reason why ultimately the prices for hydrogen and hydroxyl zeolites should not approach that of sodium zeolites. The royalty fee for hydrogen zeolites is $1 per cu ft, whereas hydroxyl zeolite is $5 per cu ft. Since the foregoing cost range parallels the relative efficiency of the materials involved so far as filtering rate and volume of material required are concerned, the cost of equivalent materials on a million gallon basis of water treated in any new moderately sized plant should be about the same as that now allowed for sodium zeolites. On the basis of present prices, however, the capital investment for moderately sized plants to remove various combinations of ions on a million gallon basis is as shown in Table 2. Obviously, smaller plants might cost 50% more, but the relative ratio of the costs between the various processes is probably the same as given in Table 2. From the standpoint of initial cost, there appears to be no real obstacle to a rather wide application for the newer processes. The data in Table 2 are based on removing ions to practically zero values. In many instances only a part of the water would be completely treated to be mixed with that ratio of raw water necessary to produce allowable values. If only one third of the water were to be treated, the cost per million gallons per day capacity would be one third of that indicated in Table 2.

TABLE 2.—RELATIVE COMPARISON OF INSTALLATION COSTS, IN DOLLARS PER MILLION GALLONS PER DAY CAPACITY, FOR VARIOUS COMBINATION REMOVALS

Description	ION EXCHANGE OR REMOVAL				
	Ca and Mg[a]	+Na	+Alkalinity	+Cl +SO_4	+Cl +SO_4[b]
First cost, thousand dollars per million gallons	35	50	53	70	100
Percentage of total solids removed	30[c]	<50	75[d]	99	99
Cost, in dollars per additional percentage of removal	1,160	750	120	680	1,000

[a] Sodium zeolite. [b] By evaporator. [c] 15% to 40%. [d] 65% to 90%.

Operation costs, however, are certain to be high in difficult cases. No general rule can be given owing to wide discrepancies in local prices of sulfuric acid and caustic soda. In July, 1941, the New York price for salt was $13.70 a ton; yet at the plant of the Metropolitan Water District of Southern California, the delivered price at the same time was $4.30 to $4.70 per ton. Sulfuric acid, although quoted in New York at $16.50 a ton, has been known to be delivered in large quantities at $11 a ton. Sodium hydroxide, quoted in New York at $46 a ton, can be delivered as low as $20 a ton at certain locations. In any particular case it would be necessary to calculate the chemical cost with amounts to be used and the cost of delivered chemicals. The amount of acid should be based on 0.2 lb of sulfuric acid per 1,000 grains of ions removed, the latter being expressed in terms of calcium carbonate, and 0.165 lb of sodium hydroxide for 1,000 grains of ions (as calcium carbonate) removed.

PRACTICAL APPLICATIONS

Waters high in bicarbonates lend themselves to high reductions in total solids through cation ion exchanges which, after aeration and mixed with raw waters, are not acid and may not require treatment by anion exchangers. The Los Angeles aqueduct supply is an example of such a water. Waters high in sulfates and chlorides usually require the addition of an alkali to produce a noncorrosive water if both cations and anions are removed in large amounts. The Colorado River water illustrates this type of water. With no thought of recommending the following schemes of treatment, but using them only to illustrate the complexity of reactions and wide variation in costs involved, illustrations follow as to how to produce water with a hardness of 85 ppm, a sodium ratio below 50%, and a sulfate content under 250 ppm from the two types of waters mentioned.

Case 1. Los Angeles Aqueduct.—In the Los Angeles Aqueduct (see Table 3) only cation treatment is necessary on 23% of the flow which, after aeration, is mixed with 77% of the untreated water. The acidity produced by the calcium, magnesium, and sodium reduction is neutralized by the alkalinity of the mixing water. This treatment requires a sulfuric acid dose of 550 lb per million gal on a chemical cost of \$3.00 to \$4.55 per million gal, which is relatively low.

It should be observed that were lime to be used to produce such a water, the sodium ratio would exceed an allowable 50%, reaching 53%.

From the standpoint of first cost, this type of treatment for this water is by far the most economical because it would be designed for only 23% of the total flow.

TABLE 3.—EFFECT OF MINERAL ADJUSTMENT
(Parts per Million)

Description	LOS ANGELES AQUEDUCT[a]		COLORADO RIVER	
	Raw water	Treated water	Raw water	Treated water
Calcium	28	21	125	24
Magnesium	10	8	26	6
Total hardness	111	85	420	85
Sodium	43	33	111	37
% sodium	46	46	36	48
Alkalinity	128	80	136	15
Sulfate	45	45	380	87
Chloride	21	21	92	37
Total solids	252	180	845	200

[a] Cation exchanger only.

Case 2. Colorado River.—Many are the possible combinations of various methods of treatment to bring the Colorado River water within the standards stated. Particular attention must be given to the increasing sodium content as hardness is removed, to reduce sulfates, and to be certain that the final water is not corrosive. Without regard to the cost involved, one logical procedure would be as follows:

1. Treatment of the raw water with excess lime to 20 ppm residual alkalinity;
2. Passing 77% of the water through a cation exchanger to a sodium content of 10 ppm and aerating to remove the increased carbon dioxide acidity;
3. Treating the cation effluent in an anion exchanger to zero sulfates and 20 ppm chlorides;
4. Mixing 23% of the excess lime-softened water with the 77% flow which passed the anion exchanger to give a hardness of 70 ppm; and
5. Addition of 15 ppm of calcium carbonate.

The raw water and treated water analyses would be as shown in Table 3, in which the water has been softened to 85 ppm, with a sodium ratio of less than 50%. In this case the sulfates have been lowered to less than 250 ppm, but that was incidental to meeting the other requirements. The chemicals necessary for such a treatment for a million gallons include 915 lb of lime, 5,500 lb of sulfuric acid, 4,700 lb of sodium hydroxide, and 125 lb of calcium carbonate, which would be in the range of \$75 to \$150 per million gal depending upon maximum freight and smallness of plant capacity.

Were the Colorado River water not to be softened to 85 ppm, but to 150 ppm or thereabouts, the problem would be considerably simplified.

Case 3. Treatment of Irrigation Water.—There are many instances in which high sodium and sodium chloride waters may be corrected to the material advantage of irrigation. For instance, a sodium water of 120 ppm with a 50% sodium ratio and a chloride content which should be reduced by 200 ppm could be treated for $15 per acre-ft. It is realized that this cost is relatively high and would be restricted to the more valuable crops; yet waters of this type determine the entire success or complete failure of important irrigation projects.

The writer is not aware of any location where water is treated to remove minerals where irrigation supplies are involved, although there are a number of instances where good and bad waters are mixed so that the mineral content of the water applied to the land shall not be detrimental for agricultural purposes. In the Southwest it has been generally conceded (8) that the sodium ratio (namely, the percentage of sodium in relation to the total calcium, magnesium, and sodium content expressed in equivalent weights) should not exceed 50% except on well-drained soils where a limit of 60 might be tolerated. Values of 70% or 80% are known to be detrimental for agricultural purposes. The presence of a high ratio of calcium and magnesium salts to sodium salts is apparently beneficial on water used for irrigation purposes. It is also known that when relatively hard waters are treated with sodium zeolites such waters used for irrigation purposes destroy lawns and shrubs. This was observed frequently by the writer at Beverly Hills, Calif., where private zeolite plants have been installed at different residences. There are many irrigation supplies in which either the sodium, chloride, or sulfate content is so high that the water cannot be used beneficially for agricultural purposes. Minor corrections in mineral constituents would go a long way toward making successes or failures of many questionable water supplies. Therefore, in the reduction of mineral contents in irrigation waters, there are many cases in which it would be desirable to reduce the sodium ratio.

Case 4. Boiler-Feed Water.—In the treatment of boiler-feed water, it has been necessary to remove solids to control scale formation and corrosion embrittlement, and to reduce carry-over to a minimum. For quite some time, the high carbonate waters have been treated with acid followed by degasifying so that sodium bicarbonate cannot later break down into carbon dioxide and sodium hydroxide in the boiler. It has been found that the increased cost of making water softening equipment acid resistant, together with the cost of acid, is more than offset in many installations by the reduction in total solids and with the resulting decrease in blowdowns (9).

Each boiler-feed problem depends upon the analyses of the water to determine the best type of treatment. Most cases require a reduction in total solids. The total solids of Los Angeles water can be reduced from 225 to 141 by 50% treatment with sodium zeolites and 50% treatment with hydrogen zeolites. High bicarbonates in the sodium zeolite effluent are used to neutralize the acidity produced by the hydrogen zeolite, and the carbon dioxide is removed by aeration. Obviously, water of any total solid content can be produced using mixtures of raw water with treated portions from cation and anion exchangers.

The cost of reducing the highly bicarbonated Los Angeles water by the sodium zeolite and hydrogen zeolite on a fifty-fifty basis is $15 per million gal and for 50% removal as contrasted with only $20 per million gal for complete removal up to 3 or 4 ppm by use of cation and anion exchanges.

Silica is not removed by positive and negative exchangers and heavy coagulation doses of 300 to 400 ppm of alum may be used prior to the zeolite treatment.

Case 5. Special Cases.—Ice plants desire water with total solids of less than 350 ppm. Beer is best when made with a "plastered" water low in sodium salts. Cartridge cases and washed surfaces, requiring clean and bright faces to prevent jamming, need practically distilled water. Color photography requires water with bicarbonates removed.

Cities with concrete sewers and with sulfates present in the water supply would stop disintegration entirely by removing sulfates. Railroad engine water supplies in the Southwest could be treated better by hydrogen and hydroxyl zeolites. Nuisances and efficiency losses at cooling towers where hydrogen sulfide is produced from mineral sulfates could be controlled by the removal of sulfates.

These and other cases indicate many applications in which solids in water may be reduced to an advantage.

CONCLUSION

The development of improved organic zeolites for the removal of positive and negative ions, completely or in part, provides superior methods for softening water, reducing sodium content, removing excess alkalinity, lowering sulfates, or reducing chlorides to produce suitable water for domestic, irrigation, and industrial uses, and in special cases where inconvenience, expense, or failure to secure an adequate working plan are involved. Although the costs of operation in the more aggravated cases may be relatively high, they are in most cases not prohibitive and are certain to be lower as increased production and availability of suitable materials develop.

APPENDIX

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(9) **"New Trends in Boiler Feed Water Treatment,"** by F. G. Straub, *Transactions*, Am. Inst. of Chemical Eng., Vol. XXXVI, 1940, p. 395

DISCUSSION

James M. Montgomery,[3] **M. Am. Soc. C. E.**—The results that are being obtained in water treatment make water-works chemists and engineers wonder what the plant of the future will be like. Mr. Goudey has presented an excellent review of the work that is being done in connection with the softening and demineralizing of water by means of base-exchange materials.

The discovery that certain synthetic resins have base-exchange properties is particularly promising for the future because of the infinite number of possible variations in the molecular structure of organic compounds, some of which will probably result in the production of improved materials. The basic principle having been established, organic research chemists no doubt will work to produce materials with greater exchange capacity, improved exchange efficiency, regeneration with less expensive materials, and the property of selective ion removal.

Water softening by base exchange, using either natural or synthetic siliceous zeolites, has been practiced for many years. As a result of this long, and in many cases sad, experience, the properties, both good and bad, of these materials are now well known and, where conditions are right, the materials are used with confidence.

On the other hand, the organic exchange materials are still in the development stage. It is true that carbonaceous zeolites have been used for several years and appear to give good results when used on either the sodium or hydrogen cycle, but the number of installations is too small and the time of service too short for the establishment of definite standards. It is known that the products of some manufacturers have shown a tendency to break down in service and produce black water. Synthetic resins are really in the development stage, and there are but few actual installations, some of which have been disappointing to both customer and manufacturer. Probably a long period of experience, experiment, and education for both manufacturer and user lies ahead before all of the difficulties incident to the use of organic zeolites have been discovered and corrected. Such materials must have a useful life of several years, and unfortunately no satisfactory accelerated tests have ever been developed.

The properties possessed by organic zeolites are remarkable enough without some of the extravagant stories that have been told about them. Contrary to information furnished by certain popular science editors, it is impractical to use them to demineralize sea water. In fact, even if the materials were thoroughly proved by experience, the cost of demineralizing Colorado River water, as shown by Mr. Goudey, is too high to warrant consideration. High exchange capacity is claimed, but it has apparently been accomplished by sacrificing economy.

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[3] (J. M. Montgomery & Co.), Los Angeles, Calif.

Messrs. Myers, Eastes, and Myers (1) have made determinations of the exchange values achieved by using quantities of salt ranging from 3.11 lb to 81.06 lb per cu ft of zeolite. The use of 3.11 lb per cu ft gave an exchange value of 8,400 grains per cu ft, and 81.06 lb per cu ft gave 29,800 grains per cu ft, with proportionate results for intermediate quantities. The exchange value of 8,400 grains was accomplished by using approximately twice the theoretical quantity of salt. This approaches the economy of good, synthetic, gel-type zeolites. The exchange value of 29,800 grains per cu ft required the use of approximately sixteen times the theoretical quantity of salt. In other words, it would cost eight times as much to operate at 29,800-grains-per-cu-ft exchange value as it would to operate at 8,400 grains per cu ft. Actually, the organic zeolites so far developed would seem to be approximately equivalent to a synthetic gel-type zeolite in both exchange capacity and regenerant consumption when operated in either the sodium, hydrogen, or anion removal cycles at economical exchange values.

When organic zeolites are used in the two-step process for removal of both anions and cations from a water such as the Colorado River supply, which contains a large amount of dissolved solids, the quantity of rinse water required to wash the excess of regenerating materials from the zeolite beds becomes a large item. Sulfuric acid, which is ordinarily used for regenerating the cation removal material, is considerably more difficult to rinse out of a zeolite bed than is salt, and both sodium carbonate and sodium hydroxide, which are usually used for regenerating the anion removal material, are even more difficult to rinse out than sulfuric acid. When the water softening plant of The Metropolitan Water District of Southern California was being designed, investigations were made of the two-step demineralizing process. Statistics furnished by the zeolite manufacturer at that time indicated that the quantity of rinse water that would have to be wasted would amount to 12½% of the total plant influent when treating average water and would increase to about 20% when treating the much harder water that is now stored in Lake Mead (Arizona-Nevada), Lake Havasu (Arizona-California), and Lake Mathews (California), as compared with 2% to 4% at the Colorado River Aqueduct softening plant.

If water could be demineralized economically, this would be an ideal process for reducing both the hardness and the total mineral content of Colorado River water. Some people fear that the high sodium percentage of the softened water will reduce its value for agricultural and horticultural purposes, although twelve years' use of softened municipal water at Beverly Hills, Calif., with a high sodium percentage, has not decreased the beauty of that city's gardens and estates. It seems probable, however, that the plan suggested by Oscar C. Magistad (8) for overcoming the effect, if any, of high sodium water would be satisfactory and practical.

In the Los Angeles (Calif.) area, more people seem to take pride in their lawns and gardens than in any other place in the United States. The beautiful landscaping seen on all sides is not produced as a result of planting and watering alone. Well-kept lawns are torn up, fertilized, and reseeded every few years, and flowers and shrubs require fertilizing frequently. If high sodium water should finally prove, after long use, to be unsatisfactory for domestic irrigation due to the removal of calcium from the soil by base exchange, it would seem that the application of calcium sulfate or possibly ground limestone along with the fertilizer would be both practical and economical. The cost would be small and probably would soon be accepted as a necessary expense incident to the maintenance of good lawns and gardens. The addition of a quantity sufficient to restore the proper calcium-sodium ratio should cost only a dollar or two per year for the average home.

Work such as Mr. Goudey is doing in investigating new base-exchange materials will help to speed the development of improved products. It is to be hoped that he will continue his researches and help to keep water-works chemists and engineers informed of future progress. The Metropolitan Water District, operating the largest zeolite softening plant in the world, also will continue its research, begun at Boulder City, Nev., in 1938. Facilities for such work at the new plant are excellent, and it is hoped that a great amount of constructive work will be done.

George L. Davenport, Jr.,[4] M. Am. Soc. C. E.—The writer's experience of ten years with the use of inorganic zeolite for treatment of water used in locomotives may be summed up as follows:

(1) It is very satisfactory for most waters of 15 grains per gal of hardness or less. The Atchison, Topeka and Sante Fe Railway Company does not usually treat water having less than 5 grains of hardness.

(2) When the hardness is much in excess of 15 grains, the zeolite process will produce treated water that will cause foaming and priming in locomotives, so the railroad must use a lime-soda treatment that will reduce the total solids.

(3) About 50% of the water treated with zeolite causes disintegration of the mineral in excess of the usual annual loss guarantee. Some softening plants run as high as 14% loss of mineral per year. Even then the replacement cost is relatively small, and is insignificant as compared with the advantages gained.

(4) Corrosion of pipe lines and other steel surfaces coming in contact with the softened water can be avoided by using about 50% more wash water than is usually recommended by the manufacturer. It is the practice of the "Sante Fe" to determine the proper quantity of wash water by a silver nitrate test which guarantees that the plant will not be back in service until the salt content in the effluent is within 5 grains of the original salt content of the raw water. This is a very important point which, as a rule, zeolite plant manufacturers do not recognize, since they do not like to admit that so much water should be wasted.

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[4] Asst. Engr., A. T. & S. F. Ry., Los Angeles, Calif.

(5) Although the "Sante Fe" has obtained estimates on organic zeolite plants, no convincing reason for their use in treatment of locomotive supplies has been presented by the manufacturers. In addition to their higher first cost, there is also to be considered the practical difficulty, in the case of hydrogen cycle plants, of handling acids as compared with salt solutions. Highly trained employees, such as are essential to a large municipal plant, are not available for operating railroad softeners. In fact, the tendency is to eliminate the human element by installing fully automatic zeolite plants, operated by electric motors, which only require occasional services of a man to keep the salt storage tank full.

(6) To summarize, the railroad softening plant is a valuable supplement to the lime-soda process, but cannot take its place for the treatment of most waters in the Western States. Some waters will cause much more loss of zeolite than manufacturers will admit is possible, but even in such cases the operating cost is reasonable and advantages far exceed disadvantages. Long life for piping and equipment using zeolite treated water can be predicted, if sufficient water is wasted in washing out the mineral during the regeneration process.

Use of inorganic zeolite does not appear practicable at the present for railroad water softening due to the high cost of apparatus and lack of skilled attendants. By spending extra money for fully automatic equipment, the ordinary zeolite plant can be made "foolproof" and will render outstanding service, if restricted to treatment of suitable waters.

ROBERT SPURR WESTON,[5] M. AM. SOC. C. E.—The excellent paper by Mr. Goudey shows how much the art of water softening has developed since Thomas Clark of Scotland, about a century ago, patented the process of softening by adding lime water, and J. H. Porter, more than fifty years ago, introduced its sequel, the lime-soda process, at Southampton, England.

Both of these processes have been modified and improved by better arrangements and devices for the coagulation of precipitates and by the use of carbon dioxide and chemical dispersers to prevent the accumulation of carbonates on filter sands and in distribution systems; but even the best of these modified processes has its limits, and reductions of hardness below 80 ppm have been impracticable and difficult in most cases.

The writer first heard of base exchange softening at a conference in Breslau, Germany, when the work of Werner Gans[6] was mentioned in connection with that city's deferrization problem. Since then, the development of zeolite softening has been extensive, and would be wider if communities would only realize, as has the Los Angeles district, that large savings in soap and washed fabrics, and less depreciation of water-using devices, will follow the use of softer water.

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[5] Cons. Engr. (Weston & Sampson), Boston, Mass.

[6] "Verbesserung von Trinkwasser u. Gebrauchswasser für hausliche u. gewerbliche Zwecke durch Alluminat-Silicate oder künstliche Zeolithe," by Werner Gans, *Mitteilungen aus der Königlichen Prüfungsanstalt für Wasserversorgung u. Abwasserreinigung*, Berlin, 1907, Heft 8.

Since his first experience with softening by natural zeolite at McKees Rocks, Pa., in 1924,[7] and his later experience at South Orange, N. J.,[8] and elsewhere, the writer has had almost no municipal cases where the residual compounds were in high enough concentrations to be intolerable. He has realized how great the need is for some method of lessening these concentrations however, notably compounds of sodium in many waters which have initial hardnesses of 500 ppm or more.

Following zeolite treatment came the combination treatment with lime and zeolite which enabled one to reduce, to almost any desired degree, the hardness of waters high in sulfates.

Then, early in 1938, there appeared on the market anion and cation exchangers by the use of which, either separately or in combination, not only the hardness but the sodium content could be reduced to zero and the total solids greatly reduced as well. An excellent classification of these zeolites appeared in the author's former paper (7).

It seems almost fanciful that there are durable, organic cation exchanging zeolites on the market which will remove hardening carbonates and bicarbonates in the form of carbon dioxide gas and which may be regenerated with sulfuric acid, which, when combined with calcium, is a superlative water hardener; or, that there are anion exchangers which will remove not only bicarbonates but sulfates, chlorides, and fluorides, with increased acidity of effluent, and which require sodium hydroxide for regeneration.

Therefore, particularly valuable is the description of the author's research work and the clear exposition of the reactions involved. Interesting indeed is the showing that hydrogen zeolites may be used to remove either calcium or magnesium, or both, and that both the hydrogen and sodium cycles may be used on portions of the same water and the two mixed afterward.

All of the great possibilities that have been made available through the improvements in the durability, economy, and efficiency of the organic zeolites should make it possible to say farewell to turbid artificial ice and to boiler scale, to wash windows with waters of the desert, and to wash wool with waters from deep wells.

One rather shrinks from handling sulfuric acid or caustic soda as a regenerator because of the extreme aggressiveness of each in comparison with brine. However, resistant apparatus is available and may be used for successful handling where efficiency and cost are at stake.

One also asks if some of the regenerating chemicals may not be saved and reused, as is part of the brine at South Orange (3), with consequent material saving in cost.

In closing, the writer pleads for a simpler expression of the salt consumption for regenerating zeolites than is involved in the use of the term "grains"; for example, the expression "1.2 lb of hardness per cubic foot" seems simpler than "8,400 grains per cubic foot"—especially to those who think of hardness in terms of parts per million or pounds per million gallons. Likewise, for salt consumption it would be simpler to state "3.0 lb, or parts, of salt per pound, or part, of hardness" than "0.42 lb per 1,000 grains." It is realized, however, that those who treat boiler waters have thought in "grains" for more than half a century.

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[7] *Journal*, A. W. W. A., Vol. 32 (1940), p. 85.

[8] *Ibid.*, Vol. 33 (1941), p. 255.

Harvey F. Ludwig[9] **and Russell G. Ludwig,**[10] **Juniors, Am. Soc. C. E.**—An interesting review of what may be accomplished in the water conditioning field, through the use of various modern zeolites, is contained in this paper. Mr. Goudey does not elaborate upon the basic mechanism by which these zeolites perform their work—namely, that of "ion or adsorption exchange"—and the writers believe that a brief but critical analysis of this exchange phenomenon would be of considerable interest. This basic mechanism has wide and important, but generally unrecognized, application in many engineering fields, particularly in sanitary engineering.

Colloid chemists have determined that the ion exchange properties of zeolitic material in aqueous suspension result from the existence of an electrical double layer surrounding the nucleus or core of each particle. At the surface of the core reside relatively fixed charges or ions ("stabilizing ions") of either positive or negative charge; these make up the inner part of the double layer. The outer section comprises an envelope of oscillating ions ("adsorbed ions" or "exchangeable ions" or "replaceable ions") of opposite charge which diffuses into the surrounding dispersion medium (water). In the ordinary hardness-removing zeolite, for example, each particle (nucleus) is negatively charged, and this is balanced by an outer layer consisting largely, in freshly regenerated zeolites, of sodium ions (Na^+). These adsorbed Na^+ ions are highly dissociated—that is, they oscillate relatively far into the dispersion medium because they are weakly adsorbed.

The degree to which a particular ion will be held or adsorbed by the inner layer may be termed its "exchange adsorbability." The magnitude of this varies inversely with the hydrodynamic size of the ion, and directly with its valence. Ions in aqueous solution are thought to be hydrated or surrounded by a water shell of diameter inversely proportional to the actual size of the ion as it occurs in the crystal lattice. The combined effect of valence and of hydrodynamic size has been expressed[11] by the term "oscillation volume," representing the effective volume occupied by the oscillating ion, or, in other words, the width of the double layer. The smaller the oscillation volume, the more strongly an adsorbed ion is held and the greater its exchange adsorbability. Thus, the cations commonly found in natural waters may be classified in increasing order of their exchange adsorbability as follows: Na^+, K^+, Mg^{++}, Ca^{++}, H^+, Fe^{+++} and Al^{+++}. The trivalent Al^{+++} and Fe^{+++}, when they exist as such, are strongly adsorbed. The action of the H^+ ion is anomalous, but usually it behaves like a weak trivalent cation. The anions may be similarly classified, but in general much less is known of their behavior than of the cations.

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[9] Asst. San. Engr., U. S. Public Health Service, Atlanta, Ga.

[10] San. Eng. Aide, San Pedro, Calif.

[11] "Properties of Colloids," by Hans Jenny, Stanford Univ. Press, Stanford University, Calif., 1938.

If in the dispersion medium there are ions other than those of the diffused outer layer, then the phenomenon of "adsorption exchange" will occur, in which ions initially present in the outer layer are displaced by those from the dispersion medium. The degree to which one ion will displace another will depend upon the relative exchange adsorbabilities and upon the relative concentrations, in accordance with the general mass law. Thus, in the softening of water with ordinary zeolites, the hardness cations, Ca^{++} and Mg^{++}, displace the less strongly adsorbed Na^+ that is initially present in the zeolite and are thereby removed from solution. In the process of regeneration, the weakly adsorbed Na^+ is made to displace Ca^{++} and Mg^{++} by leaching the zeolite with a great preponderance of Na^+. Occasionally some Al^{+++} or Fe^{+++} or H^+ are also adsorbed in the softening operations. These are so strongly held that they may not be appreciably displaced by regeneration, and in time the zeolite "wears out." As noted by Mr. Goudey, wearing out may also occur through disintegration of the zeolite structure in the presence of excessive H^+ or OH^-.

Flocculation of Turbid Waters.—The mechanism of adsorption exchange has a multitude of other applications of interest to the civil engineer. In the flocculation of turbid waters, for example, the phenomenon is particularly important, in that the stability of suspended turbidity (clay) particles results largely from their zeolitic properties. Each particle is surrounded by an electric double layer, the outer section of which consists of various cations (combinations of Na^+, K^+, Ca^{++}, Mg^{++}, and H^+) of positive charge. The particles therefore repel each other, resisting their natural tendency to aggregate; and the forces of repulsion are proportional to the effective width of the double layer. When alum (or a similar coagulant) is added to the system, a portion of this dissociates to form Al^{+++} ions. These replace the cations initially adsorbed and thus reduce the effective width of the double layer to a critical value at which the particles no longer repel each other but instead tend to agglomerate. Another portion of the added alum is converted by hydrolysis (chemical combination with water) to particles or micelles of hydrous aluminum oxide. These are excellent binder material, and when the system is properly mixed or agitated they serve to entrap or "wrap up" the suspended turbidity particles and thereby form large-sized, dense flocs, which will settle out quickly on subsequent standing. The ion exchange reactions are important in this process because by this mechanism the initially stable turbidity particles are rendered unstable, so that they may be entrapped easily by the agglomerating hydrous oxide floc. The "coagulant demand" or dosage of coagulant necessary to flocculate a given water therefore may be considered as comprising two parts—a "perikinetic demand" or that part of the alum which dissociates to form Al^{+++}, and a "binder demand" or that alum which hydrolyzes to give hydrous oxide. (The term "perikinetic" refers to the forces of attraction and repulsion associated with the suspended turbidity particles. The action of Al^{+++} on the system reduces the forces of repulsion such that the particles will tend to aggregate. The system is now "perikinetically flocculated.") The "optimum pH" for any turbid water will be that at which the added alum will dissociate and hydrolyze just as required by the relative magnitudes of the perikinetic and binder demands. These concepts of the mechanics of flocculation were originally presented by the writers in a previous publication.[12] Their studies are being continued in the Sanitary Engineering Laboratories of the University of California, at Berkeley, under the direction of Prof. W. F. Langelier.

Soil Corrosion.—Another important application of the ion exchange principle is found in the study of soil corrosion of pipes. The basic mechanism involved has been termed "contact exchange," and may be demonstrated by hanging an iron nail in a suspension of clay particles containing replaceable H^+ ions. In time it is found that the nail may disappear entirely, and that the clay colloids have acquired a brownish coloration. Evidently the H^+ ions of the clay have been displaced by iron from the nail. This phenomenon is undoubtedly important in the corrosion of metallic pipes laid in soils containing replaceable hydrogen. Similarly, the deterioration of concrete pipe laid in soils containing adsorbed Mg^{++} properly may be attributed to replacement of Ca^{++} by Mg^{++}.

The question arises here as to just what adsorbed cations might be expected to occur in the soils of a given geographical region. Both the total concentration of adsorbed cations (called the "saturation capacity" or "base exchange") and the specific types present will depend largely upon climate, particularly upon annual rainfall.[13] The adsorbed cations in soils are associated with those soil particles of colloidal size (about 1 μ and less), termed the soil colloids; these include both the inorganic colloids (clay) and the organic humus colloids. The adsorbed cations generally include Na^+, K^+, Ca^{++}, Mg^{++}, and H^+; and the saturation capacity is usually expressed as the sum of the exchangeable bases (Na^+, K^+, Ca^{++}, Mg^{++}) plus the exchangeable hydrogen (H^+), all measured in milligram equivalents per 100 grams of soil. In arid regions the soils (alkali soils) contain relatively few colloids and therefore have low saturation capacities; the replaceable cations are all exchangeable bases (about 50% Na^+ and K^+, and about 50% Ca^{++} and Mg^{++}), there being no exchangeable hydrogen. When the annual rainfall becomes appreciable (in the magnitude of 25 in.), the leaching action of the percolating water results in replacement of some of the exchangeable bases by H^+ ions derived from the free carbonic acid in the water that percolates through the soil. Simultaneously, there is an increase in the saturation capacity, because the increased percolation hastens the weathering processes by which the noncolloidal primary soil minerals are converted into colloidal particles (clay), and because the increase in vegetation associated with increased rainfall increases the concentration of the humus colloids. In such soils (chernozems) the adsorbed cations comprise from 1% to 10% each of Na^+, K^+, Mg^{++}, and H^+, with about 75% Ca^{++}. Finally, in the very humid regions, the soils (podsols) are of very high saturation capacity and comprise about 70% H^+ with the remainder mostly Ca^{++} and Mg^{++}, there being little or no Na^+ or K^+.

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[12] "Laboratory Flocculation of East Bay Sewage, and the Mechanism of Flocculation in Water Purification and Sewage Treatment Practice," by H. F. Ludwig and R. G. Ludwig, *California Sewage Works Journal*, Vol. 13 (1941), pp. 54–80.

[13] "Factors of Soil Formation," by Hans Jenny, McGraw-Hill Book Co., Inc., New York, N. Y., 1941.

It is interesting to note that the physical and chemical properties of soils are generally controlled by the magnitude and nature of the soil colloids, although these usually are only a fraction of the total soil mass (about 10% in ordinary loams, for example). Further, the properties of these colloids are largely determined by the concentration and nature of their exchangeable cations, and these represent only about 5% of the soil colloid mass. Thus, the properties of a soil are determined essentially by its exchangeable cation content, which generally comprises less than 1% of the entire soil mass.

These two examples have served to illustrate how prevalent and important ion exchange reactions may be. There are many other instances, as, for example, in the fields of sewage treatment, highway construction, irrigation, etc.

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Paper No. 2180

UNUSUAL EVENTS AND THEIR RELATION TO FEDERAL WATER POLICIES

BY W. G. HOYT,[1] M. AM. SOC. C. E.

WITH DISCUSSION BY MESSRS. EDWARD H. SARGENT, DANA M. WOOD, JAMES S. SWEET, J. L. CAMPBELL, DAVID J. GUY, ALBERT S. FRY, AND W. G. HOYT.

SYNOPSIS

Within the 10-yr period between 1930 and 1940 there were profound changes in federal water policies and more than a tenfold increase in federal activities designed to modify the runoff phase of the rainfall-runoff cycle. The total federal expenditures for all purposes increased only about two and one-half fold during the same decade. During the fiscal year 1940 alone, some \$500,000,000 or \$600,000,000 of federal money were spent on measures and practices relating to storage and appurtenant works for irrigation, power, flood control, water supply, water spreading, debris control, recreation, and wild-life refuge; upstream engineering, agricultural, and land-use measures and practices; and all related activities designed to obtain beneficial conservation, control, regulation, and use of water. During 1930 the sum of money so expended was only about one tenth as much. The purpose of this paper is to inquire into the basic reasons underlying the changes in water policies and the large increase in activities and to discuss in a general way their extent, aims, and objectives.

INTRODUCTION

Many may feel that the large increase in water conservation activities is directly in accord with the long-established trend toward centralization of governmental responsibilities and corresponding expansion in federal public works. Others feel that the increase has been mainly due to the crossing of a new and largely unexplored frontier and to a normal, though rapid, expansion in long-deferred conservation and wise utilization of soils and waters, which are unquestionably two of the vital natural resources of the United States. Still others feel that social security, which was certainly a major political consideration, was the principal motivating force. All of these explanations may be safely assumed to be valid in some degree.

NOTE.—Published in February, 1942, *Proceedings*.

[1] Hydr. Engr. (Prin.), Water Resources Branch, U. S. Geological Survey, Washington, D. C.

Although recognizing the futility of attempting to evaluate the relative importance of many contributing causes that are obviously interrelated, it would appear that the indicated increase in activities relating to the conservation and control of water can be ascribed largely to the concurrent effect of the following three types of happenings:

(1) The 1930–1940 decade was characterized by a degree of Nation-wide abnormality with respect to climatic and hydrologic conditions that has probably not been exceeded during the past century;

(2) The impact of an economic crisis with chronic symptoms that began in 1929, and the associated depression, brought about changes in political leadership as well as modifications in governmental theory; and

(3) There is a belated Nation-wide recognition that the wastage of soil and water is a genuine menace to the national welfare.

Obviously these three reasons are closely related. For example, the abnormality with respect to meteorologic and hydrologic conditions created the necessity for the rehabilitation of devastated areas, the repair of damage resulting from the unusual events, and the construction activities looking toward alleviation of future occurrences. These in turn provided an outlet for relief work and funds that were made available as a result of the economic crisis. In addition, the visible evidences of the erosion and loss of fertile soil, and its transportation and deposition associated with the widespread floods and with the conditions arising from deficiencies in water supply during droughts, undoubtedly hastened the adoption of federal policies relating to the conservation of soil and water and to the control of floods.

The unusual meteorologic conditions relate largely to droughts and floods. Since 1889 there have been eleven well-recognized drought years in the United States. It is not fully realized, perhaps, that five of these major droughts of record occurred during the 10 years 1930 to 1940, as follows:

Year	Drought area
1930	Practically all states east of the Mississippi River, California, Oregon, Washington, North Dakota, and Montana
1931	Northern Great Plains, Idaho, Utah, Colorado, and the Southeastern States
1933	Upper Mississippi Valley, Nevada and Utah, South Central Plains, and Southeastern States
1934	Practically all of the United States except the Atlantic seaboard, Southern States, and Southeastern States
1936	All of the Mississippi River Valley and northern Great Plains, and Pacific Northwest

The history of these droughts and their significance in relation to climate are so fully recorded in technical reports[2] that it seems unnecessary to do more than to list them. They had a decided impact on national affairs and were directly responsible for many new national policies.

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[2] "Droughts of 1930–34," by John C. Hoyt, *Water-Supply Paper No. 680*, U. S. Geological Survey, 1936; also "Drought of 1936," by the same author, *Water-Supply Paper No. 820*, U. S. Geological Survey, 1938.

Floods, like droughts, also affected all parts of the Nation. Stages and discharges, and resulting damage in basin after basin, broke all historic records.

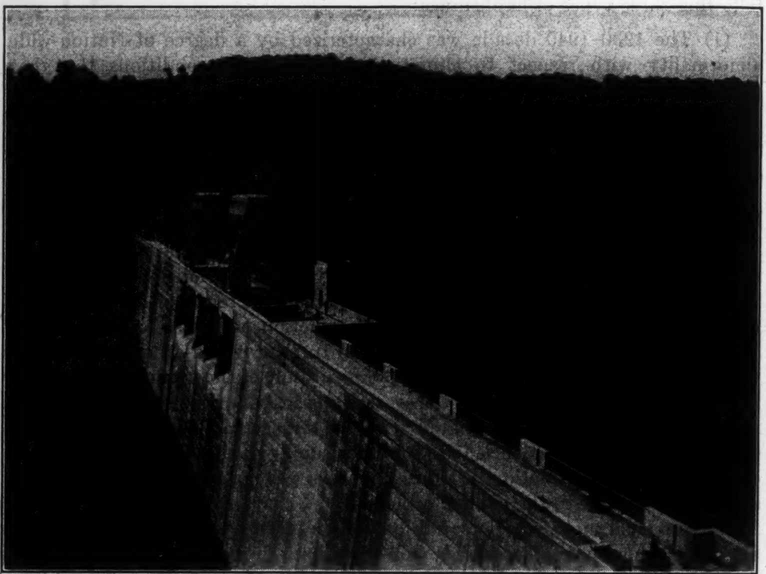

Fig. 1.—Floods, Like Droughts, Also Affected All Parts of the Nation. Mahoning Creek Dam in Western Pennsylvania Is One of Six Dams Designed as a Part of the War Department's Comprehensive Program for the Control of Floods in the Upper Ohio River Basin Above Pittsburgh, Pa.

These flood events occurred generally during the last half of the decade on river basins as follows:

Time	Region
1934:	
January	La Cañada Valley, California
1935:	
May and June	Republican and Kansas rivers in Colorado, Nebraska, and Kansas; and the Nueces and Rio Grande basins in Texas
July	Central New York
August	Muskingum River, Ohio
1936:	
March	All river basins in Northeastern United States from Ohio, West Virginia, and Virginia through to Maine
July and September	Neches, Guadalupe, Concho, and San Saba rivers in Texas

Time	Region
1937:	
January and February	Entire Ohio River Basin
March	Throughout the central upper Mississippi Valley, southern Wisconsin, eastern Iowa, and Illinois
May and June	Canadian and Pecos rivers, New Mexico
December	All of California north of Tehachapi Pass
1938:	
March	All of southern California
September[3]	Hurricane storm wave and flood in New England—believed probably to have destroyed more lives and property than any previous single natural disaster in the history of the United States
1940:	
August	South Atlantic region
November	All of eastern Texas

In addition to the foregoing major flood areas, major and record-breaking floods occurred in scattered areas on smaller streams. Although not absolutely indicative of the unusualness of the past decade with respect to floods, it seems very significant that, out of 6,792 maximum peak discharges covering all available records through September, 1938,[4] 3,137, or nearly 50%, occurred during the 8-yr period ending in 1938. With respect to the March, 1936, flood and the flood of January and February, 1937, in the Ohio River Basin, geologic evidence, suggestive rather than compelling, seems to show that in parts of the area they exceed in height any previous flood. Most of the other floods listed are clearly outstanding and unusual as shown by Myers' scale or other ratings.

It is not surprising to one acquainted with the meteorologic and hydrologic history of, and the resulting damage caused by, these events that the droughts brought home forcefully to the entire Nation the value of water in connection with its economic development or that the cataclysmic flood events made a deeper impression on policy makers, than ever before, of the need for a comprehensive program involving Nation-wide flood control with incidental benefits to power, navigation, irrigation, recreation, and related values.

Although it is somewhat difficult to ascribe definite reasons for the rapid transition of a wide popular movement or belief into an active action program, it is probably fair to state that the drought in 1930 had much to do with the initiation of land-planning and water-planning activities as a distinct federal function; that the recurring droughts during the period 1930 to 1936 resulted in an enlargement of the federal reclamation idea from one of large developments on arid lands to an all-embracing use of water for irrigation of both large and small areas in which problems of population shifts and resettlement played an important part for the first time since the elimination of the Nation's geographic frontier; that the widespread disastrous floods resulted in extension of federal flood-control activities from navigable and interstate streams to include others that affect the general welfare, regardless of the size of channel in which they occur, and, what is perhaps more significant, to include so-called "upstream engineering" as an integral part of the flood-control operations; and that the combined occurrences of both unusual floods and droughts resulted in new and extensive federal measures relating to the conservation of soil and water. No one can state definitely whether or not the unusual climatic manifestations would have resulted in new policies and related activities had they not occurred during a period when national affairs were being modified in an effort to overcome an economic depression. The droughts and floods were so unusual that, coming at any time, they would have created an urge, which no doubt would have been translated into definite action programs, at least to some extent, even under more normal economic and political conditions. The only question relates probably to the magnitude of the action program. It is sufficient for the present discussion to indicate that during 1940 hydraulic engineers in federal services were engaged on a far greater number of projects and on a far greater scale than in 1930, and that federal interests in water have grown to enormous proportions during the past decade. This interest has found expression in several different, though related, channels.

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[3] "Hurricane Floods of September, 1938," by C. G. Paulsen, B. L. Bigwood, A. W. Harrington, O. W. Hartwell, and H. B. Kinnison, *Water-Supply Paper No. 867*, U. S. Geological Survey, 1940.

[4] "Maximum Discharges at Stream-Measurement Stations Through December 31, 1941," by Gordon R. Williams and Lawrence C. Crawford, *Water-Supply Paper No 847*, U. S. Geological Survey, 1940.

NATIONAL WATER PLANNING

Prior to 1930 there was no single federal agency that devoted itself exclusively to water planning from a national standpoint. Agencies such as the Corps of Engineers, Bureau of Reclamation, and the Geological Survey had collected, compiled, and published a vast amount of basic technical information concerning the Nation's developed and undeveloped potentialities and possibilities. Moreover, just prior to the decade in question, the major floods during the spring of 1927 in the Mississippi Valley and the comprehensive report on the Tennessee River and tributaries by the Corps of Engineers were responsible, in part, for the legislation that resulted in the "308" reports by the Corps of Engineers in which was outlined technical information concerning water problems in all major basins in the United States. In these various agencies, therefore, there was a vast reservoir of technical information that made possible the early initiation of many of the major flood-control and multiple-use projects hereinafter listed.

Early in 1934 the urgent need of large federal public works projects for the unemployment relief program led Congress, on February 2, 1934, to ask the President for a comprehensive plan for the improvement and development of the rivers of the United States that would give information for the guidance of legislation to provide maximum multiple-purpose benefits. Accordingly, the President appointed promptly the secretaries of the Interior, War, Agriculture, and Labor departments to advise on the development of a water policy and on the choice of projects; and on February 20, 1934, they appointed six technical subcomitttees, each assigned to a selected geographical region. The Cabinet

Committee (referred to as "The President's Committee on Water Flow") on April 20, 1934, rendered its report to the President with qualifying or supplemental individual statements by the secretaries of War, Agriculture, and Labor. Although this composite report was made in great haste for the primary purpose of determining, quickly, construction projects of highest over-all merit in the unemployment-relief program, it is important in the record of interagency coordination in water planning.

Returning now to consideration of natural phenomena as a major influence, the droughts of the early 1930's seemed to emphasize that in addition to engineering problems it was necessary also to consider their relation to the over-all economic, social, and land problems which had then assumed such great proportions. This situation led to the appointment in 1933 of the Mississippi Valley Committee of the Emergency Administration of Public Works (PWA) which, through a series of administrative changes, became the Water Resources Committee of the National Resources Planning Board. Under the Reorganization Act of 1939 and the Presidential Reorganization Plan No. 4 (April 11, 1940), this Board became a part of the Executive Office of the President, reporting directly to him. It is perhaps significant that the first line in the first planning report—namely, "Report of the Mississippi Valley Committee" dated October 1, 1934—is a quotation credited to Charles D. Norton: "Make no little plans. They have no magic to stir men's souls."[5] No one can gainsay but that this admonition set a keynote and has been adhered to by subsequent committees. Probably no one federal agency has enlisted the services (largely voluntary) of more hydraulic engineers and hydrologists in its planning operations than has the National Resources Planning Board; and many of these men are engaged today on its forty-five drainage basin committees created to facilitate cooperation among federal and state agencies in the conduct of investigations of water in relation to navigation, power, municipal water supply, erosion, irrigation, forestry, floods, droughts, and wildlife conservation. The basin reports of the Water Resources Committee, in which are outlined water problems and programs, contain a wealth of information which has not only been of value in connection with present public works projects but which will increase in value as time goes on. Those interested in hydrology, as such, will no doubt find much of interest in two reports prepared under the auspices of the Water Resources Committee—namely, "Deficiencies in Hydrologic Data" and "Deficiencies in Hydrologic Research."

FLOOD CONTROL

As has already been indicated, federal (as distinct from non-federal agencies, which had been active in flood-control operations and which include, among others, the Miami Conservancy District, as an aftermath of the 1921 flood, and the Los Angeles Flood Control District) flood activities prior to 1930 were generally incidental to navigation and were conducted largely as a function of the Corps of Engineers. These activities were directed principally toward the control of flood waters by levees and improvement of the channels of navigable or interstate rivers. The cataclysmic flood events of the past decade, however, greatly modified and enlarged the federal policy and federal participation in relation to flood control in two notable respects: First, by the adoption of a policy which extended the scope of control of floods that affect human life and property, whatever the size of the rivers and channels; and second, by Congressional recognition[6] of a relation between land use and floods and the necessity for a coordinated land-use and water-regulation program for flood control. These two modifications or enlargements of policy not only increased the activities of the Corps of Engineers but also brought the Department of Agriculture "into the picture" as an active flood-control agency. Here again the increased and new activities demanded the effective service of hydraulic engineers. For example, under the Flood Control Act of 1936, Congress authorized some 270 flood-control projects (most of which had been investigated and reported on in the "308" reports), including many dams and reservoirs with an estimated construction cost in excess of $300,000,000 located in thirty-one states and affecting nearly every state in the Union.

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[5] Report of the Mississippi Valley Committee of the Public Works Administration, October 1, 1934, p. 3.

The Flood Control Act approved on June 28, 1938, authorized additional flood-control improvements in nineteen separate river basins, or parts of basins, and authorized the sum of $370,000,000 for the construction of the projects included in this Act. In addition to actual construction projects, these acts also authorized the Departments of War and Agriculture to make preliminary examinations and surveys for flood control at some 345 locations. One important feature of the enlarged policy was the expressed obligation for the federal government to assume a larger share of the cost of flood-control operations, which had heretofore been a local responsibility. An outstanding example relates to the reimbursement (Act of June 28, 1938) of the Muskingum Conservancy District for expenditures made by the District in acquiring lands and right of ways for the series of fourteen reservoirs in the Muskingum River Drainage Basin and the inclusion (Act of August 11, 1939) of the reservoirs in the flood-control plan for the Ohio River.

Congressional recognition of a relation between land use and floods and the necessity for a coordinated land-use program for flood control naturally greatly increased flood-control activities in the Department of Agriculture, which since 1938 has been conducting flood-control surveys and developing programs for the reduction of flood damages by the treatment of lands on which floods originate. Preliminary examinations have been completed on more than two hundred watersheds aggregating in area more than 1,000,000 sq miles. Detailed surveys have been initiated on forty-two watersheds having a total area of 265,000 sq miles. Flood-control survey reports had been completed for sixteen of these basins by the end of 1941. Treatment of one of the surveyed watersheds has been initiated. In the Department of Agriculture, especially, the hydrologists have had to play a very important part since it was necessary to evaluate not only the relation between precipitation, land use, and flood runoff but also to predict how these relationships would be modified by changes in land use.

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[6] Flood Control Act, June 22, 1936.

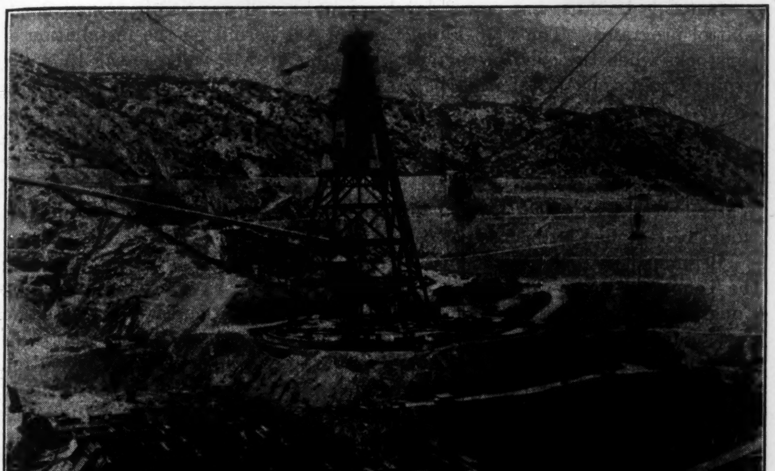

FIG. 2.—NOT ONLY LAND USE BUT POWER DEVELOPMENT WAS TIED IN TO NEW FLOOD-CONTROL PROGRAMS. CRESCENT-SHAPED SHASTA DAM WHICH IS MORE MASSIVE THAN BOULDER AND TALLER THAN GRAND COULEE IS DESIGNED TO MEET THE EVER-INCREASING DEMAND FOR POWER IN THE DEFENSE INDUSTRIES OF NORTHERN CALIFORNIA. THIS DAM, THE PRINCIPAL ENGINEERING FEATURE OF THE CENTRAL VALLEY PROJECT IN CALIFORNIA, WILL ALSO AID IN FLOOD CONTROL, NAVIGATION, IRRIGATION AND THE REPULSION OF SALT WATER FROM THE DELTA FARMING AREA

WATER POWER

Not only land use but power development as well was tied in to new flood-control programs. The 1936 Flood Control Act permitted the installation of penstocks when the chief of engineers recommended it, whereas the 1938 Act required that such installation be made when recommended by the Federal Power Commission. The latter Act made specific appropriations to be expended by the Federal Power Commission in conducting any necessary investigations or surveys. This step toward federal development of the power by-product was apparently an extension of the power policy of the Federal Water Power Act of 1920, under which all power equipment and appurtenances would be installed by licensees.

Beginning with the Boulder Dam Act of 1928, various projects, including the Tennessee Valley Authority (TVA) developments, have been approved by Congress. This indicates a disposition on the part of Congress to favor actual development by the federal government much more than theretofore. This generalization applies whether the primary purpose be navigation, flood control, or reclamation, although the greatest evidence of the changed policy is disclosed through flood-control measures which, in turn, are the outgrowth of the 1930–1940 flood situation already described.

MULTIPLE-USE PROJECTS

Prior to 1930 the operation of most public water projects related chiefly to a single purpose such as the maintenance of navigation or the storage, release, and distribution of water for irrigation. Since 1930 there has been a profound change in federal participation in multiple-use water projects. A few of these projects, however, represented the consummation of plans proposed prior to 1930 and, therefore, may not be directly related to meteorologic and hydrologic events since 1930. Two such major developments are: The Boulder Dam on the Colorado River, constructed under the Act of 1928, based on earlier reports and examinations, and the Central Valley Project of California, which had its inception in reports by the state engineer of California made prior to 1930. Most of the multiple-use projects, however, have had their inception since 1930 and are more or less related to the train of events since that date. These larger projects are (1) TVA; (2) Fort Peck Dam, in Montana; and (3) Columbia Basin Project.

(1) The all-inclusive TVA (Act of May, 1933) was born in the stress of the swiftly moving days of 1933. The initiation of its chain of navigation, flood-control, and power dams was greatly facilitated by the "308" reports of the Corps of Engineers previously mentioned. Its upstream operations embraced water conservation, erosion, forestry, and land use "in the interests of National defense and for agricultural and industrial development and to improve navigation in the Tennessee River and to control destructive flood waters in the Tennessee River and Mississippi River Basin."[7]

(2) The largest earth dam in the world at Fort Peck, on the Missouri River, was designed to store 19,000,000 acre-ft of water, with its primary engineering

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[7] From preamble, Act of May, 1933.

purpose the improvement of river navigation from Sioux City, Iowa, to the mouth of the Missouri River, a distance of 764 miles, with incidental purposes of flood control, hydroelectric power, and irrigation. In addition to the strictly engineering purposes, the dam was initiated to provide work for unemployment. More than 53,000,000 man-hr had been expended on the project to April 15, 1939. For comparison as to the work relief features, it may be of interest to state that the total man-hours by the Civilian Conservation Corps (CCC) directed by the Soil Conservation Service on forestry, soil erosion, flood control, and similar projects from the beginning of operations through December, 1940, was 67,738,591 man-hr.[8]

(3) The Columbia Basin Project and the Grand Coulee Dam on the upper Columbia River, in the State of Washington (the largest single structure ever to be constructed by man and containing enough material to build a concrete road a distance of 5,600 miles), will be used for irrigation, power, flood control, and navigation.

Other projects include the Bonneville development on the Columbia River near Portland, Ore.; the fourteen dams and reservoirs in the Muskingum River Basin, Ohio; the Conchas Dam on the South Canadian River in New Mexico; the Sardis Dam on the Little Tallahatchie River in Mississippi; the Buchanan and Marshall Ford dams on the Colorado River in Texas; the Tygart River Dam in West Virginia; the Santee-Cooper development in South Carolina; and others.

As an example of the magnitude of these and other new great developments, it may be stated that the storage capacity thus made available exceeds 80,000,000 acre-ft, or considerably more than the combined capacity of all reservoirs that had been constructed for all purposes in the United States prior to 1930. The design, construction, and operation of these structures have naturally resulted in vastly increased staffs of the Bureau of Reclamation and the U. S. Engineer Department, the two agencies most actively engaged on the projects. Problems relating to hydraulic engineering and hydrology have been paramount in their design and operation since it was necessary not only that the project works be safe, but that they should effectively perform their multiple-use purposes for navigation, flood control, power, irrigation, domestic water supply, and the real, though intangible, benefits such as recreation, wildlife conservation, National Defense, economic rehabilitation, and the like.

SOIL AND WATER CONSERVATION

Prior to 1930 soil and water conservation activities as a distinct identity separate and apart from federal irrigation, unquestionably the first major federal soil and water-conservation activity, were directed largely toward experimental research and demonstration projects in the Bureau of Agricultural Engineering, Bureau of Chemistry and Soils, and Forest Service in the Department of Agriculture. In the Interior Department, the Geological Survey, since the days of Maj. J. W. Powell, has been active in studies and measures looking toward wise use of land and water resources. Although the need for conservation of soil and water was fully realized by many, the droughts and floods provided a stimulus whereby the research activities were translated into a vast action program, first (1933) in the Interior Department under PWA and later (1935) in the Department of Agriculture. The Soil Conservation Act of April 27, 1935,[9] states in its preamble "that the wastage of soil and moisture resources on farm, grazing, and forest lands of the Nation, resulting from soil erosion, is a menace to national welfare" and declares the policy of Congress

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[8] The C.C.C.'s Eighth Birthday," by G. H. Gilbertson, *Soil Conservation*, SCS, April, 1941, p. 262.

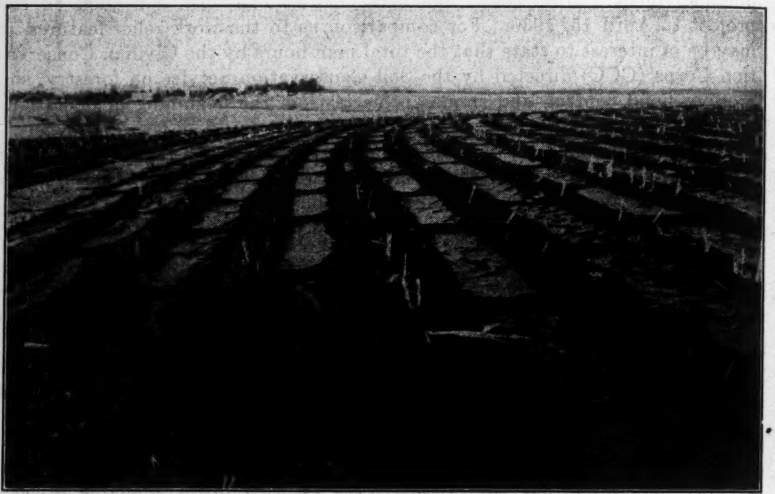

FIG. 3.—THE WASTAGE OF SOIL AND MOISTURE RESOURCES RESULTING FROM SOIL EROSION IS A MENACE TO NATIONAL WELFARE (A BASIN LISTED FIELD ARRANGED TO CONSERVE MOISTURE AND PREVENT MOVEMENT OF WATER ALONG THE FURROWS)

> "to provide permanently for the control and prevention of soil erosion and thereby to preserve natural resources, control floods, prevent impairment of reservoirs, and maintain navigability of rivers and harbors, protect public health, public land and relieve unemployment."

The unusual events also played a large part in the enactment of Soil Conservation District laws in practically every state and the establishment of Soil Conservation districts embracing more than 66,000,000 acres of land, as is evidenced by the fact that the first sentence of a letter by the President, dated February 26, 1937, to the Governors of all states, expressing the hope that state conservation laws would be adopted, reads as follows: "The dust storms and the floods of the last few years have underscored the importance of programs to control soil erosion." Prior to Reorganization Order No. 4, dated April 11, 1940, activities under the Soil Conservation Act were confined to the Department of Agriculture, but under the terms of the Presidential Order No. 4, soil and moisture conservation activities were transferred to the Department of the Interior in so far as about 280,000,000 acres of Indian, grazing, national park, and other lands under its jurisdiction are concerned.

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[9] 49 Stat. 163, 16 USC 598.

Closely allied with the purpose of the Soil Conservation Act is the so-called Taylor Grazing Act of June, 1934, which provides for the conservation of soil and water on the vast public domain of the West through the creation and administration of grazing districts. Under this act the Department of the Interior regulates grazing and other forms of land use on some 150,000,000 acres of land that theretofore were subject to entry under the public land laws. Also, under the provisions of the National Industrial Recovery Act of June 16, 1933, all public lands in the eastern and central Western States were withdrawn from entry in February, 1935, by executive order in the interests of conservation of natural resources including "control, utilization and purification of waters, prevention of soil and coastal erosion and flood control."

Thus, since 1930 the Nation has witnessed the expansion of soil and moisture conservation activities as such (and apart from irrigation) from research into a Nation-wide action and educational program by two major federal departments. In addition to providing an outlet for the employment of the CCC, Works Progress Administration (WPA), and similar "work relief" organizations, this development has increased many times the number of hydraulic engineers in the government service.

IRRIGATION AND WATER FACILITIES

Prior to 1930 federal activities in connection with irrigation were conducted chiefly by the Department of the Interior under the Act of June 17, 1902, and amendments thereof, commonly known as the Reclamation Act, and under several specific acts relating to the development of irrigation facilities on Indian lands. Under the Reclamation Act, millions of acres of arid lands of the West had been transformed into productive farms, and substantially all expenditures relating thereto had been on a basis of repayment by water users. Not including Indian lands, most of the projects involved public lands, and economic problems relating to resettlement and population shifts were not generally of primary consideration. Although the federal government was vitally interested in the relation between the adequacy of water to the individual water user on the project, it was not especially concerned with the farmer or rancher on private lands outside of federal projects.

The water shortages throughout the Great Plains, the dust storms, and the human suffering associated with the recurring droughts to which attention has been called led to the passage of three acts, all of which have materially increased federal activity with respect to water. These acts authorized the creation of the now well-known AAA, or Agricultural Adjustment Administration, the Farm Security Administration, and the Water Facilities, all in the Department of Agriculture. Under these acts, investigations, loans, general planning, and home-management guidance, relating to the construction and development of ponds, reservoirs, wells, dams, springs, pumping works, windmills, water spreaders, stock-water tanks, flood irrigation, and small irrigation systems for single farms or small groups of farms, are authorized with not more than $50,000 of federal funds to be expended on a single project.

In addition, the Wheeler-Case Act (August 11, 1939) authorized the Secretary of the Interior to construct water-conservation and utilization projects in the Great Plains area and elsewhere. This is apparently a new type of conservation legislation involving cooperation between the Bureau of Reclamation, Office of Indian Affairs, Department of Agriculture, federal relief agencies, and local state or political subdivisions, the water users (except Indians, who are dealt with by specific legislation) repaying, over a period of 40 years, all of the allotments from appropriations made for the purpose and, in addition, any amount from "relief funds" directed by the President.

In all these new activities, hydrology has played an important part, since one of the primary purposes of the activities relates to the development of dependable water supplies in regions where the supplies are not only scanty but where, in general, prior development has used the "cream," as it were, of the water. In contrast to 1930, it is now possible under existing laws for any landowner or operator under certain conditions to avail himself of financial and technical help in the development of dependable water supplies for irrigation, stock, and domestic uses.

HYDROLOGIC RESEARCH AND FACT FINDING

As of 1930, federal hydrologic research was restricted generally to a few experimental areas maintained by the Department of Agriculture under the terms of the McSweeney-McNary Forest Research Act or earlier authorization, at which the intricate relationships between rainfall and runoff were determined. The fact-finding activities related largely to the Weather Bureau in connection with the collection of rainfall, snowfall, and temperature records and to the Geological Survey in connection with the Nation-wide collection of records of stream flow, ground water, and quality of water. As a coordinated part of all of the activities previously described, there has been, since 1930, a pronounced increase in hydrologic research activities as a whole, and a substantial, although not proportional, increase in the activities of the Weather Bureau and the Geological Survey as they relate to the collection of basic meteorologic and hydrologic data. The increase in hydrologic research has taken the form, principally, of:

(*a*) Expanded scope and increase in the number of experimental areas maintained by the Soil Conservation Service and the Forest Service for the purpose of determining the intricate relations between rainfall, runoff, soil moisture, and soil movement under different climatic, physiographic, and geologic conditions, and the effect of cover and land use on such relationships.

(*b*) The investigation by the U. S. Engineer Department of approximately 1,000 major storms in the United States involving the compilation of basic precipitation from all available sources; the preparation of isohyetal maps and, for the more important storms, the construction of mass rainfall curves; and computation of duration depth-area curves by 6-hr intervals.

(*c*) Increased activity of the Geological Survey with respect to the analyses of hydrologic and meteorologic phenomena associated with floods and droughts.

(*d*) Expanded hydrologic studies by the Weather Bureau in connection with its enlarged program of flood forecasting and establishment of a hydrometeorological section in which mass rainfall curves prepared by the U. S. Engineer Department are reviewed in connection with synoptic analyses, and in which special meteorological studies are made to determine limiting precipitation rates in certain drainage basins.

(*e*) Coordinated efforts of the Soil Conservation Service and the Forest Service on hydrologic problems involved in the flood-control activities of the Department of Agriculture.

The increase in availability of basic meteorologic and hydrologic data relates principally to the increase in the number of precipitation stations, more especially the network of recording rain gages installed and maintained by the Weather Bureau in connection with its Nation-wide coverage, made possible in large part by the cooperation of the Corps of Engineers and flood-control agencies in the Department of Agriculture; the dense network in Ohio by the Soil Conservation Service in connection with its microclimatic studies; the increase in snow observations; and the increase in the number of river-measurement stations and ground-water observation wells maintained by the Geological Survey that have been occasioned by the demands of, and made possible by, the cooperation of many federal and non-federal agencies engaged on flood control, irrigation, and related problems. There also may be added numerous fact-finding projects in hydraulic laboratories and experimental areas relating to evaporation, transpiration, wave movement, debris movement, suspended load, and others either independently or jointly by the Corps of Engineers, Geological Survey, Department of Agriculture, Bureau of Reclamation, TVA, and other agencies.

CONCLUSION

In retrospect it has been made to appear, perhaps, that the water policies outlined were adopted and put into practice in a more orderly sequence and manner than was actually the case. In the war against the depression and unusual climatic and hydrologic conditions of the 1930's, however, as in the present defense activities, there was undoubtedly considerable stress and confusion. Parts of the activities were acknowledged to be experimental. It is too much to hope that under such conditions all policies and practices would ever be intrinscially sound. Future experience may be expected to show that some modifications are needed.

Engineers had considerable to do with the design and planning of projects and also with the formation of policies, although there may be considerable disagreement as to the validity of all the policies. Engineers also have had much to do with the prosecution of the newly authorized activities. Out of their combined experiences should come expressions of opinion and ultimately a knowledge that should have weight in the guidance of future changes and trends. To that extent, as well as in the accomplishment of sound results, the new situation presents a definite challenge to the profession.

ACKNOWLEDGMENT

For the photographic views in this paper the writer is indebted to three important agencies of the federal government. Fig. 1 is supplied through the courtesy of the U. S. Engineer Department; Fig. 2 is from the Bureau of Reclamation; and Fig. 3 is from the Soil Conservation Service.

DISCUSSION

Edward H. Sargent,[10] **M. Am. Soc. C. E.**—The author is to be congratulated on his splendid paper describing the basic reasons underlying the changes in the federal water policies in the decade from 1930 to 1940.

The gates of the Conklingville Dam creating the Sacandaga Reservoir, constructed by the Hudson River Regulating District on the Sacandaga River, the principal tributary of the upper Hudson River, were first closed on March 27, 1930, impounding that spring's flood waters. This reservoir has an available capacity of 760,000 acre-ft, and release of the stored water for the regulation of the flow of the Hudson River for the public welfare, including navigation, sanitary improvement, and power, began on July 1, 1930.

With reference to the author's statement (see heading "Multiple-Use Projects") that, "Since 1930 there has been a profound change in federal participation in multiple-use water projects," the writer feels that the success of the Sacandaga Reservoir in achieving these multiple purposes contributed to the changing of the current engineering opinion regarding such functioning.

Dana M. Wood,[11] **M. Am. Soc. C. E.**—This excellent paper rightly stresses the impetus given to multi-purpose planning by quite recent events. However, the trend has existed even from the earliest days of the Nation, and was strong long before the past decade. Also, this paper is based largely upon the effects of Congressional acts, whereas innumerable court decisions also have had a very definite part in broadening federal water policies. Of course, both the Congress and the courts consist of men influenced by the social impacts of droughts and floods, as well as economic and war stresses. All these factors have been present from earliest days.

At least two other papers should be read along with this one to clarify the broad background for present federal water policies. In 1938, James L. Fly[12] presented the legal development of federal policies with respect to navigation, flood control, irrigation, and finally planning for multi-purposes. Further progress has been made since that date. David E. Lilienthal and Robert H. Marquis, in 1941,[13] emphasized the expansion of governmental business activities and the advantages of the government corporation for the administration and development of multi-purpose projects. Both papers definitely indicate why recent rapid advances in policies and accomplishments have been possible.

One reason why so much progress could be made during the past decade may lie in the tremendous advance made in the arts during the previous two decades. Engineers were prepared to accomplish great things when the need became urgent.

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[10] Chf. Engr., Hudson River Regulating Dist., Albany, N. Y.

[11] Head Hydr. Engr., TVA, Knoxville, Tenn.

[12] "The Rôle of the Federal Government in the Conservation and Utilization of Water Resources," by James Lawrence Fly, *Pennsylvania Law Review*, Vol. 86, No. 3, January, 1938.

[13] "The Conduct of Business Enterprises by the Federal Government," by David E. Lilienthal and Robert H. Marquis, *Harvard Law Review*, Vol. 54, No. 4 ,February, 1941.

The writer had occasion recently to ask the U. S. Weather Bureau for its definition of a drought, and was much surprised to learn that the best it had to offer, in a hurry, was "any time when there is insufficient moisture to support the continued growth of vegetation." The author lists several years during which critically low runoffs occurred in different areas. In the comprehensive development of any single, large watershed, it will be found that consideration must be given to critically low periods lasting two years, three years, or even longer. This is important with respect to hold-over storage problems from one year to another during such periods. Considering the high degree of regulation possible after 1945 in the Tennessee Valley, indications (in advance of complete runoff records) are that the most severe drought of record has lasted from 1939 into the current year (1942), with the 2-yr period 1930–1931 holding second place, and a drought in 1903–1905 holding third place for long periods. The year 1925 was the most critical single year for storage requirements in this area and is probably controlling, even when compared with the longer periods, for regulation resulting from multi-purpose objectives. However, if operations were for maximum power production, the longer periods would probably be controlling. This is mentioned to call attention to the fact that what may be considered as a critical drought with one objective in view may be defined quite differently for other objectives.

A similar argument might be applied to floods. Regulation which might not be fully effective in a maximum flood would certainly affect the frequency of occurrence of lesser floods as well as their magnitudes.

The important effect of broadening federal water policies with respect to both droughts and floods is to lessen their damages by adequate planning on a large scale, beyond the ability of private capital or private objectives to accomplish. Since federal planning should accomplish as many social benefits as possible, many objectives should be included and the necessary compromises made between them. In the TVA this is done through the extensive use of operating rule curves for all projects, modified from time to time as experience is gained. Herein lies one of the important jobs for the engineer, mentioned by the author.

Any mention of the multi-purpose objectives of federal agencies, in the case of southern projects in particular, should include that of malaria control. The TVA, in cooperation with the U. S. Department of Public Health and local agencies, has worked on this problem from the beginning of its activities. The creation of large bodies of water where none existed previously, with associated problems of proper beach drainage and control of vegetation along the shores, places the responsibility for controlling the breeding of mosquitos directly upon the federal agency that developed the project. It is not a minor problem.

One problem that will offer increasing difficulties in its solution is that of coordinating the work of the various federal agencies engaged upon water control and development, to prevent duplication of effort; and to establish procedures for the exchange of experience data and engineering information, and perhaps even of personnel at a time when it would be most helpful. The centralized control that is so often suggested as suited to a federal Public Works Department is directly opposed to the benefits obtainable under de-centralized agencies working closely with local state agencies, the people and institutions most affected, but with proper federal checks and controls. The engineering profession could be helpful in determining the more efficient administrative theory.

If the second scheme is adopted ultimately, it will give rise to a further problem—how to make the experience and improved techniques of one agency promptly available to another. Publication of technical information is not the entire answer because frequently it is too long delayed; it is not distributed or made available where it might be most helpful either because of cost or not knowing where the need exists. There is too much published now for any one man to review thoroughly along with his active work. Engineers frequently are not interesting and concise writers; and some of the busiest men, whose ideas and experiences might be of the greatest value, cannot or will not give the necessary thought and time required to the preparation of a paper or book along with their required daily work. Perhaps, too, progress and changes are sometimes so rapid that some engineers hesitate to commit themselves in print. There are a few who do not wish to give away their "stock in trade"—a very mistaken idea, professionally.

Perhaps committees in federal engineering groups could be formed to function something like those in the former National Electric Light Association. Annual summaries of progress and experience might be issued on classified subjects, preceded by a comprehensive review of background material. If this procedure proved successful in private practice, it could be made equally so among federal agencies. The National Resources Planning Board might be the means of accomplishing this end.

James S. Sweet,[14] Assoc. M. Am. Soc. C. E.—A few papers have been written in the past describing hydraulic activities of the government; but none attempted to relate influences of social events and natural phenomena to the federal water policies. From this standpoint the paper by Mr. Hoyt occupies a unique place in the literature on the subject. Not only is the description interesting, but it also clarifies sometimes unexplainable tendencies of the national legislation in the hydraulic field. Like other writers on this subject, the author fails to indicate weak spots of the national water policy, or to offer some constructive criticism. Even the restrictions of a federal position would not bar such criticism, if it is not malicious.

Throughout his paper the author mentions various government departments in which hydraulic or hydrologic activities exist. To a layman this fact gives reason to believe that all these activities are, or could be, coordinated. Unfortunately, this fact in itself is a sign of inefficiency, for, under the law, the jurisdiction over these activities is given to the respective departments where they are placed. This makes coordination difficult. Hence, there is much duplication and delay. An attempt toward a coordinated action was made by appointment of the Water Resources Committee of the National Resources Planning Board. However, this committee has no administrative powers; and its voice, although often speaking truth and wisdom, is unheard by those to whom it is directed.

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[14] Regional Hydrologic Engr., U. S. Weather Bureau, Flushing, N. Y.

Placement of all hydrologic activities of the federal government in one bureau would be impracticable under the present circumstances. As an alternative, creation of a hydraulic coordinating joint committee seems quite feasible. This committee should consist of the executives of all branches of the government dealing with hydraulic works and hydrologic data. It should have a revolving chairmanship. It should act not only in the advisory capacity, but should also have executive powers. This would be facilitated by the fact that the head of each agency involved would serve on the committee and be its chairman at one time or another.

This joint committee should fill in gaps of inadequacy, eliminate overlapping activities, and coordinate existing and planned work of all hydrologic agencies, thus promoting efficiency and economy.

J. L. CAMPBELL,[15] M. AM. SOC. C. E.—The federal water policies developed since 1930 constitute one phase of a course of the federal government away from that American way of life which rests upon and is sustained by private initiative and enterprise toward some form of paternalism wherein, eventually, the conception that government is the servant of the people would perish, and the people would become dependents upon and servants of government. That is a momentous issue raised by some of the policies of the new order known as the New Deal.

In the first decade of such departure, beginning in 1932, certain phases of the broad issue indicated came to the fore in attempted regimentation of some of the economic life of the people by certain programs of the National Industrial Recovery Administration and the Agricultural Adjustment Administration—creatures of the new order—wherein it became evident that the people involved would not unanimously and completely voluntarily accept and comply with said programs, and that, consequently, basic objectives of the new order could be brought to full fruition only by dictatorial compulsion of all the people concerned. The clear implication of that situation is that certain ways of life in the new order could be made completely effective and successful only by government by dictatorship. In substance, that issue was brought to the Supreme Court of the United States, and the Court set up a bar against dictatorship; but that does not finally dispose of the general issue. Since that decision, the personnel of the Court has undergone an unprecedented and revolutionary change.

The activities to date of the Tennessee Valley Authority point to elimination or subordination of private initiative and enterprise and substitution of government ownership and operation. Similar implications lie in the Columbia Basin Project—the Grand Coulee extravaganza—and in other projects of the new order.

In general, projects coming to life under the federal water policies have a common characteristic vividly illustrated in Grand Coulee Dam which came to the Congress as an hydroelectric project asking for only $65,000,000—a modest New Deal sum. Then, like the beanstalk in "Jack and the Beanstalk," this project grew rapidly into the "stratosphere" and is now approaching completion as a multiple-use project costing nearly $400,000,000. This project is without the economic justification upon which private initiative and enterprise must rest in expenditure of money. Such projects have helped mightily in doubling the national debt of 1930.

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[15] Chf. Engr., Northwestern Pac. R. R. (*Retired*), Oakland, Calif.

The Grand Coulee project includes proposed reclamation of more than a million acres of desert land lying above the high-water line of the reservoir created by the dam. This requires lifting the water for irrigation several hundred feet by hydroelectric power generated at the dam. This makes Grand Coulee reclamation costly and beyond the resources of the farmers. That land can be so reclaimed and farmed only if the federal government carries for the farmers a large part of the financial load. The resulting cost of food and clothing materials so produced will be above like costs in the Mississippi valley and the United States in general.

Grand Coulee reclamation was put forward and appropriations therefor were made during the time when the federal government was proclaiming an alleged overproduction of farm products, was putting the farmers into strait jackets, and was expending millions of dollars in retirement of farm lands from production for the purpose of reducing agricultural products. The barns and granaries of the country are still overflowing with farm surpluses.

Grand Coulee illustrates extremes to which public appropriations and expenditures run under compulsion of political and other pressure groups in the unusual events of the federal water policies. It is also indicative of what comes out of the whole of the new-order philosophy. Now, more than ever, unnecessary or deferable expenditures should cease during the war.

Those who support continuance of such expenditures allege that projects of the kind indicated are necessary to the war program because of the hydroelectric power which they would provide. Accordingly, immediate authorization of construction of the St. Lawrence Seaway—a five hundred million dollar project—is urged, notwithstanding the fact that several years would elapse before power would flow from it.

If power for the war is lacking, the lack should be eliminated in the shortest way possible. Therein, time rather than cost is most vital. The quickest way is by construction of steam plants at points of power consumption. Generally, first cost of steam power is far less than cost of hydroelectric power. Moreover, in most of the war power centers of the country, there would be little difference in over-all costs of construction and operation as between up-to-date steam plants and hydroelectric plants. Finally, it would be difficult indeed for the enemy to find and destroy a multitude of relatively small steam plants, and comparatively easy to locate and destroy a few great hydroelectric plants.

Is there such lack of patriotism that pet schemes dressed in the guise of war need are delaying and crippling war effort?

A general effect of some new-order policies is discouragement of self-reliance, initiative and enterprise, and encouragement of dependence upon government for social security. That is not an American, or a healthful, trend.

Another regrettable development is the great expansion in number and size of governmental functions and agencies resulting from new-order policies and activities, upon which it is now necessary to superimpose the vast war work organizations of the government. The President suggests that many people in Washington, D. C., not in government service, should move out to give the government elbow room. Liquidation of a large number of new-order agencies would be a better remedy.

Of course, America must move forward; but she should so move in the right way, under the Bill of Rights of the individual in the charter of American liberty. A new order should not be condemned, or swallowed whole, merely because it is new. By its fruits, it should be judged. By that measure, the new order herein under criticism is partly good and partly bad. To be worthy to endure and able to serve sanely and beneficently, this new order, to the extent that it may be afflicted by them, must be purged of faults and dangers, such as are indicated in this critique.

Turning to the engineering and construction work of building the projects fathered by the federal water policies originated since 1930, the picture is a happy one wherein superb engineering design and direction and marvelous construction work have created sound and efficient structures of unprecedented magnitude and highest excellence. The engineering profession and the construction art may look in great complacence upon such achievements; but such structures are not, in themselves, justification for their existence.

Yes, under the federal water policies, there is a horn of plenty of "unusual events."

DAVID J. GUY,[16] ESQ.—Without oversimplification, the author has isolated the germs of a revolution that is taking place in the policy of the United States with respect to water resources. The combination of circumstances cited—one physical, one economic, and one altruistic—is producing (in fact has already produced) a complete reorientation in the water-utilization program. No one can challenge the fact that the decade 1930–1940 was characterized by unusually large and destructive floods. The author's statement that similar climatic and hydrologic conditions have "probably" not been exceeded during the past century is accepted.

However, for a proper evaluation of this condition as one of the combination of circumstances shaping national policy, the reader requires a fuller historical background. The impact of unusual climatic and hydrologic conditions upon national life depends upon the character of national life at the time of their occurrence. To illustrate, science reveals that there is no sound—there are only sound waves—if there is no ear to intercept the sound waves. It is so in large degree with the occurrence of unusual floods. Population density, economic wealth, and encroachment of structures upon the natural flood plains and channels of rivers are determining factors in appraising flood effects and therefore the wisdom or unwisdom of any given flood-control policy.

When levee construction along the Mississippi River began at New Orleans, La., in 1717, small earthen dikes from 4 ft to 6 ft high were all that was necessary to afford protection from floods. At that time this appeared to be a wise flood policy, although future years proved that these same levees created a bottleneck that accentuated flood dangers; and, as other sections along the river above New Orleans built dikes, robbing the river of its flood plain, the race for higher and higher levees became a flood-creating flood-preventing device, robbing Peter to pay Paul.

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[16] Hydr. Engr., Natural Resources Dept., Chamber of Commerce of the United States, Washington, D. C.

The passage of the Swamp-Land Act in 1850 established another "water-regulation policy" and that, from the backsight of today, had many unwise features. The author's paper deals with land and water, and it may also be pointed out that numerous recent land-policy discussions, appearing in print, have cast much unjust criticism at the early land-settlement policies of this country, particularly the Homestead Act. Fairness requires that the early policy makers in the United States, dealing with land and water, be credited with enacting laws and putting into effect policies deemed wisest by them in the light of the problems of their time. Even today there are lawsuits in California because of efforts to prevent river floods that overflow and replenish the ground water supply of the lower plains area.

This subject of national water policy calls for a more extensive plotting of points and circumstances before anyone can indicate a trend line with confidence. An old farmer once said he never had any trouble getting two fence posts in line, but when he had three he ran into difficulty.

Land and water policies should be founded upon scientific and engineering inquiry. Unfortunately, "The impact of an economic crisis," to which the author refers (see "Introduction"), has been more persuasive with the legislative policy makers. It is questionable whether a federal water policy shaped by the ebb and flow of unemployment can be judged engineeringly sound. It is also questionable whether the use of soil and water conservation as a Constitutional peg for economic adjustments in agriculture should be admitted by the engineering profession as a factor in shaping national water policies. This is not to say that the engineer should avoid all consideration of matters beyond those that are technical and immediately "profitable." The danger is that any water policy may be literally pulled to pieces for no other purpose than to make legal and Constitutional pegs on which to hang a variety of new social and political philosophies.

The current water policy is enmeshed in the "shelf of projects" that has been built up, awaiting the "go" sign when jobs are needed after the war ends. Many of these projects are for the purpose of conserving and using water resources. As experience gained in the past decade has shown, economic soundness does not necessarily control the order of development when the objectives sought lie outside the bounds of an integrated water policy.

Emergency relief through make-work projects was made readily acceptable by the generous use of the federal treasury. Appropriations through the various emergency relief acts of the period 1935–1942 exceed $15,000,000,000. Never before has the United States accepted the lure of debt and borrowing as witnessed in the decade 1930–1940, reviewed in the author's paper on water policy.

This decade witnessed generous use of federal loans and grants to states and municipalities for the development of water resources. The long-established policy of the Army Engineer Corps that required local contributions toward the cost of river-improvement structures was seriously threatened by the 1936 Flood Control Act, and was practically eliminated by the 1938 Flood Control Act. The complete elimination of local contributions tends to make flood control a "pork-barrel" proposition. Heretofore, the willingness to share the cost was the best index of a local need of flood control. The principle, rather than the amount, of local contribution is important. It may be, as the author states, that the great losses during the 1930–1940 decade from floods and droughts "made a deeper impression on policy makers, than ever before" (see "Introduction"), but public acceptance was achieved largely because of the federal Santa Claus. Here is one obvious reason "for the rapid transition of a wide popular movement or belief into an active action program" of which the author speaks (see "Introduction"). Here is a kind of inverted "dollar diplomacy." Dollar policy making carried with it certain objectives and created certain obligations. This has long been the situation growing out of various forms of federal aid—to wit, the beneficiary surrenders something, usually a measure of local control.

It is no surprise, therefore, that many new federal agencies were created—some by Congress, some by Executive Order—to exercise the rights and powers yielded by states and communities. Widespread federal planning came into vogue. It now embraces every possible land and water use, and extends from the smallest mountain stream to sea level. The Nation is now in a period of intensive federal water-policy making accompanied by a rapid decline in state and local control.

The extension of federal water policy does not stop at planning. It is being projected into the fields of operation and management, prompted, at times, by political rather than engineering considerations. Again the "Constitutional peg" comes into play. The licensing policy of the Federal Water Power Act established in 1920 has been abandoned for direct federal construction of water-power plants and federal marketing of the energy generated. This latter, a marketing phase, is still in a much muddled state. The political desire for public power has been of equal or greater influence in shaping present water-power policies than any circumstance of the "1930–1940 flood situation" to which the author refers.

"Multiple-use projects" has become a slogan which has political as well as engineering significance. This multiplicity of uses has greatly extended the field of federal participation and has served to divest states of much of their control of water resources. "Multiple use" presumes an ability on the part of planners to make fine economic adjustments in water-use values that, from an engineering approach, can be little more than wishful thinking. Take, for example, the question of allocating cost between flood-control and water-power benefits. Army engineers used to measure allowable flood-control expenditures by capitalizing flood losses. The best that can be said for this practice is that it served to indicate the relative importance of the regions damaged. The need for flood protection, and therefore the amount of expenditures, is determined by the measure and extent of the fear the endangered people feel, due to the recurrence of floods. It is this fear that generates the pressure on the members of Congress, state legislators, and the "city fathers" to appropriate the money for flood control. The methods of control are for the engineers to determine.

With water power, the former practice was to measure its economic value by comparison with the cost of steam power. The current policy seems to be to allocate to power only such a share of the capital investment as will make government power "cheap" in comparison with prevailing prices of private utility companies. This might be an acceptable policy if its object were for no other purpose than to assure the disposal of the government's power by-product at multiple-purpose projects. The trend of the policy, however, is to undertake federal dictation of retail rates for electric service, a long recognized state function.

The "multiple-use" policy has not developed into a true water-conservation policy such as was envisioned 40 years ago by such men as George Maxwell and the late Frederick H. Newell, M. Am. Soc. C. E. The nearest approach to water-conservation policy in recent years is found in the program for upstream development furthered by the Wheeler-Case Act and the Taylor Grazing Act. This departure in water policy, so well presented by the author, stands in marked contrast to the earlier flood-control and navigation-improvement policy which was so rigidly confined to the lower reaches of major streams.

The stimulus to fact finding and hydrologic research, which has come about not only through the demand for more and better data but also through the generous allocation or appropriation of funds to these ends, is a valuable part of the new policy from the engineering standpoint. It is to be hoped that the politicians and planners eventually will be guided more and more by this research than by politics and social philosophy.

The writer concurs with the author's concluding expectation that only future experience can show the soundness or unsoundness of the prevailing water policy and, therefore, the need for modifications. Only by gearing the planning to a practical, and, therefore, immediate future can the needs of today and the probable needs of tomorrow be balanced with the greatest measure of justice and practicability. Such planning, to be of maximum value, requires constant adjustment in the light of social and economic changes and, one may well add, in the light of what the author describes as abnormalities in climatic and hydrologic conditions. In its very best sense politics must be a matter of issues; and national water policies, in regard to both practice and planning, should be based upon economic and engineering facts and should be free of such issues.

ALBERT S. FRY,[17] M. AM. SOC. C. E.—An excellent summarization of the developments and interrelations of federal water policies during the decade between 1930 and 1940 is presented by the author. The paper is particularly noteworthy because it presents a vast amount of material in condensed form of a readable length.

Perhaps somewhat more significance should be placed on the great flood of 1927 in the Mississippi Valley as influencing a change in federal water policies, and the revolution in federal water policies might well be dated from that time rather than 1930. It was the great flood of 1927 that really demonstrated that flood control is a national problem far too big for local interests even in good times. The federal legislation that resulted in 1928, following this flood, recognized flood control for the first time as a national responsibility. Although levees had been built for many years previously along the Mississippi River, this work had been done under the guise of navigation rather than flood control. The studies of the Army engineers, embodied in the Jadwin plan, and later developments showed that the ultimate flood protection of the Mississippi Valley required something more than levees, and a comprehensive reservoir plan was developed for the Ohio River Basin. This plan was a stepping stone to the flood control acts of 1936 and 1938.

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[17] Head Engr., Hydr. Data Div., TVA, Knoxville, Tenn.

The creation of the TVA in 1933 marked not only the inception of a multiple-use project by the federal government, as the author states, but it also created for the first time a complete watershed development project with all phases of that development under one federal agency. As the author has indicated, other federal legislation has provided for cooperation between various governmental agencies, such as the Corps of Engineers and the Department of Agriculture, but only in the TVA has the complete water control of a large drainage basin, including water control both on the land and in the streams, been authorized and established under one agency. The success of this method of handling the water problems of a region has resulted in legislation being introduced into Congress to create other similar authorities.

This completely unified type of organization has facilitated the planning of water-control projects, utilizing the contributions of various sciences such as hydrology, forestry, and agriculture, and appears to have distinct advantages in the development of a complete water-control plan for a region over the more cumbersome procedure of planning by two or more separate governmental agencies, each jealous of its own particular field in the culmination of the final plan. Integral with the TVA type of organization is a recognition of the basic hydrologic information needed for complete water-control planning from all standpoints and maximum utilization of the data by the several sciences involved. Illustrative of this may be mentioned the pioneer work of the TVA in the extension of the network of rainfall stations throughout the Tennessee Valley, particularly those on higher altitudes for which no accurate information as to rainfall amounts and intensities previously existed.

A most beneficial result of recent federal water policies has been the development of closer cooperation between various government agencies concerned with water projects and the recognition of the interrelationships of the interests of these several agencies in specific problems. The TVA, for example, although organized to handle completely the water problems of its region has, from its beginning, recognized the value, and pursued a policy, of close cooperation with existing governmental agencies to avoid duplication of effort. Stream-gaging work for the TVA is done by the U. S. Geological Survey with which there has been a close and most satisfactory cooperation throughout the life of the Authority. This relationship has resulted in an increase in the scope and importance of the work of the Survey in the Tennessee Valley region. Where formerly stream-gaging stations were operated in the Tennessee Valley, and the results published years later in *Water-Supply Papers*, immediate problems of planning and construction of TVA projects have resulted in stream-flow information being obtained from an increased number of stations and kept up to date for the day-to-day needs of the Authority. The making of stream-flow measurements below a large dam on the Tennessee River in the morning to be used in connection with operations in the afternoon is quite different from collecting records to be worked up and published years later.

Another example of a cooperative relationship is that between the Weather Bureau and the TVA with respect to weather forecasting, which is utilized in the operation of the many reservoirs in the Tennessee Valley. This unique arrangement provides the TVA with weather forecasts twice daily or oftener and the TVA in turn forecasts river stages throughout the Tennessee Valley, the information on both weather and rivers being given to the public in the form of a cooperative daily bulletin. It was recognized by Weather Bureau officials that, although the Bureau is required by law to forecast river stages throughout the United States, since the TVA necessarily had to do much more detailed forecasting in connection with its reservoir operations, it would be inefficient for the Weather Bureau to duplicate this work.

W. G. Hoyt,[18] M. Am. Soc. C. E.—It was not, and is not, the intent of the writer to criticize any present water function or policy; the paper is factual and philosophical and not critical. Therefore, he will not attempt especially to point out "weak spots" as suggested by Mr. Sweet. As emphasized in the paper and discussions, water policies, at any one time, have depended in some immeasurable degree on climatic, economic, and social circumstances. When conditions change, policies respond. What may seem valid and constructive criticism currently, often looks ill-considered in retrospect. From the viewpoint of peacetime, Mr. Campbell may be justified in holding that federal water policies developed since 1930 "constitute one phase of a course of the federal government away from that American way of life which rests upon and is sustained by private initiative" and that many projects were extravagant and without economic justification.

From the point of view of peacetime, no one can yet say positively whether the projects were generally worth the cost. It is certain, however, that water planning and development have stood the Nation in good stead during the present emergency. By providing major sources of energy, many so-called extravagant developments have been the very foundation on which this country's enormous industrial expansion for war has been based. Factories were kept in operation during periods of depression, turning out equipment for use on what was often then considered "made work" incident to water, soil, and conservation projects. As a result, when it became necessary to establish army and navy bases in the four corners of the world, mobile equipment was available; many factories were tuned to produce and expand if necessary; and what was more important, many persons trained for peaceful employment in the Public Works Administration, Work Projects Administration, Civilian Conservation Corps, and other organizations, stepped into war activities without losing stride. This fortunate result is not necessarily attributable to the planners; nor does it follow that because peacetime policies proved their worth in wartime they were or are necessarily the best that could be devised. In fact, much evidence seems to indicate that major modifications in water policies are desirable.

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[18] Hydr. Engr. (Prin.), Water Resources Branch, U. S. Geological Survey, Washington, D. C.

The discussions by Messrs. Sargent, Wood, Guy, and Fry have added greatly to the treatment and development of the subject as a whole through inclusion of text and references outlining events and circumstances having to do with the water policies prior to the decade under consideration. The paper and discussions outline many of the events and circumstances by which the nation has arrived at its present water policy, a policy under which billions of dollars have been spent by federal, state, and local interests "without the benefit or direction of a consistent national water policy designed to produce the optimum coordinated development in the interests of all."[19] Now the Nation is at war, and until this war is won, there will be little modification of present policies. After the war is over, if policy makers agree with the National Resources Planning Board that "the present uncoordinated administration of laws and policies * * * cumulatively have not been sufficiently effective in producing properly balanced development of our resources for the people of the Nation, for the people of the regions or even for the people of certain individual basins,"[19] then they may well review some of the pertinent points brought out in the paper and discussions, parts of which are quoted herein.

Future policy makers must decide whether or not the best objectives will be accomplished by following the advisory leadership of the Water Resources Committee of the National Resources Board; or whether a similar committee should be vested with sufficient authority and so organized as to enable it to coordinate and direct future water policies; or whether the creation of a "hydraulic coordinating joint committee," as suggested by Mr. Sweet, "with a revolving chairmanship and executive powers" would be more appropriate. They must decide whether a basin project with all phases of that development under one federal agency, embracing a complete, unified type of organization for the planning of water-control projects and utilizing contributions of various sciences, such as hydrology, forestry, and agriculture, has more advantages than the procedure of planning by separate governmental agencies with separate functions and with Nation-wide scope, even though the planning is guided by a central committee.

In other words, should there be a Nation-wide group of authorities similar to the Tennessee Valley Authority, the diversified activities of which have been so clearly outlined by Messrs. Wood and Fry, or should developments for irrigation, power, flood control, and land-use continue to be administered by separate organizations, federal or state, the policies of which are fixed by law, or a combination of both? Should such groups (as Mr. Guy questions) embrace a federal water policy "shaped by the ebb and flow of unemployment" and use "soil and water conservation as a Constitutional peg for economic adjustment?" Future policy makers should avoid the danger that "any water policy may be literally pulled to pieces for no other purpose than to make legal and Constitutional pegs on which to hang a variety of new social and political philosophies." They should keep in mind that unusual meteorologic and hydrologic events, such as described in the paper, will recur and that in all probability the shaping of future policies will be affected by the occurrence and magnitude of these events. They should determine "whether the complete elimination of local contributions tends to make flood control a pork barrel proposition" or whether the present policy with respect to local contributions for irrigation development is the more equitable. They should determine whether or not the creation of many new federal agencies "to exercise the rights and powers yielded by states and communities" is wise and whether or not a federal water policy should be projected further into the fields of operation and management, "prompted, at times, by political rather than engineering considerations." These and many other pertinent questions raised in the discussions will necessarily have to be considered by those who will guide future policies.

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[19] "National Resources Development—Report for 1943," National Resources Planning Board, Water Resources Section.

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EFFECT OF VARIATION OF ELASTIC CHARACTERISTICS ON STATIC UNKNOWNS

By A. Hrennikoff,[1] Assoc. M. Am. Soc. C. E.

With Discussion by Messrs. A. H. Finlay, Stefan J. Fraenkel, Joseph A. Wise, and A. Hrennikoff.

Synopsis

As is well known, the values of statically indeterminate quantities are influenced by the elastic characteristics of the structure, such as cross-sectional areas of compression and tension members, moments of inertia of flexural members, and moduli of elasticity. The actual magnitudes of these elastic characteristics are likely to differ somewhat from the values assumed in analysis because of inaccuracies and accidents of manufacturing and construction, and these may result in deviation of the actual stress values from the quantities used in the design. A proportional change in an elastic characteristic in different parts of the structure in most cases will not affect the values of static unknowns, but a random variation of cross-sectional area, moment of inertia, or modulus of elasticity from point to point may influence the results considerably, depending on location, magnitude, and sign of the variation.

These discrepancies or variations of the elastic characteristics are not always negligible. Thus, a ¼-in. deficiency in the thickness of a 12-in. reinforced-concrete slab causes a decrease of more than 6% in the moment of inertia of the member. The modulus of elasticity of steel is not likely to differ more than 3% from the usual value of 29,000,000 lb per sq in., but the modulus of concrete is subject to large variations. Thus, in tests of full-sized reinforced-concrete rigid frames, conducted in 1938,[2] the values of the modulus of elasticity of concrete specimens cut from different parts of the same frame varied from 3.7 to 5.9 million pounds per square inch—that is, the extreme values differed from the average of 4.8 by 23%. In conditions of actual practice these variations are likely to be even greater.

Note.—Published in January, 1942, *Proceedings*.

[1] Asst. Prof., Civ. Eng., Dept. of Civ. Eng., Univ. of British Columbia, Vancouver, B. C., Canada.

[2] "An Investigation of Rigid Frame Bridges," by Wilbur M. Wilson, Ralph W. Kluge, and John V. Coombe, *Bulletin No. 308*, Univ. of Illinois Eng. Experiment Station, 1938, Pt. 2.

This paper presents a general method of determining the greatest possible variation in the magnitude of a statically indeterminate quantity caused by most unfavorable distribution of assumed legitimate variations in the elastic characteristics.

NOTATION

The letter symbols used in this paper are defined where they first appear and are assembled for convenience of reference in the Appendix.

MAXIMUM VARIATION OF A STATICALLY INDETERMINATE QUANTITY

Before proceeding with the mathematics of the problem, it is necessary to state definitely what variations of the elastic characteristics should be considered legitimate. It will be assumed that an elastic characteristic may deviate from its theoretical value ϵ anywhere on the structure by an amount not in excess of $i\,\epsilon$, in which i is a small, theoretically infinitesimal, quantity. The actual values of the assumed variations and their signs at different points of the structure will be selected in such a manner as to make the variations of the statically indeterminate quantity in question the greatest. A different statically indeterminate quantity will require a new arrangement of variations of the elastic characteristic, staying within the limits $\pm\, i\,\epsilon$ anywhere on the structure. It will be assumed also that the modulus of elasticity E of flexural members will vary in such a manner as to have the same values in the planes normal to the axis of the member. Thus the member will consist of short rectangular blocks, cut by sections normal to the axis, with E varying from block to block but constant in the same block. This assumption that E is constant in each block is more severe in its effect on the variation of the statically indeterminate quantity than the assumption that it changes across the axis of the member, since a given change in E over the entire block modifies flexural deformability of this particular element of the structure by a greater amount than an equal change in E extending only over a part of the block.

Consider a structure with an arbitrary number of statically indeterminate quantities acted upon by a given set of loads. For definiteness only three static unknowns X_a, X_b, and X_c will be used in the derivation, although the formulas derived will be applicable to any number of unknowns. The equations of continuity from which the unknowns are to be found may be written in the form:

$$\left.\begin{aligned} X_a\,\delta_{aa} + X_b\,\delta_{ab} + X_c\,\delta_{ac} + \delta_{ao} &= 0 \\ X_a\,\delta_{ab} + X_b\,\delta_{bb} + X_c\,\delta_{bc} + \delta_{bo} &= 0 \\ X_a\,\delta_{ac} + X_b\,\delta_{bc} + X_c\,\delta_{cc} + \delta_{co} &= 0 \end{aligned}\right\} \dots\dots\dots\dots\dots (1)$$

In Eqs. 1 the familiar quantities δ refer to the basic statically determinate structure and represent the movements of the points of application of the static unknowns in the directions of the unknowns. The first subscript always indicates the point that moves, and the second subscript, if it is o, indicates that the movement is caused by the actual loads, and, if it is a, b, or c, signifies that the movement is caused respectively by any of the forces $X_a = 1$, $X_b = 1$, or $X_c = 1$. Thus, δ_{ab}, for example, represents the movement of the point of application of X_a in the direction of X_a, caused in the statically determinate structure by the force $X_b = 1$; also δ_{co} represents the movement of the point of application of X_c in the direction of X_c caused in the statically determinate structure by the actual loading. By the law of reciprocity $\delta_{ab} = \delta_{ba}$, with two similar equalities for the other values of δ.

By solving Eqs. 1,

$$X_a = -\frac{D_{1a}\,\delta_{ao} + D_{2a}\,\delta_{bo} + D_{3a}\,\delta_{co}}{D} \qquad (2)$$

with two similar expressions for X_b and X_c.

In Eq. 2, D is the determinant composed of the coefficients of the three unknowns in Eqs. 1; and D_{1a}, D_{2a}, and D_{3a} are the corresponding minor determinants composed of the coefficients of the unknowns other than X_a—that is:

$$D = \begin{vmatrix} \delta_{aa} & \delta_{ab} & \delta_{ac} \\ \delta_{ab} & \delta_{bb} & \delta_{bc} \\ \delta_{ac} & \delta_{bc} & \delta_{cc} \end{vmatrix} \qquad (3)$$

and

$$\left.\begin{aligned} D_{1a} &= \begin{vmatrix} \delta_{bb} & \delta_{bc} \\ \delta_{bc} & \delta_{cc} \end{vmatrix} \\ D_{2a} &= \begin{vmatrix} \delta_{bc} & \delta_{cc} \\ \delta_{ab} & \delta_{ac} \end{vmatrix} \\ D_{3a} &= \begin{vmatrix} \delta_{ab} & \delta_{ac} \\ \delta_{bb} & \delta_{bc} \end{vmatrix} \end{aligned}\right\} \qquad (4)$$

The minor determinants D_{1b}, D_{2b}, etc., entering the expressions for X_b and X_c, are formed in a manner similar to Eq. 4.

It will be recalled that the various expressions δ represent integrals involving the elastic characteristics. Thus, in structures consisting of flexural members only,

$$\left.\begin{aligned} \delta_{ab} &= \int \frac{m_a\,m_b}{E\,I}\,ds \\ \delta_{ao} &= \int \frac{M_o\,m_a}{E\,I}\,ds \end{aligned}\right\} \text{(etc.)} \qquad (5a)$$

In Eq. 5*a*, I is the moment of inertia; and M_o, m_a, and m_b signify the bending moments developed in the basic statically determinate structure: By the given loading; by the force $X_a = 1$; and by the force $X_b = 1$, respectively. Likewise, in structures made up of the direct-stress members only:

$$\left.\begin{aligned} \delta_{ao} &= \sum \frac{N_o\,n_a}{E\,A}\,L \\ \delta_{ab} &= \sum \frac{n_a\,n_b}{E\,A}\,L \end{aligned}\right\} \text{(etc.)} \qquad (5b)$$

In Eq. 5*b*, L is the length of a structural member; and N and n are the direct stresses in the members corresponding to the same loading conditions as the moments M and m in Eq. 5*a*.

Suppose now that one of the elastic characteristics, such as the modulus of elasticity E, increases by an infinitesimal increment dE in a block of the length ds in any part of the structure. This causes differential increases in δ-values —$d\delta_{ao}$, $d\delta_{ab}$, etc. Since Eqs. 1 still hold for the structure, with the elastic properties modified in this manner, the values of static unknowns must readjust themselves to the new δ-values, and their corresponding increments dX_a, dX_b, etc., may be found by taking the total differentials of Eqs. 1 and solving the resultant equations for these increments. By this differentiation:

$$\delta_{aa}\, dX_a + \delta_{ab}\, dX_b + \delta_{ac}\, dX_c + (X_a\, d\delta_{aa} + X_b\, d\delta_{ab} + X_c\, d\delta_{ac} + d\delta_{ao}) = 0 \quad (6)$$

with two other similar equations.

These three equations can be solved for the unknowns dX_a, dX_b, and dX_c since all values of X and δ are known from Eqs. 2 and 5, and values of $d\delta$ are fixed by the assumed variation in the elastic characteristic.

Next, introduce the new functions:

$$\left.\begin{aligned} \Delta_a &= X_a\, \delta_{aa} + X_b\, \delta_{ab} + X_c\, \delta_{ac} + \delta_{ao} \\ \Delta_b &= X_a\, \delta_{ab} + X_b\, \delta_{bb} + X_c\, \delta_{bc} + \delta_{bo} \\ \Delta_c &= X_a\, \delta_{ac} + X_b\, \delta_{bc} + X_c\, \delta_{cc} + \delta_{co} \end{aligned}\right\} \quad (7)$$

in which the X-quantities are assumed constant and equal to the values determined from Eq. 2; and the δ-quantities, defined by Eqs. 5, are thought of as functions of the elastic characteristics E and I, considered as variables. The new expressions Δ will be recognized as the left-hand side parts of the basic Eqs. 1, and, although they vanish for the original δ-values, they are generally different from zero for the modified values of δ corresponding to new and variable values of E and I.

By differentiation of Eqs. 7

$$d\Delta_a = X_a\, d\delta_{aa} + X_b\, d\delta_{ab} + X_c\, d\delta_{ac} + d\delta_{ao} \quad (8)$$

with two similar expressions for $d\Delta_b$ and $d\Delta_c$. Substitution of these expressions in Eqs. 6 converts the latter into the following form:

$$\left.\begin{aligned} \delta_{aa}\, dX_a + \delta_{ab}\, dX_b + \delta_{ac}\, dX_c + d\Delta_a &= 0 \\ \delta_{ab}\, dX_a + \delta_{bb}\, dX_b + \delta_{bc}\, dX_c + d\Delta_b &= 0 \\ \delta_{ac}\, dX_a + \delta_{bc}\, dX_b + \delta_{cc}\, dX_c + d\Delta_c &= 0 \end{aligned}\right\} \quad (9)$$

A marked similarity may be noted between these equations, serving for the determination of the variations dX_a, dX_b, and dX_c, and the basic Eqs. 1, serving for the determination of the static unknowns X_a, X_b, and X_c. The expressions for dX_a, etc., may be written by analogy with Eq. 2:

$$dX_a = -\frac{D_{1a}\, d\Delta_a + D_{2a}\, d\Delta_b + D_{3a}\, d\Delta_c}{D} \quad (10)$$

with similar expressions for dX_b and dX_c. The determinants D, D_{1a}, etc., have the same significance in these expressions as in Eq. 2.

Eq. 10 determines the value of a small variation in the static unknown X_a produced by an assumed variation in an elastic characteristic resulting in corresponding variations in the values of δ and Δ. When the modulus of elasticity E receives an increment dE anywhere along an infinitesimal length ds of one of the members comprising the structure, the changes caused in different values of δ may be exemplified by the following expression for one of them:

$$d\delta_{aa} = d\left(\frac{m^2_a}{E\,I}\,ds\right) = -\frac{m^2_a\,ds}{I}\,\frac{dE}{E^2} \qquad \text{(11)}$$

If $\frac{dE}{E} = i'$, a small, theoretically infinitesimal, quantity, the following equation results:

$$d\delta_{aa} = -\,i'\,\frac{m^2_a\,ds}{E\,I} \qquad \text{(12}a\text{)}$$

In Eq. 12*a* the variable quantities m_a, E, and I refer to the element of structure whose modulus of elasticity E has undergone the change.

If the variation of E occurs over a finite part, or even over the entire structure, the expression for $d\delta_{aa}$ becomes:

$$d\delta_{aa} = \int -\,i'\,\frac{m^2_a\,ds}{E\,I} \qquad \text{(12}b\text{)}$$

in which the integration is extended over the appropriate lengths of the members of the structure. Variations of other values of δ have similar expressions. The term i' should be considered at the present stage as a quantity variable along the structure but, naturally, the same in all values of δ for the same length element ds.

Substituting Eq. 12*b* in Eq. 8, and combining several integrals into one, the following expression results for $d\Delta_a$:

$$d\Delta_a = \int -\,i'\left(X_a\frac{m^2_a}{E\,I} + X_b\frac{m_a\,m_b}{E\,I} + X_c\frac{m_a\,m_c}{E\,I} + \frac{M_o\,m_a}{E\,I}\right)ds \qquad \text{(13)}$$

Two similar expressions follow for $d\Delta_b$ and $d\Delta_c$.

It should be stated that if the variation in E has occurred over the entire structure, and if i' is constant, the expressions for $d\Delta_a$, $d\Delta_b$, and $d\Delta_c$ all vanish since i' appears outside of the integral signs, and the integrals themselves become the left-hand parts of the basic Eqs. 1, slightly rearranged. On the other hand, with the value of i' variable, the expressions (Eq. 13) generally do not vanish.

Substituting the expressions for $d\Delta_a$, $d\Delta_b$, and $d\Delta_c$ in Eq. 10 and combining the integrals once more, the expression for dX_a becomes:

$$\begin{aligned} dX_a = \frac{1}{D}\int i'\Bigg[\, & D_{1a}\left(X_a\frac{m^2_a}{E\,I} + X_b\frac{m_a\,m_b}{E\,I} + X_c\frac{m_a\,m_c}{E\,I} + \frac{M_o\,m_a}{E\,I}\right) \\ & + D_{2a}\left(X_a\frac{m_a\,m_b}{E\,I} + X_b\frac{m^2_b}{E\,I} + X_c\frac{m_b\,m_c}{E\,I} + \frac{M_o\,m_b}{E\,I}\right) \\ & + D_{3a}\left(X_a\frac{m_a\,m_c}{E\,I} + X_b\frac{m_b\,m_c}{E\,I} + X_c\frac{m^2_c}{E\,I} + \frac{M_o\,m_c}{E\,I}\right)\Bigg] ds \qquad \text{(14)} \end{aligned}$$

The variable term i' in this expression has been moved outside of the brackets because it has the same value in all three quantities in parentheses, at the same elements ds.

Eq. 14 can be used to evaluate the variation dX_a provided that the function i' expressing the law of variation of dE along the structure is known or has been assumed. The problem, however, is to determine the greatest possible value of dX_a.

A little thought will serve to show that, if i' is taken as a constant quantity over the entire structure, it appears as a factor outside of the integral sign, and the integral itself then vanishes, since it represents a sum of three integrals, each of which vanishes separately. This deduction is evident, of course, from general principles, because a constant i' signifies a change in E in a constant ratio over the entire structure, and such change, as has been explained, does not generally influence the values of the static unknowns. This vanishing of the integral in Eq. 14 for a constant value of i' means, of course, that the values within the brackets must be positive for some parts of the structure and negative for others. Therefore, a comparatively large variation dX_a will correspond to an assumption of a positive i' in the parts of the structure in which the values within the brackets in Eq. 14 are positive, and a negative i' in the parts in which these brackets are negative. On this assumption, the integral in Eq. 14 will represent an infinite sum of infinitesimal terms each of which is positive. Furthermore, since each of these infinitesimal terms is proportional to i', the greatest possible variation dX_a will evidently correspond to an assumption of numerically greatest value of i' (namely i), occurring over the entire structure and being positive at the points with positive values within the brackets, and negative at the negative values within the brackets. Therefore,

$$
\begin{aligned}
dX_a\,(\max) = \frac{i}{D}\int \Bigg| \Bigg[& D_{1a}\left(X_a \frac{m^2_a}{EI} + X_b \frac{m_a m_b}{EI} + X_c \frac{m_a m_c}{EI} + \frac{M_o m_a}{EI}\right) \\
& + D_{2a}\left(X_a \frac{m_a m_b}{EI} + X_b \frac{m^2_b}{EI} + X_c \frac{m_b m_c}{EI} + \frac{M_o m_b}{EI}\right) \\
& + D_{3a}\left(X_a \frac{m_a m_c}{EI} + X_b \frac{m_b m_c}{EI} + X_c \frac{m^2_c}{EI} + \frac{M_o m_c}{EI}\right) \Bigg] \Bigg|\, ds \quad \ldots\ldots (15)
\end{aligned}
$$

in which the sign | | signifies the numerical value of the expression within the brackets.

Evaluation of Eq. 15 involves a determination of the zero points of the function within the brackets by equating it to zero and by solving the resultant equation in s, exactly or approximately. The integration then is performed separately in the intervals between the terminal points and these zero points, and the numerical values of the positive and negative parts of the integral are added. The function within the brackets may also be plotted against the abscissas s, and the value of the integral may be obtained by measuring the area between the curve and the axis of s with a planimeter. In this operation the areas both above and below the axis must be taken as positive.

It will be clear from examination of Eq. 15 that the function under the integral represents a first-degree polynomial with constant coefficients with respect to the terms $\frac{M_o m_a}{EI}$, $\frac{m^2_a}{EI}$, etc. Therefore, the integration in Eq. 15 is usually no more complicated than the integrations required for a determination of various δ-values. If the axis of the structure is of a complicated shape and if the loading is complicated, or if the moment of inertia is variable, an exact integration of Eq. 15 is impossible and that formula must be replaced by a summation, a procedure similar to the usual method of determining the values of δ in a statically indeterminate analysis.

After Eq. 15 has been evaluated it is useful to state the required variation in the form

$$dX_a\,(\max) = K\,i\,X_a \ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

and to calculate the coefficient K representing the maximum percentage by which the value of X_a may change if the elastic characteristic is to vary by 1%.

The method of determining the most unfavorable effect of a variation of the moment of inertia I is similar to the foregoing, except that the symbols i' and i would in this case signify the ratio $\frac{dI}{I}$ instead of $\frac{dE}{E}$.

In the analysis of statically indeterminate reinforced-concrete structures it is customary to make a constant percentage allowance for steel reinforcement in the moments of inertia and to use the moments of inertia of uncracked sections. Taking $I = 1.1\,\frac{b\,h^3}{12}$ and differentiating this expression

$$\frac{dI}{I} = \frac{db}{b} + 3\,\frac{dh}{h} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(17)$$

It is reasonable to assume that the ratios $\frac{db}{b}$ and $\frac{dh}{h}$ can take any values up to the limits i_b and i_h anywhere on the structure, or in symbols

$$\left|\frac{db\,(\max)}{b}\right| = i_b; \quad \text{and} \quad \left|\frac{dh\,(\max)}{h}\right| = i_h \ldots\ldots(18a)$$

and

$$\left|\frac{dI\,(\max)}{I}\right| = i_b + 3\,i_h \ldots\ldots\ldots\ldots\ldots\ldots(18b)$$

Eq. 18*b* must be substituted for i in Eq. 15 to evaluate the maximum effect of variation in I on the value of the statically unknown quantity X_a. In barrel arches the moment of inertia may vary only by virtue of a variation in the depth h, which requires that $i_b = 0$.

It is also possible to assume that

$$|db\,(\max)| = C_b \quad \text{and} \quad |dh\,(\max)| = C_h \ldots\ldots\ldots(19)$$

in which C_b and C_h are constant numbers. In that case the ratio i' in Eq. 14 becomes variable along the structure and remains under the integral sign.

It is needless to state that the method described herein is equally applicable to structures composed of tension and compression members. In that case integrals in Eqs. 12 to 15 must be replaced by the finite sums of the terms involving the functions $\frac{N_o\, n_a}{E\,A}\,L$, $\frac{n^2_a}{E\,A}\,L$, etc.

The greatest effect on X_a of the simultaneous most unfavorable variations in E and I for the same condition of loading may be found by direct addition of the two separate effects of changes in elastic characteristics. This can be demonstrated easily by reference to the foregoing equations. By the rules of differentiation the total differential of any δ when both E and I vary is equal to the sum of the two $d\delta$'s corresponding to separate variations of E and I. Then it follows from Eq. 8 that the quantity $d\Delta$, with any subscript for the condition when E and I both vary, equals the sum of the functions $d\Delta$ with the same subscript found for the conditions of separate variations in E and I. Finally, the linear form with respect to various $d\Delta$'s of Eq. 10 for the total variation dX_a proves the required theorem of straight addition of the two separate variations dX_a.

Eq. 15 for the greatest value of dX_a is perfectly general with regard to the number of static unknowns in the structure investigated, with the exception of the structure with only one unknown, since in that case symbols for minor determinants D_{1a}, etc., become meaningless.

For a structure only once statically indeterminate, $X_a = -\frac{\delta_{ao}}{\delta_{aa}}$, which by differentiation gives

$$dX_a = -\frac{1}{\delta_{aa}}\left(d\delta_{ao} - \frac{\delta_{ao}}{\delta_{aa}}\,d\delta_{aa}\right) \ldots\ldots\ldots\ldots\ldots (20a)$$

and then, following the same logic,

$$dX_a\,(\max) = \frac{i}{\delta_{aa}}\int \left|\left(\frac{M_o\, m_a}{E\,I} + X_a\,\frac{m^2_a}{E\,I}\right)\right| ds \ldots\ldots\ldots (20b)$$

In a structure with two static unknowns, substitution of the values of determinants in Eq. 15 results in a simplified expression for the maximum variation:

$$dX_a\,(\max) = \frac{i}{\delta_{aa}\,\delta_{bb} - \delta^2_{ab}}\int \left|\,\delta_{bb}\left(\frac{M_o\, m_a}{E\,I} + X_a\,\frac{m^2_a}{E\,I} + X_b\,\frac{m_a\, m_b}{E\,I}\right)\right.$$
$$\left. - \,\delta_{ab}\left(\frac{M_o\, m_b}{E\,I} + X_a\,\frac{m_a\, m_b}{E\,I} + X_b\,\frac{m^2_b}{E\,I}\right)\right| ds \ldots\ldots\ldots\ldots (21)$$

MAXIMUM VARIATION OF STRESS QUANTITIES DEPENDENT ON STATIC UNKNOWNS

The greatest variations of stress quantities other than those chosen as static unknowns may be determined by two methods. In the first method different quantities are selected as static unknowns, including the stress quantities whose maximum variations are required, and then the problem is solved in the usual manner by the formulas presented herein.

In the second method, which is applicable only in cases when the variations of the original static unknowns have already been found for the same load condition, the variation of the new stress quantity is determined easily by differentiating the corresponding equation of statics. This may be best illustrated in an example.

Fig. 1 represents a fixed-ended rigid frame loaded on the horizontal part with a uniform load of intensity w. The structure is three times statically indeterminate. The bending moments at points A, B, and C, designated by X_a, X_b, and X_c, respectively, have been chosen as the unknowns, and their variations have been determined by Eq. 15. It is required to determine the greatest variation in M_E, the bending moment at the point E, the center of the horizontal part. By statics:

$$M_E = \frac{X_b + X_c}{2} + \frac{w L^2}{8} \qquad (22)$$

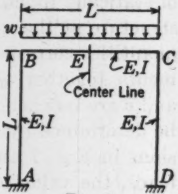


FIG. 1

Differentiating Eq. 22 and remembering that w and L are constant,

$$dM_E = \tfrac{1}{2}(dX_b + dX_c) \qquad (23a)$$

The three differentials in Eq. 23a may be considered as variations of the corresponding stress quantities. Since variations dX_b and dX_c are interdependent, it is incorrect to assign to them simultaneously the greatest values determined by Eq. 15. On the other hand, attributing to them the general integral expressions by Eq. 14 (in which the signs of the variations dE or dI in the variable terms i' at different points of the structure have not yet been specified) is quite legitimate. As a result of this substitution the following expression is obtained for the required greatest variation in M_E:

$$dM_E\ (\text{max}) = \frac{i}{D}\int \left| \tfrac{1}{2}\,[^b] + \tfrac{1}{2}\,[^c] \right| ds \qquad (23b)$$

in which the symbols $[^b]$ and $[^c]$ represent the brackets following the integral signs in the expressions for dX_b and dX_c.

MAXIMUM VARIATIONS FOR DIFFERENT LOAD CONDITIONS

Calculation of the maximum variation of a statically unknown quantity for a new load condition requires considerably less labor than a similar calculation for the first load, since the values of δ with subscripts a, b, and c, and the values of the determinants, are the same as in the first calculation. The quantities δ_{ao}, δ_{bo}, δ_{co} and the values of static unknowns X_a, X_b, and X_c are different, however.

The greatest variation dX_a, corresponding to two sets of loads acting simultaneously, may be found by superposing the results caused by separate actions of the two sets. However, it is not the numerical values of dX_a (max) that are to be superimposed but the expressions within the brackets in Eq. 15. The proof of this proposition may be derived by close inspection of the foregoing formulas. First, it should be realized that the expressions for the values of the moments M_o in the basic statically determinate structure, when both sets of loading are acting, may be obtained by superposition or algebraic addition of the values corresponding to separate sets. With this fact in mind it is easy to see that the quantities δ_{ao}, δ_{bo}, and δ_{co} (see Eq. 5*a*) also may be found by superposition, as concerns both the algebraic expressions under the integral signs and the numerical values obtained after the integration of these expressions. It is a well-known fact, evident also from Eq. 2, that the values of statically indeterminate quantities when two sets of loads are acting on the structure at the same time are equal to algebraic sums of the values of these quantities corresponding to separate actions of the loads. The last two statements, together with the fact that all δ and $d\delta$ quantities with subscripts a, b, and c are independent of the loads, make clear that Δ and $d\Delta$ values may also be determined by superposition, as may be seen from the form of their expressions in Eqs. 7 and 8. Since various determinants are also the same for all loads, the value of dX_a in Eq. 10 may likewise be found by superposition or (which is the same) the expression within the brackets of Eq. 14 for the two sets of loads presents an algebraic sum of the corresponding bracket expressions for the separate loads, which proves the required proposition.

However, the numerical value of dX_a (max) for the two sets of loads is different from the sum of numerical values corresponding to separate loads, because the zero points of the functions under the integral signs in the three cases concerned are generally different.

TWO-LEGGED FRAME OF CONSTANT MOMENT OF INERTIA

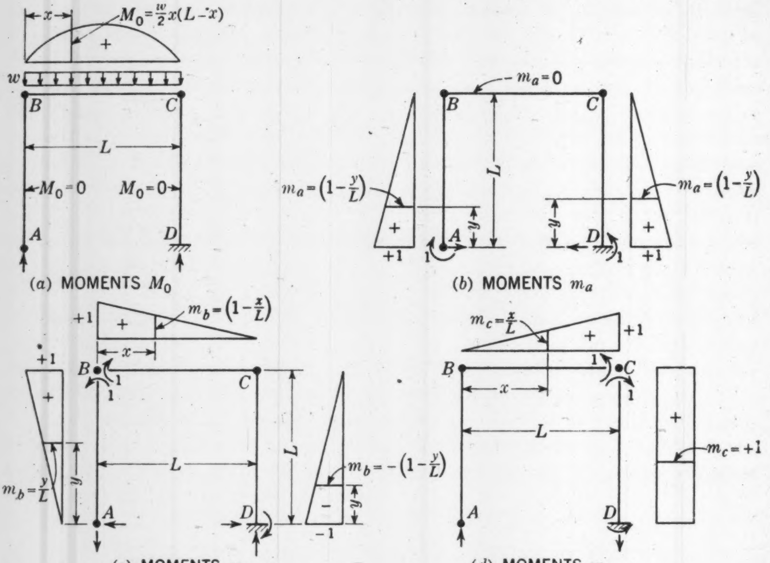
The fixed-ended frame shown in Fig. 1, with the height equal to the span and possessing constant moment of inertia, will be investigated for the greatest possible variations in the bending moments at the top end of one of the columns B and at the center of the beam part E, caused by variation in the value of the modulus of elasticity.

By introducing three hinges at points A, B, and C (Fig. 2(*a*)), the structure is made statically determinate and the corresponding determinate bending moments M_o are as shown. An unsymmetrical choice of static unknowns X_a, X_b, and X_c, representing the bending moments at the points A, B, and C, respectively, complicates the analysis but provides a means for checking the results, since the expressions for similar quantities at the points B and C will be obtained from equations in a dissimilar manner.

Moments m_a, m_b, and m_c, created in the statically determinate structure by unit moments acting at points A, B, and C, respectively, are shown in Figs. 2(*b*), 2(*c*), and 2(*d*). The sign convention for these moments, as well as for the static unknowns X_a, X_b, and X_c, is such that positive moments produce tension on the inside of the frame.

The δ-quantities are now expressed in the integral form and then evaluated. Both kinds of expression will be necessary in further calculation. The integral expressions of δ are composed of three parts, one for each leg of the frame and one for the beam part. In several δ-values, however, some of these integrals vanish because some of the moments M_o, m_a, m_b, or m_c are zero. Thus,

$$\left.\begin{array}{lllll}
\delta_{ao} = \int \frac{M_o\, m_a}{E\,I}\,ds = 0 & & & & \\
 & \overbrace{\hphantom{\int_0^L \frac{L}{E\,I}\left(1 - \frac{2\,y}{L}\right)}}^{\text{left leg}} & \overbrace{\hphantom{\int_0^L \frac{L}{E\,I}\left(1 - \frac{2\,x}{L}\right)}}^{\text{beam}} & \overbrace{\hphantom{\int_0^L \frac{L}{E\,I}\left(1 - \frac{2\,y}{L}\right)}}^{\text{right leg}} & \\
\delta_{bo} = \int \frac{M_o\, m_b}{E\,I}\,ds = & 0 & + \int_0^L \frac{w\,L^3}{2\,E\,I}\left(\frac{x}{L} - \frac{2\,x^2}{L^2} + \frac{x^3}{L^3}\right) d\left(\frac{x}{L}\right) + & 0 & = \frac{wL^3}{24\,E\,I} \\
\delta_{co} = & 0 & + \int_0^L \frac{w\,L^3}{2\,E\,I}\left(\frac{x^2}{L^2} - \frac{x^3}{L^3}\right) d\left(\frac{x}{L}\right) \quad + & 0 & = \frac{w\,L^3}{24\,E\,I} \\
\delta_{aa} = \int \frac{m_a\, m_a}{E\,I}\,ds = & \int_0^L \frac{L}{E\,I}\left(1 - \frac{2\,y}{L} + \frac{y^2}{L^2}\right) d\left(\frac{y}{L}\right) + & 0 & + \int_0^L \frac{L}{E\,I}\left(1 - \frac{2\,y}{L} + \frac{y^2}{L^2}\right) d\left(\frac{y}{L}\right) & = \frac{2\,L}{3\,E\,I} \\
\delta_{ab} = & \int_0^L \frac{L}{E\,I}\left(\frac{y}{L} - \frac{y^2}{L^2}\right) d\left(\frac{y}{L}\right) \quad + & 0 & - \int_0^L \frac{L}{E\,I}\left(1 - \frac{2\,y}{L} + \frac{y^2}{L^2}\right) d\left(\frac{y}{L}\right) & = -\frac{L}{6\,E\,I} \\
\delta_{ac} = & 0 \quad + & 0 & + \int_0^L \frac{L}{E\,I}\left(1 - \frac{y}{L}\right) d\left(\frac{y}{L}\right) & = \frac{L}{2\,E\,I} \\
\delta_{bb} = & \int_0^L \frac{L}{E\,I}\frac{y^2}{L^2}\, d\left(\frac{y}{L}\right) & + \int_0^L \frac{L}{E\,I}\left(1 - \frac{2\,x}{L} + \frac{x^2}{L^2}\right) d\left(\frac{x}{L}\right) & + \int_0^L \frac{L}{E\,I}\left(1 - \frac{2\,y}{L} + \frac{y^2}{L^2}\right) d\left(\frac{y}{L}\right) & = \frac{L}{E\,I} \\
\delta_{bc} = & 0 & + \int_0^L \frac{L}{E\,I}\left(\frac{x}{L} - \frac{x^2}{L^2}\right) d\left(\frac{x}{L}\right) & + \int_0^L \frac{L}{E\,I}\left(-1 + \frac{y}{L}\right) d\left(\frac{y}{L}\right) & = -\frac{L}{3\,E\,I} \\
\delta_{cc} = & 0 & + \int_0^L \frac{L}{E\,I}\left(\frac{x^2}{L^2}\right) d\left(\frac{x}{L}\right) & + \int_0^L \frac{L}{E\,I}\, d\left(\frac{y}{L}\right) & = \frac{4\,L}{3\,E\,I}
\end{array}\right\} \quad (24)$$


x
$M_0=\frac{w}{2}x(L-x)$
+
w
B
C
L
$M_0=0$
$M_0=0$
A
D
$m_a=0$
B
C
L
$m_a=\left(1-\frac{y}{L}\right)$
$m_a=\left(1-\frac{y}{L}\right)$
+
+
y
y
A
D
1
1
+1
+1
+1
$m_b=\left(1-\frac{x}{L}\right)$
+
x
+1
B
1
1
C
+
L
L
$m_b=-\left(1-\frac{y}{L}\right)$
$m_b=\frac{y}{L}$
y
y
A
D
−
−1
$m_c=\frac{x}{L}$
+
+1
B
1
C
1
x
+
L
$m_c=+1$
A
D

(a) MOMENTS M_0 (b) MOMENTS m_a

(c) MOMENTS m_b (d) MOMENTS m_c

FIG. 2.—MOMENT DIAGRAMS; SYMMETRICAL BENT, WITH CONSTANT MOMENT OF INERTIA

Having thus found all δ-values, it is now possible to set up the equations to determine the static unknowns and to compute the determinants required for the evaluation of dX_b (max). Eqs. 1 apply to this case as follows:

$$\left.\begin{aligned} \frac{2\,L}{3\,E\,I}X_a - \frac{L}{6\,E\,I}X_b + \frac{L}{2\,E\,I}X_c &= 0 \\ -\frac{L}{6\,E\,I}X_a + \frac{L}{E\,I}X_b - \frac{L}{3\,E\,I}X_c + \frac{w\,L^3}{24\,E\,I} &= 0 \\ \frac{L}{2\,E\,I}X_a - \frac{L}{3\,E\,I}X_b + \frac{4\,L}{3\,E\,I}X_c + \frac{w\,L^3}{24\,E\,I} &= 0 \end{aligned}\right\} \quad \text{.........}(25)$$

The value of the minor determinant corresponding to X_b in the first equation is

$$D_{1b} = \left(-\frac{L}{3\,E\,I}\right)\left(\frac{L}{2\,E\,I}\right) - \left(-\frac{L}{6\,E\,I}\right)\left(\frac{4\,L}{3\,E\,I}\right) = \frac{1}{18}\left(\frac{L}{E\,I}\right)^2 \text{..}(26a)$$

Similarly,

$$D_{2b} = \frac{23}{36}\left(\frac{L}{E\,I}\right)^2 \quad \text{and} \quad D_{3b} = \frac{5}{36}\left(\frac{L}{E\,I}\right)^2 \text{.........}(26b)$$

The major determinant, made up of all the coefficients of the unknowns, is found to be

$$D = \frac{7}{12}\left(\frac{L}{E\,I}\right)^3 \text{.......................}(26c)$$

On solving the equations the values of the static unknowns come out:

$$\left.\begin{aligned} X_a &= \frac{1}{36}\,w\,L^2 \\ X_b &= -\frac{1}{18}\,w\,L^2 \\ X_c &= -\frac{1}{18}\,w\,L^2 \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27)$$

the two latter values being equal as might be expected from symmetry.

The next step is to set up the integral expressions for the three functions Δ by Eq. 7:

$$\begin{aligned} \Delta_a = {} & \frac{w\,L^3}{36}\Bigg[\overbrace{\int_0^L \frac{1}{E\,I}\left(1-\frac{2\,y}{L}+\frac{y^2}{L^2}\right)d\left(\frac{y}{L}\right)}^{\text{left leg}} \overset{\text{beam}}{+0+} \overbrace{\int_0^L \frac{1}{E\,I}\left(1-\frac{2\,y}{L}+\frac{y^2}{L^2}\right)d\left(\frac{y}{L}\right)}^{\text{right leg}}\Bigg] \\ & -\frac{w\,L^3}{18}\Bigg[\int_0^L \frac{1}{E\,I}\left(\frac{y}{L}-\frac{y^2}{L^2}\right)d\left(\frac{y}{L}\right) +0- \int_0^L \frac{1}{E\,I}\left(1-\frac{2\,y}{L}+\frac{y^2}{L^2}\right)d\left(\frac{y}{L}\right)\Bigg] \\ & -\frac{w\,L^3}{18}\Bigg[\qquad 0 \qquad +0+ \int_0^L \frac{1}{E\,I}\left(1-\frac{y}{L}\right)d\left(\frac{y}{L}\right)\Bigg] \\ & + \qquad 0 \qquad +0+ \qquad 0 \\ = {} & \frac{w\,L^3}{36}\Bigg[\int_0^L \frac{1}{E\,I}\left(1-\frac{4\,y}{L}+\frac{3\,y^2}{L^2}\right)d\left(\frac{y}{L}\right)+0+\int_0^L \frac{1}{E\,I}\left(1-\frac{4\,y}{L}+\frac{3\,y^2}{L^2}\right)d\left(\frac{y}{L}\right)\Bigg] \quad (28) \end{aligned}$$

Within all brackets in Eq. 28 the first integrals refer to the left leg of the frame, the second (which in case of Δ_a are zero) to the beam part, and the third to the right leg. In arriving at the final expression for Δ_a it is the integrals referring to the same part of the frame that are combined, whereas the integrals referring to different parts remain separate. Similarly:

$$\left.\begin{aligned} \Delta_b = {} & \frac{w\,L^3}{36}\Bigg[\overbrace{\int_0^L \frac{1}{E\,I}\left(\frac{y}{L}-\frac{3\,y^2}{L^2}\right)d\left(\frac{y}{L}\right)}^{\text{left leg}} - \overbrace{\int_0^L \frac{1}{E\,I}\left(2-\frac{20\,x}{L}+\frac{36\,x^2}{L^2}-\frac{18\,x^3}{L^3}\right)d\left(\frac{x}{L}\right)}^{\text{beam}} \\ & \qquad - \overbrace{\int_0^L \frac{1}{E\,I}\left(1-\frac{4\,y}{L}+\frac{3\,y^2}{L^2}\right)d\left(\frac{y}{L}\right)}^{\text{right leg}}\Bigg] \\ \Delta_c = {} & \frac{w\,L^3}{36}\Bigg[\overbrace{\qquad\qquad 0 \qquad\qquad}^{\text{left leg}} - \overbrace{\int_0^L \frac{1}{E\,I}\left(\frac{2\,x}{L}-\frac{18\,x^2}{L^2}+\frac{18\,x^3}{L^3}\right)d\left(\frac{x}{L}\right)}^{\text{beam}} \\ & \qquad - \overbrace{\int_0^L \frac{1}{E\,I}\left(-1+\frac{3\,y}{L}\right)d\left(\frac{y}{L}\right)}^{\text{right leg}}\Bigg] \end{aligned}\right\}(29)$$

These expressions for various values of Δ have been checked by evaluating the integrals on the assumption that $E\,I$ is constant and by finding that all Δ's then vanish.

Moment X_b.—The expression for the greatest possible variation of X_b is found by multiplying the integral expressions of various Δ's by their minor determinants, in accordance with Eq. 15, and by recombining the integrals:

$$dX_b\,(\max) = \frac{i}{\frac{7}{12}\left(\frac{L}{E\,I}\right)^3}\left(\frac{L}{E\,I}\right)^2 \frac{w\,L^3}{36\,E\,I}$$

$$\times\Bigg[\overbrace{\int_0^L \Bigg|\frac{1}{18}\left(1-\frac{4\,y}{L}+\frac{3\,y^2}{L^2}\right)+\frac{23}{36}\left(\frac{y}{L}-\frac{3\,y^2}{L^2}\right)+\frac{5}{36}(0)\Bigg|}^{\text{left leg}}$$

$$\overbrace{+\int_0^L \Bigg|\frac{1}{18}(0)+\frac{23}{36}\left(-2+\frac{20\,x}{L}-\frac{36\,x^2}{L^2}+\frac{18\,x^3}{L^3}\right)+\frac{5}{36}\left(-\frac{2\,x}{L}+\frac{18\,x^2}{L^2}-\frac{18\,x^3}{L^3}\right)\Bigg|\,d\left(\frac{x}{L}\right)}^{\text{beam}}$$

$$\overbrace{+\int_0^L \Bigg|\frac{1}{18}\left(1-\frac{4\,y}{L}+\frac{3\,y^2}{L^2}\right)+\frac{23}{36}\left(-1+\frac{4\,y}{L}-\frac{3\,y^2}{L^2}\right)+\frac{5}{36}\left(1-\frac{3\,y}{L}\right)\Bigg|\,d\left(\frac{y}{L}\right)}^{\text{right leg}}\Bigg]$$

$$=\frac{w\,L^2}{756}\,i\,\Bigg[\overbrace{\int_0^L \Bigg|2+\frac{15\,y}{L}-\frac{63\,y^2}{L^2}\Bigg|\,d\left(\frac{y}{L}\right)}^{\text{left leg}}$$

$$\overbrace{+2\int_0^L \Bigg|-23+\frac{225\,x}{L}-\frac{369\,x^2}{L^2}+\frac{162\,x^3}{L^3}\Bigg|\,d\left(\frac{x}{L}\right)}^{\text{beam}}$$

$$\overbrace{+\int_0^L \Bigg|-16+\frac{69\,y}{L}-\frac{63\,y^2}{L^2}\Bigg|\,d\left(\frac{y}{L}\right)}^{\text{right leg}}\Bigg] \quad \ldots\ldots\ldots\ldots\ldots (30a)$$

In Eq. 30*a* each integration must be performed with respect to the numerical values of the functions under the integral signs, rather than with respect to the functions themselves, as is indicated by the sign $|\,|$. This necessitates the determination of the points at which the expressions under the integral signs become zero.

Left Leg.—On solving the quadratic it is found that the function under the integral sign becomes zero at the points $y = \frac{1}{3}\,L$ and $y = -\frac{2}{21}\,L$, of which the latter value is outside of the range considered. Therefore:

$$\int_0^L \left|2 + \frac{15\,y}{L} - \frac{63\,y^2}{L^2}\right| d\left(\frac{y}{L}\right) = \left|\int_0^{L/3} \left(2 + \frac{15\,y}{L} - \frac{63\,y^2}{L^2}\right) d\left(\frac{y}{L}\right)\right|$$

$$+ \left|\int_{L/3}^L \left(2 + \frac{15\,y}{L} - \frac{63\,y^2}{L^2}\right) d\left(\frac{y}{L}\right)\right| = \left|\frac{13}{18}\right| + \left|-\frac{220}{18}\right| = \frac{233}{18} = 12.944.$$

Beam Part.—An approximate solution of the cubic equation in x gives the following values: $x = 0.128\,L$ and $x = 0.874\,L$. Integrating separately in the three intervals between the limits 0, $0.128\,L$, $0.874\,L$, and L, and adding the numerical values of the results:

$$\int_0^L \left|-23 + \frac{225\,x}{L} - \frac{369\,x^2}{L^2} + \frac{162\,x^3}{L^3}\right| d\left(\frac{x}{L}\right)$$

$$= |-1.347| + |8.624| + |-0.272| = 10.243.$$

Right Leg.—The integration must be performed separately between the limits $0, \frac{1}{3}\,L, \frac{16}{21}\,L$, and L:

$$\int_0^L \left|-16 + \frac{69\,y}{L} - \frac{63\,y^2}{L^2}\right| d\left(\frac{y}{L}\right) = \left|-\frac{41}{18}\right| + \left|\frac{81}{98}\right| + \left|-\frac{925}{882}\right| = 4.153.$$

Summation.—Substituting these numerical values of the integrals in the expression for the required variation (Eq. 30*a*):

$$dX_b\,(\max) = \frac{w\,L^2}{756}\,i\,[12.944 + 2(10.243) + 4.153]$$

$$= 0.0497\,i\,w\,L^2 = 0.894\,i\,|X_b|\ \ (30b)$$

in which $X_b = -\frac{1}{18}\,w\,L^2$, as has been found, and 0.894 is the value of the coefficient K expressed by Eq. 16.

Parts of the frame over which the modulus of elasticity must be increased or decreased in order to cause the greatest possible variation in the numerical value of the bending moment X_b are shown in Fig. 3, the stiffened parts being indicated by heavy lines, and the ones that are made less stiff—by light lines.

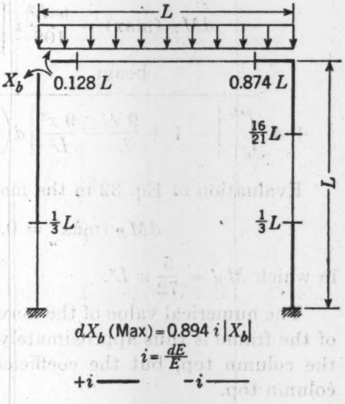


FIG. 3

Moment X_c.—A similar procedure leads to the following expression for the greatest variation in the value of the moment X_c at the top of the right column:

$$\begin{aligned} dX_c\,(\text{max}) = \frac{w\,L^2}{756}\, i \Bigg[& \overbrace{\int_0^L \left| -16 + \frac{69\,y}{L} - \frac{63\,y^2}{L^2} \right| d\left(\frac{y}{L}\right)}^{\text{left leg}} \\ & + \overbrace{2 \int_0^L \left| -5 + \frac{27\,x}{L} + \frac{117\,x^2}{L^2} - \frac{162\,x^3}{L^3} \right| d\left(\frac{x}{L}\right)}^{\text{beam}} \\ & + \overbrace{\int_0^L \left| 2 + \frac{15\,y}{L} - \frac{63\,y^2}{L^2} \right| d\left(\frac{y}{L}\right)}^{\text{right leg}} \Bigg] \quad \ldots\ldots\ldots\ldots\ldots (31) \end{aligned}$$

Comparing Eq. 31 with Eq. 30*a*, it may be noticed that the first and the third integrals in the former are identical with the third and the first integrals, respectively, in the latter. Furthermore, replacement of the variable x by $(L - x)$ in the second integral of Eq. 31 leads to an expression identical with the second integral in Eq. 30*a*. These relationships between the expressions for dX_b (max) and dX_c (max), of course, are the direct result of the symmetry of points B and C on the symmetrical frame, and they lead to an equality of the numerical values of the variations considered.

Bending Moment at the Center of the Frame.—The greatest variation of the bending moment at the center of the frame E is found by Eq. 23*b*:

$$\begin{aligned} dM_E\,(\text{max}) = \frac{w\,L^2}{108}\, i \Bigg[& \overbrace{\int_0^L \left| -1 + \frac{6\,y}{L} - \frac{9\,y^2}{L^2} \right| d\left(\frac{y}{L}\right)}^{\text{left leg}} \\ & + \overbrace{4 \int_0^L \left| -1 + \frac{9\,x}{L} - \frac{9\,x^2}{L^2} \right| d\left(\frac{x}{L}\right)}^{\text{beam}} + \overbrace{\int_0^L \left| -1 + \frac{6\,y}{L} - \frac{9\,y^2}{L^2} \right| d\left(\frac{y}{L}\right)}^{\text{right leg}} \Bigg] \quad . (32) \end{aligned}$$

Evaluation of Eq. 32 in the manner illustrated previously leads to:

$$dM_E\,(\text{max}) = 0.0460\, i\, w\, L^2 = 0.663\, i\, M_E \ldots\ldots\ldots\ldots (33a)$$

in which $M_E = \dfrac{5}{72}\, w\, L^2$.

The numerical value of the maximum variation of the moment at the center of the frame is thus approximately equal to the value of a similar quantity at the column top, but the coefficient K at the center is smaller than at the column top.

It has also been found that

$$dX_a\,(\text{max}) = 0.0447\, i\, w\, L^2 = 1.609\, i\, X_a \ldots\ldots\ldots\ldots (33b)$$

in which $X_a = \dfrac{w\,L^2}{36}$.

The greatest variation in the moment M_E in the same structure has been determined also for a different load condition—namely, for a vertical concentrated load P acting at the center of the frame. New loading creates new values of δ_{ao}, δ_{bo}, δ_{co} and of the static unknowns. The other δ's and the determinants remain the same. The results are as follows:

$$X_a = \frac{1}{24} P\,L; \qquad X_b = X_c = -\frac{1}{12} P\,L; \qquad M_E = \frac{P\,L}{6};$$

and

$$\begin{aligned} dM_E\,(\max) = \frac{P\,L}{72} i \Bigg[& \overbrace{\int_0^L \left| -1 + \frac{6\,y}{L} - \frac{9\,y^2}{L^2} \right| d\left(\frac{y}{L}\right)}^{\text{left leg}} \\ & + \overbrace{\int_0^{L/2} \left| -4 + \frac{24\,x}{L} \right| d\left(\frac{x}{L}\right)}^{\text{beam, left half}} + \overbrace{\int_{L/2}^{L} \left| 20 - \frac{24\,x}{L} \right| d\left(\frac{x}{L}\right)}^{\text{beam, right half}} \\ & + \overbrace{\int_0^L \left| -1 + \frac{6\,y}{L} - \frac{9\,y^2}{L^2} \right| d\left(\frac{y}{L}\right)}^{\text{right leg}} \Bigg] = 0.074\, i\,P\,L = 0.4444\, i\,M_E \quad (34a) \end{aligned}$$

The greatest variation in M_E in the structure loaded with both the uniform and the concentrated loads of the previous problems can be found by direct superposition of the corresponding integral expressions. Assuming, for example, that $P = w\,L$, the following expression is developed for dM_E (max):

$$\begin{aligned} dM_E\,(\max) = \frac{P\,L}{216} i \Bigg[& 5 \overbrace{\int_0^L \left| -1 + \frac{6\,y}{L} - \frac{9\,y^2}{L^2} \right| d\left(\frac{y}{L}\right)}^{\text{left leg}} \\ & + \overbrace{4 \int_0^{L/2} \left| -5 + \frac{36\,x}{L} - \frac{18\,x^2}{L^2} \right| d\left(\frac{x}{L}\right)}^{\text{beam, left half}} + \overbrace{4 \int_{L/2}^{L} \left| 13 - \frac{18\,x^2}{L^2} \right| d\left(\frac{x}{L}\right)}^{\text{beam, right half}} \\ & + \overbrace{5 \int_0^L \left| -1 + \frac{6\,y}{L} - \frac{9\,y^2}{L^2} \right| d\left(\frac{y}{L}\right)}^{\text{right leg}} \Bigg] = 0.1196\, i\,P\,L = 0.5067\, i\,M_E \quad (34b) \end{aligned}$$

in which $M_E = \frac{17}{72} P\,L$.

Two-Hinged Frame of Variable Moment of Inertia

In problems discussed thus far, expressions following the integral signs have been such that the integrations could be performed easily. In more complicated cases integrations must be replaced by summations.

In the uniformly loaded frame shown in Fig. 4 the thickness of the legs changes linearly from 24 in. at the hinge to 36 in. at the top. The thickness of the horizontal member varies from 16 in. at the center to 36 in. at the ends, following the law of a semicubic parabola:

$$h = \frac{4}{3}\,(1 + 0.01\,x^{3/2}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (35)$$

in which both h and x are expressed in feet. The structure is typical for rigid-frame reinforced-concrete bridges except for the somewhat impractical rectangular outline of the axis. However, this peculiarity of the form, assumed for simplicity of calculation, affects the structural behavior of the frame only slightly. It is required to find the greatest variations of thrust and bending moments at different points of the structure due to variation of the modulus of elasticity. The horizontal thrust H, assumed to act outwardly on the structure, is the only static unknown. If horizontal restraint is removed, the bending moment in the horizontal part of the frame becomes $M_o = \frac{w}{2}\,(25^2 - x^2)$, and no flexure occurs in the legs. When unit outward thrusts act on the structure, the moments m_a assume the values shown in Fig. 5. The frame is 1 ft wide. Expressions for δ's after simplification are as follows:

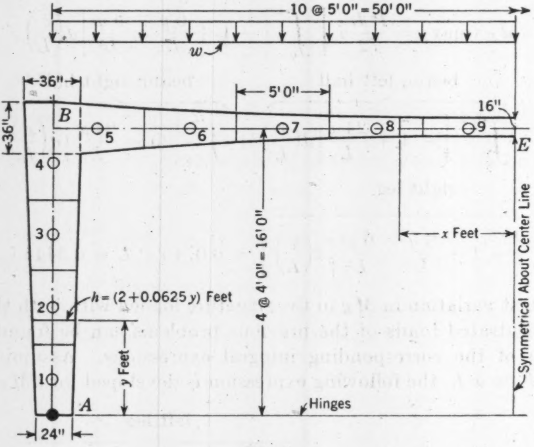


FIG. 4

$$\left.\begin{aligned} \delta_{ao} &= \overset{\text{left leg}}{0} + \overbrace{81\,w\int_0^{25}\frac{(25^2 - x^2)\,dx}{(1 + 0.01\,x^{3/2})^3\,E}}^{\text{beam}} + \overset{\text{right leg}}{0} \\ \delta_{aa} &= \overbrace{2(12)^4\int_0^{16}\frac{y^2\,dy}{(24 + 0.75\,y)^3\,E}}^{\text{two legs}} + \overbrace{2(6)^4\int_0^{25}\frac{dx}{(1 + 0.01\,x^{3/2})^3\,E}}^{\text{beam}} \end{aligned}\right\} \ldots (36)$$

For the purpose of summation, the structure is divided into eighteen sections 4 ft and 5 ft long, as shown in Fig. 4. The sectional increments of δ_{ao} and δ_{aa} (that is, the values of the expressions under the integral signs of all sections multiplied by their respective values of dx or dy) are given in Table 1, together with the values of δ_{ao} and δ_{aa} themselves. Then

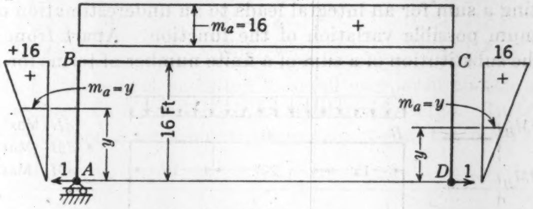


FIG. 5

$$H = -\frac{\delta_{ao}}{\delta_{aa}} = -15.54\,w \ldots\ldots\ldots\ldots\ldots\ldots (37a)$$

The greatest variation in H may be found by Eq. 20b, which for finite summation becomes:

$$dH\ (\text{max}) = \frac{i}{\delta_{aa}} \sum |\text{Increment of } [\,\delta_{ao} + H\,\delta_{aa}\,]| \ldots\ldots\ldots (37b)$$

The values of increments of the expression in brackets are stated in Table 1. All data in this table, as well as the foregoing formulas for δ_{ao}, δ_{aa}, and other values, are given in feet units, and the values in the table refer to the sums of corresponding increments for the pairs of sections located symmetrically with respect to the vertical axis, so that the summation need be extended only over one half of the structure.

Substituting the numerical values from Table 1, the greatest variation of the thrust proves to be:

$$dH\ (\text{max}) = 4.18\,w\,i$$
$$= 0.269\,i\,|H| \ldots\ldots (37c)$$

In this equation w is expressed in feet units, and so are the various stress quantities to the end of this chapter.

TABLE 1.—SOLUTION FOR dH (MAX), FIG. 4; VALUES OF THE INCREMENTS IN EQ. 37(b) IN FEET

Section	$E\,\delta_{aa}$	$\frac{E}{w}\,\delta_{ao}$	$\frac{E}{w}\,H\,\delta_{aa}$	$\frac{E}{w}\,(\delta_{ao} + H\,\delta_{aa})$
1	40	....	−620	−620
2	258	....	−4,010	−4,010
3	531	....	−8,250	−8,250
4	778	....	−12,300	−12,300
5	1,467	5,450	−22,800	−17,350
6	2,494	24,840	−38,760	−13,920
7	4,311	63,320	−67,180	−3,860
8	7,400	131,520	−115,000	16,520
9	11,537	223,080	−179,290	43,790
Σ	28,816	448,210	−448,210	0
Summation of numerical values of increments..................				120,620

The variations of the moments at the knee and at the crown are found by differentiating the equations of statics relating these moments with H. Calling moment at the knee M_B and at the crown M_E:

$$dM_B\ (\text{max}) = dM_E\ (\text{max}) = 16\,dH\ (\text{max}) = 66.91\,w\,i \ldots\ldots (37d)$$

Since $M_B = -248.65\,w$ and $M_E = 63.85\,w$, dM_E (max) $= 1.048\,i\,M_E$ and dM_B (max) $= 0.269\,i\,|M_B|$. Fig. 6 shows the parts of the frame that are modified in stiffness for the purpose of creating the greatest variations in H, M_B, and M_E.

Susbtituting a sum for an integral leads to an underestimation of the value of the maximum possible variation of the function. Apart from some error inherent in the substitution of a sum of a finite number of terms for an integral, the value of the greatest possible variation of a function may be substantially the same for integration and summation only if all the zero points of the expression under the integral sign coincide with the boundaries of sections. Since such a condition is not likely to occur, the increment of the sum for the section containing a zero point will be approximately equal to an algebraic sum of the positive and negative areas under the curve in this particular section instead of the sum of the numerical values of the two areas. That means that the value found by summation should be somewhat less than the one found by integration. The difference of the two values, however, is not likely to be great.

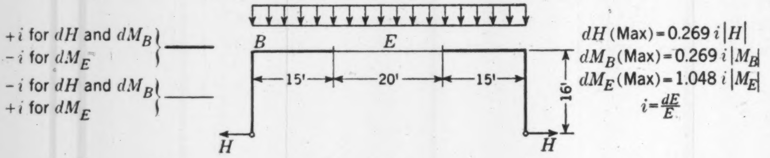


FIG. 6

The uniformly loaded rigid frame of Fig. 4 also has been analyzed, assuming the bottom ends of the columns as fixed. Thus modified, the structure becomes three times statically indeterminate. The following results have been obtained, using feet units: $H = -22.04\,w$; $M_A = 100.00\,w$; $M_B = -252.58\,w$; $M_E = 59.92\,w$; dH (max) $= 0.421\,i\,|H|$; dM_A (max) $= 1.567\,i\,M_A$; dM_B (max) $= 0.496\,i\,|M_B|$; and dM_E (max) $= 1.047\,i\,M_E$.

Comparing the fixed-ended frame with the two-hinged frame, it may be seen that dH (max) and dM_B (max) increase considerably when column supports become fixed—increase both in absolute values and in relation to the magnitudes of H and M_B themselves. On the other hand, dM_E (max) remains practically the same.

TWO-HINGED TRUSSED ARCH

As an example of a trussed structure, the two-hinged arch of Fig. 7, uniformly loaded, has been investigated for the maximum possible variations of the thrust and of the stresses in some of the members, caused by variation in E. The cross-sectional areas of all members have been assumed equal. The only static unknown, the thrust, and its greatest variation have been found to be: $H = -135.36\,w$; and dH (max) $= 0.187\,i\,|H|$, with w in feet units.

The greatest variations of stresses in the truss members have been found by differentiating the expressions for these stresses stated in terms of H. Some of the stresses and their variations are:

Member (Fig. 7)	Stress S	Maximum variation
$L_0 L_1$	$-166.77\ w$	$0.187\ i\lvert S\rvert$
$U_4 U_5$	$-103.9\ w$	$1.22\ i\lvert S\rvert$
$U_3 L_4$	$38.6\ w$	$1.21\ i\lvert S\rvert$

In member $L_0 L_1$ the stress variation is small compared to the stress itself, whereas in members $U_4 U_5$ and $U_3 L_4$ it is relatively large. In the first two members, uniform load over the entire span represents the most unfavorable condition of loading, and the same is approximately true for the third member considered. In several other members maximum stress variations prove to be even larger in proportion to their respective stresses than in the foregoing three members, but their stresses themselves are relatively small, since the loading considered is by far not the most unfavorable one.

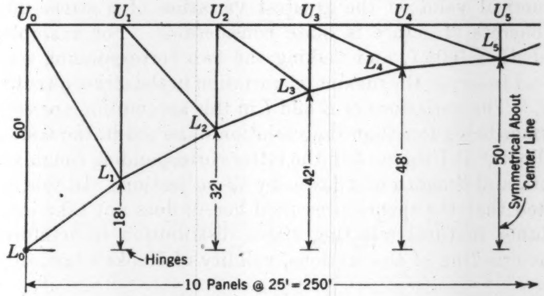


FIG. 7

CONCLUSION

It must be realized clearly that the variation of the static unknown discussed in this paper is an extreme quantity that may be attained only in an exceptional case when the variations of the elastic characteristics causing it assume their maximum possible values (positive or negative), arranged with regard to signs in the most unfavorable and unique manner along the structure. In spite of its improbability, the variation herein considered has a genuine significance as a limiting quantity that the actual deviations of stress functions from their theoretical values cannot exceed, but may approach, in unfavorable cases.

Furthermore, the maximum variation of a stress quantity may be viewed as a measure of the sensitivity of this quantity to small unforeseen and accidental changes in elastic characteristics. Coefficient K in Eq. 16 can be called the coefficient of sensitivity. The higher its value, the more sensitive is the stress quantity to small variations in E, I, or A. Thus, the moment at the crown of the frame of Fig. 4 is much more sensitive to variation in E than the moment at the knee.

From the examples solved it may be seen that the K-values frequently equal unity and occasionally are as high as 1.5 or even 2.0. On the other hand, they are often as low as 0.15 to 0.20. It is possible that there is general increase in sensitivity as the number of static unknowns increases. At least, a comparison of the results for two-hinged and fixed-ended frames suggests this possibility; but the evidence supporting this statement is inconclusive.

In theory, the coefficient of relative variation of the elastic characteristic i is an infinitesimal quantity, but in actual computation it must be given an appropriate finite value, disregarding the error that may result from such assumption.

The numerical value of the greatest variation of a stress quantity in a reinforced-concrete structure is quite considerable. For example, using $dE = 0.2\,E$ and $dI = 0.04\,I$, and making the two corresponding sensitivity coefficients equal to unity, the maximum variation in the stress quantity X proves to be $0.24\,X$. The variations of E and I in this assumption are very conservative, the former being less than the variation observed in the tests at the University of Illinois,[2] at Urbana, and the latter corresponding roughly to a ± ⅛-in. error in depth and breadth of a 12-in. by 12-in. section. In this connection it may be stated that the theory presented herein does not take into consideration such other features affecting stress distribution in reinforced-concrete structures as cracking of the sections, validity of Hooke's law, shrinkage, and plastic flow.

In the tests at the University of Illinois some of the moments at column bases differed from theory much more than the 24% found in the foregoing example. The discrepancy, however, was attributed to cracking. In steel structures the greatest variations of stress quantities from the cause herein considered are not likely to be more than 3% to 5% of their values.

The foregoing suggests the futility of using involved methods for finding the "exact" values of stress quantities in most statically indeterminate, reinforced-concrete structures, and it encourages the use of approximate methods. On the other hand, in structures of great importance the determination of maximum values of stress quantities appears insufficient, and must be supplemented by a determination of the greatest variations of these functions.

It is reasonable also to conclude that the legitimate fluctuation of stress values discussed herein should be allowed for by varying working stresses in amounts depending on the sensitivity of the function in question, unless the member is designed on the basis of the theoretical value of the stress computed, modified by the amount of its maximum variation.

APPENDIX

NOTATION

The following letter symbols, used in the paper, conform essentially to American Standard Letter Symbols for Mechanics, Structural Engineering and Testing Materials, prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1932:[3]

b = breadth of a structural member;
C = constants;
D = a determinant;
E = Young's modulus;
H = a horizontal thrust;
h = height or depth of cross section of a structural member;
I = moment of inertia;
i = a small, theoretically infinitesimal, quantity which is a measure of the greatest variation of an elastic characteristic: $i = \frac{dE}{E}, \frac{dI}{I}$, etc.
K = coefficient representing the maximum percentage by which values of X may change when elastic characteristics change by one per cent;
L = length or height of a structural member;
M = bending moment for structures consisting of flexural members:
  M_E = moment at point E, etc.;
  M_o = moment due to the externally applied loading;
m = moment due to a unit load for members subjected to flexure only:
  m_a = moment due to $X_a = 1$;
  m_b = moment due to $X_b = 1$;
N = direct stress for structures consisting of direct-stress members only;
n = direct stress due to a unit load, for structures consisting of direct-stress members only;
P = a concentrated load;
ds = a small element of a member;
w = a uniform load intensity;
X = a statically unknown force: X_a = force at point a, etc.;
ϵ = an elastic characteristic; and
δ = movement of the point of application of a static unknown in the direction of the unknown.

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[3] ASA—Z10a—1932.

DISCUSSION

A. H. Finlay,[4] **Assoc. M. Am. Soc. C. E.**—Professor Hrennikoff has presented an intriguing paper with clarity and precision. Others[5] previously have determined the effect of variations of the elastic characteristics of structures upon the statically indeterminate stress functions, but these workers have confined themselves to arbitrary distributions of such variations throughout the structure. To the writer's knowledge, the author is the first to develop a method determining the maximum effect that a variation of specified magnitude may exert upon a given stress function. In so doing he defines the "pattern" to which such a variation must conform.

The author's applications of his method to several specific structures reveal that substantial variations in the stress functions are at least within the realms of possibility and the profession is fortunate to have the problem placed upon a quantitative basis.

In order to interpret the significance of the expressions under the integral signs in the author's various equations (such as Eq. 30*a*), the writer has plotted the functions concerned for a special case—a square frame having members of uniform cross section and carrying a uniform load on the horizontal member. In Fig. 8, curves have been plotted for those expressions associated with dX_a, dX_b, and dM_E, defined, respectively, as follows:

$$dX_a\,(\text{max}) = \frac{w\,L^2\,i}{378}\,A_a \qquad \text{(38}a\text{)}$$

$$dX_b\,(\text{max}) = \frac{w\,L^2\,i}{756}\,A_b \qquad \text{(38}b\text{)}$$

and

$$dM_E\,(\text{max}) = \frac{w\,L^2\,i}{108}\,A_E \qquad \text{(38}c\text{)}$$

in which A is the area under each respective curve, without regard to sign. (The graph of the function under the integral for dX_c is similar to Fig. 8(*b*), but "opposite hand.") These areas are, of course, measures of the maximum values that dX_a, dX_b, and dM_E may attain. It should be noted that the areas under these curves with regard to sign are zero since a uniform change in an elastic characteristic (that is, with constant i) produces no change in the statically indeterminate forces. It is the fact that i may be positive in those parts of the structure where such areas are positive, and negative in those parts where the areas are negative (or vice versa), that leads to the possibility of the substantial errors found by the author. It is also apparent from the curves that considerable errors may still occur in the stress functions even though chance fails to cause a change in the sign of i in strict accordance with the pattern.

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[4] Associate Prof., Civ. Eng., Univ. of British Columbia, Vancouver, B. C., Canada.

[5] "Dependability of the Theory of Concrete Arches," by Hardy Cross, *Bulletin No. 203*, Univ. of Illinois Eng. Experiment Station, Urbana, Ill., 1930.

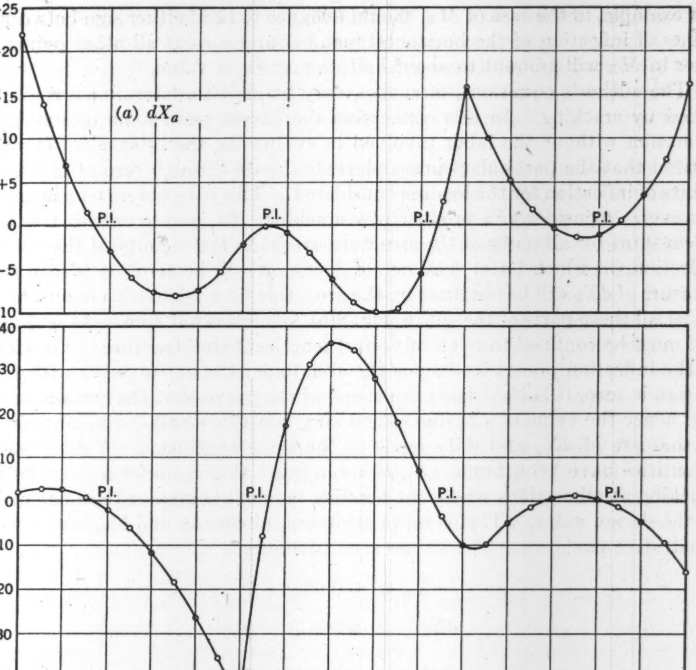

FIG. 8.—GRAPHS OF FUNCTIONS UNDER INTEGRAL SIGNS IN THE AUTHOR'S EQUATIONS (SUCH AS EQ. 30a)

For example, in the case of M_E, should i chance to be of either sign between the points of inflection of the horizontal member and zero at all other points, the error in M_E will amount to about half its maximum value.

The author's equations, of course, may be used to determine errors introduced by cracking. In this connection the curves yield some interesting information without the labor involved in evaluating the integrals. It will be noticed that the particular curves plotted all pass through zero at or near the points of inflection for the loading considered. This may not be true in general, however. Consider dX_b produced by cracking: If i, due to cracking, is taken as constant for all parts of the structure except in the vicinity of the points of inflection (in which latter sections, of course, it will be zero), it is seen that a measure of dX_b will be obtained by the area, this time taken with regard to sign, under all those parts of the curve, Fig. 8(*b*), where i is not zero. As zero values of i must be confined to relatively short lengths of the structure in the vicinity of the inflection points, and since the area under the entire curve with regard to sign is zero, it is clear from the shape of the curve that the remaining area, and hence the value of dX_b due to cracking, must be small. Similar reasoning in the case of dX_a and dM_E leads to the same conclusion. Values of these quantities have been found on the assumption that a uniform i results from cracking at all sections where the bending moment is greater than one quarter of the design value. This is quite arbitrary, of course, and has been used for illustrative purposes. The values are as follows:

$$dX_a = +\,0.0358\, i\, X_a \qquad (39a)$$

$$dX_b = -\,0.00573\, i\, |\, X_b\, | \qquad (39b)$$

and

$$dM_E = -\,0.00321\, i\, M_E \qquad (39c)$$

It should be noted that, in Eqs. 39, i carries a sign and, since cracking reduces the moment of inertia, i should be substituted in Eqs. 39 as a negative quantity. The errors are seen to be extremely small, as could have been anticipated from the curves.

Had i been assumed as a variable quantity, the curves in Fig. 8, of course, would have been different, since i could not have been left outside the integral sign. As an approximation, however, the curves as plotted could be used in such a case. The areas under various parts of them then would have to be multiplied by the value of i assumed to exist over the section in question. This, of course, is equivalent to assuming that i is constant along short sections of the structure but varies abruptly from section to section. It still seems clear, however, from the characteristics of the curves and the fact that i is everywhere of the same sign, that errors in the statically indeterminate forces due to cracking will be of a smaller order altogether than those which may arise due to chance variations in the elastic characteristics.

The writer is in complete agreement with the author's concluding paragraphs, and he also feels that the paper should prove of considerable interest to those whose lot it is to explain discrepancies between observed and calculated quantities in the field of rigid-frame testing.

Stefan J. Fraenkel,[6] **Jun. Am. Soc. C. E.**—In the writer's opinion, the method presented by Professor Hrennikoff constitutes a valuable contribution to structural analysis and is the first ever suggested along this particular line. The author is to be commended for having devised this thorough and workable solution of a problem otherwise rather beset with difficulties which often make an exact solution impracticable.

It is important to realize the limitations to which this method is subject, as far as its application to the field of design is concerned. Professor Hrennikoff rightly states that in steel structures the variation of stress quantities is very small. Even assuming that the variation amounts to 5%, the resulting additional stress would, in most structures, be smaller than stresses caused by secondary stresses, poorly maintained pin connections, etc. The main field of application, therefore, would lie in the realm of reinforced-concrete structures. Even there, enough uncertainties enter into a design to make unnecessary an investigation along the lines suggested in the paper, except in the very largest structures. It should be possible, however, to put this method to excellent use for purposes of investigation in the following manner:

During the construction of a reinforced-concrete structure, test cylinders of the material could be taken at regular intervals along the structure, and the values of E of these cylinders then could be determined. The maximum value of i, and, consequently, the maximum possible variation of any stress quantity may then be computed for the particular case. Further, the deviations of E and I from their average values within the intervals defined by the roots of Eq. 15 could be investigated, and their algebraic signs would furnish an indication of how closely the theoretical sensitivity coefficient would be approximated. Although this information undoubtedly would be of interest, it would constitute little more than a "post-mortem" examination, and its results would not be applicable to a future design.

A tabulation of sensitivity coefficients for different structures would be a helpful guide to designers, although more in a qualitative than in a quantitative way. Inasmuch as design is still to a considerable degree a matter of judgment, some indication thus would be available to the designer as to where to exercise particular caution. The recommendation that the working stresses be varied in amounts proportional to the sensitivity of the function in question, although theoretically sound, would seem to the writer to be, in effect, wasteful of materials. In framed indeterminate structures it is impossible in any case to stress all members to the allowable unit stress.

To satisfy his curiosity, the writer applied Professor Hrennikoff's method to an arch rib of uniform cross section and of uniform outline. In presenting the results of this investigation, the writer wishes to state that, unless summation is substituted for integration, it will be necessary, in the case of structures with curved center lines, to solve for the roots of algebraic polynomials of the fourth degree, or even higher, which is a tedious process. A computing machine is almost indispensable, inasmuch as the ordinates of the functions are obtained as differences of large numbers. The writer used the slide rule and had to make extensive long-hand computations in order to gain a tolerable accuracy.

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[6] Care, Eng. Dept., Pittsburgh-Des Moines Steel Co., Neville Island, Pittsburgh, Pa.

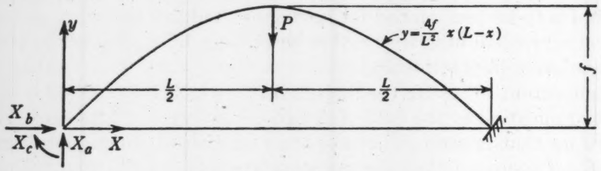

FIG. 9

The arch shown in Fig. 9 was loaded with a concentrated load P at the center, and its equation is

$$y = \frac{4\,f}{L^2}\,x\,(L - x) \qquad \text{(40)}$$

The statically indeterminate quantities are shown at the left abutment.

After a routine application of the Maxwell-Mohr method, the values of the unknowns were found to be as follows:

$$X_a = \tfrac{1}{2}\,P; \qquad X_b = \frac{15}{64}\,\frac{P\,L}{f}; \qquad \text{and} \qquad X_c = \frac{P\,L}{32} \qquad \text{(41)}$$

Eqs. 7 become:

$$\Delta_a = \frac{P}{64}\Bigg[\int_0^{L/2} \frac{1}{E\,I}\left(-\,28\,x^2 + \frac{60}{L}\,x^3 + 2\,L\,x\right)dx + \int_{L/2}^{L} \frac{1}{E\,I}\left(-\,92\,x^2 + \frac{60}{L}\,x^3 + 34\,L\,x\right)dx\Bigg] \qquad \text{(42a)}$$

$$\Delta_b = \frac{P}{64}\Bigg[\int_0^{L/2} \frac{1}{E\,I}\left(120\,f\,\frac{x^2}{L} - 352\,f\,\frac{x^3}{L^2} + 240\,f\,\frac{x^4}{L^3} - 8\,f\,x\right)dx + \int_{L/2}^{L} \frac{1}{E\,I}\left(504\,f\,\frac{x^2}{L} - 608\,f\,\frac{x^3}{L^2} + 240\,f\,\frac{x^4}{L^3} - 136\,f\,x\right)dx\Bigg] \qquad \text{(42b)}$$

and

$$\Delta_c = \frac{P}{64}\Bigg[\int_0^{L/2} \frac{1}{E\,I}\left(-\,28\,x + 60\,\frac{x^2}{L} + 2\,L\right)dx + \int_{L/2}^{L} \left(-\,92\,x + 60\,\frac{x^2}{L} + 34\,L\right)dx\Bigg] \qquad \text{(42c)}$$

Completing the process, the writer found the maximum variations of the indeterminate quantities to be:

$$dX_a\,(\max) = 0.187\,i\,X_a \qquad \text{(43a)}$$

$$dX_b\,(\max) = 1.59\,i\,X_b \qquad \text{(43b)}$$

and

$$dX_c\,(\max) = 1.92\,i\,X_c \qquad \text{(43c)}$$

Generally, in order to obtain the maximum variation, the values of the elastic variables in a parabolic arch with a concentrated center load should be increased from the left abutment to 0.1 L, decreased from 0.1 L to 0.4 L, increased from 0.4 L to 0.6 L, decreased from 0.6 L to 0.9 L, and increased from 0.9 L to the right abutment.

The sensitivity coefficients found are very high for both the horizontal thrust and the moment at the springing, and—to the writer—unexpectedly low for the vertical reaction. Altogether, the parabolic arch appears to be sensitive to elastic variations to a very considerable degree, and a close control of the concrete mixture, in the case of a masonry arch, and of the dimensions, seems advisable.

JOSEPH A. WISE,[7] M. AM. SOC. C. E.—The fundamental principles of the effect of the variation of elastic characteristics and their application to specific structures are well presented in this excellent paper. In the analysis given by the author, a statically determinate basic structure is taken, and the n redundant stresses are superimposed upon it. If, instead, a statically indeterminate basic structure is used (one having $n - 1$ redundants, such as is used in obtaining an influence line for the nth redundant), the computations are simplified and some new and interesting relationships appear. In effect, this reduces all structures to singly redundant types, assuming that the remaining redundants have been determined. To avoid introducing new and possibly confusing nomenclature, the symbols used in Professor Hrennikoff's paper will be adhered to, but with the understanding that the moments, M_o, refer to moments due to the given loading in the $(n - 1)$-degree statically indeterminate base structure, and the moments m_a are those in the same base structure due to a unit load applied in place of the nth redundant. Thus, the only difference from a singly redundant structure is that the moments M_o and m_a must be calculated for an $(n - 1)$-degree statically indeterminate structure instead of being determined by statics alone.

The analysis first will be made for structures subjected to bending alone. Then, in general:

$$X_a = -\frac{\delta_{ao}}{\delta_{aa}} \qquad \text{(44)}$$

$$dX_a = -\frac{1}{\delta_{aa}}\left(d\delta_{ao} - \frac{\delta_{ao}}{\delta_{aa}}\, d\delta_{aa}\right) \qquad \text{(20a)}$$

and

$$dX_a\,(\max) = \frac{i}{\delta_{aa}}\int\left|\left(\frac{M_o\, m_a}{E\,I} + X_a\,\frac{m^2_a}{E\,I}\right)\right| ds \qquad \text{(20b)}$$

Eq. 20*b* also can be written:

$$dX_a\,(\max) = \frac{i}{\delta_{aa}}\int\left|\frac{(M_o + X_a\, m_a)\, m_a}{E\,I}\right| ds = \frac{i}{\delta_{aa}}\int\left|\frac{M\; m_a\, ds}{E\,I}\right| \qquad \text{(45)}$$

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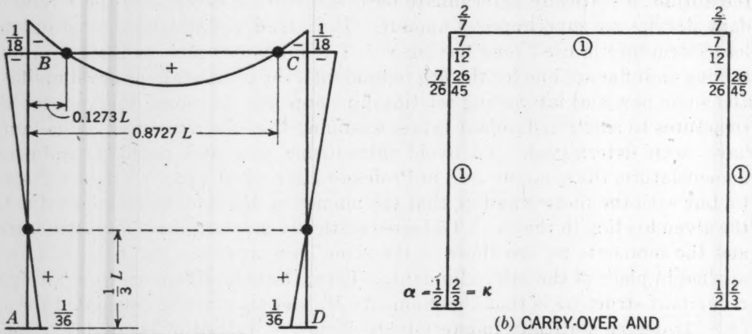
[7] Lt.-Comdr. (CEC), U.S.N.R., Care, Public Works Office, 3rd Naval Dist., New York, N. Y.; and Assoc. Prof. of Structural Eng., Univ. of Minnesota, Minneapolis, Minn. (on leave).

in which M is the final or true moment at any segment, ds, and is equal to $M_o + X_a\, m_a$. Eq. 45 shows that dX_a will have a zero value (that is, a change in $E\,I$ in a segment ds will not change X_a) when either M or m_a is zero. In integrating along the members of the structure, whenever either M or m_a passes through zero the integrand changes sign; hence, the segments between these null points are alternately increased and decreased in $E\,I$. If M and m_a should happen to have a common null point, this null point is omitted, since both change sign in passing through it. Thus, a general rule could be formulated as follows:

> If it is desired to find the maximum variation dX_a of the moment at a point in a structure, determine the actual moments M at all points. Then consider the redundant X_a to be replaced by a pair of unit couples acting in the same sense, and determine the moments m_a in the otherwise unloaded statically indeterminate base structure. Determine the null points; then compute dX_a from Eq. 45, integrating between singular null points.



(a) MOMENT DIAGRAM

(b) CARRY-OVER AND DISTRIBUTION COEFFICIENTS

FIG. 10

This analysis can be readily extended to structures with axially stressed members, as indicated by Professor Hrennikoff, so that Eq. 45 becomes:

$$dX_a = \frac{i}{\delta_{aa}} \sum \left| \frac{N\, n_a\, L}{E\, A} \right| \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (46)$$

It is obvious that members having N and n_a of like sign must be increased in area, and the other members must be decreased in area (or in $E\,A$) to obtain the greatest increase, dX_a.

To illustrate the application of the foregoing principle, the frame of Fig. 1 will be analyzed. The writer has used a precise moment-distribution method[8] in the analysis, as it gives accurate results with rapidity, but any method could be used to obtain the same results. Fig. 10 shows the actual moments M at ends of all members, and the moment diagram. The null points determined by this method are shown by heavy dots. The moment at point B is expressed by

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[8] "A Precise Moment Distribution Method," by Joseph A. Wise, *Journal*, A.C.I., November, 1938, p. 93.

$$|M_B| = \frac{w\,L^2}{12} \times \frac{3 \times \frac{2}{7}\left(1 - \frac{2}{7}\right)}{1 - \frac{2}{7} \times \frac{2}{7}} = \frac{w\,L^2}{18} \ldots\ldots\ldots\ldots (47)$$

Fig. 11 shows the analysis for m_a. Here the moment at point B is taken as the redundant X_a and the frame is analyzed for unit couples applied as shown.

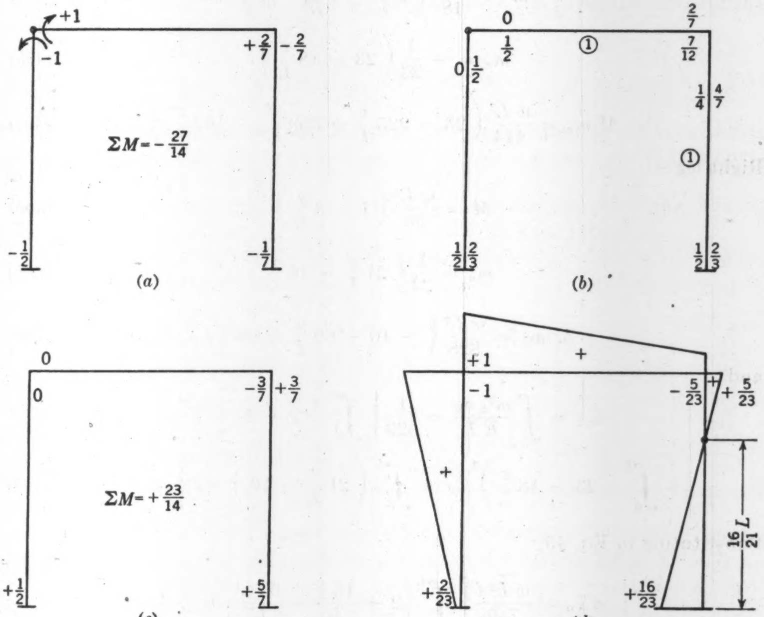
+1
-1
$+\frac{2}{7}$
$-\frac{2}{7}$
$\Sigma M = -\frac{27}{14}$
$-\frac{1}{2}$
$-\frac{1}{7}$
(a)
0
$\frac{2}{7}$
$\frac{1}{2}$
①
$\frac{7}{12}$
0
$\frac{1}{2}$
$\frac{1}{4}$
$\frac{4}{7}$
①
$\frac{1}{2}$
$\frac{2}{3}$
$\frac{1}{2}$
$\frac{2}{3}$
(b)
0
0
$-\frac{3}{7}$
$+\frac{3}{7}$
$\Sigma M = +\frac{23}{14}$
$+\frac{1}{2}$
$+\frac{5}{7}$
(c)
+
+1
-1
$-\frac{5}{23}$
+
$+\frac{5}{23}$
+
$\frac{16}{21}L$
$+\frac{2}{23}$
$+\frac{16}{23}$
(d)

FIG. 11

Fig. 11(*a*) gives the first moment distribution, assuming no sidesway; Fig. 11(*b*) gives the precise carry-over and distribution coefficients; Fig. 11(*c*) gives moments due to an assumed sidesway; and Fig. 11(*d*) gives moments corrected for sidesway. The moment at point A, for example, is $-\frac{1}{2} + \frac{27}{23} \times \frac{1}{2} = +\frac{2}{23}$. Fig. 11 locates another null point. These null points agree substantially with those determined in the paper.

If the equations for M and m_a are written for the three parts of the frame:

Left leg—

$$M = \frac{w\,L^2}{36}\left(1 - 3\,\frac{y}{L}\right) \qquad \text{(48}a\text{)}$$

$$m_a = \frac{1}{23}\left(2 + 21\,\frac{y}{L}\right) \qquad \text{(48}b\text{)}$$

$$M\,m_a = \frac{w\,L^2}{828}\left(2 + 15\,\frac{y}{L} - 63\,\frac{y^2}{L^2}\right) \qquad \text{(48}c\text{)}$$

Beam—

$$M = \frac{w\,L^2}{18}\left(9\,\frac{x}{L} - 9\,\frac{x^2}{L^2} - 1\right) \qquad \text{(49}a\text{)}$$

$$m_a = -\frac{1}{23}\left(23 - 18\,\frac{x}{L}\right) \qquad \text{(49}b\text{)}$$

$$M\,m_a = \frac{w\,L^2}{414}\left(23 - 225\,\frac{x}{L} + 369\,\frac{x^2}{L^2} - 162\,\frac{x^3}{L^3}\right) \qquad \text{(49}c\text{)}$$

Right leg—

$$M = \frac{w\,L^2}{36}\left(1 - 3\,\frac{y}{L}\right) \qquad \text{(50}a\text{)}$$

$$m_a = \frac{1}{23}\left(21\,\frac{y}{L} - 16\right) \qquad \text{(50}b\text{)}$$

$$M\,m_a = \frac{w\,L^2}{828}\left(-16 + 69\,\frac{y}{L} - 63\,\frac{y^2}{L^2}\right) \qquad \text{(50}c\text{)}$$

and

$$\delta_{aa} = \int \frac{m^2{}_a\,ds}{E\,I} = \frac{1}{529}\left[\int_0^L \left(2 + 21\,\frac{y}{L}\right)^2 dy + \int_0^L \left(23 - 18\,\frac{x}{L}\right)^2 dx + \int_0^L \left(21\,\frac{y}{L} - 16\right)^2 dy\right] = \frac{21\,L}{23\,E\,I} \qquad \text{(51)}$$

Substituting in Eq. 45,

$$dX_a = \frac{w\,L^2\,i}{756}\left[\int_0^L \left|2 + \frac{15\,y}{L} - \frac{63\,y^2}{L^2}\right| d\left(\frac{y}{L}\right) + 2\int_0^L \left|-23 + \frac{225\,x}{L} - \frac{369\,x^2}{L^2} + \frac{162\,x^3}{L^3}\right| d\left(\frac{x}{L}\right) + \int_0^L \left|-16 + \frac{69\,y}{L} - \frac{63\,y^2}{L^2}\right| d\left(\frac{y}{L}\right)\right] \qquad \text{(52)}$$

This agrees exactly with Eq. 30*a*. However, the null points have already been found, and it is only necessary to evaluate Eq. 52 between these null points. The result will be the same as in the paper.

From a study of Eq. 45 it will be seen that dX_a will be most affected by changes in $E\,I$ where the product $M\,m_a$ is a maximum and, by obtaining the maximum of each of the three terms in Eq. 52, it will be found that the most sensitive points in the frame are: Left leg, $\frac{y}{L} = \frac{5}{42}$; beam, $\frac{x}{L} = 0.386$; and right leg, $\frac{y}{L} = \frac{23}{42}$. Thus, a curve $M\,m_a$ could reveal the parts of the structure in which changes in $E\,I$ have the greatest effect on X_a. In the demolition of structures it might be quite useful to have this information. Also, if the $M\,m_a$-curve is plotted, a planimeter could be used to obtain the mechanical integration of Eq. 45.

The approximate integration by finite summation can be easily applied to this analysis. The frame of Fig. 5 is analyzed, as in the paper, by substituting the numerical values from Table 2:

$$dH = \frac{5{,}004}{1{,}201}\,w\,i = 4.17\,w\,i \qquad (53)$$

TABLE 2.—ALTERNATE SOLUTION FOR *dH* (MAX), FIG. 4

Segment	$\frac{M}{w}$	m_a	I	$\left\lvert\frac{M\,m_a\,\Delta s}{I}\right\rvert$	$\frac{m^2\,\Delta s}{I}$
1	− 31.08	2	9.6	26	2
2	− 93.24	6	13.3	168	11
3	−155.40	10	18.0	345	22
4	−217.56	14	23.7	514	33
5	−189.27	16	20.8	703	62
6	− 89.27	16	12.3	580	104
7	− 14.27	16	7.1	161	180
8	+ 35.73	16	4.16	687	308
9	+ 60.73	16	2.67	1,820	479
Total	....	..	....	5,004	1,201

In comparing the moment at the crown and the moment at the knee of this frame, it is noted that the two m_a-diagrams are the same and, since M is obviously the same for both cases, the variation of dX_a will be the same for both. However, in obtaining K, dX_a is divided by X_a for each of the respective cases, and for the moment at the crown, $|X_a| = 63.86$, whereas, for the moment at the knee, $|X_a| = 248.64$. Therefore, the K-values for these two points should be inversely proportional to the respective moments, or K (knee) is to K (crown) as 63.86 is to 248.64. The respective values for K—0.269 and 1.048—given in the paper agree with this ratio. From this it can be concluded that K tends to be largest where stresses are smallest. In fact, if one were to investigate a point of inflection in the structure, where X_a is zero, one would find K becoming infinite, as it obviously should, since any change in $E\,I$ elsewhere in the structure would result in a moment at this point where no moment had existed previously. Thus, it appears that K is not entirely satisfactory as a measure of the sensitivity of the structure to changes in the elastic constants. The designer is not much concerned about changes in moment where the moments are small and the stresses are far below the allowable working stresses. It is suggested that a factor

$$K' = K\,\frac{f_a}{f_w} \qquad (54)$$

be used as a measure of sensitivity, in which f_a is the actual maximum fiber stress in the structure at the location of the redundant and f_w is the allowable working stress. At points of inflection K' would be indeterminate, being of the form $\infty \times 0$, and would not be considered as significant.

The suggestion made by Professor Hrennikoff that sensitivity appears to increase as the number of static unknowns increase can be explained at least partly by the fact that, in general, as the redundancy of the structure increases, its stiffness increases, so that the deflections due to unit couples, δ_{aa}, decrease. Since this term appears in the denominator of Eq. 45, a decrease in it will cause an increase in dX_a.

The opinions or assertions contained herein are the private ones of the writer and are not to be construed as official or reflecting the views of the Navy Department or the naval service at large.

A. Hrennikoff,[9] Assoc. M. Am. Soc. C. E.—In Fig. 8 Professor Finlay has plotted the graphs of the functions under the integral signs in the expressions for the variations of static unknowns in the three-legged frame of Fig. 1. These curves depict the effect of the variation of an elastic characteristic, at different points of the structure, on the values of three static unknowns, and they may be called the "variation lines." "Influence lines" might be a better term, but it would be unsuitable because it would disagree with the generally accepted meaning of this term. The influence line gives the value of a stress quantity for different positions of a unit load; the variation line, on the other hand, shows the change in a statically indeterminate quantity for different locations of the variation of an elastic characteristic, while the loading remains the same. For each static unknown there is a distinct variation line for every loading.

If an elastic characteristic undergoes a change i' at some location with an abscissa s on the length ds, the corresponding variation of the unknown is equal to the product of i', ds, and the ordinate of the variation line at the abscissa s. (There is also an additional constant factor, $\frac{1}{D}$, that can be brought into the expression for the variation line, as may be seen from Eq. 14.)

Variation lines are useful not only as a pictorial representation of the component parts of the variations contributed by different parts of the structure, but also as a means of determining, with some exercise of judgment, the probable variations of the unknowns, as distinct from the extreme values expressed by Eq. 15.

Professor Finlay has made use of these curves in his investigation of possible variations of stress quantities due to the frame cracking in the vicinity of points with high bending moments. In the structure considered, this effect proved to be very low; however, as the discusser correctly states, this may not be true in general.

The writer agrees with Mr. Fraenkel that practical applicability of the method considered may be confined to reinforced-concrete structures, in which the variations of the modulus of elasticity and of the moment of inertia may be quite substantial. In this connection, Mr. Fraenkel's suggestion to investigate the actual behavior of the structure as built, on the basis of tests of concrete specimens, taken from different parts of the structure, is quite appropriate, provided the test cylinders are made truly representative of the quality of concrete in the structure itself.

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[9] Asst. Prof., Civ. Eng., Dept. of Civ. Eng., Univ. of British Columbia, Vancouver, B. C., Canada.

The writer prefers to differ with the discusser on the question of the suitability of varying the working stresses in reinforced-concrete structures as functions of sensitivity coefficients. In spite of Mr. Fraenkel's objections, the writer still thinks that even if such departure from the current practice might add to the complexity of design, the high values of sensitivity coefficients frequently occurring even in common structures, such as the one analyzed by Mr. Fraenkel, should make this measure quite in order, when designing structures of considerable importance.

Commander Wise has made an important contribution to the theory of the variation of statically indeterminate quantities by his illuminating interpretation of the significance of the expression for, what was termed herein, the "variation line." Eq. 45 is presented by Commander Wise in lieu of Eq. 15, as being applicable to structures with flexural stress. In this formula, M signifies the actual moments in the given structure with n static unknowns under the given loading, whereas m_a and δ_{aa}, respectively, represent the moments and the relative movement produced by a pair of stress quantities $X_a = 1$, in a modified structure with $(n - 1)$ unknowns, obtained from the given structure by removal of the restraint corresponding to X_a. This statement with regard to restraint should be understood in the following sense: If X_a signifies a bending moment at a point A, this modified structure should be assumed capable of a relative rotation, but not of a linear separation, of the two parts of it at this point; in other words, a pin is assumed to be inserted at the point A. On the other hand, if X_a signifies a shear, then the two parts of the structure are assumed capable of separating at the point A in the direction normal to the axis, while no tangential or angular relative movement is permitted at this point.

Eq. 45, proposed by Commander Wise, possesses the following advantages over the original Eq. 15:

1. It is simpler in form;
2. It illustrates the mechanical significance of the integrand, and thus can be computed by any suitable method, giving the values of M and m_a; and
3. It affords a better means of determining the null points of the integrand by equating to zero the expressions for M and m_a separately, rather than the product of these functions, which naturally has a more complex form.

Eq. 45 shows clearly why the variation line corresponding to the sum of two loadings acting simultaneously can be found by superposition of the two variation lines of the separate loadings. The reason for this relationship is that the expression m_a is the same in all three loadings, whereas the moment curve M of the combined loading is equal to the algebraic sum of the moment curves of the two separate loadings. The null points of the variation line of the combined loading, that come from m_a, are the same as in the separate loadings, whereas the null points produced by M are generally different in the three loadings considered.

The advantages of Eq. 45, demonstrated by Commander Wise and elaborated in the preceding paragraphs, make the inclusion of this equation into the theory of variation of static unknowns very desirable.

The counterpart of Eq. 45 applicable to structures with axial stresses has been given by Commander Wise in the form of Eq. 46. The significance of the normal stresses N and n_a in this equation is analogous to the significance of the moments M and m_a in Eq. 45.

Commander Wise suggests that the expression for the sensitivity coefficient K be modified by multiplying it by the ratio of the allowable and the actual unit stresses. The writer does not think that this modification would be of any advantage in the design, since the values of the variations need only be calculated for the stress quantities governing the design, in application to which the proposed modifying factor would always be not far from unity.

The writer is grateful to all the discussers for their contributions.

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BASIC ECONOMIC CONSIDERATIONS AFFECTING SINGLE AND MULTIPLE PURPOSE PROJECTS

By Otis M. Page,[1] M. Am. Soc. C. E.

With Discussion by G. B. Pillsbury, M. Am. Soc. C. E.

Synopsis

A practical analysis of a given condition is followed throughout the paper. The writer has defined an exemplary area and discusses data collected as a result of field investigations. The paper is subdivided into two sections in discussing single and multiple purpose projects.

SECTION I.—ANALYSIS OF DATA, FLOOD-CONTROL PROJECTS

1. Introduction

This subject is so complicated that scarcely any set rules are applicable in all cases. For this reason, an exemplification is made of what may be encountered in the collection of flood damage data, and in the determination of the benefits that may result from the construction of proposed flood protection works. The actual collection of data in the field by experienced investigators cannot be stressed too strongly. It matters not what interpretation may be placed on "direct" or "indirect" damages or benefits; if the basic data are not a true representation of facts, the ultimate answer may be much in error.

2. Direct Benefits

Direct benefits resolve themselves into one question: What benefit in dollars and cents will accrue to the owners and others affected? It does not make any difference whether the owners "cash in" now or wait and receive the benefits when such benefits materialize. They may assume many forms, depending on the adaptability of the particular ownership to benefit. A farm may have partial or total flood protection as a result of protective works; woodlands in either case may be improved from nonagricultural to agricultural lands and, likewise, properties unsuited for industries may be made proper locations for industries. The same condition holds true relative to residential and other uses. The item, "direct net flood losses prevented" on certain agricultural

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[1] Col., Corps of Engrs., U. S. Army, Care, Adjutant Gen., Washington, D. C.

properties is a function of benefit, but flood losses on properties cannot be capitalized where there are no flood losses. For example, lands may become industrial properties because of flood protection; or, as a result of flood protection, pasture or ranch lands may be converted to higher agricultural use. In many cases, woodlands and wild grass meadow pastures are benefited by occasional overflows. It may be noted, therefore, that the application of a single rule is not feasible. Different rules and methods must be applied to meet the conditions or, in other words, "every tub must stand on its own bottom." In this connection, every report should be a compendium of the actual field facts based on actual inspection by qualified field men. Guesswork, even on a preliminary examination, should not be accepted. It appears evident that direct benefits should be measured along the following general lines:

(*a*) Take credit for all direct prevented net flood losses (crops, buildings, livestock, or other tangibles), after flood protection, hereafter referred to as Item 2(*a*).

(*b*) Take credit for any additional direct net flood losses prevented that may ensue as a result of clearing or conversion of lands to a higher use after flood protection, hereafter referred to as Item 2(*b*).

Discussing these two benefits, Item 2(*a*) should be susceptible of close estimation by the use of proper field personnel making use of existing factual data; Item 2(*b*) involves an element of speculation or guess. A close study of the economic conditions, in general, in the affected area should be made. The unbiased opinions of reputable businessmen and farmers will be of great assistance. Under Item 2(*b*), abstract considerations are involved, and care should be taken to avoid the consideration of Utopian ideas. Credit should not be taken for benefits from developments that could occur only after flood protection is provided as an item of flood losses prevented. The sum total of remaining benefits in the form of flood losses prevented will be taken care of under Item 2(*b*). Simplification of considerations is important. Field men should not be asked to perform the impossible and draw fine-cut lines, thereby producing a distribution that serves no useful purpose.

3. Calculation of Losses and Benefits

Under Item 2(*a*), all flood losses may be reduced to an annual basis and this, in turn, may be capitalized for the increment of value as a result of flood losses prevented. The rate of capitalization should be variable to reflect risks accurately. For example, an area with only partial flood protection will carry a higher rate than one with complete flood protection. Furthermore, this phase of the situation deals with a farmer's or businessman's dollar and should carry a higher rate than consideration of public funds. Consideration is given now to benefits to individuals. For example, *A* applies to a Land Bank or other agency for a loan. His average annual net income is $500 from his farm or property. If $500 is capitalized at 5%, $10,000 is the value of his farm. Does the Land Bank or loaning agency value this farm or property at

$10,000? In all probability it does not, as it will use from 6% to 15% as a rate for capitalization, making the value range from $3,333 to $8,333, which is more in line for a property with a net income of $500 per year. There is an economic consideration to be given capitalization rates that cannot be ignored; and 10% appears to be normally about the correct rate to apply if abnormal hazards are not involved. This discussion refers to increased value by capitalizing the value of prevented flood losses. In applying capitalization to obtain the increment of increase in value, the entire operation must be analyzed to determine exactly what this prevented loss means to the individual in net gain on an operation. (This is a long discussion within itself and will be omitted here.)

Under Item 2(*b*), "annual flood losses prevented" do not apply, and to apply would be erroneous. The benefit must be taken care of by anticipated enhancement in value. It makes no difference whether the owner sells or holds his property, the problem is to determine the present worth of this increase in value. This will be the "benefit." It represents a "capital sum," and it can be reduced to an annual benefit. The question that may present itself is: What about prevented flood losses on lands converted to higher use as a result of flood protection? For instance, a factory or dwelling is built on property falling under Item 2(*b*), and this property is protected from flood losses as a result of the project. In the first place, the owner purchased or already possessed the land. If he purchased it, he paid the increased benefit price obtained as a result of flood protection, and the seller received a capital benefit. If he happened to be the original owner under the project, he has reaped a benefit in like measure. Pursuing the question further: The factory or house is benefiting by the flood protection. However, the house or factory owner has paid the price he would pay in a flood-free area. Therefore, any increased benefits to this house or factory as a result of prevented flood loss must be in the indirect classification. Otherwise, there will be duplication. It should be borne in mind, also, that enhancement due to change in land use that would have occurred if no protection or additional protection had taken place cannot be considered as a project benefit. Although, for the purposes of exemplification, a case is used in which the trend justifies the consideration of future development, no credit is taken nor should be taken on this development for flood losses prevented or for capital gains. Schedule *B*, the "as is, where is" value which represents the average value of the property without flood protection for the same period as the period of proposed flood protection, is used as the basis for determining the proposed project benefits. Observation has been rather conclusive that, in areas in need of flood protection, developments are usually at a standstill, or going backward. It is believed that a *status quo* condition is likely to hold true in most cases in future years as the population of the United States is expected to reach a stationary level within twenty or twenty-five years. Further, the trend toward the elimination of marginal and submarginal lands from private use will probably continue. Many other valid reasons can be enumerated.

4. SUMMARY

Direct benefits equal flood losses prevented, plus present worth of enhancement brought about by changed use or potential changed use; or, direct benefits = Item 2(*a*) + Item 2(*b*).

5. EXAMPLE OF APPLICATION

TABLE 1.—AVERAGE CLASSIFICATION (SCHEDULE *A*) OF 640 ACRES IN A PROPOSED FLOOD-CONTROL AREA

No.	No. of acres	Unit value (dollars per acre)	Value (dollars)
a	100	90.00	9,000
b	100	70.00	7,000
c	100	30.00	3,000
d	100	12.50	1,250
e	140	7.50	1,050
f	100	4.00	400
g	Buildings and improvements		6,300
	640	Total value	28,000

TABLE 2.—AVERAGE CONDITIONS OF THE PROPERTY WITHOUT FLOOD PROTECTION (SCHEDULE *B*)

No.	Description	Acres	Dollars per acre	Total value (dollars)
a	Cleared and in cultivation, subject to slight flood damage	100	100	10,000
b	Cleared and in cultivation, subject to considerable flood damage	120	75	9,000
c	Cleared and in cultivation, subject to heavy flood damage	180	45	8,100
d	Pasture land, largely meadow; cleared but too severely flooded to cultivate	100	15	1,500
e	Woodland; can be cleared (potential) but has severe flood hazard	40	10	400
f	Nonagricultural woodland; too low to drain, or has poor soil; cannot be considered agricultural land if protected; a timber-growing or pasture proposition, or both	100	5	500
g	Building and all-improvement value on entire tract	..	..	6,500
	Total value	..	..	36,000

An exaggerated case is taken to attempt to emphasize various considerations in the foregoing discussion. Taking the case of an area of 640 acres owned by an individual or group of individuals in a proposed flood-control area, the "as is, where is" classification and value of this area are given in Table 1. It is entirely probable that conditions may be such that the present "as is, where is" setup should be used as the basis for determining the proposed project benefits. As previously expressed, one is injecting an element of guess or prophecy in estimating a development trend for the area since he must know what periods in the past will be representative of the future. Fifty years hence is a long time to project a curve of this nature, with the uncertain or vague data as to what has happened in the past. Fluctuations in real estate and commodity prices are beyond accurate estimation. It might be emphasized further that agricultural development during the past eight years has been confined generally to new areas, whereas the trend points toward retrogression in the older areas. However, in the case being exemplified, the development trends exist, and they are reflected in Table 2, which represents the average conditions of the property without flood protection for the same period as the proposed project life, which is fifty years in this case. This schedule is the appraisal of the average value of the property without flood protection for the same period as the period of proposed flood protection, and it is used as the basis for determining the proposed project benefits. Since complete flood protection is offered in this case, the following conclusions are reached from a study of the area in Schedule *B*:

Item a.—All of this area, when protected, will receive benefits through direct flood losses prevented.

Item b.—Due to very favorable location, 20 acres of this area, when protected, will have great potentiality for suburban home sites and will convert to higher use. Therefore, 20 acres will receive a capital gain, leaving 100 acres to receive benefits through direct flood losses prevented.

Item c.—Due to location and shipping facilities, 20 acres of this area, when protected, will have great potentiality as industrial property. These 20 acres will receive a capital gain, and the remaining 160 acres will receive benefits through direct flood losses prevented.

Item d.—This area has only a small flood loss and, when protected, it will remain in *status quo* as pasture land, because the owner requires that much acreage in pasture. Due to the nature of the soils and to insufficient drainage facilities, pasturage is its highest and best use. By the elimination of flood hazards, an increase in the number of livestock ranging on this land can be expected. To use both of these benefits would be duplication. The greater benefit (in this case, benefit through flood losses prevented) should be taken.

Item e.—No appreciable flood losses occur on this area; however, when protected, it will receive a capital gain due to flood protection, because the protection will render it more suitable for cultivation and will improve logging conditions in case the owner retains it in woodland. Consequently, the market value will be increased, thereby effecting a capital gain.

Item f.—On this area flood losses are virtually nothing, and due to soils and the lack of drainage facilities, the land is nonpotential or nonagricultural and will remain in this classification after flood protection. However, when protected, the market value will be increases as a timber-growing proposition, due to more favorable logging conditions, and the area will be more valuable as a woods range for livestock. This area, therefore, will receive a small capital gain.

Item g.—These improvements will create benefits either as capital gains or as flood losses prevented; the greater benefit should govern. In this case, the buildings are located principally on high ground. Little use is being made of some of the buildings, and their value to the farm is little more than salvage value. When the area is protected, these improvements will convert the land to maximum use. Consequently, their value to the farm will increase; this will be a capital gain. This example is selected for exemplification, and it is, no doubt, an unusual case. In most instances, Item *g* would receive a greater benefit from flood losses prevented, putting it under Item 2(*a*), than from capital gain.

From the foregoing conclusions, the appraisal of the average value of the property for the period of protection is given in Table 3. The increase in the value of the area of Items *a*, *b*, *c*, and *d* of Schedule *C* (Table 3) over the value of the same acreage of Schedule *B* (Table 2) is due to flood losses prevented, and the increase in the value of the remaining area of Schedule *C* over the value of the same acreage of Schedule *B* is due to enhancement in value as a result of changed use or potential changed use. These differences are indicated in Table 4.

TABLE 3.—AVERAGE VALUE OF PROPERTY FOR A GIVEN PERIOD OF PROTECTION (SCHEDULE *C*)

No.	Acres	Dollars per acre	Value (dollars)
a	100	110.00	11,000
b	100	100.00	10,000
c	160	65.00	10,400
d	100	17.50	1,750
e	40	12.00	480
f	100	6.00	600
g	20	200.00	4,000
h	20	175.00	3,500
i	Buildings and improvements		7,170
	640	(Total)	48,900

The enhancement in Items *a*, *b*, *c*, and *d* of Table 4 may not always represent annual flood losses capitalized. The average annual flood losses prevented must be determined from data secured in the area under consideration. In this case, the sum is estimated to be $625.00 (see heading "Summary"). By the construction of proposed flood-control works, the area considered would receive an annual benefit of $625 through average annual flood losses prevented and a total capital gain through flood protection of $5,950. Assuming the life of the project to be fifty years, the present worth of this capital gain of $5,950 would be $5,291.27, using 4.5% rate of interest. The annual benefit of $5,291.27, at 4.5%, would be $238.11. The total direct annual benefit, therefore, would be $625.00 plus $238.11, which equals $863.11.

TABLE 4.—DIFFERENCES DUE TO FLOOD LOSSES PREVENTED, AND INCREASE IN VALUE OF REMAINING AREAS

No.	Enhancement by Prevented Flood Losses		Capital Gain	
	Acreage	Dollars	Acreage	Dollars
a	100	1,000	....	....
b	100	2,500	....	....
c	160	3,200	....	....
d	100	250	....	....
e	....	....	40	80
f	....	....	100	100
g	....	....	20	2,500
h	....	....	20	2,600
i	....	....	....	670
....	460	6,950	180	5,950

6. DIRECT FLOOD LOSSES (OTHER ITEMS)

Other items of flood damage are numerous, such as damage to livestock, seed, feed, and implements, the cost of exodus and reoccupation, damage to business operation through curtailment, inability to fulfil contracts, increased overhead, and damage to chattels of all kinds not heretofore mentioned. In some cases, there may be labor losses but, in such cases, considerable care should be taken not to duplicate this item with indirect losses applying to the area as a whole. The aforementioned losses usually may be cataloged and treated under Item 2(*a*) (flood losses prevented). However, other than farming operations, these losses can best be handled separately and may (and usually do) consist of both Item 2(*a*) (flood losses prevented) and Item 2(*b*) (outright capital gains as a result of flood protection). Other direct flood losses may consist of physical damage to highways, railroads, and utilities of all kinds and also, when justified, cost borne by federal, state or quasi-public or private agencies in flood fighting and relief work of all kinds, including exodus and reoccupation. Ordinarily, these losses will fall under Item (2*a*) (flood losses prevented).

7. Indirect Flood Losses

No doubt more care should be exercised in the evaluation of indirect flood losses than in dealing with direct losses. Certainly, more experience is required, because the appraiser must have a clear economic picture of the area as a whole. In no sense can justification be found whereby an arbitrary percentage of direct flood losses can be taken as a correct value for indirect losses. Usually, these losses are not inventoried as easily as direct losses where there are factual data available as a result of prior floods. In most cases, the principal items would consist of: (1) Inconvenience caused by disruption of the functioning of railroads, highways, and utilities with the attendant monetary losses; (2) economic losses to the community as a whole on account of business cessation or disruption not specifically included in direct losses; (3) any increased cost of medical care not included in indirect losses; and (4) labor losses as affecting the community as a whole and not cataloged under direct losses. Consideration should be given to the cost of extra labor used in rehabilitation as an offsetting factor. Contiguous and sometimes remote areas may be materially benefited. However, as a result of a project, consideration should be given to the economic effect of taking large areas out of productive use, and

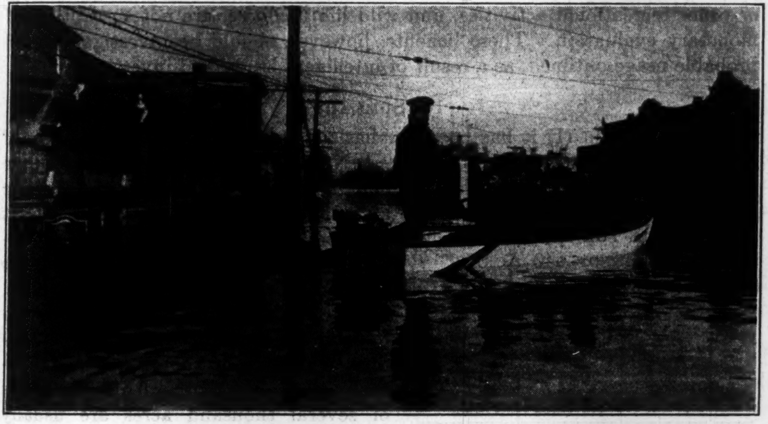

Fig. 1.—Among Indirect Flood Damages Are the Economic Losses to the Community as a Whole on Account of Business Cessation or Disruption

out of taxation. If credit is fully taken for community benefits in indirect benefits, it would appear that proper charge should be made at least for any reimbursement the federal government would make to political units as a result of property taken out of taxation. One can readily see that real field fact-finding must be resorted to in order to form any kind of intelligent estimate of this situation which, for purposes of convenience, is placed under the same heading as "Indirect Losses." The benefit is more suitably cataloged under "Capital Gains."

8. INTANGIBLE LOSSES

Certain flood losses or inconveniences are not susceptible of evaluation in dollars and cents. In this category may be loss of human life and the abnormal spread of disease affecting the health and happiness of inhabitants, disruption of schools and religious assemblies, burial of the dead, and, perhaps, many other inconveniences of more or less importance to a specific community.

9. INTANGIBLE BENEFITS

Benefits accruing to the community, county, state, or nation, as a result of project works according recreational facilities such as hunting, fishing, bathing, or other recreational activities, and wild life facilities, are not susceptible of monetary evaluation. These benefits, however, should be investigated, and probable usage outlined, as a result of intelligent study and investigation.

10. SUMMARY

In this section (I) it has been emphasized that the accurate collection of field data is of paramount importance. No detailed discussion of technique and method is outlined as this is quite an extended discussion within itself. It may be well to mention here that ordinarily a much greater zone or segment of a proposed project would be studied as an entity than is exemplified in the foregoing, depending upon varying conditions of terrain as related to topography, culture, and other features. Belts of several thousand acres are usually inspected in the field and data are compiled, using the most convenient and applicable methods. Technique and method used in the field will be largely governed by maps and other data available and the degree of accuracy required in the findings. It is emphasized that the damage and valuation figures presented in Section I must be based actually on field data obtained by competent personnel.

TABLE 5.—ANNUAL DIRECT BENEFITS FOR EACH CLASSIFICATION OF 640 ACRES EXEMPLIFIED

No.	Acreage	Average Annual Benefits Through:	
		Losses prevented	Capital gain
a	100	$ 85.00	$ 40.02
b	100	225.00	100.05
c	160	300.00	128.06
d	100	15.00	10.00
e	40	0	3.20
f	100	0	4.00
g	20	45.00	100.05
h	20	37.50	104.05
i	....	20.00	26.81
....	640	$727.50	$516.24

Annual direct benefits of $625 through flood losses prevented were determined from field data. Table 5 gives the annual direct benefits through flood losses prevented and through capital gain for each classification of the 640 acres exemplified. It shows that the benefits through flood losses prevented for Items *a*, *b*, *c*, and *d* are greater than the benefits through capital gain, whereas the benefits through capital gain for Items *e*, *f*, *g*, *h*, and *i* are greater than the benefit through flood losses prevented. It has been emphasized, throughout the analysis, that the greater benefit should govern. Consequently, the total average annual direct benefits would be as follows:

Average annual direct benefits due to:	
Flood losses prevented = Items *a*, *b*, *c*, and *d*	$625.00
Capital gain = Items *e*, *f*, *g*, *h*, and *i*	238.11
Total average annual direct benefits	$863.11

Conclusion.—On the area of 640 acres exemplified, flood protection would result in a direct annual benefit of $625 and a total capital gain of $5,950. Assuming the life of the project to be 50 years, the present worth of the capital sum enhancement of $5,950, using 4.5% rate of interest, would be $5,291.27. The annual benefit of this sum would be $238.11. The total annual direct benefit, therefore, will be $625.00 plus $238.11 which equals $863.11. Before estimating indirect benefits, all of the area affected by the project must be considered. Consequently, it is not practical to make estimates of the indirect benefits on the 640 acres.

SECTION II.—MULTIPLE PURPOSE PROJECTS

Section I discusses factors affecting flood-control projects. It is possible, of course, to encounter projects wherein flood control may not be of primary importance and the other features—irrigation, power, and navigation—may all be considered in determining economic justification. The degree of flood protection afforded must be considered, of course, when combined with irrigation, power and stream regulation or navigation. For this reason it would complicate the subject, no doubt, to pursue a synthetic analysis of the property taken for exemplification under a flood-control project as outlined in Section I. The same reasoning would apply, however, in determining Schedules *A* and *B*, Section I, as heretofore. In all probability, Schedule *C* would assume a different form, depending upon how the property would be affected by the project purposes.

11. DIRECT BENEFITS

The same principles will apply in a multiple purpose project as in a single purpose project—there may be both annual direct benefits and capital gains. It must be borne in mind that duplication should be avoided and that the aggregate of the benefits from whatever sources cannot exceed the intrinsic worth of the increased value of the property due to the construction of the project works.

12. INDIRECT BENEFITS

Indirect benefits will be more far-reaching under multiple purpose projects as greater outlying areas will be affected. Plentiful and cheap power, of course, has far-reaching benefits. Water for irrigation purposes may convert large areas of cheap range lands into areas of intensified farming, bringing in settlers with all the attendant community improvements. Numerous other indirect benefits could also be enumerated.

13. Intangible Losses and Benefits

The same general principles will apply to the consideration of intangible losses and benefits as are discussed in Section I.

14. Acknowledgment

The opinions expressed in this paper are entirely the writer's responsibility and have no relationship to the official directives of any agency of the federal government. It is desired to acknowledge the services of H. E. Cox, Land Appraiser, of the Southwestern Division, U. S. Engineer Department, in checking mathematical calculations and collaborating on economic considerations.

DISCUSSION

G. B. Pillsbury,[2] **M. Am. Soc. C. E.**—A procedure is presented in this paper for evaluating the economic benefits from proposed projects intended wholly or partly for flood prevention. The applicability of the procedure obviously depends upon the purpose that the investigation is to serve. An evaluation of flood-prevention benefits conceivably might be made for a large landholder, or group of landowners, to determine whether the construction of works for the protection of the property would be profitable as a private enterprise; or it might be undertaken for a municipality or flood control district to determine whether the local benefits would justify the local taxpayers in undertaking flood protection works; or it may be undertaken to determine the economic justification from the viewpoint of the adoption of a federal flood control or multiple purpose project. The procedure presented in the paper evidently has in view this third purpose.

A prudent, or even an imprudent, private owner would be moved principally by the anticipated enhancement of the value of his property in deciding upon the advisability of investing his own funds in flood control works. This enhancement might be credited with the flood losses prevented, which are listed as Item 2(*a*) in the paper, as well as with the more profitable utilization of the land after protection, listed as Item 2(*b*), but the latter is usually the compelling consideration. In any case, he would scarcely undertake any considerable expenditure unless at least the paper profits showed a wide margin over the estimated cost. The close figuring shown in the author's example ordinarily would not have much financial appeal.

Flood control works, whether in the nature of storm drains or more extensive enterprises, ordinarily are undertaken by municipalities or local flood control districts on broad grounds of public necessity or convenience, rather than because of a detailed evaluation of financial benefits. Generally, such works are adopted either to rectify an inconvenient or intolerable situation that has grown up because of municipal development, or to prevent the recurrence of a recently experienced flood disaster. Such works are built to satisfy an insistent public demand, and a close figuring of financial benefits is not often required. In exceptional cases, to be sure, legal requirements may entail the assessment of benefits against the properties affected, and the considerations presented by the author might be applicable to the preparation of the assessment roll.

In the investigation of a proposed federal flood control project, an evaluation of the benefits is required by the Flood Control Act of June 22, 1936, in which Congress laid down the policy:

> "* * * that the Federal Government should improve or participate in the improvement of navigable waterways or their tributaries, including watersheds thereof, for flood control purposes, if the benefits to whomsoever they may accrue are in excess of the estimated costs and if the lives and social security of people are otherwise adversely affected."

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[2] Brig.-Gen., Corps of Engrs., U. S. Army (*Retired*), Ross, Calif.; formerly Asst. Chf. of Engrs.

Sound policy undoubtedly requires that public works be undertaken only when the benefits exceed the estimated cost; but these benefits are not necessarily, nor even generally, material ones which can be evaluated by an engineering computation. The more important works contemplated by the Federal Flood Control Act are those for the prevention or mitigation of a recurrence of destructive floods which have caused loss of life as well as of property, have produced widespread inconvenience and discomfort because of the interruption of rail and highway traffic and other public services, and have inflicted suffering and misery upon the considerable number of persons who were driven from their homes. The writer does not believe it to be the intent of Congress that the justification for such projects should rest on a nice evaluation of the material benefits, but rather that the rule of reason be applied to determine, on general grounds, whether the value of benefits which cannot be definitely computed in money will exceed the estimated cost.

On the other hand, the original Flood Control Act included the authorization of a number of projects intended to remove a continuing flood hazard to agricultural lands on which the local inhabitants had exhausted their resources in attempted protection, or were unable to raise funds for the necessary works. The procedure described by the author would appear particularly applicable to the investigation of proposed projects of this character.

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ANALYTICAL AND EXPERIMENTAL METHODS IN ENGINEERING SEISMOLOGY

BY M. A. BIOT,[1] ESQ.

WITH DISCUSSION BY MESSRS. GEORGE R. RICH, N. J. HOFF, MERIT P. WHITE, N. H. HECK, FRANK NEUMANN, JACOB FELD, GEORGE W. HOUSNER, HOMER M. HADLEY, ROBERT E. GLOVER, AND M. A. BIOT.

SYNOPSIS

The arithmetical steps in the computation of the spectrum are extremely lengthy. A mechanical analyzer was developed by the author at Columbia University, in New York, N. Y., to avoid this numerical work.[2] Some of the writer's earlier work with the earthquake spectrum is reviewed briefly in this paper. It also stresses engineering applications, and presents some new results, in particular regarding the effect of the foundation. Sections 1 and 2 introduce the definition of earthquake spectrum and show the results obtained for various earthquakes with the mechanical analyzer. Section 3 is a treatment of the spectrum curves obtained with the analyzer in relation to some observed facts and to the problem of stress prediction in actual structures. Section 4 considers examples of structures with more than one degree of freedom and shows how the stresses may be computed by means of the effectiveness factor. (The expression "efficiency factor" instead of "effectiveness factor" was used in the previous paper.[2]) The danger of a phenomenon referred to as the "whip effect" is also pointed out. Some attention has been given to another aspect of the problem in section 5; the effect of the foundation on the rocking motion of a rigid structure is taken into account. It is shown that in this case the same methods using a spectrum and effectiveness factors can still be applied by introducing an additional degree of freedom and a natural period corresponding to the rocking motion on the foundation.

INTRODUCTION

To a great extent the design of earthquake resistant structures is still an art based on observational facts and experience. Development of experimental

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NOTE.—Published in January, 1942, *Proceedings*.

[1] Research Associate in Aeronautics, California Inst. of Technology, Pasadena, Calif.; Asst. Prof. of Mechanics, Physics Dept., Columbia Univ. (on leave of absence).

[2] "A Mechanical Analyzer for the Prediction of Earthquake Stresses," by M. A. Biot, *Bulletin*, Seismological Soc. of America, Vol. 31, No. 2, April, 1941, pp. 151–171.

and analytical approaches has been very slow partly because of the lack of accurate information on the accelerations of strong-motion earthquakes and partly because of the great complexity of the phenomena involved. This gap between the empirical and scientific approaches is being reduced constantly, leading to improvement in codes and rules for the design of quake-resistant structures. Information on strong-motion earthquakes has been made available in recent years by the valuable work of the U. S. Coast and Geodetic Survey. Pooling the data furnished by the earthquake accelerograms with existing seismological knowledge only marks the beginning of analytical and experimental investigation. To picture the phenomenon in all its complexity one must imagine that the structure to be analyzed is floating on a medium in which highly irregular waves are propagating. The structure has the attributes of distributed elasticity and damping effects. Like a ship in the ocean it does not participate completely in the motion of the surrounding medium, its motion being dependent upon its own rigidity and mass and on its size relative to the waves. Internal friction and yield point in the surrounding soil must have an important effect on resonance phenomena. Also the properties of the surface layers of the earth vary greatly with location and depth so that complicated reflection, refraction, and diffraction of the waves must be expected. This is also true for agglomeration of buildings in which case the structures themselves must have a considerable influence on the intensity and direction of the waves.

Instead of approaching this problem as an entirely complex matter it must rather be expected that a solution will emerge gradually from the careful analysis of simplified cases in which the influence of each individual factor is clearly defined and checked critically against observation.

One of the simplifications usually introduced is the assumption that the ground behaves as a shaking table, the horizontal motion of which is taken to be the same as that derived from the horizontal accelerogram of an earthquake. A basic analytical approach to this problem was developed by the author in 1932,[3, 4, 5] in which the concept of the earthquake spectrum was introduced. This is a curve characteristic of a given earthquake which gives some kind of periodicity content by associating a certain acceleration intensity with a given period. It was shown in the earlier work how this curve could be computed and how use of it could be made for the acceleration of earthquake stresses. To this purpose, the motion of the structure is considered as the superposition of its various modes of vibration and the maximum stress produced in each mode is made to depend on a coefficient characteristic of the structure and of that particular mode. This coefficient in the present paper is referred to as the effectiveness factor. The procedure permits the easy comparison between various earthquakes and between various types of structures or modes within these structures as to the stresses produced by a given earthquake.

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[3] "Transient Oscillations in Elastic Systems," by M. A. Biot, *Thesis No. 259*. Submitted to the Aeronautics Dept., California Inst. of Technology, Pasadena, Calif., in 1932 in partial fulfilment of the requirements for the Degree of Doctor of Philosophy.

[4] "Theory of Elastic Systems Vibrating Under Transient Impulse with an Application to Earthquake Proof Buildings," by M. A. Biot, *Proceedings*, National Academy of Science, Vol. 19 (1933), pp. 262–268.

[5] "Theory of Vibration of Buildings During Earthquakes," by M. A. Biot, *Zeitschrift für angewandte Mathematik und Mechanik*, Bd. 14, H. 4, (1934), pp. 213–223.

Applications of the concept of earthquake spectrum,[6] using the writer's analytical method, have been made by R. R. Martel, M. Am. Soc. C. E., and M. P. White, Assoc. M. Am. Soc. C. E., and experiments along the same line have been made by L. S. Jacobsen and N. J. Hoff.[7]

Notation.—The letter symbols in this paper are defined where they are first introduced and are assembled for reference in the Appendix.

I.—Effect of an Earthquake on a Structure with One Degree of Freedom

Consider a mass m connected to the ground by weightless springs (Fig. 1). The horizontal displacement of the mass relative to the ground is denoted by u, and the spring rigidity is such that a horizontal force F produces a displacement

$$u = \frac{F}{k} \qquad (1)$$

The constant k is called the "spring constant."

If the ground is given a horizontal acceleration a_0 applied very gradually so that no transient oscillation occurs, the mass will assume a constant deflection:

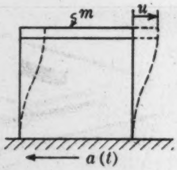

FIG. 1

$$u_0 = \frac{m\, a_0}{k} \qquad (2)$$

The total shear in the springs is then

$$V = m\, a_0 \qquad (3)$$

During an earthquake the horizontal acceleration is a function $a(t)$ of the time t. Denoting by v the displacement of the ground and neglecting the damping, the equation of motion of the mass m is

$$m \frac{d^2}{dt^2}(u + v) + k\, u = 0 \qquad (4)$$

The displacements u and v are taken positive to the right and the acceleration is taken positive to the left; hence

$$\frac{d^2 v}{dt^2} = -a(t) \qquad (5)$$

and Eq. 4 may be written

$$m \frac{d^2 u}{dt^2} + k\, u = m\, a(t) \qquad (6)$$

Eq. 6 shows that the relative displacement u obeys the differential equation of motion of a simple oscillator under the force

$$m\, a(t) = F(t) \qquad (7)$$

The earthquake is taken to start at the instant $t = 0$; the mass m being initially at rest, the relative displacement u as a function of time is given by the well-known solution[8]

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[6] "Some Studies on Earthquakes and Their Effects on Structures," by R. R. Martel and M. P. White, Rept. on Earthquake Studies for Los Angeles County, Pt. I (1939) (unpublished).

[7] *Ibid.*, Pt. II.

$$u = \frac{1}{\sqrt{k\,m}} \int_0^t F(\theta) \sin \sqrt{\frac{k}{m}}\,(t - \theta)\, d\theta \quad \text{(8a)}$$

or

$$u = \frac{T}{2\,\pi} \int_0^t a(\theta) \sin \frac{2\,\pi}{T}\,(t - \theta)\, d\theta \quad \text{(8b)}$$

in which: $F(\theta)$ = force; θ = time variable of integration; and, T, the natural period of the oscillator, equals

$$T = 2\,\pi \sqrt{\frac{m}{k}} \quad \text{(9)}$$

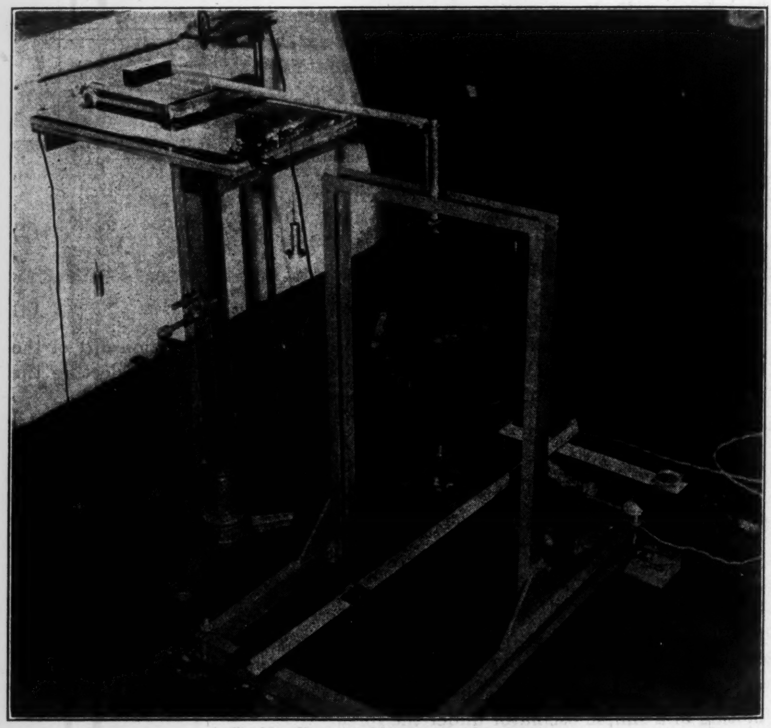

FIG. 2.—VIEW OF MECHANICAL ANALYZER

The total shear in the springs is

$$V = k\,u = m\,\frac{2\,\pi}{T} \int_0^t a(\theta) \sin \frac{2\,\pi}{T}\,(t - \theta)\, d\theta \quad \text{(10)}$$

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[8] "Mathematical Methods in Engineering," by Theodor von Kármán and M. A. Biot, McGraw-Hill Book Co., Inc., New York, N. Y., 1940, p. 404.

The quantity:

$$f(t) = \frac{2\pi}{T}\int_0^t a(\theta)\sin\frac{2\pi}{T}(t-\theta)\,d\theta \ldots\ldots\ldots\ldots\ldots (11)$$

is a function of time which gives the complete stress history of the oscillator. That is, if the integration is performed with respect to θ between the limits 0 and t, and repeated for all values of t, a function of time is obtained which, according to Eq. 10, will give the value of the total shear V at every instant t.

II.—Earthquake Spectrum Evaluated by Means of a Mechanical Analyzer

It is of special interest to find the maximum value of the stress produced by a given earthquake. Denoting by A the maximum value of Eq. 11, the maximum shear is written

$$V_{\max} = m\,A \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

Comparing with Eq. 3, one may say that, as far as the maximum shear is concerned, the effect of the earthquake is equivalent to that of a constant acceleration A applied gradually at so slow a rate that only a statical deflection is produced without the occurrence of any transient oscillation.

Of course for a given earthquake the value of A depends on the parameter appearing in Eq. 11—that is, on the natural period T of the structure. The quantity A is referred to as the "effective acceleration" of the earthquake for the period T. It will be noticed that the effective acceleration for a particular earthquake depends only on the period of the oscillator. Therefore, one may evaluate this effective acceleration for various oscillator periods and consider it to be a characteristic function $A(T)$ of the period for each particular earthquake. This function is called the "acceleration spectrum" of the earthquake. (Use has sometimes been made of the expression "equivalent acceleration" for "effective acceleration," and "oscillator response curve" or "influence line for horizontal shear" instead of "spectrum.")

The engineering significance of this concept lies in the fact that, once the spectrum is known, it is possible to write immediately the value of the maximum shear produced by the earthquake on any undamped, one-degree-of-freedom, structure. To obtain the shear produced by an earthquake in such a structure of period T the mass of the structure is multiplied by the ordinate of the spectrum for the particular value T of the abscissa. Furthermore, as will be shown, it is possible to extend the usefulness of the spectrum to structures much more complicated than the one-degree-of-freedom oscillator considered herein.

It is relatively tedious to evaluate the spectrum by analytical methods, as this would involve the calculation of the integral (Eq. 11) from a graphically given accelerogram $a(t)$ and for a great number of values of both T and t. Fortunately there are simple experimental methods by which this can be done.

A mechanical analyzer shown in Fig. 2 has been developed for this purpose. A detailed description of the apparatus was given in a previous publication.[3] It is a torsion pendulum with variable tuning whose point of suspension can be made to turn proportionally to the acceleration of the earthquake. When the pendulum is tuned for the period T, it can be shown that its maximum amplitude yields the value of A. By varying the tuning it is then possible to plot the spectrum $A(T)$ point by point. The main advantages of this type of analyzer are its low cost, its simplicity of operation, and the fact that it takes an average of eight hours to plot one spectrum curve. The use of a torsion pendulum at the Bureau of Reclamation to evaluate earthquake stresses was mentioned by J. L. Savage, Hon. M. Am. Soc. C. E., in 1939.[9]

The following earthquakes have been analyzed (their spectrum curves are plotted in fractions of gravity g against period in seconds):

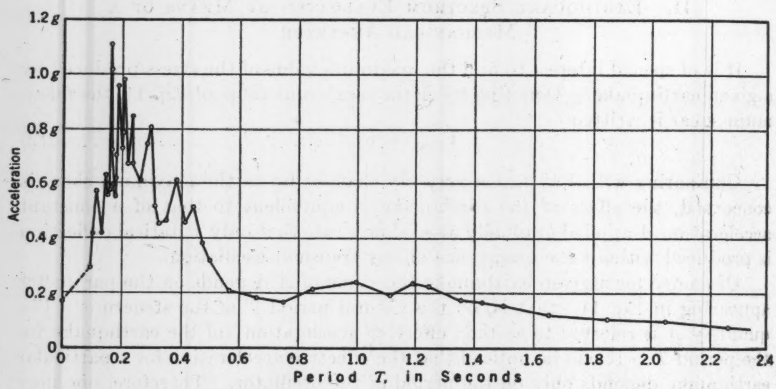

FIG. 3.—SPECTRUM OF EAST-WEST HORIZONTAL ACCELERATION OF THE EARTHQUAKE OF HELENA, MONT., OCTOBER 31, 1935

(A) Helena, Mont. (October 31, 1935) a horizontal east-west acceleration. The spectrum in Fig. 3 shows that a peak value of 1.05 g occurs for $T = 0.16$ sec. The maximum recorded acceleration of the earthquake is 0.16 g; and the amplification due to resonance is $\frac{1.05}{0.16} = 6.5$ times this value.

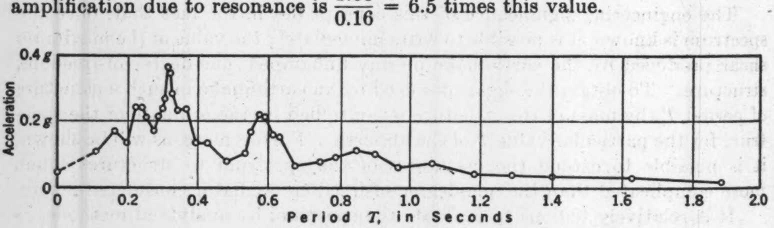

FIG. 4.—SPECTRUM OF THE NORTH-EAST HORIZONTAL ACCELERATION OF THE EARTHQUAKE OF FERNDALE, CALIF., FEBRUARY 6, 1937

(B) Ferndale, Calif. (February 6, 1937) a horizontal northeast acceleration. The spectrum in Fig. 4 shows that this is a minor earthquake. Its maximum intensity is 0.039 g, and the amplification is 9.5.

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[9] "Earthquake Studies for Pit River Bridge," by J. L. Savage, *Civil Engineering*, August, 1939, pp. 470–472.

(*C*) Ferndale (September 11, 1938) horizontal accelerations in both northeast and southeast directions. The maximum recorded intensity is 0.17 g in the northeast direction and the amplification is 6. Both spectrums are found to be analogous to (*A*).

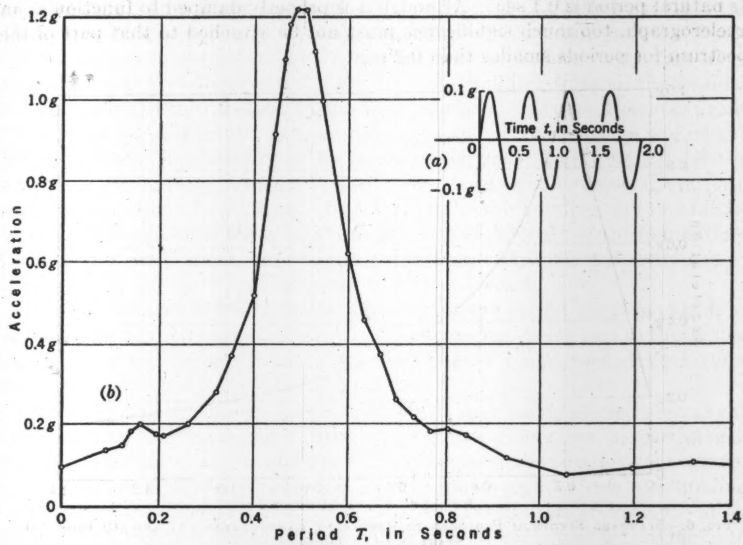

FIG. 5.—SPECTRUM OF AN ARTIFICIAL EARTHQUAKE CHARACTERIZED BY A SINUSOIDAL ACCELERATION

(*D*) A sinusoidal earthquake (Fig. 5(*a*)) of intensity 0.1 g and a duration of four cycles of 0.5 sec each was analyzed. Its spectrum is plotted in Fig. 5(*b*). The peak value of 1.23 g obtained is in good agreement with the theoretical value 1.25 g.

III.—SPECTRUM RELATED TO DESIGN AND ACTUAL BEHAVIOR OF STRUCTURES

It is apparent that, from the viewpoint of the designer, the individual sharp peaks in the spectrums are unimportant since their frequencies do not seem to be characteristic constants of the location as found by a comparison of the two Ferndale earthquakes (*B*) and (*C*) (see section II). The envelope of the spectrum, or better still, the envelope of a collection of spectrum curves obtained at the same location, constitutes the basic information for design purposes. A simple spectrum such as that plotted in Fig. 6 might well be taken to represent the Helena and Ferndale earthquakes ((*A*) and (*C*), section II). For $T > 0.2$ sec, the curve is chosen as the hyperbola

$$A = \frac{0.2\,g}{T} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

which emphasizes in quantitative form the fact that buildings of longer periods undergo smaller stresses. In general, therefore, high buildings will be less vulnerable to earthquakes than the smaller structures with shorter periods. The accelerograms from which these spectrum curves have been derived were recorded with the instrument designed by the U. S. Coast and Geodetic Survey. Its natural period is 0.1 sec. Although it is properly damped to function as an accelerograph, too much significance must not be attached to that part of the spectrum for periods smaller than 0.2 sec.

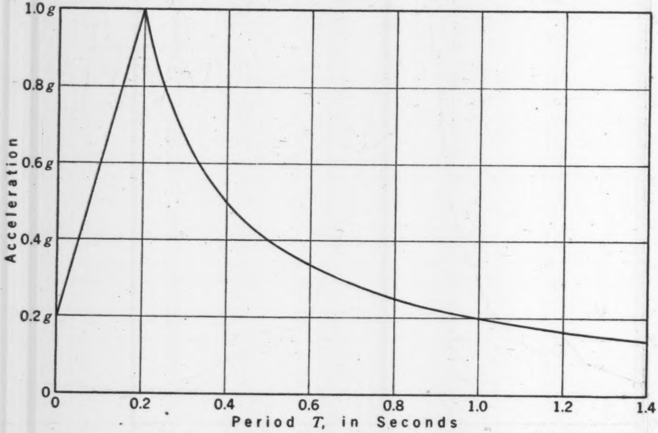

FIG. 6.—STANDARD SPECTRUM PROPOSED TO REPRESENT EARTHQUAKES (A) AND (C) FROM THE STANDPOINT OF DESIGN

Before proceeding any further it is important to give more careful consideration to the possible engineering interpretation of the foregoing results. The spectrums for earthquakes (A) and (C) indicate that an undamped structure with a period of approximately 0.2 sec would undergo a horizontal shear equal to its own weight. From observation of the effects of earthquakes this value seems rather high, but it must be remembered that it constitutes an upper limit and that actually a number of stress-reducing factors enter into play.

One of these factors, of course, is the damping influence of the structure. The magnitude of this damping effect must depend very much on the nature of the structure and the amplitude of the stresses. In fact, the damping observed with the aid of building vibrators or during minor earthquakes may very well be small, but one must be prepared to discover that in a strong earthquake the damping is considerably greater. When the amplitude of the stress reaches the yield point in some part of a structure, plastic deformation and friction will produce a high degree of energy dissipation. Assuming, for instance, that this type of damping occurs as soon as the spectrum ordinate is greater than 0.2 g, any further increase of stress by resonance will be strongly counteracted. That structural damping is an important factor agrees with the conclusion of studies by Professor White.[10]

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[10] "Friction in Buildings: Its Magnitude and Its Importance in Limiting Earthquake Stresses," by M. P. White, *Bulletin*, Seismological Soc. of America, Vol. 31, No. 2. April, 1941, pp. 93–99.

Another stress reduction is that due to the influence of the foundation. This influence is threefold. When oscillations are set up in a building, strains are produced in the foundation and energy is dissipated, by internal friction in the soil. This effect depends on looseness, shear strength, internal damping, etc. A second cause of stress reduction is the radiation of elastic waves into the soil because of the motion of the building. This phenomenon was the object of a theoretical investigation by K. Sezawa and K. Kanai.[11] By this effect the energy of the oscillations is drained from the building through the foundation and radiated into the soil in the form of elastic waves. The magnitude of this effect depends on the size and natural period of the structure and on the elastic constants and density of the surrounding soil. The elasticity of the foundation will have an influence on the stresses because it increases the natural period of vibration of buildings. In other words, the building will not follow the horizontal motion of the ground but, due to the elasticity of the foundation, it will tend to rock about its center of percussion. This effect is examined in more detail in section V, and is shown to be considerable.

Finally, it must be noted that a variation of the period with amplitude is very effective against resonance effect. The stress limitation due to this factor can be important especially in strong earthquakes when large deformations and local failures occur.

From the choppy aspect of the spectrum it may be concluded that slight differences in building periods may cause great differences in earthquake stresses. This may very well be one of the reasons for the paradoxical observation that the destructiveness of an earthquake varies greatly from one building to another at the same location. The differences are not as great as the spectrums would indicate, however, but this could be explained by taking into account the influence of the damping.

IV.—Effect of an Earthquake on a Structure with Many Degrees of Freedom

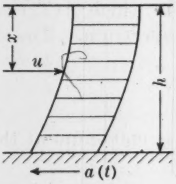

Fig. 7

The fact has been established that a number of modes of oscillation are excited by the earthquake, each of which contributes to the stress.[4, 5] It can be shown that each mode behaves like a system with a single degree of freedom and that the concept of spectrum is applicable to each mode separately. This will be demonstrated briefly by a simple example. Consider a building of height *h*, total mass *m*, and total rigidity *k*, both *m* and *k* being uniformly distributed (Fig. 7). The building is assumed to behave like a shear beam such that a unit horizontal displacement at the roof is produced by a force *k*. The shearing oscillations of the building obey the equation

$$h\,k\,\frac{\partial^2 u}{\partial x^2} + \frac{m}{h}\,a(t) = \frac{m}{h}\,\frac{\partial^2 u}{\partial t^2} \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

in which u is the displacement relative to the ground at a distance x from the roof, and $a(t)$ is the horizontal acceleration of the ground, taken positive in a direction opposite to u. Write the acceleration $a(t)$ as a Fourier series in x

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[11] "Some New Problems of Forced Vibrations of a Structure," by K. Sezawa and K. Kanai, *Bulletin*, Earthquake Research Inst., Tokyo, Vol. XII, Pt. 4, December, 1934, p. 845; also "Decay in the Seismic Vibrations of a Simple or Tall Structure by Dissipation of Their Energy into the Ground," by K. Sezawa and K. Kanai, *ibid.*, Vol. XIII, Pt. 3, September, 1935, p 681.

$$a(t) = \frac{4}{\pi}\left(\cos\frac{\pi x}{2h} - \frac{1}{3}\cos\frac{3\pi x}{2h} + \frac{1}{5}\cos\frac{5\pi x}{2h}\cdots\right)a(t)\ldots\ldots(15a)$$

and the solution $u(x,t)$ of Eq. 14 in the same form:

$$u(x,t) = U_1(t)\cos\frac{\pi x}{2h} + U_2(t)\cos\frac{3\pi x}{2h} + U_3(t)\cos\frac{5\pi x}{h}\ldots\ldots(15b)$$

The quantities $U_1, U_2\cdots$, represent the unknown roof deflections of each mode. Substituting Eqs. 15 into the differential Eq. 14 and equating the coefficients of $\cos\frac{\pi x}{2h}$, $\cos\frac{3\pi x}{2h}$, etc., to zero, the following equations are found:

$$\left.\begin{aligned} m\frac{d^2U_1}{dt^2} + \frac{k\pi^2}{4}U_1 &= \frac{4}{\pi}m\,a(t) \\ m\frac{d^2U_2}{dt^2} + \frac{9k\pi^2}{4}U_2 &= -\frac{4}{3\pi}m\,a(t) \\ \cdots\cdots\cdots&\cdots\cdots\cdots \\ m\frac{d^2U_n}{dt^2} + \frac{(2n-1)^2k\pi^2}{4}U_n &= \pm\frac{4m}{(2n-1)\pi}a(t) \end{aligned}\right\}\ldots\ldots(16)$$

These equations are analogous to Eq. 6 for the structure with a single degree of freedom. According to Eq. 8*a* the roof deflection of the *n*th mode is:

$$U_n(t) = \frac{16m}{(2n-1)^3\pi^3k}\frac{2\pi}{T_n}\int_0^t a(\theta)\sin\frac{2\pi}{T_n}(t-\theta)\,d\theta\ldots\ldots(17a)$$

in which T_n is the period of the *n*th mode. The largest value attained during the earthquake for the roof deflection in the *n*th mode can be written:

$$U_n = \frac{16m}{(2n-1)^3\pi^3k}A(T_n)\ldots\ldots\ldots\ldots\ldots\ldots(17b)$$

in which $A(T_n)$ is the ordinate at $T = T_n$ of the aforementioned acceleration spectrum. The shear stress

$$V = hk\frac{\partial u}{\partial x}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18a)$$

is maximum at the base, $h = x$; hence the maximum shear in the *n*th mode is

$$V_n = \frac{8}{(2n-1)^2\pi^2}m\,A(T_n)\ldots\ldots\ldots\ldots\ldots(18b)$$

Denoting by V_g the stress which would be produced by a statical horizontal acceleration equal to gravity

$$V_n = C_n\frac{A(T_n)}{g}V_g\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18c)$$

in which

$$C_n = \frac{8}{(2\,n-1)^2\,\pi^2} \tag{19}$$

The expression $\frac{A}{g}\,V_g$ represents the shear that would occur in a structure of one degree of freedom with the same mass and period as the particular mode considered. The factor C_n may be called an "effectiveness factor" because it represents the extent to which this shear appears in each mode of the structure with distributed mass and rigidity. A more direct method of calculating these factors is available. Assume that some restraining mechanism is used so that the building can only deflect in its fundamental mode, and apply a horizontal statical acceleration g. The deflection will be found by the energy principle. The external work, W, done by the inertia forces applied gradually is:

$$W = \frac{1}{2}\,\frac{m\,g\,U_1}{h}\int_0^h \cos\frac{\pi\,x}{2\,h}\,dx \tag{20}$$

The elastic potential energy E_p in this deformation is more conveniently evaluated as the kinetic energy in the fundamental mode of vibration; hence:

$$E_p = \frac{1}{2}\,\frac{m}{h}\left(\frac{2\,\pi}{T_1}\right)^2 U^2_1\int_0^h \cos^2\frac{\pi\,x}{2\,h}\,dx \tag{21}$$

Equating expressions 20 and 21 gives

$$U_1 = \frac{16}{\pi^3}\,\frac{m\,g}{k} \tag{22a}$$

and the shear

$$V_1 = \frac{8}{\pi^2}\,m\,g = \frac{8}{\pi^2}\,V_g \tag{22b}$$

The factor $C_1 = \frac{8}{\pi^2}$ appears in this expression, and coincides, for $n = 1$, with the value (19) already found by the more elaborate previous method. The same method is applicable to the higher modes.

The values of the coefficients for the three first excited modes are: $C_1 = 0.810$; $C_2 = 0.0900$; and $C_3 = 0.0324$. Similarly coefficients can be found for all kinds of structures provided the natural modes of vibration are known. In the earlier work[4, 5] the effectiveness factors were computed for a building with an elastic first story. Some of these values are given in Table 1 for different values of the elasticity ratio $\frac{k_1}{k}$. The rigidity k_1 of the first story is

TABLE 1.—CONSTANTS C FOR VARIOUS ELASTICITY RATIOS

$\frac{k_1}{k}$	C_1	C_2	C_3
0	1	0	0
0.556	0.993	0.0295	0.00436
2.50	0.947	0.0712	0.0155
∞	0.810	0.0900	0.0324

the force necessary to produce a unit deflection of the second floor with respect to the ground, and k is the rigidity between the second floor and the roof. Note that for a rigid first story $\left(\frac{k_1}{k} = \infty\right)$ the coefficients are the same as the foregoing for a uniform building. These coefficients make possible a comparison of the relative importance of the various modes. In a building with uniform distribution of mass and rigidity, if 100% denotes the shear that would act in a structure with one degree of freedom under an earthquake of constant spectrum the fundamental mode of the uniform building picks up 81% of the shear, whereas the second and third modes pick up only 9% and 3% of the shear. Table 1 shows that the effect of an elastic story is to decrease, further, the importance of the higher modes relatively to the fundamental.

For the case of a cable or simple truss the maximum shear stress is also given by Eq. 18c with appropriate values of C_n. The bending moment in a truss is expressed as

$$M_n = B_n \frac{A}{g} (T_n)\, M_g \ldots\ldots\ldots\ldots\ldots\ldots (23)$$

in which M_g denotes the maximum bending moment produced by a static horizontal force equal to gravity. The coefficients C_n and B_n depend on the type of structure and the particular mode of vibration. Values of these coefficients for a cable and a pin-ended uniform truss are given in Table 2. Only the symmetrical modes (n odd) are excited by the earthquake so that the coefficients for modes of even order are zero. It is seen that the importance of the higher modes for the bending moment tends to decrease more rapidly than for the shear. Comparing the importance of the various modes under the assumption that the spectrum follows a law of the hyperbolic type as in Eq. 13, the conclusion may be drawn that generally the higher modes are less dangerous than the fundamental. Exception must be made for the case of the shear in a flexural beam where it seems that each mode would carry about the same amount of shear. This is due to the fact that the effectiveness coefficients for shear in a mode of oscillation of the order n decreases as $1/n^2$ whereas the frequency of a flexural beam increases approximately as n^2. However, in practice, due to the increasing influence of damping in the higher modes, one would expect that their importance would always be less than that of the fundamental. The attention of the reader is called to the fact that the values of the effectiveness factors for the higher modes of a truss[2, 12] are erroneous.

TABLE 2.—COEFFICIENTS C_n AND B_n

Order of mode	$n = 1$	$n = 2$	$n = 3$
C_n for cable	0.810	0.090	0.032
C_n for pin-ended uniform truss	0.810	0.090	0.032
B_n for pin-ended uniform truss	1.03	0.038	0.008

Table 2 was applied to the evaluation of an upper limit for the stresses that would be produced by an earthquake of the Helena or Ferndale type in the San Francisco-Oakland Bay Bridge.[12] Conditions are found to be least favorable in the side-span truss which is treated as a pin-ended truss with a period of 3 sec. For this period the effective acceleration of the standard spectrum in Fig. 6 is $A = 0.066\,g$. The effectiveness factor B_1 for the bending moment in the fundamental mode is $B_1 = 1.03$; hence the bending moment is

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[12] *Transactions*, Am. Soc. C. E., Vol. 106 (1941), p. 1385.

$$M_1 = 0.068\,M_g \qquad (24)$$

This means that the maximum bending moment during the earthquake is not greater than that due to a static horizontal acceleration of 6.8% gravity. From the foregoing values of the effectiveness factors it seems as if the stresses in structures with more than one degree of freedom are of the same order as those in a structure with a single degree of freedom. This is not always true, however, and a particularly dangerous condition may arise from assuming that it is.

Consider, for instance, that a mass m_1 is elastically restrained to the ground with a stiffness k_1, and that a much smaller mass m_2 is restrained to m_1 with a stiffness k_2 (Fig. 8). Denoting by u_1 and u_2 the horizontal displacement of m_1 and m_2, respectively, the equations for the amplitudes of the harmonic motion are

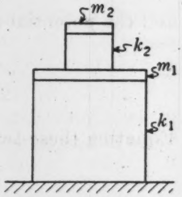

Fig. 8.—System with Two Degrees of Freedom, the Main Building and the Roof Structure

$$m_1 u_1 \omega^2 = k_1 u_1 + k_2 (u_1 - u_2) \qquad (25a)$$

and

$$m_2 u_2 \omega^2 = k_2 (u_2 - u_1) \qquad (25b)$$

in which ω is the circular frequency.

Assume that the small mass is "tuned" to the vibration of the large mass, by making

$$\frac{k_1 + k_2}{m_1} = \frac{k_2}{m_2} = \omega^2{}_1 \qquad (26)$$

Writing $\frac{\omega}{\omega_1} = \lambda$ the equations of motion become

$$(\lambda^2 - 1)\,u_1 + \frac{m_2}{m_1} u_2 = 0 \qquad (27a)$$

and

$$u_1 + (\lambda^2 - 1)\,u_2 = 0 \qquad (27b)$$

Hence, by elimination of u_1 and u_2,

$$\lambda^2 - 1 = \pm\sqrt{\frac{m_2}{m_1}} \qquad (28)$$

From Eqs. 27b and 28 the ratio of amplitudes is

$$\frac{u_2}{u_1} = \mp\sqrt{\frac{m_1}{m_2}} \qquad (29)$$

The lower sign refers to the fundamental mode. Now, in order to evaluate the effectiveness factor a restraint is assumed to force the structure to deflect in its fundamental mode while a static horizontal acceleration equal to gravity is applied.

The work done by the gravity force in deflecting the structure is

$$\frac{g}{2}(m_1 u_1 + m_2 u_2) = \frac{1}{2} m_1 g u_1 \left(1 + \sqrt{\frac{m_2}{m_1}}\right) \tag{30a}$$

and the potential energy is

$$\frac{\omega^2}{2}(m_1 u^2_1 + m_2 u^2_2) = \frac{m_1}{m_2} k_2 u^2_1 \left(1 - \sqrt{\frac{m_2}{m_1}}\right) \tag{30b}$$

Equating these two expressions yields

$$u_1 = \frac{1}{2} \, \frac{m_2 g \left(1 + \sqrt{\frac{m_2}{m_1}}\right)}{k_2 \left(1 - \sqrt{\frac{m_2}{m_1}}\right)} \tag{31a}$$

and the shear in the spring k_2 is

$$V_1 = k_2 (u_2 - u_1) = \frac{1}{2}\left(1 + \sqrt{\frac{m_1}{m_2}}\right) m_2 g \tag{31b}$$

Therefore the effectiveness factor is

$$C_1 = \frac{1}{2}\left(1 + \sqrt{\frac{m_1}{m_2}}\right) \tag{32}$$

It is seen that for small values of the mass ratio $\frac{m_2}{m_1}$ this factor becomes quite large. For instance, if this mass ratio is $\frac{1}{100}$, then $C_1 = 5.5$. If an earthquake acts on such a system the stress in the spring k_2 will be

$$V_1 = 5.5 \frac{A}{g}(T_1) V_g \tag{33}$$

or 5.5 times what it would be if the spring mass ($k_2 m_2$) were directly connected to the ground. Note that in Eq. 33, $V_g = m_2 g$ is the stress in the spring k_2 under a horizontal statical acceleration equal to gravity. This introduces an example in which the effectiveness factor C_n refers to a particular location in the structure. It is then possible to refer to the effectiveness factor "at a certain point" in the structure. The amplification effect revealed here belongs to a class of phenomena which might be designated as a "whip effect." Considered from the standpoint of wave propagation it is analogous to what happens in a whip, when a wave generated at the base is propagated to the tip. The wave may be considered as an energy lump moving from the heavy part of the whip to the lighter section at the tip. This produces at the tip a concentration of energy in a small mass and therefore gives rise to a high velocity. This effect is essentially associated with taper and explains many vibration failures in tapered beams, such as propeller tip failure or the fact that a tapered column receiving a blow at the base will fail at the tip. The case of two degrees of freedom, treated in the example, may well be considered as a simplified model of a tapered building. It points to the possibility of serious danger in roof structures and is in agreement with the observation of disproportionate damage in penthouses in the Long Beach earthquake of March, 1933. The California Building Code takes this danger into consideration by raising the lateral gravity force for which the penthouse must be designed.

The case of a structure in which the mass and stiffness both vary linearly with distance from the top was investigated by Professors Martel and White.[6] The effectiveness factor C_1 for total shear at various heights is:

Distance, $\frac{x}{h}$, from the roof	C_1
0	1.60
0.2	1.55
0.5	1.33
0.8	0.97
1.0 (base)	0.693

It is seen that these coefficients also point to the existence of a "whip" effect.

V.—Influence of Foundation on Motion of Blocks

The foregoing mathematical developments are based on the tacit assumption that the effect of an earthquake is the same as that of a shaking table. As will be shown now there is strong theoretical evidence that this concept must be modified to take care of the influence of the foundation. The elastic properties of the structure cannot be dissociated entirely from those of the ground and both must be studied simultaneously in order to predict the dynamic properties of the system.

The problem is extremely complex because it involves a complete knowledge of the propagation and properties of the seismic waves in the strongly heterogeneous surface layers of the earth as well as their diffraction and reflection by objects built on the surface. The influence of the soil on the damping of oscillations in structures was also discussed in the foregoing. An immediate answer to such a complex problem cannot be expected. It is believed, however, that the simplified problem noted in this section throws revealing light on the nature of foundation effects.

The question investigated is that of the influence of ground elasticity on the rocking motion of a block. How resistant is the surrounding soil to the rocking displacement of a foundation? What are the factors influencing this rigidity? Can this effect have a practical influence on the action of earthquakes on buildings? The problems are simplified by neglecting the mass of the soil, the internal friction in the soil, and the radiation of elastic waves due to the rocking.

Assume that the coordinate axes xy lie on the surface of the soil. A load-distribution $p(x)$-function only, of the distance x, is applied to the surface on a strip infinitely long in the y-direction and extending from $x = -L$ to $x = +L$. The distribution is asymmetric with respect to the y-axis so that

$$p(-x) = -p(x) \qquad (34)$$

It can be shown from the theory of elasticity that the soil deflection $w(x)$ is given by

$$w(x) = \frac{2}{\pi}\frac{(1-\mu^2)}{E}\int_0^L p(\xi)\log_e\left|\frac{x+\xi}{x-\xi}\right| d\xi \qquad (35)$$

in which E is Young's modulus of the soil and μ its Poisson ratio. Substitute in Eq. 35 the pressure distribution

$$p(x) = \frac{\alpha\, x}{\sqrt{L^2 - x^2}} \qquad (36)$$

Then, the deflection is a straight line:

$$w = 2\,\alpha\frac{(1-\mu^2)}{E}x \qquad (37)$$

In other words, the assumed pressure distribution (Eq. 36) is that under a rigid slab of width 2 L rocking about the y-axis. The ratio between the elastic moment M of p about the axis and the slope $\frac{dw}{dx}$ is the elastic stiffness coefficient C_r for rocking motion

$$C_r = \frac{M}{\frac{dw}{dx}} = \frac{\pi}{4}\frac{E}{(1-\mu^2)}L^2 \qquad (38)$$

If a uniform pressure distribution equal to a constant p from $x = 0$ to $x = +L$ and $-p$ from $x = 0$ to $x = -L$ is applied, the deflection of the soil is

$$w = \frac{2}{\pi}\frac{(1-\mu^2)}{E}p\,L\left(\log_e\left|\frac{L+x}{L-x}\right| + \frac{x}{L}\log_e\left|\frac{L^2-x^2}{x^2}\right|\right) \qquad (39)$$

From this solution it is possible to derive, by superposition, the deflection due to uniform asymmetric load distribution extending from $x = \pm L$ to various distances from the axis. It is found that the average rocking stiffness is given by a formula similar to Eq. 38 except that the numerical coefficient is somewhat different from $\frac{\pi}{4}$. For instance, in the case in which the load extends from

$x = 1$ to $x = \frac{4}{5}L$ and $x = -1$ to $x = -\frac{4}{5}L$ the coefficient $\frac{\pi}{4}$ is replaced by $0.231\,\pi$. It may be concluded that Eq. 38 represents a reasonable approximation, independent of the load distribution.

Eq. 38 implies the knowledge of the elastic constants E and μ of the soil. These are obtainable from tests made by M. Ishimoto and K. Iida.[13] Samples from the underground of several regions within Tokyo, Japan, at depths to 60 ft, were made into the form of rectangular prisms and submitted to vibration tests. The average values of the Poisson ratio are found to be about $\mu = \frac{1}{3}$.

Some of the values found for Young's modulus are

Material	E, in lb per sq in.
Silty clay in natural state with 50% moisture	792
Same with 42% moisture	5,750
Loam	8,200

From measured velocities of waves in soils[14, 15] the following order of magnitudes may be derived:

Material	E, in lb per sq in.
Loose sand	4,000
Clay	6,000
Sandstone	100,000

D. D. Barkan[16] has experimented, under field conditions, with various foundations weighing as much as 30 tons and having various areas at the bottom as great as 90 sq ft. The foundation was tested dynamically for rocking oscillations and also statically for the rocking rigidity of the foundation slab on the soil. The results are in good agreement with Eq. 38. A length of 21 was taken to represent the size of the square foundation slab used in the test. The values of E derived from this test are

Material	E, in lb per sq in.
Loess	12,000
Water-soaked brown loam	5,000

The order of magnitude of these values is in satisfactory agreement with the aforementioned values cited from other sources.

It is now possible to derive the rocking period of oscillation of a block lying on the soil. The results established in the preceding sections have shown that the fundamental period of oscillation of a structure is one of the essential controlling factors in earthquake stresses. The rocking period of a block is

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[13] "Determination of Elastic Constants of Soils by Means of Vibration Methods," by M. Ishimoto and K. Iida, *The Bulletin of the Earthquake Research Institute*, Vol. XIV (1936), Pt. I; also *ibid.*, Vol. XV (1937), Pt. II.

[14] "Soil Mechanics," by D. P. Krynine, McGraw-Hill Book Co., Inc., New York, N. Y., 1941.

[15] "Praktische Anwendungen der Baugrund Untersuchungen," by W. Loos, Berlin, J. Springer, 1935, p. 53.

[16] "Field Investigations of the Theory of Vibrations of Massive Foundations Under Machines," by D. D. Barkan, *Proceedings*, International Conference on Soil Mechanics and Foundation Eng., Vol. II, 1936, p. 285.

$$T = \frac{2\pi}{\sqrt{\frac{C_r}{m\,r^2} - \frac{g\,h}{r^2}}} \qquad (40)$$

in which C_r is the stiffness of the foundation according to Eq. 38, m the mass of the block, r the radius of gyration with respect to the rocking axis, and h the height of the center of gravity above the ground. The term $\frac{g\,h}{r^2}$ represents the destabilizing influence of gravity. The case in which

$$\frac{C_r}{m\,r^2} = \frac{g\,h}{r^2} \qquad (41)$$

corresponds to statical instability when the tipping moment due to gravity is equal to the restoring moment of the soil. Usually the term $\frac{g\,h}{r^2}$ may be neglected and if the weight $m\,g$ is written as $m\,g = 2\,p_s\,L$ the period becomes

$$T = 9.5\sqrt{\frac{p_s}{E}\frac{r^2}{g\,L}} \qquad (42)$$

The pressure p_s is an average load on the foundation due to the weight of the structure. Consider the case of hard clay, and use for p_s the bearing capacity given by the Boston Building Code—$p_s = 84$ lb per sq in. According to the aforecited data, a reasonable value for hard clay seems to be $E = 15{,}000$ lb per sq in. The periods given in Table 3 are then obtained. With these values of the periods the spectrum may be used to evaluate earthquake stresses as explained previously herein.

TABLE 3.—VALUES OF THE PERIOD T, IN SECONDS

$2\,L$ (ft)	VALUES OF r, IN FEET			
	5	10	50	100
5	0.39	0.78	3.9	7.8
10	0.28	0.55	2.8	5.5
50	0.12	0.24	1.2	2.4

According to the standard spectrum in Fig. 6 the stresses are inversely proportional to the period of the structure. Since the soil can have a marked effect on the period it is to be expected that it exerts also a proportional influence on the destructiveness of the earthquake. Actually, of course, structures do not behave as rigid blocks, so that their period will be influenced by both the building and the foundation rigidity. The combined periods must be used in applying the spectrum curves to evaluate earthquake stresses. According to the data in Table 3 and Eq. 42 the slenderness of a structure has a considerable effect on its vulnerability to earthquakes. This probably bears some relation to the observed fact that columns and towers sometimes show paradoxical resistance with respect to neighboring structures.

CONCLUSIONS

Experimental and analytical methods of approach have been developed for the evaluation of earthquake stresses. An earthquake may be characterized by a certain function of period, called a spectrum, which is derived from the accelerogram by means of a mechanical analyzer. By using the spectrum an upper limit can be evaluated simply for the stresses produced in undamped structures of given dynamic properties. The stresses in each mode are shown to be derived quite simply from the value of the period and a coefficient dependent on the nature of the structure and referred to as the "effectiveness factor."

The comparison of stresses calculated by this method with observed destructiveness of an earthquake on a building seems to indicate that, at least for short periods, the values of the stresses obtained are considerably higher than could be expected. The possible effect of certain stress-reducing factors is discussed, such as internal damping in the soil and in the structure, the radiation of elastic waves, etc. The influence of these factors can become very large in certain cases.

Particular attention is given to the influence of the elasticity of the foundation and it is shown that considerable stress reduction occurs through the lengthening of the natural period due to the foundation. This is equivalent to stating that rigid slender structures will have a tendency to rock about the center of percussion because of the elastic yielding of the foundation.

Attention is directed to a phenomenon referred to as the "whip" effect which increases the destructiveness of earthquakes on penthouses and the tip of tapered columns and buildings.

Spectrum curves presented herein indicate that the effectiveness of an earthquake is inversely proportional (roughly) to the period, so that increasing the period means an increase in safety. Application of the methods developed in this paper should be helpful as a guide in extending safety rules and interpreting earthquake data.

A considerable field lies open for further research, especially as regards the stress-reducing factors discussed in section III. The influence of damping, for instance, could be taken into account by using a controllable amount of damping in the mechanical analyzer. This would yield a set of "damped response curves" which would give directly the effective acceleration for a building with given period and damping. In this terminology the spectrum would correspond to the undamped response curve. Further research should also be directed toward a better knowledge of the dynamics of the foundation, the interference of structures with one another, and the influence of agglomerations on the earthquake waves themselves.

ACKNOWLEDGMENT

The investigation summarized in section V was undertaken as part of a research program directed by Professor Martel and sponsored by the Los Angeles County Department of Buildings and Safety.

APPENDIX

NOTATION

The following letter symbols, adopted for use in this paper, conform essentially to Standard Letter Symbols for Mechanics, Structural Engineering, and Testing Materials, prepared by a Committee of the American Standards Association with Society representation and approved by the Association in 1932:[17]

A = effective acceleration giving the maximum value of shear through Eq. 12: $A(T)$ = acceleration spectrum;
a = linear acceleration:
  a_0 = constant horizontal acceleration;
  $a(t)$ = earthquake acceleration as a function of time (accelerogram curve);
  $a(\theta)$ = acceleration as a function of the variable of integration θ (Eq. 8*b*);
B_n = effectiveness factor for the bending moment in the *n*th mode (see Table 2 and Eq. 23); extent to which bending moment appears in each mode of the structure;
C_n = effectiveness factor for the shear in the *n*th mode (see Tables 1 and 2 and Eq. 18*c*); extent to which shear appears in each mode of the structure;
C_r = coefficient defining the stiffness of a foundation (Eq. 38);
E = modulus of elasticity: E_p = potential energy;
F = force;
g = gravity constant;
h = height;
k = spring constant; rigidity: k_1 = the rigidity between ground and second floor;
L = length;
M = bending moment:
  M_n = maximum bending moment in the *n*th mode of a truss (Eq. 23);
  M_g = maximum bending moment produced by a static horizontal force equal to gravity;
m = mass;
p = unit pressure; uniform pressure distribution:
  p_s = average load on a foundation due to the weight of the structure;
  $p(x)$ = pressure distribution under a rocking foundation at a distance x from the axis;
  $p(\xi)$ = pressure distribution with the variable of integration ξ in the place of x;

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[17] ASA-Z10a-1932.

r = radius of gyration;

T = period of vibration; natural period of oscillation:

  $A(T)$ = acceleration spectrum;

  T_n = the period of the nth mode;

t = time;

U_n = $U_n(t)$ = largest value of roof deflection obtained during an earthquake for the nth mode, as a function of time;

u = displacement relative to the ground (u and v are taken positive to the right):

  u_0 = a constant deflection (Eq. 2);

  $u(x,t)$ = displacement at the instant t and distance x from the roof;

V = total shear in a structure:

  V_n = maximum shear in the nth mode of a cable or truss;

  V_g = maximum shear produced by a static horizontal force equal to gravity;

v = (see u);

W = work;

$w(x)$ = soil deflection;

x = distance from roof (Eq. 14); distance from rocking axis of foundation (Eq. 35);

α = (Eq. 36) a coefficient;

θ = variable of integration in Eqs. 8;

λ = ratio of natural frequency ω to ω_1 (Eq. 27*a*);

μ = Poisson's ratio;

ξ = variable of integration in place of x (Eq. 35);

ω = circular frequency: ω_1 = circular frequency defined by Eq. 26.

DISCUSSION

George R. Rich,[18] M. Am. Soc. C. E.—In modern seismic design, the use of arbitrary inertia loadings acting at the center of gravity of the structure has been superseded properly by the application of ground motions fairly representative of earthquakes to be expected at the site. Additional important economies and improved distribution of material still remain to be accomplished by increased recognition of the mitigating factors outlined by the author: First, the earthquake shock is transitory and the particular pattern required for resonance is generally repeated for only a few cycles; second, the effect of damping is controlling in cases approaching the resonant condition; and, third, a part of the seismic shock is dissipated in the propagation of elastic waves in the foundation due to the motion of the structure. Improved analytical technique is a distinct aid to progress along these lines.

Operational methods somewhat similar to the one used by the author in obtaining the seismic spectrum were evolved originally by Oliver Heaviside for studying the effect of transient electrical impulses and may be extended profitably, by means of the theory of functions of a complex variable,[19,20] in analyzing the effect of earthquake transients upon structures. These methods not only effect a great economy of time and labor by their directness and automatic elimination of integration constants, but they also afford a ready means of depicting the motion of the structure after the forced ground disturbance has been suppressed. In rigid structures with damping, this feature may be important since the response lags the transient driving impulse. Maximum stresses in such cases might occur after the ground motion ceases.

As an illustration, suppose it is desired to impress only *n* cycles of a sinusoidal ground acceleration upon Fig. 1 and to include the effect of damping. The basic differential equation is:

$$m \frac{d^2u}{dt^2} + \beta \frac{du}{dt} + k\,u = m\,a \qquad (43)$$

in which β is the coefficient of damping. For convenience, let $\beta/m = 2\,A$ and $k/m = B^2$. Then

$$\frac{d^2u}{dt^2} + 2\,A \frac{du}{dt} + B^2\,u = a \qquad (44)$$

The operational form for a sustained sinusoidal ground acceleration is $\frac{\omega\,p\,a_m}{p^2 + \omega^2}$, in which a_m is the maximum value of the ground acceleration (0.1 g in Fig. 5); ω is the angular velocity of the ground motion equal to $2\,\pi/T$, in which T is the period in seconds; and p, in the best modern terminology, represents the transform parameter in the Mellin inversion theorem.[21, 22] Heaviside called p the operator d/dt.

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[18] Chf. Design Engr., TVA, Knoxville, Tenn.

[19] "Complex Variable and Operational Calculus," by N. W. McLachlan, Cambridge Univ. Press, 1939.

[20] "Operational Methods in Mathematical Physics," by Harold Jeffreys, Cambridge Tract No. 23, Cambridge Univ. Press, 2d Ed., 1931.

The ground acceleration is suppressed after n cycles by incorporating the shift[23] operator e^{-q} (in which, to simplify typography, $q = 2\,n\,\pi\,p/\omega$), so that the operational form for the right-hand side of the differential equation becomes $(1 - e^{-q})\,\dfrac{\omega\,p\,a_m}{p^2 + \omega^2}$. From the basic differential equation, the operational form of u is:

$$u \supset \frac{(1 - e^{-q})\,\omega\,p\,a_m}{(p^2 + \omega^2)\,(p^2 + 2\,A\,p + B^2)} \ldots\ldots\ldots\ldots\ldots (45)$$

For the case of interest in seismic design, the zeros of $(p^2 + 2\,A\,p + B^2)$ are complex numbers, so, for convenience, let $\phi^2 = B^2 - A^2$; and then placing $i = \sqrt{-1}$:

$$u \supset \frac{(1 - e^{-q})\,\omega\,p\,a_m}{(p + i\,\omega)\,(p - i\,\omega)\,(p + A - i\,\phi)\,(p + A + i\,\phi)} \ldots\ldots (46)$$

By the Mellin inversion theorem:

$$u = \frac{\omega\,a_m}{2\,\pi\,i}\int_{c-i\infty}^{c+i\infty} \frac{(e^{zt} - e^{zv})\,dz}{(z + i\,\omega)\,(z - i\,\omega)\,(z + A - i\,\phi)\,(z + A + i\,\phi)} \ldots (47)$$

or

$$u = \frac{\omega\,a_m}{2\,\pi\,i}\left[\int_{c-i\infty}^{c+i\infty} \frac{e^{zt}\,dz}{(z + i\,\omega)\,(z - i\,\omega)\,(z + A - i\,\phi)\,(z + A + i\,\phi)}\right.$$

$$\left. - \int_{c-i\infty}^{c+i\infty} \frac{e^{zv}\,dz}{(z + i\,\omega)\,(z - i\,\omega)\,(z + A - i\,\phi)\,(z + A + i\,\phi)}\right] \ldots (48)$$

in which, to simplify typography, $v = t - \dfrac{2\,n\,\pi}{\omega}$.

Since the only singularities of the integrand are simple poles $z = -\,i\,\omega$; $+\,i\,\omega$; $-\,A + i\,\phi$; $-\,A - i\,\phi$; integration along the standard Bromwich contour is equivalent to $2\,\pi\,i$ times the summation of the residues[24, 25, 26, 27] at the poles. The value of the first integral inside the brackets, Eq. 48, accordingly is:

$$u = a_m\left\{\frac{(B^2 - \omega^2)\sin\omega\,t - 2\,A\,\omega\cos\omega\,t}{(B^2 - \omega^2)^2 + (2\,A\,\omega)^2}\right.$$

$$\left. + \frac{(2\,A^2 - B^2 + \omega^2)\,\omega\sin\phi\,t + 2\,A\,\phi\,\omega\cos\phi\,t}{e^{At}\,\phi\,[(B^2 - \omega^2)^2 + (2\,A\,\omega)^2]}\right\} \ldots\ldots\ldots (49)$$

for $t > 0$ and $< 2\,n\,\pi/\omega$.

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[21] "Complex Variable and Operational Calculus," by N. W. McLachlan, Cambridge Univ. Press, 1939, p. 117.

[22] "The Theory of Fourier Integrals," by E. C. Titchmarsh, Oxford Univ. Press, 1937, pp. 7, 46.

[23] "Complex Variable and Operational Calculus," by N. W. McLachlan, Cambridge Univ. Press, 1939, p. 129.

[24] "Complex Variable and Operational Calculus," by N. W. McLachlan, Cambridge Univ. Press, 1939, p. 53.

[25] "Functions of a Complex Variable," by E. T. Copson, Oxford Univ. Press, 1935 ,p. 117.

[26] "The Taylor Series," by P. Dienes, Oxford Univ. Press, 1931, p. 233.

[27] "The Theory of Functions," by E. C. Titchmarsh, Oxford Univ. Press, 2d Ed., 1939.

The value of u for all times between $t = 0$ and $t = 2\,n\,\pi/\omega$ is to be taken from this expression only, which, it will be noted, correctly gives $u = 0$ and $du/dt = 0$ when $t = 0$.

The values of u after suppression of the ground acceleration are given by the sum of both integrals within the brackets, Eq. 48:

$$u = a_m \left\{ \frac{(2\,A^2 - B^2 + \omega^2)\,\omega \sin \phi\, t + 2\,A\,\phi\,\omega \cos \phi\, t}{e^{A\,t}\,\phi\,[(B^2 - \omega^2)^2 + (2\,A\,\omega)^2]} \right.$$

$$\left. - \frac{(2\,A^2 - B^2 + \omega^2)\,\omega \sin \left[\phi\left(t - \frac{2\,n\,\pi}{\omega}\right)\right] + 2\,A\,\phi\,\omega \cos \left[\phi\left(t - \frac{2\,n\,\pi}{\omega}\right)\right]}{e^{A\,v}\,\phi\,[(B^2 - \omega^2)^2 + (2\,A\,\omega)\,]} \right\} \quad (50)$$

for $t > 2\,n\,\pi/\omega$.

The use of this combined value is valid only for times greater than $t = 2\,n\,\pi/\omega$. Emphasis is placed upon this important characteristic of the use of shift operators in general.[28] The time is referred to the time of quiescence as an origin, but the use of the sum of the two integrals is correct only subsequent to $t = 2\,n\,\pi/\omega$.

The operational method is not limited to impressing sinusoidal transients. Operational forms for a wide variety of impulses are available.[29] Among these may be noted as possible components of actual accelerograms the diminishing Bessel wave $p/\sqrt{p^2 + 1}$ and the Morse dash or hammer blow $(1 - e^{-p\tau})$. Operational methods also make it possible to combine these various elementary impulse forms; for example, the Bessel wave represented by the foregoing operator may be impressed immediately following a sine wave of $n + 1/4$ cycles, fairly approximating the characteristic graph observed on typical accelerograms. In synthesizing ground motions in this manner, care should be taken to incorporate in the operational derivation the fact that the system is quiescent at the start of the first impulse, but in motion at the start of the second component impulse.

N. J. HOFF,[30] ESQ.—Predicting the stresses in buildings due to earthquakes is probably the most complicated problem a civil engineer may encounter. In problems of stresses caused by static or steady dynamic loads the use of refined mathematical methods may be required to obtain a rigorous solution, but, as a rule, an approximate answer satisfactory for practical design purposes can be obtained by the use of only elementary mathematics. Such is not the case with earthquake vibrations. Without an involved mathematical analysis not even the order of magnitude of earthquake stresses can be predicted.

The first part of this paper presents Professor Biot's earlier complicated theoretical contributions to the solution of problems of earthquake stresses in a form easily understandable to structural engineers. The explanation of the concept of the "earthquake spectrum" is of special interest since it may lead eventually to a simple routine engineering approach to this involved problem.

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[28] "Complex Variable and Operational Calculus," by N. W. McLachlan, Cambridge Univ. Press, 1939, p. 130.

[29] *Ibid.*, p. 155.

[30] Asst. Prof., Aeronautical Eng., Polytechnic Inst. of Brooklyn, Brooklyn, N. Y.

Advantages of the mechanical analyzer in Fig. 2 are obvious for the determination of the "spectrum." It may be of interest to compare this device to the "shaking table" used in the experiments conducted in the Earthquake and Vibration Laboratory of Stanford University, Stanford University, Calif., where the writer had the opportunity to work under the direction of Professor Jacobsen. This shaking table is mounted on balls and is actuated by a cam and follower arrangement in such a way that its motion is a replica of the recorded horizontal motion of the ground during an earthquake. On the shaking table is mounted a single-degree-of-freedom vibrating system, the forced vibration of which is recorded. The spectrums of several earthquakes were established with the aid of the shaking table.

The general appearance of the spectrums obtained at Stanford University is very much the same as that of the curves in Professor Biot's paper. The maximum oscillator acceleration, however, was found to be about 0.5 g in the Stanford tests whereas Fig. 3 of the paper shows a peak value of about 1.1 g. It is suggested that the reason for this discrepancy is the intrinsically higher frictional damping of the shaking table. Systematic investigations at Stanford University showed that small changes in the friction have little effect upon the motion of the oscillator if the friction is comparatively high, but the effect is very marked when the friction is low. Of course, even the accelerations found at Stanford are much higher than those experienced by actual buildings during the same earthquake. This must be so, since the great majority of buildings designed to withstand a horizontal acceleration of 0.1 g only was not damaged. The reason may be found in the internal friction of the buildings and in the effects of the foundation as explained by Professor Biot. Nevertheless, a mechanical analyzer with low friction has theoretical advantages and a damping can always be applied to it if required.

A great advantage of Professor Biot's mechanical analyzer is its ease of operation. This is due mainly to the fact that it makes direct use of the accelerogram of the earthquake. For the investigations on the shaking table the accelerogram is integrated twice and the displacement curve so obtained is used for constructing the cam. The numerical integration is very laborious as is evident to all who ever saw an earthquake accelerogram. For this reason, Professor Martel of the California Institute of Technology suggested the use of the accelerogram for the construction of the cam. Such a procedure, however, was not found practicable since it would require a cam of prohibitively large diameter.

In the investigation of the vibrations of buildings with distributed mass and elasticity, Professor Biot made the tacit assumption that the maximum shear in the building is the greatest of the maximum shears of the different natural frequencies. In the Stanford tests it was found that in many cases the time of the occurrence of the maximum shear was approximately the same for single-degree-of-freedom models of different natural frequencies when subjected to the same ground motion. If this holds true for buildings of many degrees of freedom, the cumulative effect may cause a maximum shear greater than indicated by the spectrum. Furthermore, it is even conceivable that in such a case the maximum shear would occur somewhere else than at the base of the building. It is thought that this problem may merit some further consideration.

The calculations relative to the "whip effect" and the influence of the foundation are of great importance to practical design. It is desirable that the investigation of these problems be continued with a view to establishing a reliable and easily applicable procedure of calculation such as was achieved in the simpler cases with the aid of the concept of the earthquake spectrum. It is hoped that, in due season, Professor Biot again will be able to devote some of his time to problems of earthquake stresses in buildings.

MERIT P. WHITE,[31] ASSOC. M. AM. SOC. C. E.—Professor Biot's paper is probably one of the most significant of those which have appeared in the field of engineering seismology. Although many of the ideas presented are not new, nevertheless the fact that this is the first published attempt to present a complete picture of the earthquake problem makes it important.

As in nearly every other branch of engineering, there are two possible ways to attack the problem of earthquake resistant design. One of these represents the empirical, trial-and-error school of thought which, in general, is responsible for the present design methods. Actually, this approach can give, and has given, excellent results, partly on account of what must be called "engineering intuition." Nevertheless, the writer prefers the other approach, which may be characterized as the rational approach, which attempts to isolate and to understand the significance of the different factors involved in a problem. The rational method may be based on experiment, or on a combination of experiment and theory. Certainly, Professor Biot's paper is representative of this approach.

Mechanical Analysis of Accelerograms.—The use of the mechanical analyzer described by the author certainly will result in a saving of time when compared with the arithmetical solution for the response of an oscillator to an earthquake. However, the writer believes that the difference is not always as great as the author implies. In 1938 a comparatively rapid tabular method for solving the equation of motion was developed at the California Institute of Technology.[6] This method has the advantage that the effects of various amounts of damping can be found with little additional labor. Of course, a further advantage is the fact that such a method requires no mechanical equipment. So far as the writer knows, the first use of the mechanical analyzer for finding oscillator response to an earthquake motion was by Frank Neumann of the U. S. Coast and Geodetic Survey in 1936. In this work the earthquake displacement curve, obtained by double integration of an accelerogram, was used to govern the motion of a torsional pendulum. In 1939, Ralph E. Byrne, Jr., Jun. Am. Soc.

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[31] Palmer Physical Laboratory, Princeton Univ., Princeton, N. J.

C. E., and the writer suggested a method by which an accelerogram might be used directly to actuate a mechanical analyzer.[32] This is the principle of the author's analyzer.

Peaks Appearing in Spectrum Curve.—The physical explanation for the peaks that always appear in an earthquake spectrum is a matter of importance. If these peaks truly represent characteristics of the basic earthquake motion, then, as suggested by the author, they may account for some of the paradoxical occurrences in earthquakes. However, to the writer, a more satisfying explanation is one suggested by Hugo Benioff of the California Institute of Technology; namely, that the peaks appearing in an earthquake spectrum are due to the influence of the motion of the building housing the recorder on the record made by the recorder. To test this hypothesis, extensive forced vibration tests, in which a shaking machine was used, were made by the U. S. Coast and Geodetic Survey with the assistance of Mr. Byrne and the writer in the summer of 1938.[6] It was found: (*a*) That there was definite correlation between the natural building periods and the periods of spectrum peaks for the earthquake records made by recorders in the buildings in question; and (*b*) that forced oscillation of a building gave measurable motion not only in the basement of the structure, where recording instruments may be located, but over a surrounding area, 1,000 ft or more in extent. Hence, spectrum peaks may even be caused by neighboring structures. The building coupling effect, in which a faint but regular motion is superimposed on the basic earth motion, might not be in evidence on the seismogram, but on account of its regularity it would have a large effect on the spectrum.

Change of Building Period with Amplitude.—The author states that a change of building period with amplitude of motion is effective against resonance effect. This is generally true, although it is conceivable that the opposite may occur—that is, an oscillator which is slightly off resonance may acquire enough amplitude to cause a period change, putting it in better resonance. However, if the spectrum peaks are not characteristic of the basic earth motion, true resonance cannot exist, and a small change of building period will have no particular importance.

Effect of Foundation Yielding and of Non-Uniformity of Mass Distribution or Stiffness of Building.—The general case, in which the structure is not uniform and rests on a yielding foundation (but has vertical planes of symmetry, or near symmetry), can be treated quite simply as follows: To determine the maximum shearing force at the base of the structure, the frequencies and shapes of the lower modes of vibration and the distribution of weight along the height of the building are needed. No other information regarding the foundation or the characteristics of the structure is required (damping is neglected). The natural frequencies of any building are found easily by the use of vibration meters or by comparison with similar structures in similar locations. The mode shapes (usually only the fundamental is really important) ordinarily can be assumed with sufficient accuracy. The distribution of weight is easily found, of course.

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[32] "Model Studies of the Vibrations of Structures During Earthquakes," by Merit P. White and Ralph E. Byrne, Jr., *Bulletin*, Seismological Soc. of America, Vol. 29, No. 2, April, 1939, pp. 327–332.

As was stated by the author, the total response will be the sum of the responses of the various modes, the fundamental mode predominating. Each mode will be excited in much the same way as is a simple oscillator of the same period.

Letting $Y_n(x)$ be the shape, T_n the period, and V_n the maximum shear at the base of the building, all for the nth mode of vibration, and $\rho(x)$ the mass per unit height of structure at the height x, it can be shown that

$$V_n = A(T_n) \frac{\left[\int_0^h Y_n(x)\,\rho(x)\,dx\right]^2}{\int_0^h Y^2{}_n(x)\,\rho(x)\,dx} = m\,A(T_n)\,R_n \ldots\ldots\ldots (51)$$

in which

$$R_n = \frac{1}{m}\,\frac{\left[\int_0^h Y_n(x)\,\rho(x)\,dx\right]^2}{\int_0^h Y^2{}_n(x)\,\rho(x)\,dx} \ldots\ldots\ldots\ldots\ldots\ldots (52)$$

In Eq. 51, $A(T_n)$ is the ordinate of the author's acceleration spectrum at $T = T_n$, h is the height of structure, and m is the total mass of the building. Thus, R_n is the multiplying factor that gives the ratio between the maximum shear at the base, caused by the nth mode, and the maximum shear in a simple oscillator of equal mass and the same period. (Note that only the shape of $Y_n(x)$ is significant, and that the scale assumed has no effect.) Eq. 51 can be shown to reduce to Eq. 18*b* of the paper for the particular case in which weight distribution and stiffness are both constant.

Now, for the fundamental mode, consider the importance of foundation yielding and of mode shape on the numerical value of the multiplying ratio R, defined by Eq. 52, as demonstrated for the following cases:

(1) Weight concentrated at a point (the simple oscillator). Ratio $R = 1.0$.

(2) Structure has uniform stiffness and weight distribution; base is rigid. Ratio $R_1 = 0.81$.

(3) Weight is uniformly distributed, and the fundamental mode is linear as in Fig. 9 (this might represent a very stiff structure on a yielding base, a structure having a particular variation of stiffness, or one in which bending and shearing deflections are about equally important). Ratio $R_1 = 0.75$.

(4) Weight uniformly distributed; fundamental mode is parabolic as in Fig. 10 (this is approximately the shape for bending deflection). Ratio $R_1 = 0.555$.

The effect of non-uniform distribution of weight will be similar to the effect of variation of mode shape. It appears that for the fundamental mode the maximum shear generally will vary between the value corresponding to the ordinate of the acceleration spectrum and about one half this amount. It can be shown that the sum of all the R_n-values for any symmetric structure must equal unity. Hence, R_1 cannot exceed unity, which is its value in the case of the simple oscillator. Also, the smaller the value of R_1, the greater must be the remaining R-values and the more important will be the shears due to the higher modes.

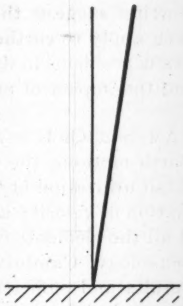

FIG. 9.—LINEAR FUNDAMENTAL MODE

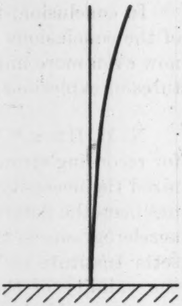

FIG. 10.—PARABOLIC FUNDAMENTAL MODE

Relative Importance of the Fundamental and the Higher Modes.—The relative importance of the fundamental and the higher modes in a particular case will depend on: (*a*) At what point of the structure shear is determined; (*b*) the values of the different R_n-values of the structure; (*c*) the periods and shapes of the different modes; and (*d*) the shape of the acceleration spectrum.

As an illustration, consider a uniform structure on a rigid base. Then, $R_1 = 0.81$; $R_2 = 0.09$; $R_3 = 0.032$ (from Table 1); and $T_1 : T_2 : T_3 = 1 : \frac{1}{3} : \frac{1}{5}$. Assume that the spectrum shape is given by Eq. 13—that is, $A(T) = \frac{0.2\,g}{T}$. Then, $A_1 : A_2 : A_3 = 1 : 3 : 5$; and, at the base of a uniform structure, $V_1 : V_2 : V_3 = 0.81 : 0.27 : 0.16$.

Variation of Shear with Elevation.—Away from the base, the situation will be somewhat different from that at the base. For example, using the same spectrum as before, at a point one third down from the roof of a uniform building, the maximum shears due to the different modes are in the ratios $V'_1 : V'_2 : V'_3 = 0.40 : 0.27 : 0.08$. Here, the second mode shear is about two thirds as great as the fundamental shear.

The effective acceleration for a section of a building, or the ratio of the maximum shear at that point to the total mass above it, is not a constant, as is generally implied in building codes, but increases with elevation. Considering only the fundamental mode of a uniform building, the following effectiveness factors are found:

Relative distance $\frac{x}{h}$ below the roof	Effectiveness factor C'_1
0.0	1.275
0.2	1.255
0.5	1.145
0.8	0.965
1.0	0.810

Thus, the maximum acceleration at the roof of a uniform building, caused by its fundamental mode, is 1.275 times the maximum acceleration of a simple oscillator of the same period, and is $\frac{1.275}{0.81} = 1.57$ times the effective acceleration for the entire structure. This is the "whip effect" mentioned by the author. It is present in any flexible structure.

In conclusion, the writer suggests that much of the information and many of the conclusions which apply to earthquakes are also applicable to what are now even more important problems in dynamics—namely, the effects on structures of explosions, and the impact of projectiles.

N. H. HECK,[33] M. AM. SOC. C. E.—With the development of the program for recording strong earth motions, the Coast and Geodetic Survey has recognized the necessity of full utilization of the results. Its principal contribution has been the determination of velocity and displacement (oscillatory) from the accelerograms so that all the elements of motion are known. The Massachusetts Institute of Technology, Cambridge, Mass., has cooperated in testing the validity of the results and sufficiency of their accuracy for engineering purposes. Little has been done in the application of the results to engineering design. However, the collection of engineering information after a destructive earthquake and the determination of building periods and the periods of other structures, in some cases at intervals during construction, constitute a close relation with practical engineering problems.

For these reasons the paper and previously reported work of Mr. Biot have been especially welcome. Its essential feature is the use of the earthquake spectrum obtained from strong-motion accelerograms with his mechanical analyzer. The spectrum gives a direct measurement of the maximum shear in a building due to a given earthquake. One thing is disconcerting—the high maxima in the relatively weak Ferndale (Calif.) shocks. The motions at El Centro, Calif., due to the Imperial Valley earthquake of March 18, 1940, were considerably greater than had been expected and accordingly one might expect very high maximum values of the spectrum for a major earthquake.

Mr. Biot states that there are a number of reasons for believing that the internal damping of structures is appreciable, and that since he has used an undamped pendulum, high resonance values are due in some measure to lack of damping in the torsion pendulum. The analysis obviously should be extended to include some forms of damped pendulums.

In general, it has been held that the tower type of building has special advantages as an earthquake resistant type, but the whip effect on the upper stories cannot be neglected.

The effect of the building on the ground is important from the viewpoint of the Coast and Geodetic Survey strong-motion program. By engineering advice all but about six of the sites adopted were in buildings and the instruments installed in the basements probably are affected by the motion of the building in the earthquake. The fact that the building at El Centro is small has enhanced the value of the conclusions from the El Centro record.

The paper emphasizes the importance of the determinations which have been made of the vibration periods of buildings, since without this knowledge the maximum stress cannot be determined for a given building. The conclusion that change of period damps out resonance is undoubtedly correct, but further investigation of the possibility that prolonged resonance in special cases may result in serious damage is needed.

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[33] Asst. to the Director, U. S. Coast and Geodetic Survey, Washington, D. C.

The paper is stimulating to thought and opens up important lines of investigation. Interest in the subject has not been diverted by the war effort, but few persons are left to work on the problems. Interest in high-acceleration high-frequency vibrations has greatly increased but progress should be made in the entire field of vibrations including the relatively low-frequency long-period vibrations characteristic of destructive earthquakes.

Frank Neumann,[34] **Affiliate Am. Soc. C. E.**—Equivalent acceleration spectrums, as well as those for velocity and displacement, open up a new and vitally important avenue of attack on the engineering aspects of the earthquake problem. The way now seems to be clear to furnish the engineer seismic data in a form that he can put to direct use. The development of spectrums by practical methods has been considered a seismological as well as an engineering problem, and workers in both fields have made important contributions toward current accomplishments.

An interesting feature of the "standardized" spectrum is that, if reduced to "equivalent maximum velocity," the curve will show a constant maximum velocity for all periods greater than 0.2 sec. This would seem to provide an "intensity factor" in standard units for any earthquake motion that has been thus analyzed. This factor, in the case of the Helena north-south component, is 2.4 times the maximum velocity of the ground motion; but the Helena "standardized" curve cannot, as Professor Biot infers,[35] be considered typical of all curves in view of the effect of epicentral distance, for one thing, on ground periods. A notable instance of this distance effect occurred in 1933 when an earthquake originating 183 miles from the San Jose (Calif.) Bank of America Building registered an acceleration, on the thirteenth floor, seven times greater than the recorded ground acceleration of 0.005 *g*. The approximate resonance period in this case was 1.6 sec. The need for families of spectrums is thus obvious if earthquake motions are to be adapted to standard patterns. It appears that, for any one earthquake, the acceleration spectrums at different epicentral distances may be restricted in magnitude to the limits of a spectrum obtained very close to the epicenter, barring all geological influences.

It is thought that the apparent multiplicity of periods in the short-period end of the spectrum may be due, in some measure at least, to difficulty in following the acceleration curve during the analyzer work with the necessary precision, especially since resonance is likely to occur between the motion of the torsion-head lever arm (which is manually operated) and the motion of the pendulum. In a similar operation in 1936[36] (to determine the equivalent response of a 5-sec pendulum with a torsion pendulum analyzer) the writer found that it required considerable skill to follow the east-west component of the Helena acceleration curve, moving with the equivalent of the middle speed used in Professor Biot's work, even though the curve was magnified seven times. It is believed that the use of acceleration curves expanded many times, as now used in integration processes in the Coast and Geodetic Survey, would work for greater precision at these critical periods. As to the general form of the spectrum, recent ground period studies[37] indicate that the period pattern changes somewhat with the orientation of the seismograph pendulums, a condition which may be expected to be reflected in the computed spectrums.

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[34] Chf., Section of Seismology, Div. of Geomagnetism and Seismology, U. S. Coast and Geodetic Survey, Washington, D. C.

[35] "A Mechanical Analyzer for the Prediction of Earthquake Stresses," by M. A. Biot, *Bulletin*, Seismological Soc. of America, Vol. 31, No. 2, April, 1941, pp. 163–164.

[36] "A Mechanical Method of Analyzing Accelerograms," by Frank Neumann, *Proceedings*, Am. Geophysical Union, Washington, D. C., May 1 and 2, 1936; also "The Simple Torsion Pendulum as an Accelerogram Analyzer," by Frank Neumann, *Publications du Bureau Central Seismologique International*, Serie A: Travaux Scientifiques, Fascicule 15, 1937.

With reference to that part of the spectrum below 0.2 sec, the error of Coast and Geodetic Survey accelerometers very seldom exceeds 15% or 20% between 0.1 and 0.2 sec. An experiment is now (1942) under way to test the practicability of shortening accelerometer pendulum periods with a view to increasing the range (downward) for recording true acceleration, and increasing the general recording capacity by decreasing the sensitivity.

It is unfortunate that some of the most valuable acceleration records are difficult to read because of overlapping of curves and occasional overdevelopment of the photographic paper. Overdevelopment is sometimes necessary to bring out the faint records of light spots moving at higher speeds and through larger amplitudes than had been anticipated in the original recorder design. Although these records do not always offer the sharpness of detail and cleanliness which Professor Biot desired in his work, it is nevertheless a fact that only one record to date has been lost to the extent that the data are not available for engineering studies. Available records include two of the Long Beach (Calif.) earthquake of 1933 yielding maximum accelerations of 0.06 *g* and 0.21 *g*, and maximum displacements of 20 cm double amplitude; also one of the Imperial Valley earthquake with maximum acceleration of 0.35 *g* and double amplitude of 40 cm. Comparing these values with the maximum of 0.16 *g* and displacement of 6 cm involved in Professor Biot's Helena spectrum, some idea may be obtained of the magnitude of the spectrums to come. Moreover, in spite of the magnitude of the recorded motion in the case of the Imperial Valley earthquake, the damage at the recording point, El Centro,[38] was "confined to walls that were not reinforced or tied, and to projecting balconies." The amplitude to be expected in an earthquake of catastrophic proportions can only be conjectured.

Concerning the inference that the torsion pendulum is[39] "a less accurate but simpler analyzer" than the proposed electrical apparatus, there will always be a doubt in the writer's mind until actual experience with both types proves the point. Any lack of accuracy in torsion pendulum analyzers would seem to lie in the manner in which the variable acceleration is transferred to the torsion head of the pendulum rather than in the functioning of the pendulum itself; and again, with reference to the statement in Section II of the paper that "it takes an average of eight hours to plot one spectrum curve," engineers are likely to be misled on the magnitude of the task which future workers in this field will face, remembering that even records of intensity VIII earthquakes present a much more formidable problem in processing than one of the Helena type which does not represent an intensity greater than VI. The Helena instrument was mounted on a limestone foundation and not on valley fill where most of the damage occurred in the 1935 earthquake.

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[37] "Analysis of the El Centro Accelerograph Record of the Imperial Valley Earthquake of May 18, 1940," *Manuscript 9*, U. S. Coast and Geodetic Survey, Washington, D. C.

[38] "Analysis of the El Centro Accelerograph Record of the Imperial Valley Earthquake of May 18, 1940," *Manuscript 9*, U. S. Coast and Geodetic Survey, Washington, D. C., p. 3.

[39] "A Mechanical Analyzer for the Prediction of Earthquake Stresses," by M. A. Biot, *Bulletin*, Seismological Soc. of America, Vol. 31, No. 2, April, 1941, p. 154.

JACOB FELD,[40] M. AM. SOC. C. E.—In characteristic fashion the author prepares a complete and logical derivation of various factors concerning the effect of earthquake vibration on physical structures. The writer is chiefly interested in that section concerning the influence of the foundation on the motion of blocks. In the approach to a solution of that factor, Professor Biot makes the assumption that the soil characteristics do not change during the vibration. Based also upon the assumption that the soil displacements within the limits of the deformations or deflections resulting from the vibrations of the block do not exceed the elastic limit strain, he deduces a formula for the rocking motion of the block, as restrained by the resisting characteristics of the soil.

Practical experience with vibrating structures embedded in soil does not confirm the assumptions made. It is well known in construction practice that, when vibrating machinery such as air compressors are placed on concrete foundations embedded in soil that has been made very heavy to dampen vibrations and thereby reduce their transference to adjacent structures, there is a change in the damping effectiveness with time. The only explanation for the change must be an alteration of the characteristics of the soil. It seems that continuous vibration of this nature increases the density of the surrounding soil. In some instances the writer has noted that a definite gap develops between the faces of the embedded concrete foundation and the adjacent soil. In such instances the vibration transference is considerably reduced, suddenly appearing again when, due to rain or other causes, the gaps are filled in. In one contract the writer designed a support that was kept entirely free from the adjacent soil and was bedded on a layer of cork. A gap was maintained on the four sides of the supporting base over a period of three years, and measurements made of the transference of vibration showed a loss of at least 90% of the compressor vibration wave. The period of the compressor vibration was 0.25 sec. Measurements were made by a vibrograph instrument which reported both horizontal and vertical components on a celluloid sheet.

Investigations of the effect of earthquakes and similar vibrations on the lateral pressure of earth have been made by Nagaho Mononobe and Haruo Matsuo.[41] These men found that the pressure of earth against a vertical retaining wall increased quite suddenly to a maximum upon the application of a vibration similar to an earthquake. The amount of increase depended upon the severity of the shock; but in each case the total pressure was found to be maximum immediately upon the application of the vibration, with a slight decrease thereafter. However, the measured pressure after vibration was larger than that before vibration, and the only explanation seemed to be that the material changed in character. This was somewhat substantiated by a series of tests with more compact fills, which did not show as large an increase in pressure from vibration. It also was found that the pressure against rigid walls was increased to a larger extent than pressure against deformable walls. This again seems to indicate that the change in pressure was caused by a change in the characteristics of the fill. Incidentally, the author concludes that the maximum earth pressure under earthquake conditions can be calculated from the usual earth-pressure formula by the use of a density equal to the resultant acceleration obtained by graphic summation of the acceleration of gravity and the maximum seismic acceleration.

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[40] Cons. Engr., New York, N. Y.

[41] "Experimental Investigation of Lateral Earth Pressure During Earthquakes," by Nagaho Mononobe and Haruo Matsuo, *Bulletin*, Earthquake Research Inst., Vol. X, 1932, Pt. 4.

GEORGE W. HOUSNER,[42] JUN. AM. SOC. C. E.—In so far as it treats the earthquake problem from the viewpoint of vibration theory, this paper should prove of interest, particularly to engineers working in regions of seismic activity. That the effect of an earthquake is dependent on the physical properties of structures has long been known, and it was early recognized that the period of vibration of a structure was a significant index of its properties. One of the first attempts to measure the intensity of strong-motion earthquakes was made by Robert Mallet (1810–1881), a famous British engineer who made important contributions to seismology. The method used by Mallet[43] was to set up a group of six cylindrical wood blocks of graduated heights. By observing which of the blocks were overturned, he hoped to be able to classify the destructiveness of the earthquake. This wood-block approach has continued to interest engineers to the present time, but the rocking motion of a block is so complex that it has not been possible to draw any conclusions from the behavior of such a set of blocks.

A more satisfactory approach was taken by the late K. Suyehiro (1877–1932), noted Japanese engineer and one-time head of the Earthquake Research Institute of Japan. Suyehiro constructed a seismic vibration analyzer[44] to measure the intensity of earthquakes for different periods. This instrument[45] consisted of thirteen oscillators of different periods of vibration ranging from 0.22 to 1.81 sec. Such an instrument determines thirteen points on an earthquake spectrum such as that presented by the author. A defect of the instrument was the insufficient number of oscillators—that is, the gaps between the periods recorded were too large. Acting on a suggestion from Professor Martel, the San Francisco office of the U. S. Coast and Geodetic Survey constructed an instrument on this principle with a larger number of oscillators whose maximum displacement is recorded.

The strong-motion earthquake records furnished by the U. S. Coast and Geodetic Survey since 1932 have made it possible to compute the response of oscillators of different periods, thus greatly increasing the scope of this phase of seismological research. This computation can be done in several ways—namely, by numerical integration as was done by M. P. White[6] at the California Institute of Technology in Pasadena, and by F. Neumann[46] of the U. S. Coast and Geodetic Survey; by actuating a shaking table and measuring the response of oscillators as was done by Professor Jacobsen[7] at Stanford University in Stanford University, Calif.; or by utilizing the torsion pendulum as was done by Mr. Savage[9] at the Bureau of Reclamation and also by Professor Biot.

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[42] U. S. Engr. Office, Los Angeles, Calif.

[43] "Dynamics of a System of Rigid Bodies," by E. J. Routh, Macmillan Co., 1913, p. 175.

[44] "A Seismic Vibration Analyzer," by K. Suyehiro, *Proceedings*, Imperial Academy II, 1926 (Tokyo), p. 286.

[45] A brief description of the instrument may be found in "Engineering Seismology: Notes on American Lectures," by K. Suyehiro, *Proceedings*, Am. Soc. C. E., May, 1932, Pt. 2.

A defect in this approach, as pointed out by the author, is that the computations are based on records of unknown accuracy. (A testing program to determine the accuracy of these strong-motion seismographs is being conducted at the Massachusetts Institute of Technology, in Cambridge, for the U. S. Coast and Geodetic Survey. The results of this investigation have not yet [June, 1942] been made public.) Any inaccuracy in the accelerogram will be reflected in the computed oscillator response curve or spectrum. It would be well for the author to call attention to the fact that the most reliable point on the computed spectrum is that for a period equal to zero. As stated by Professor Biot, the value of the computed oscillator response curve or spectrum is much increased by virtue of its application to structures more complex than a simple vibrating mass. This is particularly true since the vibration theory required has already been developed in the theory of acoustics. It is interesting to note that a building with shearing deformations is the mathematical analogue of a vibrating string, and the building with flexural deformations is the analogue of a tuning fork. This, then, is another instance in which the mathematics required by engineers has already been developed by physicists in the solution of analogous problems; even the analogue of the building with the flexible first story has been treated by them.

Since 1936, a program of earthquake research, sponsored by the Los Angeles County (California) Department of Building and Safety, has been conducted at the California Institute of Technology under the direction of Professor Martel. During this time spectra or oscillator response curves have been evaluated for the following earthquake records: Los Angeles Subway Terminal, March, 1933; Vernon, Calif., March, 1933; Los Angeles Subway Terminal, October, 1933; El Centro, Calif., December, 1934; Helena, Mont., October, 1935; and El Centro, May, 1940. In engineering terminology, these so-called spectra are in reality influence lines for maximum shear. The influence lines or spectra for the aforementioned earthquakes were found to be of two general types. These are illustrated in Fig. 11 where it will be noted that the records used were obtained from the same seismograph but for different earthquakes. The curve in Fig. 11(*a*) is similar to those presented by the author—that is, the ordinates decrease rapidly for increasing periods of vibration. In Fig. 11(*b*) this rapid decrease does not occur. It is seen that in this case the shear is not reduced, in general, appreciably by lengthening the period of vibration.

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[46] Mr. Neumann also called attention to the use of the torsion pendulum for this problem in "Special Publication 201, Earthquake Investigations in California 1934–1935," Dept. of Commerce publication.

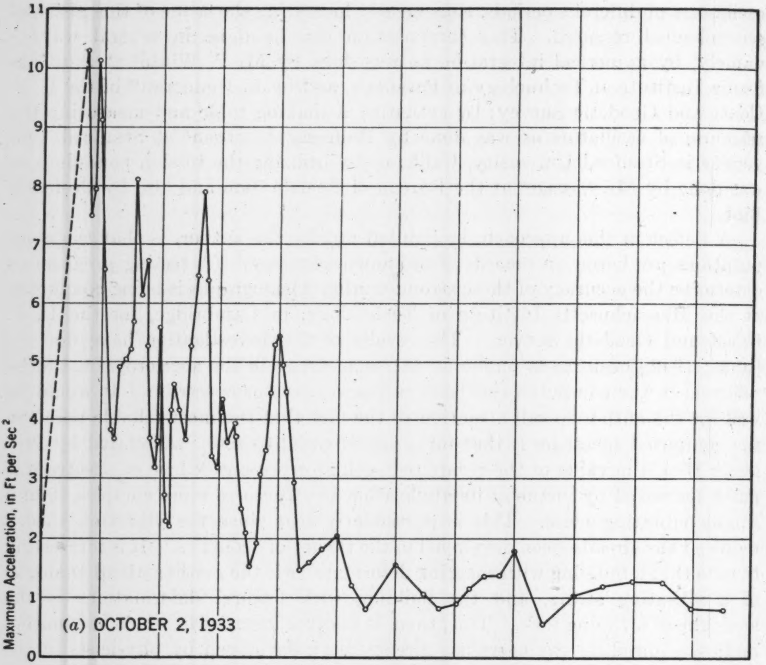

FIG. 11.—SPECTRA OF THE N 141° W COMPONENT OF THE LOS ANGELES SUBWAY TERMINAL RECORD

The problem of foundation yielding discussed by the author is a phenomenon well worth studying. It was first investigated by Suyehiro and Ishimoto[47] in 1926. These investigators constructed a special vibrograph and made measurements on the rocking oscillations of buildings.[45] Some results of their measurements are shown in Table 4. These were made on reinforced concrete buildings of very rigid construction. To compare values with the results of the author's analysis, it is necessary to recompute Table 3. In his calculations, Professor Biot has used the value $p_s = 84$ lb per sq in., apparently confusing allowable bearing pressure with actual building weight. On the basis of the dead load of typical buildings, it would be proper to take $p_s = 1.04$ lb per sq in. for each 12 ft of height. Recomputing on this basis gives the following values:

TABLE 4.—OBSERVED PERIODS OF VIBRATION

Laboratory	Stories	METERS			Ground	Period (sec)
		Length	Breadth	Height		
Aeronautics; main building	2[a]	56	15	11	Reclaimed	0.50
Mitsubishi Laboratory; main building	2[b]	60	14	10	Loam	0.35
Mitsubishi Laboratory; metallurgical room	1[a]	11	9	5	Loam	0.20

[a] No basement. [b] With basement.

Values of r (ft)	For $2L = 50$ ft, period T, in seconds, equals:
16	0.04
50	0.35
100	0.99

Only the structure 50 ft wide has been computed and the value $r = 16$ represents a height of 12 ft. As recomputed, it is seen that the calculated periods of the rocking oscillations are much shorter than the corresponding periods of the first modes of vibration would be. These computations would indicate that foundation yielding does not cause an appreciable stress reduction due to period lengthening.

A very extensive program of vibration measurements is reported in "Special Publication 201."[46] The periods of 212 buildings were measured and, of these, four showed evidence of possible foundation yielding. These four were low, rigid buildings on soft ground. The building vibrations (due to microseisms) measured by Suyehiro and Ishimoto[47] would also indicate that foundation yielding may be of importance for low, rigid structures on soft ground. It is of interest to note that Suyehiro concluded that the rocking oscillations were nonlinear since the period was dependent on the amplitude. The approach used by the author would not be applicable to nonlinear vibrations, although in itself nonlinearity would seem to be beneficial in reducing maximum shears.

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[47] "Vibrations of Buildings," by K. Suyehiro and M. Ishimoto, *Proceedings*, Third Pan-Pacific Science Cong., Tokyo, 1926, p. 1482.

It appears that more experimental research is required to determine the fundamental characteristics of rocking oscillations of buildings.

HOMER M. HADLEY,[48] ASSOC. M. AM. SOC. C. E.—The gist of Professor Biot's scientific approach to the problem of earthquake resistant construction and to the improvement of codes appears to lie in his advocacy that recognition be given to the varying responses which different structures having different periods of vibration give to the same earthquake and, further, that recognition be given to varying types of foundations with their differing elastic properties and consequent differing effects upon structures built upon them. That such differences exist cannot be questioned, and it would seem entirely reasonable and logical to include consideration of these matters in design. When specific application is attempted, however, the engineer finds himself embarrassingly deficient in reliable data upon which to proceed. A sage saying which the writer heard years ago in Alaska was this: "Now, if we only had some ham, we'd have some nice ham and eggs for breakfast—if we only had some eggs." In short, for ham and eggs one must have both ham and eggs. Even so, for the application of the scientific method, the engineer must have reliable data for its basis. What is definitely known about earthquakes, about foundations, about the structures themselves?

One scientific fact about an earthquake always faithfully reported is the location of the epicenter. Not much else may be learned of it, but the location of the point on the earth's surface above the subterranean origin is readily determined. From the accumulation of these data engineers have come to know what regions are most subject to earthquakes, but beyond this there is no particular application of the knowledge made to earthquake resistant construction. Even the position of the epicenter is subject to question at times. At Long Beach, Calif., on March 10, 1933, the epicenter was first reported in the ocean bed some 15 miles off Newport Beach, Calif., where only slight damage occurred.[49] It seemed somewhat anomalous that the severe damage should have occurred at places much farther removed from the epicenter than Newport Beach. Later observations[50] led to the conclusion that there had been a series of epicenters at various points along the Inglewood, Calif., fault, there being a series of successive origins. Since these later epicenters were adjacent to the districts of severe damage, that much empirical confirmation of their existence is found.

As to the earthquakes themselves, the various records show certain common characteristics: An initial period of slight movement; then the period of severe, violent movement; and then the final period of diminishing, dying movement. As to their intensities, it is known that they vary widely. The accelerograms of the two Ferndale, Calif., earthquakes of February 6, 1937, and September 11, 1938,[2] are quite different. Who, on March 1, 1937, for instance, having seen the accelerogram and spectrum of the February 6, 1937, earthquake and of all preceding earthquakes at Ferndale, could have said that a more severe earthquake would occur on September 11, 1938, or, on any particular date, could have foretold what its accelerogram or spectrum would look like? Of necessity, such a prophet would have to employ "art based on observational facts and experience" rather than "science" as the basis of his prediction; so it is, with Ferndale, Calif., or with any Ferndale, large or small, upon this earth. Somewhere in his writings, Omori discussed the earthquake peculiarities of certain places and stated that there were individual characteristics and features to be observed; but except for a certain directional similarity there was nothing strikingly individualistic in these characteristics. Now, and probably for a long time in the future, engineers must be empirical in judging the earthquake to be assumed in design.

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[48] Regional Structural Engr., Portland Cement Assn., Seattle, Wash.

[49] "Preliminary Report on the Long Beach Earthquake," by Harry O. Wood, *Bulletin*, Seismological Soc. of America, April, 1933.

[50] "The Determination of the Extent of Faulting with Application to the Long Beach Earthquake," by Hugo Benioff, *ibid.*, April, 1938.

As to the influence of the foundations on the motion of the structure, Professor Biot clearly states the complexity of the problem (see heading "V.—Influence of Foundation on Motion of Blocks"): "* * * it involves a complete knowledge of the propagation and properties of the seismic waves in the strongly heterogeneous surface layers of the earth as well as their diffraction and reflection by objects built on the surface." This undoubtedly is true and undoubtedly presents an obstacle to "science." It is stated that Eq. 38 implies the knowledge of the elastic constants E and μ of the soil but that "These are obtainable from tests made by M. Ishimoto and K. Iida.[13]" This is a happy solution of the difficulty: See what Messrs. Ishimoto and Iida give for these values. Nevertheless, it is to be noted in their quoted findings that a change of moisture content of silty clay from 42% to 50%—that is, a change of 8%—changes E from 792 lb per sq in. to 5,750 lb per sq in. Who is to know whether what he terms silty clay is the same as the silty clay defined by Messrs. Ishimoto and Iida, or what the moisture content of that silty clay may be when a foundation is subjected to an earthquake? Again, what manner of bedrock underlies the silty clay? Those who have driven the highways of Southern California and looked upon the confused, contorted, heterogeneous geologic structure revealed in cuts and excavations must feel mistrust and misgivings in whatever assumptions they may make as to the values of E and μ for scientific calculations.

As to the elastic properties of the structure that is being designed, it should be possible from measurements, past and future, of existing comparable structures to assume values fairly approximate to those that will develop. With values of T decided on, with what spectrum shall they then be used? Apparently there is only a very limited number in existence. Fig. 6 is a "standard" spectrum of two earthquakes only—that of Helena, Mont., on October 31, 1935, and the Ferndale earthquake of September 11, 1938. It shows that a natural period of 0.5 sec gives an acceleration of 0.4 g; a natural period of 1.0 sec gives an acceleration of 0.2 g. From the text one learns (see heading "III.—Spectrum Related to Design and Actual Behavior of Structures") that "The differences [in the behavior of buildings of different periods] are not as great as the spectrums would indicate." Apparently "Science" must be empirically modified, at times.

There is a further matter of engineering seismology to be considered, as well as the matter of intensity of stresses, which can be modified and altered by change of sections. This is the matter of greater deformations that accompany longer periods. The Marunouchi Building in Tokyo, Japan, the largest steel-frame building in that city, was under construction in 1922 when a considerable earthquake occurred. As a result of the distortions it sustained at that time, it was felt that the building needed stiffening. Consequently, a number of panels of diagonal bracing were introduced, extending from basement to roof, and these were covered with metal lath and plaster. When the great earthquake of September 1, 1923, occurred, these bracing panels were loaded terrifically at the outset and either stretched amazingly or ruptured. The natural period of the building then reverted to its original value. A number of different people who were in the building told the writer that they did not enjoy the experience. Partitions were cracked and shattered, marble trim in the corridors loosened and fell, and pendant lighting fixtures swung so violently that their reflectors knocked plaster off the ceilings. After it was all over the owners—perhaps not scientifically, yet empirically—rebuilt the exterior and interior walls and introduced bracing walls, for the definite purpose of making the building stiff—that is, giving it a low natural period of vibration. States a well-known advertisement: "Ask the man who owns one."

It must not be thought that making a building stiff will prevent an earthquake from being felt in it. On the contrary, whatever movement occurs at the foundations will be transmitted with little damping to the roof, and the building's contents will be shifted and perhaps thrown to the floor quite as if they stood upon the ground. On the other hand, after the earthquake has passed, a well-designed, rigid-type building will be found very satisfactory to all concerned. It may be added that all the comment made herein is limited to buildings of moderate height. As to skyscrapers, the writer has no knowledge other than that such buildings successfully passed through the San Francisco, Calif., earthquake of 1906.

Yet another aspect of the earthquake problem is the matter of occurrence. Fortunately, severe earthquakes are not frequent. It may well be questioned whether provision against them warrants highly elaborate and refined engineering design. The severe damages which have repeatedly occurred in the past have not resulted from failure to design for this or that value of shearing stresses in the structures involved but from completely ignoring earthquakes and all horizontal forces they entail. A definite observed deficiency in present design or proof that current practise is unwarrantedly too severe should constitute the basis of change and progress.

There is a widespread feeling among American engineers that some special virtue is imparted to the solution of highly complex, not to mention uncertain and unknowable, problems by going through mathematical processes in reaching a decision regarding them. They prefer to assume E and μ and all the various other variables that may be involved and to "figure" themselves to an answer rather than to accumulate empirical data and proceed simply and directly from them. There is much to be said for the mathematical process, but it contributes no more truth, no more science, than is put into its basic assumptions. The writer would conclude with the words of the late David A. Molitor, M. Am. Soc. C. E., used in the "Synopsis" of his paper "Wave Pressures on Sea-Walls and Breakwaters":[51] "Since all factors contributing to the solution of the problems herein considered are only approximately knowable, no theoretical approach was deemed advisable."

ROBERT E. GLOVER,[52] ESQ.—The author develops a method of using a torsion pendulum to determine the amount of resonance to be expected when a structure of known frequency is subjected to an earthquake. This ingenious procedure makes use of an accelerometer record obtained during the quake. A torsion pendulum operation based on the use of a displacement curve, derived by double integration of an accelerometer record, has been used by the writer for a similar purpose. A comparison of results obtained by the two methods is shown in Table 5, which is based on the East-West component of the Helena quake of October 31, 1935.

TABLE 5.—COMPARISON OF ACCELERATION RESULTS

Structure period (sec)	MAXIMUM ACCELERATION	
	Author's data[a]	Writer's data[b]
0.20	0.73 g	0.79 g
0.25	0.67 g	0.69 g
0.40	0.50 g	0.52 g
0.50	0.32 g	0.27 g
0.80	0.19 g	0.20 g
1.00	0.24 g	0.26 g
2.00	0.10 g	0.06 g

[a] Scaled from author's spectrum curve. [b] Based on an integration made by the U. S. Coast and Geodetic Survey.

The slight differences are probably due to the choice of values for the constants of integration used in computing the displacement curve. Since the accelerometer instruments are generally arranged to be put in motion by a starter, which responds to ground movement, a small part of the record near the beginning is lost, and there is, therefore, some room for exercise of judgment when the values of these constants must be selected. The results obtained by the two methods appear, however, to be in essential agreement.

M. A. BIOT,[53] ESQ.—Mr. Rich presents an application of the operational calculus to a simplified case. Since the response of structures to earthquakes is essentially of a transient character, it is natural to apply the methods specifically designed for the treatment of such phenomena. However, the difficulty in applying the Heaviside methods to the full extent is the non-analytical and random character of the seismogram. The simplified procedure derived by the writer avoids the computation of the actual motion and makes possible the direct evaluation of the maximum stress by the use of a standardized spectrum. However, the operational method can be used, as such, in combination with the spectrum, as a convenient way of deriving the effectiveness coefficients or any desired characteristic in the structure. This is especially true when the information needed is restricted to a particular variable or location in the structure. The operational method is also useful in investigating the response to simplified theoretical earthquakes, as shown in the examples set forth by Mr. Rich. The latter method acquires special value if a great number of cases are solved and if general principles are uncovered as to the comparative behavior of structures under standardized earthquakes. This approach might truly be called a mathematical, experimental method.

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[51] *Transactions*, Am. Soc. C. E., Vol. 100 (1935), p. 984.

[52] Senior Engr., Bureau of Reclamation, U. S. Dept. of the Interior, Denver, Colo.

[53] Research Associate in Aeronautics, California Inst. of Technology, Pasadena, Calif.; Asst. Prof. of Mechanics, Physics Dept., Columbia Univ. (on leave of absence).

Professor Hoff mentions that the results obtained at Stanford University show the influence of the friction on the oscillator response. It is interesting to note the experimental fact that the response is very sensitive to changes in friction when the latter is small. It is probable that the differences in peak spectrum accelerations between the Stanford results and those of the writer are partly due to errors on the small-period components of the displacement cam. The question of the possible accumulation of the stresses in each mode by the simultaneous occurrence of their maximum values at a given point is well worth investigating. A conservative estimate can be made theoretically by adding the maximum values associated with each mode at various locations in the structure.

As stressed by Professor White, the idea of an automatic analyzer to avoid the numerical work in evaluating the response of a structure is not new. In the writer's earlier work in 1932, the mathematical groundwork was laid for showing the possibility of using the accelerogram directly without the process of evaluating the displacement. An electric model was suggested, of which the present torsional pendulum is the mechanical analogue. An electric model has been developed by Arthur C. Ruge, Assoc. M. Am. Soc. C. E., at the Massachusetts Institute of Technology in Cambridge, Mass.

Professor White raises an interesting and important point regarding the nature of the peaks in the spectrum. The writer called attention to the lack of correlation between the peaks in the two Ferndale earthquakes, (*B*) and (*C*), as supporting evidence that these peaks are not characteristic of the location. It is possible that some peaks might be characteristic of the location, or of a group of buildings, and that these would stand out particularly in earthquakes of small intensity where the effects of resonance seem to be more acute. More probably the hatched appearance of the spectrum is due to random transients superposed on a basic periodicity distribution. Professor White makes the statement that the whip effect, although less pronounced, also appears in structures that are not necessarily tapered. He also suggests the application of the present results to the effect of explosions and projectile impact on structures. The writer wishes to mention another possible application—determining the dynamic effect of wind on structures by introducing the spectrum of atmospheric turbulence.

Mr. Heck mentions the importance attached by the U. S. Coast and Geodetic Survey to the recording of high accelerations throughout the frequency range of earthquakes. The results already accomplished in this direction were an absolute prerequisite to the writer's work.

Mr. Neumann rightly warns against using a particular spectrum as a basis for final standardization. The writer uses the standardized Helena spectrum merely as an example. The spectrum peak seems to be correlated both in intensity and frequency to the distance of the epicenter. As stated by Mr. Neumann, it is probably true that the torsion pendulum analyzer is not necessarily less accurate than a more intricate electrical apparatus. The writer's statement on this matter was perhaps somewhat conservative, since the spectrum curves could be duplicated closely in spite of the rather crude design of the analyzer and the use of non-magnified seismographic records.

Mr. Feld recalls the change of damping characteristics with time in a vibrating soil. As stated by the writer, the treatment of a foundation is restricted to its elastic properties. The results are in good agreement with tests on actual foundations made by Mr. Barkan.[16] The magnitude of structure and soil damping is a factor left to future experimental investigation. The methods of the paper are applicable if it is desired to introduce the effect of this damping on the amplitudes of oscillation. However, in seeking to establish experimental data, it will be well to keep in mind that the vibrator itself can have a considerable effect on the properties of the soil.

The two spectra computed by Mr. Housner are interesting, in that they show the same general appearance as those obtained by the writer. The ratio of the spectrum peak to the maximum acceleration is also of the same order. The spectrum of the March 10, 1933, earthquake shows relatively high equivalent accelerations (of the order of 20% gravity) for the large periods. Mr. Housner presents an application of the writer's simplified formula for the rocking period of blocks. His results are based on a much lower bearing pressure than that used by the writer, and the rocking periods in this case are relatively short. Taking into account the possibility of the soil being considerably more spongy than assumed in the computations, it should be concluded that the effect of the foundation is important or not, depending on the magnitude of such factors as soil elasticity, foundation size, and height and rigidity of the building. It would seem, however, that in the case of towers or chimneys the rocking effect should be preponderant. In regard to the relations between acoustics and engineering seismology, a theorem[4,54] was originally developed by the writer which states that the energy accumulated by an oscillator depends only on the intensity of the impulse spectrum for the particular frequency. This theorem forms the basis of the present method and justifies the use of the spectrum.

Mr. Hadley's pimented discussion is enjoyable reading. His argument reduces to a confession of man's total ignorance of the subject, and the defeatist viewpoint that nothing can be done about it "Now, and probably for a long time in the future." Mr. Hadley, however, is anxious not to discredit the esoteric art of earthquake-proof design by contributing the following statement:

> "* * * after the earthquake has passed, a well-designed, rigid-type building will be found very satisfactory to all concerned. * * * As to skyscrapers, the writer has no knowledge other than that such buildings successfully passed through the San Francisco, Calif., earthquake of 1906."

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[54] "Acoustic Spectrum of an Elastic Body Submitted to a Shock," by M. A. Biot, *Journal*, Acoustical Soc. of America, Vol. V, January, 1934, p. 207.

This, said the medieval doctor, is a sleeping potion; it makes you sleep because it contains a soporiferous virtue. It has been the writer's experience that a feeling of uneasiness usually develops when proselytes of the empirical method are brought in contact with symbols such as E and μ, even if they do not represent an essential part of the argument. This is entirely unfortunate and unjustified, as there is no fundamental conflict between empiricism and science. Mr. Hadley's objections might well have been raised three hundred years ago against the basic principles of Newtonian mechanics with their abstract and elusive concepts of force and mass.

Mr. Glover's results are in good agreement with those of the writer. They point to the comforting fact that satisfactory duplication of results is possible even when different methods and instruments are used.

In conclusion, the writer wishes to thank all those who have taken time to contribute to the present discussion for their constructive criticism.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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TRANSACTIONS

Paper No. 2184

HYDRODYNAMICS OF MODEL STORM SEWER INLETS APPLIED TO DESIGN

By G. S. Tapley,[1] Assoc. M. Am. Soc. C. E.

With Discussion by Messrs. Ralph W. Powell, and G. S. Tapley.

Synopsis

A technique is developed in this paper by which it is possible to apply experimental results from $\frac{3}{32}$-scale models of side-opening storm sewer inlets to the design of full-size prototypes. The method is applied to design in a single case, the development and presentation of design data for various other shapes, slopes, and dimensions being reserved for the future.

The phenomenon of flow into an inlet is characteristically gravitational in type, but the small depths of flow in the model suggest that the effect of viscosity is also important. Lateral deflection of the flow is an additional characteristic.

An equation was thus sought and obtained describing the portion of the flow entering the inlet in terms of Froude's and Reynolds' numbers and the moment of pressure and moment of momentum. The development of the form of the equation was facilitated by an analytical determination of boundary conditions corresponding to infinite and zero approach velocity.

Total differentiation yielded an expression in which Reynolds' number has an arbitrary value with respect to viscosity. A Reynolds' number of prototype magnitude, but with model velocity and linear dimension, was thus selected, the value of the coefficient of viscosity being computed. Integration by means of planimeter and the use of flow-distribution relationships yielded a result which represents flow in the model similar to flow in a prototype with regard to both Froude's and Reynolds' laws.

In developing the basic data important facts regarding flow distribution in the channel upstream from the inlet were discovered.

Notation.—The letter symbols in this paper are defined where they first appear and are assembled for reference in Appendix *B*.

Note.—Published in March, 1942, *Proceedings*.

[1] Junior Civ. Engr., Bureau of Eng., City of Los Angeles, Los Angeles, Calif.

I.—General

The action of prototypes can often be predicted from the observed operation of models together with the application of a single law of similarity, such as that of Froude or Reynolds. In other instances, such as in open-channel flow and ship-model work, no single law can be considered as being exclusively applicable.

The present case deals with the development of a technique for applying to the design of prototypes the experimental results from $\frac{3}{32}$-scale models of side-opening storm-sewer inlets of a type shown in Fig. 1. Flow approaching the inlets is turbulent and gravity is the principal force; Froude's law evidently is important. On the other hand, depths of flow in the model varied from about $\frac{1}{2}$ in. at the curb to film thickness at the outer edge of the stream; serious doubts would exist as to the validity of results in which the effect of Reynolds' number had been ignored. Flow approaching the inlets (except on streets with slopes flatter than about 0.005 ft per ft) is further characterized as being of the shooting type (velocity is greater than wave velocity). The flow, moreover, is wide in relation to depth and undergoes a lateral deflection in entering the inlet; a change of angular momentum is involved.

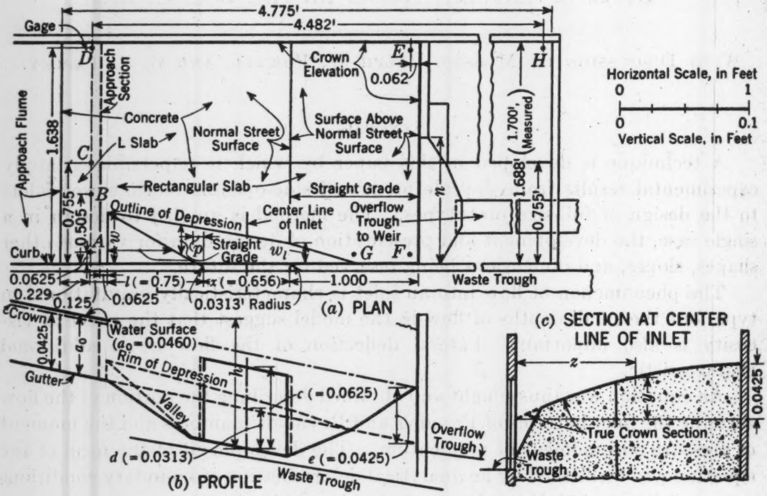
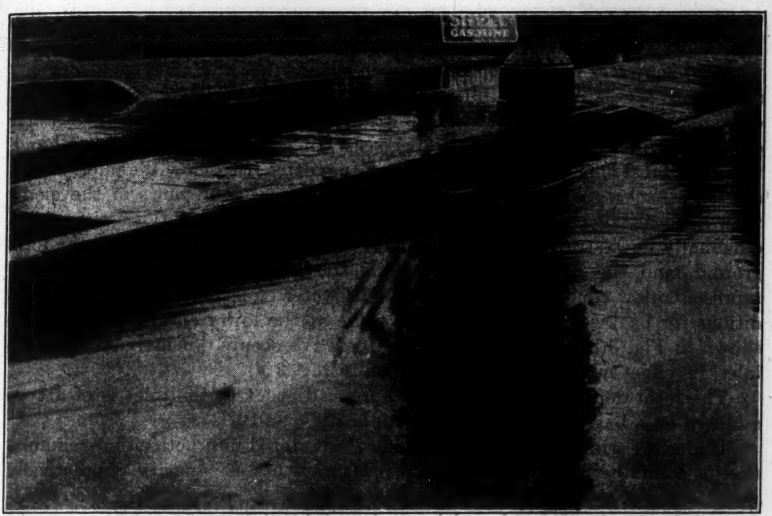
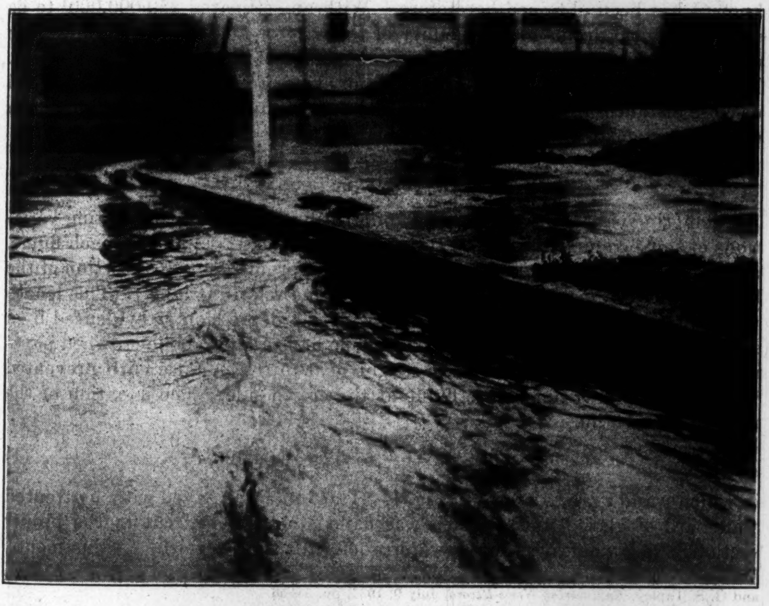

4.775'
4.482'
Gage
D
E
H
Crown Elevation
0.062
Approach Section
Approach Flume
1.638
Concrete
Normal Street Surface
Surface Above Normal Street Surface
1.700', Measured
Horizontal Scale, in Feet
0
1
0
0.1
Vertical Scale, in Feet
C
L Slab
B
Rectangular Slab
Straight Grade
1.688'
0.755
0.505
0.755'
n
Outline of Depression
Center Line of Inlet
Overflow Trough to Weir
w
p
Straight Grade
w_l
G
F
A
Curb
0.0625
0.229
0.125
0.0625
l (=0.75)
x (=0.656)
1.000
Waste Trough
0.0313' Radius
(a) PLAN
Crown
Water Surface (a_0=0.0460)
0.0425
a_0
Gutter
Rim of Depression
Valley
h
c (=0.0625)
d (=0.0313)
e (=0.0425)
Overflow Trough
Waste Trough
(b) PROFILE
(c) SECTION AT CENTER LINE OF INLET
z
y
0.0425
True Crown Section
Waste Trough

Fig. 1.—Storm-Sewer Inlet; 3/32-Scale

The development of technique described in this paper makes use only of the data necessary. Development of the remaining experimental data covering a range of shapes of depression, slopes of street, lengths of inlet, etc., remains for the future. The data utilized include only cases involving "free flow" at the lower end of the inlet. In this type of flow that portion of the water, if any,



(*a*) No Backwater Effects



(*b*) Considerable Backwater Effect

FIG. 2.—VIEWS OF FULL-SIZE INLETS SHOWING RELATIVE BACKWATER EFFECTS

which passes the lower end of the inlet has a velocity greater than wave velocity. Free flow is differentiated from "backwater effect," a condition that occurs when a backwater wave extends upstream of the lower end of the inlet. Fig. 2 shows two views of actual full-size inlets. The inlet in Fig. 2(*a*) has no local depression. There is a small gutter rise below the inlet, and no backwater effect. The flow, Q_t, is 0.9 cu ft per sec. In Fig. 2(*b*), on the other hand, there is a local depression and a 6-in. or 8-in. gutter rise below the inlet. Considerable backwater effect is evident. Fig. 3 shows four views of the model with various stages of backwater effect. For example, Fig. 3(*a*) is a view facing diagonally downstream toward the inlet. The approach flume is in the foreground, and the overflow notch is directly below the hammer. Fig. 3(*b*) represents flow conditions for a heavy backwater effect; and the limiting conditions for backwater effect, with the wave at the lower end of the inlet, are shown in Fig. 3(*c*). When the backwater wave occurs downstream from the inlet, its effect misses the inlet, as shown in Fig. 3(*d*).

The local depression or approach apron adjacent to the inlet is of a type adopted on a basis of experiments made in 1926 and 1927.[2]

Under the wide variety of conditions encountered in practice the accumulation of a number of isolated full-size measurements of capacity is practically worthless, and the advantage of a small model must be evident. The recentness of the development of model experimental technique, no doubt, accounts, in part, for the fact that no satisfactory solution of the problem (so far as is known) has been obtained heretofore. With an estimated $50,000,000[3] to be spent for storm drains in the vicinity of the City of Los Angeles, Calif., alone over a period of years, the economic importance of the problem is apparent. Moreover, a reexamination of the designs of existing installations probably will disclose instances in which the benefit of a part of investments already made in conduit systems is not being realized because of insufficient inlet capacity.

II.—APPARATUS AND TECHNIQUE

The principal apparatus consisted of a tilting framework (permitting variations of slope from zero to 10%) supporting a stilling tank, an approach flume, and the model depression and inlet. The plan, profile, and cross section of the latter are shown in Fig. 1. The approach flume was 10 ft long. (This length was sufficient to insure "developed" flow at the depression as evidenced by a series of tests.) It represented, to scale, one half of a parabolic-crowned pavement, 36 ft wide, having a crown height of 0.45 ft. A false curb prevented water from spilling out of the flume at the crown side. The equation of the model cross section is

$$(z - 1.688)^2 = -67.0\,(y - 0.0425) \qquad (1)$$

in which y and z are as shown in Fig. 1. The flume was paved with a patented wallboard of glazed finish, which was nailed to templates cut to the proper section. The whole was made waterproof with spar varnish. Adjustments,

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[2] See "Increasing Stormwater Inlet Capacity by Improved Gutter Approaches," by L. W. Armstrong and G. S. Tapley, *Engineering News-Record*, July 9, 1931, pp. 54–56.

[3] *Los Angeles Times*, June 10, 1939, p. 6.

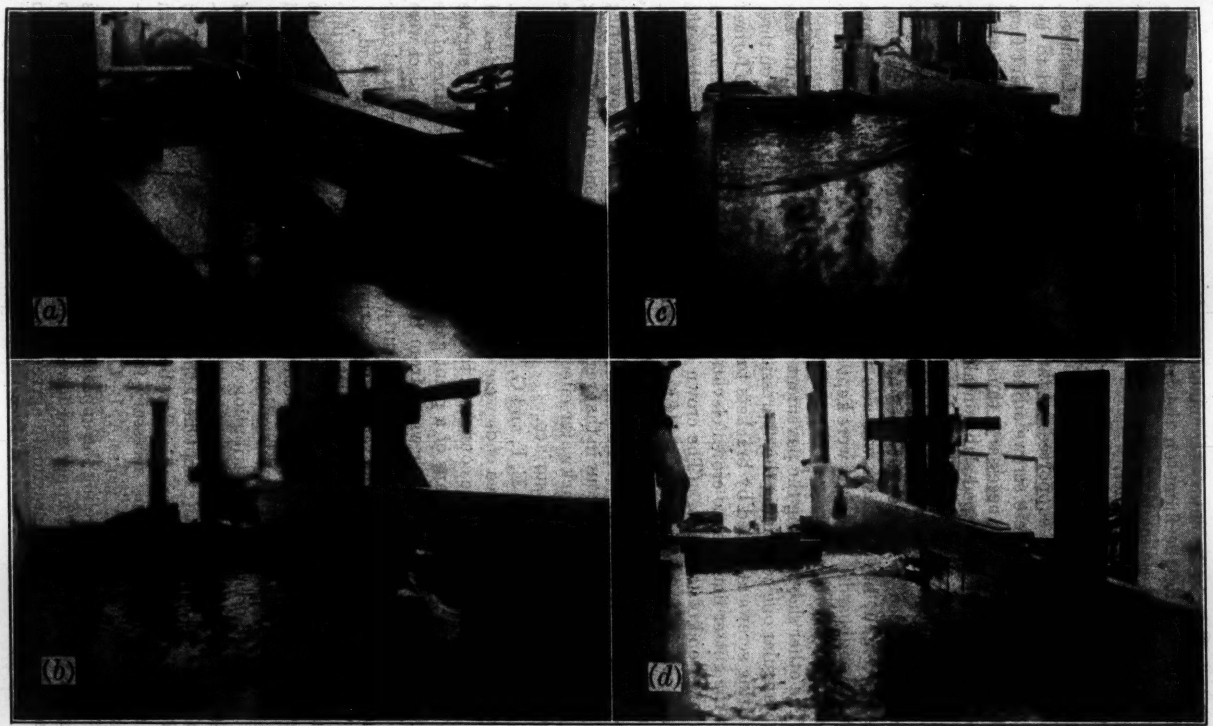

FIG. 3.—VIEWS OF MODEL INLET DEMONSTRATING BACKWATER EFFECTS

for warp and sag in the flume, were made by means of the adjusting screws and wedges supporting the flume on the framework. Local adjustments in the flume pavement were made by sandpapering high areas and filling in low ones with lacquer putty or lacquer.

The pavement of the lower end of the apparatus which contained the depression and inlet was formed from two concrete slabs. The outer, L-shaped, "permanent" slab was supported on three adjusting screws attached to the tilting framework. These screws adjusted the slab to a slope different from that of the flume, when necessary, thus varying the approach velocity in the experiments. The L-shaped slab contained the orifices for gages A, B, C, D, and E. The slab itself was formed to normal section throughout, the part having a surface above normal (Fig. 1) being built up with plaster of Paris. This was chipped off and rebuilt when models of different length were installed. Gage E, Fig. 1, in such cases was also moved to keep the standard distance from the endboard. All joints were kept smooth and watertight with a mixture of beeswax and resin.

The other slab forming the pavement of the lower end of the apparatus was a rectangular one containing the depression. This slab was supported on three adjusting screws attached by brackets to the L-slab. The slab was so formed that the gutter line rises to crown elevation at a point 12 in. downstream from the lower end of the inlet. (The crown elevation is 0.0113 ft higher than the normal elevation, which corresponds to an actual street with its curb face 8 in. normally and 4 in. at the center of the curb return.) For the present case—free flow—the contour of the pavement below the inlet has no effect on flow entering the inlet. Indeed, the contour in that region has little influence on flow entering the inlet even in case of backwater effect, the principal factor then being the elevation of the backwater surface.

Depth gages A, B, C, D, E, F, and G, Fig. 1, consisted of copper tubes, $\frac{3}{16}$ in. inside diameter, whose orifice ends were carefully set flush with the surface of the concrete, and which were connected to stilling wells where the water surface elevations were read by means of a point gage. The zero-flow readings of the gages were obtained by wiping water from the orifices until the menisci were perfectly flat, thus avoiding the effect of surface tension. Usually an average of at least three readings of the zero points during a day's run was obtained.

Total flow, Q_t, approaching the inlet was measured by means of an orifice meter calibrated by weighing. The excess flow, Q_e, or flow passing the inlet was measured by means of a 15° V-notch weir (also calibrated by weighing) and the inlet flow was found by subtraction.

Data on horizontal Q-distribution were obtained in a separate series of experiments by intercepting the flow in the normal channel between thin vertical vanes set 2 in. apart (except near the crown where the spacing was wider) and volumetrically measuring the flow between each pair of vanes over a given time interval. The depths necessary for computing velocities from the discharges were measured by gages A, B, C, and D, Fig. 1.

The procedure for leveling the apparatus was first to set the flume by means of grade marks near the upper and lower ends of the flume, then carefully to level the L-slab into approximate agreement with the flume. The next step was to adjust the lower end of the flume so that the joint was flush, ignoring the small departure from the lower flume marks. Adjustments in leveling the concrete were made until the difference in elevation between gages A and D, and between D and H, was correct to within 0.001 ft. This quantity was used as a criterion for acceptable accuracy throughout the model and lower end of the approach flume.

Readings of the gages were taken at time intervals of sufficient length to insure the existence of steady conditions, the weir being the controlling factor. Five minutes between weir readings was found to be a satisfactory interval. The maximum total flow in each set of readings was about 0.1 cu ft per sec, and the minimum about 0.02 cu ft per sec. The usual procedure was to turn on a flow of about 0.12 cu ft per sec for about 5 min, decrease the flow to about 0.10 cu ft per sec, wait for 2.5 min, locate the valley wave (a wave near the valley), read the orifice meter, read gages A, B, C, D, and E, and last, read the weir gage. Flow was then decreased in 6 or 8 stages at 5-min intervals to the minimum discharge. By this procedure the uneven effect of capillarity and surface tension on previously unwetted surfaces was avoided.

III.—Theory

For a fixed depth, a_0, and a known flow distribution in the channel upstream from the inlet, the ratio

$$K = \frac{\bar{z}\,P}{U} \qquad (2)$$

describes the flow within a distance z_i from the curb and represents a relationship between pressure and velocity patterns. In Eq. 2, $\bar{z}\,P$ is the moment of pressure, and U is the moment of momentum per second, or angular momentum per second, at the approach section, taking moments about the curb, corresponding to a filament of flow within a distance z_i from the curb.

The nature of K may be purely "distributional," or without regard to inlet flow; or it may be "gravitational," in which case the outer boundary of the filament of flow included coincides with the outer boundary of the inlet flow. In the latter case it must be evident that the value of K will depend upon the forces that act on the fluid as it passes from the approach section to the inlet lip. These forces may be described by means of a function of Froude's and Reynolds' numbers, and shape factors:

$$K = \phi\,(\bar{v}, h, \rho, g, \mu, a_0, d, t, c_s) \qquad (3a)$$

in which $\bar{v}\left(= \dfrac{G}{\rho\,Q}\right)$ is the average velocity of the water particles within a distance z_i from the curb; G = momentum per second; ρ = mass per cubic foot; and h is the fall of the water in passing from the approach section to the inlet lip. (The approach section was located 0.0625 ft upstream from the local depression, and thus a small part of the total fall represents fall in the approach channel. Depths were measured at a section 0.125 ft upstream from the depression. For cases in which the approach flume had a slope different from that of the model or the depth in the approach flume was greater than critical depth a small error in h is present.) Standard dimensional-analysis procedure now yields

$$K = \psi(\mathbf{F}, \mathbf{R}, A_0, D, T, C_s) \qquad (3b)$$

in which Froude's number is

$$\mathbf{F} = \frac{\bar{v}^2}{g\,h} \qquad (4a)$$

Reynolds' number:

$$\mathbf{R} = \frac{\bar{v}\,h}{\nu} \qquad (4b)$$

the approach-depth number:

$$A_0 = \frac{a_0}{h} \qquad (4c)$$

the depth-of-depression number:

$$D = \frac{d}{h} \qquad (4d)$$

the roughness number:

$$T = \frac{t}{h} \qquad (4e)$$

and

$$C_s = \frac{c_s}{h} \qquad (4f)$$

which is the number representing that part of h due to the normal slope of the model.

Variations in the ratio A_0 cause changes in the geometrical shape of the

TABLE 1.—EXPERIMENTAL DATA REQUIRED FOR AN ANALYSIS OF FLOW

(The Approach Depth $a_0 = 0.0460$; and Area = 0.0298)

Symbol	Remarks	LONGITUDINAL SLOPE, S_f, OF APPROACH FLUME:					
		0.01	0.02	0.04	0.06	0.02	0.02
d	....	0.0313	0.0313	0.0313	0.0313	0.0	0.0625
h	....	0.1001	0.1001	0.1001	0.1001	0.0688	0.1313
Q_t	Fig. 4	0.0436	0.0613	0.0864	0.1063	0.0613	0.0613
Q_s	Fig. 5	0.0197	0.0377	0.0617	0.0835	0.0487	0.0236
Q	$Q_t - Q_s$	0.0239	0.0236	0.0247	0.0228	0.0126	0.0377
Q/Q_t	....	0.549	0.385	0.286	0.214	0.206	0.615
z_i/z_t	Fig 6	0.217	0.141	0.102	0.076	0.072	0.252
z_i	$(z_i/z_t) \times z_t$	0.367	0.239	0.172	0.129	0.122	0.4265
Q^2_t	....	0.00190	0.00375	0.00747	0.01130	0.00375	0.00375
G_1	Fig. 7	42.9	30.1	22.2	16.8	15.7	47.7
G	$G_1 \times Q^2_t$	0.0816	0.1129	0.1657	0.191	0.0589	0.1790
$\bar{v}$	$G/\rho\,Q$	1.761	2.468	3.461	4.31	2.412	2.447
v'_t	Q_t/area	1.46	2.05	2.89	3.55	2.05	2.05
$\mathbf{F}$	$\bar{v}^2/g\,h$	0.963	1.892	3.720	5.81	2.632	1.421
$1/\mathbf{F}$	....	1.038	0.5285	0.2688	0.172	0.3797	0.703
$\bar{z}\,P$	Fig. 8	0.00254	0.00132	0.00076	0.00043	0.00039	0.00312
U_1	Fig. 7	7.47	3.51	1.93	1.11	1.00	9.27
U	$U_1 \times Q^2_t$	0.01419	0.01316	0.0144	0.01254	0.00375	0.03472
K	$\bar{z}\,P/U$	0.1790	0.1003	0.0527	0.0343	0.1040	0.0899
$\nu \times 10^6$	....	14.9	13.7	14.37	14.37	12.7	13.3
$\mathbf{R}$	$\bar{v}\,h/\nu$	(11,810)	(18,030)	(24,040)	(29,970)	(13,070)	(24,150)
$K\,\mathbf{F}$	....	0.172	0.189	0.196	0.199	0.274	0.128
a_i	....	0.0296	0.0349	0.0379	0.0397	0.0401	0.0273
$a_i\,g/2\,\bar{v}^2$	....	0.153	0.0919	0.0508	0.0343	0.111	0.0731
$(1/\mathbf{F})^{0.5}$	....	1.017	0.727	0.518	0.415	0.616	0.838
$(1/\mathbf{F})^{1.5}$	....	1.062	0.383	0.139	0.0712	0.233	0.589
$(1/\mathbf{F})^2$	....	1.08	0.279	0.0723	0.0297	0.144	0.494
$a_i/2\,h$	....	0.148	0.1745	0.1895	0.1985	0.292	0.1041
$\bar{v}/Q_t$	....	40.4	40.3	40.1	40.5	39.3	39.9
$Q_t - Q$	....	0.0197	0.0377	0.0617	0.0835	0.0497	0.0236

approach-flow cross section. Analysis can best be made, therefore, by covering the desired range of values of the ratio in a series of steps, each step of which represents a constant value of A_0.

Moreover, c_s is constant so long as S, x, and l are constant. When, also, d is constant, h is constant, for,

$$h = a_0 + d + c_s \qquad (5)$$

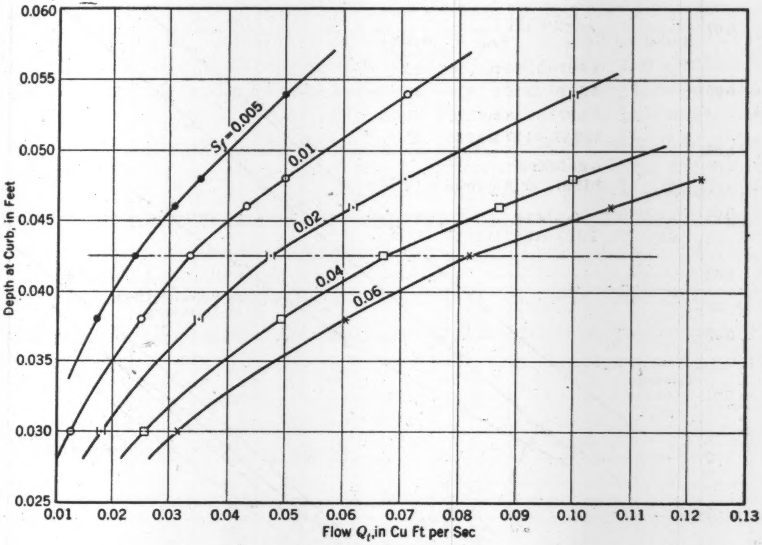

FIG. 4.—DEPTH AND DISCHARGE DATA

The magnitude, t, of the roughness protuberances was not varied in the model. (The results in this case apply to a prototype whose roughness particles are geometrically similar to those of the model. When, in the model, h is varied by varying d or a_0 the relative roughness varies although t itself is constant.) Under these conditions Eq. 3b reduces to

$$K = \theta(\mathbf{F}, \mathbf{R}) \qquad (6a)$$

or, in product form,

$$K = C_{AD}\, \alpha(\mathbf{F})\, \beta(\mathbf{R}) \qquad (6b)$$

This cannot be reduced further to simpler form, since (in the absence of varying ν or g) **F** and **R** vary simultaneously. The next step after preparing data in suitable form is to insert values in the equations and determine the forms of the functions, $\alpha(\mathbf{F})$ and $\beta(\mathbf{R})$.

IV.—PREPARING THE DATA

The experimental data required for an analysis of flow at the approach depth $a_0 = 0.0460$ ft are given in Table 1. Comparison can be made in the table between the average velocity $\bar{v}$ and the apparent average velocity $v'_t = \dfrac{\text{total discharge}}{\text{total area}}$.

Values of Q_t in the table were taken from Fig. 4 which represents the averaged result of a large number of depth, Q_t, observations. The averaging of depth-Q_t data was made necessary by the considerable dispersion of values. In the averaging process a cross slope of the water surface, which seemed to be systematic and to increase with S_f, the slope of the approach flume, was eliminated, and the resulting surface was made transversely level. The consistency of relationships described subsequently indicates acceptable accuracy of the resulting average. Slight discrepancies exist between tabular values of

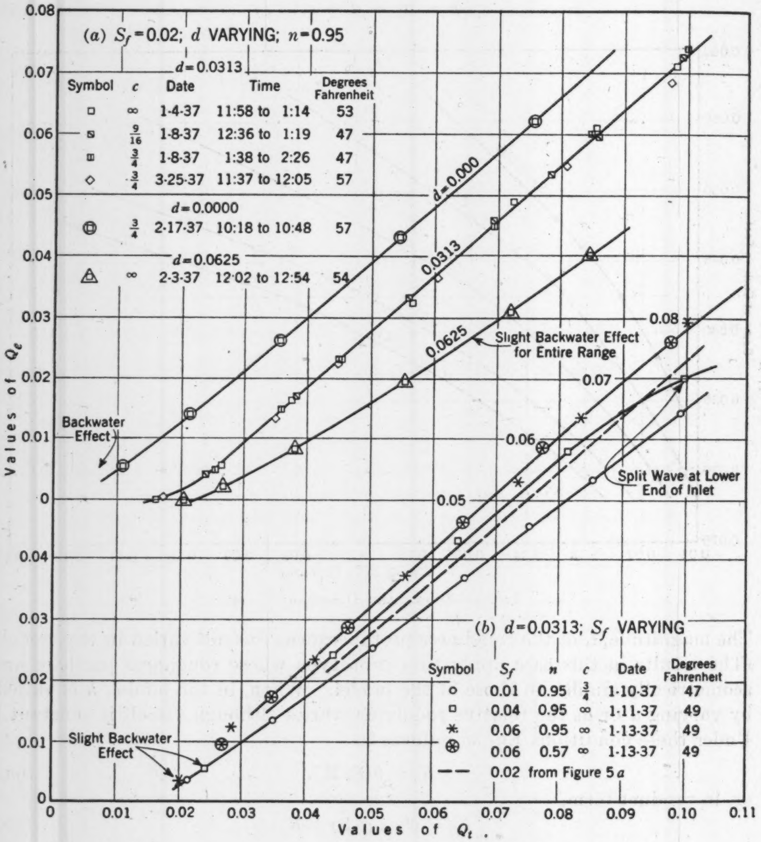

Fig. 5.—Experimental Flow Relationships

Q_t and those of Fig. 4 because of recomputing and redrafting the figure. Slight backwater effect existed in the cases: $d = 0.0313$, $S_f = 0.01$; and $d = 0.0625$, $S_f = 0.02$. The process of obtaining the average depths in Fig. 4 is described in a comprehensive manuscript entitled "Depth and Distribution Data for Flow in Model Street," by the writer. This material, with supporting experimental data, has been placed on file in the Engineering Societies Library. The latter manuscript also contains the derivation of the distribution of Q/Q_t (Fig. 6) and hydrodynamic quantities shown in Figs. 7, 8, 16, and 17.

Values of Q_s in Table 1 may be taken from Fig. 5(*a*), representing a constant value of S_f and three values of *d*, and from Fig. 5(*b*), representing a constant value of *d* and various values of S_f selected to vary the approach velocity. The slope, *S*, of the model itself was held constant for the present set of data. The small dispersion of data representing the several separate settings of the apparatus indicated by the different symbols along the curve for $d = 0.0313$ in Fig. 5(*a*) indicates the good consistency which was characteristic of such data. The apparatus was set at a large number of different slopes and adjustments in the period between the two extreme dates given in Fig. 5(*a*).

Values of *Q*, the inlet flow, were obtained by subtracting values of Q_s from those of Q_t. Values of $\frac{Q}{Q_t}$ were computed. The values of $\frac{Q}{Q_t}$ in Table 1 represent gravitational quantities as distinguished from distributional ones.

Fig. 6 shows distributional values of $\frac{Q}{Q_t}$ for $a_0 = 0.0460$. The graph has been made dimensionless by using values of the ratio $\frac{z_i}{z_t}$ for abscissas. In working up data for the figure it was found that the ratio $\frac{Q}{Q_t}$ for a given value of $\frac{z_i}{z_t}$ is constant for all values of Q_t to within a very small limit of error. $\left(\text{It is also true that } \frac{v}{Q_t} \text{ is constant for constant values of } a.\right)$ Hence, let

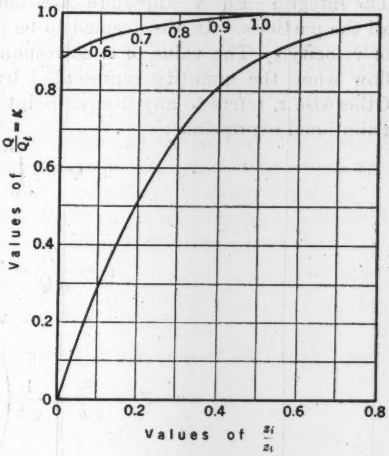

FIG. 6.—DISTRIBUTION VALUES OF Q/Q_t FOR $a_0 = 0.0460$

$$\frac{Q}{Q_t} = \kappa \qquad (7)$$

a constant for a given value of z_i.

The consistency found in the ratio $\frac{Q}{Q_t}$ for different values of Q_t and given values of a_0 confirms, to some extent, the accuracy of the depth-Q_t results obtained by the averaging process mentioned. The fact that the value of the ratio is known for all values of Q_t, and hence for $Q_t = 1.000$, makes it possible to use a single set of distributional curves in calculations.

Designating by the subscript 1 quantities corresponding to $Q_t = 1.000$, it will be shown that the required quantities G and U can be expressed in terms of G_1 and U_1, respectively, and Q^2_t. It is convenient to introduce at this point distributional expressions for the various quantities:

$$G = \rho \int_0^{z_i} v^2\, a\, dz \quad \text{(8)}$$

In Eq. 8, v represents the velocity at any value of z within the limits of the integral, a is the depth at that point, and z_i is the distance from the curb to the outer boundary of the filament of flow considered. (Actually v represents the apparent average velocity in a vertical line and not the velocity at a point. The integral—Eq. 8—therefore does not strictly represent the momentum of all the particles. It was deemed to be permissible to ignore vertical variation of velocity.) The value of z_i corresponds to the outer boundary of the inlet flow when the quantity represented by the integral is a gravitational one. Otherwise z_i refers to any desired point in the channel. Continuing with distributional expressions:

$$Q = \int_0^{z_i} v\, a\, dz \quad \text{(9a)}$$

$$\bar{v} = \frac{G}{\rho\, Q} = \frac{\int_0^{z_i} v^2\, a\, dz}{\int_0^{z_i} v\, a\, dz} \quad \text{(9b)}$$

$$\mathbf{F} = \frac{\bar{v}^2}{g\, h} = \frac{1}{g\, h}\left(\frac{\int_0^{z_i} v^2\, a\, dz}{\int_0^{z_i} v\, a\, dz}\right)^2 \quad \text{(9c)}$$

$$\mathbf{R} = \frac{\bar{v}\, h}{\nu} = \frac{h}{\nu}\,\frac{\int_0^{z_i} v^2\, a\, dz}{\int_0^{z_i} v\, a\, dz} \quad \text{(9d)}$$

$$U = \rho \int_0^{z_i} z\, v^2\, a\, dz \quad \text{(9e)}$$

$$\bar{z}\, P = \frac{\gamma}{2} \int_0^{z_i} z\, a^2\, dz \quad \text{(9f)}$$

By Eq. 7, $Q = \kappa Q_t$; so that

$$\frac{dQ}{dz} = Q_t \frac{d\kappa}{dz} \qquad (10)$$

A differential of discharge can be written

$$dQ = v\,a\,dz \qquad (11)$$

It follows, then, that

$$v = \frac{1}{a}\frac{dQ}{dz} \qquad (12a)$$

Hence, by Eq. 10,

$$v = \frac{1}{a} \times Q_t \frac{d\kappa}{dz} \qquad (12b)$$

in which the quantity

$$\frac{d\kappa}{dz} = \frac{d\left(\frac{Q}{Q_t}\right)}{dz} \qquad (13)$$

represents the slope of a graph like Fig. 6, but having z_i instead of $\frac{z_i}{z_t}$ for abscissas. The value of v in Eq. 12*b* can thus be determined by a process that is theoretically exact. The accuracy of the result depends only on the accuracy of the data and the accuracy with which the slopes of the graph can be scaled. Therefore, the method does not involve the systematic error which results when, for example, it is assumed that the velocity at the midpoint between a pair of the vanes used in the distribution experiments referred to equals the increment of discharge divided by the increment of area between the vanes.

Putting the value of v from Eq. 12*b* in Eq. 8, keeping Q_t outside the integral sign (since Q_t is not a function of z) and canceling a, the result is

$$G = \rho\, Q^2_t \int_0^{z_i} \frac{1}{a}\left(\frac{d\kappa}{dz}\right)^2 dz \qquad (14)$$

The quantity under the integral sign is purely a function of z so that, when $Q_t = 1.000$, Eq. 14 becomes

$$G_1 = \rho \int_0^{z_i} \frac{1}{a}\left(\frac{d\kappa}{dz}\right)^2 dz \qquad (15a)$$

Hence,

$$G = G_1\, Q^2_t \qquad (15b)$$

From Eqs. 12*b* and 9*e*,

$$U = \rho\, Q^2_t \int_0^{z_i} \frac{z}{a}\left(\frac{d\kappa}{dz}\right)^2 dz \qquad (16a)$$

so that

$$U = U_1\, Q^2_t \qquad (16b)$$

By Eq. 12*b*, when $Q_t = 1.000$,

$$v_1 = \frac{1}{a}\frac{d\kappa}{dz} \qquad (17)$$

Values of $\frac{d\kappa}{dz}$ were scaled from a curve similar to Fig. 6. A graph of v_1 is shown in Fig. 7. Values for the graphs of G_1 and U_1 in Fig. 7 were obtained by plotting values of

$$\frac{dG_1}{dz} = \frac{\rho}{a}\left(\frac{d\kappa}{dz}\right)^2 \dots\dots\dots\dots\dots\dots (18a)$$

and

$$\frac{dU_1}{dz} = z\,\frac{dG_1}{dz} \dots\dots\dots\dots\dots\dots (18b)$$

against values of z and measuring the areas under the curves by means of a planimeter. Values of Q_1 are represented by the ordinates in Fig. 6, and, using the values of G_1 from Fig. 7, values of $\bar{v}_1$ are obtainable by dividing G_1 by $\rho\,Q_1$, and are plotted in Fig. 7. Values of $\bar{z}\,P$ in Fig. 8 were likewise obtained by measuring the areas under the graph of $z\,\frac{dP}{dz}$ as a function of z.

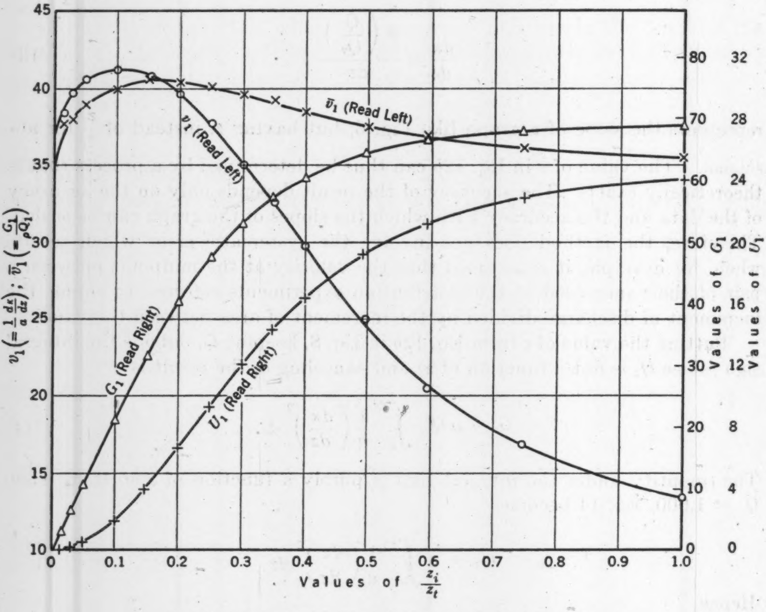

FIG. 7.—DISTRIBUTION DATA FOR a_0 = 0.0460, Q_t = 1.000

From the experimental values of $\frac{z_i}{z_t}$ in Table 1 and the graphs in Figs. 5 to 8, the other necessary gravitational data for beginning analysis can be obtained, and these are shown in Table 1.

V.—ANALYSIS

Form of the Function of **F**.—Since the phenomenon of flow into an inlet is principally gravitational in character, and since the flow is turbulent rather than laminar, K, in Eq. 6*b*, should vary principally with **F**, and to a minor extent with **R**. The dependent variable K is shown as a function of $\bar{v}$ and

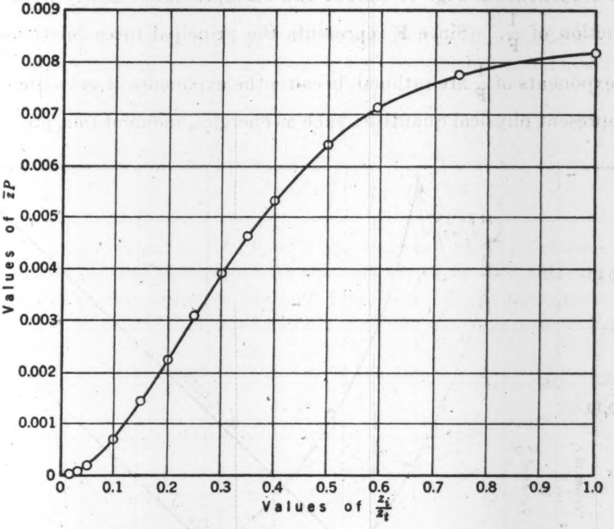

FIG. 8.—DISTRIBUTION OF MOMENT OF PRESSURE ($a_0 = 0.0460$)

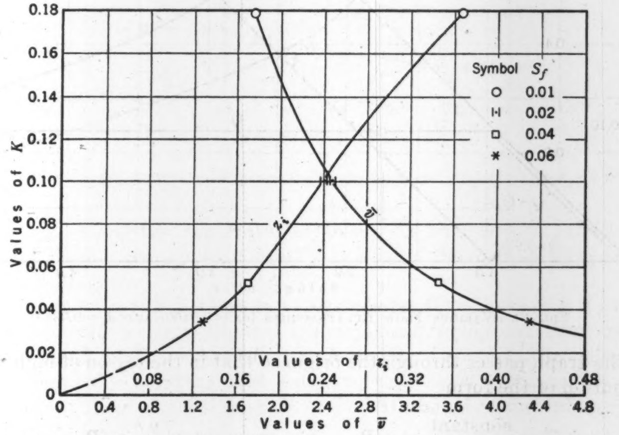

FIG. 9.—CORRESPONDING VALUES OF K, $\bar{v}$, AND z_i FOR $a_0 = 0.0460$ AND $d = 0.0313$

z_i in Fig. 9. Various flow relationships are shown in Fig. 10 and, in Fig. 11, K is shown as a function of **F**, and of $\left(\frac{1}{\mathbf{F}}\right)^{0.5} \frac{1}{\mathbf{F}}, \left(\frac{1}{\mathbf{F}}\right)^{1.5}$, and $\left(\frac{1}{\mathbf{F}}\right)^{2}$.

It is evident from Fig. 11 that K can be represented nearly by a straight-line function of $\frac{1}{\mathbf{F}}$. (Since **F** represents the principal force relationship only simple exponents of $\frac{1}{\mathbf{F}}$ are rational, because the exponents of velocities involved must represent physical quantities such as energies, momentums, etc.)[4] Moreover, the graph passes through the origin so that in the region of high velocities an equation of the form

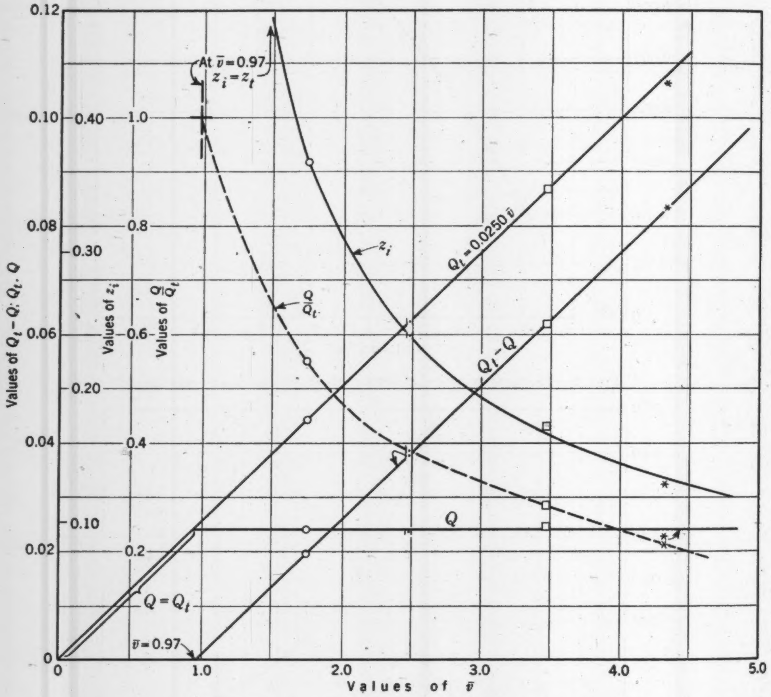

FIG. 10.—VARIOUS FLOW RELATIONSHIPS ($a_0 = 0.0460$; AND $d = 0.0313$)

$$K = \frac{\text{constant}}{\mathbf{F}} \times \phi(\mathbf{R} \ldots) = \text{constant} \times \frac{g\,h}{\bar{v}^2}\,\phi(\mathbf{R} \ldots) \ldots\ldots (19)$$

is correct. This form is also correct when g varies, because the left-hand member is

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[4] "Engineering Mathematics," by Charles P. Steinmetz, McGraw-Hill Book Co., Inc., 1917, p. 213.

$$K = \frac{\bar{z}\,P}{U} = \frac{0.5\,\rho\,g \int z\,a^2\,dz}{U} \quad \dots\dots\dots\dots\dots\dots\dots (20)$$

In the region of low velocities, for an inviscid fluid, the right-hand member of Eq. 19 approaches infinity. By Eqs. 9*f* and 16*a*, the left-hand member is the distributional expression

$$K = \frac{\bar{z}\,P}{U} = \frac{\frac{\gamma}{2}\int_0^{z_i} z\,a^2\,dz}{\rho\,Q^2_t \int_0^{z_i} \frac{z}{a}\left(\frac{d\kappa}{dz}\right)^2 dz} \quad \dots\dots\dots\dots\dots\dots (21)$$

When $\bar{v}$ approaches zero, Q_t (the flow in the approach section between the curb and the crown) approaches zero. As the velocity, $\bar{v}$, of flow entering the inlet decreases, the corresponding value of z_i increases. If the maximum value of z_i is finite (as in actual fact it must be), then the denominator of Eq. 21 approaches zero while the numerator is finite. In this case, *K*, the left-hand member of Eq. 19, also approaches infinity so that the form of this equation is again satisfactory.

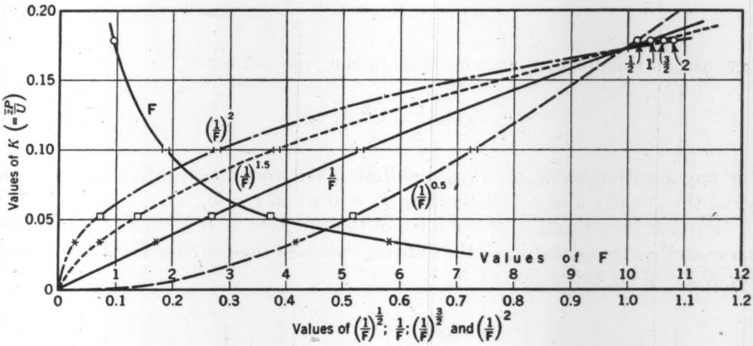

FIG. 11.—*K* AS A FUNCTION OF VARIOUS POWERS OF F (a_0 = 0.0460; AND d = 0.0313)

Theoretically, z_i can increase to values greater than those of z_t if the crown board is removed and the channel is extended laterally at crown elevation. Since the depth in the extended part is constant ($= a_c$), the velocity will become practically constant and equal to v_c a short distance beyond the crown. The values of the integrals in Eq. 21 then become infinite so that $K = \frac{\infty}{0 \times \infty}$. This indeterminate quantity can be evaluated by differentiating under the integral signs. (Each of the integrals is a function of z_i, its upper limit, considered as a variable. Moreover, z can be substituted for z_i so that, for example,[5] $\int_0^{z_i} z\, a^2\, dz = f(z)$, and $\frac{df(z)}{dz} = z\, a^2$.) Eq. 21 then becomes

$$K = \frac{\frac{\gamma}{2} z\, a^2}{\rho\, Q^2{}_t\, \frac{z}{a}\left(\frac{d\kappa}{dz}\right)^2} = \infty \quad\dots\dots\dots(22)$$

when $Q_t = 0$. In either case, therefore, K becomes infinite as $\bar{v}$ approaches zero $\left(\frac{1}{\mathbf{F}} \text{ approaches infinity}\right)$; for the time being it can be assumed that the form of function of **F** in Eq. 19 is correct.

Form of Function of **R**.—The form of function of **R** can now be studied by investigating the variation of $K\,\mathbf{F}$ with respect to **R**, since, by Eq. 19,

$$K\,\mathbf{F} = f(\mathbf{R}) \quad\dots\dots\dots(23)$$

It can be shown at once that the function of **R** is not of power form, such as

$$f(\mathbf{R}) = C\,\mathbf{R}^n = C\left(\frac{\bar{v}\,h}{\nu}\right)^n \quad\dots\dots\dots(24)$$

because, in this form, when $\nu = 0$, as in the "perfect fluid,"

$$K = \frac{C}{\mathbf{F}}\left(\frac{\bar{v}\,h}{\nu}\right)^n = \infty \quad\dots\dots\dots(25)$$

for any positive value of n—a manifest incongruity. Negative values of n yield the equally absurd relationship $K = 0$ when $\nu = 0$.

Further investigation of the form of the function of **R** will be facilitated by an analytical investigation of the limiting values of the distributional expression for $K\,\mathbf{F}$. This expression is

$$K\,\mathbf{F} = \frac{\frac{\gamma}{2}\int_0^{z_i} z\, a^2\, dz \left(\frac{\int_0^{z_i} v^2\, a\, dz}{\int_0^{z_i} v\, a\, dz}\right)^2}{\rho\, g\, h \int_0^{z_i} z\, v^2\, a\, dz} = \frac{1}{2}\, \frac{\int_0^{z_i} z\, a^2\, dz \left[\frac{\int_0^{z_i} \frac{1}{a}\left(\frac{d\kappa}{dz}\right)^2 dz}{\int_0^{z_i} \frac{d\kappa}{dz}\, dz}\right]^2}{h \int_0^{z_i} \frac{z}{a}\left(\frac{d\kappa}{dz}\right)^2 dz} \quad\dots(26)$$

canceling Q_t. Hence $K\,\mathbf{F}$ is independent of Q_t, a fact also confirmed by experimental data. In Eq. 26, if v is uniform,

$$K\,\mathbf{F} = \frac{\int_0^{z_i} z\, a^2\, dz}{2\,h \int_0^{z_i} z\, a\, dz} = \frac{\bar{a}}{2\,h} \quad\dots\dots\dots(27)$$

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[5] "Advanced Calculus," by Edwin Bidwell Wilson, Ginn & Co. (Copyright 1911, 1912), p. 27.

in which $\bar{a}$ is a weighted or average value of a, and this value depends only upon the geometrical shape of the cross section and on the value of the limit z_i of the integrals. The locus of $K\,\mathbf{F}$ is thus, in this case, simply a geometrical curve. In Eq. 27, if the depth is constant $(= a)$, then $K\,\mathbf{F}$ has the constant value $\frac{a}{2\,h}$ for all values of z_i.

On the other hand, when v varies with z, $K\,\mathbf{F}$ is a function of v. Moreover, in Eq. 26, z_i decreases as the approach velocity increases, and when z_i approaches zero $K\,\mathbf{F}$ takes on the indeterminate form: $K\,\mathbf{F} = \frac{0}{0} \times \frac{0}{0}$. Resorting, as before, to differentiation, and substituting for a the quantity a_0, which it approaches in value,

$$K\,\mathbf{F} = \frac{\dfrac{z\,a^2}{2}\left[\dfrac{\dfrac{1}{a}\left(\dfrac{d\kappa}{dz}\right)^2}{\dfrac{d\kappa}{dz}}\right]^2}{h\,\dfrac{z}{a}\left(\dfrac{d\kappa}{dz}\right)^2} = \frac{a_0}{2\,h} = \frac{A_0}{2} \quad \ldots\ldots\ldots\ldots\ldots (28)$$

when $\mathbf{R}$ approaches infinity.

In like manner, when $\bar{v}$ approaches zero, or z_i approaches infinity, a approaches a_c, and $K\,\mathbf{F}$ takes on the indeterminate form: $K\,\mathbf{F} = \frac{\infty}{\infty} \times \frac{\infty}{\infty}$. Again, by differentiation and substitution, when $\mathbf{R} = 0$,

$$K\,\mathbf{F} = \frac{a_c}{2\,h} = \frac{A_c}{2} \quad \ldots\ldots\ldots\ldots\ldots (29)$$

Eqs. 28 and 29 furnish valuable additional data for supplementing the limited range of experimental values.

It is noteworthy that in the theoretical case of uniform velocity throughout the cross section and in the two cases representing limiting values of approach velocity K takes on the form of a Froude number: By Eq. 27, with v uniform—

$$K = \frac{\bar{a}}{2\,h} \times \frac{g\,h}{\bar{v}^2} = \frac{1}{2}\,\frac{\bar{a}\,g}{\bar{v}^2} \quad \ldots\ldots\ldots\ldots\ldots (30a)$$

by Eq. 28, with $z_i = 0$—

$$K = \frac{a_0}{2\,h} \times \frac{g\,h}{\bar{v}^2} = \frac{1}{2}\,\frac{a_0\,g}{\bar{v}^2} \quad \ldots\ldots\ldots\ldots\ldots (30b)$$

and, by Eq. 29, with $z_i = \infty$—

$$K = \frac{a_c}{2\,h} \times \frac{g\,h}{\bar{v}^2} = \frac{1}{2}\,\frac{a_c\,g}{\bar{v}^2} \quad \ldots\ldots\ldots\ldots\ldots (30c)$$

Derivation of the Equation.—For the experimental case being investigated $a_0 = 0.0460$, $a_c = 0.0035$, and $h = 0.1001$. Substituting these values in Eq. 28 and Eq. 29, there results: For $z_i = 0$,

$$K_0\,\mathbf{F}_0 = \frac{0.0460}{2 \times 0.1001} = 0.230 \quad \ldots\ldots\ldots\ldots\ldots (31a)$$

and, for $z_i = \infty$,

$$K_c\,\mathbf{F}_c = \frac{0.0035}{2 \times 0.1001} = 0.0175 \ldots\ldots\ldots\ldots\ldots (31b)$$

Moreover, it was observed in developing distribution curves that $K\,\mathbf{F}$ has the constant value $K_t\,\mathbf{F}_t = 0.133$ for all values of Q_t. These values are plotted, together with the four experimental ones, in Fig. 12. From the shape of the curve and the physical fact that $K\,\mathbf{F}$ must approach a constant value asymptotically when $\mathbf{R}$ approaches infinity, it is suggested that an hyperbola should fit the data. It may be a mere coincidence that the value of $K_t\,\mathbf{F}_t$ falls on the graph at a point on a line which bisects the angle between the asymptotes.

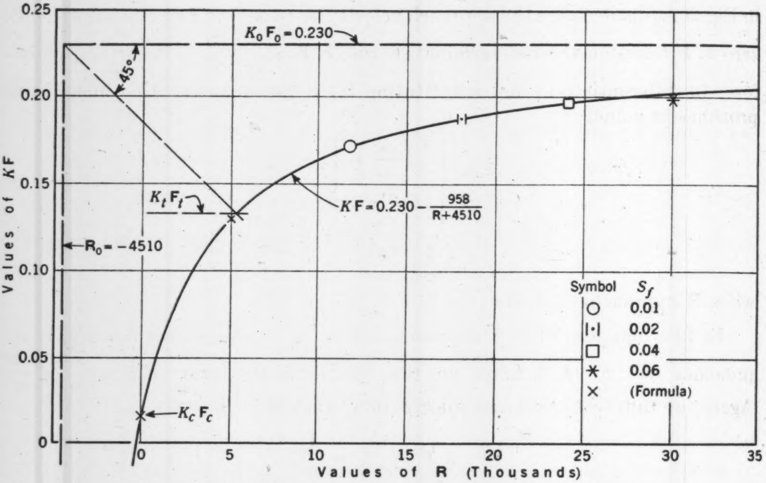

FIG. 12.—$K\,\mathbf{F}$ AS A FUNCTION OF $\mathbf{R}$ ($a_0 = 0.0460$; AND $d = 0.0313$)

The three-constant form of the equation for an hyperbola can be written

$$(\mathbf{R} - \mathbf{R}_0)\,(K\,\mathbf{F} - K_0\,\mathbf{F}_0) = -\,C_0 \ldots\ldots\ldots\ldots\ldots (32a)$$

in which $\mathbf{R}_0$ and $K_0\,\mathbf{F}_0$ are the asymptotes and C_0 is a constant of the hyperbola, or

$$K\,\mathbf{F} = 0.5\,A_0 - \frac{C_0}{\mathbf{R} - \mathbf{R}_0} \ldots\ldots\ldots\ldots\ldots (32b)$$

This is in the form of Eq. 23.

By Eqs. 29 and 32*b*, when $\mathbf{R} = 0$,

$$\frac{C_0}{\mathbf{R}_0} = -\,\frac{a_0 - a_c}{2\,h} = -\,\frac{y_c}{2\,h} = -\,0.5\,Y_c \ldots\ldots\ldots\ldots\ldots (33)$$

in which y_c is the crown height.

Only one constant, $\mathbf{R}_0$, remains to be determined; any one of the experimental sets of data is sufficient for its evaluation. Computation of a value corresponding to each set will provide comparative data by means of which the reliability of the method and the accuracy of the data can be judged. Solving for $\mathbf{R}_0$,

$$\mathbf{R}_0 = \frac{(K\,\mathbf{F} - 0.5\,A_0)\,\mathbf{R}}{K\,\mathbf{F} - 0.5\,A_c} \qquad (34)$$

Values of $\mathbf{R}_0$ for $d = 0.0313$ are computed in Table 2. The values in the last row in the table represent the arithmetical average values of $K\,\mathbf{F}$ and $\mathbf{R}$ for $S_f = 0.02$ and 0.04. These two sets are believed to be the most reliable. Hence, the value $\mathbf{R}_0 = -4{,}510$ will be considered satisfactory. By Eq. 33 $\left(\text{since } Y_c = \dfrac{0.0425}{0.1001} = 0.425\right)$, $C_0 = -0.5 \times 0.425\,(-4{,}510) = +958$. The equation representing $a_0 = 0.0460$, $d = 0.0313$ is then, by Eq. 32*b*:

$$K\,\mathbf{F} = 0.230 - \frac{958}{\mathbf{R} + 4{,}510} \qquad (35)$$

The check computations in Table 3 are obtained by inserting experimental values of $\mathbf{R}$ in Eq. 35. Thus, in all cases the values of $K\,\mathbf{F}$ computed from Eq. 35 are within 1.7% of the observed values of $K\,\mathbf{F}$. Comparison of the discrepancies on a basis of percentage of the small observed range (= 0.027)

TABLE 2.—VALUES OF $\mathbf{R}_0$ FOR $d = 0.0313$

S_f (1)	$K\,\mathbf{F}$ (2)	$\mathbf{R}$ (3)	$K\,\mathbf{F} - 0.5\,\mathbf{A}_0$ (4)	Col. 4 × $\mathbf{R}$ (5)	$K\,\mathbf{F} - 0.5\,\mathbf{A}_c$ (6)	$\mathbf{R}_0$ (7)	$\mathbf{R}_0$ (aver.) $- \mathbf{R}_0$ (8)	Discrepancy (% of aver.) (9)	$-4{,}510 - \mathbf{R}_0$ (10)	% of $-4{,}510$ (11)
0.01	0.172	11,810	−0.058	−685	+0.1545	−4,430	−175	+ 3.8	− 80	+ 1.8
0.02	0.189	18,030	−0.041	−739	+0.1715	−4,310	−295	+ 6.4	−200	+ 4.4
0.04	0.196	24,040	−0.034	−818	+0.1785	−4,580	− 25	+ 0.5	+ 70	− 1.6
0.06	0.199	29,970	−0.031	−928	+0.1815	−5,100	+495	−10.8	+590	−13.1
Aver., S_f = 0.01, 0.02, 0.04, 0.06				....	....	−4,605	....	....	....	....
Aver.[a]	0.1925	21,035	−0.0375	−790	+0.1750	−4,510	[a] Averages for S_f = 0.02 and 0.04			

TABLE 3.—COMPUTATIONS TO CHECK EQ. 35

S_f	$K\,\mathbf{F}$	Discrepancy	% of observed $K\,\mathbf{F}$	% of observed range
0.01	$0.230 - \dfrac{958}{11{,}810 + 4{,}510} = 0.1713$	− 0.0007	− 0.40	− 2.59
0.02	$0.230 - \dfrac{958}{18{,}030 + 4{,}510} = 0.1875$	− 0.0015	− 0.79	− 5.56
0.04	$0.230 - \dfrac{958}{24{,}040 + 4{,}510} = 0.1965$	+ 0.0005	+ 0.25	+ 1.85
0.06	$0.230 - \dfrac{958}{29{,}970 + 4{,}510} = 0.2022$	+ 0.0032	+ 1.61	+ 11.85
($\mathbf{R}$ = 5,000)	$0.230 - \dfrac{958}{5{,}000 + 4{,}510} = 0.1293$[a]	....	....	....

of values of $K\,\mathbf{F}$ yields less than 6% for the first three slopes and less than 12% for $S_f = 0.06$. In the last case the value of z_i represents a distance less than that between the curb and the first vane by which the flow distribution was investigated, and a region, moreover, where the velocity profile curves rapidly. The data for $S_f = 0.06$, therefore, should be accredited less weight than the others. The results, in general, are considered satisfactory.

The equation for K is then

$$K = \frac{1}{\mathbf{F}}\left(0.230 - \frac{958}{\mathbf{R} + 4{,}510}\right) \quad \text{(36a)}$$

or, in general terms,

$$K = \frac{1}{\mathbf{F}}\left[\frac{a_0}{2\,h} + \frac{\dfrac{a_0 - a_c}{2\,h}\mathbf{R}_0}{\mathbf{R} - \mathbf{R}_0}\right] = \frac{0.5}{\mathbf{F}}\left(A_0 + \frac{Y_c\,\mathbf{R}_0}{\mathbf{R} - \mathbf{R}_0}\right) \quad \text{(36b)}$$

The Function When D Varies.—It can now finally be demonstrated that the form of function of $\mathbf{F}$ in Eq. 19 is consistent when D (and, therefore, h) varies, because, by Eq. 36*b*,

$$K = \frac{a_0\,g}{2\,\bar{v}^2}\left(1 + \frac{a_0 - a_c}{a_0}\,\frac{\mathbf{R}_0}{\mathbf{R} - \mathbf{R}_0}\right) \quad \text{(37)}$$

but,

$$\frac{\mathbf{R}_0}{\mathbf{R} - \mathbf{R}_0} = \frac{\mathbf{R}_0}{\mathbf{R}} + \left(\frac{\mathbf{R}_0}{\mathbf{R}}\right)^2 + \left(\frac{\mathbf{R}_0}{\mathbf{R}}\right)^3 + \cdots \quad \text{(38a)}$$

in which each term $\dfrac{\mathbf{R}_0}{\mathbf{R}}$ can be written

$$\frac{\mathbf{R}_0}{\mathbf{R}} = \frac{\dfrac{\bar{v}_0\,h}{\nu}}{\dfrac{\bar{v}\,h}{\nu}} = \frac{\bar{v}_0}{\bar{v}} \quad \text{(38b)}$$

Hence, when ν is the same for different values of h,

$$K = \frac{a_0\,g}{2\,\bar{v}^2}\left(1 + \frac{a_0 - a_c}{a_0}\,\frac{\bar{v}_0}{\bar{v} - \bar{v}_0}\right) \quad \text{(39)}$$

In Eq. 39, h does not appear in the right-hand member, and since, by Eq. 21, it is also absent from the left-hand member, it must be concluded that the form (Eq. 19) is correct. The small variation of K with h, which can be noted in Table 1, is in conformity with the small variation of $\bar{v}$ with z_i when Q_t is constant as shown in Fig. 7.

Shape Parameters.—In Eq. 36*b*, A_0, Y_c, and $\mathbf{R}_0$ are parameters whose values depend upon shape factors. The physical meaning of this equation is made more apparent when it is noted that, for high values of $\mathbf{R}$ (high velocities and small widths of inlet flow), K depends principally upon A_0; whereas, for lower values of $\mathbf{R}$ (lower velocities and wider filaments of inlet flow), K depends to an increased extent upon Y_c and $\mathbf{R}_0$.

The values of A_0 and Y_c depend upon the shape of the approach flow (if d is constant). When that shape is constant, other form factors such as length of inlet, width of depression, etc., govern the value of $\mathbf{R}_0$. Evidently K will be especially affected by variations of the latter form factors when approach velocities are low.

TABLE 4.—VALUES OF $\mathbf{R}_0$

d	h	D	$0.5\,A_0$	$0.5\,A_e$	KF	$\mathbf{R}$	$KF - 0.5\,A_0$	$(KF - 0.5\,A_0)\mathbf{R}$	$KF - 0.5\,A_e$	$\mathbf{R}_0$
0.0000	0.0688	0.000	0.334	0.0254	0.274	13,070	−0.060	− 783	0.249	−3,140
0.0625	0.1313	0.476	0.175	0.0133	0.128	24,150	−0.047	−1,135	0.115	−9,870

Values of $\mathbf{R}_0$ *for* $D = 0.000$ *and* $D = 0.476$.—The values of $\mathbf{R}_0$ in Table 4 were obtained by substituting experimental data in Eq. 34. Various relationships between $\mathbf{R}_0$, D, d, and $f(D)$ are shown in Appendix *A*.

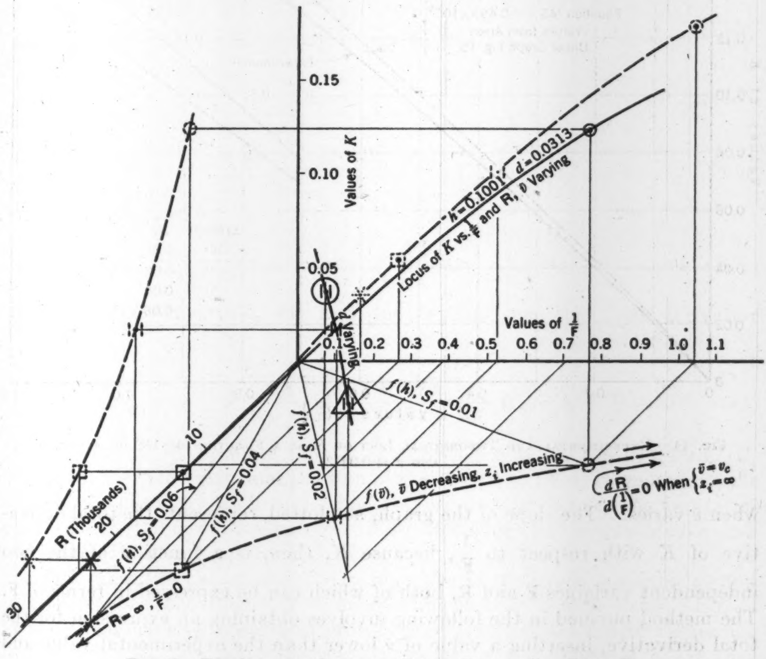

FIG. 13.—EXPERIMENTAL LOCUS OF K AS A FUNCTION OF F AND R

VI.—APPLICATION TO DESIGN

Eqs. 36 describe the experimental locus of K as a function of F and R. The locus is shown in three coordinates in Fig. 13. It is now desired to develop an expression by means of which the model results can be applied to a full-size prototype. This requires a separation of the effects of $\mathbf{F}$ and $\mathbf{R}$ so that a value of $\mathbf{R}$ can be used which will represent viscous conditions similar to those of a prototype. For model size and velocities this involves substituting a smaller value of ν.

Experimental values of K are plotted as a function of $\frac{1}{\mathbf{F}}$ (solid graph) in Fig. 14. It is evident that the values of K in this figure represent the combined effect of $\mathbf{F}$ and $\mathbf{R}$. This follows from the fact that $\mathbf{F}$ and $\mathbf{R}$ vary simultaneously when $\bar{v}$ varies. The slope of the graph, as plotted, represents the total derivative of K with respect to $\frac{1}{\mathbf{F}}$, because K, then, is a function of the two independent variables $\mathbf{F}$ and $\mathbf{R}$, both of which can be expressed in terms of $\mathbf{F}$. The method pursued in the following involves obtaining an expression for the total derivative, inserting a value of ν lower than the experimental value and then integrating. The result is shown as the broken graph in Fig. 14.

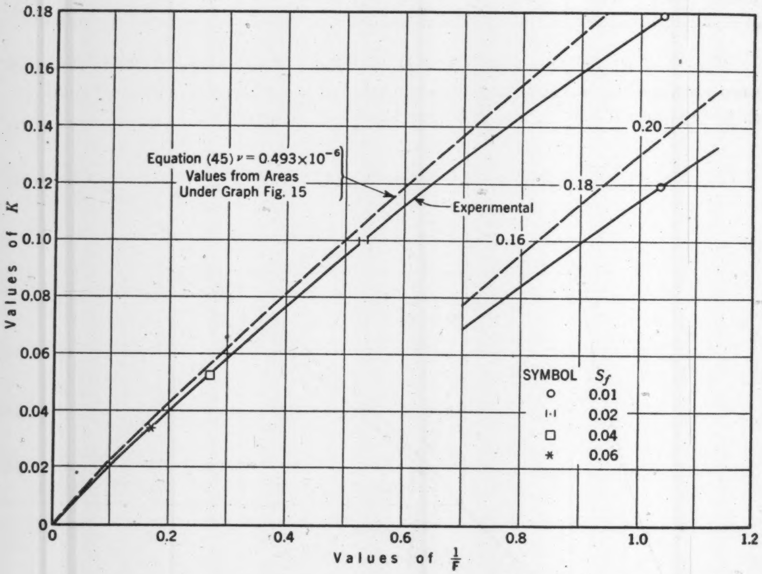

Fig. 14.—Experimental and Theoretical Loci of K as a Function of $1/\mathbf{F}$ (a_0 = 0.0460; and d = 0.0313)

An expression for the total derivative is:

$$\frac{dK}{d(1/\mathbf{F})} = \frac{\partial K}{\partial(1/\mathbf{F})} + \frac{\partial K}{\partial \mathbf{R}} \frac{d\mathbf{R}}{d(1/\mathbf{F})} \qquad \text{(40a)}$$

From Eq. 36*b*,

$$\frac{\partial K}{\partial(1/\mathbf{F})} = \frac{a_0}{2\,h} + \frac{\dfrac{a_0 - a_c}{2\,h}}{\mathbf{R}_M - \mathbf{R}_0}\mathbf{R}_0 \qquad (40b)$$

Note by comparison with Eq. 36*b* that this equals *K* **F**. The subscript *M* in Eq. 40*b* indicates the constancy of **R** ($\bar{v}$, *h*, and ν must have the model values—see Fig. 14—at the point concerned). Moreover,

$$\frac{\partial K}{\partial \mathbf{R}} = -\frac{1}{\mathbf{F}_M}\left[\frac{\dfrac{a_0 - a_c}{2\,h}\mathbf{R}_0}{(\mathbf{R}_\nu - \mathbf{R}_0)^2}\right] \qquad (41)$$

the subscript ν indicating that $\mathbf{R}_\nu$ is arbitrary with regard to ν, but in other respects it must be consistent with the constancy in **F**.

Reynolds' number may be expressed in terms of $\frac{1}{\mathbf{F}}$ in the following manner:

$$\bar{v}^2 = \frac{g\,h}{1/\mathbf{F}} \qquad (42a)$$

and

$$\mathbf{R} = \left(\frac{g\,h}{1/\mathbf{F}}\right)^{0.5}\frac{h}{\nu} \qquad (42b)$$

Thus

$$\frac{d\mathbf{R}}{d(1/\mathbf{F})} = \frac{g^{0.5}\,h^{1.5}}{\nu}\left[-\frac{1}{2}\left(\frac{1}{\mathbf{F}}\right)^{-1.5}\right] \qquad (43a)$$

Eq. 43*a* can be written

$$\frac{d\mathbf{R}}{d(1/\mathbf{F})} = -\frac{1}{2}\frac{\bar{v}\,h}{\nu} \times \frac{\bar{v}^2}{g\,h} = -\frac{1}{2}\mathbf{F}_M\,\mathbf{R}_\nu \qquad (43b)$$

Then inserting Eqs. 40*b*, 41, and 43*b* in Eq. 40*a*, the total derivative ($\bar{v}$ varying), is

$$\frac{dK}{d(1/\mathbf{F})} = \frac{a_0}{2\,h} + \frac{\dfrac{a_0 - a_c}{2\,h}\mathbf{R}_0}{\mathbf{R}_M - \mathbf{R}_0} + \frac{1}{2}\frac{\dfrac{a_0 - a_c}{2\,h}\mathbf{R}_0}{(\mathbf{R}_\nu - \mathbf{R}_0)^2}\mathbf{R}_\nu = \mathrm{A} + \mathrm{B} + \mathrm{C} \qquad (44)$$

If the value of $\mathbf{R}_\nu$ becomes infinite (ν approaches zero), the last term (C) approaches zero in value, and the sum of the other two (A + B) represents the slope of the graph for an inviscid fluid. Moreover, the sum of the two terms equals the partial derivative as given in Eq. 40*b*, which, as before stated, equals the value of *K* **F** as given by Eq. 36*b*.

With regard to the effects of viscosity, flow in the model is similar to flow in a prototype when Reynolds' number is the same in both systems. Eq. 44 thus may be made to represent a prototype by inserting the proper values of ν in the terms $\mathbf{R}_\nu$, but using experimental values of $\bar{v}$. Values of *K* are then given by

$$K = \int \frac{dK}{d(1/\mathbf{F})}\,d\!\left(\frac{1}{\mathbf{F}}\right) = \int\left[\frac{a_0}{2\,h} + \frac{\dfrac{a_0 - a_c}{2\,h}\mathbf{R}_0}{\mathbf{R}_M - \mathbf{R}_0} + \frac{1}{2}\frac{\dfrac{a_0 - a_c}{2\,h}\mathbf{R}_0\,\mathbf{R}_\nu}{(\mathbf{R}_\nu - \mathbf{R}_0)^2}\right]d\!\left(\frac{1}{\mathbf{F}}\right) \qquad (45)$$

The integral can be evaluated by measuring the area under the graph of the integrand, $\dfrac{dK}{d(1/\mathbf{F})}$, as a function of $\dfrac{1}{\mathbf{F}}$.

To obtain a value of ν, the case representing $S_f = 0.02$ is taken as an average one. For this case, $\mathbf{R} = 18{,}030$. Let it be assumed that the value of $\mathbf{R}$ for a prototype is about 500,000. (The exact value is not important, because the rate of variation of K with $\mathbf{R}$ in the region of $\mathbf{R} = 500{,}000$ is very small; $\mathbf{R}_\nu$ could be considered infinite without great error. The last term in Eq. 45 represents viscous effect.) The total flow, by Chézy's formula, in a prototype channel whose dimensions are $\dfrac{32}{3}$ times those of the model ($a_0 = 0.0460$), and for which Kutter's coefficient of roughness is $n = 0.013$, is $Q_{tP} =$ area $\times\, C_P\,(R_P\,S)^{0.5} = 3.40 \times 79\,(0.184 \times 0.02)^{0.5} = 16.3$.

Using subscripts M and ν to denote model and prototype (both to be simulated with model size, however), the relationship between Reynolds' numbers and values of ν (h and $\bar{v}$ being constant) is given by

$$\frac{\mathbf{R}_M}{\mathbf{R}_\nu} = \frac{\dfrac{\bar{v}\,h}{\nu_M}}{\dfrac{\bar{v}\,h}{\nu_\nu}} = \frac{\nu_\nu}{\nu_M} \qquad (46)$$

Since $\nu_\nu = \nu_M \dfrac{\mathbf{R}_M}{\mathbf{R}_\nu}$, $\nu_\nu = 13.7 \times 10^{-6} \times \dfrac{18{,}030}{500{,}000} = 0.493 \times 10^{-6}$.

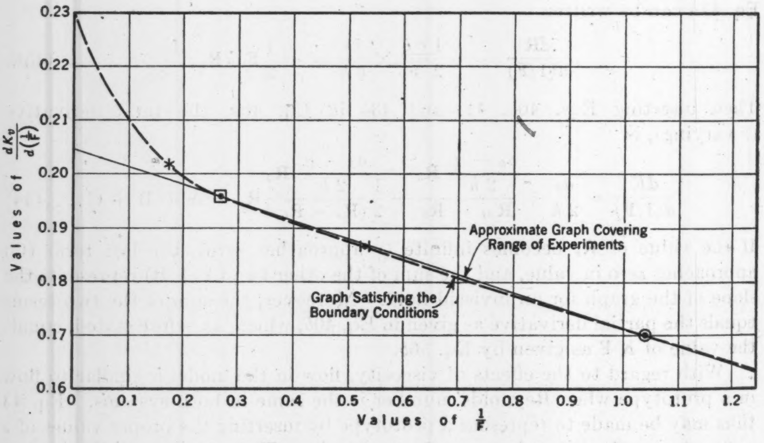

FIG. 15.—VALUES OF $\dfrac{dK_\nu}{d(1/\mathbf{F})}$ CORRESPONDING TO $\nu = 0.493 \times 10^{-6}$ ($a_0 = 0.0460$)

Values of $\dfrac{dK_\nu}{d(1/\mathbf{F})}$ corresponding to the value of $\nu = 0.493 \times 10^{-6}$ are calculated in Table 5 and have been plotted in Fig. 15 for the four experimental cases and the theoretical limiting case, $\frac{1}{\mathbf{F}} = 0$; that is, $\frac{dK}{d(1/\mathbf{F})} = \frac{a_0}{2\,h}$ (= 0.230). Fig. 15 also holds for $\nu = 0.322 \times 10^{-6}$ (see Table 5).

TABLE 5.—CALCULATION OF VALUES OF $\frac{dK_\nu}{d(1/\mathbf{F})}$

$$\left(\frac{a_0}{2\,h} = 0.230;\ \frac{a_0 - a_c}{2\,h}\mathbf{R}_0 = -\,958;\ \text{and}\ \mathbf{R}_0 = -\,4{,}510\right)$$

S_f	$\mathbf{R}_M$	$\frac{1}{\mathbf{F}_M}$	$\bar{v}$	$\mathbf{R}_\nu$ $\left(=\frac{\bar{v}\,h}{\nu_\nu}\right)$	$\mathbf{R}_\nu - \mathbf{R}_0$	$(\mathbf{R}_\nu - \mathbf{R}_0)^2$	$\mathbf{R}_M - \mathbf{R}_0$	Quantity B, Eq. 44	Quantity C, Eq. 44	$\frac{dK_\nu}{d(1/\mathbf{F})}$ (Eq. 44)	K_ν (Fig. 14)	$K_\nu\,\mathbf{F}$
$\nu_\nu = 0.493 \times 10^{-6}$												
0.01[a]	11,810	1.038	1.761	0.357[b]	0.362[b]	131[c]	16,320	−0.0587	−0.00131	0.1700	0.196	0.189
0.02	18,030	0.5285	2.468	0.501[b]	0.506[b]	256[c]	22,540	−0.0424	−0.00094	0.187	0.106	0.200
0.04	24,040	0.2688	3.461	0.702[b]	0.707[b]	498[c]	28,550	−0.0335	−0.00068	0.196	0.0562	0.209
0.06	29,970	0.172	4.31	0.873[b]	0.878[b]	770[c]	34.480	−0.0278	−0.00054	0.202	0.0367	0.213
$\nu_\nu = 0.392 \times 10^{-6}$												
0.01	11,810	1.038	1.761	0.449[b]	0.454[b]	206[c]	16,320	−0.0587	−0.00104	0.170	....	....
0.02	18,030	0.5285	2.468	0.629[b]	0.634[b]	402[c]	22,540	−0.0424	−0.00075	0.187	....	....
0.04	24,040	0.2688	3.461	0.883[b]	0.888[b]	788[c]	28,550	−0.0335	−0.00054	0.196	....	....
0.06	29,970	0.172	4.31	1.098[b]	1.103[b]	1,220[c]	34,480	−0.0278	−0.00043	0.202	....	....

[a] For $S_f = 0.01$, furthermore, $\frac{z_{t\nu}}{z_{t\nu}}$ (Fig. 16) = 0.140; and $\frac{Q_\nu}{Q_{t\nu}} = 0.385$. [b] Multiply by 1,000,000. [c] Multiply by 1,000,000,000.

For comparison, values of $\frac{dK_\nu}{d(1/\mathbf{F})}$ are also shown in Table 6 together with experimental values of $K\,\mathbf{F}$ and calculated values of $\frac{\partial K}{\partial(1/\mathbf{F})}$ (Eq. 40*b*), of

TABLE 6.—COMPARISON OF VALUES OF $\frac{dK_\nu}{d(1/\mathbf{F})}$

S_f	$\frac{dK_\nu}{d(1/\mathbf{F})}$ ($\nu = 0.493 \times 10^{-6}$)	$K\,\mathbf{F}$ (Observed)	$\frac{\partial K}{\partial(1/\mathbf{F})}$ (Eq. 40*b*)	$\frac{dK_M}{d(1/\mathbf{F}_M)}$ (Eq. 44)	$\frac{a_i}{2\,h}$	K (Eq. 36*b*)	K (Observed)	K_ν (Fig. 14)	$\frac{a_i}{2\,h} \times \frac{1}{\mathbf{F}}$
0.01	0.170	0.172	0.171	0.150	0.148	0.177	0.179	0.196	0.153
0.02	0.187	0.189	0.188	0.170	0.175	0.0989	0.100	0.106	0.0919
0.04	0.196	0.196	0.196	0.182	0.190	0.0527	0.0527	0.0562	0.0508
0.06	0.202	0.199	0.202	0.190	0.199	0.0348	0.0343	0.0367	0.0343

$\frac{dK_M}{d(1/\mathbf{F}_M)}$ (Eq. 44, $\nu_\nu = \nu_M$), of $\frac{a_i}{2\,h}$, and of $\frac{a_i}{2\,h} \times \frac{1}{\mathbf{F}}$. $\Big($Note that $\frac{a_i}{2\,h} \times \frac{1}{\mathbf{F}} = \frac{1}{2\,\bar{v}^2/(a_i\,g)}$, in which $\frac{\bar{v}^2}{a_i\,g}$ is a Froude number.$\Big)$ It may be noted that throughout the experimental range the corresponding values of $\frac{dK_\nu}{d(1/\mathbf{F})}$, $K\,\mathbf{F}$, and $\frac{\partial K}{\partial(1/\mathbf{F})}$ are substantially equal and that, for $S_f = 0.06$, values of the three quantities practically equal that of $\frac{a_i}{2\,h}$. Values of the total derivative, $\frac{dK_M}{d(1/\mathbf{F})}$, representing the actual experimental value of ν are approximately equal to values of $\frac{a_i}{2\,h}$, especially for the lower values of S_f (greater values of z_i).

In Fig. 15 a straight line represents, with sufficient accuracy, the three points representing $S_f = 0.01$, 0.02, and 0.04, but passes through neither the limiting point $\frac{dK}{d(1/\mathbf{F})} = 0.230$ at $\frac{1}{\mathbf{F}} = 0$, nor the other extreme, $\frac{dK}{d(1/\mathbf{F})} = \frac{a_e}{2\,h}$ ($= 0.0175$) at $\frac{1}{\mathbf{F}} = \infty$, obtainable from Eq. 44, when $\bar{v} = 0$—that is, when $\mathbf{R}_M$ and $\mathbf{R}_\nu$ equal zero.

Values of the integral Eq. 45 have been obtained from the areas under the dashed curve, Fig. 15. These are given in Table 7.

TABLE 7.—VALUES OF THE INTEGRAL, EQ. 45

$\frac{1}{\mathbf{F}}$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.9	1.1
ΔK_ν	...	0.0219	0.0204	0.0197	0.0193	0.0189	0.0186	0.0182	0.0355	0.0342
K_ν	0	0.0219	0.0423	0.0620	0.0813	0.1002	0.1188	0.1370	0.1725	0.2067

Values of K_ν ($= \Sigma\Delta K_\nu$) from Table 7 are plotted in Fig. 14 and for the four experimental points their values are shown in Table 5, and in Table 6 together with the experimental and calculated values of K. Values of $K_\nu\,\mathbf{F}$ have also been calculated. Fig. 16 gives the corresponding values of $\frac{z_i}{z_t}$. It may be noted that except for the data representing $S_f = 0.01$, the values of $K_\nu\,\mathbf{F}$ are greater than the observed distributional values of $K\,\mathbf{F}$. The explanation of this is that the values of the equations derived herein depend upon conditions beyond the frictional effect of the curb. Such conditions have been represented in the region of the curb by the broken line in Fig. 16. The matter is not of practical importance since corresponding prototype values of z_i are smaller than are usually encountered.

Determination of the Model Case Corresponding to a Given Prototype.—In the approach channels of model and prototype, let the ratio of corresponding flows be the same. (This is in accordance with the usual assumption of model analysis technique.) Then

$$\frac{Q_P}{Q_{tP}} = \frac{Q_M}{Q_{tM}} \qquad \text{(47a)}$$

Assume, moreover, that for corresponding inlet flows the total flows follow Froude's law, or,

$$Q_{tP} = \left(\frac{32}{3}\right)^{2.5} \times Q_{tM} = 372\,Q_{tM} \qquad \text{(47b)}$$

Fig. 10 then yields the value of $\bar{v}$ $\left(\text{and, hence, of } \frac{1}{\mathbf{F}}\right)$ representing the inlet flow corresponding to the prototype total flow.

Design Example.—Let it be required to determine the quantity of flow that will enter an inlet 7 ft long, with a local depression 4 in. deep. The pavement is 36 ft wide, with a crown height of 0.45 ft and a longitudinal slope of 2%. The total flow between the curb and the crown is 16.3 cu ft per sec—the conditions corresponding to the model. (The flow in such a prototype, assuming Kutter's $n = 0.013$, is probably actually greater than 16.3, for the true mean depth is greater than an hydraulic radius determined by dividing the cross sectional area by the wetted perimeter.)

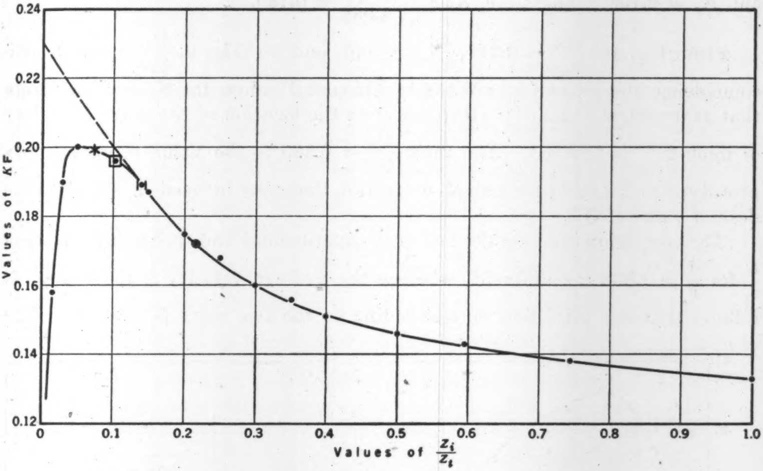

FIG. 16.—DISTRIBUTION OF KF FOR ALL VALUES OF Q_t ($h = 0.1001$)

The design process is as follows: By Eq. 47*b*, $Q_{tM} = \frac{16.3}{372} = 0.0438$. (Note that the experimental value representing $S_f = 0.01$ is $Q_{tM} = 0.0436$.) By Fig. 10, for this amount of total flow in the model, the average velocity of the part entering the inlet is $\bar{v}_M = 1.76$; so that: $\frac{1}{\mathbf{F}_M} = \frac{32.2 \times 0.1001}{(1.76)^2} = 1.038$. The value of ν representing the prototype value of $\mathbf{R}$ with model dimensions and velocity now can be computed. Since Froude's number is the same in model and prototype, and since $h_P = 1.07$, $\bar{v}_P = \left(\frac{32.2 \times 1.07}{1.038}\right)^{0.5} = 5.76$. Hence, for $\nu_P = 13.7 \times 10^{-6}$ (the experimental value for $S_f = 0.02$), $\mathbf{R}_P = \frac{5.76 \times 1.07}{13.7 \times 10^{-6}} = 450{,}000$. If Reynolds' number for model dimensions and velocities is to equal this, $\nu_\nu = \frac{1.76 \times 0.1001}{450{,}000} = 0.392 \times 10^{-6}$. Values of $\frac{dK_\nu}{d(1/\mathbf{F})}$ for this value of ν are shown in Table 5, and, since these agree with the values representing $\nu = 0.493 \times 10^{-6}$, the corresponding graph (Fig. 14) is satisfactory for design use in the present case.

For $\frac{1}{\mathbf{F}} = 1.038$, Fig. 14 yields $K_M = 0.179$; so that $K_M\,\mathbf{F} = \frac{0.179}{1.038} = 0.172$; and $K_\nu = 0.196$. Therefore, $K_\nu\,\mathbf{F} = \frac{0.196}{1.038} = 0.189$.

From Fig. 16: $\frac{z_{iM}}{z_{tM}} = 0.217$; $\frac{z_{i\nu}}{z_{t\nu}} = 0.140$; and, by Fig. 6, $\frac{Q_\nu}{Q_{t\nu}} = 0.385$. By coincidence the latter two are the experimental values for $S_f = 0.02$. Note that geometrical similarity with respect to the location of the outer boundary of inlet flow is ignored. The ratio $\frac{Q_\nu}{Q_{t\nu}} = 0.385$ is the value representing a prototype and, except for a small correction that may be needed, will yield the desired value of Q_P.

The correction involves the fact that experimental and theoretical values of $\frac{z_i}{z_t}$ for $\frac{1}{\mathbf{F}} = 1.038$ are different, whereas the average velocity is the same. It follows that the total flow corresponding to the two cases is different. The relationship between the two flows is found as follows: Two values, $\bar{v}_{\nu 1}$ and $\bar{v}_{M1}$, representing unit total flow, can be obtained for the two values of $\frac{z_i}{z_t}$ from Fig. 17. The ratio of these velocities equals the ratio of two other average velocities representing two different total flows, but a given value of $\frac{z_i}{z_t}$. This

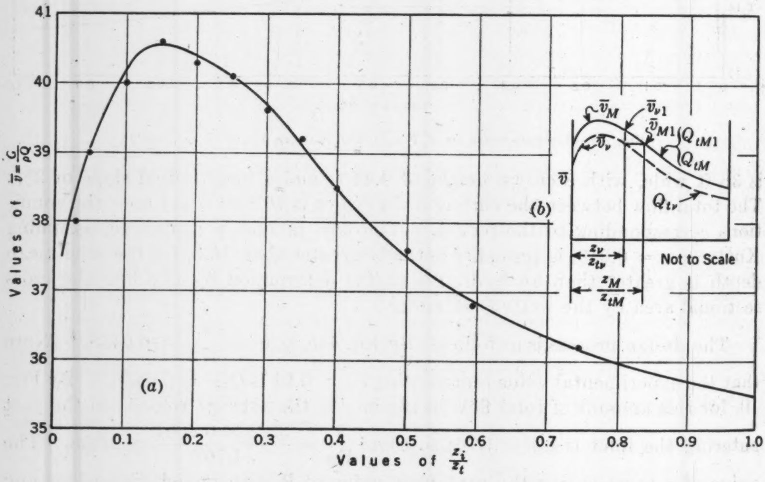

FIG. 17.—DISTRIBUTION OF $\bar{v}$ FOR UNIT TOTAL FLOW ($a_s = 0.0460$)

is shown schematically in Fig. 17(*b*). Hence

$$\frac{\bar{v}_{M1}}{\bar{v}_{\nu 1}} = \frac{\bar{v}_{\nu}}{\bar{v}_{M}} = \frac{Q_{t\nu}}{Q_{tM}} \quad \text{......................} \quad (48)$$

Fig. 17(*a*) yields, for $\frac{z_{iM}}{z_{tM}} = 0.217$ and $\frac{z_{i\nu}}{z_{t\nu}} = 0.140$, $\bar{v}_{M1} = 40.28$, and $\bar{v}_{\nu 1} = 40.55$; so that by Eq. 48 $\frac{Q_{t\nu}}{Q_{tM}} = \frac{40.28}{40.55} = 0.994$. Hence,

$$Q_{t\nu} = 0.994\, Q_{tM} \quad \text{......................} \quad (49)$$

Since $\frac{Q_{\nu}}{Q_{t\nu}} = 0.385$, $Q_{\nu} = 0.385 \times 0.994 \times 0.0438 = 0.0167$ (the correction is negligible in this case, and may be found to be so for all cases). The ratio $\frac{Q_{\nu}}{Q_{M}} = \frac{0.0167}{0.0239} = 0.699$ indicates that the flow that may be expected to enter a prototype inlet is about 70% of that which would be obtained by direct application of the model results.

Since, for equal values of **F**, $\frac{Q_P}{Q_{tP}} = \frac{Q_{\nu}}{Q_{t\nu}} = 0.385$, $Q_P = 16.3 \times 0.385 = 6.3$ cu ft per sec, the desired value of inlet flow in the prototype.

Where values of $\frac{Q}{Q_t}$ in the prototype are appreciably different from those in the model, it is necessary that the flow distribution be known. The variables K and $\bar{v}$ were selected with such a contingency in mind, but demonstration as to whether Froude's law applies in the case is not available at this time.

VII.—Conclusion

The result, $Q_P = 6.3$ cu ft per sec, obtained in the preceding example, is a reasonable quantity on a basis of experience, and in the absence of an error in the analytical method must be assumed to be correct. Nevertheless, confirmation of the result by means of an exactly similar larger, or full-scale, model is desirable.

The writer is aware that the method of solution used is not the only one, and, quite possibly, not the simplest, nor the best. Further investigation may disclose, for example, that the maximum velocity can be used as the characteristic velocity instead of $\bar{v}$. This would have the advantage of removing any doubt as to whether $\bar{v}$ is actually entirely independent of K. The maximum velocity can easily be obtained from Fig. 7, and it has the value

$$v_{\max} = 41.2\, Q_t \quad \text{......................} \quad (50)$$

The method of solution presented in this paper is intended to be used in future development of data for purposes of design. The necessary experimental data covering a fairly wide range of lengths of inlet, slopes of street, etc., have already been obtained.

Acknowledgments.—The experimental work upon which this paper is based was completed at the California Institute of Technology in 1937. The work was initiated and conducted mainly by the Bureau of Engineering, City of Los Angeles, through L. W. Armstrong, M. Am. Soc. C. E., the writer being in direct charge. The County of Los Angeles, through C. E. Arnold, Assoc. M.

Am. Soc. C. E., cooperated by furnishing an assistant. Valuable criticisms and suggestions have been contributed by R. T. Knapp, M. Am. Soc. C. E., V. A. Vanoni, Assoc. M. Am. Soc. C. E., and James Daily; and Hunter Rouse, Merit P. White, and W. J. Manetta, Assoc. Members, Am. Soc. C. E.

APPENDIX *A*

THE ASYMPTOTE, $\mathbf{R}_0$, AND THE FUNCTION OF *D*

The distributional expression for **R** given in Eq. 9*d* represents a range of values from $v = \infty$ and $z_i = 0$ to $v = 0$ and $z_i = \infty$. The value of **R** $(= \mathbf{R}_0)$ representing the asymptote, however, corresponds to a condition in which v has decreased to zero and beyond to minus infinity. Negative velocity represents a reversed direction of flow, that is, flow entering the street from the inlet. Although it may be possible by analytical means to derive the expression for $\mathbf{R}_0$ in a manner similar to that used in determining $K_0\,\mathbf{F}_0$, no such method has suggested itself.

The asymptote $\mathbf{R}_0$ does not occur at the value $\mathbf{R} = 0$, for at that point $K\,\mathbf{F} = \frac{a_c}{2\,h}$, a finite quantity, whereas at an asymptote the quantity must be infinite. This, moreover, is a reason that the three-constant form of hyperbolic equation is required rather than the two-constant form representing the "Second Hyperbolic Law":[6]

$$\frac{\mathbf{R}}{K\,\mathbf{F}} = c_1\,\mathbf{R} + c_2 \quad \text{.......................(51)}$$

THE FUNCTION OF *D*

In Eq. 36*b* the quantity within parentheses represents a product of functions of **R** and various dimensions. Dividing the coefficient of the term involving **R** by the constant term, $\frac{a_0}{2\,h}$, of the function, and rearranging, then yields (see Eq. 5)

$$\frac{K\,\mathbf{F}}{1 + \frac{a_0 - a_c}{a_0}\left(\frac{\mathbf{R}_0}{\mathbf{R} - \mathbf{R}_0}\right)} = \frac{a_0}{2\,h} = \tfrac{1}{2}\,\frac{h - d - c_s}{h} = \tfrac{1}{2}\,(1 - D - C_s) \quad \text{..(52)}$$

Hence,

$$\frac{2\,K\,\mathbf{F}}{1 + \frac{a_0 - a_c}{a_0}\left(\frac{\mathbf{R}_0}{\mathbf{R} - \mathbf{R}_0}\right)} + C_s = 1 - D \quad \text{............(53)}$$

The locus of values of the left-hand member plotted against values of *D* thus should yield a straight line. In Eq. 53 $\mathbf{R}_0$ is not constant. Values of the left-hand member for the three experimental cases are calculated in Table 8(*a*). When these values are plotted against values of *D*, the result is a straight line, confirming the arithmetic in calculations of the value of $\mathbf{R}_0$.

If, on the other hand, it is assumed that $\mathbf{R}_0$ is constant and has the values $-4{,}310$ ($d = 0.0313$, $S_f = 0.02$), and $-3{,}140$ ($d = 0.0000$, $S_f = 0.02$), then values of the left-hand member of Eq. 53, as calculated in Tables 8(*b*) and 8(*c*), produce slightly curved graphs when plotted against D. When plotted against D, R_0 is itself a function of D, and it is practically a straight-line function of D^4 and of d^2.

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[6] "Intermediate Calculus," by Percy F. Smith and William Raymond Longley, Ginn & Co. (Copyright 1931), p. 203.

TABLE 8.—COMPUTATION OF VALUES OF FUNCTION OF D

No.	d (1)	h (2)	D (3)	C_s (4)	$2KF$ (5)	R (6)	R_0 (7)	$R - R_0$ (8)	$\frac{R_0}{R - R_0}$ (9)
					(*a*) $\frac{a_0 - a_e}{a_0} = \frac{0.0425}{0.0460} = 0.924$; AND $c_s = 0.0228$				
1	0	0.0688	0.000	0.332	0.548	13,070	−3,140	16,210	−0.1937
2	0.0313	0.1001	0.313	0.228	0.378 0.385	18,030 21,035	−4,310 −4,510	22,340 25,545	−0.1928 −0.1764
3	0.0625	0.1313	0.476	0.1738	0.256	24,150	−9,870	34,020	−0.290

Table 8.—(*continued*)

No.	Col. 9 × 0.924 (10)	1 + Col. 10 (11)	$2KF$ ÷ Col. 11 (12)	Col. 12 + C_s (13)	Col. 6 + 4,310 (14)	− 4,310 ÷ Col. 14 (15)	− 3,982 ÷ Col. 14 (16)	1 − Col. 16 (17)	$2KF$ ÷ Col. 17 (18)	Col. 18 + C_s (19)
	(*a*) (*continued*)				(*b*) $R_0 = -4{,}310$; AND $\frac{R_0}{a_0}(a_0 - a_e) = 0.924 \times (-4{,}310) = -3{,}982$					
1	−0.1792	0.821	0.668	1.000	17,380	−0.248	−0.229	0.771	0.711	1.043
2	−0.1783 −0.1631	0.822 0.837	0.460 0.460	0.688 0.688	22,340	−0.1927	−0.1783	0.822	0.460	0.688
3	−0.268	0.732	0.350	0.524	28,460	−0.1513	−0.1397	0.860	0.298	0.472

Table 8.—(*continued*)

No.	D^2 (see Col. 3) (20)	D^4 (21)	3,140 + R (see Col. 6) (22)	− 3,140 ÷ Col. 22 (23)	− 2,901 ÷ Col. 22 (24)	1 − Col. 24 (25)	$2KF$ ÷ Col. 25 (26)	Col. 26 + C_s (27)
	(*b*) (*continued*)		(*c*) $R_0 = -3{,}140$; AND $\frac{R_0}{a_0}(a_0 - a_e) = 0.924 \times (-3{,}140) = -2{,}901$					
1	0	0	16,210	−0.194	−0.179	0.821	0.667	0.999
2	0.0981	0.00962	21,170	−0.148	−0.137	0.863	0.438	0.666
3	0.226	0.0512	27,290	−0.115	−0.106	0.894	0.286	0.460

APPENDIX *B*

NOTATION

The following symbols are defined where they first appear and are assembled here for convenience of reference (see Fig. 1 for designation of dimensions):

$A_0 = \frac{a_0}{h}$, the approach-depth number;

a = depth at a distance, z, from the curb, in feet:

$\bar{a}$ = a weighted or average value of a;

a_0 = depth at curb at the approach section, in feet;

a_c = depth at the crown, or center line of street, in feet;

C = a coefficient:

C_0 = an hyperbolic constant;

$C_s = \frac{c_s}{h}$;

c = a coefficient; height of inlet opening, in feet: c_s = fall due to the normal slope of the model between the approach section and the center of the inlet, in feet;

$D = \frac{d}{h}$, the depth-of-depression number;

d = depth of depression, in feet;

e = subscript denoting excess flow, or overflow; height of overflow sill above normal gutter line produced, in feet;

$\mathbf{F}$ = Froude's number, $\frac{\bar{v}^2}{g\,h}$: $\mathbf{F}_M$ = Froude's number representing actual model flow;

h = head; the characteristic dimension; difference in elevation between the water surface at the approach section and the lip of the inlet at its midlength, in feet: h_P = head for prototype;

i = subscript indicating "inlet" or "integral";

K = the dependent variable, $\frac{\bar{z}\,P}{U}$:

$K_M = \frac{\bar{z}\,P}{U}$ for actual model flow;

$K_\nu = \frac{\bar{z}\,P}{U}$ for model dimensions and velocity, but a calculated value of ν;

l = length of upper end of depression (see Fig. 1);

n = width of overflow notch, in feet; an exponent;

R = hydraulic radius, in feet;

$\mathbf{R}$ = Reynolds' number, $\frac{\bar{v}\,h}{\nu}$:

$\mathbf{R}_M$ = Reynolds' number for actual model flow;

$\mathbf{R}_P$ = Reynolds' number for prototype flow;

$\mathbf{R}_\nu$ = Reynolds' number of prototype magnitude, but model dimension and velocity (ν calculated);

S = longitudinal slope of the model, in feet per foot: S_f = longitudinal slope of the approach flume, in feet per foot;

T = the roughness number, $\frac{t}{h}$;

t = height of roughness protuberances, in feet; subscript denoting "total";

$Y_c = \frac{y_c}{h}$, the crown height number;

y_c = the crown height;

z = distance from curb:

z_i = distance from curb to outer boundary of a filament of flow;

z_t = distance from curb to crown;

z_{iM} = distance, z_i, for model;

z_{tM} = distance, z_t, for model;

f, α, β, θ, etc. = function of;

$\kappa = \dfrac{Q}{Q_t}$.

DIMENSIONAL FORMULAS

G = momentum per second at the approach section of flow within the distance z_i from the curb; $M\,L\,T^{-2}$

g = acceleration of gravity, in feet per second per second $\left(= \dfrac{\gamma}{\rho}\right)$; $L\,T^{-2}$

P = hydrostatic pressure acting at the approach section on flow within a distance z_i from the curb, in pounds; $M\,L\,T^{-2}$

Q = rate of flow, in cubic feet per second at the approach section of flow within the distance z_i from the curb: $L^3\,T^{-1}$

Q_e = excess flow, or overflow;

Q_t = "total" flow or flow between the curb and the crown;

Q_P = flow, Q, in a prototype;

Q_M = flow, Q, in the model;

U = moment of momentum per second within a distance z_i from the curb at the approach section; $M\,L^2\,T^{-2}$

v = the apparent average velocity in a vertical line at a distance z from the curb, in feet per second: $L\,T^{-1}$

$\bar{v}$ = average velocity, or momentum per second per unit mass per second of flow within the distance z_i from the curb at the approach section $\left(= \dfrac{G}{\rho\,Q}\right)$; $L\,T^{-1}$

$\bar{v}_M$ = average velocity, $\bar{v}$, for the model;

$\bar{v}_P$ = average velocity, $\bar{v}$, for the prototype;

x = length of inlet;

$\bar{z}\,P$ = moment of hydrostatic pressure, P; $M\,L^2\,T^{-2}$

γ = specific weight, in pounds per cubic foot ($= \rho\,g = 62.4$ lb per cu ft); $M\,L^{-2}\,T^{-2}$

ν = kinematic coefficient of viscosity, in square feet per second $\left(= \dfrac{\mu}{\rho}\right)$: $L^2\,T^{-1}$

ν_P = actual kinematic viscosity for a prototype;

ν_r = kinematic viscosity representing a Reynolds number of prototype magnitude, but of model velocity and dimension;

μ = coefficient of viscosity; $M\,L^{-1}\,T^{-1}$

ρ = mass (slugs) per cubic foot $\left(= \dfrac{\gamma}{g} = 1.938\right)$. $M\,L^{-3}$

DISCUSSION

Ralph W. Powell,[7] **M. Am. Soc. C. E.**—A first reading of this paper gives one the impression that large caliber mathematical artillery has been brought to bear on a problem that should have succumbed to simpler methods. In default of a simpler method to suggest, perhaps such a criticism is ungracious; and this method of "distributional" analysis seems quite worth studying on its own account, entirely apart from the problem to which it is here applied. As the writer understands it, Mr. Tapley's method (omitting some of the details) is equivalent to the following. He has computed the kinematic viscosity of an imaginary, less viscous liquid (which would have produced Froudian similarity in his model) and the Reynolds number resulting from using the actual size and velocity of his model, but this imaginary viscosity instead of the actual viscosity. By running tests at several different slopes, he obtained enough variation in his mean velocities to give an appreciable variation in the Reynolds and Froude numbers and to derive a formula for the coefficient which he was studying in terms of **F** and **R**. His range of observed Reynolds numbers was only from 11,810 to 29,970, but by a very clever mathematical device (Eqs. 26 to 29), the validity of which the writer is not in a position to discuss at present, the range is extended to the entire field from zero to infinity. Therefore, the value of the coefficient for the aforementioned fictitious Reynolds number is obtained and the problem is solved. The writer wishes to make just a few observations.

It would seem that Eq. 6*b* is only one of the forms that Eq. 6*a* might take. For example, might not the equation be expressed as

$$K = C_{AD}\left[\alpha\,(\mathbf{F}) + \beta\,(\mathbf{R})\right] \ldots\ldots\ldots\ldots\ldots\ldots (54)$$

Of course, the experimental results seem to answer this particular question in the negative, but in general it must not be assumed that $\theta\ (\mathbf{F}, \mathbf{R})$ always can be put in the form $\alpha\ (\mathbf{F})\ \beta\ (\mathbf{R})$.

To the writer, the most interesting point of the paper is the use of a "momentum average" velocity instead of the usual "discharge average" in computing **F** and **R**—that is, instead of defining average velocity as discharge divided by area, the author has defined it as momentum divided by mass. This gives about 20% higher velocities in the examples of the paper. Since the ratio of $\bar{v}$ to v' for the different columns of Table 1 is quite constant, it would seem that this innovation would hardly affect the final result in this particular problem. However, it raises the very interesting question as to whether, in general, it would be an improvement over the ordinary usage. It would seem that it could hardly be expected ever to replace, entirely, the ordinary definitions of **F** and **R** because a knowledge of the velocity distribution would be required before either could be computed. It also raises the question as to whether one should ever use an "energy average"—the square root of twice the kinetic energy divided by the mass.

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[7] Associate Prof. of Mechanics, Ohio State Univ., Columbus, Ohio.

The ratio $\frac{\bar{v}}{v'}$ has been variously denoted by β, α, and K_m by various authors. J. W. Johnson,[8] Assoc. M. Am. Soc. C. E., and M. P. O'Brien,[8,9] G. H. Hickox,[9] and R. W. Powell,[10] Members, Am. Soc. C. E., used β; S. M. Woodward, M. Am. Soc. C. E., and C. J. Posey,[11] Assoc. M. Am. Soc. C. E., used α; and Hunter Rouse[12] and Victor L. Streeter,[13] Assoc. Members, Am. Soc. C. E., used K_m. The value $\beta = 1.20$, which is indicated in Table 1, is quite high, but the true value is still higher because, as the author admits, he has neglected the variation of velocity in the vertical direction. Since this would be considerable, the true value of β was probably between 1.25 and 1.30—almost as much as for laminar flow. This must be due to the unsymmetrical cross section for, as the work of Professor Streeter,[13] shows, any reasonable assumption as to the frictional resistance would not account for such a high value if the section were symmetrical.

It should also be noted that in Eq. 8, and the other equations derived from it, special attention is required if any of the velocities are negative (that is, in the upstream direction). This has been pointed out by Professor Posey.[14]

A minor criticism of the paper is that the meaning of h in the definitions of **F** and **R** is not clear to the casual reader. Another is that, in a model this small, it would seem that, among the factors listed in Eq. 3*a*, surface tension would have to be included. In fact, it is the writer's opinion (without any rational reason to back it up) that the model should have been at least twice as large—that is, $\frac{3}{16}$ scale instead of $\frac{3}{32}$ scale.

G. S. TAPLEY,[15] ASSOC. M. AM. SOC. C. E.—The problem attacked in the paper is of a type upon which little has yet appeared in literature; features believed to be novel were included. It is regretted, therefore, that criticism, either for or against, was not more widespread.

Professor Powell's comments are very much to the point in several particulars. His statement that the method of distributional analysis seems worthy of study is believed to be particularly pertinent when, as in the case of storm-sewer inlets, the lateral distribution is so characteristically a feature of the phenomenon.

The study of distribution itself is also an active field at present. Attention is invited to the observed, practically perfect independence of the values of κ from the total flow Q_t in the model channel. Table 9 shows, for comparison, values of κ representing $S_f = 0.02$, or $Q_t = 0.0615$, and corresponding values of κ representing the average for $S_f = 0.01$, 0.02, 0.04, and 0.06, or

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[8] "Velocity-Head Correction for Hydraulic Flow," by M. P. O'Brien and J. W. Johnson, *Engineering News-Record*, August 16, 1934, p. 215.

[9] "Applied Fluid Mechanics," by M. P. O'Brien and G. H. Hickox, McGraw-Hill Book Co., Inc., New York, N. Y., 1937, p. 271.

[10] "Mechanics of Liquids," by R. W. Powell, Macmillan Co., 1940, p. 125.

[11] "The Hydraulics of Steady Flow in Open Channels," by S. M. Woodward and C. J. Posey, John Wiley and Sons, Inc., New York, N. Y., 1941, p. 47.

[12] "Fluid Mechanics for Hydraulic Engineers," by Hunter Rouse, McGraw-Hill Book Co., Inc., New York, N. Y., 1938, p. 54.

[13] "The Kinetic Energy and Momentum Correction Factors for Pipes and for Open Channels of Great Width," by Victor L. Streeter, *Civil Engineering*, April, 1942, pp. 212–213.

[14] "Variations in Correction Factors for Pipes and Open Channels," by C. J. Posey, *ibid.*, July, 1942, p. 398.

[15] Junior Civ. Engr., Bureau of Eng., City of Los Angeles, Los Angeles, Calif.

Q_t = 0.0434, 0.0615, 0.0870, and 0.1066. All data represent $a_0 = 0.0460$, an average obtained from a large number of observations.[16]

Inasmuch as Professor Powell calls attention to the writer's use of the momentum average, the following remarks upon average velocities may be in order: The so-called average velocity, "discharge divided by area," has been aptly called the "apparent average" velocity. It should be kept in mind that this velocity is an average one only in the uninteresting case where velocity does not vary throughout the cross section. The following mathematical expression shows this:

TABLE 9.—COMPARISON OF VALUES OF κ

Description	Values of z_i							
	0.168	0.339	0.508	0.674	0.841	1.003	1.257	1.700
κ (S_f=0.02)........................	0.278	0.517	0.690	0.805	0.882	0.928	0.969	1.001
κ (average)........................	0.280	0.518	0.693	0.810	0.886	0.932	0.970	1.000

$$v' = \frac{\int_0^A v\,dA}{\int_0^A dA} \qquad (55)$$

in which A represents area. Moreover, the physical conception is not clear; a particle of fluid cannot be described satisfactorily by its area of cross section, dA.

On the other hand, the expression for the momentum-average velocity, $\bar{v}$, is one for which the physical conception is grasped readily:

$$\bar{v} = \frac{\int v\,dM}{\int dM} = \frac{\int v\,\rho\,dQ}{M} = \frac{\rho \int_0^A v^2\,dA}{M} \qquad (56)$$

in which M represents mass per second, and Q represents volume per second. Each elementary mass of fluid is weighted by its particular velocity. The average, $\bar{v}$, is the velocity of all the particles moving as one mass when the corresponding momentum equals the aggregate of the momentums of all the individual masses moving at velocity v.

The expression for the energy-average velocity, $\bar{\bar{v}}$, may be obtained from

$$\bar{\bar{v}}^2 = \frac{\frac{1}{2}\int v^2\,dM}{\frac{1}{2}\int dM} = \frac{\int v^2\,\rho\,dQ}{\int \rho\,dQ} = \frac{\rho \int_0^A v^3\,dA}{\int_0^Q \rho\,dQ} \qquad (57)$$

---

[16] Values in Table 9 are from "Depth and Distribution Data for Flow in Model Street," by G. S. Tapley, Table 19. The original manuscript is on file at Engineering Societies Library, 29 West 39th Street, New York, N. Y.

Hence,

$$\bar{\bar{v}} = \frac{\left(\rho \int_0^A v^3\, dA\right)^{0.5}}{\left(\int_0^Q \rho\, dQ\right)^{0.5}} \qquad (58)$$

In Eq. 58 the physical conception is again not so clear as in the case of $\bar{v}$, because the denominator represents the square root of a mass per second. It is required, then, to state: $\bar{\bar{v}}$ is the velocity at which the square root of the mass per second moves when the energy of the total mass equals the sum of the energies of the individual particles.

Assuming that both energy-average and momentum-average velocities exist, the writer believes that the criterion for choice between them should be: Use the energy-average velocity when the phenomenon is properly and conveniently expressible in terms of work and energy; and use the momentum-average velocity when momentum and force are the proper mediums for expression.

Reference should be made to a bulletin of the National Bureau of Standards, dated July, 1942, regarding energy and momentum methods.[17] The bulletin calls attention to the long-standing nature of the controversy over the methods, and states that Boussinesq advocated the momentum method, whereas Coriolis advocated the energy method, Coriolis' view generally prevailing. The bulletin further gives the view of G. H. Keulegan that either method is correct, provided the proper interpretation is placed upon the operation, but that, practically, it is preferable to use the momentum method.

It should be emphasized that the ratio $\frac{\bar{v}}{v'}$, mentioned by Professor Powell, represents the ratio of the momentum-average velocity of the inlet flow to the discharge-over-area average velocity of the total flow. These quantities, therefore, are not comparable in the manner Professor Powell evidently believes they are, since they do not represent the same cross sections of flow.

In an investigation of backwater effect,[18] the writer found that the relationships in Table 10 hold for the total width of cross section, the area of cross section being 0.0298 sq ft. The ratio is thus constant, but less in magnitude than computed by Professor Powell. The writer did not feel warranted in ignoring a constant discrepancy even as small as 6%, because the quantity under investigation (Reynolds' number) was expected to make itself evident by only a feeble effect within the model range.

TABLE 10.—RELATIONSHIP BETWEEN $\bar{v}$ AND v'

Description	Values of S_f			
	0.01	0.02	0.04	0.06
$\bar{v}$	1.55	2.18	3.07	3.77
v'	1.46	2.06	2.90	3.57
$\bar{v}/v'$	1.06	1.06	1.06	1.05

Professor Powell inquires whether Eq. 54 could not have been used as well as Eq. 6*b*. The following arguments seem to prove that it could not, although some of the facts involved could not be anticipated without regard to the experimental data. All quantities of physics are products,[19] and any set of observations can be represented by a potential series.[20] Moreover, a product of a series is only another series. Eq. 54 evidently would apply to a series in which there were no terms such as **FR**; such terms are required in the experimental case because **F** and **R** vary simultaneously. In a similar case, Morrough P. O'Brien and George H. Hickox,[21] Members, Am. Soc. C. E., state that the unknown function should be considered as the product of two infinite series in the dimensionless groups.

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[17] *Technical News Bulletin*, National Bureau of Standards, July, 1942, p. 50.

[18] "Inlet Flow with Backwater Effect," by G. S. Tapley, Table 9. The manuscript is filed with Engineering Societies Library, 29 West 39th Street, New York, N. Y.

Professor Powell is correct in stating that the meaning of *h* in connection with its use in **F** and **R** was not sufficiently emphasized. The dimension, of course, is the one selected as being descriptive of the characteristic phenomenon, gravitation, and it represents the fall of the water during its transit over the effective part of its course. Had viscosity been the principal consideration, another dimension would have been selected. When both gravity and viscosity are considered important, a single dimension is used in both the Froude and Reynolds numbers, but the dimension best representing the most important force is the one selected.

The writer agrees that it might have been preferable to use a larger model, but for the reason that difficulty was experienced in measuring depths accurately at the approach section, and not because of fear regarding the effect of surface tension. Considerable attention was given to the possible effect of surface tension because of the small size of the model, but it was concluded that the effect was inappreciable except in a few minor cases, such as when the overflow was approaching zero magnitude. However, in the case of submerged inlets (a subject not included in the paper), where the inlet operates as an orifice, surface tension was deemed to be an important consideration, and in an unpublished report entitled "Submerged Curb-Opening Storm Drain Inlets," the writer included it in the analysis by means of which the orifice coefficient was derived. The principal argument against a larger model, of course, is that time and expense items increase rapidly with increase in size. With a larger model it is possible, moreover, that it would have been difficult to have determined the form of the function of **R**, because the effect of viscosity becomes feebler with increase in size.

Backwater Effect.—Subsequent to completion of the analysis of free flow, an investigation of backwater effect[18] was undertaken. The following results of the latter investigation are submitted here as being substantiative, to some extent, of the correctness of the former analysis, and also for the value they may have in extending the range of conditions covered.

Following Eq. 36*a*, let

$$J = \frac{K\,\mathbf{F}}{0.230 - \dfrac{958}{\mathbf{R} + 4{,}510}} = \phi(\bar{v}, b, g, h) \ldots\ldots\ldots\ldots\ldots (59)$$

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[19] "Mathematics of Modern Engineering," by Robert E. Doherty and Ernest G. Keller, John Wiley & Sons, Inc., New York, N. Y., Vol. 1, 1936, pp. 140 and 145.

[20] "Engineering Mathematics," by Charles P. Steinmetz, McGraw-Hill Book Co., Inc., New York, N. Y., 1917, p. 212.

[21] "Applied Fluid Mechanics," by Morrough P. O'Brien and George H. Hickox, McGraw-Hill Book Company, Inc., New York, N. Y., 1937, p. 99.

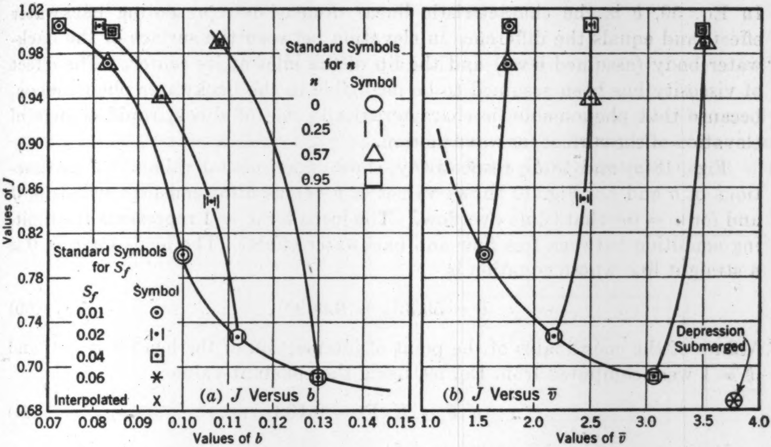

FIG. 18.—VALUES OF J AS FUNCTIONS OF b AND $\bar{v}$ ($a_s = 0.0460$)

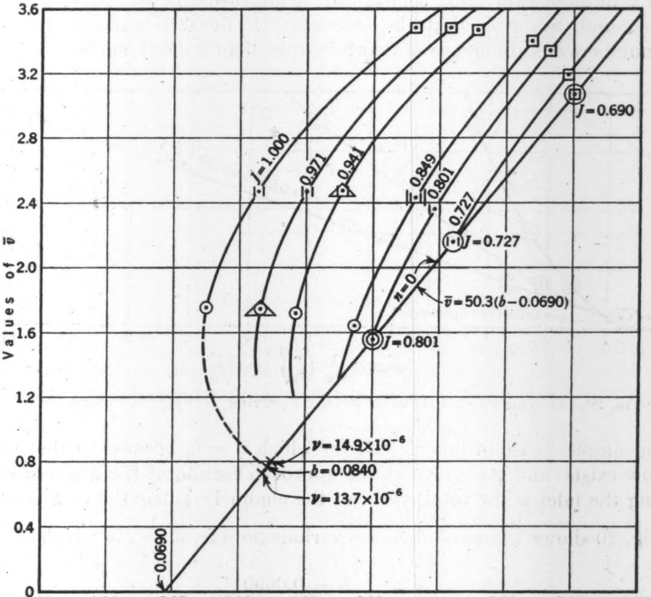

FIG. 19.—EXPERIMENTAL VALUES OF $\bar{v}$ AS A FUNCTION OF b, FOR CONSTANT VALUES OF J AND FOR $n = 0$

In Eq. 59, b is the characteristic linear dimension representing backwater effect, and equals the difference in elevation between the surface of the backwater body (assumed level) and the lip of the inlet at its center. The effect of viscosity has been assumed to be negligible in the backwater phenomenon, because that phenomenon is characteristically one of shock, rapid change of elevation of the surface, or wave motion.

Figs. 18(a) and 18(b), respectively, show experimental values of J as functions of b and $\bar{v}$. Fig. 19 shows values of $\bar{v}$ versus b for constant values of J and for $n = 0$—that is, no overflow. The locus for $J = 1$ represents the limiting condition between free flow and backwater effect. The locus for $n = 0$ is a straight line whose equation is

$$\bar{v} = 50.3\,(b - 0.0690) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (60)$$

Values of the coordinates of the point of intersection of the loci for $n = 0$ and $J = 1$ were computed from Eq. 59, using the constant value

$$K\,\mathbf{F} = K_t\,\mathbf{F}_t = 0.133 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (61)$$

and an average experimental value of ν.

It can be reasonably anticipated that the arguments in the function ϕ will combine in such a manner that J is expressed as a function of a Froude number and of a number expressing a geometrical dimension of the backwater body. It was found, however, that the characteristic linear dimension that enters these numbers is a difference of depths rather than a depth such as b.

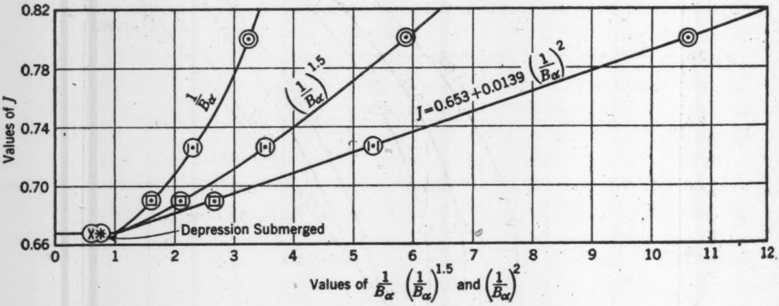

FIG. 20.—EXPERIMENTAL VALUES OF J AS A FUNCTION OF VARIOUS POWERS OF $\frac{1}{B_\alpha}$

The simplest case of flow is that for which $n = 0$, because in that case no overflow exists, and the width at the approach section of the filament of flow entering the inlet is the total width of the channel—z_i has the constant value z_t. Fig. 20 shows values of J versus various powers of $\frac{1}{B_\alpha}$, in which

$$B_\alpha = \frac{b - 0.0690}{h} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (62)$$

In Eq. 62, 0.0690 is the value of b when $\bar{v} = 0$ and $n = 0$, as shown in Fig. 19.

The equation for J in terms of B_α has the simple form

$$J = 0.653 + 0.0139\left(\frac{1}{B^2_\alpha}\right) \ldots\ldots\ldots\ldots\ldots\ldots (63)$$

Note that in Fig. 20 the data representing $S_f = 0.06$ correspond to the condition wherein the backwater wave has reached a point upstream far enough to submerge the depression completely, so that the velocity distribution at the approach section is no longer normal for the value of S_f.

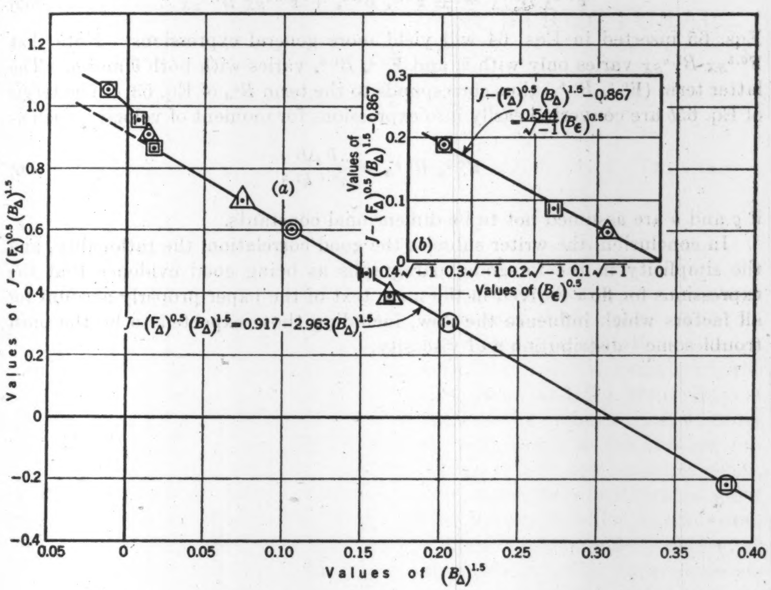

FIG. 21.—FLOW FACTORS FOR $b > 0.0840$ AND $b < 0.0840$ ($a_0 = 0.0460$)

It was found that, for conditions of flow other than those representing $n = 0$, other terms than B_α are necessary. The results are shown in Fig. 21 and are expressible by means of the linear relationships:

$$J - \mathbf{F}^{0.5}_\Delta\, B^{1.5}_\Delta = 0.867 - 2.963\,(B^{1.5}_\Delta - 0.0170) \ldots\ldots\ldots (64a)$$

for values of b greater than 0.0840, and

$$J - \mathbf{F}^{0.5}_\Delta\, B^{1.5}_\Delta = 0.867 + 0.544\,(-B_\epsilon)^{0.5} \ldots\ldots\ldots\ldots (64b)$$

for values of b less than 0.0840. In Eqs. 64: $\mathbf{F}_\Delta = \frac{\bar{v}^2}{g\,\Delta b}$; $\Delta b = b - 0.0774$; $B_\Delta = \frac{\Delta b}{h}$; $B_\epsilon = \frac{\epsilon\, b}{h}$; and $\epsilon\, b = b - 0.0840$. (Note that, by Eq. 61, the quantity $0.867 = 1 - K_t\, \mathbf{F}_t$.)

The difference in the parameters 0.0774 and 0.0840 is 0.0066, which is the fall in half the length of the inlet due to the normal slope of the model. This fact suggests the manner in which the length of inlet and slope of street enter the backwater function and should be useful when, later, these quantities are investigated as variables. It can be shown readily that

$$B_{\Delta} = B_{\epsilon} + B_{SX} = B_{\epsilon} + \frac{0.0066}{h} \qquad (65a)$$

and

$$\mathbf{F}^{0.5}_{\Delta}\, B^{1.5}_{\Delta} = \pm\, \mathbf{F}^{0.5}_{\epsilon}\, B^{1.5}_{\epsilon} + \mathbf{F}^{0.5}_{SX}\, B^{1.5}_{SX} \qquad (65b)$$

Eqs. 65 inserted in Eqs. 64 will yield more general expressions. Note that $\mathbf{F}^{0.5}_{SX}\, B^{1.5}_{SX}$ varies only with $\bar{v}$, and $\mathbf{F}^{0.5}_{\epsilon}\, B^{1.5}_{\epsilon}$ varies with both $\bar{v}$ and b. The latter term ($\mathbf{F}^{0.5}_{\epsilon}\, B^{1.5}_{\epsilon}$) thus corresponds to the term B^{2}_{α} of Eq. 63. The terms of Eq. 65*b* are converted easily into expressions for moment of velocity, such as

$$\mathbf{F}^{0.5}_{\Delta}\, B^{1.5}_{\Delta} = \frac{\bar{v}\, \Delta b}{g^{0.5}\, h^{1.5}} \qquad (66)$$

if g and h are assumed not to be dimensional constants.

In conclusion, the writer submits the good correlation, the rationality, and the simplicity in the foregoing paragraphs as being good evidence that the expressions for flow derived in the main text of the paper properly account for all factors which influence the flow, including those expected to be the most troublesome—distribution and viscosity.

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Paper No. 2185

MECHANICS OF CREEP FOR STRUCTURAL ANALYSIS

BY JOSEPH MARIN,[1] ASSOC. M. AM. SOC. C. E.

WITH DISCUSSION BY MESSRS. A. HRENNIKOFF, C. A. P. TURNER, RALPH W. STEWART, AND JOSEPH MARIN.

SYNOPSIS

Under certain conditions in a stressed structural material, a continuous plastic deformation occurs with time for a constant stress value. This deformation is called "creep." The importance of creep in a particular design depends upon various factors as, for example, the material considered, the allowable stress used, the temperature of operation, and the desirability of maintaining a small deformation. For this reason the designer must consider the effect of creep in many applications, such as occurs in the turbine, oil-refining and automotive industries. In cases where high temperatures occur, changes in the stress distribution and deformations are sometimes so great that the need for considering effects of creep has been realized for several years. For normal temperatures, on the other hand, little has been done to determine the influence of creep as a factor in structural design. This paper is an attempt to analyze this problem. In developing methods of analysis for statically indeterminate structures it was necessary first to select a creep-stress law that could be used in place of Hooke's law. For this purpose a new creep-stress law is proposed in this paper. With a basic creep-stress relation for simple tension, theories for bending stresses and deflections in beams are developed. These theories form the basis for the slope-deflection and moment-distribution procedures developed for the analysis of statically indeterminate structures in the case of creep.

The importance of creep in structural analysis is twofold: First, there is the possibility that the creep deflection might be excessive; and second, there is, in the case of statically indeterminate structures, a change in the values of reactions and resisting moments when Hooke's law is replaced by a creep law. The significance of the creep deflection or the change in stress distribution in

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NOTE.—Published in May, 1942, *Proceedings*.

[1] Prof., Eng. Mechanics, Pennsylvania State College, State College, Pa.

a particular design naturally will depend upon the material considered. Calculations indicate that in some cases a consideration of creep is important.

Some preliminary tests are reported in this paper in support of theories developed. It should be emphasized, however, that these theories will give only approximations, since the creep law is an approximate one. It must also be noted that there is great need for experimental work in this new field of study.

The importance of creep in structural analysis is particularly significant if stresses beyond the yield point are permitted or if such stresses are produced due to overloading. This is the case because there is evidence of creep for such stresses. A consideration of creep is necessary, therefore, for some structural materials and applications at both normal and elevated temperatures. Its importance is also magnified in view of the recent interest shown in designing structural parts stressed to the plastic range.

INTRODUCTION

When they are subjected to stress, engineering materials sometimes develop plastic deformations with time, called "creep deformations." At high temperatures these deformations are of such magnitude that they definitely influence design. Considerable test data have been obtained at high temperatures, particularly for materials used in the turbine and oil-refining industries. (See, for example, "Compilation of Available High Temperature Creep Characteristics of Metals and Alloys," A. S. M. E.—A. S. T. M., 1938.) Many attempts have been made to interpret these data for the purpose of selecting working stresses.[2] To the structural engineer, the behavior of materials at normal temperatures is of greater interest, but for such temperatures there is little information available. A brief review of the information on creep in simple tension and at normal temperatures will be made first. This study leads to the selection of a new creep-stress law in place of Hooke's law for the development of a mechanics of creep.

NOTATION

The letter symbols used in this paper are defined where they first appear and are assembled for reference in the Appendix.

SIMPLE TENSION; CREEP-STRESS LAW

Usually the simple creep-tension test is used for obtaining basic data on the creep behavior of engineering materials. In this test a specimen is subjected to a stress that remains constant, and the creep deformations are measured with time. With this information creep-time relations can be plotted for each stress value. Fig. 1 shows such plots for Aluminum Alloy 3S-3/4H tested at room temperature. These experiments in tension and others in pure bending have been reported in detail.[3] For many problems in design where creep is significant, it is necessary to select a working stress based on an allowable creep deformation in the estimated life of the machine or structural member. Industry cannot wait for lifetime test results, and it becomes necessary, therefore, to extrapolate test data that will apply for lengths of time not covered by the test results. This has led to a number of methods for interpreting the test data. For this purpose it is desirable to express a relationship for the creep deformation e_c in a time t for a unit stress s. When elevated temperatures must be considered, there is the additional variable of temperature.

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[2] "Interpretation and Use of Creep Results," by J. J. Kanter, *Transactions*, A. S. M., Vol. 24, No. 4, December, 1936, p. 870; see also "A Comparison of the Methods Used for Interpreting Creep Test Data," by J. Marin, *Proceedings*, A. S. T. M., Vol. 37, 1937, Pt. II, Technical Papers, p. 258.

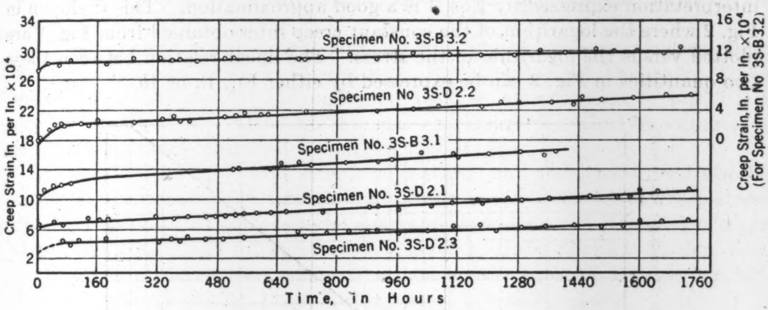

FIG. 1.—TENSION CREEP CURVES

Some of the methods for interpretation developed for elevated temperatures have been presented elsewhere.[2] More recently, procedures for interpretation of data have been proposed by S. H. Weaver,[4] C. R. Soderberg,[5] A. Nádai,[6] and others. A consideration of these methods shows that, although some of the methods fit a particular set of test data better than other methods, they do so by using a more complicated formula. If the creep-stress law is not a simple one, it leads to mathematical difficulties in developing a mechanics of creep, or the expressions deduced become too complicated for practical purposes. A creep law that satisfies the condition of simplicity and has considerable experimental support for many engineering materials at elevated temperatures is the one called the log-log creep-stress relationship. By this method, a constant creep rate is assumed. (See, for example, the many tests reported in "Compilation of Available High Temperature Creep Characteristics of Metals and Alloys," A. S. M. E.—A. S. T. M., 1938. These test results are plotted on a log-log basis and show an approximate linear plot.) In addition, a straight-line relationship is also assumed between the logarithm of the creep rate and the stress—that is,

$$C_n = a\,s^n \qquad (1a)$$

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[3] "Creep of Aluminum Subjected to Bending at Normal Temperatures," by J. Marin and L. E. Zwissler, *Proceedings*, A. S. T. M., Vol. 40, 1940, p. 937.

[4] "The Creep Curve and Stability of Steels at Constant Stress and Temperature," by S. H. Weaver, *Transactions*, A. S. M. E., Vol. 58, 1936, p. 745.

[5] "The Interpretation of Creep Tests for Machine Design," by C. R. Soderberg, *ibid.*, p. 733.

[6] "The Influence of Time Upon Creep"—The Hyperbolic Sine Creep Law, Stephen Timoshenko 60th Anniversary Vol., by A. Nádai, Macmillan Co., 1938, pp. 155–171.

in which C_n = constant creep rate = $\frac{e_c}{t}$, and a and n are experimental constants. Eq. 1a also can be written:

$$e_c = a\, s^n\, t \qquad (1b)$$

Considering the behavior of engineering materials at normal temperatures, there are only a few experiments upon which to base a creep-stress law. It is desirable to review briefly some of these tests.

Experiments in tension[3] on Aluminum Alloy 3S-3/4H show that the log-log interpretation expressed by Eqs. 1 is a good approximation. This is shown in Fig. 2 where the logarithm of the constant creep rates obtained from Fig. 1 are plotted versus the logarithm of the stress. The linear relation between these two quantities in Fig. 2 can be expressed by either Eq. 1a or 1b.

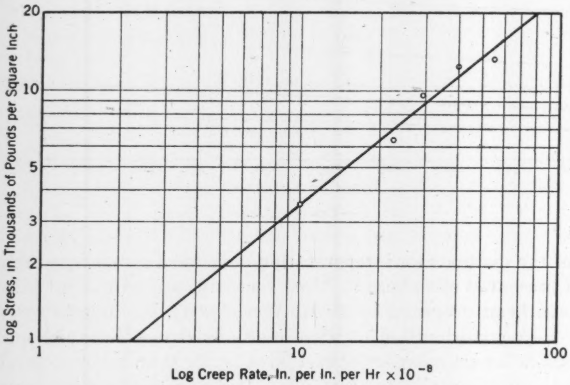

FIG. 2.—RELATION BETWEEN CREEP RATE AND STRESS

Creep tests on various engineering materials at normal temperatures were reported by R. G. Sturm, Assoc. M. Am. Soc. C. E., C. Dumont, and F. M. Howell.[7] These experiments were made on several aluminum alloys, soft and hard copper, and two steels. The material was in the form of sheets and wires. These tests show a log-log plot between the creep deformation and the time. Accordingly, the creep law proposed can be expressed essentially by the equation

$$e_c = F\, t^m \qquad (2)$$

in which F and m are experimental constants. By proposing Eq. 2, no attempt is made to give a general expression involving a variable stress.

Creep experiments on five steels were reported by H. J. French, H. C. Cross, and A. A. Peterson,[8] in which some of the tests were made at room temperature. These experiments were made on:

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[7] "A Method of Analyzing Creep Data," by R. G. Sturm, C. Dumont, and F. M. Howell, *Transactions*, A. S. M. E., Vol. 58, 1936, p. A-63.

[8] "Creep in Five Steels at Different Temperatures," by H. J. French, H. C. Cross, and A. A. Peterson, *Paper No. 362*, Technical Papers of the Bureau of Standards, Vol. 22, 1927–1928.

One 0.24% carbon boiler-plate steel specimen;
One structural alloy steel specimen;
One tungsten-vanadium high-speed steel specimen;
One chromium "stain-resisting" steel specimen; and
One nickel chromium "austenitic" steel specimen.

Although sufficient experiments were not made for purposes of establishing a creep-stress law, these tests are significant in giving evidence of creep deformation in structural steel. They show that, for the structural steel tested, creep occurs when the stress is only slightly above the "proportional limit" of the material.

Creep of commercial leads at normal temperatures has been investigated by A. J. Phillips[9] and by H. F. Moore, B. B. Betty, and C. W. Dollins.[10] These experiments indicate that the log-log creep-stress law expressed by Eqs. 1 is a good approximation. Although this material is not of general importance, it is of interest in showing that the log-log plot applies.

Concrete is an important structural material and therefore should also be considered, if possible, in formulating a creep-stress law. Many creep-stress experiments on concrete in compression have been made. A summary of some of the test results and interpretations is given by J. R. Shank,[11] M. Am. Soc. C. E. The recommended creep-stress law on the basis of a study by Mr. Shank,[11] in terms of Eqs. 1 and 2, is

$$e_c = a\, s\, t^m \qquad (3)$$

An examination of the foregoing creep-stress equations shows that a general expression for these equations might be written

$$e_c = a\, s^n\, t^m \qquad (4)$$

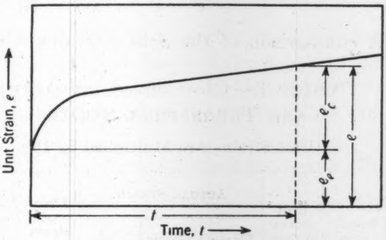

Fig. 3.—Creep-Time Curve

Eq. 4 seems to have considerable experimental support and has the advantage of simplicity when compared to some that have been proposed. One disadvantage of most creep-stress relationships, as in the case of Eq. 4, is that the elastic or initial deformation is not included. Referring to the deformation-time relation of Fig. 3, the total unit deformation e at a time t is made up of the elastic or initial deformation e_e plus the creep deformation e_c—that is,

$$e = e_e + e_c \qquad (5a)$$

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[9] "Some Creep Tests on Lead and Lead Alloys," by A. J. Phillips, *Proceedings*, A. S. T. M., Vol. 36, 1936, Pt. II, Technical Papers, p. 170.

[10] "The Creep and Fracture of Lead and Lead Alloys," by H. F. Moore, B. B. Betty, and C. W. Dollins, *Bulletin No. 272*, Univ. of Illinois Eng. Experiment Station, Urbana, Ill., 1935; see also "The Investigation of Creep and Fracture of Lead and Lead Alloys for Cable Sheathing," by the same authors, *Bulletin No. 306, ibid.*, 1938.

[11] "The Mechanics of Plastic Flow of Concrete," by J. R. Shank, *Proceedings*, A. C. I., 1936, pp. 149–180.

The value of the elastic or initial deformation will be assumed equal to

$$e_c = B\,s^p \qquad (5b)$$

For a material in which Hooke's law applies, $B = 1/E$ and $p = 1$, in which E = the modulus of elasticity of the material. Substituting the values of the deformations from Eqs. 4 and 5*b* in Eq. 5*a*, the unit deformation is

$$e = B\,s^p + a\,s^n\,t^m \qquad (6a)$$

Or, for a material for which Hooke's law applies,

$$e = \frac{s}{E} + a\,s^n\,t^m \qquad (6b)$$

Eqs. 6 define the total deformation in terms of the unit stress and the time. In addition, there are five experimental constants. This creep-stress law is used in the next section to develop a theory for stresses and deflections in the case of bending. It is found, however, that this leads to cumbersome expressions, making the formulation of structural mechanics in the case of creep very difficult. For this reason an approximation to Eq. 6*a* is suggested. This involves assuming the elastic or initial deformation to be expressed by $B\,s^n$ and determining B for the maximum stress—that is, B is determined from the equation $B\,s^2_{\max} = s_{\max}/E$. The total strain then will be assumed as

$$e = s^n\,(B + a\,t^m) \qquad (7a)$$

An indication of the accuracy obtained in using Eq. 7*a* is given by an analysis of the test data shown in Fig. 1. Using the results for specimens B3.2 and B3.1, $n = 1.22$, $m = 0.20$, $a = 3.75 \times 10^{-9}$, and $B = 1.04 \times 10^{-8}$. The value of B was determined on the basis of Hooke's law, as explained. The expression for the total strain as given by Eq. 7*a* then becomes

$$e = s^{1.22}\,(1.04 \times 10^{-8} + 3.75 \times 10^{-9}\,t^{0.20}) \qquad (7b)$$

A comparison of the actual strains with the theoretical values given by Eq. 7*b* is shown in Table 1. In determining the actual strains, the elastic strains were added to the creep strains. These deformations are for a time $t = 1{,}760$ hr. The percentage error in the value of the strains as given by Eq. 7*b* indicates that for these test data it is a good approximation. The approximation made in obtaining Eq. 7*a* produces a smaller error than is indicated in the test data. This is because the time covered by the tests is only a small fraction of the estimated life of the structure—that is, the test data cover a period of approximately two months, whereas the limiting deflection would be based on a time on the order of ten or twenty years. The elastic deformation is a very

TABLE 1.—COMPARISON OF ACTUAL AND THEORETICAL STRAINS

(Unit Strains Are Multiplied by 10^5)

Specimen	Unit stress (lb per sq in.)	ACTUAL STRAIN			Theoretical strain (Eq. 7*b*)	% error
		Creep strain (e_c)	Elastic strain (e_e)	$e_c + e_e$		
B3.2	3,472	35	29	64	56	−12
D2.3	6,227	70	52	122	115	− 6
D2.1	9,655	90	81	171	163	− 5
D2.2	12,321	110	102	212	271	−28
B3.1	13,130	174	109	283	284	0

small fraction of the creep deformation, and the error produced in using $B\,s^n$ for the elastic strain is therefore very small.

MEMBERS SUBJECTED TO BENDING

For a straight beam subjected to a pure bending moment M, as shown in Fig. 4, a theory for the stress distribution can be developed as in the case of the elastic theory. To do this it is only necessary to replace Hooke's law in the development, using the elastic theory with the creep-stress law expressed by Eq. 6*a*—that is, the following assumptions will be made:

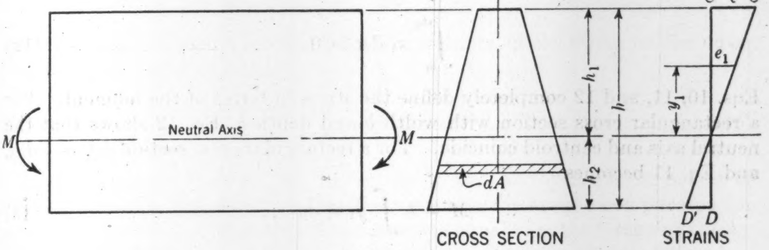

FIG. 4

(1) A transverse cross section of the beam originally plane and normal to the center line of the member remains plane after bending;

(2) For any fiber the strains are defined as in the case of tension by Eq. 6*a*;

(3) The creep-stress relations are identical for fibers in tension and compression at equal distances from the neutral axis; and

(4) The lateral compression between the fibers is negligible.

Assumption (1) has been verified by some experiments in bending.[12] By means of the foregoing assumptions and the conditions of equilibrium, a theory for the stresses and deflections in bending can be obtained. By the creep-stress law of Eq. 6*a*, the strains at distances y_1 and h_1 from the neutral axis are, respectively,

$$e_1 = B\,s^p{}_1 + a\,s^n{}_1\,t^m \qquad \text{(8a)}$$

and

$$e = B\,s^p + a\,s^n\,t^m \qquad \text{(8b)}$$

Assuming that plane sections remain plane, the strains are proportional to the distances from the neutral axis and

$$\frac{e_1}{e} = \frac{y_1}{h_1} \qquad \text{(9)}$$

Placing values of the strains from Eqs. 8 in Eq. 9,

$$y_1 = h_1\left(\frac{B\,s^p{}_1 + a\,s^n{}_1\,t^m}{B\,s^p + a\,s^n\,t^m}\right) \qquad \text{(10)}$$

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[12] "Stresses and Deformations in Pipe Flanges Subjected to Creep at High Temperatures," by J. Marin, *Journal*, Franklin Inst., Vol. 226, November, 1938, p. 645.

For equilibrium the external and resisting moments are equal, or

$$M = \int y_1\, s_1\, dA \qquad (11)$$

In order to integrate Eq. 11, the position of the neutral axis must be defined. This is given by noting that the summation of stresses over the cross section of the beam must be zero, or

$$\int_{h_1}^{h_2} s_1\, dA = 0 \qquad (12)$$

Eqs. 10, 11, and 12 completely define the stress in terms of the moment. For a rectangular cross section with width *b* and depth *h*, Eq. 12 shows that the neutral axis and centroid coincide. For a rectangular cross section $dA = b\, dy_1$, and Eq. 11 becomes

$$M = b \int y_1\, s_1\, dy_1 \qquad (13)$$

To evaluate the integral in this equation it is necessary to determine dy_1 from Eq. 10. Placing the values of dy_1 and y_1 in Eq. 13 and integrating, the resulting equation is

$$M = \frac{b\, h^2}{2\,(B\, s^p + a\, s^n\, t^m)^2} \times \left(\frac{p\, B^2\, s^{2p+1}}{2\,p+1} + \frac{2\,p\,n\,a\,B\, s^{p+n+1}\, t^m}{p+n+1} + \frac{n\, a^2\, s^{2n+1}\, t^{2m}}{2\,n+1} \right) \qquad (14)$$

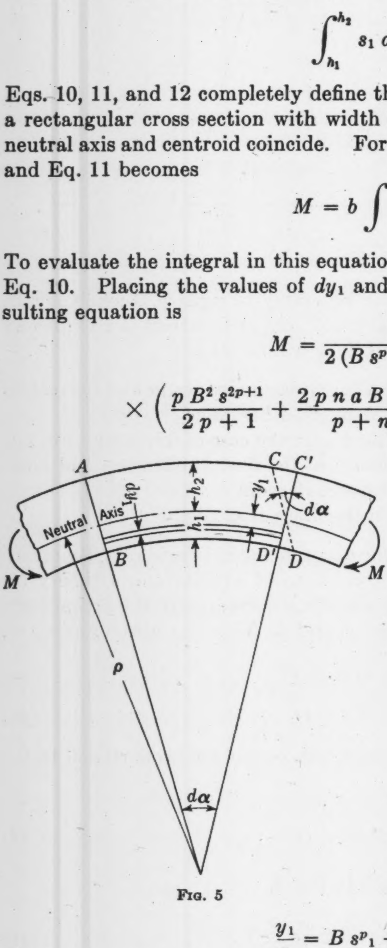

Fig. 5

The stress *s* on the outer fiber is defined by Eq. 14 in terms of the moment *M*, dimensions of the beam, and constants of the material.

In order to obtain an expression for the deflection in a beam subjected to pure bending, consider a section of a beam as shown in Fig. 5. The strains at distances y_1 and h_1 from the neutral axis in terms of the curvature are

$$e = \frac{h_1}{\rho} \text{ and } e_1 = \frac{y_1}{\rho} \qquad (15a)$$

in which ρ is the radius of curvature of a curved beam. Placing the values of e_1 from Eq. 8*a* in Eq. 15*a*, the curvature is defined by

$$\frac{y_1}{\rho} = B\, s^p{}_1 + a\, s^n{}_1\, t^m \qquad (15b)$$

Using the condition of equilibrium as given by Eq. 13, and placing the values of y_1 and dy_1 as obtained from Eq. 15b in this equation,

$$\frac{1}{\rho} = \frac{2}{h}(B\,s^p + a\,s^n\,t^m) \qquad (15c)$$

For small deflections the relation between the deflection y and the curvature is

$$\frac{1}{\rho} = \frac{d^2y}{dx^2} \qquad (15d)$$

The differential equation for the deflection in terms of the stress on the outer fiber is, by Eqs. 15c and 15d,

$$\frac{d^2y}{dx^2} = \frac{2}{h}(B\,s^p + a\,s^n\,t^m) \qquad (16)$$

A consideration of Eqs. 14 and 16 for the stress and deflection in beams of rectangular cross section shows that these expressions are too complicated for practical purposes. If the value of p is assumed equal to n, the foregoing theory is simplified. This is equivalent to assuming the approximate creep-stress law given by Eq. 7a. Placing $p = n$ in Eq. 14, the expression for the maximum stress is

$$s = \left(\frac{M\,h}{I}\right)\left(\frac{2\,n+1}{6\,n}\right) \qquad (17)$$

in which $I = b\,h^3/12$ = the moment of inertia of the cross section.

The differential equation for the deflection is given by substituting the value of $p = n$ in Eq. 16. Then

$$\left(\frac{D}{B + a\,t^m}\right)\frac{d^2y}{dx^2} = M^n \qquad (18)$$

in which

$$D = \frac{(2\,b)^n\left(\frac{h}{2}\right)^{2n+1}}{\left(2+\frac{1}{n}\right)^n} \qquad (19)$$

For the case where $t = 0$ and $n = 1$, Eq. 18 reduces to the equation given by the elastic theory. The value of D can be obtained in a manner similar to the foregoing for symmetrical sections other than rectangular. Eq. 18 can be used for determining deflections in beams in a manner similar to the usual double integration method. It is only necessary to integrate Eq. 18 twice and to obtain the constants of integration by using the boundary conditions of the particular problem. For example, in a simply supported beam of length l, subjected to a center concentrated load P, the deflection at a distance x from one end is

$$y = \frac{B + a\,t^m}{D}\left[\frac{x^{n+2}}{(n+1)(n+2)} - \frac{\left(\frac{l}{2}\right)^{n+1}}{(n+1)}x\right]\frac{P^n}{2^n} \qquad (20)$$

For a value of $x = l/2$, the maximum deflection is, by inspection or by Eq. 20,

$$y_{\max} = \frac{B + a\,t^m}{D} \times \frac{l^{n+2}}{2^{2(n+1)}} \times \frac{P^n}{n+2} \ldots\ldots\ldots\ldots\ldots (21)$$

The accuracy of the foregoing theory is indicated by a comparison between the measured and theoretical deflections in aluminum alloy beams subjected to pure bending.[3] These beams were subjected to pure bending moments of values shown in Table 2. They were of rectangular cross section of width $b = 0.50$ in. and depth $h = 1.00$ in., and the deflections were measured over a gage length $l = 8$ in. By Eq. 18 the expression for the deflection becomes

$$y = \frac{B + a\,t^m}{D} \times \frac{l^2}{8} \times M^n. \quad (22)$$

Using the values for the experimental constants obtained from the tension tests of Fig. 1, the theoretical values of the deflections, as given by Eq. 22, are shown in Table 2. The experimental values of these deflections are also listed. A comparison between the theoretical and experimental values shows that the theory gives a good approximation. In making this comparison the preliminary nature of these tests in this new field of study must be considered.

TABLE 2.—COMPARISON OF EXPERIMENTAL AND THEORETICAL DEFLECTIONS

Specimen No.	Moment (in-lb)	Theoretical deflection (in.)	Actual deflection (in.)	% error
D5	687	0.018	0.013	28
A3	996	0.029	0.022	24
C3	1,333	0.040	0.031	22
A4	1,667	0.053	0.046	13

STATICALLY INDETERMINATE STRUCTURES

Knowing the relationship between the stresses and deformations in the case of simple tension and pure bending, theories for the analysis of statically indeterminate structures can be developed. It is convenient to discuss first statically indeterminate beams of one span.

(*1*) *Double Integration Method—Beams of One Span.*—For beams that are statically indeterminate, the values of the reactions and deflections can be found in a manner similar to the double integration method used for the elastic case—that is, essentially it is only necessary to integrate the differential Eq. 18 twice, first substituting the value of the moment M in terms of x for the particular problem. The constants of integration and the statically indeterminate reactions are then found by using the particular boundary conditions of the problem. The difficulties in this procedure are mathematical because in some cases the integration is not direct and in others considerable work is involved. These difficulties arise since the moment M in Eq. 18 is raised to a power n that is usually different from one, and the simplifications afforded in the elastic case by using the methods of superposition can no longer be used in this analysis. Some examples of statically indeterminate beams will now be discussed and the maximum moments compared with the "elastic" values.

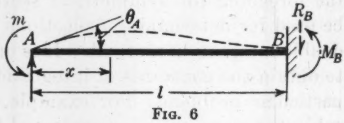

FIG. 6

Example 1.—A beam fixed at one end, simply supported at the other, and subjected to an end moment m is shown in Fig. 6. By Eq. 18 the deflection equation is

$$C_n \frac{d^2y}{dx^2} = M^n = (R_A\,x - m)^n \ldots\ldots\ldots\ldots\ldots\ldots (23)$$

in which R_A = the reaction at point A, and

$$C_n = \left(\frac{D}{B + a\,t^m}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$$

Integrating Eq. 23 twice,

$$C_n \frac{dy}{dx} = \frac{(R_A\,x - m)^{n+1}}{R_A\,(n+1)} + c_1 \ldots\ldots\ldots\ldots\ldots\ldots (25a)$$

and

$$C_n\,y = \frac{(R_A\,x - m)^{n+2}}{R^2_A\,(n+1)\,(n+2)} + c_1\,x + c_2 \ldots\ldots\ldots\ldots (25b)$$

in which c_1 and c_2 are constants of integration. Using the boundary conditions that the deflections are zero at both ends of the beam and the slope is zero at point B, the value of the reaction R_A is determined by the equation

$$R_A\,l\,(n+2)\,(R_A\,l - m)^{n+1} - (R_A\,l - m)^{n+2} + (-\,m)^{n+2} = 0 \,.\,. (26a)$$

Noting that $R_A\,l - m = M_B$, Eq. 26*a* can be written in terms of M_B—that is,

$$M^{n+2}{}_B + \left(\frac{n+2}{n+1}\right) m\,M^{n+1}{}_B + \left(\frac{1}{n+1}\right)(-\,m)^{n+2} = 0 \ldots (26b)$$

The value of the moment M_B as defined by Eq. 26*b* will depend upon the value of n. For different values of n the variation in M_B is represented in Table 3.[12a] The table also shows that there is a 44% increase in the moment at B for $n = 7$ over the value determined by the elastic theory. Values of n at elevated temperatures for steel have been reported as high as 9 so that it is possible to have an appreciable increase in the value of the moments over those values obtained by elasticity. The value of the bending stress, as determined by Eq. 17 for a rectangular section, is also modified, depending upon the value of n. In estimating the resultant effect it is necessary, therefore, to consider the influence of the change in moments as well as the change in stress distribution. Another important consideration is that of limiting the maximum deflection obtained from Eq. 25*b*.

TABLE 3.—VALUES OF M_B

Value of n	M_B	% increase
1 (elastic case)	0.5 m	0
3	0.65 m	30
5	0.67 m	34
7	0.72 m	44

Example 2.—A uniformly loaded beam, fixed at one end and simply supported at the other, is shown in Fig. 7. The differential equation for this problem, using Eq. 18, is

$$C_n \frac{d^2y}{dx^2} = \left(R_A\,x - \frac{w\,x^2}{2}\right)^n \ldots\ldots\ldots\ldots\ldots\ldots (27)$$

[12a] For discussion of errors, see closing comments on p. 479.

in which w is the load per unit of length. Proceeding in a manner similar to Example 1, an equation for the reaction R_A at point A can be determined. For a value of $n = 3$ the equation for R_A is

$$R^3_A - (1.25\,w\,l)\,R^2_A + (0.535\,w^2\,l^2)\,R_A - 0.078\,w^3\,l^3 = 0 \quad\ldots\ldots(28)$$

From Eq. 28 the value of R_A is $0.40\,w\,l$. For the elastic case, $R_A = 0.375\,w\,l$. The moment at the support for $n = 3$ is, by statics, $M_B = 0.10\,w\,l^2$ as compared to $0.125\,w\,l^2$ by the elastic case.

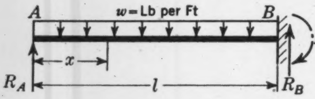

Fig. 7

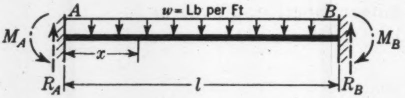

Fig. 8

Example 3.—For a beam fixed at both ends and subjected to a uniform load, as shown in Fig. 8, Eq. 18 becomes

$$C_n\,\frac{d^2y}{dx^2} = \left(\frac{w\,l\,x}{2} - M_A - \frac{w\,x^2}{2}\right)^n \quad\ldots\ldots\ldots\ldots\ldots\ldots(29)$$

Again following the procedure as explained for Example 1—namely, integrating Eq. 29 twice and applying the boundary conditions—an equation defining the value of M_A can be determined. For $n = 3$ this relation is

$$M^3_A - (0.25\,w\,l^2)\,M^2_A + (0.025\,w^2\,l^4)\,M_A - 0.00088\,w^3\,l^6 = 0 \quad\ldots(30)$$

Solving for this moment, its value is $M_A = 0.074\,w\,l^2$ as compared with $M_A = 0.0833\,w\,l^2$ for the elastic case.

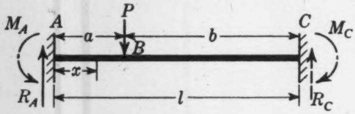

Fig. 9

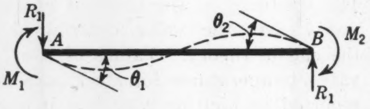

Fig. 10

Example 4.—For a beam fixed at both ends and subjected to a concentrated load (Fig. 9), it is necessary to write equations for both parts AB and BC of the beam. This leads to equations that are too complicated to present in this paper. If the load is considered at the center, $R_A = P/2$ and Eq. 18 need be written for only one half the beam. Thus, for AB,

$$C_n\,\frac{d^2y}{dx^2} = \left(\frac{P}{2}\,x - M_A\right)^n \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(31)$$

Integrating this equation twice and applying the necessary boundary conditions, an equation for M_A can be found—namely,

$$\left(\frac{P\,l}{4} - M_A\right)^{n+1} - (-\,M_A)^{n+1} = 0 \quad\ldots\ldots\ldots\ldots\ldots(32)$$

Using this relation, $M_A = 0.13\,P\,l$ for $n = 3$, and $M_A = 0.14\,P\,l$ for $n = 5$. These values are higher than that of $0.125\,P\,l$ as found for the elastic case.

The foregoing examples show that the values of the statically indeterminate reactions and moments may be appreciably different from the magnitudes obtained for the elastic case. It may be necessary, therefore, to consider this difference in the design of statically indeterminate beams in which creep occurs.

(*2*) *Slope-Deflection Method.*—The slope-deflection method is one of the classical methods for structural analysis. For the condition of creep, equations similar to those for the elastic case can be derived. To obtain these equations, consider the beam AB in Fig. 10 simply supported at points A and B and subjected to moments M_1 and M_2 at the ends. By the slope-deflection method these moments are determined in terms of the slopes or rotations θ_1 and θ_2 at the ends of the beam. To do this, Eq. 18 can be used. Then

$$C_n \frac{d^2y}{dx^2} = M^n = (R_1\,x - M_1)^n \tag{33}$$

Integrating this equation twice,

$$C_n \frac{dy}{dx} = \frac{(R_1\,x - M_1)^{n+1}}{R_1\,(n+1)} + c_1 \tag{34a}$$

and

$$C_n\,y = \frac{(R_1\,x - M_1)^{n+2}}{R_1^2\,(n+1)\,(n+2)} + c_1\,x + c_2 \tag{34b}$$

Using the conditions that the deflections are zero at both $x = 0$ and $x = l$, the constants of integration c_1 and c_2 are determined. Then the slopes θ_1 and θ_2 are the values of dy/dx at $x = 0$ and $x = l$, respectively, as found by Eq. 34*a*. Noting also that $R_1 = (M_1 + M_2)/l$, the slope-deflection equations expressing the slopes in terms of the moments are:

$$\begin{aligned} C_n\,\theta_1\,(M_1 + M_2)^2\,(n+2)\,(n+1) &= (M_1 + M_2)\,(-\,M_1)^{n+1}\,l\,(n+2) \\ &\quad + l\,(-\,M_1)^{n+2} - (M_2)^{n+2}\,l \end{aligned} \tag{35a}$$

and

$$\begin{aligned} C_n\,\theta_2\,(M_1 + M_2)^2\,(n+2)\,(n+1) &= (M_1 + M_2)\,(M_2)^{n+1}\,l\,(n+2) \\ &\quad + l\,(-\,M_1)^{n+2} - (M_2)^{n+2}\,l \end{aligned} \tag{35b}$$

The solution of Eqs. 35 simultaneously for M_1 and M_2 offers difficulties not present in the elastic case. The conclusion must be, therefore, that the slope-deflection method is not suitable and that the modified moment-distribution method should be used.

(*3*) *Moment-Distribution Method.*—A study of the moment-distribution method of structural analysis as applied to members subjected to creep shows again the fundamental nature of the Hardy Cross method.[13] The basic character of this classic method remains unchanged for applications where creep must be considered, but it is necessary to determine the beam constants—namely, the fixed-end moments, the carry-over factors, and the distribution factors.

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[13] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 1.

The determination of the fixed-end moments has been explained herein in the section on the double integration method. To determine the carry-over and distribution factors it is necessary to determine the slope at the left end of the beam in Example 1, as shown in Fig. 6. This slope is the value of dy/dx in Eq. 23 for the value $x = 0$ or

$$C_n \theta_A = m^n \left\{ \frac{l\left[(-1)^{n+1} - (k)^{n+1}\right]}{(k+1)(n+1)} \right\} \qquad (36)$$

or

$$m^n = \frac{C_n (k+1)(n+1)}{l\left[(-1)^{n+1} - (k)^{n+1}\right]} \theta_A = J\, \theta_A \qquad (37)$$

in which

$$k = \frac{M_B}{m} \qquad (38a)$$

and

$$J = \frac{C_n (k+1)(n+1)}{l\left[(-1)^{n+1} - (k)^{n+1}\right]} \qquad (38b)$$

Eq. 37 expresses the moment m at A in terms of the slope θ_A at A. If the arrangement of members shown in Fig. 11 is considered, then, by Eq. 37,

$$M^n_{ab} = J_{ab}\theta_a; \quad M^n_{ac} = J_{ac}\theta_a; \quad M^n_{ad} = J_{ad}\theta_a; \quad \text{and} \quad M^n_{ae} = J_{ae}\theta_a \qquad (39)$$

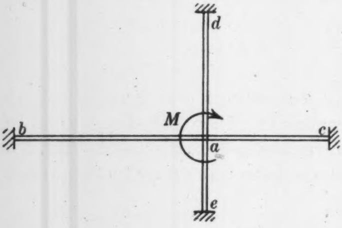


FIG. 11

For equilibrium,

$$M_{ab} + M_{ac} + M_{ad} + M_{ae} + M = 0 \qquad (40)$$

Substituting values from Eq. 39 in Eq. 40, and solving for θ_a,

$$\theta_a = -\frac{M^n}{(\Sigma J^{1/n})^n} \qquad (41)$$

Placing this value of the slope θ_a in Eq. 39, the ratios of the moments in the members to the applied moments—called the distribution factors—are:

$$\frac{M_{ab}}{M} = \frac{J^{1/n}{}_{ab}}{\Sigma J^{1/n}}; \quad \frac{M_{ac}}{M} = \frac{J^{1/n}{}_{ac}}{\Sigma J^{1/n}}; \quad \frac{M_{ad}}{M} = \frac{J^{1/n}{}_{ad}}{\Sigma J^{1/n}}; \quad \text{and} \quad \frac{M_{ae}}{M} = \frac{J^{1/n}{}_{ae}}{\Sigma J^{1/n}} \qquad (42)$$

or

$$D_f = \frac{J^{1/n}{}_p}{\Sigma J^{1/n}} \qquad (43)$$

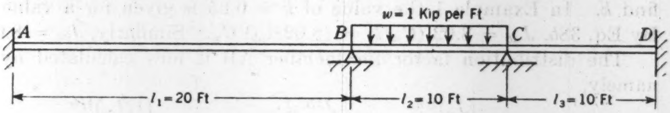
in which D_f is the distribution factor for any member.

The carry-over factor can be obtained by reference to Fig. 10 and Eq. 35b, which give the slope at point B in the beam AB. If this slope, $\theta_2 = \theta_B$, is placed equal to zero, the moment M_2 is then the moment at the fixed end of a beam AB, fixed at point B, simply supported at A, and subjected to a moment M_1 at A. The ratio M_2/M_1 is then the carry-over factor.

By Eqs. 35b, with $\theta_2 = 0$, the carry-over factor, $C_f = M_2/M_1$, is defined by the equation:

$$C^{n+2}{}_f + \left(\frac{n+2}{n+1}\right) C^{n+1}{}_f - \left(\frac{1}{n+1}\right) = 0 \qquad (44)$$

For values of n from 1 to 7 the carry-over factor varies from 0.5 to 0.72. With the fixed-end moments, distribution factors, and carry-over factors determined, the solution of some particular problem can be found, as illustrated by the following example.



(*a*) Computation for Moments at Supports

C_f	0.61			0.61		0.61
D_f		0.44	0.56	0.50	0.50	
Fixed-End Moments			−7.40	−7.40		
		−3.26	+4.14	+3.70	−3.70	
	+1.98		−2.26	−2.53		+2.26
		−1.00	+1.26	+1.26	−1.26	
	+0.61		−0.77	−0.77		+0.77
		−0.34	+0.43	+0.38	−0.38	
	+0.21		−0.23	−0.26		+0.23
		−1.0	+0.13	+0.13	−0.13	
Moments at Supports	+2.80	−4.70	−4.70	−5.47	−5.47	+3.26

(*b*) Comparative Computations for the Elastic Case

C_f	0.5			0.5		0.5
D_f		0.33	0.67	0.50	0.50	
Fixed-End Moments			−8.83	−8.83		
		−2.78	+5.55	+4.16	−4.16	
	+1.39		−2.08	−2.78		+2.08
		−0.68	+1.40	+1.39	−1.39	
	+0.34		−0.70	−0.70		+0.70
		−0.23	+0.47	+0.35	−0.35	
	+0.12		−0.18	−0.24		+0.18
		−0.06	+0.12	+0.12	−0.12	
Moments at Supports	+1.85	−3.75	−3.75	−6.02	−6.02	+2.96

FIG. 12.—CONTINUOUS BEAM OF THREE SPANS

Example 5.[12a]—A continuous beam of three spans is shown in Fig. 12. The moments at the supports will be determined by the moment-distribution method for a value of $n = 3$, and these values will be compared with those for the elastic case.

In order to determine the moments at the supports, the beam constants must be obtained. The fixed-end moments for a uniform load and a value of

$n = 3$ are determined in Example 3. These moments are $M = 0.074\, w\, l^2 = 0.074 \times 1 \times 100 = 7.40$ kip-ft. The carry-over factor for a value of $n = 3$ is determined by Eq. 44. Its magnitude is $C_f = 0.61$. Finally, the distribution factors are found by Eq. 43, using Eq. 38*b* to determine J_p, and Eq. 38*a* to find k. In Example 1 the value of $k = 0.65$ is given for a value of $n = 3$. By Eq. 38*b*, $J_{ab} = 8.02\,(C_n/l) = (8.02/l_{ab})\,C_n$. Similarly, $J_{bc} = 8.02\,(C_n/l_{bc})$.

The distribution factor for member AB is now calculated by Eq. 43—namely,

$$D_{ab} = \frac{(J_p)^{1/n}}{\Sigma(J^{1/n})} = \frac{J^{1/n}{}_{ab}}{J^{1/n}{}_{ab} + J^{1/n}{}_{bc}} = \frac{(1/l_{ab})^{1/n}}{(1/l_{ab})^{1/n} + (1/l_{bc})^{1/n}} \quad \ldots\ldots (45)$$

or $D_{ab} = 0.44$; and similarly, $D_{bc} = 0.56$. Using the beam constants as given in Fig. 12(*a*), the moments at the supports are found. In Fig. 12(*b*) these moments are determined for the elastic case for the purpose of comparison. A comparison of these moments indicates that there is sufficient difference in the moments to warrant consideration in design. Other types of statically indeterminate structures can be analyzed in a similar manner.

A consideration of the energy methods of structural analysis shows that these methods cannot be applied in the case of creep because they require that the material be elastic. Other fundamental methods of structural analysis used in the elastic theory do not apply in the case of creep as, for example, the use of the law of superposition and the determination of influence lines by mechanical means. On the other hand, secondary stresses in the case of creep can be determined since the moment-distribution method can be applied.

Conclusion

In this paper a new creep-stress law is proposed for materials in which creep at both normal and elevated temperatures is important. Some experimental evidence is given and other data are cited that show the validity of the creep law assumed. A theory for stresses and deflections in members subjected to bending is developed and the values compared with creep test results. Theories are then derived for the analysis of statically indeterminate structures in cases where creep is present. Comparison of results with the elastic case shows that there may be an appreciable difference in the values of reactions and moments. It should be noted that there is considerable need for experimental work on this problem. In the meantime, however, this paper shows, in general, how structural analysis is modified when creep occurs in engineering materials.

APPENDIX

Notation

The following letter symbols, used in this paper, conform essentially to American Standard Letter Symbols for Mechanics, Structural Engineering and Testing Materials, prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1932:[14]

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[14] ASA—Z10a—1932.

A = area; dA = element of area;
a = experimental constant; also B, F, m, n, and p;
B = (see a)
b = breadth;
C_n = constant creep rate: C_f = carry-over factor;
c_1 and c_2 = constants of integration;
D = a dimension constant: D_f = distribution factor;
E = modulus of elasticity;
e = strain, unit deformation;
F = (see a);
h = height or depth of beam:
  h_1 = distance from neutral axis to top of beam;
  h_2 = distance from neutral axis to bottom of beam;
I = moment of inertia;
J = (see Eq. 38b): J_p, J_{ab}, etc., refer to given members of a structural arrangement;
k = a substitution constant (Eq. 38a);
l = span length: l_1, l_2, etc., denote lengths of adjacent spans 1, 2, etc.;
M = bending moment, subscripts denoting points about which moments are taken;
m = (see a);
n = (see a);
P = concentrated load;
p = (see a);
R = reaction at the point denoted by subscript;
s = unit stress;
t = time;
w = uniformly distributed load;
x = coordinate distance from the left support;
y = ordinate to an elastic curve; deflection of a beam;
α = angle of curvature between two points on a curved beam;
θ = slope of the elastic curve at points designated by subscripts; and
ρ = radius of curvature of beam.

DISCUSSION

A. Hrennikoff,[15] Assoc. M. Am. Soc. C. E.—The writer wishes to call attention to two mistakes made in the derivation of the theory of creep as presented in the paper of Professor Marin.

Creep Law.—For the basis of his theory the author (see Eq. 7*a*) selects the following relationship between the unit stress s and the unit strain e: $e = s^n (B + a\,t^m)$, in which the exponent n depends on the properties of the material and may assume any integral or fractional positive value above unity.

In order that a stress-strain relationship could be used to advantage in the derivation of a theory of flexure it should be valid both for the positive and the negative values of e and s—that is, both for the state of tension and the state of compression, a requirement which, of course, is satisfied by the familiar formula that expresses Hooke's. This condition, however, is true with regard to Eq. 7*a* only when the exponent n signifies an odd integer. On the other hand, when n assumes the value of an even integer or of a fraction, this equation under the conditions of negative stress becomes inconsistent, since the left-hand side of it is then negative, and the right-hand side positive when n is an even integer and imaginary when n is a fraction.

Eq. 7*a* would be correct, of course, from a mathematical point of view if e and s were used in the sense of the numerical values of stress and strain. This, however, was not the intention of the author, and with this interpretation Eq. 7*a* would scarcely be suitable for the purpose intended.

The error discussed herein leads to inconsistencies in further derivations. Thus, the basic differential equation (Eq. 18) relating the value of the bending moment in the beam with the value of the curvature of the elastic line $\frac{d^2y}{dx^2}$ is incorrect when n is an even integer or a fraction, since according to this equation a negative bending moment bends the beam in the same sense as a positive bending moment when $n = 2, 4$, etc., or produces an imaginary curvature when n is a fraction.

The same error is naturally carried into the solutions of the statically indeterminate structures, discussed later in the paper. Take, for instance, Example 1 depicted in Fig. 6. The symbols m and M_B in this figure denote the numerical values of the end moments, since the arrows representing these quantities are shown in their true directions. The relationship between these moments is expressed by Eq. 26*b*, an inspection of which shows that, should n be taken as a positive even integer, no positive values of m and M_B may be found to satisfy this equation—that is, to satisfy the conditions of the problem.

The author has been unlucky in selecting for the illustration of his theory in Table 3 the numerical values of $n = 1, 3, 5$, and 7—that is, those specific values for which his theory still holds. Had he used any other values of n the fallacy of his theory would have manifested itself in the absurdity of the numerical results.

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[15] Asst. Prof., Civ. Eng., Dept. of Civ. Eng., Univ. of British Columbia, Vancouver, B. C., Canada.

Moment-Distribution Method.—The peculiarity of the stress-strain law used by the author makes superposition of deflections in beams caused by different loadings inadmissible. As an illustration of this principle, imagine a simply supported beam of a rectangular cross section loaded with a concentrated couple M_A on the end A and free from any other applied loads. The couple M_A causes rotation of the end A through an angle θ_A. Imagine now a new couple of the same magnitude M_A applied to the same end in the same direction. The second couple doubles the original moments caused by the first couple, as follows directly from statics, and doubles the original stresses anywhere in the beam, as may be concluded from inspection of Eq. 17. The strains, however, are more than doubled in view of the nature of the stress-strain law assumed, because the beam fibers are working now on the flatter parts of the stress-strain curve than they were before. Larger strains result in larger deformations, and the additional angle of rotation of the end A, produced by the second couple M_A alone, is larger, therefore, than the original angle θ_A; that is, the presence of the first state of stress in the beam has made the beam more flexible on the additional loading.

This behavior of the beam has an important bearing on the validity of the moment-distribution procedure, and it makes evident the fact that in the beam just considered the distribution factor on the second loading is smaller than on the first loading, provided that the other members attached to the same joint were still unstressed before the second loading. This dependence of the distribution factor (and of the carry-over factor as well) on the state of stress already existing in the beam may be demonstrated also for the fixed-ended beam by going through the necessary algebraic derivation. The author did not detect this dependence because his beam had been unstressed previous to the application of the end rotation. Dependence of the value of distribution factor on the state of stress in the beam makes the version of the distribution procedure presented by the author erroneous.

A modified moment-distribution procedure allowing for the state of stress existing in the beam is conceivable theoretically but scarcely possible in practice. It would be too laborious, since each step in the course of distribution would require a new set of values of the factors in each span of the structure.

The conclusion reached concerning the applicability of the method of moment distribution should not be surprising if one realizes that this method is based on the principle of superposition.

Summarizing the discussion it may be stated that the theory of creep as presented by Professor Marin is valid, from the mathematical point of view, only in the following two cases:

1. When the exponent n is an odd integer; and
2. When the signs of the bending moments are known beforehand in all parts of the structure. The value of n under these conditions may be arbitrary, but some modification in the author's formulas will be necessary if the sign of the moment is not the same everywhere in the structure.

The moment-distribution procedure is inadmissible even in cases when the theory holds.

It must also be kept in mind that the theory presupposes constant loads and seems to be inapplicable to simultaneous action of loads of different duration.

C. A. P. TURNER,[16] M. AM. SOC. C. E.—Professor Marin is to be congratulated for bringing into question the validity of treating Hooke's hypothesis as a law of nature. It may be doubted whether Hooke was first to present the suggestion that strain is approximately proportional to stress, since it appears in the Mariotte-Leibnitz theory of flexure following the publication of Galilei's problem.

At that early date, caloric or heat, light, electricity, and magnetism were regarded as imponderable chemical elements. This viewpoint (that of complete ignorance of intermolecular forces which oppose elastic change in form) warranted the adoption of the infinitesimal point hypothesis of Boscovich, which embodied a complete disregard of the temperature change incident to the application of the load.[17]

Now this temperature change which takes place upon the application of the load furnishes a measure of the magnitude of the load, as described by the writer in 1902.[18] The first practical recognition appears in the work of testing at the rolling mill where, by varying the speed of the machine in testing, the operator could satisfy engineers who wanted soft steel or others who wanted medium steel from the same metal with a different speed in testing. Thus, from the same bar, it was possible to develop, for soft steel, a 40,000-lb to 41,000-lb yield point with an ultimate strength of 58,500 to 60,000 lb per sq in., thus satisfying the college-trained engineer who wanted soft steel; and again, by increasing the speed in pulling the bar, a yield point of 44,000 to 45,000 lb per sq in., and an ultimate strength of 60,000 to 62,000 lb per sq in., could be developed, thus satisfying the engineer who wanted medium steel!

The energy of this temperature change upon application of the load depends on the final state and is independent of intermediate states as opposed to the external work, which is measured kinetically by the mean force times the deformation (by "kinetic," the initial equilibrium of the external and internal forces is meant).

The shift of internal energy resisting the external force is kinetically double the external work, and from this relation the physical properties of the metal may be coordinated.[19] As the elastic strain thus involves not only elongation but temperature change as well, it is apparent that the kinetic or instantaneous balance of equilibrium of external force and internal resistance will shift to the static balance as the thermo-elastic temperature change is dissipated by radiation. It further appears that to the extent the specimen is homogeneous the temperature change will be proportional to the applied force and, when this thermal limit of elasticity is exceeded, a range of imperfect elasticity is encountered before the development of the yield point of the metal. As this range of imperfect elasticity is developed, the cooling effect of tension is reduced until a radical heating effect is developed, with flow of the metal.

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[16] Cons. Engr., Columbus, Ohio.

[17] "An Elementary Treatise on the Mathematical Theory of Perfectly Elastic Solids, with a Short Account of Viscous Fluids," by William J. Ibbetson, Macmillan and Co., London, England, and New York, N. Y., 1887.

[18] *Transactions*, Am. Soc. C. E., Vol. XLVIII, August, 1902, p. 140.

[19] "Elasticity, Structure, Strength and Chemical Action of Materials," by C. A. P. Turner, Pt. III.

The shift from kinetic to static equilibrium was discovered by Kelvin, who attributed the effect to viscosity instead of the radiation of temperature change incident to the application of the load. Kinetically, the internal energy shift is double the external work, but this kinetic external work is but five sixths of the static, as may be demonstrated in many ways by different formulas. Hooke's hypothesis is in no sense a law of nature; it is inexact unless the time element is taken into consideration.

In his work on the manufacture of steel, H. H. Campbell[20] found that by applying the load on the structural specimen at the rate of 0.1 in. per min, a yield point of 39,000 lb per sq in. was developed, whereas with the fast speed of the machine a yield point of 45,000 lb per sq in. was developed.

To cover the phenomena of viscosity within the thermal limit of elasticity, one may state that, if the load remains in place at a constant intensity until the temperature change on its application has been dissipated by radiation, the static elongation is proportional to the load applied and there is no creep or increased elongation. The situation is altered by residual strain from cold bending or other harsh treatment.

Residual strain presents a kind of elasticity of its own, best illustrated by cold bending some ⅜-in. square rods in a half circle with a 3-in. radius; then placing the horseshoe type specimen in a vise and sawing kerfs ⅛ in. deep and ½ in. apart, the distance between the ends of the bent piece will be decreased when taken from the vise. Reversing the process and sawing the kerfs on the outside of the bend, the ends will again spring in or will present a sharper curvature; then sawing the kerfs on both sides, the horseshoe will open out and decrease its curvature. This change in form is caused by upsetting the internal equilibrium between the internal residual and elastic areas of stress.[21]

When extensometer measurements of stresses in concrete floors were a fad with student investigators, it often happened that they measured, according to Hooke's law, residual changes in a form which they misinterpreted as elastic strains one and one-half to one and three-quarters times as great as the yield point of the metal they were investigating. (Measurements across cold bends or kinks in small rods give rise to erratic results.) This kind of residual creep has never received the attention it deserves.

Prior to Wohler's[22] experiments on repeated stress, the elastic limit of iron and steel was considered little less than the yield point of the metal. With range of stress from tension to compression exceeding ⅝ of the primitive yield point, failure occurred under a sufficiently large number of repetitions.

Instead of recognizing a range of imperfect elasticity beyond the natural elastic limit, the metaphysical range of stress formulas of Launhart and Weyrauch was developed based upon the assumption of fatigue within this natural limit where none was developed by experiment.

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[20] "Manufacture and Properties of Structural Steel," by H. H. Campbell, 1896.

[21] "Elasticity, Structure, Strength and Chemical Action of Materials," by C. A. P. Turner, Pt. I, p. 55.

[22] *London Engineering*, Vol. XI, 1871.

With the passage of time these ranges of stress formulas have been discarded.

The writer's experiments on thermo-elastic measurement of stress place the thermal limit of elasticity as approximately five eighths of the yield point, in harmony with the work of Wohler. Creep or increased deformation will occur with time under stresses beyond the thermal limit of elasticity because, when this limit is exceeded, work has been done in displacing the molecular arrangement.

Recalescence points in melting or cooling of metals are recognized as points where the molecular formation undergoes modification. For example, soft ingot metal of the cubic system presents molecular kernels of eight atomic spheres. Upon rolling, this structure is broken down, and body-centered, nine-atom kernels appear. Again, drawing into fine wire, the body-centered formation changes to a face-centered formation, and the increase in strength under the law of attraction is as the square of the number of atoms in the kernel.

High-Temperature Creep.—At higher temperatures, a shift of molecular structure may take place in a manner similar to that occurring at the recalescent points of the metal, developing the so-called phenomena of creep at high temperature.

No natural phenomenon has been so much misrepresented by metaphysical hypotheses as that of elastic reaction. Volume, rather than elongation, is the correct measure of the effect of an axial force. Tension increases volume and compression reduces volume, and, although the same force causes equal volumetric changes whether from tension or compression, the temperature changes differ less in magnitude for tension and more for compression. In accordance with the law of volume, increase of volume is endothermic and decrease of volume is exothermic. This fundamental physical law destroys the basis of Lorentz and Einstein's calculation[23] of the interconvertibility of energy and mass and proves the irrationality of that characteristic energy of matter developed mathematically from a literal expression of Hooke's hypothesis. In a conservative system, the distribution of energy depends on the relation of masses and is irrespective of the individual unit, as demonstrated in the philosophical treatment of *The Encyclopædia Britannica.*[24]

The fantastic idea that one-quarter gram of matter has an intra-atomic energy equivalent to that developed by the engineer in burning 5,000 tons of coal is a metaphysical dream.[25] Not only has the elastic action of solids been confused in theory, but, in elastic resistance of gases to compression, physicists have invented a perfectly imaginary ideal gas in which the molecules have no volume and occupy no space. Because all gases have been liquefied and liquids are almost incompressible, one needs only to deduct the volume of the liquefied gas from the gross volume to find the net volume capable of compression in accordance with the laws of Charles, Boyle, and Gay-Lussac.

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[23] *Webster's New International Dictionary* (unabridged), 2d Ed., 1934, p. 847—$m = \frac{E}{c^2}$; "Problems of Modern Physics," by H. A. Lorentz, Ginn and Co., Boston, Mass., and New York, N. Y., 1927; "World of Atoms," by A. E. Haas and H. S. Uhler, Van Nostrand, New York, N. Y., 1937, pp. 112–113; and "Structure of Matter," by Cranston, p. 9.

[24] *The Encyclopædia Britannica*, 9th Ed., p. 207; also "Elasticity, Structure, Strength, and Chemical Action of Materials," by C. A. P. Turner, Pt. III, pp. 49, 51, and 141.

[25] "A Comprehensive Treatise on Inorganic and Theoretical Chemistry," by Joseph W. Mellor (Longmans, Green), London, England, 1922.

Professor Marin's creep theory for beams embodies the assumption that a plane section before bending remains plane after bending. Among the infinite number of sections in the length of the beam, one only in a simple beam remains plane after bending. That section divides the moment area in equal parts. As applied to the restrained and continuous beam, notwithstanding the creep correction of Professor Marin, the maximum tensile stresses at midspan and those over the support will be calculated anywhere from 15% to 30% greater than actual and the maximum compressive stresses will be approximately correct. The deflections will be much less than those computed in the restrained and continuous beams and greater than those computed by the ordinary formula for the simple beam.

The most curious inconsistency of the "Ut tensio sic vis" (extension proportional to force) lies in the fact that, under longitudinal compression, lateral extension occurs without any force or component of the applied force to account for it. Some have endeavored to explain this by a misconception of shearing stress; but the shears, as in beam action, can be developed only by principal stresses of opposite signs, and one of these principal stresses has no external force in its direction of action to account for its development.

The elastic temperature creep (that due to radiation of temperature change on application of the load) discloses the irrationality of the dynamic stress theory because the kinetic external work is but five sixths of the static; doubling it would be one and two thirds. Because the yield point before radiation is six fifths that after radiation is complete, the general effect of the suddenly applied load in developing the yield-point value is but one and one-third times the gradually applied static load, instead of twice the load, as taught in handbooks and textbooks of today.

Initial tests of newly cured reinforced-concrete beams disclose that steel stress times its lever under moderate loads is customarily 20% to 30% less than the applied moment. This inequality creeps out of existence under repeated loading. Textbook authors point to it as evidence of the operation of tensile resistance of concrete; but, by turning the concrete beam bottom side up, evidence of its insignificant value of concrete tension in flexure appears, requiring a more logical explanation.

As the concrete matrix crystallizes, it expands, inducing tensile stresses in the steel of a magnitude of 4,000 to 6,000 lb per sq in. The reaction to these steel stresses induces a state of residual compression in the concrete about the metal and, because the reaction to residual strain is antipodal to its elastic counterpart as shown by the cold-bent horseshoe bars, the resisting moment is that of steel stress times its lever plus residual compression in the concrete times the same arm, with the joint effect equilibrating the external moment.

Some deformed reinforcing bars indicate their top and bottom surface under their final pass through the rolls. If, on an 18-in. specimen, saw kerfs are made in the vise at ½-in. intervals, the bar will curve up slightly because the residual strain from rolling has been thrown out of balance by the reduction of sectional area. As in the example of the initially cured concrete beam, such residual strains are dissipated by repeated loading. After years of painstaking investigation, Bauschinger[26] concluded that sound metal, rather than being fatigued or brought nearer to rupture by alternating stresses from tension to compression within the natural elastic limits, was improved in elasticity, ductility, and tenacity.

An interesting example of apparent creep or downward motion of the neutral plane of reinforced-concrete beams was developed by the National Bureau of Standards[27] under the direction of the late Richard L. Humphrey, M. Am. Soc. C. E., in connection with the joint committee of the Society. In locating the position of the neutral plane, Mr. Humphrey and his committee assumed that this locus would be found where the longitudinal compression passed through zero, as in homogeneous beams, but that this locus crept downward as the load increased. In the reinforced-concrete beam, the shears between the tension of the steel and compression in the concrete are equilibrated at the steel instead of at the locus of the centers of rotation.

In the stress mechanism of a homogeneous beam, the lines of greatest compression above the neutral plane pair with the lines of greatest tension below it and intersect the neutral plane at 45°. In the concrete beam, because the shears are equilibrated at the bottom about the steel, the states of right shearing stress as centers of rotation are twisted 22.5° from their position in the homogeneous beams. This causes the lines of greatest compression to pair with the lines of greatest horizontal shear, intersecting the neutral plane at 22.5° instead of 45° so that the neutral plane is compressed longitudinally in proportion to the applied load, causing the locus of zero compression to creep downward as the load is increased. Elastic deflection of a concrete beam, therefore, increases more rapidly than the load because of this compression and shortening of the neutral plane.

Failure to grasp this relation has led to the unscientific conception of parabolic stress distribution in the compression zone and general confusion and mystery regarding the co-action of metal and concrete, limiting the conception of the field of investigation to that of concrete ribs functioning as steel beams would do.

No progressive achievement resulted from the deliberations of the Humphrey committee because they failed to grasp the idea of monolithic action in economic construction and the nature and distribution not only of shearing stresses in beams but of twisting stresses in plates. The stress mechanism of both homogeneous and reinforced-concrete beams and plates has been developed by the writer.[28]

The theory of elastic reaction as treated under Hooke's hypothesis is inconsistent with the statics of abstract forces because lateral extension occurs under longitudinal force with no component of that force to account for its development. In thermo-elastic theory this inconsistency disappears and

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[26] "Concrete-Steel Construction," by H. T. Eddy and C. A. P. Turner, Minneapolis, Minn., p. 109.

[27] *Technological Bulletin No. 2*, National Bureau of Standards' experiments on concrete beams.

[28] "Elasticity, Structure, Strength and Chemical Action of Materials," by C. A. P. Turner, Pt. III, plates VIII and IX.

Young's modulus may be coordinated with specific heat; coefficient of expansion; unit weight; atomic weight and number; strength; and melting and boiling temperature through J (the mechanical equivalent), g (the gravitational constant), the gas-volume coefficient (constant temperature), or the gas-pressure coefficient (constant volume).

Thermo-elastic theory, treating actual rather than imaginary substances, embodies nothing more complicated than common school arithmetic, which should render it popular with the poor student who has labored with the tedious differential equations of mathematical theory, unrelated to the physical properties of actual substances.

The college textbook theory of the nature of tension and compression embodies the spheres of exclusion developed in 1650, and the foundation of mathematical elasticity is the infinitesimal point hypothesis of Boscovich of that same modernistic date and generation.

Ralph W. Stewart,[29] M. Am. Soc. C. E.—The gradual increase in permanent deformation of a highly stressed structure, which the author calls "creep," has been given little attention in civil engineering literature. In reading this paper the writer was impressed with the fact that it contained valuable information on the subject.

In the writer's early practice he noticed that the 20-yr-old wooden joists in certain heavily traveled bridges had acquired a pronounced sag due to creep. For long beams continuous over a number of supports, the effect of creep obviously would tend to equalize the positive and negative moments after the manner of the plastic flow in an overstressed structure. In fact, the distinction between creep and plastic flow is narrow and seems to depend on the time element.

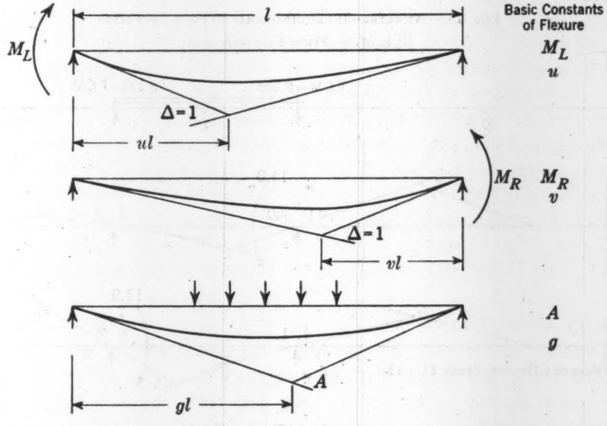

Fig. 13

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[29] Engr. of Bridge and Structural Design, City of Los Angeles, Los Angeles, Calif.

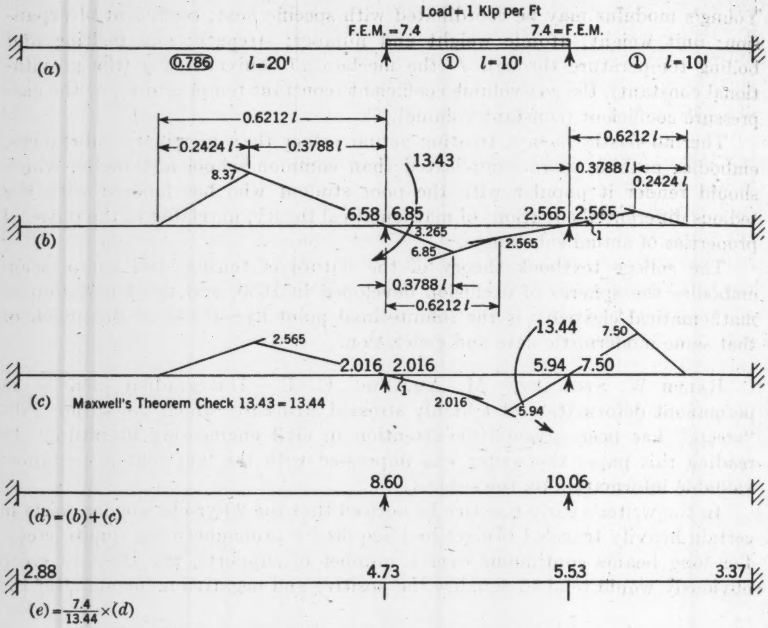
Load = 1 Kip per Ft
F.E.M. = 7.4
7.4 = F.E.M.
(a)
0.786
l = 20'
①
l = 10'
①
l = 10'
0.6212 l
0.2424 l
0.3788 l
8.37
13.43
6.58 6.85
3.265
6.85
2.565 2.565
2.565
0.3788 l
0.2424 l
0.6212 l
(b)
2.565
2.016 2.016
13.44
7.50
5.94 7.50
2.016
5.94
(c) Maxwell's Theorem Check 13.43 = 13.44
8.60
10.06
(d) = (b) + (c)
2.88
4.73
5.53
3.37
(e) = 7.4/13.44 × (d)

FIG. 14.—ANALYSIS INCLUDING THE EFFECT OF CREEP (F.E.M. = Fixed-End Moment)

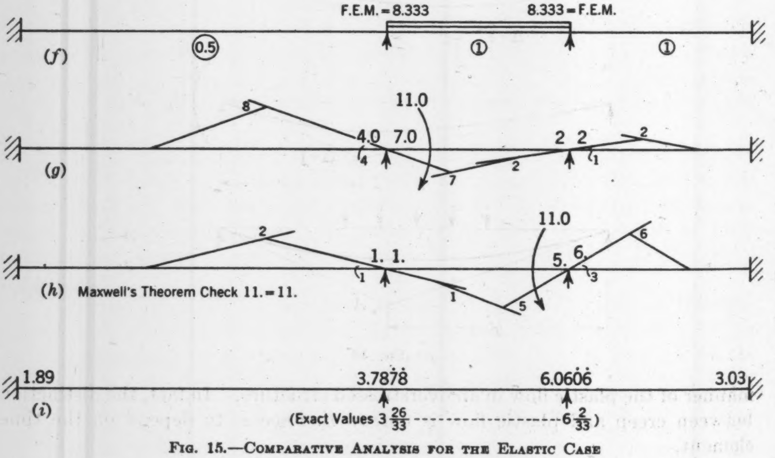
F.E.M. = 8.333
8.333 = F.E.M.
(f)
0.5
①
①
8
11.0
4.0 7.0
4
7
2
2 2
1
2
(g)
2
11.0
1. 1.
1
1
5
5. 6.
3
6
(h) Maxwell's Theorem Check 11. = 11.
1.89
3.7878
6.0606
3.03
(i)
(Exact Values 3 26/33 ------------ 6 2/33)

FIG. 15.—COMPARATIVE ANALYSIS FOR THE ELASTIC CASE (F.E.M. = Fixed-End Moment)

In designing machinery there is evidently justification for including the effect of creep in the stress analysis, but in designing continuous bridges and building frames the neglect of shear deformations may produce greater inaccuracy than the neglect of creep.

One difficulty in including the effect of creep in a complicated structure is that at some points the stresses would be sufficient to cause creep but that at other locations (except on rare occasions) the stresses might be too light to cause creep. This would make the stresses due to creep discontinuous, whereas the elastic stresses would be continuous.

In two places the author indicates conceptions and conclusions that the writer considers to be erroneous—namely, the reference to end-moment distribution as "basic" in character and "fundamental" in nature, and the statement that it should be used for computing moments which include the effect of creep.

The basic constants of beam flexure for any beam are illustrated by Fig. 13 which shows the elastic curves, their tangents, and their long chords for the three different conditions that yield the basic constants of flexure. The illustrations can be considered as depicting railway curves or any geometrical curves, and the properties may be considered as geometrical rather than mechanical. The basic constants of flexure read from these figures are shown in Fig. 13. They are the best and the minimum number of constants necessary to determine the geometry of the triangles which circumscribe the elastic curves.

The end-moment distribution constants are not basic but are "derived" constants. For tapering members or other special conditions they become complex.

The writer would define a solution of basic character as one which uses the basic constants of flexure and which does not use any form of converging approximations. Such a solution is shown in Figs. 14 and 15. It involves fewer arithmetical operations than the author's solution. The arithmetic is simpler, in that there are no long columns of figures to add, and its result is more accurate than the author's. It will compute moments to any predetermined degree of accuracy in one operation. It does not involve the use of simultaneous equations. Except for the last operation, it has an automatic check by Maxwell's theorem of reciprocal deflections. The writer therefore contends that the author's statements regarding methods of computation are incorrect and that they tend to perpetuate a now common practice of failing to select the most effective method for analyzing a large class of structures.

JOSEPH MARIN,[30] ASSOC. M. AM. SOC. C. E.—The writer is greatly indebted to Professor Hrennikoff for pointing out two errors in this paper. With reference to the first one, it appears that the limitation in using only odd integer values for n is not a serious one. This fact is realized when the approximations involved in formulating a creep-stress law are considered and when the necessity of extrapolation is noted. If the test data on the material warrant a more accurate analysis than that indicated by the use of odd integers for n, it is always possible to solve the problem for two successive odd values of n and to interpolate for a particular value of n. Regarding the second error noted by Professor Hrennikoff, the writer realized that the law of superposition does not apply, and noted this in the paper (see heading, "Statically Indeterminate Structures: Example 5"). The general theory given in the paper ("Statically Indeterminate Structures: Moment-Distribution Method") appears to the writer to be correct. The application of this theory in Example 5 is incorrect as Professor Hrennikoff points out. That is, a new set of values for the carry-over and distribution factors should have been determined at each step in the moment-distribution procedure.

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[30] Prof., Eng. Mechanics, Pennsylvania State College, State College, Pa.

The comments of Mr. Turner are of considerable interest and importance in the general problem of behavior of materials subjected to stresses. The validity of the assumption that plane sections remain plane is based on creep-deformation measurements made on beam specimens.[3]

The discussion by Mr. Stewart regarding the justification for considering creep and the difficulties encountered in applying creep-stress analysis to structures is significant. In many problems, creep may not be important, and in some cases it may be desirable to develop methods of analysis. The use of the words "basic" and "fundamental" seem to the writer questions of definition and point of view. However, the analysis given in Fig. 15 has considerable merit. This analysis, applied to the case of creep, gives similar results to Example 5 and is therefore incorrect.

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PROFILE CURVES FOR OPEN-CHANNEL FLOW

By Dwight F. Gunder,[1] Esq.

With Discussion by Messrs. Carl Rohwer, C. J. Posey, J. C. Stevens, Boris A. Bakhmeteff and Nicholas V. Feodoroff, Hunter Rouse, and Dwight F. Gunder.

Synopsis

Certain irregularities appearing in the surface profile curves for gradually varied flow are discussed in this paper. These irregularities occur under conditions in which the depth of flow is less than both the normal and critical depths. Further, the equation of gradually varied flow in a wide rectangular channel is integrated for the case where the Chézy coefficient is given by the Manning relation $C = 1.486 \frac{R^{1/6}}{n}$. As a consequence of the contents of this paper, the curves for a variable Chézy coefficient given by the Manning relation are sketched, and it is concluded that:

(1) Perhaps the backwater curves in most texts are based on certain qualitative features of the equation of gradually varied flow, plus a knowledge on the part of the writers of what these curves should look like;

(2) The use of the differential equation of gradually varied flow at depths below both the critical and normal depths should be restricted, in that the Manning relation for the Chézy coefficient should not be used, although in most cases, from a theoretical standpoint, the Bazin or Ganguillet-Kutter relations are satisfactory; and

(3) The differential equation of gradually varied flow with the Chézy coefficient given by the Manning relation is integrated for the case of the broad rectangular channel.

***Notation.*—The letter symbols in this paper are defined where they first appear in the text and are assembled for convenience of reference in the Appendix.**

In calculating surface profile curves for gradually varied flow in open channels, using the customary step-by-step method in combination with the

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Note.—Published in April, 1942, *Proceedings*.

[1] Research Engr. and Associate Prof., Math. and Civ. Eng., Colorado State College, Fort Collins, Colo.

differential equation of gradually varied flow, the writer found certain irregularities in the results. These irregularities consisted of deviations from the general shape for such curves as are found in the standard texts treating the subject of gradually varied flow. The deviations occurred in the case where the flow was at depths less than either the critical depth y_c or the normal depth y_o. (Hereafter, in this paper, curves satisfying these conditions of flow will be indicated by the subscript 3;[2,3] that is, in the case of a mild slope, $y_c < y_o$, that part of the profile of the surface of flow for depth less than y_c will be indicated by the symbol M_3.) A point of inflection appears in the calculated curves of this type (3), whereas such a point is not shown in the usual textbook graphs. Although not so stated in most of these texts, perhaps the surface profiles for gradually varied flow have been sketched from a combination of certain qualitative features of the equation, plus a knowledge of the forms which these curves should take. In this paper the usual differential equation will be re-derived, giving the result in a lucid form, and the correct profile curves corresponding to this equation will be drawn, assuming a Chézy coefficient varying according to the Manning, Bazin, or Ganguillet-Kutter relations. Further, the related academic problem of integrating the differential equation of gradually varied flow with a variable Chézy coefficient as given by the Manning relation will be solved for the case of a wide rectangular channel.

It is recalled that the general equation of varied flow is obtained as follows:[2] The mean total head E_w at a cross section is given by,

$$E_w = \frac{V^2}{2\,g} + y + h_o \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which V is the velocity of flow, g is the gravitation constant, y is the depth of flow, and h_o is the height of the bottom of the channel above an arbitrary horizontal datum. Differentiation of this equation with respect to x, the distance measured in the direction of flow, gives:

$$\frac{dE_w}{dx} = \frac{d}{dx}\left(\frac{V^2}{2\,g}\right) + \frac{dy}{dx} + \frac{dh_o}{dx} \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

In this equation the expression $\frac{dE_w}{dx}$, representing the rate of change of mean total head, is assumed to be the same as if the flow were uniform, and hence, by the Chézy relation, will be equal to $-\frac{V^2}{C^2\,R}$, C being the Chézy coefficient and R the hydraulic radius for the section of the channel under discussion. If b is the breadth, A the cross-section area, and Q the total discharge across the section of the channel, then the expression $\frac{d}{dx}\left(\frac{V^2}{2\,g}\right)$ can be modified by making $V = \dfrac{Q}{A}$ and noting that $dA = b\,dy$; whence,

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[2] See, for example, "Fluid Mechanics for Hydraulic Engineers" by Hunter Rouse, McGraw-Hill Book Co., Inc., New York, N. Y., and London, England, 1938, 1st Ed., pp. 288 *et seq.*

[3] "Hydraulics of Open Channels," by Boris A. Bakhmeteff, McGraw-Hill Book Co., Inc., New York, N. Y., 1932 (see types 1 and 2).

$$\frac{d}{dx}\left(\frac{V^2}{2\,g}\right) = -\frac{Q^2\,b}{g\,A^3}\frac{dy}{dx} \qquad (3)$$

If S_o is the slope of the channel bottom (positive for a downward slope in the direction of flow), then $\dfrac{dh_o}{dx} = -S_o$, and Eq. 2 becomes,

$$-\frac{V^2}{C^2\,R} = -\frac{Q^2\,b}{A^3\,g}\frac{dy}{dx} + \frac{dy}{dx} - S_o \qquad (4)$$

or

$$\frac{dy}{dx} = \frac{S_o - \dfrac{V^2}{C^2\,R}}{1 - \dfrac{Q^2\,b}{A^3\,g}} \qquad (5)$$

Since in the remainder of this paper the writer is interested only in the case of the broad rectangular channel, the general Eq. 5 will now be simplified accordingly. It follows at once that for such a channel $R = y$, and that for a unit width of the channel $A = y$ (Q will be retained but is understood now to be the discharge per unit width of channel).

Remembering that the height H of the energy line above the channel bottom is given by $H = y + \dfrac{V^2}{2\,g}$, that $V = \dfrac{Q}{y}$, and that $\dfrac{dH}{dy}$ is equal to zero at the conditions of critical flow:

$$\frac{dH}{dy} = \frac{d}{dy}\left(y + \frac{Q^2}{2\,g\,y^2}\right) = 1 - \frac{Q^2}{g\,y^3} \qquad (6)$$

Hence, at critical conditions of flow,

$$y^3_c = \frac{Q^2}{g} \qquad (7)$$

y_c being the critical depth. This reduces the denominator of the right member of Eq. 5 to $1 - \dfrac{y^3_c}{y^3}$.

In a similar manner, by noting that at the normal depth y_o the energy loss $-\dfrac{dE_w}{dx} = -S_o$, it is seen that, for normal conditions of flow,

$$S_o = \frac{V^2}{C^2_o\,y_o} = \frac{Q^2}{C^2_o\,y^3_o} \qquad (8)$$

or

$$y^3_o = \frac{Q^2}{S_o\,C^2_o} \qquad (9)$$

in which C_o is the value of C when flow is at normal depth. Whence the numerator of Eq. 5 becomes:

$$S_o - \frac{V^2}{C^2 y} = S_o - \frac{Q^2}{C^2 y^3} = S_o - \frac{S_o C^2_o y^3_o}{C^2 y^3} = S_o \left(1 - \frac{C^2_o y^3_o}{C^2 y^3}\right) \ldots \text{(10)}$$

and Eq. 5 becomes:

$$\frac{dy}{dx} = \frac{S_o \left(1 - \dfrac{C^2_o y^3_o}{C^2 y^3}\right)}{1 - \dfrac{y^3_c}{y^3}} \ldots\ldots\ldots\ldots \text{(11}a\text{)}$$

In Eq. 11a the relation of the slope of the surface profile to the critical and normal depths appears in easily recognized form. This equation is valid when S_o is positive, negative, or zero. In the latter case, the equation simplifies to:

$$\frac{dy}{dx} = \frac{-g\, y^3_c}{C^2 (y^3 - y^3_c)} \ldots\ldots\ldots\ldots \text{(11}b\text{)}$$

The factor C may now be replaced by any one of the three expressions for it as given by Manning, Bazin, or Ganguillet-Kutter. For the moment the Manning relation

$$C = 1.486 \frac{R^{1/6}}{n} \ldots\ldots\ldots\ldots \text{(12)}$$

will be used. This value, when substituted in Eq. 11a, gives at once,

$$\frac{dy}{dx} = S_o \left[\frac{1 - \left(\dfrac{y_o}{y}\right)^{10/3}}{1 - \left(\dfrac{y_c}{y}\right)^3} \right] \ldots\ldots\ldots\ldots \text{(13)}$$

From Eq. 13 it is seen at once that the surface is vertical (a theoretical condition only, since the postulated uniform flow condition is not satisfied here) not only at the critical value but also at $y = 0$. Consequently, there must be a point of inflection in the type 3 curve (except when y_c is greater than, or equal to, y_o). To find the location of this point of inflection, Eq. 13 is differentiated with respect to x and the result is equated to zero. This gives:

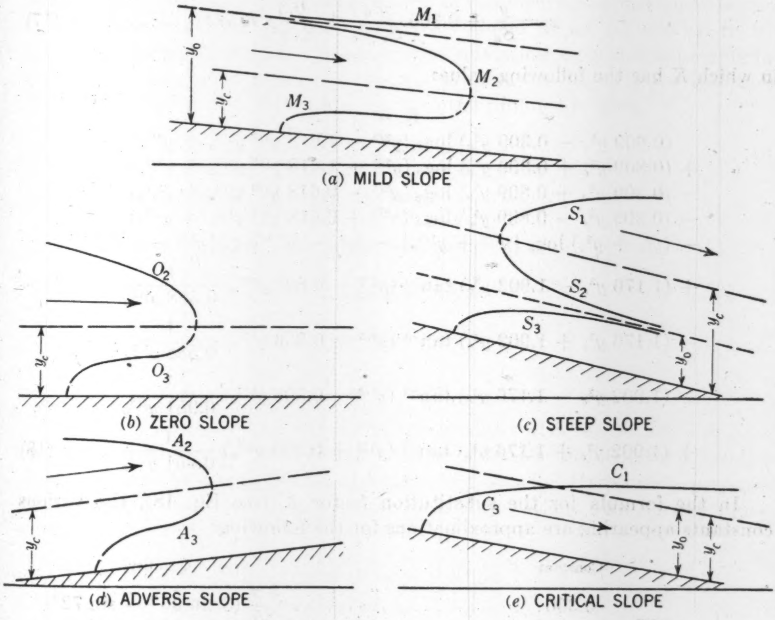
$$9\, y^3_c\, y^{10/3} - 10\, y^{10/3}_o\, y^3 + y^{10/3}_o\, y^3_c = 0 \ldots\ldots\ldots\ldots \text{(14)}$$

The solution of Eq. 14 for y is seen to depend upon both y_o and y_c. If y_c is much less than y_o, the resulting value of y is small and this point of inflection is near the bottom of the channel; but, if y_c approaches y_o, the value of y also approaches y_o. Even for the value $y_c = \frac{y_o}{2}$, the inflection occurs at about $y = \frac{y_o}{4}$.

However, if the value of C, as given by either the Bazin or Ganguillet-Kutter relations, is used in Eq. 11a, the inflection point, although still existing, is so close to the bottom of the channel that any error introduced by ignoring it is practically negligible. For the case of $y_c = \frac{y_o}{2}$, the point of inflection occurs at a value of y that is less than 1% of y_o. From the foregoing discussion two conclusions can be drawn at once:

(1) If step-by-step integration is used in the range of type 3 curves, the Chézy coefficient C should be determined by either the Bazin or Ganguillet-Kutter relation and not by the Manning relation; and

(2) Perhaps the backwater curves as drawn in the standard texts are based upon a constant Chézy coefficient C rather than upon one given by the usual Manning, Bazin, or Ganguillet-Kutter relations.[3a]



(*a*) MILD SLOPE

(*b*) ZERO SLOPE

(*c*) STEEP SLOPE

(*d*) ADVERSE SLOPE

(*e*) CRITICAL SLOPE

FIG. 1.—SURFACE PROFILES FOR GRADUALLY VARIED FLOW $C = 1.486 \frac{R^{1/6}}{n}$

In order to illustrate the contrast between the actual forms of the surface profiles for gradually varied flow in a broad rectangular channel, using a variable Chézy coefficient as determined by the Manning relation, and the forms of these profiles as sketched in texts for a constant Chézy coefficient, the sketches of the former curves are presented in Fig. 1 to be contrasted with the latter curves as found in textbooks.

[3a] See corrected conclusion in the closing discussion, p. 504.

The differential Eq. 13 is not difficult to integrate. In fact it may be rewritten in the form:

$$S_o\, dx = \frac{y^{1/3}\,(y^3 - y^3_c)\, dy}{y^{10/3} - y^{10/3}_o} \quad \dots\dots\dots\dots\dots\dots\dots (15)$$

or

$$S_o\,(x + C_1) = y + \int \frac{-\,y^{1/3}\, y^3_c + y^{10/3}_o}{y^{10/3} - y^{10/3}_o}\, dy$$

$$= y + \frac{1}{2\, y^{5/3}_o} \int \frac{-\,y^{1/3}\, y^3_c + y^{10/3}_o}{y^{5/3} - y^{5/3}_o}\, dy - \frac{1}{2\, y^{5/3}_o} \int \frac{-\,y^{1/3}\, y^3_c + y^{10/3}_o}{y^{5/3} - y^{5/3}_o}\, dy \quad \dots (16)$$

Although the details are somewhat tedious, each of these integrals can be evaluated by partial fractions. The result is:

$$x = \frac{y}{S_o} + 0.306\, \frac{y^3_c}{S_o\, y^2_o} + \frac{0.3}{S_o\, y^2_o}\, K \quad \dots\dots\dots\dots\dots (17)$$

in which K has the following value:

$$\begin{aligned} K = {} & \\ & (0.809\, y^3_c - 0.309\, y^3_o) \log_e (y^{2/3} - 1.618\, y^{1/3}\, y^{1/3}_o + y^{2/3}_o) \\ & + (0.809\, y^3_c + 0.309\, y^3_o) \log_e (y^{2/3} + 1.618\, y^{1/3}\, y^{1/3}_o + y^{2/3}_o) \\ & - (0.309\, y^3_c + 0.809\, y^3_o) \log_e (y^{2/3} - 0.618\, y^{1/3}\, y^{1/3}_o + y^{2/3}_o) \\ & - (0.309\, y^3_c - 0.809\, y^3_o) \log_e (y^{2/3} + 0.618\, y^{1/3}\, y^{1/3}_o + y^{2/3}_o) \\ & - (y^3_c + y^3_o) \log_e |y^{1/3} + y^{1/3}_o| - (y^3_c - y^3_o) \log_e |y^{1/3} - y^{1/3}_o| \\ & + (1.176\, y^3_c - 1.902\, y^3_o) \tan^{-1} (y^{1/3} - 0.809\, y^{1/3}_o)\, \frac{1}{0.588\, y^{1/3}_o} \\ & - (1.176\, y^3_c + 1.902\, y^3_o) \tan^{-1} (y^{1/3} + 0.809\, y^{1/3}_o)\, \frac{1}{0.588\, y^{1/3}_o} \\ & - (1.902\, y^3_c - 1.176\, y^3_o) \tan^{-1} (y^{1/3} - 0.309\, y^{1/3}_o)\, \frac{1}{0.951\, y^{1/3}_o} \\ & + (1.902\, y^3_c + 1.176\, y^3_o) \tan^{-1} (y^{1/3} + 0.309\, y^{1/3}_o)\, \frac{1}{0.951\, y^{1/3}_o} \quad \dots\dots (18) \end{aligned}$$

In the formula for the substitution factor K (see Eq. 18), the various constants appearing are approximations for the following:

Constant	Quantity
0.306	$\frac{3\pi}{25}$ (3 sin 36° − sin 72°)
0.809	cos 36°
0.309	cos 72°
1.618	2 cos 36°
0.618	2 cos 72°
1.176	2 sin 36°
1.902	2 sin 72°
0.588	sin 36°
0.951	sin 72°

All these constants arise from the integration and the determination of the constant of integration so that the origin will be at the intersection of the type 3 curve with the bottom of the channel. It is readily seen that several of the terms appearing in Eq. 18 could be combined; but the given form lends itself somewhat more readily to calculation. Furthermore, the logarithms have been left to the base e to facilitate work with the slide rule, by which satisfactory results can be obtained quite quickly.

The foregoing solution enables one to draw quantitative graphs of the various profile curves for the channel type considered here. Further, although it is not practical for accurate calculations of flow in natural stream beds, the equation is convenient for making rough approximations in such cases and for use in certain artificial stream beds. It is to be noted, of course, that this result, depending upon the Manning relation for C, is subject to the same restriction in the region of the type 3 curve as was mentioned previously herein.

The integration for zero and negative channel slopes is included in the integrated forms 17 and 18. However, the indeterminate forms appearing for the case of zero slope make it advisable to integrate Eq. 11*b* for this case. This integration is elementary and leads to the profile equation,

$$7.87\, g\, n^2\, y^3_c\, x = 13\, y^{4/3}\, y^3_c - 4\, y^{13/3} \quad \text{.............} (19)$$

in which n is the Manning roughness factor.

APPENDIX

NOTATION

The following letter symbols conform essentially with the Society's *Manual of Engineering Practice No. 11*, on "Letter Symbols and Glossary for Hydraulics," and American Standard Letter Symbols for Hydraulics, prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1942:[4]

A = the area of the cross section of the flow;
b = the breadth of the channel;
C = the Chézy coefficient: C_o = the Chézy coefficient for normal depth of flow;
E_w = mean total head;
g = the gravitation constant;
H = the height of the specific energy line above the channel bottom;
h_o = the height of the channel bottom above an arbitrary horizontal datum;

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[4] ASA—Z10.2—1942.

K = a simplifying, substitution factor (Eqs. 17 and 18);
M_3 = that part of the surface profile for a mild slope and depth of flow less than the critical depth;
n = Manning's roughness factor;
Q = total quantity of flow:
R = the hydraulic radius;
S_o = the slope of the channel bottom with respect to the horizontal;
V = the velocity of flow;
x = the distance measured parallel to the channel bottom;
y = the depth of the flow at any section:
  y_c = the depth of flow for critical flow conditions; and
  y_o = the normal depth of flow.

DISCUSSION

Carl Rohwer,[5] **Assoc. M. Am. Soc. C. E.**—There is considerable difference between the shape of certain types of profile curves as generally shown in textbooks and the shape obtained by the solution of the backwater formula. This is demonstrated clearly by Professor Gunder. The question immediately arises as to which of these is the true shape of the profile in a stream or canal under the conditions producing this type of backwater curve. Obviously, both cannot be correct. If there is a point of inflection in the curves, as stated by the author, then the curvature will change from concave upward to concave downward and the length of the surface profile would be materially reduced, in consequence. If this change in the shape of the profile actually occurs in nature, it is strange that this fact has not been revealed before. Such a change scarcely could have passed unnoticed by engineers taking levels on profiles under the conditions necessary to produce this phenomenon. It seems more plausible to the writer that this discrepancy is caused by some error in the assumptions involved in the derivation of the backwater equations rather than in the failure to observe the physical facts. The discovery of this anomaly in the general equation of surface profiles leads one to wonder if the application of the equation to the solution of the shapes of other types of profile curves may not result in errors that are not so readily detected.

Although the study of surface profiles in open channels for gradually varied flow is of special interest to engineers concerned with the problem of determining the rise in the water level in streams above dams and locks, it is of importance, also, to engineers engaged in the design of irrigation structures. The writer's experience is limited to the special backwater curve produced when a Parshall flume[6] is installed in a canal. Since the Parshall flume constricts the channel, there is an increase in the depth of the canal above the flume. The amount of this increase has been determined by tests on flumes of various sizes for a wide range of discharges. This increase in depth extends upstream in the canal and, since the head gate and diversion dam are usually only a short distance from the Parshall flume, the problem is to determine whether the rise in the canal will be sufficient to make it necessary to increase the height of the diversion dam.

Canal cross sections are relatively narrow and deep; consequently, it cannot be assumed that the hydraulic radius is equivalent to the depth. To determine the effect of backwater, it is therefore necessary to divide the canal into sections for which the length, corresponding to an arbitrarily chosen change in depth, can be computed. This is a tedious process. It is hoped that the author will derive a simple equation, applicable to the surface profile produced in this special case.

C. J. Posey,[7] **Assoc. M. Am. Soc. C. E.**—Upon reading this paper, one wavers between two conclusions—that Professor Gunder is wrong, or that the backwater curves shown in standard texts are seriously in error. Fortunately, neither conclusion is correct. The principal difference lies in the shape of the type 3 curves in channels of mild, zero, adverse, or critical slope. According to Fig. 1, the M_3, O_3, and A_3 curves start perpendicular to the bottom of the channel and are concave downward for some distance before becoming concave upward. Previously published sketches of these curves have shown them as concave upward from end to end. The C_3 curve in Fig. 1 starts upward perpendicular to the bottom and is concave downward all the way to the critical depth line. Heretofore it has been shown as a simple horizontal line.

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[5] Irrig. Engr., Div. of Irrig., SCS, U. S. Dept. of Agriculture, Colorado Experiment Station, Fort Collins, Colo.

[6] "The Parshall Measuring Flume," by R. L. Parshall, *Bulletin No. 423*, Colorado Experiment Station, 1936.

[7] Associate Engr., Iowa Inst. of Hydr. Research; Associate Prof., Hydraulics and Structural Eng., State Univ. of Iowa, Iowa City, Iowa.

Actually, the discrepancies are not as serious as one might infer from examination of Fig. 1. The author himself states that the point of inflection lies near the bottom of the channel—very close to it if the Bazin or Ganguillet-Kutter expressions are used for the Chézy coefficient. Even if the Manning relationship is used the discrepancies are not important. In drawing small-scale illustrations such as Fig. 1, the true shape of the curves often must be modified. That Fig. 1 has been somewhat distorted is shown by the fact that the curves start up from the bottom at right angles to it, whereas, according to the text of the paper, they should start up vertically. This peculiar feature of the curves, which is the most surprising result of Professor Gunder's studies, and which makes the reversal of curvature necessary, can easily be shown to be consistent with the assumptions ordinarily made in computing backwater curves.

The rate of loss of head through friction, with respect to distance along the channel, is assumed to be given by Chézy's formula

$$S_f = \frac{Q^2}{C^2\, y^3} \qquad (20)$$

In the range of infinitesimal depths close to the bottom of the channel the velocity head is very great, and changes at a rapid rate, whereas the elevation head is scarcely changing at all. In a rectangular channel the rate of change of velocity head, with respect to distance along the channel, is given by

$$\frac{d}{dx}\left(\frac{V^2}{2\,g}\right) = \frac{d}{dx}\left(\frac{Q^2}{2\,g\,y^2}\right) = -\frac{Q^2}{g\,y^3}\frac{dy}{dx} \qquad (21)$$

Since the change of elevation head is comparatively insignificant, the friction loss is derived almost entirely from the change of velocity head. Equating the values given by Eqs. 20 and 21, and changing signs to allow for the implicit negative sign of S_f, one finds that for depths close to zero

$$\frac{dy}{dx} = \frac{g}{C^2} \qquad (22)$$

From this expression it is evident that if any formula is used in which C approaches zero as y approaches zero, the corresponding water-surface slope will be infinite. The limit is entirely theoretical, of course, representing a flow condition impossible of attainment.

The evaluation of friction in the region near this limit, where the velocity of flow is many times the Belanger critical, is very uncertain. It is customary to assume that the rate of loss of head by friction, in variable flow, is the same as in uniform flow at the same depth and velocity. The rate of loss in uniform flow is then computed by one of the standard open-channel formulas. It is doubtful, however, whether any open-channel formula has a proven range extending to velocities greater than twice the critical. Observations of extremely high velocities are difficult, and flow on slopes steep enough to sustain such velocities is almost certain to become pulsating or "slug" flow rather than uniform flow.[8] Although uniform flow at the high velocities existing in the lower portions of the type 3 curves is nearly impossible, variable flow in this range can exist. Examples of computed profiles for a common case are shown in Fig. 2. A discharge of 507 cu ft per sec flows out from under a gate into a channel with horizontal bottom. Flow is from left to right in the figure, and two possible conditions of flow starting from different gate heights are shown. The Manning formula is used, with *n* taken as 0.016. If the bottom of the gate is 10 ft above the horizontal bottom of the channel, the water is backed up to a depth of 50 ft behind the gate. If the bottom of the gate is 4 ft above the bottom of the channel, the depth upstream becomes 254 ft. In either case the water-surface profile computed by Eq. 19 follows the same curve, crossing the critical depth vertically at a point 3,370 ft downstream from the gate when the opening is 10 ft, or 6,250 ft downstream when the opening is 4 ft.

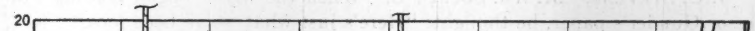

FIG. 2.—WATER-SURFACE PROFILES OF THE O_3 TYPE, BASED ON VARIABLE OR CONSTANT CHÉZY COEFFICIENTS

Water-surface profiles computed using the constant value of *C* which corresponds to the median value of *y* are also shown in Fig. 2. The curve downstream from the 10-ft gate reaches the critical depth in a distance of 3,500 ft, 4% in excess of the value given by Eq. 19. The curve starting at the 4-ft gate reaches the critical depth 6,760 ft downstream, a distance 7½% greater than that given by Eq. 19. Although it is evident that the discrepancy would be greater for smaller gate openings, the 4-ft gate opening is already beyond the range of values likely to be encountered in practice. The discrepancies could be decreased by using a value of C corresponding to an estimated average value of y rather than the median value. For example, with the gate at 10 ft, a value of C corresponding to a depth of 13 ft gives a result differing from that computed by Eq. 19 by less than 1%.

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[8] "A Criterion for Instability of Flow in Steep Channels," by G. H. Keulegan and G. W. Patterson, *Transactions*, Am. Geophysical Union, 1940, Pt. II, p. 594.

With regard to simplicity, it is not much easier to use a constant value of C in computing the O_3 curves than to use Eq. 19, but the use of a constant value of C in computing M_3 and A_3 curves permits the use of Bresse's function,[9] with great saving in labor over that required in using Eqs. 17 and 18.

From a practical viewpoint, there seems to be little choice between results obtained using a variable Chézy coefficient and those obtained using a constant coefficient. The effect of channel shape will ordinarily be of greater importance. Until a comparison has been made with data from variable high-velocity flow in wide rectangular channels, neither method can be considered to have superior accuracy. The author's prediction of a reversal of curvature in the type 3 curves, a phenomenon which may be observable in channels of any shape, should help to stimulate interest in obtaining actual measurements.

J. C. STEVENS,[10] M. AM. SOC. C. E.—When the writer began reading Professor Gunder's paper, he thought: "Here's just what we've been looking for"; and, when he had read as far as his general differential equation for the surface profile (Eq. 13), it appeared so innocent that he thought: "We'd better turn real mathematicians loose on some of our other problems." Then, on turning the page, that innocent looking thing the Professor states "is not difficult to integrate" loomed out of the darkness like the iceberg did to the *Titanic* and wrecked everything. It is suggested that the Professor be sentenced to prepare a table of values of his K-factor, in terms of y, y_c, and y_o, varying by tenths of feet to 50 ft. One wonders how large a volume it would make.

However, it does prove that, defining the friction factor by the Manning relation (which is purely empirical and dimensionally incorrect), the surface profiles, determined for a unit width of a wide channel (which does not exist), contain points of inflection in the range where depths are less than either normal or critical, in which range the Manning formula does not apply! Professor Gunder did quite well with his disclaimers, but he is challenged to beat that one.

After all, it appears as though hydraulic engineers will just have to struggle along with their old step-by-step method of computing water-surface profiles.

BORIS A. BAKHMETEFF,[11] M. AM. SOC. C. E., AND NICHOLAS V. FEODOROFF,[12] ESQ.—A serious error that has remained unnoticed in hydraulic literature for years has been disclosed by Professor Gunder. As correctly surmised by the author, the shape of the so-called M_3 surface curve, traditionally presented in textbooks, dates back to the past century, when varied flow profiles were first classified by considering the special case of broad rectilinear channels, with the further simplifying premise of a constant average Chézy factor C applied over the entire range of depths. Under these special conditions, the outline of the M_3-curve remains concave $\left(\frac{dy}{dx} > 0; \frac{d^2y}{dx^2} < 0\right)$ for all the depths, tending toward $\frac{dy}{dx} = \infty$ as y approaches the critical value. A further circumstance to be taken into account is that within the customary reaches the M_3-profiles happen to be very "flat," exhibiting (except near y_c) only slight curvature and, on the whole, approximating a straight line. This means that the changes in curvature, as disclosed by Professor Gunder, were not conspicuous and the observed appearances did not upset accepted notions. Therefore, the over-all concave M_3-profile, initially deducted for specific conditions, was inadvertently extended and assumed to apply in all cases, without due investigation of the special effects devolving from a variable friction factor. For such unpardonable negligence, the senior writer, as the author of a manual especially devoted to varied flow, presents his humble apologies. It should be stated, nevertheless, that at least in some texts[13] the tracings of the M_3-curve in the conventional surface profile diagrams are "stopped" at a certain distance from the bottom, indicating that in the vicinity of $y = 0$ the deductions drawn from the general varied flow equation cannot apply.

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[9] Tables of Bresse's function for adverse as well as mild and steep slopes are given in "Hydraulics of Steady Flow in Open Channels," by Sherman M. Woodward and Chesley J. Posey, John Wiley & Sons, New York, N. Y., 1941.

[10] Cons. Hydr. Engr. (Stevens & Koon), Portland, Ore.

[11] Prof., Civ. Eng., Columbia Univ., New York, N. Y.

[12] Research Associate, Civ. Eng. Dept., Columbia Univ., New York, N. Y.

Professor Gunder refers to the features disclosed by him as "irregularities." It is only proper to qualify previous ideas as plainly erroneous, and to accept the author's discoveries as regular characteristics of the M_3 varied flow curves, especially as these properties seem to have a broader significance than that suggested by the paper, and as they will be found to obtain quite generally, irrespective of the particular formula used for the C-factor.

The following analysis complements Professor Gunder's pioneer findings, which from now on should replace all former ideas relating to M_3-curves. Also, his discoveries are interpreted in the light of experimental evidence.

(1) The first deduction, drawn by Professor Gunder from Eq. 13, is that the surface curve for broad rectilinear channels with Manning coefficients theoretically becomes vertical at $y = 0$. As a matter of fact, this property is not to be limited to any particular form of channel or to any specific friction formula. Indeed, it constitutes a general characteristic which applies in all instances. In the general varied flow equation[14]

$$\frac{dy}{dx} = S_o \frac{1 - \dfrac{A^2_o\, R_o\, C^2_o}{A^2\, R\, C^2}}{1 - \dfrac{Q^2\, b}{g\, A^3}} \qquad \ldots\ldots\ldots\ldots\ldots (23)$$

when the depth approaches zero, the unity factors both in the numerator and denominator become small by comparison to the second members, and can be omitted. Accordingly, with $S_o\, A_o\, R_o\, C_o = Q$, the limit of the surface gradient

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[13] "Hydraulics of Open Channels," by Boris A. Bakhmeteff, McGraw-Hill Book Co., Inc., New York, N. Y., 1932, p. 76.

[14] *Ibid.*, Eq. 21, p. 31.

$$\text{Limit}\left(\frac{dy}{dx}\right)_{y\to 0} = \frac{\dfrac{Q^2}{A^2\,R\,C^2}}{\dfrac{Q^2\,b}{g\,A^3}} = \frac{g}{C^2}\,\frac{p}{b} = \sigma \ldots\ldots\ldots\ldots (24)$$

in which σ is the critical slope.[15] For small depths, $\frac{p}{b}$ tends toward unity. Hence,

$$\text{Limit}\left(\frac{dy}{dx}\right)_{y\to 0} = \frac{g}{C^2} = \frac{c_f}{2} \ldots\ldots\ldots\ldots (25)$$

in which c_f is the dimensionless friction drag coefficient in the average wall stress expression

$$\tau_o = \rho\, c_f\, \frac{V^2}{2} \ldots\ldots\ldots\ldots (26)$$

By reason of its physical essence, as well as theoretically in the customary Bazin, Ganguillet-Kutter, Manning, etc., relations, the C-factor for $y \to 0$ tends to become zero, making the critical slope $\left(\text{and thus } \frac{dy}{dx} \text{ in Eq. 25}\right)$ infinite. Note that this feature does not depend on the value of the channel slope S_o. In other words, all surface curves, both of the M_3 and the S_3 class, tend to be theoretically vertical at $y = 0$.

(2) As Professor Gunder infers, properties derived from the $y \to 0$ region represent purely theoretical conditions. The important practical effect, nevertheless, is that in the reach of small depths the M_3-curve must be convex $\left(\frac{d^2y}{dx^2} < 0\right)$, and that therefore at some depth y_i, between $y = 0$ and $y = y_c$, there is an inflection of curvature, with the surface profile changing from convex to concave. The paper conveys the impression that the eventual location of y_i is particularly affected by friction coefficients expressed through the Manning formula. In fact, the paper suggests (line 12 following Eq. 14) that "* * * in the range of type 3 curves, the Chézy coefficient C should be determined either by the Bazin or Ganguillet-Kutter relation and not by the Manning relation." The recommendation apparently is actuated by the desire to avoid the noted "irregularities."

One should not try to escape facts. What really matters is whether the property in question is actual or not; and under all circumstances the selected formula should be the one that is known to comply best with established reality. Unfortunately, very little is known about resistances in rapid (shooting) motion. In such a region investigators use formulas and coefficients deduced from observations in tranquil motion ($y > y_c$) without any direct assurance that they apply equally to supercritical ($V > V_c$) flow. Moreover, for "slowly varied (parallel) motion,"[16] they assume that the friction rate for a certain discharge and depth is substantially equal to what the resistance would have been under uniform conditions. In the M_3-case, flow is divergent and the divergence at times becomes quite pronounced. Analogous to closed conduits, one may anticipate a complementary loss connected with the restoration of kinetic energy into potential energy. Again, there is no material available at present that would evidence the magnitude of such restoration losses in open-channel flow.

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[15] "Hydraulics of Open Channels," by Boris A. Bakhmeteff, McGraw-Hill Book Co., Inc., New York, N. Y., 1932, p. 48.

[16] *Ibid.*, pp. 8, 24, and 29.

Accordingly, in many respects the precise answers to the questions raised by the paper will have to await evidence from future research. In the meantime, one is forced to seek approximate solutions by recourse to one or another of the existing friction formulas, and it is here that the writers cannot quite follow Professor Gunder's conclusions, for their investigations seem to show that the choice of any relation for C does not materially affect the position of the inflection point.

(3) The location of the "inflection depth" y_i on the M_3-curve, where the curvature changes from convex to concave, can be expressed conveniently in dimensionless terms by the ratio

$$\frac{y_i}{y_c} = \frac{\text{inflection depth}}{\text{critical depth}} = (y'_c)_i \dots\dots\dots\dots\dots\dots (27)$$

Generally, the y_i-value will be influenced by the shape of the channel, the discharge, the slope S_o, and the wall roughness. For two-dimensional motion (broad, rectangular channel), the shaping agencies are reduced to the parametric ratio

$$\frac{y_c}{y_o} = \frac{\text{critical depth}}{\text{normal depth}} = \eta_c \dots\dots\dots\dots\dots\dots (28)$$

and to the form of the function

$$C = f(y) \dots\dots\dots\dots\dots\dots\dots\dots (29)$$

which depicts the change of the Chézy factor C with the varying depth of flow.

To apprehend the possible effect of the different formulas for C, it is necessary to make clear the physics that actually underlie the structure of all empirical relations. In all formulas, a "surface roughness type" is characterized by a proper numerical term, such as the n-factor in the Ganguillet-Kutter and the Manning expressions. Such a friction term is held to be invariant for all conduits involving a particular type of surface roughness. Each formula offers the friction term with a mathematical expression deemed to represent best the factual change of C with the conduit size. In all instances C increases with the latter, the implied physical meaning being that the friction term n exemplifies a certain roughness size, and that the increase of C with the hydraulic radius R reflects the eventual reduction of the friction drag with the decrease of the relative roughness. In fact, relative roughness appears in the empirical relations in the form of a ratio of the friction term to an appropriate power of the hydraulic radius. Thus, the Bazin and Kutter expressions are built around the ratio of the friction term to $\sqrt{R}$, whereas in the Manning formula the shaping ratio is $\frac{n}{R^{1/6}}$. In general, the aim of "empirical" research in the past was to evolve a mathematical form between C and R, which for a certain "n" would best average and fit the maze of observational data assumed to respond to the same roughness type. In essence, the purpose was essentially the same, as answered more recently by the Prandtl-Kármán semi-rational theory, which expressed the interrelation between $\frac{1}{\sqrt{c_f}} = \frac{C}{\sqrt{2\,g}}$ and the relative roughness $\frac{e}{R}$ in logarithmic terms.

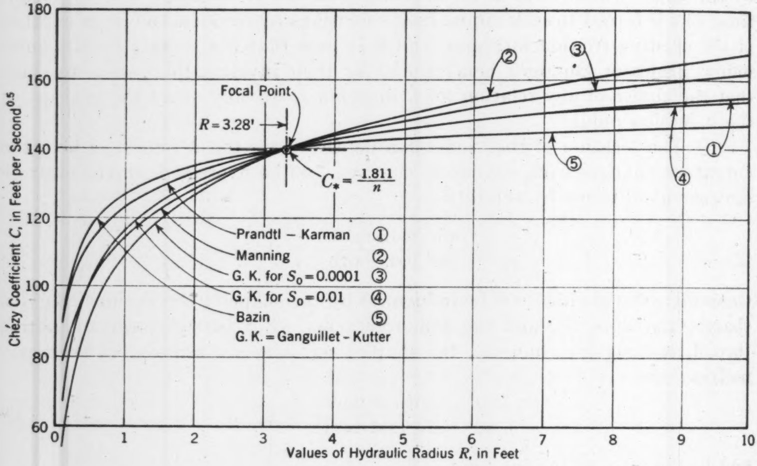

FIG. 3

A comparison of the different suggestions is given in Fig. 3, which shows a group of $C = f(R)$-curves computed by the different formulas with the Ganguillet-Kutter term $n = 0.013$. It should be remembered that, in the otherwise rather complex and obviously artificial Ganguillet-Kutter structure, the value of C_*, for $R = 3.28\text{ ft} = 1\text{ m}$, constitutes a focal point at which the Chézy factor is assumed to be unaffected by the slope S_o, and in metric measures is numerically equal to $C_* = \frac{1}{n}$. For feet, the focal value is $C_* = \frac{1}{n}\sqrt{3.28} = \frac{1.811}{n}$. It is the same focal point that conditions the Manning curve, the metric form for which is

$$C = C_*\,R^{1/6} = \left(\frac{1}{n}\right) R^{1/6} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (30a)$$

which is expressed in feet as

$$C = \left(\frac{1.811}{n}\right)\left(\frac{R}{3.28}\right)^{1/6} = \left(\frac{1.49}{n}\right) R^{1/6} \ldots\ldots\ldots\ldots (30b)$$

On the other hand, the logarithmic expression

$$C = (C_* - 16.5) + 32.1 \log R = \frac{1.811}{n} - 16.5 + 32.1 \log R \ldots (31)$$

represents, in foot units, the Prandtl-Kármán curve, customarily referred to as the Nikuradze formula, adapted to pass through the Ganguillet-Kutter focal point for the roughness term n. (In the well-known expression for rough pipes,

$$1 \sqrt{f} = A_r + 2 \log \frac{r_o}{e} \ldots\ldots\ldots\ldots (32)$$

replace $\frac{1}{\sqrt{f}} = \frac{1}{\sqrt{4\,c_f}}$ by $\frac{C}{\sqrt{8\,g}}$, and r_o by 2 R. This gives

$$C = \sqrt{8\,g}\left(A_r + 2 \log \frac{2}{e}\right) + 2\sqrt{8\,g} \log R \ldots\ldots (33a)$$

or, in foot units,

$$C = A_c + 32.1 \log R \ldots\ldots\ldots\ldots (33b)$$

For R = 3.28 ft, C must equal $C_* = \frac{1.811}{n}$. Hence, the constant $A_c = C_* - 32.1 \log 3.28 = \frac{1.811}{n} - 16.5$.) Finally, the Bazin curve $C = \frac{158}{1 + \frac{M}{\sqrt{R}}}$ is fitted to pass through the focal point by making the friction term $M = 158\,n - 1.811$.

In Fig. 4 the C-values are plotted logarithmically. As seen, the curvature of the outlines becomes rather unpronounced, and one is naturally tempted, for example, to replace the complex Ganguillet-Kutter relation by a suitable straight line answering the exponential relation $C = C_* R^p$, passing through the focal point C_*, and adjusted, so far as the incline p is concerned, to approximate best the $C = f(R)$-curve in the desired region. This explains the origin and the significance of the Manning formula, which is fitted to conform with $p = \frac{1}{6}$.

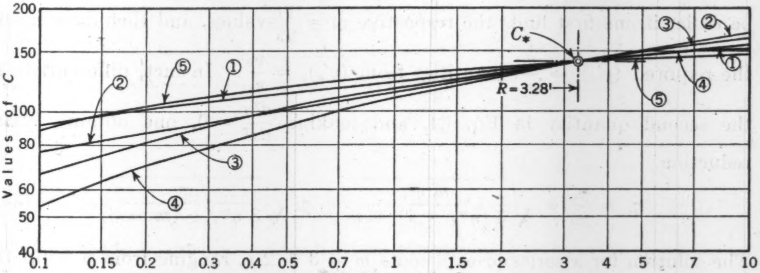

FIG. 4

More generally, in the light of Fig. 4, one may state that the various relations of C versus R differ by the relative steepness of their incline. Therefore, if an exponential expression were to be used for approximating the $C = f(R)$-function in that or any other reach of the curve, the respective formulas could be said to differ by a larger or smaller value of the exponent. The particular effect on Eq. 23 would be the substitution, in the numerator, of the ratio $\left(\frac{R_o}{R}\right)^{2p}$ for $\left(\frac{C_o}{C}\right)^2$.

(4) In the particular instance of a broad rectangular channel (Eq. 11*a*), one obtains, with such substitution,

$$\frac{dy}{dx} = S_o \frac{1 - \left(\frac{y_o}{y}\right)^{3+2p}}{1 - \left(\frac{y_c}{y}\right)^3} = S_o \frac{1 - \left(\frac{y_o}{y}\right)^m}{1 - \left(\frac{y_c}{y_o}\right)^3 \left(\frac{y_o}{y}\right)^3}$$

$$= S_o \frac{1 - \left(\frac{y_o}{y_c}\right)^m \left(\frac{y_c}{y}\right)^m}{1 - \left(\frac{y_c}{y}\right)^3} \dots\dots\dots\dots\dots\dots (34)$$

The possible influence of friction, as appraised by different formulas, depends on determining the value of $(y'_c)_i = \frac{y_i}{y_c}$, which in Eq. 34 would correspond to the different values of the "hydraulic exponent," $m = 3 + 2p$, subject to the parametric values of $\eta_c = \frac{y_c}{y_o}$. One could proceed directly by using the third quantity in Eq. 34; but the numerical computations prove to be easier if one first finds the respective $\eta_i = \frac{y_i}{y_o}$-values, and then determines the required $(y'_c)_i = \frac{y_i}{y_c}$-quantities from $(y'_c)_i = \frac{\eta_i}{\eta_c}$. In fact, differentiating the second quantity in Eq. 34, and making $\frac{d^2y}{dx^2} = 0$, one obtains, after reduction,

$$\eta_c = \frac{y_c}{y_o} = \sqrt[3]{\frac{m\,\eta^3_i}{3\,(\eta^m_i - 1) + m}} = \eta_i \sqrt[3]{\frac{m}{3\,\eta^m_i + (m - 3)}} \dots\dots (35)$$

The solution for a series of exponents $m = 3 + 2p$, ranging from $p = 0.1$ to $p = 0.4$, thus embracing all the possible slopes in Fig. 4, is presented in Fig. 5(*a*). Fig. 5(*b*) shows the final values of $(y'_c)_i = \frac{y_i}{y_c}$ versus $\eta_c = \frac{y_c}{y_o}$. In particular, the values of $(y'_c)_i$ for the vertical $\eta_c = 0$, which corresponds to flow over a horizontal bed, are found from Eq. 11*b*:

$$\frac{dy}{dx} = -\frac{g\,y^3_c}{C^2\,y^3\left(1 - \frac{y^3_c}{y^3}\right)} = -\frac{g\,y^3_c}{C^2_*\,y^m\left(1 - \frac{y^3_c}{y^3}\right)} \dots\dots\dots (36)$$

which, when differentiated and solved for $\frac{d^2y}{dx^2} = 0$, gives

$$(y'_c)_i = \left(\frac{y_i}{y_c}\right)_{S_o=0} = \sqrt[3]{\frac{m-3}{3}} \ldots\ldots\ldots\ldots\ldots\ldots (37)$$

As seen, the $(y'_c)_i$ versus η_c outline is rather flat, except when η_c approaches unity—that is, when the slope S_o becomes close to the critical, and the normal depth tends to become critical. Therefore, under ordinary circumstances, for $y_c \lesssim 0.6\, y_o$, the simpler expression (Eq. 37) can be conveniently used for quick appraisals. On the other hand, for all possible values of the exponent, the inflection depth y_i remains within 0.4 to 0.6 of the critical, showing remarkable steadiness in its position. Thus, a rule of thumb, assuming inflection at about half the critical depth, seems to be a justifiable approximation.

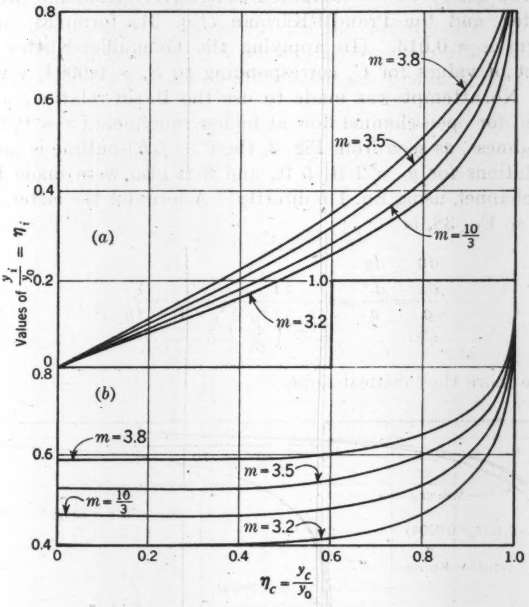

FIG. 5

(5) The purpose now is to disclose in a more direct fashion the exact possible effect of that or some other friction formula. Also, in view of the "flatness" of the M_3-curves, it is desirable to know more about their eventual outline in the regions adjoining the inflection point. The general procedure would be cumbersome and ineffective, so recourse must be had to practical examples. Calculations were performed for a wide range of circumstances, assuming two-dimensional flow at $y_o = 10$ ft, and making y_c successively 2 ft,

5 ft, and 8 ft. Using Eq. 11*a* in the form

$$\frac{\frac{dy}{dx}}{S_o} = \frac{1 - \left(\frac{C_o}{C}\right)^2 \left(\frac{y_o}{y_c}\right)^3 \left(\frac{y_c}{y}\right)^3}{1 - \left(\frac{y_c}{y}\right)^3} \qquad (38)$$

one determined the dimensionless ratio $\left(\frac{dy}{dx}\right): S_o$ versus $y'_c = \frac{y}{y_c}$, with the values for the Chézy factor computed successively from the Manning, Ganguillet-Kutter, and the Prandtl-Kármán (Eq. 31) formulas, and with the friction term $n = 0.013$. (In applying the Ganguillet-Kutter formula, an identical set of values for C, corresponding to $S_o = 0.0004$, was used in all instances. No attempt was made to use the Bazin relation. The latter is well adapted for open-channel flow at higher roughness [$n = 0.020$ to 0.025]. At low roughness, as seen from Fig. 3, the $C = f(R)$-outline is much too flat.)

Computations for $y_c = 2$ ft, 5 ft, and 8 ft also were made for flow in a horizontal channel, using Eq. 11*b* directly. A form for the latter, which would correspond to Eq. 38, is

$$\frac{\frac{dy}{dx}}{\frac{g}{C^2}} = \frac{\frac{dy}{dx}}{\sigma} = \frac{1}{1 - \left(\frac{y}{y_c}\right)^3} = \frac{1}{1 - (y'_c)^3} \qquad (39)$$

with σ once more the "critical slope."

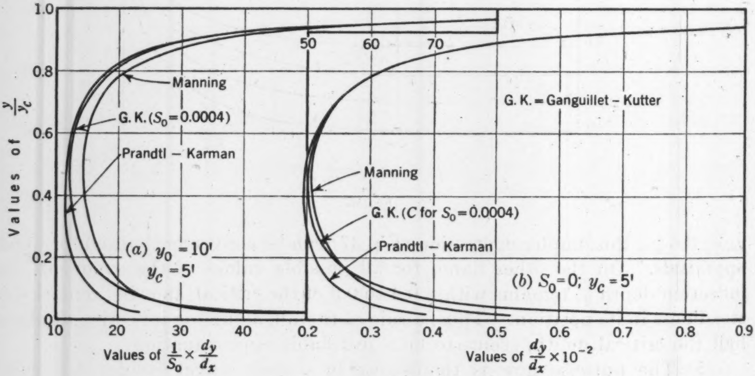

FIG. 6

Curves for the case of $y_c = 5$ ft are presented in Fig. 6. The outlines as shown are typical of the plottings obtained in all instances. All the $\frac{dy}{dx}$-curves exhibit only slight change over a comparatively wide middle reach, revealing the previously emphasized flatness of the M_3-curve. The curvature as derived from Eqs. 38 and 39 becomes pronounced only near $y = y_c$, and close to the bottom, where the computations based on the Bélanger premises of parallel motion with negligible curvature cease to be applicable. Again the curves corresponding to the different formulas vary only in detail, without changing the major features of the picture.

(6) It behooves one now to compare the results of the theoretical computations with actual observations. In practice, M_3 surface curves occur at the foot of spillways and chutes, where the steep incline changes to a mild slope channel, or at the point where water flows from beneath a sluice into a horizontal or mildly sloped bed. In either case the initial surface, in the transition reach, is ipso facto concave. Accordingly, if the initial depth is less than y_c, the subsequent flow must exhibit a portion of convex surface between two concave reaches. Such a succession of forms in practice is highly improbable.

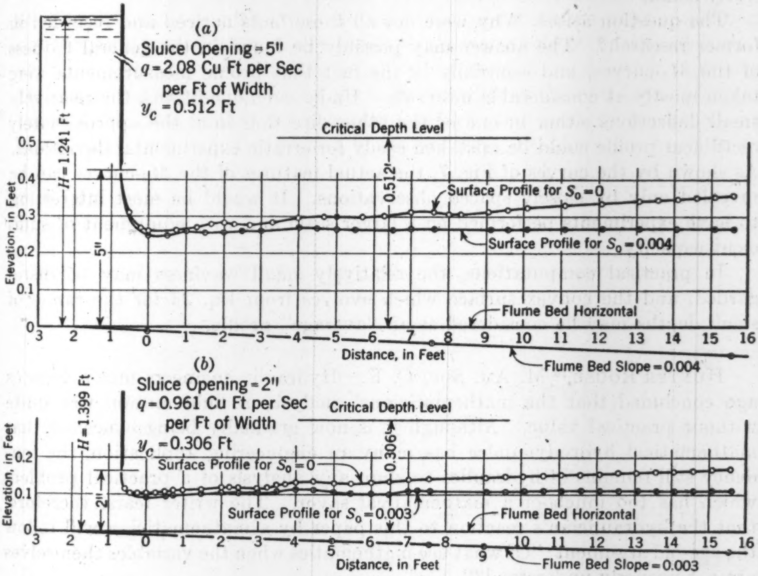

FIG. 7

Moreover, whenever, in hydraulics, a certain equation results in two distinct branches, as in Eqs. 38 and 39 in Fig. 6, one is certain to anticipate that the respective branches reflect altogether different physical regimen, or that flow on one of the branches is unstable and in reality presents wavy features. An example is the double-branched curve[17] which features the height of the jump sustained by flow from under a sluice, and in which the branch for lower initial kinetic energies corresponds to jumps of the "undular" type. Thus, with respect to the M_3-curves (Fig. 6), one could reasonably expect that the lower and theoretically convex part of the profile will prove in reality to be more or less undulated. Special experiments in the 6-in. tilting flume of the Fluid Mechanics Laboratory of Columbia University in New York, N. Y., confirm such anticipation. The profiles shown in Fig. 7 are typical of the results obtained for a wider range of sluice openings and initial heads. In each instance, the surface profile in the reach succeeding the initial "contracted" section exhibits distinct undular outlines. The waves are not large, however, and furthermore their size decreases with the increase of the depth, as the curve evolves into the final, practically linear, form. Most characteristically, the effect of waviness was particularly accentuated in the bottom pressures. Apparently, due to the high prevailing velocities, substantial centrifugal effects were exercised even when the curvature, caused by the waves, was comparatively small.

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[17] "Hydraulics of Open Channels," by Boris A. Bakhmeteff, McGraw-Hill Book Co., Inc., New York, N. Y., 1932, p. 248.

The question arises: Why were not all these facts noticed and recorded in former research? The answer may possibly be found in the general flatness of the M_3-curves, and especially in the fact that profile measurements were taken mostly at considerable intervals. Under such conditions, the relatively small deflections either in one or the other direction from the approximately rectilinear profile could be mistaken easily for erratic experimental deviations. As shown by the curves of Fig. 7, the actual features of the M_3-curves can be revealed only by closely spaced observations. It would be most interesting to have experiments performed on a larger scale and with equipment of sufficient capacity.

In practical computations, the relatively small waviness may be disregarded, and the convex surface which evolves from Eq. 23 for the range of small depths may be considered as an "average" profile.

HUNTER ROUSE,[18] M. AM. SOC. C. E.—Hydraulic engineers many decades ago concluded that the mathematicians' analysis of fluid motion was quite without practical value. Although it is now gradually being conceded that mathematical hydrodynamics has many an engineering application, the tendency still remains in hydraulics to shun any analysis of a practical problem which has too much of a mathematical savor. The writer fears, therefore, that the hydraulician's reaction to this paper by a mathematician will follow the age-old argument: "Of what use mathematics when the variables themselves are not properly understood?"

The writer feels responsible to some extent for Professor Gunder's attack upon this problem, for it was during the writer's lectures on advanced fluid mechanics at Fort Collins, Colo., in July, 1940, that the author clarified the consistent failure of computed surface curves below the critical depth to agree with the traditional curves in the writer's text. Comments by the writer on the obvious difficulty of integrating the varied-flow equations then led Professor Gunder to perform this integration, largely to satisfy his own curiosity. The results were finally submitted to the Society, not—so far as the writer is aware—

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[18] Prof., Fluid Mechanics, State Univ. of Iowa, and Assoc. Dir. in Charge of Laboratory, Iowa Inst. of Hydr. Research, Iowa City, Iowa.

in the guise of a final solution to the varied-flow problem, but simply to place on record the information contained therein.

It is scarcely to be expected, of course, that a member of one profession should at once feel as much at home in another profession as in his own. It is the writer's experience, however, that mathematicians have far greater facility in grasping the salient features of an engineering problem than have engineers in mastering the mathematics necessary to its solution. For this reason, the writer most cordially welcomes Professor Gunder's initial contribution to the hydraulic engineering field, regardless of the merit of the paper itself. This profession, as already stated, still suffers from many misunderstandings of the past century, and can only profit from a mathematician's ability to cope with functional relationships which are beyond the normal engineering grasp.

As to the actual merits of this paper, the writer believes that the conclusions would be far more significant if freed from the handicap of being based upon resistance formulas which are in themselves none too trustworthy. Although the Manning formula is not necessarily worse than those of Bazin and Kutter, as implied by the author in his second conclusion, it is obviously more convenient than the other two, else the author would not have chosen it for purposes of integration; that it yields results which differ from those of the other two is surely not sufficient reason to condemn it. On the other hand, it has been shown both experimentally and analytically that the actual resistance due to wall roughness varies with the logarithm, rather than some power, of the relative roughness. Since logarithmic and power functions can approximately coincide over only a limited range, the author's curves necessarily reflect the discrepancies to an ever-increasing degree as they approach an upper or lower limit.

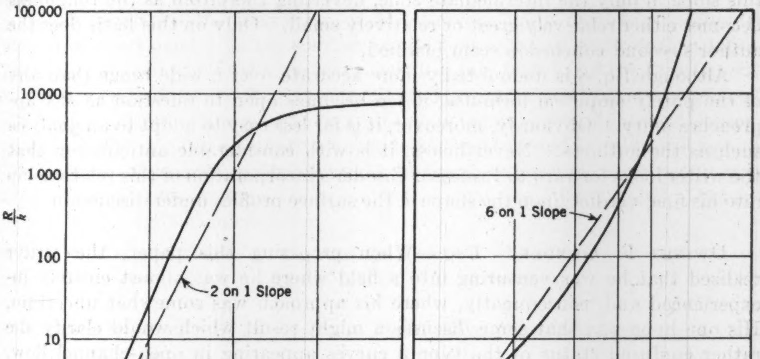

FIG 8

To be explicit, for the case of flow in a very wide channel in which the resistance is independent of viscosity, the Chezy C may be expressed with very close approximation in the Kármán-Prandtl form:

$$\frac{C}{\sqrt{8\,g}} = 2.1 + 2 \log_{10} \frac{R}{k} \qquad (40)$$

in which the factor k is a linear measure of the boundary roughness. (It is to be noted that the dimensionless factor 2.1 is a function of cross-sectional form, and will vary with the proportions of the channel section.) Like R/k, the Bazin parameter $m/R^{1/2}$ and the Manning parameter $R^{1/6}/n$ are measures of relative roughness, the Bazin m in American units obviously having the dimension of $(\text{feet})^{1/2}$, and the Manning n the dimension of $(\text{feet})^{1/6}$; what dimension to ascribe to the Kutter n is a problem, owing to the two different dimensions it appears to have in the Kutter formula. Were Eq. 40 and the Bazin formula both correct, a logarithmic plot of R/k against $R^{1/2}/m$ for identical values of C would appear as a straight line having a slope of 2 on 1; as seen from Fig. 8(*a*), the actual relationship has approximately the required slope only in the range of moderate to excessive roughness. Similarly, were Eq. 40 and the Manning formula in agreement, a similar plot of R/k against $R^{1/6}/n$ would yield a straight line having a 6 on 1 slope, whereas the curve in Fig. 8(*b*) displays this slope in only the intermediate zone, deviating therefrom as the roughness becomes either relatively great or relatively small. Only on this basis does the author's second conclusion seem justified.

Although Eq. 8 is undoubtedly more accurate over a wide range than any of the purely empirical formulas, it too becomes open to question as R/k approaches unity. Obviously, moreover, it is far less easy to adapt to an analysis such as the author's. Nevertheless, it is with considerable anticipation that the writer looks forward to Professor Gunder's incorporation of this relationship into his final verdict upon the shape of the surface profiles under discussion.

Dwight F. Gunder,[19] **Esq.**—When preparing this paper, the writer realized that he was venturing into a field where he was almost entirely inexperienced and, consequently, where his approach was somewhat uncertain. His one hope was that some discussion might result which would clarify the rather confused status of the type 3 curves appearing in open-channel flow. The excellent contributions of the discussers have certainly justified this hope.

Before considering these discussions individually, the author wishes to acknowledge an error in the paper. The point of inflection for the Bazin and Ganguillet-Kutter relations by the straightforward but extremely tedious method of determining the point of inflection was located by finding $\frac{d^2y}{dx^2}$ and solving this equation by approximation. In so doing, a serious error of computation was committed. (Few mathematicians, if such the writer could be called, can do arithmetic.) The much more elegant treatment of this part of the problem by Messrs. Bakhmeteff and Feodoroff clearly showed a discrepancy. In light of this and the experimental findings of the same two discussers, the second of the conclusions of the "Synopsis" should be written as follows:

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[19] Research Engr. and Associate Prof., Math. and Civ. Eng., Colorado State College, Fort Collins, Colo.

(2) Use of the differential equation of gradually varied flow at depths below both the normal and critical depths should be made with reservations until more experimental work has been done in this region of high velocity flow. The author deeply regrets the inclusion of this error but is glad that it was caught in the discussions.

The suggestions of Messrs. Rohwer and Rouse that certain other specific cases be considered are excellent although they provide rather severe difficulties unless one is to adopt approximate methods which, in turn, are essentially those now used. Further, it seems from the evidence presented by Messrs. Bakhmeteff and Feodoroff that further refinement in the analysis should await additional experimental data relative to the various roughness factors and to the profile curves themselves. As soon as some one obtains some rather complete and reliable data on this particular region of flow, it may be advisable to attack the problem using, say, the Kármán-Prandtl expression for the Cheźy *C*, as is suggested by Professor Rouse. The mechanical difficulties attendant upon attaining a closed solution to this problem are great but not necessarily prohibitive.

With regard to the discussion of Mr. Stevens, only one point probably should be mentioned. In the solution of the flow equation, as Professor Rouse has stated so well, the final form was presented only for what it might be worth to those who had use for it. As a result, this equation was stated in terms of the original variables and parameters with no attempt at simplification, leaving such simplifications or alternative forms to those interested in using the result. The feature which the writer thought obvious, but which apparently Mr. Stevens (and hence perhaps others) missed, is that the differential Eq. 15 can be put in the dimensionless form

$$d\xi = \frac{\eta^{1/3}\,(\eta^3 - \eta^3_c)}{\eta^{10/3} - 1}\,d\eta \ldots\ldots\ldots\ldots\ldots\ldots (41)$$

in which $\eta = \frac{y}{y_o}$, $\eta_c = \frac{y_c}{y_o}$, $\xi = \frac{S_o\,x}{y_o}$. Eq. 17 then reduces to

$$\xi = \eta + 0.306\,\eta^3_c + 0.3\,K_1 \ldots\ldots\ldots\ldots\ldots\ldots (42)$$

in which K_1 is dimensionless and a function of η_c and η only. It is thus seen that the plot of the ξ,η-curves, which give the x,y-curves directly, is a one-parameter family with the parameter η_c. Furthermore, K_1 may be separated into two parts, one of which (say, K_2) is independent of η_c, and the other of which (say, K_3) consists of three or four relatively simple terms involving η and η_c. Thus, the construction of the table which Mr. Stevens recommended as a sentence for the writer could be accomplished in quite a restricted space. It appears as though the "professor" had overestimated the ability of one of his "students." This situation seems inseparable from the learning process.

The writer wishes to acknowledge especially the fine work of Messrs. Bakhmeteff and Feodoroff in correlating the discussion of the various resistance factors and correcting the writer's error, as well as in giving some intimation by experimental studies of the true state of affairs as regards the type 3 curves in open-channel flow.

Finally, the author feels that the excellent discussions have contributed more to the problem than did the original paper, the only merit of which is perhaps that it brought forth these discussions.

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THE GREASE PROBLEM IN SEWAGE TREATMENT

By Almon L. Fales [1] and Samuel A. Greeley,[2] Members, Am. Soc. C. E.

With Discussion by Messrs. Arthur D. Weston, Harvey F. Ludwig, W. S. Mahlie, Denis Dickinson, Charles R. Velzy, F. M. Dawson, and Almon L. Fales and Samuel A. Greeley.

Synopsis

Until recent years, municipalities in the United States, situated on the seacoast, have had relatively little difficulty in disposing of their sewage with scant, if any, treatment. In general, the most that was required of such municipalities was the construction of intercepting sewers (to prevent the discharge of sewage from numerous small outlets at the shore) and one or more outfall sewers extending to deep water for disposal by dilution. With increasing population and higher standard of cleanliness, it has become increasingly difficult to dispose of sewage in this manner without creating conditions objectionable to bathing, boating, fishing, and the like.

The offensive conditions have been due primarily to deposits of settling suspended matters and to floating sewage matters, including grease and oil which form "sleek" on the surface. The result in many cases has been that steps have been taken to provide sewage treatment plants for clarification and grease removal. In view of the important relation of grease to the problem of sewage treatment and disposal, the purpose of this paper is to discuss the problems of grease removal in sewage.

Definition of Grease

A standard concept of grease in sewage is that it "includes fats, waxes, free fatty acids, calcium and magnesium soaps, mineral oils, and other non-fat materials" (1).[3] When "grease" is mentioned in connection with sewage and its treatment, almost any combination of these substances may be meant. Sometimes, however, the liquid portions are called "oil" and the solid portions

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[1] Cons. Engr. (Metcalf & Eddy), Boston, Mass.

[2] (Greeley & Hansen), Chicago, Ill.

[3] Numerals in parentheses, thus: (1), refer to corresponding items in the Bibliography (Appendix).

"grease." The term "grease" is generally applied to all these materials whether of animal or vegetable origin.

SOURCES OF GREASE IN SEWAGE

In a broad sense, there are three sources of the grease and oil in sewage—domestic, industrial, and automotive. Domestic sources of grease include dwellings, hotels, restaurants, clubs, and institutions, the grease originating in human feces and kitchen slops. Feces contain fatty acids, soap, and neutral fat, whereas kitchen slops contain not only the unused fats from foodstuffs, such as lard, suet, butter and olive oil, but also soaps. The latter include both the sodium soaps and the calcium and magnesium soaps, or mineral soaps, which result from the reactions due to the hardness of the water supply.

Industrial wastes may contain large quantities of grease and oil, particularly those from abattoirs, packing houses, wool-processing plants, rayon plants, tanneries, oil refineries, gas works, canning plants, dairies, bakeries, and laundries.

Automotive sources of oil and grease include garages, repair shops, and automobile-washing plants. Although the larger garages may collect such material and prevent it from entering the sewerage system, many individual owners discharge waste of this kind into the sewers.

CHARACTERISTICS OF GREASE

True fats are compounds of the alcohol glycerol (commonly called glycerin) with the fatty acids, such as oleic, palmitic and stearic, the resulting glycerids being designated as olein, palmitin and stearin (2). Other glycerids are also known. The three mentioned are found in animal fats—the first being a liquid and the others solids.

Fats contain carbon, hydrogen, and oxygen in varying proportions, the chemical structure of the molecules being relatively simple. Fats are among the more stable of organic compounds. The mineral acids attack them, with the result that glycerin and fatty acid are formed. In the presence of alkalis, such as sodium hydroxide, glycerin is liberated and alkali salts of the fatty acids are formed. The latter are known as soaps and, like the fats, are of stable character. Common soaps, soluble in water, are made in the manner indicated, by saponification of fats with sodium hydroxide. Mineral soaps, on the other hand, are insoluble and are precipitated. Soaps in sewage are largely of the latter kind.

The grease in sewage may be either "free" or "combined." The free grease is partly of mineral origin, such as gasoline, kerosene, and fuel oil, and partly of organic origin, such as lard, butter, and cottonseed oil. Most of the combined grease in sewage is soap. Although oil and water are immiscible, one can be dispersed in the other with the aid of a third substance, or emulsifying agent, to form an emulsion. In sewage, oil and water form emulsions, with the aid of soaps and, in this manner, much of the "free" grease is held in a colloidal state.

From a physical standpoint, grease in sewage may be classified as capable of floating, when in emulsion or suspension, or of settling. Soap curds and the lighter oils and greases tend to float. Grease in meat-packing wastes may partly sink and partly flow along in emulsion or in suspension, although some of the material floats. Apparently these types of material tend to rise as the liquid chills. Gas wastes and similar substances may contain relatively heavy tarry residue (which sinks) as well as lighter oils that form a thin film on the surface.

EFFECTS OF GREASE ON RECEIVING WATERWAYS

When sewage is discharged without treatment into rivers or tidal waters, the normal quantity of grease and oil contained in the sewage tends to give a smooth, silky appearance, called "sleek," to the surface of the body of water near the point of discharge. Small quantities of oil can cover large surfaces and, in a general way, the extent of the sleek field in the vicinity of a given outlet depends upon the grease and oil content of the sewage. Such sleek may be detrimental to boating, due to the formation of oily deposits on the sides of pleasure crafts passing through it. Where sleek fields are sufficiently extensive to reach bathing beaches, the use of the beaches may be out of the question, on account of the deposition of oil and grease.

In the case of rivers, an oil film may interfere with natural purification by shutting off the air needed for re-aeration. Spawning places for fish may be damaged by oil, and fish food injured. Oils may impart unpleasant tastes and odors to water supplies. Furthermore, if the water is used for irrigation, oil films may tend to fill up the pores of the soil.

Another ill effect of discharging untreated sewage into bodies of water is the formation of grease balls. According to Edward Wright, M. Am. Soc. C. E. (3),

> "* * * the amount of fats in the sewage from one of the large tannery districts in Massachusetts, which is discharged into the sea, has been so great as to cause the formation of numberless grease balls, varying in size from a large marble to a cantaloupe. They are particularly offensive in connection with the recreational uses of adjacent waters for bathing, * * * Such balls, presumably from the Boston Harbor outfall sewers, were found at an isolated point 66 miles from the harbor."

EFFECTS OF GREASE ON SEWAGE TREATMENT PLANTS

The troubles caused by grease at sewage treatment plants are many and varied, some of them being of rather common occurrence. In fine-screening plants, grease tends to clog the screens and increase the cost of maintenance. In other types of treatment plants, fatty substances may form unsightly scum on the surface of the sewage in channels and tanks, as well as on the side-walls at the surface of the sewage. In sedimentation tanks, grease may interfere with the settling of suspended solids by coating particles of organic matter and causing them to rise to the surface as scum. In some cases, it has been found that grease digested with difficulty and caused trouble in the digestion chambers of Imhoff tanks.

Fats are a deterrent to the biological processes taking place in intermittent sand filters, trickling filters, and activated-sludge plants, in that they coat with a stable film surfaces that otherwise would be subject to microbial action. They may also cause clogging of filters and of trickling-filter nozzles.

Accumulations of grease are likely to cause offensive odors at sewage treatment plants. They may also destroy paint, as shown by experience at the

Marina sewage pumping plant at San Francisco, Calif. According to Joseph Corrao (4),

> "When oil or grease become emulsified with water they very effectively destroy the paints they come in contact with. They dissolve the vehicle, soften the film and make it susceptible to mechanical damage thereby exposing the steel to attack by water, hydrogen sulfide, and other destructive agents."

Oil and grease in sewage may also cause difficulties in sludge disposal. For example, it was reported in 1931 that, at the Calumet works of the Sanitary District of Chicago, Ill., the activated sludge, which was being heat-dried, showed a marked tendency to ignite in the dryer, when oil was prevalent (5). More recently, at the Southwest Works of Chicago, the distillation of grease and other volatile substances in the flash-drying process has caused accumulations on the forced-draft fans and air ducts, which seriously interfere with the operation of the drying system. Furthermore, large amounts of oil and grease tend to make sludge dewatering more difficult. They depreciate the commercial value of sludge as a fertilizer.

Another possible source of difficulty is the entrainment of large quantities of air in sewage before it reaches the sewage treatment plant. In the early months of 1938, at the Wards Island sewage treatment plant in New York, N. Y., thick scum formed on the surface of the preliminary clarifiers, often to a depth of 4 in. (6). The trouble was traced to the fact that air under pressure was entering the sewer system feeding the plant. Scum formation was eliminated by raising the sewage level at the Manhattan and Bronx grit chambers, thus preventing air from being entrained in the raw sewage.

A difficulty of a similar nature occurred at the North Toronto sewage treatment plant in Toronto, Ont., Canada, where the sewage falls 80 ft in a drop shaft before entering the plant. Large quantities of air, entrained in the raw sewage during this drop, caused the grease in the sewage to form thick froth and scum on the surface of the influent channels. The difficulty was overcome by installing water sprays to beat down the scum in these channels or force it into the preliminary sedimentation tanks.

TESTS FOR DETERMINING GREASE IN SEWAGE

The quantity of fats and oils in sewage is commonly determined as ether-soluble matter, by liberating with acid the fatty acids, after evaporating the water, and dissolving them in ether which is then driven off, leaving behind the ether-soluble matter which can be weighed.

Any one of three solvents may be used in the determination of grease, as shown by the following quotation (1):

> "Of the three common fat solvents, petroleum ether, ethyl ether and chloroform, the last extracts the greatest amount of grease, and petroleum ether the smallest. These solvents may extract additional material after the sample has been acidified. This is particularly true of petroleum ether, and to a much smaller extent also true of chloroform. More data are needed regarding the chemical composition of the material extracted by the different solvents, and until this is available it seems best to recommend

> petroleum ether, ethyl ether and chloroform. In reporting results the extracting agent should always be stated. On the insufficient data available, chloroform appears to be the most satisfactory solvent. However, it has not received sufficiently widespread use to warrant discarding the other solvents at this time."

Data on the extraction of lipids from sewage sludge by four different solvents have been furnished by O. J. Knechtges, W. H. Petersen, and F. M. Strong (7). The term "lipids" is intended to designate fatty acids, substances yielding fatty acids on hydrolysis, and other substances that are physically of a fat-like character and can be extracted with fat solvents from biological materials. Four solvents were compared with respect to the quantity of lipid extracted from domestic sludge and packing-house sludge. The results (7) are given in Table 1.

TABLE 1.—PERCENTAGE OF DRY SOLIDS EXTRACTED FROM DRIED SLUDGE BY DIFFERENT FAT SOLVENTS
(Sludge Not Acidified)

Sludge	Petroleum ether	Ethyl ether	Isopropyl ether	Chloroform
Domestic	24.7	23.6	31.1	42.5
Packing-house	18.4	21.3	30.3	38.8

Chloroform extracted the largest percentage of material, and petroleum ether the smallest. In fact, without acidification, chloroform extracted approximately twice as much material as did petroleum ether. Even with acidification of the sludge, petroleum ether removed less material than chloroform did without acidification.

Fractionation of the material extracted from five sludges by petroleum ether and chloroform showed that a fraction totaling from 6% to 19% of the extracts was soluble only in chloroform. This fraction was a black, tar-like material with rather acidic properties. The matter was not investigated further, but it was thought that this material might not be lipoidal in character.

Data on the quantity of grease in sewage, as determined by five different solvents, have been furnished by G. J. Wiest from laboratory studies at Lancaster, Pa. (8). Determinations were made on catch samples of raw sewage from the North sewage treatment plant, the South sewage treatment plant, and the Maple Grove pumping station—the sample from the latter point being representative of strictly domestic sewage. In each case the dried solids were acidified before extraction with the solvent. The results of the Lancaster studies on raw sewage are summarized in Table 2. These results show that, in each sample, chloroform extracted a larger quantity of material than petroleum ether or ethyl ether. Furthermore, chloroform extracted more material than

TABLE 2.—QUANTITIES OF GREASE IN RAW SEWAGE AT LANCASTER, PA., AS DETERMINED WITH FIVE DIFFERENT SOLVENTS

Solvent	North sewage treatment plant	South sewage treatment plant	Maple Grove pumping station
Chloroform	37.0	147.4	39.2
Petroleum ether	7.9	17.2	12.0
Ethyl ether	31.8	70.0	33.4
Carbon tetrachloride	18.7	74.2	42.2
Benzene (benzol)	23.7	62.6	43.0

either carbon tetrachloride or benzene, except in the case of the sample of strictly domestic sewage.

The data presented, although rather meager, are sufficient to indicate that the quantity of grease found in sewage by the recommended methods (1) is dependent mostly upon the solvent used for extraction. According to D. A. Okun, Jun. Am. Soc. C. E., and E. Hurwitz and F. W. Mohlman, the literature indicates that chloroform and ethyl ether extract a larger proportion of non-fatty substances than does petroleum ether, which is more specific for fats and oils (9). Chloroform is said to dissolve resins and waxes in addition to strictly fatty substances. It is evident, therefore, that comparisons of results of grease determinations must be made with caution and with due regard for the solvents used in extraction.

The method of grease determination recommended in "Standard Methods for the Examination of Water and Sewage" (1) has not proved entirely satisfactory, aside from the flexibility that it allows in the solvent to be used for extraction. The evaporation of large quantities of sewage required by this method delays the determination considerably and may cause the loss of some of the grease, which is volatile. Furthermore, according to Mr. Mohlman, evaporation to dryness, after acidification to release fatty substances from soaps, may produce low results because of re-saponification of part of the fats on concentration of the solution (9).

To avoid the necessity for lengthy evaporation, as well as possible re-saponification of fats, Richard Pomeroy and C. M. Wakeman have suggested wet extraction (10). Their method is to agitate the acidified sample with the solvent and separate the two layers in a separatory funnel. Two extractions complete the removal of the soluble material.

TABLE 3.—GREASE IN THE SEWAGE OF VARIOUS CITIES, AS DETERMINED BY EXTRACTION WITH ETHER

City[a]	Sewage flow (gal)[b]	Grease		Year	Observation period (days)
		ppm	Lb per capita daily		
Alliance, Ohio	131	37	0.040	1920	12
Atlanta, Ga. (Intrenchment Creek plant)	167	104	0.145	1920	11
Baltimore, Md.	106	40	0.035	1920	12
Canton, Ohio	64	62	0.033	1920	9
Cleveland, Ohio (Westerly plant) (12)	...	66	....	1934	196
Columbus, Ohio	94	100	0.078	1920	10
Dayton, Ohio (2)	55	47	0.021	1923	1
Fitchburg, Mass.	88	63	0.046	1920	11
Gloversville, N. Y. (2)[c]	145	489	0.591	1921–22	640
Houston, Tex. (North Side plant)	75	39	0.024	1920	11
Reading, Pa.	101	25	0.021	1920	10
Rochester, N. Y. (Irondequoit plant)	137	34	0.039	1920	12
Toronto, Ont., Canada[d]	142	33	0.039	1931	14

[a] Reference (11) except as noted. [b] Gallons per capita daily. [c] Industrial wastes from tanneries present in large quantities. [d] Data from Toronto.

Mr. Mohlman prefers to pass the sample through a specially prepared filter, dry the material retained, and extract with the solvent in a Soxhlet extraction apparatus (9). The filtrate is shaken with solvent, and the solvent, with whatever grease it may contain, is then added to the extraction flask. In most cases, especially with sludges, the grease in the filtrate may be disregarded.

Quantities of Grease in Sewage

Table 3 contains data on the quantities of grease in the sewage of various cities, as determined by extraction with ether. Similar data for Massachusetts cities are given in Table 4.

TABLE 4.—Grease in the Sewage of Massachusetts Cities, as Determined by Extraction with Ether

Description (1939 data from the Massachusetts Department of Health)	Attleboro	Brockton	Fitchburg	Gardner		Leominster	North Adams	Pittsfield	Worcester
				Gardner area	Templeton area				
Grease, in parts per million..	46	79	45	104	62	43	40	54	80
Period of observation, in days..................	4	5	7	4	7	7	7	7	5

According to these data, the quantity of grease in city sewage ranges from 25 ppm at Reading, Pa., to somewhat more than 100 ppm at Atlanta, Ga., and Gardner, Mass. In cities where the sewage contains large quantities of industrial wastes, the grease content may be very high. Such is the case at Gloversville, N. Y., where, due to large quantities of tannery wastes in the sewage, nearly 500 ppm of grease were found in the sewage in 1921–1922. From Table 3 it will be seen that, on a per-capita basis, the quantity of grease in sewage ranges from 0.02 lb per day at Dayton, Ohio, Houston, Tex., and Reading, to 0.15 lb at Atlanta, and 0.59 lb at Gloversville.

Additional data on the grease content of sewage have been compiled by W. S. Mahlie, but unfortunately the analytical methods used were not recorded (13).

Methods of Removing Grease from Sewage

There are two general methods of separating grease from sewage: First, by the installation of separate tanks or other devices especially designed to retain grease and other floating matters; and second, by the collection of such materials as they float on the surface of sedimentation tanks or settle to the bottom with sludge. Equipment that may be installed for the first method includes skimming tanks, skimming-detritus tanks, and aerated grease-separating tanks. To aid in the subsequent separation of grease by skimming, the sewage may be agitated with air or by mechanical means, or it may be treated with a combination of air and chlorine.

Plain Skimming.—Probably the earliest method of grease removal at sewage treatment plants was that of skimming the grease by hand from the surface of sedimentation tanks or grit chambers, after arresting the floating grease by means of suitable baffles. In a few cases plain skimming tanks have been installed for the purpose of separating scum from the sewage.

A skimming tank is arranged in such a way that floating matter rises in it to the surface of the sewage and remains there until it is skimmed off, while the liquid flows out continuously through deep outlets or under partitions, curtain walls, or deep scum boards. The few American skimming tanks in existence are either elliptical or circular in shape and furnish detention periods of from 1 to 15 min. The submerged outlet is situated opposite the inlet and at a lower elevation, to promote flotation of the scum and removal of any solid matter that may settle. By means of such skimming tanks material such as corks and bits of wood, as well as grease and oil, is removed.

In a few places, tanks have been built for the separation of the heavy, readily settling solids and of the light floating material in single units, called skimming-detritus tanks. In such a tank at Akron, Ohio, floating wooden booms skim the grease from the surface and collect it in front of an adjustable weir, over which it is pushed on to a draining bed. Revolving plows move the grit and other solids settling from the sewage to a central hopper, whence they are pumped with about 20% of the average sewage flow to grit chambers, where the final separation of grit takes place. After passing fine screens, this "underflow" joins the main body of sewage that has passed through the skimming-detritus tanks. There are two tanks in the Akron plant, 55 ft in diameter and 12 ft in maximum depth, which furnish an average detention period of 12.6 min. There are skimming-detritus tanks of a similar type at Allentown, Pa., and Dayton.

At the West Side works of the Sanitary District of Chicago, the sewage, before entering the Imhoff tanks, passes through rectangular tanks with an average detention period of about 15 min. These tanks are equipped with sludge-removal equipment of the scraper-conveyer type, the return flights on the surface serving as skimmers. About 15% of the flow, containing the settled solids and grit, is diverted to the grit chambers.

Although a considerable quantity of objectionable scum may be removed from sewage by skimming tanks and skimming-detritus tanks, it is doubtful whether such tanks alone are capable of removing more than a small proportion of the grease contained in normal sewage.

Sedimentation and Skimming.—A somewhat larger proportion of the grease is often removed from sewage in sedimentation tanks, especially where such tanks are equipped with skimming devices. In this case, part of the grease rises to the surface and is skimmed off, while another portion settles to the bottom and forms part of the sludge.

In installations equipped with mechanical apparatus for sludge removal, this apparatus, or an attachment to it, may also be used to remove scum. The scraper-conveyer type of mechanism has already been mentioned. Another device adapted to the removal of floating matter is a commercially patented mechanism that consists of a bridge traveling on rails along the longitudinal walls of the tank, with a scraper attached to the bridge by hinged arms. The scraper is used to force the sludge into hoppers at the influent end of the tank and, when raised to the surface of the sewage, it is used to skim the tank.

In tanks where the sludge-removal mechanism cannot be readily adapted for use in skimming, a system of water piping with small nozzles, as used by H. B.

Allen in the Imhoff tanks at Fitchburg, Mass., may be of value. The nozzles—horizontal hack-saw cuts in vertical pipes—discharge jets of water almost horizontally over the surface, driving the accumulated grease and oil to points where they can be collected. An installation of this type in the preliminary sedimentation tanks of the North Toronto plant has given good results.

In some plants grease is removed from the surface of sedimentation tanks by hand. It may be skimmed off by means of dippers, drawn off by lowering inverted sluice gates, or pushed into troughs or bays by squeegees.

Since the determination of fats is not ordinarily made in settled sewage, the available data on the removal of grease from sewage by sedimentation tanks are not plentiful. At the Westerly plant in Cleveland, Ohio, according to the late George B. Gascoigne, M. Am. Soc. C. E., a study of ether-soluble matter was made on 28 weekly composites from January to June in 1934, when the grease catchers were not in operation (12). This study showed the average grease content of the raw sewage to be 66 ppm and that of the settled sewage 46 ppm, with a removal of 20 ppm, or 30%. In experiments on sewage of the Boston Main Drainage system in 1935, the results indicated that about 40% of the 46 ppm of grease present in the raw sewage, as determined by ether extraction, could be removed by skimming and sedimentation in a 2-hr detention period, leaving 28 ppm in the effluent (14). Similar results were obtained by the same investigators in experiments on sewage of the North Metropolitan sewerage system in 1935 and 1936.

At the Sewage Treatment Works of Buffalo, N. Y., determinations have been made of the grease in weekly composites of acidified daily samples of sewage leaving the grit chambers and of the effluent from the sedimentation tanks, by petroleum-ether extraction of solids removed by filtering through absorbent cotton. The conduits, through which the sewage passes to the sedimentation tanks, are aerated to prevent deposition of solids and keep the sewage fresh. The sedimentation tanks have a displacement period of 80 min at rated capacity. They are provided with mechanical equipment for removing the sludge and scum.

TABLE 5.—GREASE REMOVAL BY SEDIMENTATION TANKS AT BUFFALO, N. Y., SEWAGE TREATMENT WORKS DURING 1940

Month	GREASE BY PETROLEUM-ETHER EXTRACTION OF FILTERED SOLIDS				Average daily flow of sewage (mgd)
	Grit chamber effluent (ppm)	Sedimentation tank effluent (ppm)	Removed by sedimentation tanks (ppm)	Removal (%)	
Jan.	50.2	42.7	7.8	15.5	127
Feb.	54.7	46.2	8.5	15.6	141
March	53.8	48.8	5.0	9.3	183
April	45.8	43.3	2.5	5.4	174
May	34.4	27.2	7.2	21.2	129
June	37.1	31.5	5.6	15.1	127
July	39.9	23.1	16.8	42.0	118
Aug.	23.4	15.6	7.8	33.3	128
Sept.	38.2	29.7	8.5	22.2	125
Oct.	40.5	24.1	16.4	40.5	113
Nov.	29.8	29.0	0.8	2.8	161
Dec.	23.1	16.9	6.2	26.8	202
Aver.	39.2	31.5	7.7	19.6	144

A monthly summary of data on grease removal by the sedimentation tanks at Buffalo during 1940 is given in Table 5. (Grease content was determined in weekly composites of daily samples acidified with sulfuric acid. The averages were not weighted. The data were furnished by C. R. Velzy, M. Am. Soc. C.

E.) The monthly average grease content of the sewage, as determined in the grit-chamber effluent, varied from 54.7 to 23.1 ppm, averaging 39.2 ppm, for the year, and the proportion of grease removed by the sedimentation tanks varied from 42.0 to 2.8%, averaging 19.6% for the year. There does not seem to have been any definite relation between the quantity of grease in the sewage and the percentage removed.

Data on the removal of grease by settling tanks at sewage treatment plants of Massachusetts cities (15) are presented in Table 6. The data for fats, or grease, are averages for 1939 as determined by ether extraction of the solids of composite samples of raw and settled sewage, collected in most cases at approximately bi-monthly intervals. Table 6 shows that of 43 to 80 ppm of grease in the raw sewage at these plants, from 8% to 72% was removed by the settling tanks. There appears to be no definite relation between detention period and removal of grease at the different plants.

TABLE 6.—AVERAGE GREASE REMOVAL BY SETTLING TANKS AT SEWAGE TREATMENT PLANTS OF MASSACHUSETTS CITIES DURING 1939

City	Detention period (hr)	FATS BY ETHER EXTRACTION (PPM)			Fats removed (%)
		Raw sewage	Settled sewage	Removed	
Brockton	2.0[a]	79	56	23	29
Fitchburg	6.9[b]	45	24	21	47
Gardner[c]	....[a]	62	34	28	45
Leominster	1.9[a]	43	27	16	37
Marlborough	6.9[a]	76	34	42	55
North Adams	2.4[a]	40	37	3	8
Pittsfield	3.3[a]	54	15	39	72
Worcester	2.8[b]	80	42	38	47

[a] Sedimentation tank. [b] Imhoff tank. [c] Templeton area.

AERATION PRIOR TO SKIMMING

The separation of grease and oil from sewage by flotation is facilitated by a short period of pre-aeration, although, as stated in the report of the Boston experiments of 1935–1936 (14), there is some doubt whether pre-aeration materially increases the over-all removal of grease from sewage by settling tanks. The assistance of aeration in flotation may be due in part to the de-emulsifying of the oil, which is thus separated from the sewage, and in part to physical action, as the grease adheres to the small air bubbles and is carried to the surface with them.

Several systems have been developed for the removal of grease and oil by aeration. Karl Imhoff, M. Am. Soc. C. E., has built combined aerating and skimming tanks at several plants in the Ruhr District in Germany, using detention periods of from 3 to 5 min. Tanks of this type have been installed at Pasadena, Calif., and Springfield, Ill. The tank in the latter city, which was placed in operation in 1931, is 16 ft by 35 ft in plan, with a double row of air-diffuser plates through the center at the bottom and two longitudinal baffles, which form a stilling chamber along each side for the accumulation of grease. The detention period is 8 min at the average flow of 7.5 mgd (16).

At Whittier, Calif., there is an aeration chamber, 40 ft long, 10 ft wide, and 9 ft deep, set in front of a skimming chamber, 10 ft square, in which the scum is collected in a trough while the effluent is drawn off at the bottom. The aeration system is of the ridge-and-furrow type. At Hamilton, N. Y., there is an aerated grease-separating tank, in which the grease settles in pockets at one end and is removed by means of 2-in. suction pipes (17).

Recently tanks or channels have been installed at some sewage treatment plants, to provide a short period of aeration preceding treatment in sedimentation tanks equipped with skimming devices. After passing through the pre-aeration tanks, the grease, oil, and other floating matter are collected and removed by the skimming devices within the sedimentation tanks. For example, at the District of Columbia plant the sewage passes through three aerated grease separating tanks, each 15 ft wide, 78 ft long, and 15.5 ft in maximum water depth. Each tank is equipped with two rows of diffuser plates along one side. With all three tanks in operation, the detention period in the grease-separating units is designed to be 4 min at an average flow of 130 mgd. After the grease-separating treatment, the sewage flows into twelve sedimentation tanks with an average detention period of 2 hr, equipped with clarifier mechanisms for sludge removal and skimming baffles for removing scum.

At San Antonio, Tex., and Lancaster, Pa., air-diffuser plates have been installed in the distribution channels leading to the preliminary sedimentation tanks. The sewage is subjected to a short period of aeration and the oil, grease, and other floating matters are skimmed mechanically from the surface of the tanks.

In experiments at Los Angeles, Calif., aeration was found to aid flotation of grease and oil and form a thick, tenacious scum that was easily removed from the grease skimmer (18). Aeration was accomplished by discharging small streams of screened sewage into the inlet channel at high velocity, the nozzles being placed 2 in. above the surface of the sewage. Removals of 50% to 75% of the free grease were obtained with detention periods of from 2 to 3 min.

At Rochelle, Ill., there is a combined grease-separating and flocculation tank, equipped with an "American" downdraft mechanism, in which the sewage is treated with diffused air in the presence of gentle agitation. After passing through this tank, which provides a detention period of 20 to 30 min, the sewage enters two preliminary sedimentation tanks, equipped with skimming mechanisms. A large proportion of the separated grease is skimmed directly from the surface of the separating tank. Removals of grease, determined by ether extraction, ranging from 48% to 69%, have been reported for the grease-separating and primary sedimentation tank units at this plant.

At the Richmond-Sunset sewage treatment plant in San Francisco, the sewage passes through two "grit-grease" tanks—aerated Imhoff-type tanks—which provide a detention period of 10 min (19). The grease is carried by water sprays to a movable trough and thence to a sump. Following the "grit-grease" tanks are two mechanical flocculation tanks with a detention period of 30 min, the effluent from which flows into two sedimentation tanks with a detention period of 1.5 hr at the average flow. Some of the grease is liberated in the inlet channel to the sedimentation tanks. Scum forming on the sedimentation tanks themselves is removed by water sprays. During an investigation in January, February, and March, 1940, comprising 29 tests on day sewage, the removal of grease, as determined by chloroform extraction, ranged from about 57% for raw sewages containing 115 ppm of grease to about 46% for those containing 60 ppm of grease.

According to Mr. Gascoigne, there were two grease catchers in operation at the Westerly plant in Cleveland during 1933, treating 5 to 20 mgd (12). One of these tanks was of the Imhoff type and the other was similar in design, except for the inside baffling arrangement. Air was supplied to the influent channels and tanks at a rate of 0.023 cu ft per gal. A study of the ether-soluble matter at this plant for two months in 1933 showed the following results (in parts per million):

FIG. 1.—AT THE RICHMOND-SUNSET SEWAGE TREATMENT PLANT IN SAN FRANCISCO, THE SEWAGE PASSES THROUGH TWO GRIT-GREASE TANKS

Total ether-soluble matter in raw sewage	70
Total ether-soluble matter in settled sewage	46
Removal (34%)	24

This grease was removed in the following forms:

Skimmings from Imhoff tanks	10.5
Skimmings from grease catchers	2.5
Attached to sludge particles	11.0
Total	24.0

Data on the removal of grease at the sewage treatment plant in Dodge City, Kans., during four months of 1932 have been furnished by F. C. Roe (20). At this plant the aeration period was designed to be approximately 13 min. Based on four monthly composite samples, the removal of grease ranged from 44% to 63%, while the quantity of grease in the raw sewage ranged from 25 to 216 ppm.

AERO-CHLORINATION

To improve the recovery of grease from wool-washing wastes, Jean deRaeve used chlorine (21). He found that the fats were in an emulsified state, protected by a gelatinous nitrogenous colloid, and that chlorine destroyed this protective colloid, allowing the liberated grease to be recovered by flotation and skimming.

An experiment on the use of chlorine as an aid in the removal of grease from sewage was undertaken by H. A. Faber at the sewage treatment plant in Woonsocket, R. I. (22). Here the sewage has a high grease content because it receives wastes from such industries as wool scouring, rayon manufacture, and weaving processes. Gaseous chlorine was injected directly into the compressed air line serving the grease-removal tank. The experiment consisted of aeration with chlorinated air for one week and aeration in the usual manner for the following week. At the end of each 24-hr period the total grease collected in the form of scum on the tank was removed and weighed. The results indicated that a gaseous chlorine dose averaging less than 1.5 ppm effected increased grease removal by skimming, the quantity removed with chlorine and air ranging from 189% to 442% of that removed with air alone.

A group of similar experiments conducted by C. E. Keefer, M. Am. Soc. C. E., and E. C. Cromwell at Baltimore, Md., likewise showed increased grease removals by skimming where chlorine was used. With chlorine injected into the air pipe serving the grease-removal tank, removals ranged from 149% to 847% of the removals when air alone was used (23).

At the South plant in Lancaster, Mr. Wiest tried a form of aero-chlorination under plant-scale conditions in 1938 (24). A chlorinator was set up to feed chlorine in solution directly to the raw sewage just ahead of air diffusion in the pre-aeration channel. The sewage flow was maintained at 4.0 mgd and chlorine was applied at 2.0 ppm. Three tests were run for at least five days each—in the first, neither air nor chlorine was used; in the second, aeration alone was used; and in the third, aero-chlorination was utilized. The results were gaged by determining the grease content of the raw sewage and the effluent from the primary sedimentation tanks, using petroleum ether as a solvent. The removal of grease by sedimentation and skimming during the three tests was as follows:

Description	Percentage removal of grease
Without aeration or chlorination	38
With aeration alone	50
With aero-chlorination	80

During 1939 the aero-chlorination process was placed in operation at the Westerly plant in Cleveland, Ohio, as well as at the sewage treatment plant in Bellefontaine, Ohio. At Dallas, Tex., a treatment plant containing a grease-removal unit was placed in operation in 1940 (25). The grease-removal unit provides a total detention period of 8 min for design flows. During the first 4 min air alone is applied, and the second 4 min consist of aero-chlorination. The ridge-and-furrow type of aeration was selected. The separated grease is swept by flat water sprays to an area of the grease-removal unit, from which it is automatically removed to a scum box at time-clock intervals.

At present (1941) insufficient operating data are available to form a basis for a sound conclusion in regard to the value of aero-chlorination in the over-all removal of grease from sewage.

Other Methods of Grease Removal

For many years sewage treatment plants for residential and institutional sewage have been protected by grease traps, for collecting grease and soap from kitchen and laundry wastes before they are discharged to the sewer. The method is to pass the wastes through a reservoir of sufficient capacity to reduce the temperature of the hot wastes, thus permitting the grease to separate and rise to the surface, whence it may be removed by skimming. During the previous World War, many grease traps were constructed at army camps, where the practice was to clean each trap once a week or every other week. At many of the present National Defense projects, a grease interceptor is being installed for each kitchen sink within the building. The size of the grease interceptor is based on the rate of flow to be passed through it.

The removal of grease from sewage by acid treatment has been practiced at Bradford, England, for many years. Here the sewage contains large amounts of grease due to wool scouring wastes discharged into the sewers. The grease is recovered and put upon the market.

The Miles process of sewage treatment consists of the addition of acid to precipitate solids in the form of sludge which can be dried and degreased, thereby producing for sale both fertilizer and grease. The process has not been adopted in the United States for treatment of domestic sewage, although acid precipitation is used for the treatment of certain kinds of industrial wastes. Very high removal of fats by acid precipitation was reported by R. S. Weston, M. Am. Soc. C. E., and E. S. Dorr in experiments at Boston, Mass., the ether-soluble matter in the sewage being 303 lb per million gallons and in the sludge 430 lb per million gallons (26). This increase was attributed to the decomposition of soaps by the acid. In experiments at New Haven, Conn., C. E. A. Winslow and F. W. Mohlman recorded a removal of approximately 100% of the grease by acid treatment (27).

Disposal of Grease

Grease is removed from sewage in two forms—namely, sludge and scum. Except in the case of acid treatment, the grease which settles with the sludge is disposed of with the latter. Grease removed from sewage by acid treatment may be recovered. At Bradford, England, part of the grease is converted into such products as soap powders, metallic soaps, lubricants, and pitch (28). Results obtained with a pilot plant led to the design of a full-scale plant, which was under construction in 1938, to handle about one third of the output of raw grease.

Grease and oil removed from sewage in the form of scum may be disposed of by burial, incineration, or digestion with other sewage solids. In some places the oil skimmed from sewage has been used on roads.

The materials collected on the surface of the skimming-detritus tanks at Akron are drawn to small draining beds, and the drained material is disposed of by burial with the dewatered Imhoff sludge. At the plant of the Rahway Valley Joint Meeting in New Jersey, the grease removed from the settling tanks is flushed to a grease-separation chamber, in which the grease is separated from its carrying water by flotation (29). It is then scraped into buckets and buried, although it may be burned in the screenings incinerator. The scum from the skimming-detritus tanks at Allentown and Dayton is incinerated with screenings.

Although troubles of various kinds have been reported, due to the disposal of grease and oil by digestion with sludge (2) (5) (13), the fact remains that it is now common practice to discharge grease, removed from sewage in the form of scum, into separate sludge-digestion tanks for disposal. Among the sewage treatment plants where this method of disposal is used are those for the District of Columbia, and New Rochelle, N. Y.; and the Terminal Island plant at Los Angeles and the Richmond-Sunset plant at San Francisco.

At the District of Columbia plant, where grease is digested with sludge in heated digestion tanks, equipped with fixed covers, no trouble has been experienced in operation. At Rochelle, Ill., where grease is introduced into a heated sludge-digestion tank, equipped with a floating cover, some trouble has been caused by formation of a thick layer of scum in the tank, at one time reaching a thickness of 9 ft. The operator has found that trouble can be avoided by drawing supernatant liquor from a very high elevation at frequent intervals.

At the Terminal Island plant, grease was formerly discharged upon sand beds, dried and burned with the aid of surplus gas from the digester (30). Since this method of disposal proved cumbersome and laborious, it has been superseded by that of discharge into the digester. At the Richmond-Sunset plant, grease is discharged into the sludge-thickening tanks, whence the combined ground screenings, grease, and sludge are pumped into the primary unit of a two-stage sludge-digestion system, equipped with turbo-mixers, heating coils, and a stationary steel dome (19).

CONCLUSION

The grease content of the sewage is an important consideration in the problem of sewage treatment and disposal.

The quantity of "grease" found in sewage by the present standard method of analysis is largely dependent upon which of the three standard solvents is used for extraction. Differences in the solvent used may lead to confusion when an attempt is made to compare the grease content of sewages of different cities and, in some cases, that of the sewage of a given city for different years. Furthermore, the standard method of determination of grease in sewage has not proved entirely satisfactory, aside from the choice of solvent which is allowed. Consequently, the following steps appear to be desirable:

(1) Agreement upon an answer to the question as to what substances should be included in the term "grease";

(2) Adoption of a standard method which is most suitable for determination of grease in sewage as thus defined; and

(3) Accumulation of data on grease in sewage, effluents, and sludge based upon this standard method.

The data presented indicate that a considerable proportion of the ether-soluble matter in sewage can be removed by sedimentation and plain skimming. Moreover, the available data show, in general, that a somewhat larger proportion can be removed if the sewage is aerated prior to sedimentation and skimming. Experiments indicate, furthermore, that aero-chlorination prior to sedimentation and skimming accomplishes a still larger removal of grease. At the present time, however, insufficient operating data are available to form a basis for a sound conclusion relative to the value of pre-aeration and of aero-chlorination in the over-all removal of grease from sewage.

APPENDIX

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DISCUSSION

Arthur D. Weston,[4] M. Am. Soc. C. E.—The need for adequate means of grease removal in the partial treatment of sewage is well illustrated by the discharge into Boston Harbor and its tidal estuaries of some 244 mgd of sewage from a contributing population of about 1,840,000 persons in the Boston metropolitan area from three different outlets. The sewage from these outlets is chiefly of domestic origin and during the period 1935–1936 had a total suspended solids content from about 119 to 256 ppm, or 0.13 to 0.28 lb per capita per 24 hr; a 5-day B.O.D. from about 119 to 249 ppm, or 0.13 to 0.28 lb per capita per 24 hr; and ether-soluble fats from about 34 to 83 ppm, or 0.04 to 0.09 lb per capita per 24 hr. The diluting body of water into which this sewage is discharged is typical of the salt water of the harbors along the New England coast, is reasonably free from pollution except that from the sewers of the Metropolitan Sewerage districts and the Boston Main Drainage District, and has a chloride content of about 17,000 ppm. Oxygen tests of the harbor waters have shown ample oxygen. The sewage from the Boston Main Drainage District is discharged at the surface on the second and third hours of the outgoing tide from storage tanks, whereas the sewage from the North and South Metropolitan Sewerage districts is discharged continuously through outlets located from 30 to 50 ft below mean low tide. Definite nuisances have prevailed from bacterial pollution of the waters by the sewage and from slick or sleek fields.

Studies made under the direction of the writer (31) in 1935 and 1936 showed sleek areas as large as 2,940 acres in connection with the Boston Main Drainage System, 2,050 acres in connection with the North Metropolitan Sewerage System, and 6,820 acres in connection with the South Metropolitan Sewerage System. These areas are not only esthetically objectionable because of the use of the harbor waters for yachting and recreation, but the floating solids, consisting of wood, oil, grease, soap, and other debris, especially the grease and oil (which are characteristic of sewage and commonly found along the edges of the sleek areas), have been known to reach shore areas highly developed as summer resorts and bathing beaches, and areas normally well suited for the growing of shellfish. The authors cite Mr. Wright (3) as reporting grease balls, presumably from the outfall sewers of Boston, found at an isolated point 66 miles from Boston Harbor (see heading "Effects of Grease on Receiving Waterways").

With a view to remedying these conditions, numerous studies have been conducted to ascertain the most practicable means of treating and disposing of the sewage from the three sewerage systems. These studies included certain experiments to determine methods of removing grease, oil, and other matters from the sewage but with no thought in mind of salvaging the fats. Salvaging matters from sewage was investigated by the Boston Main Drainage District under Edgar S. Dorr and Robert Spurr Weston in 1911 (as mentioned by the authors under "Other Methods of Grease Removal"), and by the Massachusetts

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[4] Director and Chf. Engr., Mass. Dept. of Public Health, Boston, Mass.

Department of Public Health under a resolve of the Massachusetts Legislature in 1917, particularly in regard to a patented process that involves (32) the addition of sulfuric or sulfurous acid to the sewage to precipitate the solids. H. W.

TABLE 7.—AVERAGE RAW SEWAGE ANALYSES
(Parts per Million)

Sewerage system	No. of samples	Suspended solids	B.O.D.	Kjeldahl nitrogen	Oxygen consumed	Albuminoid ammonia	Free ammonia	Chlorides	Fats
Boston Main Drainage:									
Calf Pasture, 1935	230	162	157	9.1	53	4.1	21.5	4,080	50
Moon Island, 1936	72	119	144	8.6	50	....	18.1	3,478	34
North Metropolitan:									
Deer Island, 1935	81	256	249	16.4	81	7.5	34.1	3,280	83
Deer Island, 1936	105	148	175	11.2	69	5.3	24.4	3,151	41
South Metropolitan:									
Nut Island, 1936	64	156	119	9.6	50	3.71	20.3	198	49

Clark, at that time director of the Water and Sewage Laboratories and the Lawrence Experiment Station of the Massachusetts Department of Public Health, concluded that the recovery of grease from sewage was not economical and that the removal of organic matter by plain sedimentation was nearly as good as by the aforementioned patented process, and considerably cheaper.

TABLE 8.—RESULTS OF PLAIN

Sewage of North (N) and South (S) Metropolitan Sewer-

Description	One-Hour Detention			One-Hour Detention			Two-Hour Detention		
	N	B	S	N	B	S	N	B	S
No. of Samples[a]	14	19	10	16	9	6	15	9	10
Suspended Solids:									
Raw (ppm)	170	153	195	176	159	168	171	190	190
Effluent (ppm)	97	108	73	97	92	62	83	93	59
% reduction	42.9	29.4	62.6	44.9	42.2	63.1	51.4	50.9	68.9
B.O.D.:[a]									
Raw (ppm)	218	173	160	224	173	118	183	161	134
Effluent (ppm)	181	143	117	167	143	81	138	130	80
% reduction	17.0	17.3	26.9	25.4	17.3	31.4	24.6	19.3	40.3
Kjeldahl Nitrogen:									
Raw (ppm)	13.0	8.9	11.4	12.5	8.7	10.6	11.9	8.7	10.7
Effluent (ppm)	9.8	7.4	6.9	8.5	7.7	6.9	8.0	6.8	6.4
% reduction	24.6	16.9	39.5	32.0	11.5	34.9	32.8	21.9	40.2
Albuminoid Ammonia:									
Raw (ppm)	....	4.21	4.98	....	4.41	3.63	....	4.23	4.00
Effluent (ppm)	....	3.40	2.99	....	3.68	2.70	....	3.34	1.98
% reduction	....	19.3	40.0	....	16.5	25.6	....	21.0	50.5
Oxygen Consumed:									
Raw (ppm)	72	56	54	73	61	54	71	56	59
Effluent (ppm)	62	48	39	61	46	34	58	42	38
% reduction	13.9	14.3	27.8	16.5	24.6	37.0	18.3	25.0	35.6
Fats:									
Raw (ppm)	49	54	59	40	39	46	49	46	48
Effluent (ppm)	30	37	38	28	22	28	28	28	24
% reduction	38.8	31.5	35.6	30.0	43.6	39.1	42.9	39.1	50.0

[a] For exceptions to the number of samples,

The studies (31) made in 1935 and 1936 on the removal of fats from the sewage included small-scale experiments with the use of tanks on the sewage of the

North and South Metropolitan Sewerage districts and the sewage of the Boston Main Drainage District, and large-scale experiments on the sewage of the Boston Main Drainage District. The latter experiments made use of one of the tanks of this district that is normally used to store the sewage for discharge on the second and third hours of the outgoing tide.

The small-scale experimental plant consisted of small tanks 2 ft by 5 ft in plan and 2 ft deep, equipped for fine screening, grease removal, and plain sedimentation. (The tank for screening had a channel 6 in. wide at the influent and effluent ends, which flared to a width of 18 in. at the center where screens of slotted yellow brass 18 in. by 21 in. could be placed.) The screens used in the experiments were of three sizes of slots: $\frac{1}{16}$ in., $\frac{3}{32}$ in., and $\frac{1}{8}$ in. by 2 in. long.

The small tank used for plain fill-and-draw sedimentation tests and for grease removal with aeration was 4 ft by 5 ft in plan and 5 ft deep. This tank was fitted on all sides except the influent side with sloping false bottoms 30 in. on 20 in. It had parallel wooden baffles spaced 1 ft 4 in. apart, dividing the tank into three channels and extending within a foot of the effluent end where a wooden cross baffle spread across the tank. All baffles extended to within about 6 in. of the bottom. The parallel longitudinal baffles had $\frac{1}{8}$-in. metal baffles that could be adjusted to the level of sewage in the tank. The cross baffle at the effluent end extended above the sewage flow. When aeration was required, two removable porous diffuser plates were placed inside the center at the bottom of the center channel, and the sewage entering the tank at the end of

SEDIMENTATION EXPERIMENTS

age Districts and Boston Main Drainage District (B)

THREE-HOUR DETENTION			FOUR-HOUR DETENTION			FIVE-HOUR DETENTION			SIX-HOUR DETENTION		
N	B	S	N	B	S	N	B	S	N	B	S
16	18	9	16	12	7	10	11	2	10	9	5
187	156	150	166	136	136	192	172	121	147	173	145
80	77	55	66	70	51	58	82	28	55	65	58
57.2	50.6	63.3	60.3	48.5	62.5	69.9	52.3	76.9	62.6	62.5	60.0
250	192	126	198	186	115	178	203	84	185	174	114
181	142	76	148	136	73	120	139	42	139	115	71
27.6	26.1	39.7	25.3	26.9	36.5	32.6	31.5	50.0	24.9	33.9	37.7
13.3	9.6	9.2	12.5	8.7	9.1	11.5	9.3	6.0	12.1	9.0	8.3
8.8	6.6	5.8	8.2	6.2	6.0	7.4	6.9	3.8	8.2	5.3	5.8
33.8	31.3	37.0	34.4	28.8	34.1	25.6	25.8	36.7	32.2	41.2	30.1
....	4.40	3.64	....	4.28	3.27	....	4.47	2.15	....	4.27	2.74
....	3.14	2.28	....	2.78	2.04	....	3.18	1.00	....	2.66	2.18
....	28.6	37.4	....	35.0	37.6	....	26.6	53.5	....	37.7	20.4
73	58	51	75	53	47	66	59	41	76	56	43
58	47	33	58	46	32	49	48	28	61	42	31
20.6	19.0	35.3	22.7	13.2	31.9	25.8	18.7	31.7	19.8	25.0	27.9
58	52	57	36	45	51	41	57	25	42	62	51
36	29	30	17	28	31	21	34	15	21	26	36
38.0	44.2	47.4	52.8	37.8	39.2	48.8	40.3	40.0	50.0	58.0	29.4

in the case of B.O.D. and fats, see text.

the center compartment was forced over the longitudinal baffles and under and into the center channel again by a spiral motion. The grease collected on the surface was retained by the cross baffle. Air in the amount of about 0.1 cu ft per gal of sewage was applied.

The larger tank used for plain sedimentation, both for intermittent and continuous-flow experiments, was 4 ft by 7 ft in plan and 6 ft deep. Sewage entered this tank through two elbows that diverted the sewage back against the inlet end of the tank. About 1 ft from the influent end a cross baffle extended to within 1 ft of the tank bottom. A semicircular trough 6 in. in diameter, located about 1 ft from the effluent end of the tank and protected by a skimming baffle, was used to collect the settled sewage. Weir boxes 6 in. by 6 ft in plan and 6 in. deep were fitted with three baffles to smooth the flow at the entrance of the tank, and a triangular brass weir was used to measure the flow. The sewage flow during continuous-flow tests varied from 9 to 35 gal per min.

In the continuous-flow sedimentation tests, samples of the influent were collected as near as practicable to the influent pipe, and samples of the effluent were collected at the weir box. These samples were composite, consisting of ½-pt portions collected every 0.5 hr over a period of 8 hr. The dissolved-oxygen tests during the continuous-flow experiments were on samples collected every 2 hr. The samples collected during the intermittent sedimentation tests consisted of composite samples of the influent collected during the period of filling of the tank and a 2-qt sample of the settled sewage collected at the end of the detention period. The dissolved-oxygen samples during the intermittent sedimentation tests were taken at the completion of the tank filling and at the end of the detention period. During the pre-aeration sedimentation tests, composite samples of ½-pt portions were collected every 0.5 hr over a period of 8 hr from the influent to the degreasing tank, the influent to the settling tank, and the final effluent. Dissolved-oxygen samples were taken at these locations every 2 hr. Table 7 shows the results of average analyses of the samples of raw sewage collected during the tests from the three sewerage systems in question.

Table 8 shows the results of the analyses made during the plain sedimentation tests. This table includes only those samples having an initial suspended solids content of 100 to 300 ppm. The number of samples listed varied, in the case of B.O.D. and fats, as follows (compare Table 8):

Detention time	B.O.D.	Fats
1 hr	17	18
1 hr	..	8
2 hr	7	..
3 hr	17	17
4 hr	10	..

The results of these small experimental plant tests showed that, although the fine screens removed the larger suspended and floating solids, they removed only a small amount of grease or fats, one of the principal causes of complaint of the discharge of sewage into Boston Harbor. Only 5% to 8% of the suspended solids were removed by fine screens as compared with about 50% by sedimentation. The authors mention these tests (see "Methods of Removing

Grease from Sewage: Sedimentation and Skimming") and refer to the raw sewage of the Boston Main Drainage System where about 40% of the 46 ppm of grease could be removed by skimming and sedimentation (14). Pre-aeration of the sewage was not successful in the over-all removal of fats and would indicate that the use of air alone, before sedimentation, in connection with these sewages would not be justified.

One of the existing storage tanks was used for the large-scale experiments on the sewage of the Boston Main Drainage District. This tank, which is trapezoidal in shape with sloping ends, is 162 ft wide and has an average length of 612 ft. In order that it might serve as well as practicable as a sedimentation basin, it was equipped with a dike and skimming baffle, and gates were installed to prevent sewage from flowing out at the bottom of the tank. Five of the outlets at the end of the tank were closed by bulkheads, and the two end openings were closed by sliding wooden sluice-gates. The dike was a triangular-shaped wooden structure about 9 ft high and 16 ft on the two equal sides, and extended symmetrically from the end wall of the basin to enclose one of the existing arched openings. The opening was the only outlet of the tank after bulkheading and gating the other openings, and it acted as an outlet for the tank during the experiments. The top of the dike formed the crest of two sharp-edged weirs each 13 ft long. The skimming baffle was a floating vertical wooden structure 3 ft deep, of which 2 ft were submerged, held 6 ft out from, and parallel to, the sides of the dike by spreaders. The spreaders were so connected as to permit the baffle to rise and fall with the level of the sewage in the tank.

In the first experiments the settled sewage was stored in the other three remaining tanks used for the storage of the sewage for discharge on the second and third hours of the outgoing tide. In later experiments the settled sewage was permitted to discharge into the harbor continuously. Analyses were made of the raw and settled sewage during the experiments and, in the case of the continuous discharge, aerial photographs were taken to show the difference in extent and character of the sleek areas in the harbor resulting from the intermittent discharge of stored raw sewage, and the continuous discharge of settled sewage. Measurements made of the flow of sewage through the experimental tank showed that the average flow was about 60 mgd and that the detention period varied from 1.7 to 5.8 hr, with an average of 2.8 hr. Table 9 shows the results of these settling tests.

TABLE 9.—SETTLING TESTS, MOON ISLAND, BOSTON, MASS.

(Averages of 72 Samples)

Description	PARTS PER MILLION		% reduction
	Raw sewage	Settled sewage	
Free ammonia......	18.1	18.8	....
Chlorides...........	3,478	3,437	....
Oxygen consumed...	49.6	39.7	19.9
Kjeldahl nitrogen ...	8.58	5.94	30.8
Fats...............	34.2	20.0	41.5
B.O.D..............	144	107	25.6
Suspended solids....	119	51.5	56.7

It will be seen from Tables 7 to 9 that the raw sewage during these large-scale experiments was somewhat weaker than the sewage of the small-scale experiments. The samples collected from hour to hour showed that the sewage was strongest during the afternoon and early evening and weakest during the early morning hours. The purification obtained in the larger tank compared favorably with that of the smaller units.

It was found that the amount of scum, garbage, and other floating matter retained by the skimmers was much greater than expected, and rough estimates indicate that as much as 40 cu ft of such material per million gallons of sewage were held back.

Although he was not entirely satisfied with the results of the large-scale experiments (because the tank used was not well suited for sedimentation), the writer was convinced that a comparison of the results of the various experiments with the aerial photographs of sleek areas shows that proper settling and skimming of the sewage will result in improved conditions in the harbor by greatly diminishing the sleek areas.

Tests made during 1940 relative to clarification of the sewage at Nut Island included the use of certain coagulants. The raw sewage in this case had the following average composition:

Description	ppm
Albuminoid ammonia	3.73
Oxygen consumed	33
Turbidity	130
Suspended solids	115
Fats	27

The result of these experiments is given in Table 10. Tests made by stirring some of this sewage for 15 min, followed by 1 hr of settling, showed a 27% decrease in turbidity over settling alone. Later tests were made to determine the effect of stirring the sewage prior to settling. In this case, since the sewage was received at different times, there was a decided difference in its strength, and each run was treated separately. Turbidity, suspended solids, and fats were determined for the raw sewage; for sewage that had settled 1 hr; for raw sewage that had been stirred 15 min and settled 1 hr; and for sewage that had been stirred 15 min with 8% by volume of sludge settled out of the five bottles, and settled 1 hr. Stirring the raw sewage gave an appreciable reduction in suspended solids over that due to settling alone. Stirring with additional sludge showed little effect. The use of the modified, rapid type of filter gave the best results (see Table 11, items 1 to 18). In the tests shown in Table 11, items 19 to 36, samples stirred with 10% (by volume) of returned sludge gave poorer results than stirring the raw sewage alone.

TABLE 10.—PERCENTAGE REDUCTION BY VARIOUS TREATMENTS OF SOME CONSTITUENTS IN RAW SEWAGE AT NUT ISLAND

Description	1-hr settling	4 grains[a] alum	5 grains[a] alum	10.8 grains[a] CaO	14.5 grains[a] CaO	12.6 grains[a] H_2SO_4	Modified rapid type filter
Turbidity	28	78	91	78	88	94	66
Suspended solids	55	75	88	..	..	..	83
Fats	47	80	..	..	..	..	90
Coli-Aerogenes	..	..	90	..	90	90	..

[a] Grains per gallon, plus sedimentation.

Tests were made on samples collected on February 27 and 28, 1940, of the raw sewage and of the sewage after 1 hr of settling. The sludge collected from five 1-gal bottles was settled and elutriated several times with tap water. About 10% by volume of this sludge was added to portions of the different sewages, aerated 10 min, and then stirred 5 min and settled 1 hr.

Another sludge was prepared by coagulating a gallon of sewage from Lawrence, Mass., with alum and collecting the precipitate. This precipitate was then mixed with another gallon of Lawrence sewage and the resulting sludge collected and concentrated; 10% of this sludge by volume was stirred 15 min with the different Boston sewages and then settled 1 hr. This last sludge was intended to show the effect of a return sludge from the precipitation of Boston sewage with alum. Lawrence sewage, however, was used in preparing this sludge because there was not enough sludge from the Boston sewage.

On the whole, stirring Boston sewage with added sludge has shown very little beneficial action over stirring the raw sewage alone, but with some lots of sewage the sludge did show improved results. It was conceded, however, that the action was so uncertain as to make its value doubtful, and there was a tendency for such sludge, after it had gathered up colloidal or fine suspended matter, to give it back later into the sewage being treated. This condition is similar to the condition found at the Lawrence Experiment Station from stirring ferric and manganese hydroxides in water. Table 11, items 37 to 54, shows the results of these experiments.

It was the opinion of George O. Adams, chief chemist at the Lawrence Experiment Station, that more satisfactory results could probably be obtained from these experiments if the period of stirring were longer, preferably twice as long. It also is his opinion that the reason for greater removal by plain sedimentation of the sewage from the South Metropolitan Sewerage System than is found in some other places is due to the presence of salt water in this sewage, resulting in the precipitation of the soluble soap as insoluble calcium soaps, thereby increasing the fats removed by plain sedimentation. He was of the opinion that 20% more fat could be removed by aeration than by stirring, these fats to be removed chiefly in surface scum which could be skimmed off. It was interesting to note that in the plain sedimentation tests there was an average removal of 71% of the suspended solids in 15 min, whereas 15 min of stirring increased the removal to 81%. It was also his opinion that aero-chlorination would be desirable, provided that the results warranted the cost; but it was his opinion that aero-chlorination would have little effect on the subsequent removal of the suspended solids.

Experiments were made in 1934 by the Massachusetts Department of Public Health in connection with the disposal of sewage from the South Essex Sewerage District for the removal of fats. The sewage of this district is from Salem and Peabody, Mass., where there are many tanneries and other industries having. wastes containing considerable quantities of fats; from the City of Beverly, Mass., where the sewage is chiefly domestic in character; and from a large mental health hospital and certain other institutions. The average flow of sewage from this district at the time the tests were made was about 13 mgd (with average flows during weeks of minimum flow of 7.27 mgd and average flows during weeks of maximum flow of 25.80 mgd). It is estimated that the total population served at that time was 94,850. The sewage disposal works of

TABLE 11.—EFFECT OF STIRRING PRIOR TO SETTLING

No.	Description	Raw sewage (ppm)	PERCENTAGE REDUCTION			
			Settling only[a]	Stirred[b]	Return sludge[c]	Rapid filter[d]
	9:00 a.m., February 7:					
1	Turbidity	150	43	61	46	75
2	Suspended solids	150	65	67	63	87
3	Fats	53	58	..	75	83
	11:00 a.m., February 7:					
4	Turbidity	100	37	56	42	72
5	Suspended solids	95	48	62	61	83
6	Fats	30	53	..	67	73
	1:00 p.m., February 7:					
7	Turbidity	135	53	64	63	77
8	Suspended solids	127	71	76	85	86
9	Fats	32	4	..	62	78
	3:00 p.m., February 7:					
10	Turbidity	200	35	55	65	74
11	Suspended solids	179	59	74	84	89
12	Fats	63	8	..	79	72
	4:15 p.m., February 7:					
13	Turbidity	150	38	56	65	77
14	Suspended solids	137	63	72	83	87
15	Fats	70	26	..	84	87
	Average of Five Samples (Items 1 to 15):					
16	Turbidity	147	41	58	56	75
17	Suspended solids	138	61	70	75	86
18	Fats	50	30	..	73	78
	10:00 a.m. to 1:00 p.m., February 13:					
19	Turbidity	130	49	59	42	71
20	Suspended solids	207	81	87	81	87
21	Fats	23	61	48	26	78
	2:00 p.m. to 5:00 p.m., February 13:					
22	Turbidity	130	43	56	48	66
23	Suspended solids	143	78	83	76	78
24	Fats	31	58	77	71	84
	6:00 p.m. to 9:00 p.m., February 13:					
25	Turbidity	260	45	58	57	64
26	Suspended solids	226	78	86	83	85
27	Fats	45	40	53	47	89
	10:00 p.m. to 4:00 a.m., February 14:					
28	Turbidity	205	37	51	54	59
29	Suspended solids	222	75	83	83	86
30	Fats	44	27	55	64	70
	6:00 a.m. to 9:00 a.m., February 14:					
31	Turbidity	155	45	55	55	62
32	Suspended solids	188	80	83	81	84
33	Fats	70	79	79	76	79
	Average of Five Samples (Items 19 to 33):					
34	Turbidity	176	43	56	52	64
35	Suspended solids	197	78	85	82	84
36	Fats	43	56	65	60	79
	9:00 a.m. to 12 Noon, February 27:					
37	Turbidity	105	38	60	45	14
38	Suspended solids	117	71	80	65	55
39	Fats	33	67	42	61	76
	1:00 p.m. to 4:00 p.m., February 27:					
40	Turbidity	150	40	60	59	57
41	Suspended solids	188	76	87	77	74
42	Fats	37	51	84	76	70
	5:00 p.m. to 8:00 p.m., February 27:					
43	Turbidity	270	35	84	56	44
44	Suspended solids	333	80	93	83	80
45	Fats	106	43	82	59	65
	9:00 p.m. to 3:00 a.m., February 28:					
46	Turbidity	300	38	52	67	65
47	Suspended solids	544	83	90	93	87
48	Fats	99	38	38	54	59
	5:00 a.m. to 8:00 a.m., February 28:					
49	Turbidity	225	37	54	64	59
50	Suspended solids	232	66	83	81	78
51	Fats	130	75	78	78	76
	Average of Five Samples (Items 37 to 51):					
52	Turbidity	210	38	62	58	48
53	Suspended solids	282	75	87	82	79
54	Fats	81	55	65	66	69

[a] Raw sewage, settled 1 hr. [b] Stirred 15 min and settled 1 hr. [c] Stirred with return sludge and settled 1 hr.
[d] Raw sewage passed through modified, rapid type of filter.

the district are such that the sewage from Beverly and the institutions is pumped into the same outfall sewer that receives the sewage from Salem and Peabody. The point of outfall is in the outer limits of the Salem-Beverly harbors, about 8,300 ft from shore, where the water at low tide is about 30 ft deep.

The Salem and Peabody sewage combined had a suspended solids content equivalent to about 1.33 lb per capita per day and a fat content of about 1.15 lb per capita per day. The combined sewage from the City of Beverly and the institutions had a suspended solids content of 0.14 lb, and a fat content of 0.05 lb, per capita per day.

The experiments for the removal of fats from the sewage were in connection only with the sewage from Salem and Peabody and consisted of tests to determine the amount of fats removed by fill-and-draw sedimentation, by aeration followed by sedimentation, and by continuous flow with the use of a skimming baffle. Two circular wooden tanks of 2,000-gal and 400-gal capacity were used, into which the sewage was pumped from the trunk sewer of the system at a point about 1 ft below the normal flow line in the sewer. The connecting piping was so arranged that the tanks could be operated in series or in parallel. When air was applied, the average rate of diffusion was 0.18 cu ft per gal of sewage through a filtros plate set in the bottom of the tank.

The fill-and-draw, plain, sedimentation tests without the use of air showed that, with a detention period of 7 to 10 min, the removal of fats was 57.9%, or about 400 lb per million gallons of sewage treated. The amount of fats removed by skimming after 7 min of sedimentation was equal to 0.6% of the total fats in the sewage treated. Tests made with the application of air for a period of 7 min, followed by quiescent sedimentation for an additional 7 min, showed a removal of fats of 57.3%, or 425 lb per million gallons, and the fats removed by skimming were equal to about 2.0% of the total fats in the raw sewage. It should be stated that the tests with and without the use of air are not wholly comparable as the sewage to which the air was applied had a higher fat content than the sewage which received no air application.

In the continuous-flow experiments, a detention period of 37 min was provided by throttling the pump delivering sewage to the tank. Here the fat content of the sewage was 108 ppm as compared with 83 ppm in the fill-and-draw tests and 89 ppm in the test with air. This continuous-flow test showed a removal of fats of 54%, which is equivalent to 484 lb of fats per million gallons of sewage.

In these experiments an average of ten samples of the sludge that settled out in the tanks during various forms of sedimentation contained 16.2% of fats. Tests also were made in connection with the sewage of Salem and Beverly to determine the separation of fats from the sewage by the use of salt and chlorine.

In the salt tests, sodium chloride was added in amounts of 6.75 lb per 100 gal of sewage. The sewage in this case had a fat content of only 46 ppm. After thorough mixing, followed by a quiescent period of 7 min, the effluent had a fat content of 18 ppm, showing a fat removal of 60.9%, of which 2.2% was represented in the skimmings. Using the raw sewage with a fat content of 46 ppm, adding salt at the same rate, and aerating for a period of 7 min with 0.18 cu ft of air per gallon of sewage, followed with a quiescent period of 7 min, the fat content was reduced to 12 ppm, a reduction of 74%, of which 4.0% was represented in the skimmings.

Chlorine in amounts of 10 ppm applied to sewage containing 90 ppm of fats, followed by a quiescent period of 7 min, reduced the fats in the effluent to 34 ppm, a reduction of 62.2%, of which 0.9% only was in the skimmings. Chlorine in amounts of 10 ppm applied to a sewage containing 91 ppm of fats, followed by aeration for 7 min at the rate of 0.18 cu ft of air per gallon of sewage and a quiescent period of 7 min, reduced the fat content to 35 ppm, a reduction of 61.6%, of which 2.4% was in the skimmings. The addition of 46.5 ppm of chlorine to a sewage having a fat content of 30 ppm, followed by aeration for 7 min at the rate of 0.18 cu ft of air per gallon of sewage and a quiescent period of 7 min, resulted in a fat removal of 86.7%, of which 2.6% was in the skimmings.

TABLE 12.—SUMMARIZED RESULTS OF EXPERIMENTS

(Parts per Million)

Treatment	No. of runs	Detention period (min)	Fats				Fats skimmed (% of total)	Suspended Solids				Total Chlorine[a]		Air applied (cu ft per gal)
			PPM		Removed			PPM		Removed				
			Raw	Effluent	%	Lb per million gal		Raw	Effluent	%	Lb per million gal	Raw	Effluent	
(a) Averages														
Sedimentation[b]	23	7	83	35	57.9	400	0.6	569[c]	238	58.2	2,760	...	...	...
Aeration[d]	18	7	89	38	57.3	425	2.0	..	..	...	...	...	...	0.175
Skimming[e]	6[f]	37	108	49	54.4	484	...	633	293	53.7	2,840	...	...	....
(b) Miscellaneous Runs														
Sedimentation[b]	1	10	124	55	55.7	575	...	645	235	63.7	3,420	...	...	....
Plus salt[g]	1	7	46	18	60.9	233	2.2	..	..	...	...	3,600	7,500	....
Plus chlorine[h]	2	7	90	34	62.2	466	0.9	..	..	...	...	...	...	....
Aeration[i]	1	7	30	4	86.7	217	2.6	..	..	...	...	...	...	0.22
Plus chlorine[j]	2	7	91	35	61.6	466	2.4	..	..	...	...	...	...	0.18
Plus salt[k]	1	7	46	12	74.0	283	4.0	..	..	...	...	3,600	6,800	0.18

[a] Chlorides. [b] Fill-and-draw sedimentation. [c] Determined in only two runs. [d] Aeration followed by sedimentation. [e] Continuous flow, with skimming baffle. [f] Excluding certain preliminary runs. [g] Sedimentation with 6.75 lb (per 100 gal of sewage) of common salt added. [h] Sedimentation with 10 ppm of chlorine. [i] Sedimentation before and after aeration, with 46.5 ppm of chlorine added. [j] Sedimentation after aeration, with 10 ppm of chlorine added. [k] Sedimentation before and after aeration, with 6.75 lb of salt per 100 gal of sewage added.

The results of the analyses of the samples collected during these experiments are shown in Table 12 and confirm the results of the experiments made on sewage discharged into Boston Harbor, in that the additional amount of removal of fats by aeration did not appear to warrant the addition of aeration to plain sedimentation of the sewage without further treatment for the removal of fats.

Harvey F. Ludwig,[5] **Jun. Am. Soc. C. E.**—A comprehensive review of the grease problem in sewage treatment is presented in this paper. The authors have necessarily limited their discussion of new analytical procedures for the determination of grease; but they do emphasize that there is considerable current interest in the possibility of revising and improving the old standard methods procedure. The writer recognizes that detailed discussions of the various analytical operations making up the proposed new procedures are of interest chiefly to chemists. However, a brief consideration of the fundamental principles involved in these methods should be of considerable interest to the engineer. Such consideration promotes an understanding of the basic characteristic properties and nature of greasy substances, an understanding obviously prerequisite to an intelligent solution of an engineering grease-disposal problem.

Considerable confusion exists in the literature dealing with the analytical determination of grease, and perhaps most of this arises from the lack of an exact definition of just what the term "grease" should include. Some have questioned, for example, that both organic derivatives (lard, butter, soaps, etc.) and mineral derivatives (kerosene, fuel oils, etc.) should be included; others suggest differentiation between "greases" and "oils," according to whether they are solid or liquid at room temperature. Again, a purely theoretical definition of grease on the basis of chemical constituents leads to confusion because certain substances included by such definition have little or no grease-like qualities. Further, such a variety of substances of greatly varying properties have some grease-like qualities that it is obviously impossible to devise a single test that will recover them all.

What interests the sanitary engineer are those substances which are likely to occur in appreciable quantities in domestic or industrial wastes and which will be of sanitary significance—that is, those substances the particular properties of which are important in sewage treatment and disposal practice. "Substances of sanitary significance" again include a multitude of substances of multi-varied properties. These substances generally have one property in common, however, in that they tend to be removed from suspension and to agglomerate, usually at the surface of the medium (hence their "sanitary significance") in which they were originally dispersed in colloidal state. The writer is of the opinion that this property should form the basis of the accepted method of grease determination; and, further, that having adopted such a method, grease should be defined thereafter as just those substances which are recovered by the method.

The old standard methods (1) procedure for determination of grease consists essentially of strongly acidifying the sample with hydrochloric acid, evaporating the sample to dryness, and then extracting the grease from the dry residue with some solvent such as chloroform or petroleum ether. A Soxhlet apparatus is generally used for making the extraction. The procedure is based on the following assumptions: (1) The operation of acidification converts any non-solvent-soluble greases (calcium and magnesium soaps) to solvent-soluble greases (hydrogen soaps or fatty acids); and (2) the operation of evaporating the sample to dryness (which conveniently removes the water, leaving a dry residue on which the extraction is easily made) does not appreciably alter the state or nature of the grease materials in the system.

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[5] Asst. San. Engr., U. S. Public Health Service, Atlanta, Ga.

Various investigators (9) (33), particularly Messrs. Pomeroy and Wakeman (34), recently have made a critical study of the standard methods procedure, and have discovered that numerous errors result from the operation of evaporating to dryness, because changes in state do occur. These changes may be classified, according to Messrs. Pomeroy and Wakeman, as follows: (*a*) Volatilization of grease; (*b*) reversion of fatty acids to insoluble soaps; (*c*) production of ether-soluble matter (from combined fatty acid radicals without greasy characteristics, as egg yolk); and (*d*) treatment of oils by oxidation and polymerization. Messrs. Pomeroy and Wakeman subsequently devised a novel "wet extraction" procedure (similar to that commonly used in analysis of oil-field wastes and of milk) which eliminates the troublesome evaporation operation. The solvent is added directly to the acidified sample, and the entire mass is shaken thoroughly. After the solvent separates from the remainder of the medium (special technique may be necessary to induce this separation), it is withdrawn, filtered, and evaporated, leaving the residue to be weighed. By this procedure the aforementioned errors are largely eliminated. It should be noted, however, that in both the standard methods procedure and in the Pomeroy-Wakeman method, all of the solvent-soluble materials originally present are recovered, regardless of their grease-like qualities or sanitary significance.

While employed as engineering assistant for the East Bay Cities Sewage Disposal Survey (35), the writer, under the direction of Charles Gilman Hyde, Harold Farnsworth Gray, and A. M. Rawn, Members, Am. Soc. C. E., developed a procedure (33) based on the tendency of grease materials to be removed from suspension and to collect on the surface. This procedure, again an adaptation (being commonly used in soap chemistry), has been called the "boiling-chilling" procedure by the writer, because these two operations are of basic importance. Essentially, the method comprises acidifying the sample, boiling for a few minutes, chilling in the refrigerator (overnight), filtering, drying the filter paper and retained solids, and extracting the grease from these with a suitable solvent, either directly or by means of a Soxhlet. The operation of boiling serves a dual function: (1) To insure complete conversion of insoluble soaps to soluble fatty acids (it is the experience of soap chemists that, in a cold process, conversion is incomplete even under highly acid conditions); and (2) to concentrate the grease in a surface layer or in surface globules (if desired, a large part of the underlying liquid may be drawn off at this point). The operation of chilling serves to congeal the grease globules so that they may be separated later from the system by the operation of filtration. Any solvent-soluble materials which are present in the liquid drawn off after boiling or which are not retained on the filter paper during filtration thus will not appear in the final grease residue.

Messrs. Okun, Hurwitz, and Mohlman (9) subsequently tested this procedure, and developed an improved technique by utilizing certain inert substances to increase mechanically both the rate of filtration of the chilled liquid and the rate of extraction of grease from the dry retained solids. It is interesting to note that the method of grease determination tentatively approved by the Committee on Research of the Federation of Sewage Works Associations in its recent report (36) is the aforementioned boiling-chilling procedure, as improved by Messrs. Okun, Hurwitz, and Mohlman, the final extraction being made in a Soxhlet apparatus with either petroleum ether or hexane.

The question as to which solvent is best suited for extracting the grease from the dry retained solids is a problem in itself. Here again the question of just what is included in the term "grease" is pertinent. The best solvent would be that which most nearly extracts these substances to the exclusion of others. Messrs. Pomeroy and Wakeman have studied the extracting properties of a number of solvents, and recommend hexane as extracting a minimum of non-grease materials and yet having sufficient solvent properties to extract most of the commonly recognized greases.

W. S. MAHLIE,[6] ESQ.—For years sanitary engineers have accepted the fact that there is grease in sewage and that it hinders plant operation. There have been a number of instances in which grease was considered a factor in plant design, but by and large it was not given a very intensive study, and the paper, therefore appears at an opportune time. The question of sewage treatment at army cantonments has focused some attention on this grease problem. Considering the large percentage of grease in raw or primary sludge, it is rather surprising that more work has not been done on the effects of grease in sludge digestion. Messrs. Fales and Greeley comment on the lack of a more standardized method of grease determination. The latest edition of "Standard Methods" contains reference to this subject. The authors are to be complimented on their excellent paper, particularly the part dealing with the methods of grease removal.

DENIS DICKINSON,[7] ESQ.—The question as to which solvent should be used for the determination of grease in sewage is associated closely with the definition of "grease." If the determination is made in order to estimate the difficulties that may arise in the purification of the sewage and the subsequent disposal of the sludge, then there would appear to be sufficient reason for the adoption of chloroform as the solvent, since it is affirmed that this liquid also extracts resinous substances that are likely to be more difficult to break down biologically than the less complex fatty acids. Especially does this apply to those cases in which disposal is left to auto-purification in the sea or in a water course. On the same grounds, chloroform should be adopted for testing the efficiency of any grease removal plant. On the other hand, if it is desired to assess the practicability of grease recovery, then a more selective solvent is preferable.

It is interesting to note that the ratio of the grease as determined by chloroform to that determined by petroleum ether is frequently constant for sludges from any one sewage, although the data available are not sufficient to justify any generalization on this point. It would appear that there is much to be learned from a study of the characteristics of the greases extracted by the various solvents.

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[6] Chemist in Chg. of Water and Sewage Purification, Fort Worth, Tex.

[7] Campden, Gloucester, England; formerly Asst. Chemist, Stockport (England) Sewage Works.

The recovery of grease has been the practice at several works in England for many years, and there are other cases where floating fat is skimmed off (usually by scoops), drained, and sold for the manufacture of wagon grease, etc. The process for the recovery of grease by solvent extraction of the sludge has a further advantage in that the sludge is finally grease-free and weed-free, all seeds having been destroyed during the extraction process (37). The latter is a very important point as it increases considerably the market value of the sludge as a fertilizer.

The practice of burying grease, which is mentioned in the paper, cannot be condemned too strongly, as it is to be feared that in many cases it merely defers the day on which final disposal must be undertaken.

A very large amount of grease is often discharged from slaughterhouses and meat products factories. Much of this can be eliminated by intelligent recovery at the factory. There is a tendency to make factory grease-traps too small; wherever possible they should be of a size such that the intermittent discharge of hot effluent cannot melt and emulsify the fat which normally floats on the surface at the outlet end of the trap.

CHARLES R. VELZY,[8] M. AM. SOC. C. E.—A comprehensive description of the general problems resulting from grease in sewage is presented in this paper, which includes methods of grease determination and removal, the effect upon receiving waterways, and a brief statement regarding methods of disposal. In sewage treatment works, grease disposal is always a problem to the operator. Its magnitude depends upon the size of the plant, the quantity and characteristics of the grease in the sewage, and the facilities of the treatment plant. Brief comments on the handling and disposal of grease at sewage treatment works are offered.

Although skimming of grease from sedimentation tanks is reasonably successful, the skimming equipment becomes coated with grease, it is unsightly, and there is a tendency to produce odors. The equipment is commonly cleaned by scraping. On larger installations special cleaning equipment using a combination of steam and a cleaning compound may be used effectively and with a saving of labor. Frequent cleaning will keep the equipment in satisfactory condition.

After grease has been removed from skimming tanks or sedimentation tanks, it must be transported to the point of disposal. In small plants this can be done by hand. In large plants pumping is probably the most satisfactory method. The difficulties involved result from the characteristic of grease to stick to whatever it comes in contact with, and from the presence of floating trash which comes off with the grease. This combination defies screening, tends to clog pipes and pump valves, and forms a stiff mat when allowed to stand.

Although these conditions cannot be eliminated, a number of things can be done to minimize the difficulties. Trash can be reduced by good screening of raw sewage. Grease can be kept more or less liquid by moving it frequently with sufficient water, using jets or sprays if necessary.

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[8] Asst. Engr., Greeley & Hansen, Norfolk, Va.

In some cases grease will coat the interior of piping through which it flows to such an extent as to increase friction head greatly and necessitate removal. This has occurred at the Buffalo, N. Y., plant and there the application of steam inside the sludge line has kept the line open. It has been necessary to apply steam at intervals of from four to six months. It has been introduced through an injector with water so that the grease is actually melted and removed by hot water. Temperatures of approximately 180° F have been required.

The final disposal of grease will vary with the size of the treatment plant and other conditions. In the small plant where grease is handled manually it can be buried. In the moderate size plants it is sometimes piped to low areas where it is drained and burned. It is common practice to pump it with sludge to digestion tanks. This has the advantage of reducing the quantity, as some types will digest, and also adding to the amount of gas produced. Some grease, however, does not digest, or at least is very slow to digest, in which case it accumulates in the top of the digestion tank. If the accumulation becomes several feet thick, it tends to form a stiff mass, and takes up volume in the tank needed for digestion. This mass of floating material consists of trash with varying proportions of grease and is frequently called scum. If it is of excessive depth, it may become necessary to remove it. If the treatment plant employs incineration for sludge disposal, the obvious procedure is to burn the grease.

There are at least two problems involved in this method of disposal. One is that of handling the grease in this condition and the other is the control of temperature in the incinerator. At Buffalo the scum or grease in the sludge tanks is softened with a spray of supernatant liquor until it will flow and can be pumped. It is then drawn off and mixed with sludge for filtering and incineration. Temperature in the incinerator is controlled by regulating the the proportion of scum and sludge.

In general, problems of handling and disposing of grease must be solved at each sewage treatment plant according to the facilities of the plant and the characteristics of the grease obtained at the plant. Some procedures that have been used with varying degrees of success have been indicated. Grease in sewage, from the standpoints of determination, removal, handling, and disposal, constitutes one of the problems of sewage treatment requiring continued study and research.

F. M. DAWSON,[9] M. AM. SOC. C. E.—In discussing the various methods of grease removal from sewage, the authors made brief mention of removing grease at the source of the sewage by use of interceptors. The writer believes that this method of grease removal merits greater emphasis and consideration than hitherto it has received. This is especially true at present, when it is so vital to save all greases and oils. Naturally, reclaimed grease and oil will be in the best condition for further use if they are removed from sewage as soon as possible.

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[9] Dean and Prof. of Eng., Univ. of Iowa, Iowa City, Iowa.

The common type of grease interceptor has not proved entirely satisfactory in the past, owing to the fact that proper consideration was not given to its design, installation, and cleaning. Hitherto a fixture was installed frequently because some code or regulation required it, and then it was more or less forgotten. However, the basic principle of separating the grease from the water as close to the source as possible is sound. All that was lacking was good engineering in the design and installation of interceptors, and proper maintenance once they were installed.

Contrary to popular belief, the grease is not separated from the incoming waste water, as a result of a reduction in temperature as the water flows through. The grease must be separated in a liquid state since sufficient time is not available for cooling, and also since the temperature of the interceptor is almost always, when in operation, above the congealing temperature of grease. The separation is obtained by securing a proper reduction in velocity of the entering water by use of baffling and by preventing the occurrence of large-scale turbulence. Proper separation depends mainly upon the density difference between the grease and the water. Well-designed interceptors will maintain an average efficiency of 90% for a flow rate equal to 1 gal per min for each 1.0 to 1.5 gal of interceptor capacity. The proper design of interceptors, however, involves the application of considerable experience with, and knowledge of, fluid flow. It is well known that the general problem of separating fluids of different densities is one of importance in many manufacturing plants. The particular case of grease and water is in principle little different from many others, but it is greatly complicated by the many types of wastes carried from the kitchen sink or other sources of waste water.

The government is installing commercial-type interceptors in the kitchens of all cantonments and many other army and navy establishments. To insure that the interceptors will perform properly, each type should have a laboratory test certificate. The Iowa Institute of Hydraulic Research at the University of Iowa, in Iowa City, in cooperation with the Office of the Chief of Engineers and manufacturers of grease interceptors, has developed a laboratory-test procedure and a rating method which indicate quite well the field performance of interceptors. By this procedure interceptors are rated according to the rate of flow, in gallons per minute, which they can handle and still retain, on an average, 90% of the incoming grease. From tests on a great many interceptors, it was decided also that, for the stated flow rate, an interceptor should maintain an average efficiency of 90% up to a grease-retaining capacity equal, in pounds, to twice the flow rate in gallons per minute. Thus, a 25-gal-per-min interceptor should maintain an average efficiency of 90% with a grease capacity of 50 lb.

It is hoped that the war needs for grease and oil will emphasize the many advantages of separating grease and oil at the source, and thus interest more hydraulic and sanitary engineers in the proper design and installation of grease and oil interceptors. All restaurants, hospitals, hotels, and similar types of commercial kitchens should have interceptors near their sinks. For the ordinary restaurant kitchen sink, the flow-rate capacity of any interceptor should not be below 15 gal per min, and preferably not below 20 gal per min. Regular cleaning, at least once a week, is of prime importance.

ALMON L. FALES[10] AND SAMUEL A. GREELEY,[11] MEMBERS, AM. SOC. C. E. —Mr. Weston's discussion of this paper is the result of a large amount of analytical work and experimentation based on a forward looking conception of the problem in Boston Harbor. The data presented illustrate the elusiveness of the grease problem. Table 8, for instance, indicates a near-maximum percentage removal of fats with a 4-hr detention period. For large installations this is a relatively long period and perhaps was one of the reasons for the experiments in stirring, aeration, return sludge, chlorine, and salt. The discussion indicates a leaning toward plain sedimentation for the removal of fats.

Mr. Ludwig has presented an up-to-date discussion regarding the analytical determination of grease. His statement that the determination of grease in the laboratory should be related to design and operation, and should indicate substances of sanitary significance, is a good suggestion for future consideration. It might be well to realize that often much grease is removed with the sludge as well as with the scum. In general, this discussion prompts the suggestion by the writers that operating records could well be kept in the form of a grease balance in which the grease in the raw sewage would be distributed to the grease recovered in scum and sludge plus the grease in the effluent.

Note is made of the discussion by Mr. Mahlie which prompts the suggestion by the writers that those studying the grease problem should refer to his excellent paper (13).

The discussion by Mr. Dickinson states that the practice of burying grease cannot be condemned too strongly. In the opinion of the writers this statement does not apply to the smaller sewage treatment plants with sufficient adjacent land area for disposal by burial. This is a simple method which avoids some of the operating problems of grease disposal in larger plants.

Such operating problems are well described in the discussion by Mr. Velzy. The removal of greasy scum from sedimentation tanks, the operation of pumps for handling the skimmings, and the operations required to keep scum and sludge discharge pipes free of grease, are typical problems requiring attention in design. The formation of scum in sludge digestion tanks is another operating problem. Such problems are accentuated by the presence of small pieces of trash in the scum and sludge which Mr. Velzy indicates can be reduced by good screening. Another problem discussed by Mr. Velzy is the control of temperature in burning greasy scum. These problems indicate that operating analyses or research studies might include determinations of viscosity, congealing characteristics, and the Btu-content.

Dean Dawson calls attention to the grease interceptor which is a device for removing grease at the source of the sewage. The writers doubt that such a method has general applicability in municipal practice, although the device is applicable at places where relatively large amounts of grease are likely to enter the sewers; as, for instance, garages and large kitchens, such as those at hotels and restaurants. Such interceptors appear also to have a place at the relatively large kitchens of army cantonments. Several municipalities have adopted ordinances requiring installation of interceptors at garages and large kitchens. In general, when the disposal of the intercepted grease is difficult, the maintenance of interceptors requires careful and thorough inspection.

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[10] Cons. Engr. (Metcalf & Eddy), Boston, Mass.

[11] (Greeley & Hansen), Chicago, Ill.

Since February, 1942, when the paper was published in *Proceedings*, considerable progress has been made toward a standard method for the determination of grease in sewage and it may be that the next edition of "Standard Methods for the Examination of Water and Sewage" (1) will present such a method. The war year, however, has tended to retard the accumulation of data, on an operating scale, regarding grease in sewage, effluents, and sludge.

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DESIGN OF ST. GEORGES TIED ARCH SPAN

By J. M. Garrelts,[1] Assoc. M. Am. Soc. C. E.

With Discussion by Messrs. R. W. Abbett, Jacob Karol, Alexander Dodge, Carlos A. Bejarano, A. A. Eremin, A. M. Freudenthal, C. H. Gronquist, Harold E. Langley, and J. M. Garrelts.

Synopsis

The procedure used to determine the preliminary and final designs for a tied arch bridge is presented in this paper. Although the type of structure is not new, having been used in Europe, it is believed that the design introduces a new type in American practice. The structure differs from the usual tied arch in that the ratio of the moment of inertia of the tie girder to that of the arch rib is approximately 13 to 1. With these relative physical properties, the total bending moment taken by the rib and girder together is divided between these two elements, the major part being resisted by the girder. (The distribution of this bending moment has been discussed elsewhere by Nathan M. Newmark,[2] Assoc. M. Am. Soc. C. E.)

Introduction

The St. Georges bridge (Fig. 1) is designed to carry the DuPont Highway over the Chesapeake and Delaware Canal at St. Georges, Del. It consists of a 540-ft span over the canal flanked on either side by steel plate-girder and rolled-beam approaches, the over-all length between abutments being approximately 4,200 ft. Selection of a suitable type of structure for the main channel span involved both the essential provision for clearance and a solution that would be satisfactory from the esthetic standpoint.

Provision for full clearance below the floor level for the entire span limited the possible types of structure. Studies made by the architect and the engineers resulted in the selection of a tied arch. With this type, it was possible to extend the main line of the approach girders through the tie girder of the arch and thus, through this esthetically dominating line in the design, to give a

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Note.—Published in December, 1941, *Proceedings*.

[1] Associate Prof., Civ. Eng., Columbia Univ., and Designing Engr., Waddell & Hardesty, New York, N. Y.

[2] *Transactions*, Am. Soc. C. E., Vol. 103 (1938), p. 73.

continuity, a feeling of unity, to the entire construction. Further emphasis of this main element of the esthetic composition was secured by designing the arch rib itself relatively thin, making it appear more as a reinforcement for the tie girder than as the main supporting member.

The writer realizes that this basic idea in the esthetics of the structure may be questioned—other approaches were possible—but, after due consideration, this plan of treatment seemed to be the most promising. It is not the purpose of this paper, however, to develop this phase of the St. Georges design in greater detail.

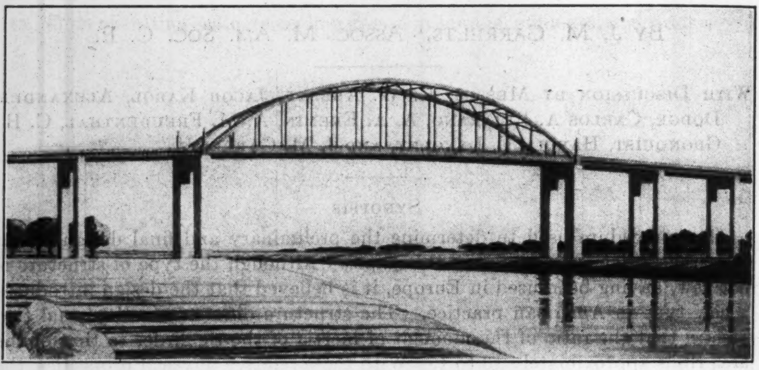


FIG. 1.—ST. GEORGES HIGHWAY BRIDGE

In line with this adopted plan, the tie girder was made the same depth as the approach girders—about 9 ft; and a depth of 3.5 ft was selected for the arch rib. With these dimensions, the tie girder will resist the major part of the bending moments produced by the live load.

Such a structure is multiply indeterminate, and it is the purpose of this paper to outline the method used in arriving at the final design and to describe the completed structure.

NOTATION

The letter symbols in this paper are defined where they first appear and are assembled in the Appendix for convenience of reference.

PRELIMINARY DESIGN

By making some reasonable assumptions which did not introduce an appreciable error in the analysis, it was possible to simplify the problem considerably. In the particular structure being considered, the hangers were rather long and slender, and it was assumed that they would not resist bending moment but would carry only direct stress.

As a first approximation, in the preliminary analysis, it was further assumed that the arch rib resisted no bending moment—that it was subjected only to compression. This assumption implies that the tie girder resists the tension produced in the tie, and also a moment at any vertical section that is equal to the difference between the bending moment due to the external loads and the moment equal to the horizontal component (H) of force in the rib and girder times the vertical distance between the neutral axes of the girder and rib. This can be written as follows (see Fig. 2(a)):

$$M = M' - H\,y \tag{1}$$

in which: H = horizontal component of the rib compression = horizontal component of the tie tension; M = bending moment to be resisted by the tie girder; M' = simple beam bending moment at any vertical cross section A–A (Fig. 2(a)) produced by the external forces. The horizontal component of the rib compression is taken as the redundant. To determine this force H, a statically determinate substitute structure, obtained by cutting the rib at the center C (Fig. 2(b)), is assumed. The principle of virtual work can be used to determine H.

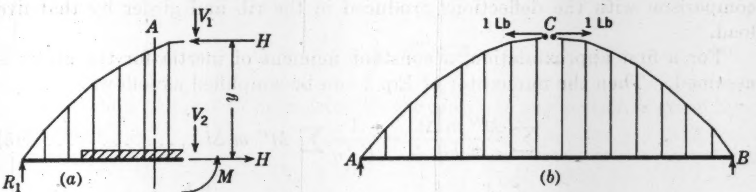


FIG. 2

Let: Δ'_C = relative horizontal displacement of the two cut faces at C, due to the applied load, with $H = 0$; Δ_{1C} = relative horizontal displacement of the two cut faces at C, due to $H = 1$; then

$$H = -\frac{\Delta'_C}{\Delta_{1C}} \tag{2}$$

in which

$$\Delta'_C = \sum \frac{M''\,m\,\Delta l}{E\,I} + \sum \frac{s''\,u\,l}{A\,E} \tag{3a}$$

and

$$\Delta_{1C} = \sum \frac{m^2\,\Delta l}{E\,I} + \sum \frac{u^2\,l}{A\,E} \tag{3b}$$

M'' being the bending moment and s'' the direct stress in any member of the substitute structure ($H = 0$), produced by the applied loads. The symbol m is the bending moment and u the direct stress in any member of the structure due to $H = 1$; l and Δl are the length of any member and an element of length of a member, respectively, used in the summation; E is the modulus of elasticity; A, the cross-sectional area; and I, the moment of inertia of the cross-sectional area of any member.

The formula for H can be written:

$$H = -\frac{\sum \dfrac{M''_G\,m\,\Delta l}{E\,I_G}}{\sum \dfrac{m^2\,\Delta l}{E\,I_G} + \sum \dfrac{u^2\,\Delta l}{A_G\,E} + \sum \dfrac{u^2\,\Delta l}{A_R\,E}} \tag{4}$$

in which A_G and I_G are the area and the moment of inertia of the cross section of the girder, respectively, and A_R is the area of the rib.

To be consistent with the assumption that compression alone acts in the rib, the hanger loads are assumed to be equal, and the load points or points of intersection of the hangers and rib are selected so that they lie on a parabolic curve.

The effects of any deformations in the hangers are neglected. This approximation introduces very small error because the hangers can be fabricated so that they will have the correct length under dead load; and at any section where the moment is appreciable the elongation of a hanger due to live load is small in comparison with the deflections produced in the rib and girder by that live load.

For a first approximation, a constant moment of inertia for the girder is assumed. Then the numerator of Eq. 1 can be simplified as follows:

$$\sum \frac{M'' \, m \, \Delta l}{E \, I_G} = \frac{1}{E \, I_G} \sum M'' \, m \, \Delta l \, \dots\dots\dots\dots\dots (5)$$

Due to $H = 1$, the load in the hangers is equivalent to $\frac{8\,h}{L^2}$ per unit of length. Then

$$m = -\frac{4\,h}{L}\,x + \frac{8\,h\,x^2}{L^2\;2} = -\,y \, \dots\dots\dots\dots\dots (6)$$

using the coordinates indicated in Fig. 3. For the girder $u = 1$, and for the rib $u = 1 \sec \alpha$.

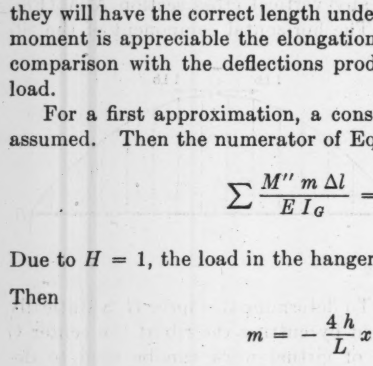


FIG. 3

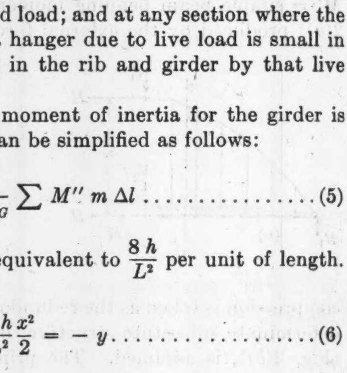


FIG. 4

The summations indicated in Eq. 4 reduce to the following values:

$$\sum \frac{m^2 \, \Delta l}{E \, I_G} = \frac{1}{E \, I_G} \sum y^2 \, \Delta l = \frac{8}{15} \frac{h^2 \, L}{E \, I_G} \dots\dots\dots\dots (7a)$$

$$\sum \frac{u^2 \, \Delta l}{E \, A_G} = \frac{1}{E \, A_G} \sum u^2 \, \Delta l = \frac{L}{E \, A_G} \dots\dots\dots\dots (7b)$$

and

$$\sum \frac{u^2 \, \Delta l}{E \, A_R} = \frac{1}{E \, A_R} \sum \left(1 \times \frac{\Delta l}{\Delta x} \right)^2 \Delta l = \frac{L + 8 \frac{h^2}{L}}{E \, A_R} \dots\dots\dots (7c)$$

In Eq. 7c, A_R is assumed to be constant for this preliminary analysis.

From Eq. 4:

$$H = \frac{\int M'' \, y \, dl}{\frac{8}{15} h^2 L + \frac{I_G L}{A_G} + I_G \frac{\left(L + 8 \frac{h^2}{L}\right)}{A_R}} \qquad (8a)$$

The denominator of Eq. 8*a* depends only on the dimensions of the structure, and an examination of the terms indicates that the first term is large in comparison with the other two. Using only the first term, a simplified formula for *H* is obtained:

$$H = \frac{15}{8 \, h^2 \, L} \int M'' \, y \, dl \qquad (8b)$$

The bending moment to be resisted by the girder at any section is given by the equation

$$M_G = M'' - H \, y = M'' - \frac{15}{8} \frac{y}{h^2 \, L} \int M'' \, y \, dx \qquad (9)$$

Using Eq. 8*b*, influence values for *H* were determined by placing a unit load at each panel point in turn, from which the maximum values for the direct stresses in the rib and girder were computed. Influence values for M_G due to unit load at each panel point were computed, using Eq. 9, and with the aid of these coefficients the maximum live-load moments in the girder were determined. With these preliminary direct stresses and moments, sections were determined which were used as a basis for making the more exact analysis.

The total bending moment *M*, which was assumed to be resisted entirely by the tie girder in the preliminary analysis, is actually taken by both the rib and girder. It is necessary then to develop a method for determining the proportion of this total moment taken by the rib and by the girder.

In order to proportion this bending moment at any section between the arch rib and tie girder, the relation between the vertical components of deflection of the rib and girder at any section will be used.

At any point *P* in the arch rib (Fig. 4), let *v* represent the component of deflection along a normal to the arch rib curve at that point. From the differential equation for the deflection of a curved beam, the normal component of deflection *v* is

$$\frac{d^2 v}{dl^2} + \frac{v}{\rho^2} = \pm \frac{M_R}{E \, I_R} \qquad (10a)$$

in which *dl* is an element of length of the arch rib; ρ is the radius of curvature of the rib at point *P*; M_R is the moment taken by the rib; and I_R is the moment of inertia of the rib.

The second term of Eq. 10*a* can be neglected without appreciable error, and the equation reduces to

$$\frac{d^2 v}{dl^2} = \pm \frac{M_R}{E \, I_R} \qquad (10b)$$

The vertical component of the deflection of the arch rib is equal to the vertical component of v, if the tangential component of deflection at P (which is small) is neglected; then

$$\eta = v \cos \alpha \text{ .... } (11)$$

in which η is the vertical component of v, and α is the angle between the tangent at P and the x-axis.

The first term of Eq. 10b can be replaced by an expression involving η, x, and α as follows: From Eq. 11—

$$\frac{d\eta}{dl} = \frac{dv}{dl} \cos \alpha - v \frac{d\alpha}{dl} \sin \alpha; \qquad \text{and} \qquad \frac{d^2\eta}{dl^2} = \cos \alpha \frac{d^2v}{dl^2}$$

$$- 2 \frac{dv}{dl} \frac{d\alpha}{dl} \sin \alpha - v \frac{d^2\alpha}{dl^2} \sin \alpha - v \left(\frac{d\alpha}{dl} \right)^2 \cos \alpha$$

$$= \cos \alpha \frac{d^2v}{dl^2} - 2 \frac{dv}{dl} \frac{1}{\rho} \sin \alpha + \frac{v}{\rho^2} \frac{d\rho}{dl} \sin \alpha - \frac{v}{\rho^2} \cos \alpha \text{ .... } (12)$$

Neglecting the last three terms of Eq. 12:

$$\frac{d^2v}{dl^2} = \sec \alpha \frac{d^2\eta}{dl^2} \text{ (approximately) .... } (13)$$

Using $dx = ds \cos \alpha$, Eq. 13 reduces to

$$\frac{d^2v}{dl^2} = \cos \alpha \frac{d^2\eta}{dx^2} \text{ .... } (14a)$$

or

$$\pm \frac{M_R}{E\,I_R} = \cos \alpha \left(\frac{d^2\eta}{dx^2} \right) \text{ (approximately) .... } (14b)$$

For the girder (Fig. 4)

$$\pm \frac{M_G}{E\,I_G} = \frac{d^2y'}{dx^2} \text{ .... } (15)$$

in which M_G is the moment taken by the girder, and I_G is the moment of inertia of the girder.

The total moment M_T is equal to $M_R + M_G$; and, neglecting the change in lengths of the hangers, $\frac{d^2\eta}{dx^2} = \frac{d^2y'}{dx^2}$. Therefore, using Eqs. 14b and 15:

$$M_T = M_G \frac{I_G + I_R \cos \alpha}{I_G}$$

Then

$$M_G = M_T \left(\frac{I_G}{I_G + I_R \cos \alpha} \right) \text{ .... } (16a)$$

and

$$M_R = M_T \left(\frac{I_R \cos \alpha}{I_G + I_R \cos \alpha} \right) \ldots\ldots\ldots\ldots\ldots (16b)$$

The formula for H becomes

$$H = \frac{\sum \frac{M' \, m \, \Delta l}{I_{tr}}}{\sum \frac{m^2 \, \Delta l}{I_{tr}} + \sum \frac{\Delta l}{A_G} + \sum \frac{\sec^2 \alpha \, \Delta l}{A_R}} \ldots\ldots\ldots\ldots\ldots (17)$$

in which $I_{tr} = I_G + I_R \cos \alpha$.

The Final Design

The span of the arch is 540 ft, with fifteen panels at 36 ft, and the rise is 100 ft. To perform the summations indicated in Eq. 17 the structure was divided into sections, each having a length equal to a panel length. With this equation, influence lines for H were determined as follows:

Load at panel point	Value of H
1	0.2358
2	0.4595
3	0.6625
4	0.8372
5	0.9772
6	1.0772
7	1.1300

The tie girder, instead of being straight, follows the vertical curve of the roadway, and the factor m in Eq. 17 is given by $m = y' - y$, in which y is the ordinate to the arch rib and y' is the ordinate to the girder.

The moment M_T to be resisted by the girder and arch rib together is given by the expression (see Eq. 1)

$$M_T = M' - H(y - y') \ldots\ldots\ldots\ldots\ldots (18)$$

in which M' is the simple beam bending moment. The curves of Fig. 5 show the influence lines for M_T as computed for the final design sections. The maximum live-load moment was computed for the final design sections, and the proportionate part taken by both the rib and the girder determined in accordance with Eqs. 16. Shears were calculated but were found to be very small.

The dead load used for design was 15.1 kips per ft of length of bridge and the H-20 live load was used. The dimensions of the structure are shown in Fig. 6 and an example of the arrangement of the design stresses and sections is shown in Figs. 7 and 8. A dead-load moment of 500 ft-kips was included in the stresses at each point to allow for the fact that there may be some dead-load moment in the structure after it is under full dead load.

The total weight of steel in the span is 2,166 tons, classified as follows:

Members	Tons
Girders	711
Ribs	547
Hangers	123
Bracing	186
Floor beams	210
Stringers	291
Diaphragms and railing	98

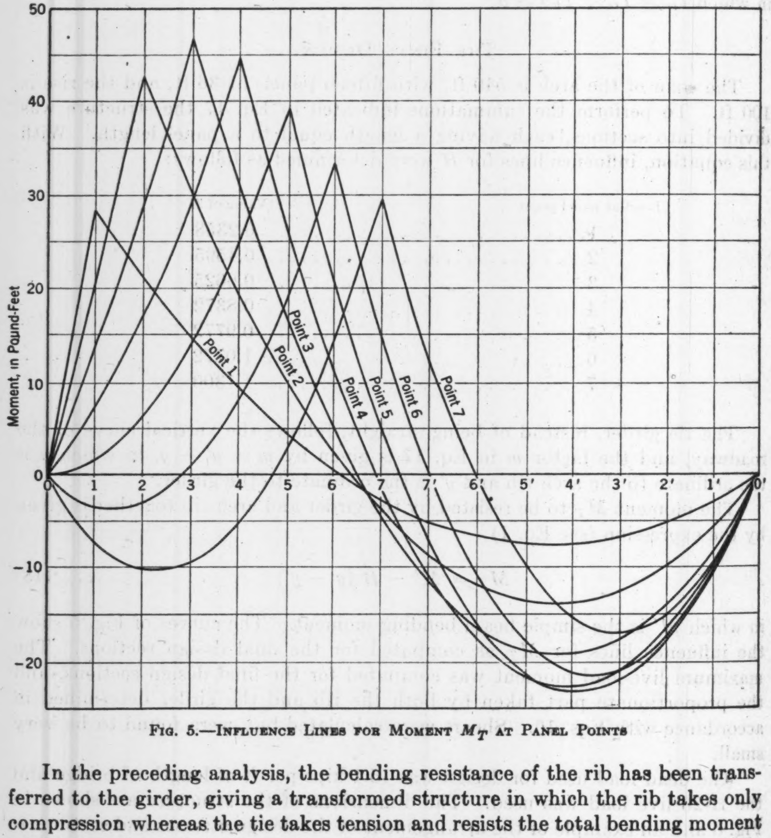


Fig. 5.—Influence Lines for Moment M_T at Panel Points

In the preceding analysis, the bending resistance of the rib has been transferred to the girder, giving a transformed structure in which the rib takes only compression whereas the tie takes tension and resists the total bending moment with a transformed moment of inertia. Theoretically, it would also be possible to analyze the structure as one in which the bending resistance of the girder is transferred to the rib. This would result in a transformed structure in which the tie girder would take only tension, while the rib would take compression

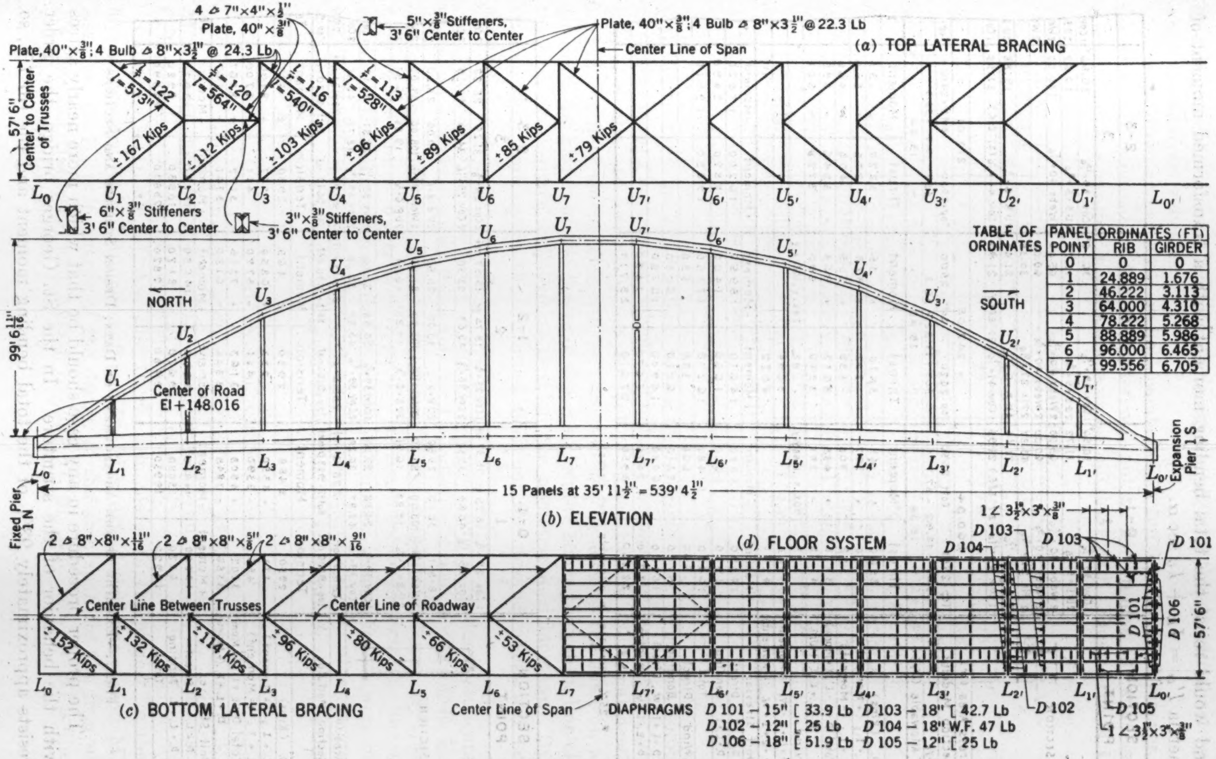
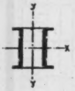

(a) TOP LATERAL BRACING
Plate, 40"×$\frac{3}{8}$"; 4 Bulb ∠ 8"×3$\frac{1}{2}$" @ 24.3 Lb
4 ∠ 7"×4"×$\frac{1}{2}$"
Plate, 40"×$\frac{3}{8}$"
5"×$\frac{3}{8}$" Stiffeners, 3' 6" Center to Center
Plate, 40"×$\frac{3}{8}$"; 4 Bulb ∠ 8"×3$\frac{1}{2}$" @ 22.3 Lb
Center Line of Span
57' 6" Center to Center of Trusses
l/r = 122
l = 573"
l/r = 120
l = 564"
l/r = 116
l = 540"
l/r = 113
l = 528"
±167 Kips
±112 Kips
±103 Kips
±96 Kips
±89 Kips
±85 Kips
±79 Kips
L_0 U_1 U_2 U_3 U_4 U_5 U_6 U_7 $U_{7'}$ $U_{6'}$ $U_{5'}$ $U_{4'}$ $U_{3'}$ $U_{2'}$ $U_{1'}$ $L_{0'}$
6"×$\frac{3}{8}$" Stiffeners 3' 6" Center to Center
3"×$\frac{3}{8}$" Stiffeners, 3' 6" Center to Center
(b) ELEVATION
NORTH
SOUTH
99' 6$\frac{11}{16}$"
Center of Road El+148.016
U_1 U_2 U_3 U_4 U_5 U_6 U_7 $U_{7'}$ $U_{6'}$ $U_{5'}$ $U_{4'}$ $U_{3'}$ $U_{2'}$ $U_{1'}$
L_0 L_1 L_2 L_3 L_4 L_5 L_6 L_7 $L_{7'}$ $L_{6'}$ $L_{5'}$ $L_{4'}$ $L_{3'}$ $L_{2'}$ $L_{1'}$ $L_{0'}$
Fixed Pier 1 N
Expansion Pier 1 S
15 Panels at 35' 11$\frac{1}{2}$" = 539' 4$\frac{1}{2}$"
(c) BOTTOM LATERAL BRACING
2 ∠ 8"×8"×$\frac{11}{16}$"
2 ∠ 8"×8"×$\frac{5}{8}$"
2 ∠ 8"×8"×$\frac{9}{16}$"
Center Line Between Trusses
Center Line of Roadway
±152 Kips
±132 Kips
±114 Kips
±96 Kips
±80 Kips
±66 Kips
±53 Kips
L_0 L_1 L_2 L_3 L_4 L_5 L_6 L_7
Center Line of Span
(d) FLOOR SYSTEM
1 ∠ 3$\frac{1}{2}$"×3"×$\frac{3}{8}$"
D 103
D 104
D 103
D 101
D 101
D 106
57' 6"
$L_{7'}$ $L_{6'}$ $L_{5'}$ $L_{4'}$ $L_{3'}$ $L_{2'}$ $L_{1'}$ $L_{0'}$
D 102
D 105
1 ∠ 3$\frac{1}{2}$"×3"×$\frac{3}{8}$"
DIAPHRAGMS
D 101 – 15" [33.9 Lb
D 102 – 12" [25 Lb
D 106 – 18" [51.9 Lb
D 103 – 18" [42.7 Lb
D 104 – 18" W.F. 47 Lb
D 105 – 12" [25 Lb

TABLE OF ORDINATES

PANEL POINT	ORDINATES (FT)	
	RIB	GIRDER
0	0	0
1	24.889	1.676
2	46.222	3.113
3	64.000	4.310
4	78.222	5.268
5	88.889	5.986
6	96.000	6.465
7	99.556	6.705

FIG. 6.—DIMENSIONS OF THE STRUCTURE

and would resist the total bending moment with a transformed moment of inertia $I'_{tr} = I_R + I_G \sec \alpha$.

SECTION	0-1		.1-2		2-3		
POINT	1		1		3		
SECTION 	1 Cov. 39 × 15/16	36.56□"	1 Cov. 39 × 15/16	36.56□"	1 Cov. 39 × 15/16	36.56□"	1 Cov
	2∟ 5 × 5 × 5/8	11.72	2∟ 5 × 5 × 5/8	11.72	2∟ 5 × 5 × 9/16	10.62	2∟5
	2 Webs 40 × 13/16	65.00	2 Webs 40 × 13/16	65.00	4 Webs 40 × 13/16	130.00	2 We
	2 Webs 40 × 3/4	60.00	2 Webs 40 × 3/4	60.00			2 W
	2 Pls. 27 × 13/16	43.87	2 Pls. 27 × 9/16	30.38			
	2∟ 8 × 6 × 13/16	21.44	2∟ 8 × 6 × 13/16	21.44	2∟ 8 × 6 × 3/4	19.88	2∟
	1 Cov. 41 × 13/16 (27" Eff.)	21.95	1 Cov. 41 × 13/16 (27" Eff.)	21.95	1 Cov. 41 × 13/16 (27" Eff.)	21.95	1 Co
Gross Area, (□")	260.54		247.05		219.01		
Section Modulus	Top · 2705	Bott. 2530	Top · 2620	Bott. · 2500	Top · 2550	Bott. · 2410	Top ·
l/r	527 ÷ 13.92 · 37.7		503 ÷ 13.82 · 36.3		482 ÷ 13.67 · 35.3		
f Allowable (K/□")	19.34		19.40		19.44		
	Thrust	Moment	Thrust	Moment	Thrust	Moment	T
D. (Kips)	3560	± 43	3410	± 35	3270	± 33	
L. + I.; (+ Moment)	210	+ 392	202	+ 328	276	+ 566	
L. + I.; (- Moment)	558	- 308	530	- 255	458	- 468	
Total D + L. + I., (+ Mom.)	3770	+ 435	3612	+ 363	3546	+ 599	
D. + L. + I.; (- Mom.)	4118	- 351	3940	- 290	3728	- 501	
Wind	173	90x-x; 1088y-y	82	74x-x; 1088y-y			
D. + L. + I. + W. (+ Moment)	3943		3694				
D. + L + I + W. (- Moment)	4291		4022				
Unit Stresses	+ Moment	- Moment	+ Moment	- Moment	+ Moment	- Moment	+
D. + L. + I. Direct	14490	15820	14610	15950	16200	17020	
D. + L. + I. Bending	1930	1660	1660	1400	2820	2490	
Total	16420	17480	16270	17360	19020	19510	
Wind Direct	660	660	330	330			
Bending	5780	5810	6010	6030			
D. + L + I. + W., Direct	15150	16480	14940	16280			
D. + L. + I. + W., Bending	7710	7470	7.670	7430			
Total *	22,860	23,950	22,610	23,710			

* 25% increase in allowable unit stress.

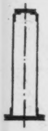
SECTION	0-1		1-2		2-3		
POINT	. 1		2		3		
SECTION 	1 Cov. 25 × 3/4	18.75 gr., 15.37 nt.	1 Cov. 25 × 3/4	18.75 gr. 15.37 net	2 Cov. 25 × 5/8	31.25 25.62	2 Co
	2 Cov. 25 × 13/16	40.63 33.30	2 Cov. 25 × 13/16	40.63 33.30	2 Cov. 25 × 11/16	34.39 28.20	2 Co
	2∟ 8 × 8 × 1	30.00 25.50	2∟ 8 × 8 × 1	30.00 25.50	2∟ 8 × 8 × 1	30.00 25.50	2∟
	2 Webs 108 × 1/2	108.00 81.00	2 Webs 108 × 1/2	108.00 81.00	2 Webs 108 × 1/2	108.00 81.00	2 W
	2∟ 8 × 6 × 3/4	19.88 16.51	2∟ 8 × 6 × 3/4	19.88 16.51	2∟ 8 × 6 × 3/4	19.88 16.51	2∟
	1 Cov. 39 × 5/8	24.38 20.16	1 Cov. 39 × 5/8	24.38 20.16	1 Cov. 39 × 11/16	26.81 22.17	1 Co
	2 Cov. 39 × 9/16	43.87 36.28	2 Cov. 38 × 9/16	43.87 36.28	2 Cov. 39 × 5/8	48.75 40.31	2 Co
Area (□")	285.51 gr.	228.13 net	285.51 gr.	228.13 net	299.08 gr.	239.31 net	29
Section Modulus	Top · 9100 net	Bott. · 9130 net	Top · 9100 net	Bott. · 9130 net	Top · 9640 net	Bott. · 9770 net	Top · 9
	Tension	Moment	Tension	Moment	Tension	Moment	Te
D. (Kips)	2930	± 458	2930	± 465	2930	± 467	
L. + I.; (+ Moment)	173	+ 4350	233	+ 6850	247	+ 8100	
L. + I.; (- Moment)	456	- 3390	411	- 5720	411	- 6700	
Total D. + L. + I.; (+ Mom.)	3102	+ 4808	3163	+ 7315	3177	+ 8567	
D. + L. + I., (- Mom.)	3386	- 3848	3341	- 6185	3341	- 7167	
Unit Stresses	+ Moment	- Moment	+ Moment	- Moment	+ Moment	- Moment	+
D. + L. + I. Direct	13600	14850	13860	14650	13280	13970	
D. + L. + I. Bending	6320	5080	9620	8170	10530	8920	
Total	19920	19930	23480	22820	23810	22890	

Fig. 7.—Example of the Arrangement of Data: Design Stresses and Sections

The proper procedure in any case should be that which more nearly agrees with the behavior of the structure. In the St. Georges bridge, the girder resists approximately 93% of the total bending moment and the rib 7%, so that the procedure used is more nearly in agreement with the structural action which the analysis indicates.

The tie girder and arch rib are both cambered for full dead load. However, the dead-load moment in the tie girder will not be zero unless the span is riveted up in the position it will assume under full dead load. Due to the fact that the riveting up will be done before the concrete deck is poured, some method of closing is necessary which will induce a moment in the girder at the time of riveting to neutralize the moment produced by the additional dead load of the floor slab. This will be accomplished by closing the span on a pin in the tie girder, at the center of the span located above the center line of the girder a distance sufficient to develop the predetermined moment. By riveting this joint up in the distorted position, the desired result will be obtained.

HANGERS

MEMBER	U_1L_1	U_2L_2	U_3L_3 U_4L_4 U_5L_5	U_6L_6 U_7L_7
D. (Kips)	230	230	230	230
L.	79	79	79	79
I.	6	6	6	6
D. + L. + I.	315	315	315	315
l/r	204 ÷ 5.10 = 40	516 ÷ 5.14 = 101	994 ÷ 5.93 = 168	1112 ÷ 6.69 = 166
f Allowable	24000 #/□"	18000 #/□"	18000 #/□"	18000 #/□"
Section Area Req.	13.12 □" net	17.50 □" net	17.50 □" net	17.50 □" net
Section Used	2-12" Ship ⊏ 30.9	2-12" Ship ⊏ 30.9	2-13" Car ⊏ 31.8	2-15" ⊏ 33.9
	2 Pls. 23 × ½ (17" Eff.)	2 Pls. 23 × ½ (17" Eff.)	2 Pls. 23 × ½ (17" Eff.)	2 Pls. 23 × ½ (17" Eff.)
Area □"	35.00 gr. 26.80 nt.	35.00 gr. 26.80 nt.	35.60 gr. 27.85 net	36.80 gr. 28.10 net
Material	Silicon	Carbon	Carbon	Carbon

NOTE: Hangers participate partially with rotation of floorbeams due to Live Load.

FIG. 8.—EXAMPLE OF ARRANGEMENT OF DATA: CHARACTERISTICS OF HANGERS

A model analysis was made of the St. Georges arch by graduate students at Princeton University at Princeton, N. J., under the direction of Prof. E. K. Timby, Assoc. M. Am. Soc. C. E. The "Beggs Deformeter" was used to measure the strains in a celluloid model. Influence values for both the direct stress and the moment computed from the deformeter measurements were in close agreement with those calculated by the theoretical analysis. The results of these tests serve to confirm the correctness of the assumptions made in the design analysis.

ACKNOWLEDGMENTS

The St. Georges bridge is a project of the Corps of Engineers of the U. S. Army under the supervision of Lt.-Col. Harry B. Vaughan, Jr., M. Am. Soc. C. E., district engineer of the Philadelphia District.

Maurice N. Quade, M. Am. Soc. C. E., is in charge of the project for the firm of Parsons, Klapp, Brinckerhoff and Douglas; Alfred Hedefine, Assoc. M. Am. Soc. C. E., is in charge of the design of the arch span for the firm of Waddell and Hardesty; and Aymar Embury II, M. Am. Soc. C. E., is the consulting architect.

APPENDIX

NOTATION

The following letter symbols, used in this paper, conform essentially to American Standard Letter Symbols for Mechanics, Structural Engineering, and Testing Materials, prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1932:[3]

A = area of cross section:
  A_G = area of girders;
  A_R = area of rib;
E = modulus of elasticity;
H = horizontal component of rib compression and tie tension;
h = rise of the arch rib;
I = moment of inertia:
  I_G = of girder;
  I_R = of rib;
L = length of span;
l = length of panel, or other structural member; Δl = a segment of l;
M = bending moment to be resisted by the tie girder:
  M' = simple beam moment at any vertical cross section (section A–A, Fig. 2(a)) produced by the external forces;
  M'' = moment in any member of a substitute structure;
  M_G = moment to be resisted by the girder at any section;
  M_R = moment resisted by the rib;
  M_T = total moment;
m = bending moment in any member of a structure due to $H = 1$;
s'' = stress in the substitute structure ($H = 0$), produced by the applied loads;
u = direct stress in any member of a structure, due to $H = 1$;
v = normal component of the deflection;
x = horizontal coordinate distance to any point on arch rib or girder;
y = vertical ordinate to any point on arch rib;
y' = vertical ordinate to any point on tie girder;
Δ = total deformation = relative horizontal displacement of two cut faces:
  Δ' = displacement when $H = 0$;
  Δ_1 = due to $H = 1$;
η = vertical component of v;
α = slope of the arch rib at any point = angle between the tangent at point P, and the x-axis;
ρ = the radius of curvature of the rib at point P.

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[3] ASA—Z10a—1932.

DISCUSSION

R. W. Abbett,[4] **M. Am. Soc. C. E.**—A clear and concise statement of the basic theory of the design of an unusual type of bridge superstructure is presented in this paper. The fact that the principles of mechanics that are involved are not new to structural engineers does not detract from the usefulness of the explanation of their unique application. In all probability, however, the author's greatest contribution lies in the original procedure that he has developed for the preliminary design. The dimensioning of a statically indeterminate structure to initiate a stress analysis is a critical matter as far as labor and time are concerned, and the systematic procedure now available will be most helpful to future designers of similar arches.

As far as the writer is aware, the St. Georges arch represents the first of its type in the United States, but several such structures were built in Europe before the beginning of the present war. It has certain definite architectural advantages when combined with deep girder approach spans, such as were possible and economical in this instance. A great deal of consideration was given to the appearance of the bridge because of its dominating location on Delaware's main highway.

As to the economy of the design, a comprehensive study of the cost of the arch span was made in comparison with other possible layouts for the St. Georges crossing, but few data are available for other span lengths. For spans of the order of 540 ft, the cost was estimated to be about the same as that of a Warren-type simple truss of the same length, and when combined with the girder approach spans this layout produced the least over-all cost of any considered for the high-level design. A low-level crossing with a vertical-lift main span, similar to the Buzzards Bay Bridge over Cape Cod Canal promised to be the least expensive of all, but this design was rejected by the client because of the heavy traffic on DuPont Highway and because of the frequent openings required for ship passages through the canal. A through cantilever truss and a self-anchored suspension bridge were considered for the main span. The costs of these two types were about equal and considerably greater than that of the arch or simple truss, principally because of the economy of short, simple-girder, flanking spans in lieu of the relatively heavy and expensive anchor spans required for both the cantilever and the suspension type. A tunnel was investigated and found to cost more than all bridge types. This was still the case when a two-lane tunnel was compared with four-lane bridges.

The fact that all piers were built on dry land effected the economy of the design in that substructure costs were reduced and shorter approach spans were possible in accordance with the economic principle of balancing the cost of substructure against cost of superstructure.

Jacob Karol,[5] **Assoc. M. Am. Soc. C. E.**—An interesting description of the design of a tied arch span is contained in this paper. This type of structure contains approximately the same amount of metal as a simple truss span carrying the same loads, and is certainly more pleasing in appearance than any type of through truss. Hence, it should find increasing use in locations where esthetic considerations are of some importance.

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[4] Lt., CEC, U.S.N.R., Bureau of Yards and Docks, Washington, D. C.

[5] Structural Engr., Airplane Div., Curtiss-Wright Corp., Lambert Field, Robertson, Mo.

The writer wishes to indicate certain simplifications in the preliminary design procedure. If the integration is performed in Eq. 8*b*, the expression for the influence line for H becomes

$$H = \frac{5\,l}{8\,h}\,(k^4 - 2\,k^3 + k) \qquad (19)$$

in which k is the proportionate distance from the left end of the span to the unit load. Table 1 shows the comparison between the values of H from this formula and the final values of H given by the author.

TABLE 1.—COMPARISON OF VALUES OF H

Load at panel point	By formula	Author
1	0.240	0.2358
2	0.468	0.4595
3	0.674	0.6625
4	0.850	0.8372
5	0.987	0.9772
6	1.080	1.0772
7	1.128	1.1300

Thus, it is seen that a very close approximation to the final influence line for H, and hence to the final direct stresses, can be obtained if only the span length and the center rise are known. Having the influence line for H, influence lines for M may be determined from Eq. 9. It should be assumed, however, that the live-load moments determined from these influence lines represent the total moments resisted by the arch rib and tie girder.

The next step is to determine the division of the total moment between the girder and the rib. The author's procedure, using Eq. 16, requires that the moments of inertia of the rib and the girder be known in advance. However, this division can be obtained approximately if only the depths of the rib and the girder are known. From the flitched-beam concept, the angle changes along the rib are equal to those along the girder (except for the change in length of the hangers, which effect will be neglected). Hence,

$$\frac{M_G\,dx}{E\,I_G} = \frac{M_R\,ds}{E\,I_R} = \frac{M_R\,dx}{E\,I_R\cos\alpha} \qquad (20)$$

from which

$$\frac{M_R}{M_G} = \frac{I_R\cos\alpha}{I_G} \qquad (21)$$

If the sections of the rib and girder are assumed to be proportional to their direct stresses, then

$$A_G = C_1\,A_R\cos\alpha \qquad (22)$$

in which C_1 = the ratio of the unit stresses due to the direct loads in the rib and girder, respectively, $= \frac{f_{R(d)}}{f_{G(d)}}$.

The value of C_1 may be obtained as follows: Assume a value of the fiber stress due to bending in the girder. In the structure under consideration this value is about 8,000 lb per sq in. on the gross section. This value may be determined approximately at a particular section, using the direct loads and moments previously found in the following equations:

$$f_G = \frac{H}{A_G}\left(1 + \frac{2.5\,e}{d_G}\right) = f_{G(d)}\left(1 + \frac{2.5\,e}{d_G}\right)\ldots\ldots\ldots\ldots(23)$$

in which f_G = allowable unit stress in girder on gross section; H = tension due to dead load and live load; $e = \frac{M}{H}$; and M is the moment due to dead load and live load.

Since $f_G = f_{G(b)} + f_{G(d)}$, by substitution in Eq. 23,

$$f_{G(d)} = \frac{f_G\, d_G}{d_G + 2.5\,e}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(24a)$$

and

$$f_{G(b)} = \frac{f_G \times 2.5\,e}{d_G + 2.5\,e}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(24b)$$

For the case presented in the paper, assume that the allowable stress in the girder $f_G = 80\% \times 24{,}000 = 19{,}200$ lb per sq in. on the gross section; also, for the allowable stress in the rib, $f_R = 19{,}400$ lb per sq in. Since the unit stresses in the girder and rib due to bending are approximately proportional to their respective depths,

$$f_{R(b)} = f_{G(b)} \times \frac{d_R}{d_G}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25)$$

Substituting numerical values, $d_R = 40$ in. and $d_G = 108$ in., and $f_{R(b)} = 8{,}000 \times \frac{40}{108} = 3{,}000$ lb per sq in. Hence, the unit stresses due to direct loads are $f_{G(d)} = 19{,}200 - 8{,}000 = 11{,}200$ lb per sq in. in the girder and $f_{R(d)} = 19{,}400 - 3{,}000 = 16{,}400$ lb per sq in. in the rib, from which $C_1 = \frac{16{,}400}{11{,}200} = 1.46$. Substituting in Eq. 22, $A_G = 1.46\, A_R \cos\alpha$.

The moments of inertia of the girder and rib may be written

$$I_G = C_2\, A_G\, d^2_G\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26a)$$

and

$$I_R = C_3\, A_R\, d^2_R\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26b)$$

Since the girder and rib are both box sections, assume $C_2 = C_3$ for simplicity. Then

$$\frac{I_R}{I_G} = \frac{C_3\, A_R\, d^2_R}{1.46\, C_2\, A_R\, d^2_G \cos\alpha} = \frac{d^2_R}{1.46\, d^2_G \cos\alpha}\ldots\ldots(27a)$$

and

$$\frac{M_R}{M_G} = \frac{d^2_R \cos\alpha}{1.46\, d^2_G \cos\alpha} = 0.685\,\frac{d^2_R}{d^2_G}\ldots\ldots\ldots\ldots(27b)$$

Substituting numerical values, $\frac{M_R}{M_G} = 0.685 \times \frac{(40)^2}{(108)^2} = \frac{1}{10.6}$; $\frac{M_R}{M_T} = \frac{1}{10.6 + 1} = \frac{1}{11.6} = 8.6\%$; and $\frac{M_G}{M_T} = \frac{10.6}{11.6} = 91.4\%$.

This agrees very well with the distribution indicated by the author. It may be stated that, by assuming $C_1 = 1$, without making any calculations whatever to determine its value, the distribution of the total moment to the rib and the girder is 12.1% and 87.9%, respectively.

Hence, the foregoing procedure results in a preliminary determination of the division of the total moment between the rib and the girder, and of the direct stresses in the rib and the girder, with sufficient accuracy that the sections determined therefrom should be very near the final sections. It is worth emphasizing that these results can be obtained when only the span length, the center rise, and the depths of the rib and girder are known. Having thus obtained sections on which to base an analysis, the final analysis would follow the author's procedure, using Eqs. 16, 17, and 18.

ALEXANDER DODGE,[6] ESQ.—Professor Garrelts undertakes to design a multiply indeterminate structure by a simplified method. A design of this nature is usually very cumbersome, and in practice various short cuts frequently are permitted. However, the introduction of various assumptions by the author, without proof of their correctness, has made his paper inconclusive. The disputable parts of the analysis are as follows:

1. In the preliminary analysis, by assuming that the arch rib has resisted no bending, as well as by omitting the term $\sum \frac{M''_R\, m\, \Delta l}{E\, I_R}$ from Eq. 4, the author assumes that the rib is composed of a number of members pin-connected at the springing and at each hanger point. How the condition of continuity in the structure was finally restored has not been shown.

2. To find the distribution of bending moment between arch rib and girder, their relative deflections were supposed to be used, but the differential equation (Eq. 10*a*) holds true only for a thin curved bar of a constant cross section with circular axis and when subjected to transverse loading only.[7] A small initial curvature makes a great change in the effect of longitudinal forces on the deflection of the arch rib. The correct differential equation for deflection of an arch rib has been published by S. Timoshenko.[8]

The paper does not give full information regarding properties of the structure such as cross section and moment of inertia of the arch rib and the tie girder. Nevertheless, Figs. 6 and 7 give some indications on which the writer bases the following deductions:

The distribution of moment of inertia in an arch rib is probably not very different from that commonly used in practice, variation being proportional to sec α. The average area of tie used in computing deformations is probably about 265 sq in., and the moment of inertia at the crown, I_C, about 48,000 in.^{4} These values will be used in the discussion that follows, as well as the given ratio of tie girder to arch rib (13 : 1). The table of ordinates in Fig. 6 shows that the arch rib is parabolic in shape with a ratio of $\frac{h}{L} = 0.185$.

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[6] Associate Engr., U. S. Engr. Office, Portland, Ore.

[7] "Strength of Materials," by S. Timoshenko, D. Van Nostrand, 1930, Pt. II, p. 457.

[8] *Ibid.*, pp. 460–467; see also, "Theory of Elastic Stability," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York and London, 1936, pp. 230–236.

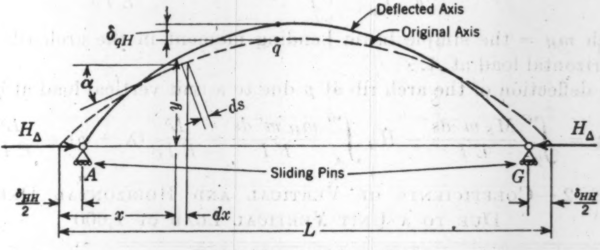

Deflected Axis
Original Axis
δ_{qH}
q
α
y
ds
H_Δ
H_Δ
A
G
Sliding Pins
$\frac{\delta_{HH}}{2}$
$\frac{\delta_{HH}}{2}$
x
dx
L

FIG. 9.—ISOLATED ARCH RIB

Assuming a pin condition at the springing and neglecting rib shortening, the deflection of an arch rib (Fig. 9), at a point q due to H_Δ, is

$$\delta_{qH} = \frac{H_\Delta}{E}\int_A^G y\,\frac{ds}{I}\,m' = \frac{H_\Delta}{E\,I_C}\int_A^G y\,m'\,dx = \frac{H_\Delta\,h\,L^2}{E\,I_C}\,\epsilon \;\ldots\ldots (28)$$

In Eq. 28, $y = \frac{4\,h}{L}\left(x - \frac{x^2}{L}\right)$; $I = I_C \sec \alpha$; I_C = minimum moment of inertia (at the crown); m' = the simple beam bending moment in the arch rib due to a unit vertical load at point q, when $H = 0$; and ϵ = the numerical coefficient of the vertical deflection.

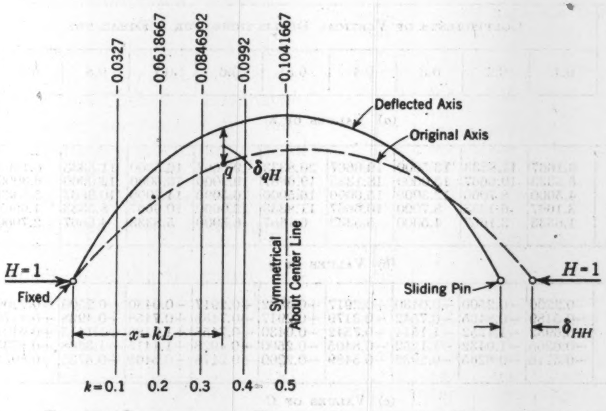

-0.0327
-0.0618667
-0.0846992
-0.0992
-0.1041667
Deflected Axis
Original Axis
q
δ_{qH}
Symmetrical About Center Line
H=1
Fixed
H=1
Sliding Pin
δ_{HH}
x=kL
k=0.1
0.2
0.3
0.4
0.5

FIG. 10.—COEFFICIENTS OF VERTICAL DEFLECTIONS, ϵ, DUE TO $H = 1$

By Maxwell's Law of Reciprocal Deflections δ_{Hq} = the horizontal deflection at A due to a vertical load P at $q = \delta_{qH}$, or $\delta_{Hq} = \delta_{qH} = \frac{H_\Delta\, h\, L^2}{E\, I_C}\,\epsilon = \frac{P\, h\, L^2}{E\, I_C}\,\epsilon$ provided the horizontal load H_Δ is equal to P.

The horizontal deflection δ_{HH} of the point A due to H_Δ is

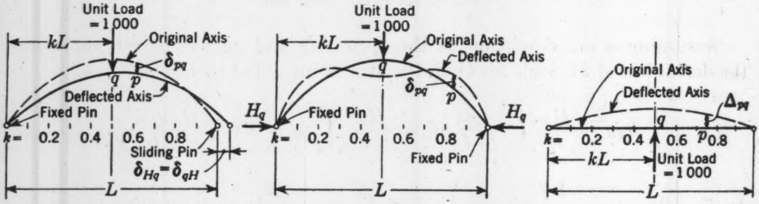
$$\delta_{HH} = \frac{H_\Delta}{E}\int_A^G \frac{m_H\, m_H\, ds}{I} = 0.53333\,\frac{h^2\, L\, H_\Delta}{E\, I_C} \ldots\ldots\ldots\ldots (29)$$

in which m_H = the simple beam bending moment in the arch rib due to a unit horizontal load at A.

The deflection of the arch rib at p due to a unit vertical load at q is

$$\delta_{pq} = \int_A^G \frac{M_S\, m'\, ds}{E\, I} - H_q \int_A^G \frac{m_H\, m'\, ds}{E\, I} = \frac{L^3}{E\, I_C}\,(\lambda - \phi) = \frac{L^3}{E\, I_C}\,\eta \ldots (30)$$

TABLE 2.—COEFFICIENTS OF VERTICAL AND HORIZONTAL DEFLECTIONS DUE TO A UNIT VERTICAL LOAD OF 1,000


Unit Load = 1 000
kL
Original Axis
δ_{pq}
q
p
Deflected Axis
Fixed Pin
k= 0.2 0.4 0.6 0.8
Sliding Pin
$\delta_{Hq} = \delta_{qH}$
L
Unit Load = 1 000
kL
Original Axis
Deflected Axis
q
δ_{pq}
p
H_q
Fixed Pin
H_q
k= 0.2 0.4 0.6 0.8
Fixed Pin
L
Original Axis
Deflected Axis
Δ_{pq}
q
p
k= 0.2 0.4 0.6 0.8
kL
Unit Load = 1 000
L

(Note: Downward deflections positive)

Position of load (tenth point)	COEFFICIENTS OF VERTICAL DEFLECTIONS, FOR k EQUAL TO:									Coefficients of horizontal deflection
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	
	(*a*) VALUES OF λ									VALUES OF 1,000 ϵ
0.5	6.1667	11.8333	16.5000	19.6667	20.8333	19.6667	16.5000	11.8333	6.1667	104.1667
0.6	5.5333	10.6667	15.0000	18.1333	19.6667	19.2000	16.5000	12.0000	6.3000	99.2000
0.7	4.5000	8.7000	12.3000	15.0000	16.5000	16.5000	14.7000	10.9667	5.8333	84.6992
0.8	3.1667	6.1333	8.7000	10.6667	11.8333	12.0000	10.9667	8.5333	4.6667	61.8667
0.9	1.6333	3.1667	4.5000	5.5333	6.1667	6.3000	5.8333	4.6667	2.7000	32.7000
	(*b*) VALUES OF η									
0.5	−0.2200	−0.2500	−0.0430	+0.2917	+0.4882	+0.2917	−0.0430	−0.2500	−0.2200	....
0.6	−0.5489	−0.8405	−0.7542	−0.3179	+0.2917	+0.7488	+0.7458	+0.4928	+0.2178	....
0.7	−0.6932	−1.1252	−1.1514	−0.7542	−0.0430	+0.7458	+1.2486	+1.1415	+0.6401	....
0.8	−0.6265	−1.0432	−1.1252	−0.8405	−0.2500	+0.4928	+1.1415	+1.3568	+0.8735	....
0.9	−0.3716	−0.6265	−0.6932	−0.5489	−0.2200	+0.2178	+0.6401	+0.8735	+0.6951	....
	(*c*) VALUES OF C									
0.5	−0.474	−0.910	−1.269	−1.513	−1.603	−1.513	−1.269	−0.910	−0.474	....
0.6	−0.426	−0.821	−1.154	−1.395	−1.513	−1.477	−1.269	−0.923	−0.485	....
0.7	−0.346	−0.669	−0.946	−1.154	−1.269	−1.269	−1.131	−0.844	−0.449	....
0.8	−0.244	−0.472	−0.669	−0.821	−0.910	−0.923	−0.844	−0.656	−0.359	....
0.9	−0.126	−0.244	−0.346	−0.426	−0.474	−0.485	−0.449	−0.359	−0.208	....

in which: M_S = the simple beam moment in the arch rib due to a unit vertical load at q; H_q = the horizontal reaction at the springing due to a unit vertical load at q, for an arch, with fixed pins to prevent horizontal movements; λ = the numerical coefficient of the vertical deflections for an arch acting as a simple beam; ϕ = the numerical coefficient of the vertical deflection due to H_q alone = $H_q\,\epsilon$; and η = the numerical coefficient of the final vertical deflection. Coefficients ϵ, λ, and η are obtained by performing the necessary integrations. The coefficients of deflections were computed by the writer and are recorded in Fig. 10 and Table 2. The inspection of these coefficients will indicate immediately the importance of the longitudinal force (horizontal thrust) on the deflection of the arch rib, as has been stated previously in this discussion.

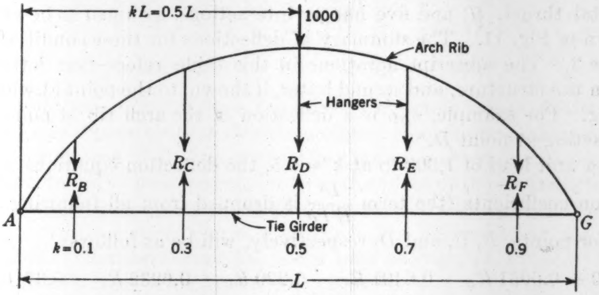


FIG. 11.—TIED ARCH

TABLE 3.—SUMMARY OF DEFLECTIONS DUE TO VERTICAL LOADS

Unit external load at k	(a) ARCH RIB DEFLECTIONS				
0.5	δ_B	δ_C	δ_D	δ_E	δ_F
Unit interaction load at k	**(b) ARCH RIB DEFLECTIONS**				
0.1	δ_{BB}	δ_{CB}	δ_{DB}	δ_{EB}	δ_{FB}
0.3	δ_{BC}	δ_{CC}	δ_{DC}	δ_{EC}	δ_{FC}
0.5	δ_{BD}	δ_{CD}	δ_{DD}	δ_{ED}	δ_{FD}
0.7	δ_{BE}	δ_{CE}	δ_{DE}	δ_{EE}	δ_{FE}
0.9	δ_{BF}	δ_{CF}	δ_{DF}	δ_{EF}	δ_{FF}
Unit interaction load at k	**(c) TIE GIRDER DEFLECTIONS**				
0.1	Δ_{BB}	Δ_{CB}	Δ_{DB}	Δ_{EB}	Δ_{FB}
0.3	Δ_{BC}	Δ_{CC}	Δ_{DC}	Δ_{EC}	Δ_{FC}
0.5	Δ_{BD}	Δ_{CD}	Δ_{DD}	Δ_{ED}	Δ_{FD}
0.7	Δ_{BE}	Δ_{CE}	Δ_{DE}	Δ_{EE}	Δ_{FE}
0.9	Δ_{BF}	Δ_{CF}	Δ_{DF}	Δ_{EF}	Δ_{FF}

Deflection of the tie girder, acting as a simple beam, due to a unit vertical load is

$$\Delta = \frac{C'\,L^3}{E\,I_G} = \frac{C\,L^3}{E\,I_C} \ldots (31)$$

in which C = the coefficient of the vertical deflection = $\frac{C'}{13}$ (see Table 2(*c*)). The ratio $\frac{I_G}{I_C} = 13$ is assumed to hold true at the crown.

The behavior of the St. Georges tied arch has been studied by means of coefficients of deflections on the assumption that five hangers only were acting, each hanger having a gross area of 35 sq in. The unknowns are a horizontal thrust, H, and five hanger interactions, assumed to be all tensions, as shown in Fig. 11. The summary of deflections for these conditions is given in Table 3. The subscript notations in this table refer—first letter, to the point on the structure, and second letter, if shown, to the point at which a force is acting. For example, δ_{BD} is a deflection of the arch rib at point B due to a load acting at point D.

For a unit load of 1,000 lb at $k = 0.5$, the deflection equations in terms of deflection coefficients (the term $\frac{L^3}{E\,I_C}$ is dropped from all following computations) for points, B, C, and D, respectively, will be as follows:

$$\begin{aligned}&- 0.220 + 0.6951\,R_B + 0.6401\,R_C - 0.220\,R_D - 0.6932\,R_E - 0.3716\,R_F\\&\quad + 0.208\,R_B + 0.449\,R_C + 0.474\,R_D + 0.346\,R_E + 0.126\,R_F = 0 \quad (32a)\end{aligned}$$

$$\begin{aligned}&- 0.043 + 0.6401\,R_B + 1.2486\,R_C - 0.043\,R_D - 1.1514\,R_E - 0.6932\,R_F\\&\quad + 0.449\,R_B + 1.131\,R_C + 1.269\,R_D + 0.946\,R_E + 0.346\,R_F = 0 \quad (32b)\end{aligned}$$

and

$$\begin{aligned}&+ 0.4882 - 0.220\,R_B - 0.043\,R_C + 0.4882\,R_D - 0.043\,R_E - 0.220\,R_F\\&\quad + 0.474\,R_B + 1.269\,R_C + 1.603\,R_D + 1.269\,R_E + 0.474\,R_F = 0 \quad (32c)\end{aligned}$$

By the law of symmetry $R_B = R_F$ and $R_C = R_E$. Solution of these equations yields (in pounds):

$$\begin{aligned}R_B &= (0.2467)\,(1{,}000) = 247\\R_C &= (0.2983)\,(1{,}000) = 298\\R_D &= (-\,0.6432)\,(1{,}000) = -\,643\end{aligned}$$

Taking these interactions into consideration, the horizontal thrust $H = 1{,}052$ lb. The coefficients of true deflections are $-\,0.015$, 0.000, and $+\,0.040$ for points B, C, and D, respectively.

Due to $H = 1$, the elongation of the tie girder is $\tau_{HH} = \frac{L}{A\,E}$. The horizontal movement of the arch rib at the springing is $\delta_{HH} = \frac{0.53333\,h^2\,L}{E\,I_C}$, and $\frac{\tau_{HH}}{\delta_{HH}} = \frac{I_C}{0.53333\,A\,h^2} = \frac{48{,}000}{(0.53333)\,(265)\,(1{,}200)^2} = 0.000236$. Therefore, the horizontal thrust that would exist, were the arch rib fixed against the horizontal movement, will be reduced, due to stretch in the girder, only 0.0236 of 1%.

For various positions of the 1,000-lb load, the reduction in horizontal thrust, H_r, due to the elongation of the tie girder, is as follows:

k	H_r (lb)
0.5	$1{,}055 \times 0.000236 = 0.249$
0.3	$857 \times 0.000236 = 0.202$
0.1	$331 \times 0.000236 = 0.078$

Taking the effect of this reduction into account, and with the aid of Fig. 10, Eqs. 32 will be changed to:

$$0.6627\,R_B + 0.7551\,R_C + 0.2621\,R_D - 0.220 = 0 \quad\ldots\ldots(33a)$$
$$0.7551\,R_B + 2.2084\,R_C + 1.2471\,R_D - 0.043 = 0 \quad\ldots\ldots(33b)$$

and

$$0.5242\,R_B + 2.4942\,R_C + 2.1169\,R_D + 0.4882 = 0 \quad\ldots\ldots(33c)$$

The solution of these equations yields practically the same values of interactions as have been found previously in this discussion. Hanger elongations similarly had been incorporated in deflection equations, and the effect of these deformations has been found to be even smaller than the effect of the tie girder stretch.

So far, the girder has been assumed to be perfectly straight, but actually it follows the vertical curve of the roadway. The deformations due to this initial parabolic curvature may be taken into account as follows: Due to $H = 1$, the moment in the tie girder is equal to

$$y' - \Delta = \frac{4\,r}{L}\left(x - \frac{x^2}{L}\right) - \Delta \quad\ldots\ldots(34)$$

and the differential equation of deflection is

$$\frac{d^2\Delta}{dx^2} = \frac{4\,r}{E\,I_G\,L}\left(x - \frac{x^2}{L}\right) \quad\ldots\ldots(35)$$

Therefore,

$$\Delta = \frac{r\,L^2}{3\,E\,I_G}\,(-\,k + 2\,k^2 - k^4) = \frac{L^3}{E\,I_C}\,C'' \quad\ldots\ldots(36)$$

in which r = maximum vertical ordinate of the tie girder = 6.8 ft; Δ = final deflection of the girder, which, being obviously insignificant, has been omitted from the right side of Eq. 35; and C'' = the numerical coefficient of deflection, expressed by

$$C'' = \frac{r\,(-\,k + 2\,k^3 - k^4)}{(3\,L)\,(13)} \quad\ldots\ldots(37)$$

These equations are written with the origin at the springing. Coefficients C'' are recorded in Fig. 12. By means of these coefficients and Fig. 10, the effect of the tie girder deflections due to a horizontal thrust is easily incorporated into the analysis, and Eqs. 33 are changed to:

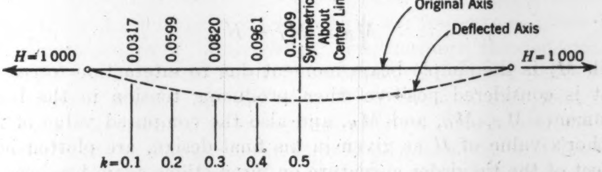


FIG. 12.—COEFFICIENTS OF VERTICAL DEFLECTIONS, C_1, DUE TO $H = 1{,}000$ (Note: $C_1 = 1{,}000\ C''$)

$$0.6417\,R_B + 0.7007\,R_C + 0.2287\,R_D - 0.220 = 0 \quad\ldots\ldots(38a)$$
$$0.7007\,R_B + 2.0678\,R_C + 1.1606\,R_D - 0.043 = 0 \quad\ldots\ldots(38b)$$

and

$$0.4574\, R_B + 2.3212\, R_C + 2.0105\, R_D + 0.4882 = 0 \ldots\ldots (38c)$$

The solution of Eqs. 38 indicates no material changes in the magnitude of interactions that have been determined previously.

Two other factors—interaction moment at the springing and rib shortening—similarly could be taken into consideration in the analysis. (The writer leaves it to students of arch design to prove that the interaction moment at the springing has an insignificant influence on design of this tied arch.) As it stands, the arch rib is considered as being pin-connected to the tie girder. The rib shortening, if taken into consideration, will reduce the value of the horizontal thrust. This in turn would increase the discrepancy between the writer's and the author's H-values.

Similarly, a new set of deflection equations has been made for interactions at every tenth point of the span. The summary of results from this study, for a unit vertical load of 1,000 applied to the girder at $k = 0.5$, is recorded in Table 4.

TABLE 4.—SUMMARY OF INTERACTION FORCES, MOMENT RATIOS, AND COEFFICIENTS OF DEFLECTION

(Horizontal Thrust = 1,054 Lb)

Description	$k = 0.1$	$k = 0.2$	$k = 0.3$	$k = 0.4$	$k = 0.5$
Hanger tension	162	142	148	140	224
M_R/M_T	−1%	+25%	−22%	+10%	+9%
M_G/M_T	+101%	+75%	+122%	+90%	+91%
Final deflection coefficients	−0.0161	−0.0177	−0.0025	+0.0214	+0.0360

The total moment

$$M_T = M' - H\,(y - y') \ldots\ldots (39)$$

is distributed to the girder and arch rib thus:

$$M_G = M' - M_I + H\,y' \ldots\ldots (40a)$$

and

$$M_R = M_I - H\,y \ldots\ldots (40b)$$

in which M_I is the simple beam moment due to interaction forces only; the moment is considered positive when producing tension in the lower fiber. The moments M_T, M_G, and M_R, and also the computed value of M_T using the author's value of H as given in his final design, are plotted in Fig. 13. The effect of the tie girder curvature on interactions again has been found to be quite negligible, varying from -2% to $+1.5\%$.

This concludes a study of the various factors entering into the analysis of the St. Georges tied arch span. After this step the analysis is shifted to an examination of the author's values. It has been found that, assuming the author's influence lines for H as being correct and using the distribution of bending moment, $M_T = M' - H\,(y - y')$, to rib and girder as 7% and 93%, respectively, the resulting values of interactions would not be very different from the values shown in Table 4. Treating these interactions as loads on the isolated arch rib, and allowing neither elongation in the tie girder nor rib shortening, the value of the horizontal thrust becomes $H = 1{,}047$ lb.

Finally, all fourteen interactions for a unit load of 1,000 lb at point 7 were similarly determined by means of Eq. 18, using the given values of $H = 1{,}130$ lb. These interactions, treated as externally applied loads on the isolated arch rib, produce the horizontal thrust $H = 1{,}049$ lb only. Unless the assumption of the elastic properties of the structure was grossly in error, the writer does not believe that the value of the horizontal thrust for the load at point 7 could be greater than 1,049 lb. This limiting value of H is for the case of an arch rib receiving no support from the tie girder. Using the author's value of the horizontal thrust ($H = 1{,}130$ lb), and the resulting values of the interactions, the deflection curve of the arch rib was found to be quite different from the deflection curve of the tie girder. Furthermore, the deflection of the crown point of the arch was found to be negative (upward deflection with respect to original position of the rib). To move the unit load upward, however, the work must be done against gravity. Therefore, some other external force must be acting on the structure besides the unit load. This external force is the discrepancy between the writer's and the author's values of the horizontal thrust. With the value of $H = 1{,}130$ lb, the ends of the arch rib at the springing actually move inward with respect to their original position, which is contrary to the physical behavior of the structure.

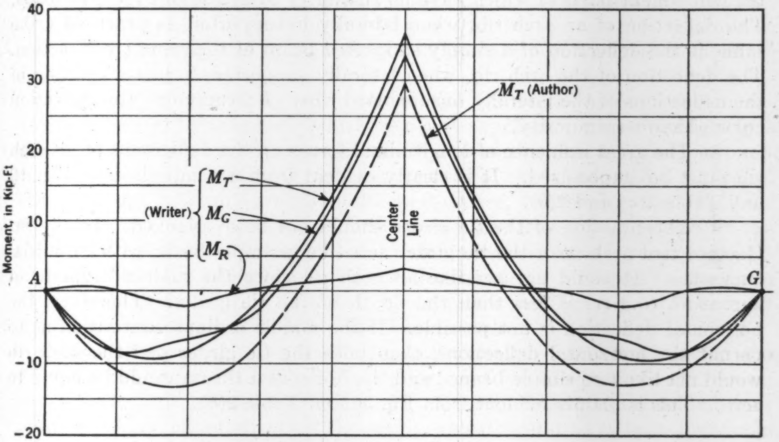


FIG. 13.—MOMENTS M_T, M_R, AND M_G

No analysis was made by means of the coefficients of arch deflections for the other positions of the load. It is possible that the actual properties of the structure, complete data for which are not included in the paper, might influence somewhat the results of this analysis. The writer hopes that, in his concluding discussion, the author will give the missing information, as well as the magnitude of his interactions and deflections.

Conclusion.—This discussion may be summarized under four items, as follows:

(1) In the design of the St. Georges tied arch span the author has compared the deflections of the tie girder, Table 2(*c*), with the deflections of the laterally unsupported arch rib, Table 2(*a*). From the foregoing discussion it is evident that the arch rib is practically fixed laterally, and the design should be based on the comparison of deflections of Tables 2(*a*) and 2(*b*). Therefore, the derivation of the author's formulas beyond Eq. 3 is not justified.

(2) The flitched-beam idea is not applicable for analysis of these structures, the component parts of which have no similarity to deflections for unit loads. The deflection of an arch rib, when laterally unsupported, is practically the same as the deflection of a simply supported beam of the same cross section. The deflection of the arch rib, when laterally supported, is just a fraction of the deflections of the laterally unsupported ribs. Furthermore, the deflection curves have no similarity.

(3) The great influence of longitudinal forces on the deflections of an arch rib must be emphasized. It is clearly evident from a comparison of Fig. 10 and Table 2(*a*) and 2(*b*).

(4) The function of the tie girder should not be overlooked. In the St. Georges tied arch span the tie girder acts like a simple beam with an initial curvature. It could not act like an arch rib since the middle ordinate of curvature (6.8 ft) is less than the depth of the tie girder. Therefore, the horizontal deflection is not possible. If the middle ordinate were such as to permit the horizontal deflections, then both the tie girder and the arch rib would act like two simple beams, and the horizontal thrust would be equal to zero. This is plainly evident from Eq. 30 and Table 2(*a*).

CARLOS A. BEJARANO,[9] JUN. AM. SOC. C. E.—The derivation of the method of analysis of a novel type of arch has been derived most interestingly and soundly by Professor Garrelts.

In arriving at a simplified expression for the distribution of moments between the arch rib and tie girder, the last three terms of Eq. 12 have been omitted, assuming that the error produced thereby is insignificant. The amount of error introduced by this step can be approximated by assuming reasonable limits to the respective factors, but the exact difference may be calculated after the stress analysis has been completed, using the actual computed values for each factor. This condition may be investigated at panel point 4, where the effect is probably the greatest, since it is close to the quarter point of the span where the stresses approach a maximum.

The radius of curvature of the span may be determined by considering the arch rib in the form of a parabola. The vertical deflections may be computed from the moment influence coefficients, applying the principle of virtual work.

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[9] Parsons, Klapp, Brinckerhoff & Douglas, New York, N. Y.

Using the computed values for both the deflection and radius of curvature, the last three terms of Eq. 12 will then assume the following values: $-\cos\alpha\,\frac{v}{\rho^2} = -0.00000173$; $\frac{v}{\rho^2}\,\frac{d\rho}{dl^2}\sin\alpha = 0.00000179$; and $-2\,\frac{dv}{dl}\,\frac{1}{\rho}\sin\alpha = -0.00000185$. It may be observed that each of these three terms is of the same magnitude, but inasmuch as one of them is of opposite sign, practically the total error will result from either of the other terms. The algebraic sum of the three terms gives -0.00000179.

In order to obtain the precision of the computation, the term $\cos\alpha\,\frac{d^2v}{dl^2}$ may be taken equal to $\frac{d^2\eta}{dl^2}$, which in turn is equivalent to $\pm\,\frac{M}{E\,I_R}$. For maximum positive moment the error is -3.1%, and for maximum negative moment, -3.4%. Since this is probably the largest error involved in Eq. 13, and also since it is of negative sign, the simplified procedure of analysis is therefore justified.

The question may arise as to whether this will be the amount of error involved wherever this method of analysis may be applied. The foregoing values apply specifically to the St. Georges tied arch, but they are a fairly good indication for any other similar structure since none of the factors in Eq. 12, except the curvature, will vary greatly in different arches. Assuming, then, that the third and fourth terms cancel each other and that the total error lies in the omission of the last, one may observe that its magnitude is inversely proportional to the radius of curvature. Thus, the precision is governed by the shape of the arch, and it can be quite accurately determined for any other arch by comparing its curvature with the St. Georges arch.

Under the heading "The Final Design," Professor Garrelts has noted the influence values for H, in which he has included the effects of rib shortening and tie extension, as indicated in Eq. 17. Actually these latter considerations produce a very minor effect on the result, and it would have been justifiable to omit them from the computations, simplifying the work to some extent. The following is a comparison of the values of H with and without the effects of direct stress:

Load at panel point	Values of H With	Without
1	0.2358	0.2380
2	0.4595	0.4639
3	0.6625	0.6687
4	0.8372	0.8452
5	0.9772	0.9865
6	1.0772	1.0874
7	1.1300	1.1408

Professor Garrelts has suggested an alternate method of analyzing the arch by using a transformed section for the rib, instead of for the tie, which would resist all the bending moment and compression while the tie would act only as a tension member. This would be more in conformance with the regular procedure in the design of an ordinary tied arch, and for this particular reason it probably would be preferable to the original scheme. The transformed moment of inertia for the rib would be $I'_{tr} = I_R + I_G \sec \alpha$. When the total bending was transferred to the tie girder, it was resisted by a transformed section whose moment of inertia was $I_{tr} = I_G + I_R \cos \alpha$. The geometric relation that exists between the two transformed sections is quite apparent. Therefore the preliminary analysis yielding the redundant reactions would be similar in each case. However, in the analysis of the loading conditions to produce maximum fiber stresses at any given point, the two methods will not necessarily yield the same results. The maximum fiber stress is the combined effect of the thrust, moment, and shear due to any given loading. Thus, for maximum total fiber stress, the moment must be taken about the kern points of each section, rather than through the apparent neutral axis. The additional moment ($\pm \Delta M$) is a function of the properties of the section in question and may be applied as a correction to the moment influence coefficients in determining the loading conditions for maximum stress at any particular point.

If the moment of inertia of the rib differs from that of the tie, the value of ΔM for the larger section obviously will be greater than for the smaller section. The effect of this difference may be noticeable at those panel points where a reversal of stress has taken place; that is, where the bending moment has changed from positive to negative (or vice versa) under a given loading. This may be observed from the moment influence coefficients shown in Fig. 5. For example, notice the effects in the determination of the maximum positive stress at panel point 1. The moment influence coefficients for point 1 when the unit load is at 6 and at 7 are, respectively, + 0.49 and − 2.69. The additional moment at 7 on considering the kern point of the arch rib is 1.00 and, when the kern point of the tie girder is considered, ΔM is equal to 3.11, which is sufficient to produce a positive moment at point 1. In the latter case, then, an additional panel would be loaded to produce maximum positive fiber stress at point 1.

The number of times that this condition might occur depends on the particular characteristics of the structure. It may be noticed that the difference in each case will involve either the loading or unloading of one panel—that is, a small fraction of the live load, which will not produce a very large discrepancy.

To reduce the errors to a minimum, it would be advisable to make an analysis of that transformed structure which more nearly approximates the true behavior of the bridge.

A. A. EREMIN,[10] ASSOC. M. AM. SOC. C. E.—Analysis of stresses in the arch rib tied with a stiffening girder is an extremely complicated problem. In the technical literature very little can be found on this subject. In computing the stresses in the St. Georges arch bridge, Professor Garrelts has adopted various assumptions to simplify the computations. The writer proposes to clarify, further, the limitations of assumptions adopted in the light of economic aspects of tied arch bridges with stiffening girders.

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[10] Associate Bridge Engr., Bridge Dept., Div. of Highways, State Dept. of Public Works, Sacramento, Calif.

From Fig. 6 it is evident that the stiffening girder in the St. Georges bridge is connected to the arch rib by a rigid joint, whereas, in the development of Eq. 3 for the computation of stresses in the arch rib, the author assumed that the stiffening girder is connected to the arch rib by a frictionless hinge. This assumption has a negligible effect on computation of the horizontal thrust in the arch rib, especially for ratios $\frac{h}{L}$ equal to or greater than $\frac{1}{10}$. However, the equations developed with this assumption are not adequate for the design of the arch rib sections at the ends and their connections to the stiffening girder.

The axial stresses in the stiffening girder, produced by the arch rib thrust, are generally small compared with the bending moment stresses. Therefore, sections of stiffening girder are usually varied according to variations of the bending moment stresses along the girder axis. In developing Eq. 17 for the horizontal thrust in the arch rib, Professor Garrelts, however, assumed that the stiffening girder section is constant for the entire length. Evidently with this approximation the refinement of considering that the arch rib section varies directly as cos α (Eqs. 16) has doubtful meaning.

In the design of tied arch rib bridges, an important consideration is the design of the lateral stiffening system. Often economy in material gained by careful computations of the stresses in the arch rib and stiffening girder may easily be lost in the wasteful approximate design of the lateral stiffening system. It is to be hoped that, in his closing comments, the author will discuss a method of designing the lateral stiffening system and will present some principles to govern the selection of panel lengths.

The author is to be commended for making a valuable contribution to this subject.

A. M. FREUDENTHAL,[11] ASSOC. M. AM. SOC. C. E.—When considering the formulas for the design of the tied arch derived by Professor Garrelts, the question immediately arises as to whether the validity of these formulas is not subject to the same limitations as the formulas of the elementary theory of the suspension bridge, particularly taking into consideration the relatively large span of the structure actually designed. High deformability is a common feature of all structures in which a shallow girder suspended from, or supported by, a flexible member provides the bending resistance for unevenly distributed live load. It is this deformability that renders the methods of the elastic theory of structures ineffective in the design of suspension bridges as well as of arches, particularly those with flexible ribs or of long spans.[12]

In general, designers are quite familiar with the fact that considerable differences are to be expected between the bending moments in the stiffening truss of a suspension bridge, computed by the elastic theory, and the actual moments. They are less familiar with the effects of the deflections upon the bending moments in arches. Therefore, a few quite general remarks regarding the limitations imposed upon the validity of elementary formulas by the flexibility of a structure may not be devoid of interest. In no respect do these comments deprecate the method of design presented by the author which, incidentally, is practically unaffected by considerations of deflections.

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[11] Cons. Engr., Tel Aviv, Palestine.

[12] "Deflection Theory for Arches," by A. M. Freudenthal, Publications of the International Assn. of Bridge and Structural Engineering, Zürich, Vol. 3, 1935, p. 100.

In non-tied flexible systems (suspension bridges and arches) the bending moment due to live load can be expressed by

$$M = M'_p - H_p\, y \mp (H_w + H_p)\,\eta \ldots\ldots\ldots\ldots\ldots (41)$$

in which p and w denote statical values due to live load and dead load, respectively, and η denotes the deflection of the cross section of the girder considered. Hence, the moments in the stiffening girder computed by the elastic theory are either reduced (suspension bridges) or increased (arches) by the amount $(H_w + H_p)\,\eta$. Owing to the considerable value of the dead-load thrust H_w for large-span structures, the deviation from the elastic theory may attain a comparatively high proportion of the total bending moment.

It can be easily shown that in self-tied structures the bending moment in the girder (Fig. 14),

$$M = M'_w + M'_p - (H_w + H_p)\,(y \pm \eta) \mp (H_w + H_p)\,\eta = M'_p - H_p\, y \,.\,. (42)$$

is not affected by deflection if the flexible member (the arch rib or the cable) has no bending resistance. However, if the bending resistance cannot be neglected (as in most arches), the effect of the deflections upon the value of the bending moments makes itself felt in the rib moments. The real values of

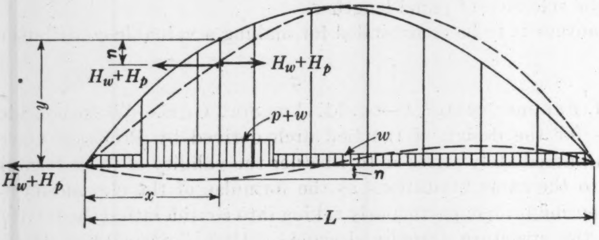

FIG. 14

these moments, therefore, may exceed, considerably, those obtained by simple distribution of the total bending moments between rib and girder according to the respective moments of inertia, as proposed by Professor Garrelts. It is particularly the buckling resistance of the rib that may be impaired by a bending moment exceeding the design value.

C. H. GRONQUIST,[13] ASSOC. M. AM. SOC. C. E.—Preliminary design of the St. Georges tied arch span was made by assuming the 3-ft arch rib as entirely flexible, leaving the 9-ft tie girder to resist the live-load bending moments without assistance from the rib. Under this assumption the action of the structure is exactly analogous to that of the self-anchored suspension bridge, inverted. The flexible arch rib, acting only in direct compression, corresponds to the cable of the self-anchored suspension bridge, whereas the tie girder, acting both in direct tension and in bending, corresponds to the stiffening girder of the self-anchored suspension bridge, which must also transmit the horizontal component of cable stress as a thrust over the entire length of the bridge.

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[13] Associate Engr., Robinson & Steinman, Cons. Engrs., New York, N. Y.

The action of the two structures being analogous, the equations and method of analysis of the self-anchored suspension bridge can be applied with only slight modification to the analysis of the tied arch with a flexible rib. One point of difference in the action of the two structures is to be found in the effect of vertical curve of roadway and tie girder or stiffening girder. The vertical curve of the stiffening girder will decrease stresses in the self-anchored suspension bridge, but the stresses in the tied arch are increased by the vertical curve. Since both structures are of the "closed" type, resembling simple trusses to that extent, the change in dimensions produced by deflection has no important effect on the stresses determined by the elastic theory using the dead-load dimensions, in the case of either the self-anchored suspension bridge or the tied arch. This is in distinction to the effect of deflection on both the externally anchored suspension bridge and the arch bridge.

A more complex but similar structure consisting of a 3-span continuous girder with the center span reinforced by a flexible arch rib tied to the girder is analogous to the self-anchored suspension bridge with unloaded side spans, except that in the case of the tied arch bridge the side-span girders will not act in direct stress. Again, however, the analysis of the two structures could be made by application of the same methods and equations.

Professor Garrelts gives no derivation for Eq. 17 for the final analysis of the St. Georges tied arch span. Although this may have been assumed to be obvious, the value of this excellent paper may perhaps be increased by clarifying somewhat the theoretical basis for the final analysis. Hanger elongation under live load was considered negligible, as were the tangential components of deflection. Consequently, the angle between the rib and girder at the end of the rib, as well as elsewhere, is not changed during deflection of the structure, and, therefore, there can be no appreciable moment in the rib at its end connections with the girder. The span can be made determinate by cutting the rib at the girder connections so as to eliminate H, the horizontal component of direct stress in the rib and girder. The structure then will consist of two simple beams, the arched rib, and the girder, which are connected by the inextensible hangers and which must participate in proportion to their effective moments of inertia in carrying the live load.

With only the value of the horizontal component of direct stress to be determined, the structure is singly indeterminate, as for the preliminary design. The moment in the structure for a unit value of this redundant is $m = y - y'$, this moment also being divided between the rib and the girder in proportion to their effective moments of inertia. The direct stress in the rib for a unit value of H is $u = \sec \alpha$, assuming the hanger loads to be equal throughout, and the direct stress in the girder is of course $u = 1$. Eq. 17

for H then follows from the substitution of the foregoing moments and direct stresses in the conventional Eq. 2 for a single redundant.

Final analysis of the previously mentioned tied-arch continuous girder can be made by this same method. By taking the basic determinate system as the combination of the continuous girder and the center span simple arched beam, there would remain only the value of H, the horizontal component of direct stress, to be determined by Eq. 17. The actual values of M' in this equation, however, could be taken for the simple span condition, and therefore only for the center span. Similarly, the first term in the denominator could be written $\frac{\Sigma\, m\, m_0\, \Delta l}{I_{tr}}$ in which m represents the moment due to a unit value of H in the basic continuous structure, and m_0 represents the moment due to $H = 1$ in the center span as a simple structure. This follows from the fact that in computing the deflection of an indeterminate structure by the method of the dummy unit load, the value of either M', the moment for the basic determinate condition, or m, the moment produced by the unit load, may be taken for a statically determinate condition. In applying Eq. 18 to obtain the final moments, M', of course, would be taken for the basic continuous system.

HAROLD E. LANGLEY,[14] ESQ.—The tied arch design adopted for the St. Georges bridge is certainly an unusual type in the United States. There seems no doubt that, prior to World War II, there has been a definite trend in design toward the selection of structures which meet esthetic as well as utilitarian requirements. A pleasing design is usually the product of the successful solution of perplexing engineering problems. Professor Garrelts' paper, therefore, is an important contribution to the engineering profession.

Formulas based on the elastic theory seem to have been accurate enough for the design of this bridge, as the results agree with those secured by a model analysis. Inasmuch as both the girder and arch rib are relatively slender members, considerable deformation under load is likely to occur. Because of this tendency, the writer feels that formulas based on the deflection theory would prove more satisfactory, especially if the means of checking the results by a model analysis is not available. In the case of suspension bridges, a design based on the elastic theory requires more material than one based on the deflection theory. The deflection theory does not permit the use of influence lines and, therefore, is somewhat more laborious to apply. Because of the unusual proportions of the bridge under discussion, the writer feels that, if he were called upon to design a similar structure, he would apply the deflection theory.

Although no reference was made to secondary stresses in Professor Garrelts' paper, they were undoubtedly considered in the design. Relatively small bending moments in the arch ribs, in combination with the axial loads, may produce buckling stresses which approach serious proportions. The writer found this to be true in the design of a two-hinged arch, as well as a tied steel arch, each of which had a 425-ft span. As the tied arch had ribs and ties of more conventional proportions than those of the St. Georges bridge, the buckling effect in the latter design might be of major importance. This appears to be very probable, as Fig. 6 indicates that the girder ties and the arch ribs are rigidly connected. If this is so, considerable bending must be transmitted to the ribs from the ties, due, at least, to live load.

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[14] Engr. of Design, Highway Dept., State of New Hampshire, Concord, N. H.

These comments are not intended in any way as a criticism of a very excellent paper, but should be regarded as a brief résumé of some design features that are likely to be slighted but which are of sufficient importance to merit a thorough investigation. Professor Garrelts is to be commended for bringing the attention of the profession to an unusual but pleasing type of bridge and an interesting solution of its design.

J. M. **Garrelts**,[15] **Assoc. M. Am. Soc. C. E.**—The relative economy of the tied arch form for this structure, compared with other possible types, is discussed by Professor Abbett. Professor Abbett made several economic studies of this kind in connection with this project and thus writes from experience. His discussion also emphasizes the attention given to the problem of appearance in the St. Georges design.

Mr. Karol has shown the close approximation of the value of H used in the preliminary design to the value resulting from the final computations. The difference is almost entirely accounted for by the effect of rib shortening and the extension of the tie girder. The values of H, obtained by a formula similar to Eq. 17 but with the last two terms of the denominator eliminated, or

$$H = \frac{\sum \frac{M^1 \, m \, \Delta l}{I_{tr}}}{\sum \frac{M^2 \, \Delta l}{I_{tr}}} \qquad (43)$$

are as follows:

Load at point:	Values of H
1	0.240
2	0.467
3	0.671
4	0.847
5	0.986
6	1.084
7	1.136

The procedure, suggested by Mr. Karol, for proportioning the total moment between the arch rib and girder will be an aid in calculating the approximate stresses and in determining approximate sections that can be used as the basis for the final analysis.

Mr. Dodge first questions the procedure of neglecting the bending resistance of the arch rib in the preliminary design, although he offers no suggestion as to how the first or preliminary stresses can be computed. In designing any statically indeterminate structure, the designer either can make a guess as to the sections, can obtain them by comparison with a similar structure, or can apply an approximate stress analysis from which trial sections can be obtained. Then, a more accurate analysis is made and the sections are revised as necessary. This latter procedure was used in the design of the St. Georges bridge. The bending resistance of the arch rib later was included in the analysis, as explained in connection with Eqs. 10 to 17.

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[15] Associate Prof., Civ. Eng., Columbia Univ., and Designing Engr., Waddell & Hardesty, New York, N. Y.

The objection of Mr. Dodge to the use of Eq. 10*a* may be likened to the objection which is sometimes raised against using $\frac{d^2y}{dx^2} = \pm \frac{M}{E\,I}$ for an originally straight beam, when the correct equation is

$$\frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} = \pm \frac{M}{E\,I} \qquad (44)$$

Using the reference cited by Mr. Dodge, values are found,[16] which result in Eq. 10*a*.

Mr. Dodge states that "A small initial curvature makes a great change in the effect of longitudinal forces on the deflection of the arch rib." The fact is that the change in curvature is essentially a function of the bending moment.

On the assumption of a parabolic rib and only five hangers, Mr. Dodge attempts to compute the forces in the structure. In the St. Georges bridge there are fourteen hangers and the rib is polygonal, being straight between hanger connections, as shown in Fig. 6. He contends that the writer's values for *H* are too high. The following values were determined from the model analysis made at Princeton University at Princeton, N. J., under the direction of Prof. E. K. Timby, Assoc. M. Am. Soc. C. E.—

Load at point:	Measurement of *H* at a point in member:	
	$L_7\,L'_7$	$U_7\,U'_7$
1	0.226	....
2	0.450	....
3	0.662	0.661
4	0.820	0.838
5	0.971	0.971
6	1.081	1.081
7	1.135	1.132

These values, which are in agreement with the theoretical values used, are probably more nearly correct than those computed by Mr. Dodge, using the assumptions he has made; and the increase in the moment M_T in Fig. 13 does not actually exist, if the writer's values of *H* are accepted as correct.

To summarize: Mr. Dodge's conclusions (1) and (2) are the result of a misconception of the writer's procedure in applying the theory used in the paper. Furthermore, his conclusions (3) and (4) appear to be based on a mistaken conception of the writer's development. His statement that "The great influence of longitudinal forces on the deflections of an arch rib must be emphasized" means nothing more than that the correct bending moment must be taken into account. If the deformations due to direct stresses are neglected, the deflection of an arch rib is almost entirely due to bending moment, and the analysis used by the writer certainly gives full attention to the moment. Study of the writer's development will show that there is no assumption that involves a horizontal displacement of the ends of the arch rib with respect to the tie girder.

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[16] "Strength of Materials," by S. Timoshenko, D. Van Nostrand, 1930, Pt. II, pp. 468–469.

Mr. Bejarano has kindly evaluated certain terms of Eq. 12 which the writer neglected in later formulas. From his result, the procedure seems to be justified. Although the effects of rib shortening and tie extension are small, they should probably be considered because a small change in H affects the moment by an amount equal to the change in H times y-y'.

The procedure of calculating the structure by transforming the tie to produce a rib with a transformed moment of inertia I'_{tr} also has been discussed by Mr. Bejarano. He shows that the general results by the two analyses would be almost the same but that in the details there are certain differences which indicate an advantage in assuming that action which is more nearly in accord with the structural behavior.

According to Mr. Eremin, the procedures used by the writer are not adequate for the design of the end connections of the arch rib to the girder. In the first place, the hanger $U_1 L_1$ has practically no change in length so that in triangle $U_1 L_1 L_0$ angle $U_1 L_0 L_1$ is changed only by the moment at U_1 and the moment at L_1. Since these two moments will have the same sense, that is, positive or negative, and both will be small, any resulting moment at L_0 also will be quite small.

The writer does not agree with Mr. Eremin's statement regarding the assumption of constant stiffening girder section in Eq. 17. Actually the summation in the equation indicates that the girder is considered made up of segments over each of which the section is constant. Actually fifteen sections, each one panel in length, were used in the analysis.

The design of the lateral system was made in a manner similar to that used for the lateral system of a curved-chord through-truss bridge. In the arch-rib lateral system, the arch rib forms the chords, and the laterals and struts, the web system, with a portal extending from U_2 to L_0. The top laterals and struts are determined in most cases on the basis of minimum $\frac{l}{r}$ rather than by stress.

The writer is grateful to Mr. Freudenthal for his discussion of the general deflection theory of arches and suspension bridges; also for calling attention to the effects of deflection on self-tied structures.

During the design of the St. Georges tied arch span, it was often spoken of as an inverted, self-anchored suspension bridge. Mr. Gronquist has pointed out this analogy. He has also called attention to the action of these two self-tied structures as contrasted with those externally anchored. In addition, Mr. Gronquist has discussed the use of the tied arch with a continuous girder and has outlined the procedure to be used in solving this type of structure.

The writer considered the development of Eq. 17 to be the same as that for Eq. 4, and therefore did not repeat it. The only difference in the two cases is that, in Eq. 17, the tie girder is considered as a transformed section which includes the bending resistance of the arch rib. Actually, in the assumed singly indeterminate structure, the statically determinate substitute structure can be selected in several different ways.

Mr. Langley's statement regarding formulas based on the deflection theory already seems to have been answered by Mr. Freudenthal and Mr. Gronquist. In this self-tied structure, the deflection theory agrees with the analysis given because the arch rib and tie girder deflect the same amount.

The buckling resistance of the arch rib was considered and was found to be ample; and, although the hangers are connected to the arch rib by rigid connections, the bending moment transmitted by the hangers is very small because the hangers are quite flexible.

In conclusion, the writer wishes to express his thanks to those who have discussed this paper.

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Paper No. 2189

RELATIVE ANGULAR, LINEAR, AND TRAVERSE ACCURACIES IN CITY SURVEYS

By S. A. Bauer,[1] Assoc. M. Am. Soc. C. E.

With Discussion by Messrs. Cleveland B. Coe, W. H. Rayner, Philip Kissam, George D. Whitmore, C. C. Miner and W. O. Byrd, G. Brooks Earnest, and S. A. Bauer.

Synopsis

The methods, procedures, and accuracies of the private practicing surveyor in the past were determined by the algebraic sum of a number of purely local and personal conditions. Each individual locality had its own standards and each surveyor within the locality was an authority unto himself.

Standardization of title search and mortgaging practices throughout the United States now combines with an awakened professional consciousness among the surveyors to demand a standardization of procedure, accuracy, and practice. Much is being accomplished on the standards of practice, but on the question of accuracy (compatible with the peculiarities of the profession) there is little information.

Existing investigations into survey accuracy fall generally into (1) studies of surveys of high precision for specialized purposes, and (2) studies of purely instrumental accuracies under carefully controlled conditions. Although both are interesting, neither indicates accuracies the surveyor can depend on under the conditions and with the equipment that the economics of the profession force upon him.

This investigation was made in that direction. The study is made of actual work that was performed over a period of fifteen years, for clients, in the regular private practice of surveying. The methods and equipment that have been used by the writer for that period are indicated merely for record and are not held out as necessarily good or advisable.

Accuracies as obtained in active practice (angular, linear, and traverse) are tabulated. The factors contributing to accuracy are investigated and certain conclusions that seem to be indicated by the facts, relative to methods and

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Note.—Published in May, 1942, *Proceedings*.

[1] (Bauer Surveys Co.), Cleveland, Ohio.

equipment, are advanced. It is the hope of the writer that similar studies will be made in other localities with other methods and equipment so that a true standard of obtainable accuracy can be determined for the practicing surveyor.

INTRODUCTION

The purpose of the study was to determine the accuracies that were obtained in survey practice by the office of the writer, a surveyor in the metropolitan area of Cleveland, Ohio, with the ordinary equipment of the profession, under conditions imposed by business necessity, traffic and weather, and economic considerations. The equipment used was: 1-min transits of standard American manufacture; 100-ft steel tapes (graduated throughout into feet, tenths, and hundredths), ¼ in. wide and 0.015 in. thick; plumb bobs; hand levels; thermometers graduated to 2° F; sighting rods (6-ft and 8-ft lengths of ½-in. round solid steel); and spring balances graduated in half-pound intervals from 0 to 30 lb.

OBSERVATION METHODS

Angular observations were made in the usual manner. The instrument was set carefully over the point by plumb bob, and stations being observed were marked by sighting rods, which were generally plumbed by balancing in the hands of the rodman and checked by instrument. If the day was windy, or if only a small part of the rod was visible to the instrumentman, the rodman plumbed his rod by plumb bob. When a rod was "planted" instead of being held during the observation, the "planting" was done by instrument.

The angle was generally observed in a clockwise direction, sighting from the center of one rod to the center of the other. All angles were turned six times for a set. The first and sixth turns were read and recorded to the nearest 30″ by estimation. The final reading was divided by six for the value of the set and angle. Only one vernier was read for economy of time. If the mean of six turns was more than 30″ removed from the first reading, the angle was re-observed. In most cases, all six turns were made with the telescope in one position. Where a re-observation was made, the telescope was inverted. (Subsequently this procedure was changed to three turns with telescope direct and three reversed.)

In taping, the tape was generally held suspended throughout. The difference in level between tape ends was observed by hand level and added to the amount necessary to clear obstructions. Points were transferred to the ground by plumb bob.

Tape lengths were marked by arrows drawn with lumber crayon when the taping was over pavement or sidewalk. In other cases the ground was leveled with the foot and a small headed nail was used to mark the point.

Only in recent years has a spring balance been used. About 80% of the measurements under study herein were made without the use of a spring balance. In those cases the tension was estimated with a tendency toward over-pull in order to compensate for taping errors which in general have the effect of a short tape. The tape was lined in by eye by the rear tapeman, the line ahead being marked by a sighting rod. In all cases, after setting a point for a tape end mark, a check measurement was made before proceeding.

It is not the intent of this paper to advocate the foregoing procedure as being either good or proper. The writer merely catalogs the methods and equipment that produced the data herein studied.

Extent of Tabulations and Working Conditions

Since the purpose of this study was to determine accuracies actually obtained in practice, no lines, angles, or traverses were run particularly for this study. The observations used herein represent all field notes and calculations, of the period from 1925 to 1940, applying to specific jobs that this office was hired to perform.

The work included in this study is almost exclusively confined to metropolitan Cleveland. About 60% of the work is within the areas of city development and about 40% in the outlying country. About 75% of the city work is in flat territory with grades that rarely exceed 4%, and the remaining 25% is in territories where the grades generally vary between 8% and 12%.

The outlying country work, involving about 40% of the work studied herein, is distributed as follows: 40% in flat country with grades up to 5%; 20% in rolling country with grades between 5% and 10%; the remaining 40% in rough country ranging from moderately rough terrain to extremely rough where distances had to be measured by slope measurements and vertical angle.

The lengths of traverse sides and of other measured lines vary from 50 ft to 4,000 ft with a mean length of 700 ft for all lines. Traverses vary in total length from 500 ft to 18,000 ft and from 4 to 37 sides per traverse. The average length of traverse is 4,300 ft.

The work studied herein was performed during all months of the year. All climatic conditions are represented except heavy rain or snow, the temperature ranging from 15° F to 95° F.

Methods Used in This Study

To compare angular and linear accuracies and the effect of these accuracies on traverse closures, a common unit for their measurement was determined; that is, the error was expressed as one part in x thousands. This quantity was obtained in linear observations by dividing the calculated probable error of a single measurement into the length of the line. In traverse closures it was obtained by dividing the error of closure into the length of the traverse.

In angular observations the probable error of each single set of six turns was calculated in terms of seconds of angle. The tangents of the angular errors converted to one part per x thousands were then determined and used as the expression indicating the linear effect of the angular error.

In most of the graphs which accompany this study, the vertical coordinates represent the accuracies in terms of one part in x thousands; the horizontal coordinates represent the percentages of the various accuracies. By a reciprocal method, the accuracies in terms of one part in x thousands can also be expressed as y parts per hundred. This latter expression of accuracy is hereinafter used for purposes of establishing mean values.

The probable error of a single observation is expressed by the formula

$$r_s = 0.6745 \sqrt{\frac{\Sigma v^2}{N - 1}} \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which r_s is the probable error of a single observation, v is the residual from the mean, and N is the number of observations. When $N = 2$, Eq. 1 becomes,

$$r_s = 0.9539\, v \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Further, the probable error of the mean of two or more observations is expressed by the following formula:

$$r_m = 0.6745 \sqrt{\frac{\Sigma v^2}{N\,(N - 1)}} \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

which, when $N = 2$, becomes

$$r_m = 0.6745\, v \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

TABULATION OF ANGULAR READINGS

To determine the accuracy of a single reading of a 1-min transit, 176 sets of angles were tabulated. In each case the first reading and the mean of six turns were recorded, together with the difference between the first reading and the mean of six. These results were as follows:

Difference, in seconds, between the first reading and the mean of six (equal to, or less than)	Percentage of total (176) readings
0	15
10	41
20	63
30	83
40	89
50	96
60	98
70	100

The mean difference of the entire 176 angles tabulated was found to be 21″.

Sixty-six angles re-observed from the same instrument setup were then tabulated. The probable error of each single set (the mean of six turns) of each pair of sets was calculated and tabulated and the percentages of the various accuracies determined. The results are plotted as curve 1 in Fig. 1. The mean probable error of the 66 single angles studied in this classification is 3.8″. It is to be noted that all of the angles represented in this classification were re-observed from the same instrument setup, with no appreciable time lapse between observations.

In Fig. 1, the mean value is plotted as a short horizontal line intersecting curve 1 at 3.8″. In order that later in this study the angular and linear accuracies may be compared, curve 1 was then converted into curve 3, shown in Fig. 2, by plotting the tangents (expressed in terms of one part in x thousands) of all angular values of curve 1, thereby giving an angular-error curve in the common unit of one part in x thousands.

After a time lapse ranging from a few days to ten years, 70 angles were then re-observed and studied. Again, the probable error of each single set (the mean of six turns) of each pair of sets was calculated and tabulated and the

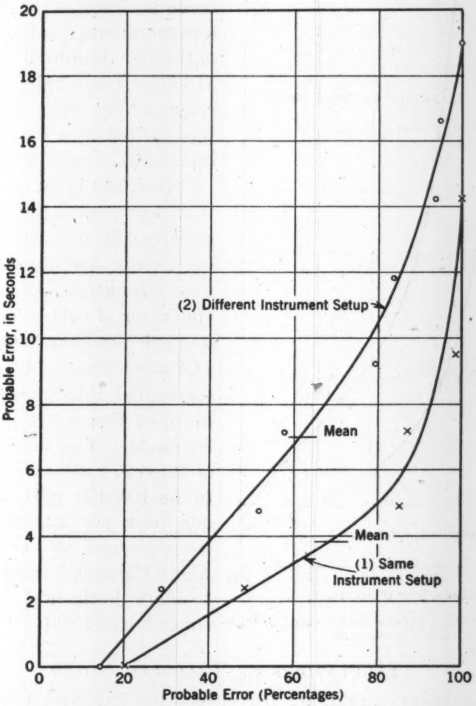

FIG 1.—ANGULAR ACCURACIES IN SECONDS OF ARC; PROBABLE ERRORS OF SINGLE SETS OF SIX TURNS

percentages of the various accuracies determined. The results are plotted as curve 2 in Fig. 1. The mean probable error of the 70 single angles studied in this classification is 7.0″. The mean value was plotted as a short horizontal line intersecting curve 2 at 7.0″ (see Fig. 1).

Again, in order to compare angular and linear accuracies, curve 2, Fig. 1, was converted into curve 4, Fig. 2, by plotting the tangents (expressed in terms of one part per x thousands) of all angular values of curve 2, giving this angular-error curve in the common unit of one part in x thousands.

TABULATIONS OF LINEAR READINGS

For this study 120 lines, retaped by the same party within the same day, were tabulated. The probable error of each single measurement of the pairs of measurements was calculated and converted into the unit of one part per x thousands. The mean of the probable errors and the percentages of the various accuracies were determined. The mean probable error of these 120 line measurements was found to be 0.00319 parts per hundred, or one part in 31,300. These values were plotted as curve 5, Fig. 3(a). The mean value was plotted as a short horizontal line intersecting curve 5 at 31,300.

Next, 274 lines, remeasured during the 15-yr period, but with time lapses between the two measurements varying from a few days to twelve years, were tabulated. As before, the probable error of each single measurement was calculated and translated into the unit used herein; the accuracies were segregated into percentages, and the mean of the probable errors was determined. The mean probable error of these 274 measurements was found to be 0.00439 part per hundred, or one part per 22,800. These values were plotted as curve 6, Fig. 3(a). Again the mean value is indicated as a short horizontal line intersecting curve 6 at 22,800.

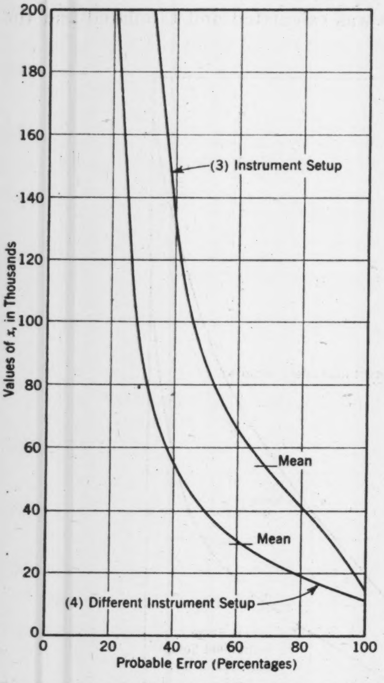

FIG. 2.—ANGULAR ACCURACIES EXPRESSED AS ONE PART IN x THOUSANDS

TABULATIONS OF TRAVERSE NOTES

During the 15-yr period, 92 traverses were run and tabulated by their errors of closure expressed in terms of one part in x thousands. (Due to lack of a precise control in this territory, all traverses herein are closed upon themselves as a complete loop.) The percentage of the various accuracies was determined as in previous investigations of angular and linear accuracies, and the mean of the traverse errors was determined. The mean traverse closure error of the 92 traverses under consideration was found to be 0.00442 part per hundred or one part in 22,600. These values were plotted as curve 8, Fig. 4, the value of the mean error again being indicated by a short horizontal line at 22,600.

FINDINGS OF FACT FROM TABULATIONS

To analyze more clearly the facts determined by this investigation, curves 4 and 6 are plotted in Fig. 4, with curve 8. These are the accuracy curves of angles observed at different times, lines measured at different times, and traverse closures. The mean value of each curve is also indicated in the manner previously described, as well as the levels of first-order, second-order, and third-order accuracies, according to the Society's *Manual of Engineering Practice No. 10* ("Technical Procedure for City Surveys"). Examination of the curves in Fig. 4 and the data from which they were plotted discloses the following facts:

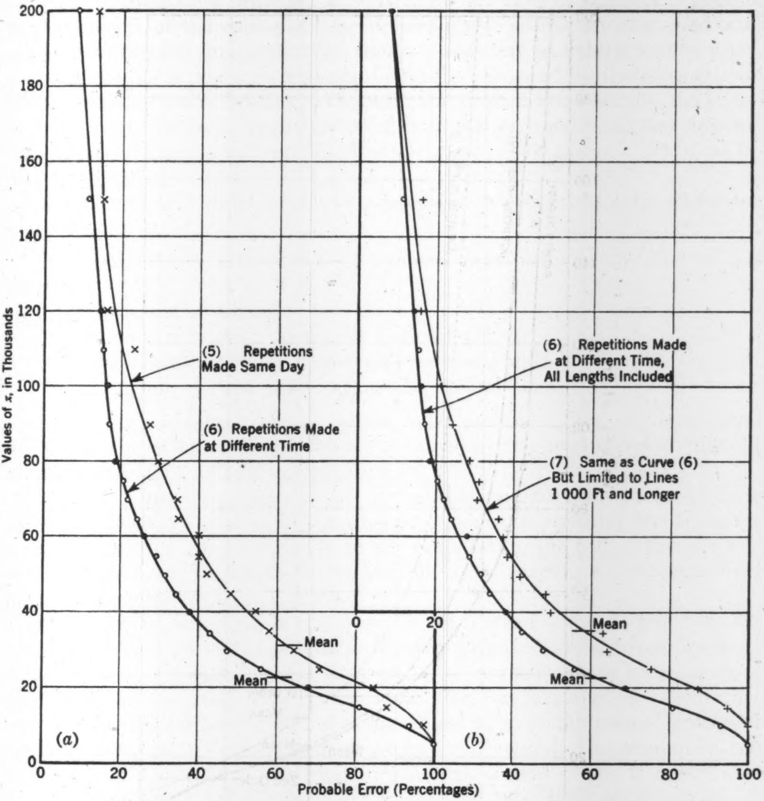

FIG. 3.—LINEAR ACCURACIES, EXPRESSED AS ONE PART IN x THOUSANDS

(*a*) All traverses run in the 15-yr period close with a third-order accuracy or better;

(*b*) 83% of the traverses close with a second-order accuracy or better;

(*c*) 39% of the traverses close with a first-order accuracy or better;

(*d*) The mean of angular accuracy exceeds the mean of linear accuracy by approximately 30%;

(*e*) The curve of traverse closure accuracy is practically identical with the curve of linear accuracy in the normal ranges;

(*f*) The probable error in a single reading of a 1-min transit is ±25″. The mean difference between a single reading of a 1-min transit in a group of six readings, and the mean of that group is ±21″;

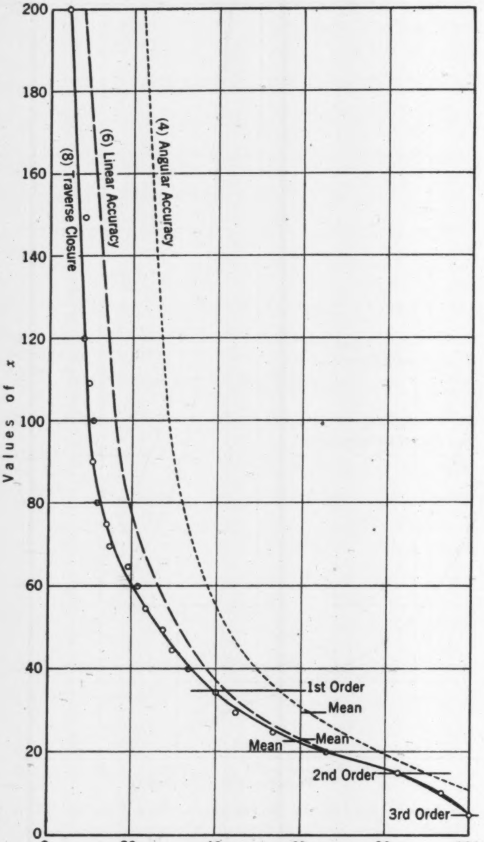

FIG. 4.—TRAVERSE, MEASUREMENT, AND ANGLE ACCURACIES, EXPRESSED AS ONE PART IN x THOUSANDS

(*g*) The mean probable error of an angle re-observed after an appreciable time interval with a 1-min transit turned six times is 7″, or a mean difference of 15″ between the two observations;

(*h*) The mean accuracy of angles re-observed at the same time from the same instrument setup is 84% greater than the mean accuracy of angles re-observed after an appreciable time lapse; and

(*i*) The mean accuracy of lines double taped at the same time is 37%

greater than the mean accuracy of lines remeasured after an appreciable time lapse.

Factors of Accuracy

The foregoing data provide information on the basis of which it is possible to study some of the causes of inaccuracies under actual working conditions with a view toward determining the limits of accuracy and the extent to which accuracy can be increased within the limitations imposed by the economics of the profession, traffic, weather, and local conditions. For example, the variations in the accuracies of angles observed from the same setup can be compared with those of angles observed at different times—a difference of 84%, as indicated previously.

The factors that differ in the two categories which might account for the variation are:

(1) Variations in instrument adjustment, principally in the line of collimation;
(2) Movement of points;
(3) Errors in centering the transit over the point;
(4) Errors in the marking of observed stations; and
(5) Refraction of light due to atmospheric conditions.

It is to be noted that, except for the movement of points, all factors tend to indicate the lower accuracy (the one obtained by re-observations of different times) as the more probably correct one.

Item (1), the adjustment of the instrument, would seem to be of little significance in the variation of the accuracies, since in inverting the telescope for the re-observation of angles from the same setup, the collimation error is corrected in much the same degree that it is by varying adjustments over different time intervals.

In Cleveland, monuments are generally of marked stone or iron bars set below grade and protected by cast-iron boxes in paved street areas; or stone monuments or iron bars set at or near grade but not enclosed in boxes in areas other than pavements. This system rates high for permanence. Nevertheless, experience indicates measurable movement in even the best of protected street monuments. To determine the effect and possibly the amount of the movement of points, and the extent to which movement of points contributes to the difference of accuracies under study, the tabulation of accuracies of angles observed at different times was broken down according to the time interval between the two observations:

Time interval (yr)	Mean probable error (sec)
>0.5	7.3
0.5 to 2	6.4
2 to 3	9.2
3 to 4	6.5
4 to 5	6.6
5 to 10	5.3
10 to 15	7.6
Mean error for period	7.0

It seems clear, from the absence of any pattern in the foregoing and from the lack of consistent change with increasing time lapse, that either the movement over a long period becomes a compensating factor, or the error is smaller than the other contributing errors and, therefore, is lost among them. With Items (1) and (2) eliminated as determining factors in the variation of accuracies under discussion, only Items (3), (4), and (5) are left as determining factors.

The minimum difference between the two mean errors is 3.2″. Then three factors—Item (3) error of centering, Item (4) error of rod position, and Item (5) refraction—contribute to an increased mean error of 3.2″. It seems clear, then, that under the conditions of work stated herein there is a basic minimum error of angular observation of 3.2″.

It is to be noted that none of the three contributing factors to this basic minimum error are altered by (*a*) the calibration of the transit plate, or (*b*) the number of times that the angle is repeated at one setup. Therefore, when the basic error of 3.2″ is reached, neither precision of instrument graduation nor number of repetitions affects the actual result. Then, for city survey work that must be performed during all but the most extreme of weather conditions (assuming the use of sighting rods for pointings and plumb bob centerings of instruments), it would seem that the only advantage to a transit calibrated more closely than 1 min would lie in the fact that the limit of accuracy can be reached more quickly—that is, with fewer repetitions.

This leads to the question: How many repetitions are necessary to reach the ultimate accuracy fixed by external conditions, with a 1-min transit, that ultimate being, according to these tabulations, ±3.2″?

Previously herein it has been shown that the mean probable error of a single observation observed twice from the same setup (in each case an observation being the mean of six turns) is 3.8″. The mean probable error of the mean of the two observations from the same setup is then 2.7″, since the error of the mean is $r_m = \frac{r_s}{\sqrt{N}}$. It is to be emphasized that these are not presented as actual accuracies but merely the relationship of accuracies of angles observed by six turns as against angles observed by twelve turns, and all from the same setup.

It has also been shown that there is a 3.2″ minimum fixed error for the conditions and type of work studied herein, which error establishes the ultimate accuracy that can be obtained.

Since, as noted, the calculated probable error of six turns is 3.8″, and that of twelve turns is 2.7″, it should then require approximately nine turns to equal the limit of accuracy obtainable from one setup, with a 1-min transit, under the foregoing conditions and type of work.

In the work described herein, the average length of line and traverse side is 700 ft. The effect of the centering and rod position errors can be reduced by longer sights, but that is apart from the type or accuracy of transit used. An optical device such as is provided on some Swiss instruments, however, should materially reduce the centering error.

In a manner similar to that used in the investigation of the variation of the foregoing angular accuracies, the variation of linear accuracies is now studied. As noted, the calculated accuracies of lines retaped at the same time are 37% greater than the accuracies of lines remeasured after an appreciable time interval.

The factors that might contribute to this variation are: Differences in tape lengths, movement of points, errors in tension on tape, and errors in determining temperature of tape. Again, except for possible movement of points, all factors tend to make the lower accuracy (lines remeasured with a time lapse) the more probably correct one.

On the accuracy of the tapes used, the writer has few authentic data. There should be a marked uniformity in tape lengths due to the type of tape used, but the manufacturer's statement of uniformity and accuracy has been the principal assurance. One dependable test was made, which showed an error of 0.0005 ft at 68° F, and 20-lb pull when suspended from the ends. Frequent tests by ordinary taping methods of new tapes against those which have been in service have never disclosed a variation of more than the thickness of a division mark, which is about 0.002 ft.

After these tapes have been in use for several months, however, they develop an exaggerated stretch for the standard pull when fully suspended, although they will continue to check for length when supported throughout at 10 lb. This is unquestionably due to the fact that the thickness of the tape has diminished through wear. When this error becomes noticeable, the tape is no longer used for anything but minor measuring.

To determine the extent to which the movement of points contributes to the variations under consideration, a breakdown of the measurement tabulation was made on the basis of the elapsed time, as in the study of angular accuracy variations. The results are as follows:

Time interval (yr)	Mean probable error, in parts per hundred
>0.5	0.00467
0.5 to 2	0.00405
2 to 5	0.00334
5 to 10	0.00389
10 to 15	0.00484
Mean error of period	0.00439

Again no pattern or variation is discernible in relation to time. It would seem, then, that the time element beyond an immediate remeasurement has no deciding effect upon the accuracy of remeasurements, and, therefore, the movement of monuments is either a smaller error than the other errors involved, or the movement is compensating over a period of time.

Errors of tension are now being corrected in a large measure by the use of a spring balance. Insufficient data are available to determine the amount of this error prior to the use of the spring balances.

Unquestionably the largest source of taping error, and probably the greatest contributor to the variations of accuracy under discussion, is the difficulty of determining tape temperature on sunny days. During the summer months pavements are often 15° warmer than the air only 4 ft above them. A tape that has been in the sun on a hot day will become too hot to hold. In winter a tape will absorb heat from the direct rays of the sun in spite of low air temperatures.

No thermometer that the writer has used, which will stand daily use, can be depended upon for more than either air or pavement temperature. In general, when there is a wide discrepancy between the two, an attempt is made to estimate the temperature at tape level and tape temperature. Where a measured line passes through an area that varies widely as to sun and shade, some attempt is made to estimate the mean temperature for the line.

As previously indicated, the mean probable error of a single measurement of a pair of measurements made at the same time is 0.00319 part per hundred, and the mean probable error of a single measurement of a pair of measurements made at different times is 0.00439 part per hundred. Furthermore, it follows that the mean probable error of the mean of two measurements made after an appreciable time interval is 0.00310 part per hundred. From this it seems clear that a remeasurement at the same time will in actual fact not increase accuracy (omitting from consideration at this time an actual mistake of reading, marking, etc.), since the probable error of a single measurement is already within 3% of the accuracy of the mean of two measurements made at separate times.

Curve 6, Fig. 3(*b*), represents the accuracies of lines of all lengths that were repeated after an appreciable time lapse. Curve 7, Fig. 3(*b*), was derived from the same tabulation but was limited to lines of 1,000 ft and longer. In each case, the mean value of the probable errors included in the curves is indicated by a short horizontal line intersecting the curve at the level of the mean error. As determined previously, the mean probable error of curve 6 is one part in 22,800. The mean probable error of curve 7 is one part in 35,500. It is apparent from Fig. 3(*b*) that accuracies are approximately 50% greater for lines of 1,000 ft and longer than for lines of all lengths.

Examination of Fig. 4 indicates a marked similarity in linear and traverse accuracies in the normal ranges of precision. As noted, angular accuracies are about 30% higher. The similarity between the traverse accuracies and the lesser of the two contributing accuracies is in accordance with the probabilities. Although it is impossible to predict how linear and angular accuracies will combine within a single traverse (there are sixty-four possible combinations of the errors in a triangular traverse), nevertheless, in a large number of traverses the mean result can be determined.

To illustrate, the linear error of a traverse side and the angular error affecting that side can either be in such a direction that they are added, or in opposite directions and therefore subtracted. In a large number of traverses, the traverse errors are made up of large numbers of linear and angular errors, some of which compensate each other as the difference between the two, and some of which combine to the total of the two. The probability of one is equal to the probability of the other; therefore, the mean of the two extremes will represent the resulting traverse error for a large number of cases.

If the angular error is represented as E_a, the linear error as E_l, and the traverse error as E_t, then the traverse error (E_t) of a single traverse will lie between the two extremes, $E_l + E_a$, and $E_l - E_a$, assuming E_l is numerically greater.

The traverse error of a large number of traverses will be the mean of the two extremes, since the probabilities of each are equal:

$$E_t = \frac{(E_l + E_a) + (E_l - E_a)}{2} = E_l \ldots\ldots\ldots\ldots\ldots (5)$$

—if E_l is the larger of the two contributing errors.

On this basis then, with the same angular accuracy as shown by curve 4, Fig. 4, and traverse sides of 1,000 ft or more as shown by curve 8, the traverse accuracy should increase, making the angular accuracy the determining factor of traverse accuracy.

As a check against this hypothesis, the traverses included in the tabulation for curve 8 were segregated as to the mean length of traverse sides. Twenty of the traverses in the tabulation had a mean length of side of 1,000 ft or more. The mean error of these traverses was found to be 0.00350 part per hundred or one part in 28,600. The mean value of curve 4 is one part in 29,400. The writer's hypothesis—that traverse accuracy is determined by, and equal to, the least accurate of the two contributing factors—is substantiated in this test by within 3%.

It would seem, then, that with the same equipment and methods now in use in the writer's office and described previously herein, a mean traverse closure of one part in about 30,000 can be maintained without undue consideration of climatic conditions, provided that traverse sides could be maintained at 1,000 ft and more.

It is of further interest to note that, for various accuracies, the percentage of the total number of traverses in rough terrain was as follows:

Traverse accuracies	Percentage of total in rough terrain
1 : 9,000 or less	100
1 : 14,000 or less	71
1 : 19,000 or less	55
1 : 20,000 or more	25

The traverses were studied for variations of accuracy based upon the over-all length of the traverse. Nothing conclusive was found, and the variations wavered back and forth about the curve for all lengths.

The dominating factor is length of side rather than length of traverse. The length of side in turn is determined by conditions of terrain. The tabulation does not contain a sufficient number of traverses of the same traverse side lengths to permit a study of the effect of variation of traverse over-all length.

CONCLUSIONS

This study has established certain findings of fact as to accuracies that have been obtained in actual survey practice. A number of these were itemized under the heading "Findings of Fact from Tabulations" (see Items (*a*) to (*i*)). Three other items follow from the discussion presented under the heading "Factors of Accuracy":

(*j*) The mean probable error of remeasurements of the same line with ordinary equipment is 0.00439 part per hundred, or expressed differently, the difference between remeasurements of the same line averages 0.00920 part per hundred;

(*k*) The accuracy of measurement of lines of 1,000 ft and more is greater than that of lines shorter than 1,000 ft; and

(*l*) Accuracies of traverses with sides having a mean length of 1,000 ft or more are greater than those with sides having mean lengths under 1,000 ft, other factors being equal.

All the foregoing items support the following conclusions:

(1) There is a definite limit to the accuracy that can be obtained from a single setup of a transit. This limit is independent of the fineness of the transit's plate calibration, and of the number of repetitions in the observation; and it should be reached, by a good 1-min transit under average working conditions, at nine turns.

(2) Retaping of a line, forward and back, under average conditions, adds nothing to actual accuracy (except to locate mistakes in taping).

(3) In the long run, accuracies of traverses are determined by, and equal to, the lesser accuracy of the two contributing factors of taping and angular observation.

The foregoing findings (Items (*a*) to (*l*)) and reasoned conclusions ((1) to (3)) pose the following questions:

A. Present specifications for survey accuracy stress the error of closure primarily. In areas that lack a precise control, are such specifications adequate to assure a desired accuracy?

B. Is the trend toward transits of ever-smaller plate graduations justified for city work?

C. Might not accuracies of transits for city surveying be maintained at their present level, and work of setup and reading be facilitated by changes in design?

DISCUSSION

Cleveland B. Coe,[2] **M. Am. Soc. C. E.**—The author of this paper joins a small and select group—those who offer something for the advancement of the professional knowledge and ability of the surveyor. In any issue of almost any technical publication, structural, hydraulic, and sanitary engineers may find results of investigation and analysis which will bear more or less upon their daily problems, but the land surveyor is the "forgotten man" of the profession, although not of the general public, to whom he is the "engineer."

Truly, the phrase used by Mr. Bauer is a masterly one (see "Synopsis"): "Under the conditions and with the equipment that the economics of the profession force upon him." The economics of the profession have forced considerable shoddy practice on many men who knew what should be done to give reliable results to their clients. The clients neither know nor care how the surveyor arrives at his results; nor are they interested in the accuracy attained, in many cases.

The writer surveyed farm lands for many years, in competition with day labor of the crudest type. Very frequently he went on a job alone, and assistance was provided by the client, in the form of his sons and nephews. This was, and is, a condition of very common occurrence and will continue as long as registers of deeds are permitted by law to accept the descriptions and plats made by unlicensed surveyors. Much has been done to correct this condition, but much educational work remains before the landowner becomes willing to pay the price which efficient help and accurate measurement of land should reasonably cost.

The paper deals entirely with city surveys and concerns itself with property of considerable value. The writer's experience was with mountain and farm land, worth from $10 to $300 per acre, and the conditions and equipment forced upon him by the economics of the profession were certainly inadequate, even for so low a valuation. If similar studies are presented, as hoped by the author, perhaps someone will present the rural point of view.

The survey of farm lands by the private surveyor, as distinguished from cadastral surveys of large areas by government or corporation surveyors, is generally in one of the following categories:

(1) To establish a dividing line or a disputed corner;
(2) To ascertain true acreage; or
(3) To subdivide property for inheritance or sales purposes, either in large tracts or small lots.

Case (1), in the writer's experience, comes more under the head of an art than a science. In Tennessee, there are no applications of the law of township and section, but all property traces back to original grants, mixed up in the wildest confusion. In no case has the procedure been clear-cut and capable of solution entirely by scientific methods and careful application of angular and linear measurement. Diplomacy is an important tool of the surveyor, and practically all final results are the result of compromise, either known or unknown to the parties to the dispute. Under such conditions, repeating angles and tension in a tape play no part.

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[2] Lt. Col., Corps of Engrs. U. S. Army, Knoxville, Tenn.

Case (2) requires closer adherence to the rules and is the one in which old notes can be reviewed and rerun. Unfortunately for such studies, it is very seldom that a corner monument can be exactly occupied. Nearly all lines are offset lines, with temporary monuments, tied to the corners by side shots. Here the parsimony of the owner plays a great part, not only by preventing the employment of skilled help but by making it necessary to run the offsets in places requiring the least clearing, not following the boundary as closely as it should be. Here, also, the rural objection to measurement in a horizontal plane manifests itself, not infrequently making it necessary to measure vertical angles by any means at hand and to correct slope measurements in the office without the knowledge of the client.

Case (3) is an extension of case (2), with the boundary survey checked by numerous crosslines. In this case, a much closer adjustment is possible.

Unfortunately, the writer's notes are stored "for the duration." It would be of interest for some country surveyor to produce a tabulation of his work, showing the accuracies attained under fire (sometimes literally), as set forth against the value of the land surveyed. All credit is due the many honest and efficient men in all parts of the United States who give reasonable results under very unfavorable conditions and thereby aid in the upbuilding of the profession, which has been sadly disgraced at different times and places by men who are neither efficient nor honest.

W. H. RAYNER,[3] M. AM. SOC. C. E.—Careful accounts of surveying procedures, and analyses of the results obtained, are meager and it is gratifying to have such a report by Mr. Bauer.

It well exemplifies the fact that, by proper attention to the more important sources of error, the accuracy of surveys, using ordinary equipment, and at small additional expense, can be greatly increased over the crude results so often obtained.

There are no very definite criteria for comparing the merits of the different surveying instruments, and it has become customary to use the least count of the vernier to designate the character of the instrument used. This is unfortunate because there are wide differences between transits having the same vernier count; also, many transits having 1′ verniers are superior to others with 30″ verniers.[4] The better instruments yield better results with such a saving in time that it is poor economy to use any but the best quality on surveys of any considerable extent.

The use of the "probable error" for two observations as computed by Eqs. 2 and 4, and as described under the heading "Tabulations of Linear Readings," is scarcely justified. This quantity is based on the assumption that a normal distribution of the accidental errors (residuals) of a series of observations is under investigation, and such a distribution can be obtained only where a considerable number of observations are used. It is certainly dubious practice to apply the formulas to a series of less than ten observations.

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[3] Associate Prof., Civ. Eng., Univ. of Illinois, Urbana, Ill.

[4] "Two Investigations on Transit Instruments," by W. H. Rayner, Eng. Experiment Station, Univ. of Illinois, Urbana, Ill., 1938.

It is stated that "The probable error in a single reading of a 1-min transit is ± 25″ (see heading "Findings of Fact from Tabulations," Item (*f*)), but it is not clear under just what conditions this value was found. In particular, an important factor is the reading lens that was used. The writer found in his investigations[4] that, in comparing the results of reading angles with and without a reading lens, the probable error of single readings with the naked eye was reduced about one half when a three-diameter lens was used. It was also found that a large part of the error of single readings of angles is caused by the inability of the observer to set the vernier precisely at zero. This condition explains, in part, why the errors of reading angles with ordinary equipment are so greatly reduced when the method of repetitions is used.

The analysis of results evidently has not differentiated sufficiently between the effects of accidental and systematic errors. For example, all traverses were run as loops, wherein the effects of systematic errors do not become evident. Thus, if a tape were of incorrect length, all sides would be affected proportionately, and the calculated error of closure would in no way be affected by this condition. Also, a constant error in the measurement of the temperature would produce a like effect. Further, the duplicate measurement of a line under the same conditions will not reveal the presence of any systematic errors. This fact probably accounts for the fact that the average accuracy of lines measured in different years was less than that for lines measured forward and back on the same day (see Fig. 3).

Accidental errors are compensative in their effects, and hence the errors as expressed in this report should be smaller for a large number of observations than for a few. This is exemplified by the fact that the accuracy found in taping long lines was greater than for short lines (curves 6 and 7, Fig. 3). This latter consideration is not verified, however, in the case of traverse closures, for no significant difference was found for traverses of different lengths. This unusual result seems to call for some explanation. It is very probable that the computed errors of closure would be larger if the traverses had closed upon each other, instead of being run as closed loops.

It would seem that, where so much care was used in measuring the distances, reference would be made to the U. S. Bureau of Standards to determine the true lengths of the tapes used. The fact that two tapes are of the same length is not complete evidence that they are correct. Their true length is never known until they have been compared with the United States standard, or with a "standardized" tape which has thus been compared.

The effects of the combined errors in distance and angle measurements in traversing are scarcely as simple as that indicated by Eq. 5. The writer attempted an analysis[5] of these effects in 1928. The relations between the linear and angular errors are complex and, in general, their effects must augment each other, so that the total error of closure is larger than the effect of either one taken separately.

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[5] "Specifications for Transit Traversing and Stadia Leveling," by W. H. Rayner, *Transactions*, Am. Soc. C. E., Vol. 94 (1930), p. 679.

Considering the foregoing remarks, the following comments may be made about the conclusions offered by the author:

(*a*) The analysis in the sixth paragraph preceding Eq. 5 leads to conclusion (2)—namely, that "Retaping a line, forward and back, under average conditions, adds nothing to actual accuracy." This analysis is faulty in that it disregards the effects of systematic errors that would appear in measurements made at different times but not in repeated measurements made under the same conditions. In other words, it is probable that the true errors in the measurements made on the same day are larger than the errors indicated by the duplicate measurements. If so, conclusion (2) cannot be justified by the evidence and analysis used. Clearly, as regards the accidental errors involved, the error of the mean of two measurements will be less than that for a single measurement.

(*b*) Conclusion (3) is based on the oversimplified analysis of the relations of the various errors in traversing. It is conceivable that, in a given traverse, the effects of angular errors will reduce the effects of the linear errors, or vice versa, but in general, where the laws of accidental errors are operating, the total error of closure will be comprised of both angular and linear errors and will be greater than the error produced by either class alone.

Philip Kissam,[6] M. Am. Soc. C. E.—Very definitely, the paper by Mr. Bauer is a contribution to the art of surveying. As the author states, too little is known of the actual accuracies obtained in the field in the usual property surveys, or just what can be done to obtain the desired accuracies at the minimum cost. If more surveyors would follow Mr. Bauer's lead by compiling data as he has, there would be no question whatever that accuracies, methods, and instruments would be improved and costs reduced.

Errors in Transit Results.—To analyze Mr. Bauer's conclusions, the writer has classified the errors of the transit as follows:

Major errors in scale reading—
(1) Errors in determining what the scale reads (probable error = r).
Minor errors in scale reading—
(2) Errors in graduation.
(3) Errors due to play between the inner and outer spindles.
Major errors in pointing—
(4) Errors due to lateral refraction.
(5) Errors due to the positions of the instrument and signals (including phase). The probable error of Items 4 and 5 is h. The two types cannot be separated using the data given.

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[6] Associate Prof., Civ. Eng., Princeton Univ., Princeton, N. J.

(6) Errors due to not pointing the cross hair on the apparent signal (probable error = p).

Minor errors in pointing—

(7) Errors due to temperature changes in the instrument and tripod.

(8) Errors due to lack of adjustment of the instrument.

(9) Errors due to play in the slide and the trunnions and errors in optics.

(10) Errors due to lack of verticality of vertical axis in setup.

(11) Errors due to play between the outer spindle and the leveling head.

This classification depends on the assumption that the transit is in good working order and that proper operational procedure is used.

In this discussion only the major errors will be considered as the minor errors are negligible under the conditions stated.

Mr. Bauer has determined the accuracy of a single reading of a 1′ transit by tabulating 176 sets of angles. In each case the first reading and the mean of six turns were recorded, together with the difference between the first reading and the mean of six. The mean difference of the entire 176 angles tabulated was found to be 21″. According to the theory of errors the probable error would be: 0.848 × 21″ = 17.8″.

It will be seen that this error E_6 is made up as follows:

Referring to Items 1, 4, 5, and 6, "h" is not present as each measurement was made at one time and with the same positions for the signals and the instrument. Since each pair of observations was made at the same time it is assumed that the lateral refraction was also constant; thus:

$$E_6 = \tfrac{1}{6}\,(k\,r_1 + r_6 + p_1 + p_2 + \cdots + p_{12}) \qquad (6)$$

in which E_6 is the actual error of the mean of six repetitions, and k is the ratio between the error in setting the vernier at zero and the error of reading a 1′ transit to 30″. If E_1 is the actual error of the first reading:

$$E_1 = k\,r_1 + r_2 + p_1 + p_2 \qquad (7)$$

and

$$E_1 - E_6 = \tfrac{5}{6}\,k\,r_1 + r_2 + \tfrac{5}{6}\,p_1 + \tfrac{5}{6}\,p_2 - \tfrac{1}{6}\,(r_6 + p_3 + p_4 + \cdots + p_{12}) \qquad (8)$$

The probable value of Eq. 8 is:

$$17.8 = \tfrac{1}{6}\,\sqrt{(25\,k^2 + 37)\,r^2 + 60\,p^2} \qquad (9)$$

Mr. Bauer finds the probable error of a set of six turns to be 3.8; hence:

$$3.8 = \tfrac{1}{6}\,\sqrt{k^2\,r^2 + r^2 + 12\,p^2} \qquad (10)$$

Solving Eqs. 9 and 10 simultaneously:

When k =	r =	and p =
0.3	16.2	4.4
0.5	15.5	4.3
0.8	14.0	4.0
1.0	13.0	3.9

Assume the third set of values as correct ($k = 0.8$).

It is possible to determine h. Mr. Bauer has determined the probable error of an angle measured by six repetitions when the measurements were made at different times. Such an error includes h. The value determined is 7.0″. Therefore, $7.0^2 = 3.8^2 + h^2$; and $h = 5.9''$. If there is no lateral refraction and the observer can assume that the error in position of the instrument is one half that of the signal, the angular error due to a setup s is $5.9 = \sqrt{2\,s^2 + 2\,(2\,s^2)}$; and $s = 1.89''$.

Since the average length of line is 700 ft, the probable error in position is expressed as follows: $d = \sin 1.89'' \times 700 = 0.0066$ ft, in which d is the error in setting up the transit.

This computation results in the following probable errors in angle determination:

r (reading scale) = 14.0″
0.8 r (setting, zero) = 11.0″
h (position of transit and signals at 700′) = 5.0″
p (pointing on apparent signal) = 4.0″
d (transit position) = 0.0066 ft
2 d (signal position) = 0.0132 ft

These depend on two assumptions. The first is that the vernier can be set at zero with 0.8 as the error of reading the scale. The second is that the error of the signal position is twice the error of the transit position.

There are no data in Mr. Bauer's paper from which can be deduced the decrease in the probable error that would result from reading the minute transit to 15″ or the effect of finer graduations. It is the opinion of the writer that:

1. There is a considerable reduction in probable error when the vernier of a 1′ transit is read to 15″. This procedure has been followed since 1933 by the New Jersey Geodetic Control Survey with considerable reduction in rejected angles.

2. The fineness of graduation has little effect on the accuracy, the accuracy being chiefly a function of the clarity of the graduation as presented to the observer.[7]

Accuracy of Linear Measurement.—Mr. Bauer has given the probable errors of linear measurement for immediate retaping and for retaping at a later date. As he states, this is not based on comparisons with precisely measured high-accuracy surveys. As the accuracy of taping for a given field procedure depends chiefly on tension, temperature, slope, and tape calibration, retaping by the same party on the same day with the same spring balance gives a poor determination of the accuracy, and, as Mr. Bauer points out, is of little practical value except to pick up blunders. The probable error for this condition is given as 1 : 22,800 or 0.0000439, which corresponds to an angle of 9″.

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[7] "Accuracy of Transit Work Depends on Clearer Circle Graduations," by Philip Kissam, *Engineering News-Record*, April 10, 1941, p. 51.

From the theory of errors (which Mr. Bauer quotes) the following is evident: Probable error $= 0.954 \frac{\text{Total error}}{2}$. Hence, the total error or accuracy for linear measurement is 1 : 10,880.

Traverse Accuracies.—It would seem that when traverses are measured the field procedure should be such that the probable error of the transit would be about equal to that of the linear measurements, as Mr. Bauer suggests. If the angles were measured once with the telescope direct and the measurement repeated once with the telescope reversed (indicated herein as 1 DR); and if both verniers were read each time, the expression for the probable error of angular measurement would be $\frac{1}{2}\sqrt{\frac{1}{4}(k^2 + 3)\, r^2 + 4\, p^2}$. Substituting the values computed, the probable error is 7.78″.

Introducing the errors of position of signals and transit, the resulting accuracy is $\sqrt{7.78^2 + 5.9^2} = 9.8''$. This probable error corresponds very closely with the probable error of linear measurement found by Mr. Bauer, which is the equivalent of an angular accuracy of 9.0″. Thus, where an average accuracy of 1 : 10,000 for a traverse is required, the angles should be measured 1 DR, both verniers should be read, and the usual methods of distance measurement used. Such traverses would have an accuracy of between 1 : 5,000 and 1 : 15,000.

If higher accuracies are desired, the method of taping should be improved. If it were desired to obtain traverses having an accuracy of 1 : 25,000, a probable error of 1 : 52,000 would be necessary. The use of the plumb bob would have to be eliminated. Probably this could be obtained in commercial work by using a 300-ft tape fully supported on some surface, like a sidewalk or street pavement, using a thermometer and spring balance, and running a rough profile over the tape for slope corrections. When such a procedure is used the accuracy of measurement will usually attain better than 1 : 25,000, having a probable error of 1 : 52,000.

A probable error of 1 : 52,000 corresponds to about 4″ of arc. To obtain a probable error of 4″ in the transit work would require a considerable improvement in the centering of signals and possibly in the transit itself.

The desired results could be obtained if the signals were designed to have no phase and the work could be performed at any time when there was at least some haze or cloudiness so that the lateral refraction would be reduced, and if the signals and the transit were set so that the probable error of their positions was 0.005 ft. Under these conditions $h = \sqrt{4\, s^2} = 2\, s$; but, at 700 ft, $s = \frac{0.005}{700} = 1.40''$, and $h = 2.8''$.

Transit accuracy also would have to be maintained so that a lower probable error would result than Mr. Bauer found to be the rule.

To obtain this accuracy it is suggested that one set of 3 DR be used, reading both verniers at the beginning and end of the set. Then, the probable error is $\frac{1}{6}\sqrt{\frac{1}{4}(k^2+3)\,r^2+12\,p^2} = 3.20''$. Combining the error of the transit operation and the error of the positions of the transit and the signals, the probable error of angle determination would be $\sqrt{(3.20)^2+(2.8)^2} = 4.2''$. This is practically low enough.

This error would be reduced, of course, by using lines of more than 700 ft so that a reasonably good balance between transit accuracy and taping accuracy would be maintained. Assuming that transit errors and taping accuracy affect the traverse equally, the probable error of such a traverse would be about 1 : 50,000, and the accuracy about 1 : 24,000.

Conclusions.—1. Assuming that Mr. Bauer's traverses represent the usual traverses run by good land surveyors, the accuracy to be expected is between 1 : 5,000 and 1 : 15,000.

2. The transit work is of a much higher degree of accuracy than the taping and so long as the present method of taping persists there is no advantage in turning the angle more than 1 DR, using both verniers.

3. If high-grade traverse work is required, an accuracy of 1 : 24,000 can be obtained with little, if any, increase in cost if a 300-ft tape is used and the angles turned 3 DR reading both verniers.

These recommendations are based on the assumption that a better target is used than is ordinarily found in practice and that the noon hours on clear days be avoided for angle measurement.

It will be noted that in determining the order of accuracies of traverses, Mr. Bauer has assumed that *Manual No. 10* uses a "probable" error in specifying the order of accuracy. It is the belief of the writer that by "error of closure" is meant the total error divided by the total length, a value which is much more difficult to obtain. For example, the average position closure required for first-order traverse is 1 : 35,000. To obtain this accuracy the average probable error would have to be 1 : 73,500.

GEORGE D. WHITMORE,[8] M. AM. SOC. C. E., C. C. MINER,[9] ASSOC. M. AM. SOC. C. E., AND W. O. BYRD,[10] ESQ.—An interesting approach to the complex subject of traverse accuracies is contained in this paper. The writers have long felt that too much emphasis has been placed on the "just-happened" closure of the entire loop, and not enough on designing measuring instruments and measuring procedures that will insure uniform precision for each and every tangent of the entire traverse. Experience shows that traverses having circuit closures of high order, such as 1 : 10,000 or 1 : 20,000, may conceal errors, in the individual courses, of the order of 1 : 1,000. The writers believe that the accuracy specified for a traverse loop or line should be an indication also of the approximate precision of each individual course in the entire line.

Traverse does not have as many rigid geometric checks available as does triangulation, and for that reason safeguards against blunders are imperative. Within the limits of accuracy desired in the traverse, known systematic errors should be eliminated, and measuring procedures should be used which will minimize the unknown systematic and accidental errors. Measuring instruments should be used that will provide the required accuracies most economically.

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[8] Chf. of Surveys, Maps and Surveys Div., TVA, Chattanooga, Tenn.

[9] Civ. Engr., Maps and Surveys Div., TVA, Chattanooga, Tenn.

[10] Associate Mathematician, Maps and Surveys Div., TVA, Chattanooga, Tenn.

Unless consideration is given to some of the systematic taping errors, third-order traverse accuracy can be attained only under exceptional circumstances. For instance, any one of the following conditions would result in systematic errors that would equal the limits for third-order traverse accuracy (1 : 5,000):

(*a*) If the actual length of a 100-ft tape differed by as much as 0.02 ft from the true length, a systematic error of 1 : 5,000 would result. Such a change in length may be caused by stretching or kinking.

(*b*) Under certain conditions a systematic error of 5 lb in the tension applied to the tape would result in a systematic error of 1 : 5,000.

(*c*) If the actual temperature of the tape is in error by 30° F, a systematic error of 1 : 5,000 would result. Such an error may result from applying no temperature correction, or, under extreme conditions, from applying the temperature of the air rather than of the tape itself.

There are, of course, other systematic taping errors, the accumulated effect of which may be appreciable.

The need for reducing systematic taping errors is felt immediately when the traverse runs between two previously fixed points, such as high-accuracy triangulation stations, or is part of a traverse net run under varying weather conditions by two or more parties using different measuring equipment and procedures. These errors, obviously, will not ordinarily be felt when dealing only with a single traverse that is run in a loop, originating and terminating on the same point, such as described by the author.

The attempt to eliminate systematic errors, both linear and angular, will influence, to a large extent, the measuring procedures and instruments used; a study of surveying accuracies is incomplete until the effects of such errors are considered. Systematic angular errors fortunately can be largely eliminated by proper observing procedures and by distributing angular closures before computing latitudes and departures.

The writers believe that accidental angular errors have a considerable effect on the quality of a traverse, but that accidental taping errors, because of their more nearly complete compensation, have a much lesser effect and account for only a small part of the total traverse closure. The effect of accidental angular errors can be reduced, however, by means of any or all of the following safeguards: (1) Frequent azimuth control lines through triangulation; (2) frequent astronomical observations; and (3) use of long azimuth control lines, which span several of the shorter, directly taped courses.

The writers would like to add the following thoughts to the three questions posed at the end of the paper:

A. The limiting of the closure error is not adequate to assure a desired traverse accuracy. The writers believe that the specifications should include the measuring procedures and instruments to be used, so that each individual tangent will be of the required accuracy.

B. Smaller plate graduations on transits for city work are justified primarily for reasons of economy, because the desired angle accuracy can be obtained with a fewer number of turns.

C. The writers find little to criticize in the design of the newer transits now on the market. Of course, improvements will continue to be made; however, they will probably be in the direction of greater accuracies and more stability.

G. BROOKS EARNEST,[11] M. AM. SOC. C. E.—Far too often the question uppermost in the mind of the practicing, private surveyor is that of profit alone, rather than profit plus self-satisfaction resulting from assurance of relative accuracy of the individual surveys. Obviously, engineering economy dictates the essential equipment, together with the methods of operation and order of procedure for various classifications of surveys executed by the private surveyor. Therefore, specifications are usually neglected—which can be contributed largely to the existing multifarious public and private control surveys.

When one undertakes to make an exhaustive search of survey records through an accumulation of field books covering a practice of fifteen years or more, he is to be commended highly. Would that other practicing surveyors might similarly correlate their work, if for no other reason than to determine their budget of accuracies throughout stages of their practice.

In all probability, there would be little similarity between published accuracy curves of a score of territorily scattered, private surveyors. Many reasons contribute to this fact; such as: Instrumentation, individual (unique) methods of operation, topography, existence or nonexistence of precise control, taping procedure, and climate. Nevertheless, the underlying thought in such research is to inculcate "precision thinking" among those in the field of surveying—not with the intent that an estate survey in the suburbs should be executed with the same degree of precision as a "downtown" survey where property is worth several thousand dollars per front foot, but wholly with the idea of impressing the private surveyor with the value of uniformity of specifications for individual types of surveys.

The author's results represented by curves, (3) through (8) (Figs. 2, 3, and 4), are indicative of experienced personnel and painstaking field execution. However, to one who is experienced in traverse surveys of first-order and second-order nature and with the knowledge of the pitfalls encountered even with the use of 10″ repeating instruments, taping bucks, etc., to comply with the control precision specified, it is difficult to conceive the respectable percentage of high-order results from commonly called "third-order" methods of field operations. Of course, without precise control, loop closures must be relied upon to compute the probable error of closure, and there is the possibility of compensating errors fortunately producing occasional, exceptional order of precision.

Relative to angle observations and taped distances between two monuments of the common "garden variety" street intersection type, taken several years apart, there would naturally be no pattern of mean probable error (especially north of the Mason and Dixon Line). Furthermore, geological conditions, frost conditions, and depth of monument would affect such comparison measurements differently in cities widely separated in latitude. Such "lapse time-measurement" curves, therefore, conform strictly to individual conditions and are meaningless unless observations are between so-called permanent monuments that have their support below the frost line and are low in susceptibility to frost action.

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[11] Associate Prof., Eng. Surveying, Case School of Applied Science, Cleveland, Ohio.

Uniformity is the biggest factor in making related studies of the private, practicing surveyor. The Society's Surveying and Mapping Division recognized this fact when, in 1928, it formed the Committee on City Surveys and directed this Committee to draft a manual "for the purpose of describing the various divisions of the complete City Survey, to furnish specifications for it, and to make detailed recommendations as to accepted technical methods." The result of this Committee's study and research was published in 1934 as *Manual of Engineering Practice No. 10,* entitled "Technical Procedure for City Surveys." The scope of this manual dwelt chiefly on control surveys of highest order. Later, in 1940, due to increased importance in the subject, *Manual of Engineering Practice No. 20* was published entitled "Horizontal Control Surveys to Supplement the Fundamental Net." These manuals represent approximately ten years of active committee work in drafting the universally recommended procedures for the several orders of control. Therefore, it behooves the practicing surveyor, who desires to stay abreast of the field, to conform to procedures recommended in these manuals and thereby make possible tangible comparisons of individual accomplishments in the future.

S. A. BAUER,[12] ASSOC. M. AM. SOC. C. E.—The writer is indebted to Colonel Coe; Professors Earnest, Kissam, and Rayner; and Messrs. Whitmore, Miner, and Byrd for their kind discussions of this paper. The discussions have been helpful in suggesting new approaches to this study of traverse accuracies.

Professor Rayner is correct that Eq. 5 (which attempts to view the overall picture) tends to oversimplify the problem in relation to the individual traverse. Curves 4, 5, and 8, Fig. 5, offer a truer picture of the problem as applied to the individual traverse than that previously presented in Fig. 4, since the conditions within a specific traverse are more nearly met by curve 5 than by curve 6. In Fig. 5, curve 9 represents the value of the square root of the sum of the squares of the linear and angular errors (reduced to the common unit of one part in x thousands) at the various frequencies. In the normal ranges of accuracy, there is a marked agreement between calculated curve 9 and traverse curve 8. This would seem to indicate a compensating effect of the linear and angular errors within a traverse.

However, the linear errors referred to are the probable errors in the measurement of single lines, and the angular errors similarly are the probable errors in the observation of single angles. Curve 9 then might be the probability curve of accuracy for the setting of a single point by angle and distance as well as the curve for traverse accuracy. If so, then the number of sides, and therefore the length of a traverse, plays no part in the accuracy of closure.

This contradicts much that has been written, and it is particularly contradictory to Professor Rayner's 1928 study of the problem,[5] wherein the accuracy of traverse closure is given as a function of the length of traverse.

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[12] (Bauer Surveys Co.), Cleveland, Ohio.

The writer finds no relationship of traverse accuracy to length of traverse but only to mean side length. Professor Rayner properly asks for explanation. On this point, further investigation has proceeded as follows: Seven cases of from two to six connecting traverse loops with the same general conditions were studied; in none of the seven cases was the over-all or outside loop closure better than the best smaller inside loop closures of which it consisted; in three cases, the outside loop closure was equal or inferior to the poorest inside loop closure; in seven cases, the outside loop closure was better than certain individual loops within itself; and, in eleven cases, the outside loop closure was inferior to certain individual loops within itself.

Six other cases of two or more interlocking traverses, having the same general topographic conditions but without an over-all outline closure, were examined. In two cases the longest loop of its group had the best closure of the group; four cases occurred in which the longest loop of a group had the poorest closure of the group; in eight cases, the longest loop had a better closure than some of the smaller loops within its group; and, in five cases, the longest loop of a group had poorer closures than others within the group.

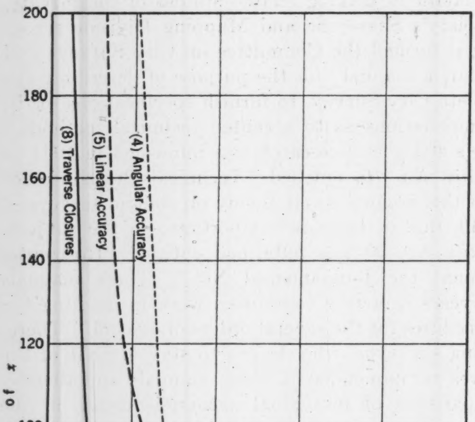

FIG. 5.—TRAVERSE MEASUREMENT AND ANGLE ACCURACIES (EXPRESSED AS ONE PART IN x THOUSANDS)

The foregoing study of thirteen groups of traverses (each traverse being comparable to the others within its group as to terrain, side length, etc.) indicates no relationship between traverse closure and traverse length. This study would seem to indicate, also, that the compensation of errors does not extend throughout a traverse, that increased length of traverse will not help the closure, that the length of the traverse side, rather than the traverse length, affects the closure, that the compensation of errors occurs within a single side, but that in a traverse the side error becomes systematic.

The writer is not ready to state that this relationship extends to all instrumentation and procedures. These findings only can be attributed to the equipment and procedure used in this paper, since the limit of accuracy of the instrumentation may have been reached at approximately the length of a mean line and that additional length may not add to accuracy any more than unlimited reobservation of a measurement will increase the accuracy of the measurement indefinitely. (It will increase only the accuracy of the value of the probable error.)

Expressed differently, the hypothesis is offered that, within a traverse, errors combine as compensating errors to a point (probably determined by the instrumentation and procedure) and then become fixed as a systematic error, increasing with the length of the traverse, and thereby establishing an accuracy, expressed in terms of error per traverse length, fixed at the point where compensation ceases.

Professors Earnest, Kissam, and Rayner in various manners raise the problem of the difference between loop closures, which close upon themselves, and position closures between fixed stations previously established by precise control.

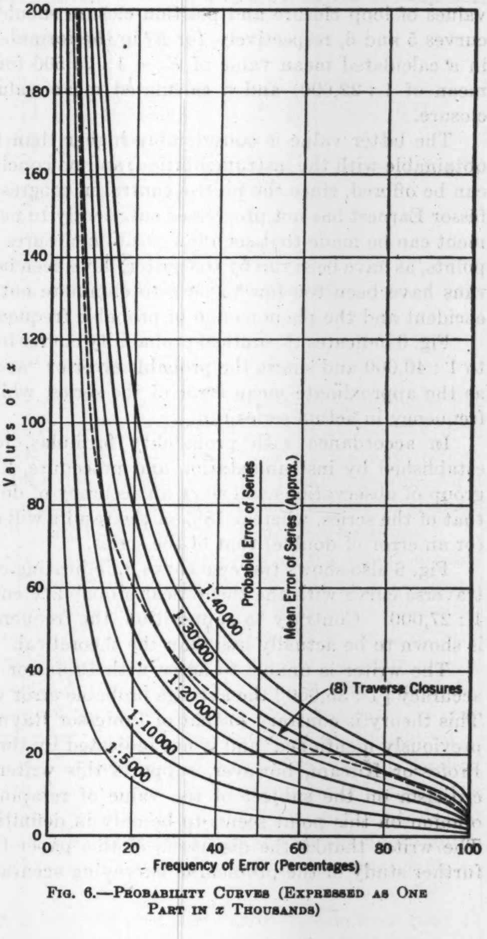

FIG. 6.—PROBABILITY CURVES (EXPRESSED AS ONE PART IN x THOUSANDS)

Although this paper is primarily concerned with actual accuracies that have been obtained in the regular business of a practicing surveyor in a territory as yet not covered by a precise control, it should not be improper (beyond the bounds of possibility) to project the same reasoning into channels indicated by the facts disclosed.

Therefore, if (as seems to be the case) traverse closures are in general equal in value to the square root of the summation of the squares of the mean linear and angular errors of the instrumentation and procedure used, the relative values of loop closure and position closure should be indicated by the use of curves 5 and 6, respectively, for E_1 in the formula $E_t = \sqrt{E^2_a + E^2_1}$ resulting in a calculated mean value of $E_t = 1 : 21{,}500$ for loop closures (an observed mean of 1 : 22,600) and a calculated mean value of 1 : 18,000 for position closure.

The latter value is considerably higher than that generally recognized as obtainable with the instrumentation, and no conclusive proof of the hypothesis can be offered, since the precise control in progress under the direction of Professor Earnest has not progressed sufficiently to permit such proof. The statement can be made that such few position closures between established transfer points, as have been run by the writer, have been better than 1 : 20,000. These runs have been too few to date to eliminate entirely the possibility of mere accident and the phenomenon of probable frequency of error.

Fig. 6 indicates theoretical probability curves for accuracies of from 1 : 5,000 to 1 : 40,000 and shows the probable error or "accuracy" of each series as well as the approximate mean error of the series, which occurs at about the 60% frequency in actual series run.

In accordance with probability formulas, for any prescribed accuracy established by instrumentation and procedure, 27% of the results of a large group of observations will show an accuracy of double (or an error of one half) that of the series, whereas 18% of the results will show an accuracy of one half (or an error of double) that of the series.

Fig. 6 also shows traverse curve 8, indicating close conformity of the actual traverse curve with the theoretical probability curve for an accuracy of about 1 : 27,000. Contrary to supposition, the frequency of extremely small errors is shown to be actually less than the theoretical.

The writer is unable to agree with Professor Kissam that "to obtain this accuracy [1 : 35,000] the average probable error would have to be 1 : 73,500." This theory is contrary to that of Professor Rayner as elaborated in his paper previously mentioned[5] and is not indicated by the results of these tabulations. Professor Kissam, however, supports this writer against Professor Rayner's criticism on the subject of the value of retaping a line. The difference of opinion on this point seems to be only in definition of terms and not in fact. The writer thanks the discussers of this paper for new thought to apply to further study of the problem of surveying accuracies.

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EARLY CONTRIBUTIONS TO MISSISSIPPI RIVER HYDROLOGY

By C. S. Jarvis,[1] M. Am. Soc. C. E.

With Discussion by Messrs. Robert Follansbee, R. W. Davenport, Charles Senour, John C. Hoyt, George O. Guesmer, H. T. Cory, and C. S. Jarvis.

Synopsis

Recent research in runoff characteristics of the Mississippi River basin has disclosed certain values among the early records which had apparently been neglected, if indeed they had ever been recognized outside the group responsible for assembling them. These data generally are to be found in some form or other in old reports and records of the War Department, mainly in publications of the Corps of Engineers and the Mississippi River Commission.

The fragmentary hydrologic data of the earliest record periods are assembled in this paper. They are extended by derivation, estimation, or comparison with related data from neighboring stations, and tentatively integrated into a continuous record covering 122 years, ending in 1938. Incorporated within the record are the discharge determinations for the 33 years ending with 1860, as published by the late A. A. Humphreys, Hon. M. Am. Soc. C. E., and Henry L. Abbot, with such adjustments as were found necessary to conform to the hydrographs of calendar years. Likewise, other results, official or unofficial, published or unpublished, were taken into consideration in compiling these basic hydrologic data. A sharp distinction has been made between the official quantities and those derived or estimated. However, such controls and cross references as are available have been used freely, so that the final results, as here submitted, are as nearly beyond challenge as it seems practicable to obtain at this time.

Trends of both precipitation and the resulting runoff depths from year to year, or from decade to decade or other record segment, are disclosed, with fair correlation generally for the 5-yr or longer period means, and to a lesser extent, on an annual or even a monthly basis.

Note.—Published in March, 1942, *Proceedings*.

[1] With Office, Chf. of Engrs., Washington, D. C.

LOWER MISSISSIPPI RIVER DISCHARGE

Heretofore, the oft-repeated inquiry concerning the actual discharge of the Mississippi River in its lower course has brought unsatisfactory results according to the writer's observation; yet the observed, computed, and estimated discharge quantities have been recorded for 33 of the 43 years, 1818 to 1860, inclusive.[2] The fact that these discharge quantities are expressed in cubic feet per year, and thus involve trillions, should be no barrier against their use. Division by the number of seconds in a calendar year readily reduces the quantities to yearly average discharge in cubic feet per second, as shown in Table 1. The use of hydrographs and tabulations of gage heights assisted in the apportionment of discharge, month by month, to conform substantially with the respective yearly quantities. Moreover, both large-scale hydrographs and daily discharge quantities are available for practically the entire years 1851, 1858, and 1859, representing the most notable flood years of the Mississippi River, within the detailed record period up to that time, at two or more stations on the lower river. Likewise, the same information is available for 1852, and for parts of other years at various stations on the main river and its tributaries, in addition to some 30 years of river-stage records, 1817 to 1860, inclusive, depicted on smaller scale for publication in *Professional Paper No. 13*, and recently generalized and further reduced for inclusion in Fig. 1. This record was compiled from several sources:[3, 4] Reports of the Mississippi River Commission;[5] records of rainfall, river-stage and flood phenomena, compiled

TABLE 1.—MISSISSIPPI RIVER DISCHARGE AT NATCHEZ, MISS.

Date[a]		Humphreys and Abbot[b]	Thousands of Cubic Feet per Second	
From Nov.:	To Oct.:		Computed average	Calendar year[c]
1818	1819	15,400	489	498
1822[a]	1822	20,500	650	636
1822	1823	27,300	866	864
1823	1824	21,200	670	701
1824	1825	18,200	577	561
1827	1828	26,400	835	847
1828	1829	13,700	434	458
1829	1830	20,700	656	593
1830	1831	17,600	558	547
1833	1834	20,300	644	589
1834	1835	17,200	546	611
1835	1836	21,400	677	673
1836	1837	15,500	492	488
1837	1838	15,300	485	484
1838	1839	11,500	365	359
1839	1840	18,900	598	645
1840	1841	21,400	679	675
1843	1844	29,300	927	947
1844	1845	19,000	603	597
1845	1846	15,300	485	516
1846	1847	21,300	676	670
1848	1849	27,000	856	843
1849	1850	24,000	761	727
1850	1851	20,600	653	668
1851	1852	17,800	563	602
1852	1853	22,000	698	642
1853	1854	17,000	539	559
1854	1855	11,000	349	384
1855	1856	14,800	468	461
1856	1857	15,100	479	500
1857	1858	26,000	825	832
1858	1859	21,000	666	701
1859	1860	15,200	481	481
Totals		640,200	20,250	20,359
Means		19,400	614	617
Mean, last 10 years		18,050	572	583

[a] From November to October, inclusive, in each case except the second item, which was January, 1822, to December 1822, inclusive. [b] See Footnote 2, in the text; billions of cubic feet per year. [c] Derived from all available hydrographs and from temperature, rainfall, and discharge records.

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[2] "Report on the Physics and Hydraulics of the Mississippi River," by A. A. Humphreys and Henry L. Abbot, *Professional Paper No. 13*, Corps of Engrs., U. S. Army, 1876, p. 130.

[3] *Ibid.*, Plates V to XVIII and tabulations.

[4] "Rainfall and Runoff of the Mississippi River Basin," by Harry Larson, ms. filed for reference in Engineering Societies Library, New York, N. Y.

[5] Progress Report of the Mississippi River Commission, 1881, Appendix F, Plate 3.

by the U. S. Weather Bureau; and the Water-Supply papers of the U. S. Geological Survey.

The key to various lines and symbols in Fig. 1 is as follows:

——— Observed River Stages at Natchez, Miss.
— - — - Derived and Estimated Gage Heights at Natchez, Miss.
·········· Monthly Maximum Gage Heights at St. Louis, Mo.
〰〰 Gage Heights at Memphis, Tenn.
····—····— Gage Heights at Donaldsonville, La.
Λ̇ Yearly Maximum Gage Heights at Natchez, Miss.
○ Flood-Peaks at Pittsburgh, Pa.
Λ V High and Low River Stages at Cincinnati, Ohio
+ Divergencies of Maj. Harry Larson's Compilations.
from Recent Rating Curves.

The meticulous care with which the observers recorded information such as wind and weather conditions that might possibly affect the operation of the primitive velocity-measuring apparatus (mainly multiple floats and early types of current meter), the faithful portrayal of results showing the influence of the rising as opposed to the falling stage of each major flood event, the recognition of inherent limitations as to attainable accuracy for individual measurements—all are earnest approaches toward, if not the stepping-stones leading to, the more rigorous standards attainable with improved apparatus and methods developed during recent years. Quoting Messrs. Humphreys and Abbot:[6]

> "It is evident that the condition of the river, whether rising or falling, makes a great difference in discharge at any given stand; but it is equally evident that a mean line between these two extremes can be drawn that shall form the basis of a table by which the annual discharge can be deduced from the recorded gauge-readings. For any given day, its indication will be erroneous, but for the entire year, which includes both the rising and the falling branches of the curve, it will be sufficiently accurate * * *."

The difficulties encountered with surface floats, subject to deflection or acceleration due to wind velocity, naturally led to the adoption of multiple floats, largely submerged, and of rods so weighted and adjusted as to maintain them nearly submerged and vertically suspended, supporting small flags above the surface for observation. By such means, Messrs. Humphreys and Abbot developed information regarding depth-velocity relations closely in accordance with the most advanced thought and adopted theories of recent years, as evidenced by the parabolic velocity curves presented by them.[7] Likewise, they presented evidences of a rigorously scientific approach to determinations of both mean and maximum velocities in terms of surface velocities or of those observed some 5 ft below; even a forerunner of the familiar reduction factor of 0.80 or thereabouts, by which observed surface velocity may lead to an approximate value of mean velocity for the station.[8] Abundant proof is to be

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[6] "Report on the Physics and Hydraulics of the Mississippi River," by A. A. Humphreys and Henry L. Abbot, *Professional Paper No. 13*, Corps of Engrs., U. S. Army, 1876, p. 128.

[7] *Ibid.*, Plate XI.

[8] *Ibid.*, p. 213.

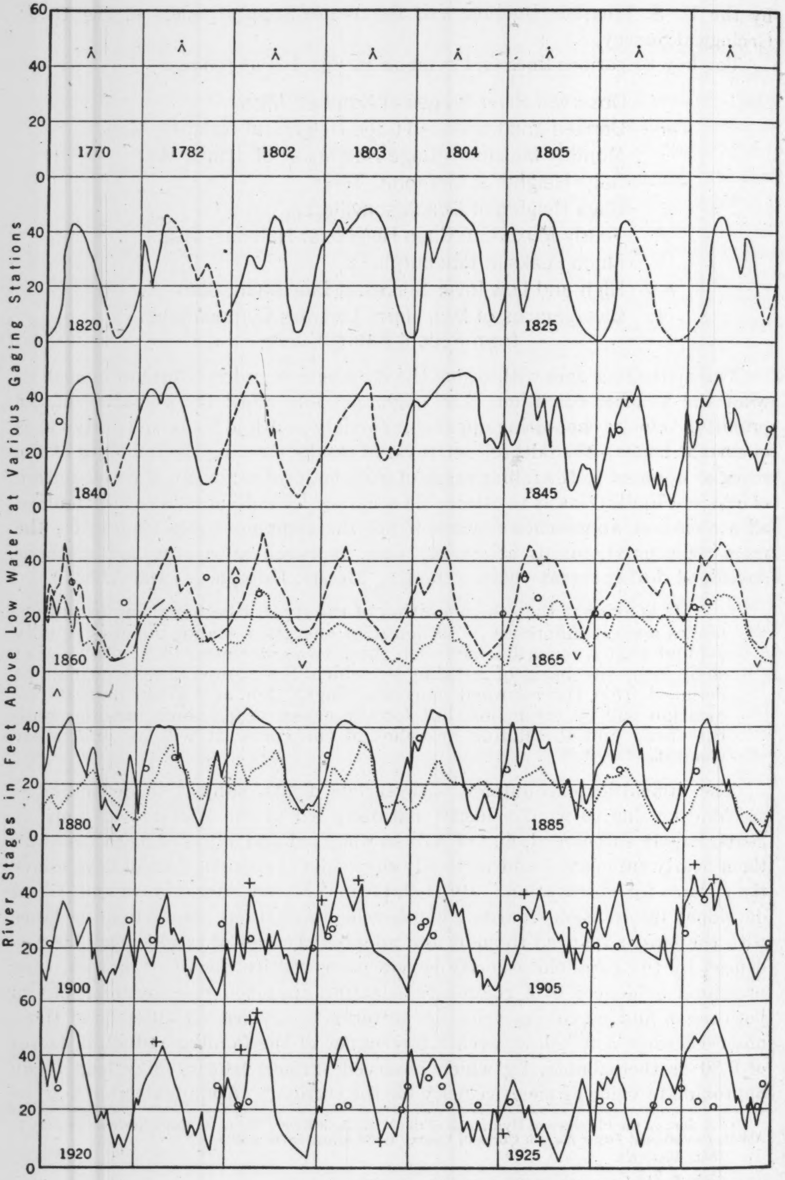
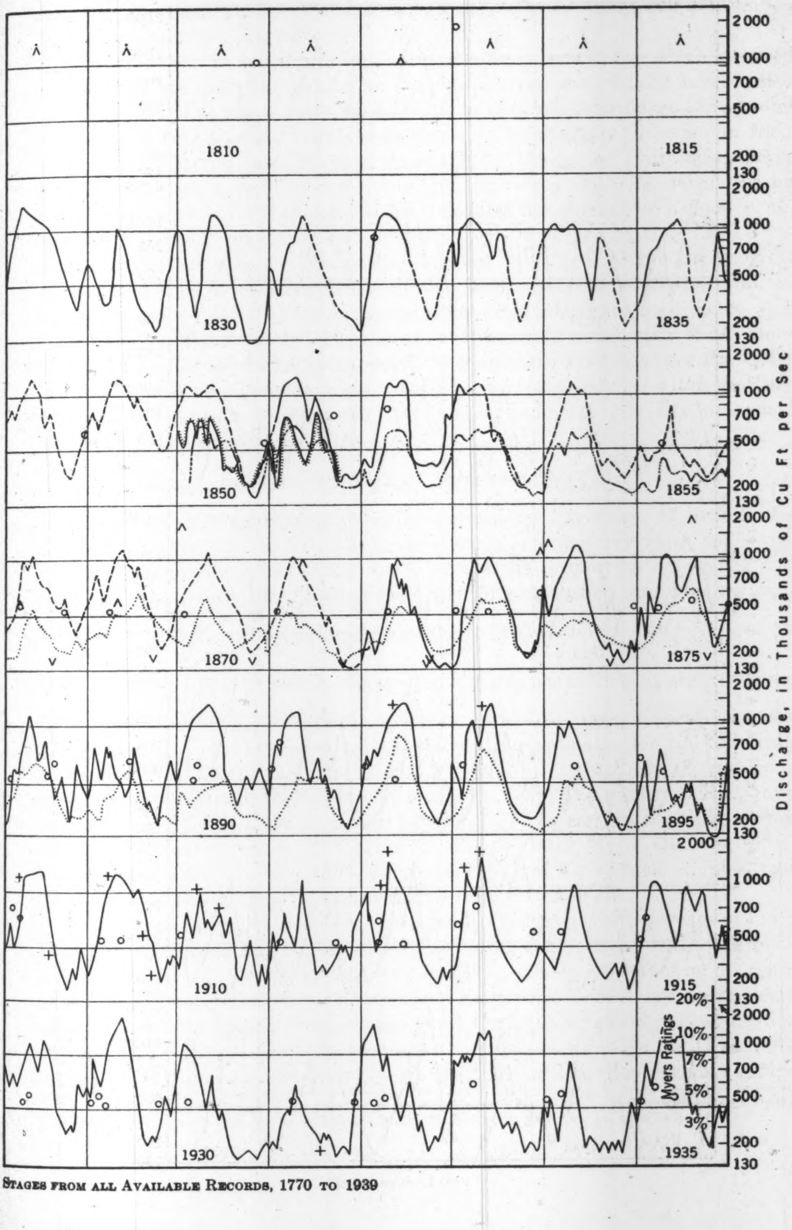


FIG. 1.—MISSISSIPPI RIVER HYDROGRAPHS AND LIMITING



STAGES FROM ALL AVAILABLE RECORDS, 1770 TO 1939

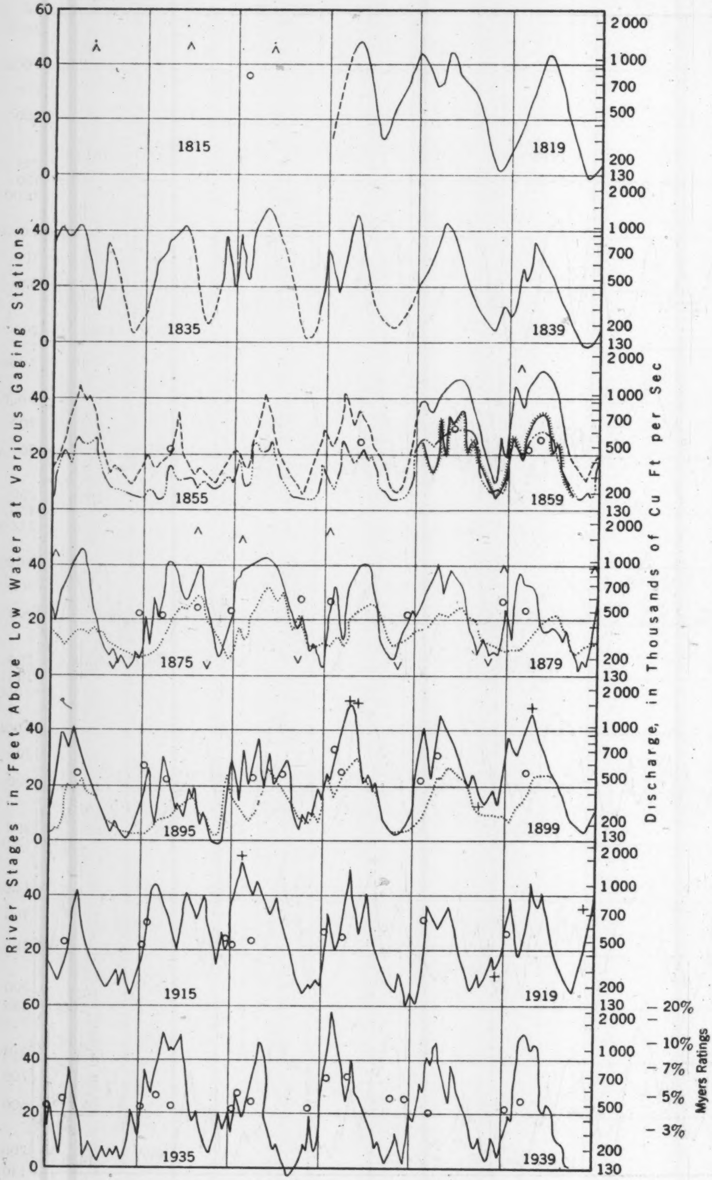


FIG. 1.—*Continued*

found in the associated text of fair agreement between quantities determined by formula and by observations.

A review of the specific examples in the same report for which detailed calculations are recorded should not fail to impress an unbiased investigator with the idea that those early measurements of stream discharge are beyond challenge, within the reasonable limitations as to accuracy, inherent in their equipment, methods, and working conditions. Whenever hydrologists have had to rely upon quantities determined by slope-area methods, depending on high-water marks observed some days following the passage of a flood peak, whether the event was last month, last year, or 100 years ago, they have occasion to regret the absence of actually observed velocities. Even the progress of uprooted trees in midstream—or better, largely-submerged floats, properly spaced, and observed by theodolites over measured courses in accordance with those early practices—might give results at least more convincing, if not more reliable, than those ordinarily expected by slope-area methods; yet the latter in skilful hands may provide fairly acceptable approximations—often the only available data as to some of the flood-peak discharges officially published during recent years.

SEDIMENTATION AND RELATED STUDIES

Other basic problems considered in *Professional Paper No. 13* include the development and effects of cutoffs, levee systems, storage reservoirs, spillways, and lateral overflow areas. Also included are the proportion by weight and the distribution of both bed load and sediment in suspension, the latter varying, for the 12-month period beginning February, 1851, from $\frac{1}{681}$ to $\frac{1}{6,383}$, or an average of $\frac{1}{1,808}$ for the year.[9] Expressed in parts per million by weight, these fractions reduced to 1,467 ppm, 157 ppm, and 553 ppm, respectively, for Carrollton, La., as contrasted with the tentative estimates for the following year—namely, 1,748 ppm, 116 ppm, and 690 ppm. Furthermore, observations extending from March to November, inclusive, 1858, at Columbus, Ky., gave results[10] reducible to 1,493 ppm, 140 ppm, and 757 ppm, whereas those at the Mississippi mouths, taken by George G. Meade (later Major General, U. S. Army) from April to June, 1838, averaged 638 ppm for surface samples, and 785 ppm for subsurface, at depths ranging from 6 ft to 90 ft.

The foregoing observations as to suspended sediment during years of more than normal rainfall and runoff afford interesting comparisons with data published by the U. S. Geological Survey[11] in 1909. These are based on 15 years of sediment determinations by the Corps of Engineers, U. S. Army, and 5 years of determinations by the New Orleans (La.) Water and Sewerage Board, prior to 1907, and one year of determinations at various stations by the U. S. Geological Survey, presumably during years of somewhat less than normal average runoff, such as prevailed in the period preceding 1907. The report shows (see Table 2)[11] that suspended sediment above and below the Missouri River mouth, at Quincy, Ill., and St. Louis, Mo. (Jefferson Barracks), respectively, was increased some eight-fold, or from 119 to 964 ppm, while the dissolved solids remained nearly constant (see Table 2).

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[9] "Report on the Physics and Hydraulics of the Mississippi River," by A. A. Humphreys and Henry L. Abbot, *Professional Paper No. 13*, Corps of Engrs., U. S. Army, 1876, pp. 147 and 510.

[10] *Ibid.*, p. 139.

[11] *Water-Supply Paper No. 234*, U. S. Geological Survey, Washington, D. C., 1909, p. 87.

TABLE 2.—SUMMARIZED OBSERVATIONS ON DISSOLVED AND SUSPENDED SOLIDS OF THE MISSISSIPPI RIVER

(Values in Parentheses Were Not Published in Original Tabulation, But Were Derived from Data Contained Therein)

Station	Drainage area, sq miles	RUNOFF		SOLIDS (PPM)		TONS OF SOLIDS PER SQ MILE PER YR		YEARS REQUIRED FOR 1-IN. DENUDATION		
		Cu ft per sec per sq mile	Cu ft per sec	Dissolved	Suspended	Dissolved	Suspended	Dissolved	Suspended	Total
(Above) Minneapolis, Minn.	19,600	0.608	(11,920)	200	7.9	(119)	(4.7)	(1,600)	(40,000)	1,600
Quincy, Ill.	135,500	0.538	(73,000)	203	119	108	63	(1,767)	(3,030)	(1,116)
St. Louis, Mo.	700,700	0.263	(184,300)	206	964	53	250	(3,600)	(764)	(630)
Menard, Ill.	711,900	0.263	(187,000)	269	634	70	164	(2,720)	(1,164)	(816)
Memphis, Tenn.	941,000	0.546	(514,000)	202	519	109	279	(1,750)	(684)	(492)
New Orleans, La.	1,261,000	(0.560)	(706,000)	190	600	(105)	(330)	(1,820)	(580)	(440)
At mouths	1,265,000	0.560	(719,000)	(190)	(600)	105	331	(1,820)	(580)	(440)

DENUDATION—LOCAL AND BASIN-WIDE

Probably the brief chapter on "Denudation," from which Table 2 was obtained,[11] represents the most comprehensive and scientific approach to the problem of which there is evidence, up to the date of its publication. Among the items of interest and value are those purporting to represent the average river discharges and proportions of solids throughout the periods of available record. The fairly consistent relationships between such discharge and sediment computations or estimates, and those developed in connection with this paper and associated research projects, afford at least partial confirmation from officially accepted and published quantities.

One of the most serious errors that might readily result from a casual use of such data would be the assumption that it requires centuries of erosion and runoff to denude the soil a single inch, when it is well known that a day of intense rainfall or of rapid thawing, or a series of storms, may displace recently disturbed, unstable, unprotected soil on moderate slopes to a depth of not only an inch, but occasionally several inches under only normal concentrations of flow. It is well known that soil on sloping shale or other smooth surfaces may slide in its entirety when the plane of cleavage becomes saturated, so that the volume of solids moved locally, for short distances, may far exceed the volume of water releasing it. Thus, the actual depth of local denudation in a semi-arid region, on steep, relatively bare slopes, might represent a large percentage of the actual rainfall and might far exceed the depth of runoff from the entire basin, which usually represents only small percentages of rainfall.

It follows from the foregoing that the solids transported by the Mississippi River and delivered into the Gulf of Mexico are but finely divided samples of the total erosional products—the dissolved or flocculent or other readily suspended portions of the disturbed soil mantle. The heavier and coarser materials, along with varying percentages of the finer elements, are strewn along the flood plains, valley floors, and network of drainage channels, awaiting the next turbulent flow to exact a further toll of easily transported elements.

TABLE 3.—ANNUAL MAXIMUM GAGE READINGS ON THE MISSISSIPPI RIVER AT CARROLLTON, LA.

Year	Maximum gage readings (ft)	Discharge (cu ft per sec)
1811	14.87	1,110,000
12	14.22	1,030,000
13	15.22	1,130,000
14	14.50	1,070,000
15	15.30	1,140,000
16	14.53	1,070,000
17	14.58	1,075,000
18	14.26	1,035,000
19	14.80	1,100,000
1820	14.22	1,030,000
21	14.72	1,095,000
22	14.62	1,085,000
23	15.26	1,135,000
24	15.12	1,120,000
25	14.80	1,100,000
26	14.64	1,090,000
27	14.05	1,017,000
28	15.26	1,135,000
29	13.20	950,000
1830	14.66	1,090,000
31	14.57	1,080,000
32	14.55	1,080,000
33	13.80	1,000,000
34	13.64	990,000
35	14.12	1,020,000
36	15.05	1,120,000
37	14.47	1,070,000
38	14.00	1,015,000
39	12.14	885,000
1840	15.03	1,120,000
41	14.47	1,070,000
42	14.57	1,075,000
43	14.76	1,100,000
44	15.05	1,120,000
45	14.86	1,110,000
46	14.86	1,110,000
47	15.05	1,120,000
48	15.10	1,125,000
49	15.21	1,130,000
1850	13.80	1,000,000
51	15.40	1,150,000
52	14.10	1,020,000
53	15.00	1,115,000
54	14.70	1,095,000
55	9.50	700,000
56	12.80	920,000
57	13.10	945,000
58	15.10	1,120,000
59	15.60	1,170,000
1860	13.40	970,000

RUNOFF FROM RAINFALL AND TEMPERATURE DATA

Without those well-organized approaches to river gagings during the early decades of the nineteenth century, engineers would have been deprived of much helpful hydrologic information as well as scientific observations and opinions in related fields. A fair-minded review of the mass of data yielding the average yearly discharges of the Mississippi River at Natchez, Miss., for record periods of early decades[12] cannot fail to impress one with the magnitude and difficulties of the undertakings, and the foresight, determination, and scientific as well as practical accomplishments of those early observers and authors.

The writer's review of those early records disclosed such consistent relationships between rainfall, temperature, and runoff, all in fair agreement with the same observed phenomena during recent years of record, that they provided access to requisite basic data and indicated a method for deriving probable values of Mississippi River monthly average discharge for the years of rainfall record, including the intermittent and fragmentary observations. One can scarcely use the term "non-record" in this connection, inasmuch as some basic data at important gaging stations are available for every year beginning with 1817. Maximum annual stages at Natchez date back intermittently to 1770, and continuously after 1801; and available records of maximum gage heights at Carrollton began in 1811, as shown in Table 3.[13]

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[12] *Water-Supply Paper No. 234*, U. S. Geological Survey, Washington, D. C., p. 131.

Table 4 shows some interesting relationships between departures from mean rainfall and runoff, year by year, and month by month, all expressed in inches of depth over the entire drainage area above Natchez. (Runoff was either measured or estimated for the Mississippi River at Natchez. Temperature observations were made at St. Paul, Minn., from 1819 to 1837, and at St. Louis, from 1838 to 1877. Precipitation was observed at Marietta, Ohio, from 1817 to 1829, and at additional stations thereafter as they became available with desirable distribution. In 1850, the total was 15 stations; in 1860, 22 stations; in 1868, 27 stations; and in 1873, 45 stations.) The two representative years selected for inclusion were 1850 (above normal), and 1860, with subnormal rainfall and runoff. Thus a yearly excess of 5.5 in. in precipitation produced an excess runoff depth of 1.61 in., whereas the deficiency of 5.8 in. produced a departure of − 1.33 in. in runoff. For the first 61 years of the record, the mean rainfall and runoff departures yearly were + 0.50 and + 0.30 in., respectively, from which it may be logically assumed that the same quantities with opposite signs would apply for the latter half of the record, if worked out on a comparable basis. Results actually computed on a yearly basis, but with the addition of several more rainfall station records, were − 0.35 in. and − 0.27 in. for the 61-yr period, 1878 to 1938.

TABLE 4.—DEPARTURES OF OBSERVED TEMPERATURE, PRECIPITATION, AND MISSISSIPPI RUNOFF FROM RECORD MEANS

(ΔT, ΔP, and ΔR = Departure of Temperature, Precipitation, and Runoff, Respectively)

Month	1850			1860			AVERAGES, 1817 TO 1877		
	ΔT (degrees F)	ΔP (in.)	ΔR (in.)	ΔT (degrees F)	ΔP (in.)	ΔR (in.)	ΔT (degrees F)	ΔP (in.)	ΔR (in.)
January	+6.1	+2.0	+0.36	+2.7	−0.4	+0.06	+0.1	−0.34	+0.01
February	+2.4	+1.0	+0.37	+2.8	+0.4	−0.06	+0.7	+0.17	−0.03
March	−0.3	+0.6	+0.14	+15.7	−2.2	+0.27	+0.3	−0.14	+0.01
April	−6.4	+0.9	+0.09	+2.9	−0.6	−0.35	+0.6	−0.00	−0.01
May	−3.6	−0.8	+0.23	+5.1	−1.4	−0.29	+0.2	−0.04	+0.05
June	+1.8	+1.3	+0.23	+1.2	−0.8	−0.32	+0.8	+0.24	+0.05
July	+2.0	+0.6	+0.09	+3.3	−0.7	−0.25	+0.7	+0.30	+0.04
August	+4.0	+0.5	+0.03	+0.8	−0.7	−0.11	+0.1	+0.13	+0.07
September	−0.5	−1.4	−0.03	−2.8	−0.3	−0.09	−1.2	+0.01	+0.04
October	−2.2	−0.9	−0.03	+0.1	+0.1	−0.05	−2.2	+0.01	+0.02
November	+2.4	+0.3	+0.05	−4.5	+0.9	−0.07	−1.2	+0.18	+0.02
December	−3.1	+1.4	+0.08	−4.4	−0.1	−0.07	−1.1	−0.02	+0.03
Total	+0.2	+5.5	+1.61	+1.9	−5.8	−1.33	−0.2	+0.50	+0.30

For the month of March, 1860, the influence of the 2.2-in. deficiency in rainfall was more than offset by the unseasonable temperature, the + 15.7° F having released the ice sufficiently to produce an excess depth of 0.27 in. in runoff for that month. Likewise, a rainfall deficiency of 0.8 in. for May, 1850, in combination with a temperature departure of − 3.6° produced a runoff excess of 0.23 in. In many instances the time lag between rainfall and runoff accounts for the excess or deficiency occurring in the following month or year instead of being in the same period as the rainfall.

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[13] "Report on the Physics and Hydraulics of the Mississippi River," by A. A. Humphreys and Henry L. Abbot, *Professional Paper No. 13*, Corps of Engrs., U. S. Army, 1876, p. 443 and Plate XIV.

Table 5 lists the average rainfall depths for the available stations, to the number of 50, for the entire 122 years of record, together with the departures in both precipitation and runoff, all measured in inches over the drainage basin above Natchez. In general, about 3 in. of rainfall departure produced 1-in. departure of the same sign in runoff. However, there are wide variations from that ratio, and a number of sign reversals.

THE VICKSBURG-NATCHEZ RECORD OF MISSISSIPPI RIVER DISCHARGE

In spite of the 4,900 sq miles excess of drainage area at Natchez as compared with that at Vicksburg, Miss. (representing 0.43% difference), the annual yields should be within 1% of equality. Therefore, they should be almost interchangeable for the early records, if not for the later ones, in view of the admitted uncertainties of stream measurements at their best, requiring a tolerance of some multiple of that difference, according to conditions. Where stream gagings are numerous and of a high standard, it may appear reasonable to take the difference of 1% into account, as has been done for the Vicksburg-Natchez record. Thus, the 122-yr means have reduced to 593,000 cu ft per sec and 599,000 cu ft per sec, respectively. From Table 1, the 33-yr record mean for Natchez was either 614,000 or 617,000 according to the two separate determinations, thus differing by 0.5%, which happens to be the same as the percentage of record-period represented by the two-month difference between those water years and the calendar years. According to Table 6, the first 50-yr mean discharge at Natchez was the maximum attained, or 626,000 cu ft per sec, as compared with 605,000 for the first 100 years, or 602,000 cu ft per sec for the first 110 years. (Table 6 was compiled from the Humphreys-Abbot report; from stages and discharge observations of the Lower Valley of the Mississippi River, January 1 to June 30, 1937; from annual reports of the Mississippi River Commission and the Corps of Engineers; and from Water-Supply papers of the U. S. Geological Survey.)

CONSUMPTIVE USE OF WATER

Obviously the increased consumptive use of water in connection with initial fillings of large storage reservoirs, together with the subsequent diversion for irrigation and other purposes, may account for some of the difference between the first 50-yr mean and the mean discharges for later periods at Natchez. It has been computed and estimated that, as of 1930, some 13,000 cu ft per sec were withheld from the Lower Mississippi River for irrigation alone, exclusive of initial storage and attendant losses.[14]

RELATIVE PERMANENCY OF GAGING SECTIONS

Opinions expressed by Messrs. Humphreys and Abbot as to the aggradation, degradation, or relative permanence of Mississippi River channel sections were evidently based on more than those early determinations, covering some 44 years of intermittent observations and records at various stations. Their several years devoted primarily to the investigation, including residence and field work, must have been supplemented prior to the second edition in 1876, because they were still actively engaged in important professional work, and thus were able to participate in the additions, with nearly a 60-yr record before them. Therefore, it seems evident that they had considerable basis for their opinions, repeatedly expressed in their report, to the effect that in general the Mississippi River bed in its lower course was neither building up nor cutting

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[14] *Proceedings*, Am. Soc. C. E., October, 1941, p. 1502, Table 14.

TABLE 5.—Comparisons of Annual Precipitation

(Units Are

Year of record	Stations	Precipitation[b]			Runoff excess[a,c]	Year of record	Stations	Precipitation[b]			Runoff excess[a,c]
		Average total record	Annual observed rainfall	Excess[a]				Average total record	Annual observed rainfall	Excess[a]	
1817	1	42.13	(52.00)	9.87	1.42	1852	17	41.59	44.30	2.71	0.11
18	1	42.13	50.92	8.79	0.71	53	16	41.75	36.94	−4.81	0.58
19	1	42.13	36.30	−5.83	−1.13	54	17	41.59	41.00	−0.59	−0.40
20	1	42.13	39.11	−3.02	−1.45	55	16	40.99	41.43	0.44	−2.49
1821	1	42.13	43.32	1.19	0.45	1856	21	40.56	35.91	−4.65	−1.57
22	1	42.13	43.38	1.25	0.51	57	22	40.91	39.25	−1.66	−1.11
23	1	42.13	49.34	7.21	3.22	58	24	41.09	49.15	8.06	2.84
24	1	42.13	(53.00)	(10.87)	1.28	59	24	41.09	42.56	1.47	1.28
25	1	42.13	(37.00)	(−5.13)	−0.38	60	22	40.23	34.01	−6.22	−1.33
1826	1	42.13	41.60	−0.53	−1.87	1861	20	38.54	39.76	1.22	0.21
27	1	42.13	41.46	−0.67	−0.49	62	19	37.95	40.00	2.05	0.55
28	1	42.13	49.50	7.37	3.02	63	19	37.95	36.90	−1.05	0.06
29	1	42.13	39.52	−2.61	−1.61	64	21	38.54	32.62	−5.92	−0.56
30	2	41.45	32.35	−9.10	0.00	65	21	38.54	47.64	9.10	1.50
1831	3	41.45	45.50	4.05	−0.55	1866	21	38.54	43.85	5.31	1.44
32	3	41.45	44.56	3.11	1.05	67	27	37.85	37.58	−0.27	1.16
33	3	41.45	37.12	−4.33	0.80	68	27	37.85	40.87	3.02	0.23
34	3	41.45	35.63	−5.83	−0.05	69	27	37.85	40.42	2.57	1.01
35	4	41.08	41.82	0.74	0.21	70	27	37.85	34.57	−3.28	−0.11
1836	6	42.80	42.87	0.07	0.95	1871	27	37.85	35.67	−2.18	−0.96
37	8	36.23	36.88	0.65	−1.25	72[d]	27	37.85	35.08	−2.77	−2.37
38	9	36.53	31.33	−5.20	−1.30	73[d]	45	36.30	38.86	2.56	−0.11
39	10	38.67	33.82	−4.84	−2.78	74[d]	45	36.30	37.64	1.34	−0.14
40	13	42.00	41.55	−0.45	0.62	75[d]	45	36.30	42.42	6.12	0.43
1841	13	42.00	41.60	−0.40	0.98	1876[d]	45	36.30	41.20	4.90	1.51
42	13	42.00	39.30	−2.70	−0.07	77[d]	45	36.30	40.25	3.95	−0.30
43	15	42.44	44.53	2.09	1.04						
44	15	42.44	41.92	−0.52	4.22	Subtotals				45.89 −33.40	12.91 −11.45
45	15	42.44	42.20	−0.24	0.05						
1846	14	40.18	44.29	4.11	−0.92						
47	14	40.18	45.29	5.11	0.92	1872	40	35.74	33.31	−2.43	−2.37
48	14	40.18	43.79	3.61	1.77	73	40	35.74	36.77	1.03	−0.11
49	15	41.71	48.51	6.80	2.98	74	40	35.74	36.02	0.26	−0.14
50	15	41.71	47.08	5.37	1.59	75	42	34.95	38.80	3.85	0.43
1851	14	41.90	38.54	−3.36	0.89	1876	42	34.95	38.74	3.79	1.51
						77	42	34.95	38.30	3.35	−0.30
Subtotals				82.26 −54.76	28.68 −13.85						

[a] A minus sign denotes "deficit."

[b] Values in parentheses were not published in the original tabulation, but were derived from data contained therein.

[c] Average discharge of the Mississippi River at Natchez, Miss., for 122 years (1817 to 1938, inclusive) was about 599,000 cu ft per sec, representing an average yearly runoff depth of 7.06 in.

[d] Using the month-by-month analysis from 27 or 45 rainfall stations for comparison with the succeeding six items.

down, but that the indurated bluish clay through which the channel is carved seems resistant to erosion so long as it remains saturated. They found that the sediment was being swept consistently forward during moderate and high flows, to extend the delta, although local deposition and erosion might occur alternately along the channel. This opinion is certainly confirmed by authoritative observations and expert findings of later date.[15] Such data show only a slight tendency toward scouring the river bed, ranging from 0 to 4%

AND RUNOFF, MISSISSIPPI RIVER
Inches)

Year of record	Stations	Precipitation[b]			Runoff excess[a,c]
		Average total record	Annual observed rainfall	Excess[a]	
1878	42	34.95	36.76	1.81	−0.44
79	42	34.95	34.93	−0.02	−1.74
80	44	33.66	38.11	4.45	−0.46
1881	44	33.66	36.62	2.96	−0.02
82	44	33.66	39.11	5.45	1.63
83	44	33.66	39.62	5.96	0.62
84	43	34.13	39.35	5.22	0.99
85	43	34.13	34.63	0.50	0.20
1886	43	34.13	32.45	−1.68	−0.45
87	42	34.52	31.47	−3.05	−1.56
88	42	34.52	37.37	2.85	−0.25
89	42	34.52	31.29	−3.23	−1.55
90	42	34.52	37.04	2.52	2.06
1891	42	34.52	34.91	0.39	0.04
92	42	34.52	36.02	1.50	1.31
93	50	34.73	32.61	−2.12	−0.01
94	50	34.73	28.22	−6.51	−2.07
95	50	34.73	29.21	−5.52	−3.61
1896	50	34.73	34.63	−0.10	−1.63
97	50	34.73	33.77	−0.96	0.47
98	50	34.73	37.66	2.93	0.35
99	50	34.73	32.17	−2.56	−0.05
1900	50	34.73	34.45	−0.28	−1.69
1901	50	34.73	30.95	−3.78	−2.17
02	50	34.73	36.48	1.45	−1.01
03	50	34.73	33.63	−1.10	1.67
04	50	34.73	29.93	−4.80	−0.81
05	50	34.73	38.20	3.47	−0.55
1906	50	34.73	35.76	1.03	0.07
07	50	34.73	34.20	−0.53	1.75
08	50	34.73	33.66	−1.07	1.48
09	50	34.73	38.25	3.52	0.62
10	50	34.73	29.39	−5.34	−1.41
1911	50	34.73	35.40	0.67	−1.58
Subtotals				46.68 −42.65	13.26 −23.35
1912	50	34.73	34.41	−0.32	0.77
13	50	34.73	34.36	−0.37	0.30
14	50	34.73	31.32	−3.41	−2.57
15	50	34.73	38.36	3.63	0.99
1916	50	34.73	33.07	−1.66	1.55
17	50	34.73	30.77	−3.96	−0.29
18	50	34.73	33.44	−1.29	−2.14
19	50	34.73	37.55	2.82	0.67
20	50	34.73	34.39	−0.34	1.34
1921	50	34.73	35.32	0.59	−0.57
22	50	34.73	33.76	−0.97	0.35
23	50	34.73	36.70	1.97	0.23
24	50	34.73	30.87	−3.86	−0.07
25	50	34.73	30.43	−4.30	−2.57
1926	50	34.73	35.25	0.52	−0.70
27	50	34.73	38.73	4.00	3.72
28	50	34.73	35.30	0.57	1.23
29	50	34.73	36.90	2.17	2.09
30	50	34.73	27.16	−7.57	−1.71
1931[e]	50	34.73	32.34	−2.39	−3.55
31[f,g]	28	37.85	35.95	−1.90	−3.40
32	50	34.73	34.91	0.18	−0.41
33	50	34.73	33.73	−1.00	+0.11
34	50	34.73	29.30	−5.43	−3.21
35	50	34.73	35.89	1.16	1.00
1936	50	34.73	29.30	−5.43	−2.20
37	50	34.73	34.28	−0.45	0.38
38	50	34.73	35.43	0.70	−0.25
Subtotals				18.31 −42.75	14.77 −20.37
1926[g]	28	37.85	39.57	1.72	−0.70
27[g]	28	37.85	45.40	7.55	3.72
36[g]	28	37.85	31.43	−6.42	−2.20
37[g]	28	37.85	38.21	0.36	0.38
122-yr totals				193.14 −173.56	69.62 −69.02
122-yr means				1.58 1.42	0.57 0.57

[e] Last of a 40-yr series evaluated by Maj. Harry Larson (see footnote 5). [f] Incorporating U. S. Geological Survey data on July 1, 1931. [g] Using the month-by-month analysis from 28 rainfall stations for comparison with corresponding preceding items.

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[15] See "Basic Data, Mississippi River," *Annex No. 5*, H. R. Doc. No. 798, 71st Cong., 3d Session, 1931, plates 8, 9, and 10.

TABLE 6.—SUMMARY OF ANNUAL AVERAGE DISCHARGES, MISSISSIPPI RIVER, AT NATCHEZ, MISS.

(Units, 1,000 Cu Ft per Sec; Drainage Area, 1,149,400 Sq Miles; P and R = Precipitation and Runoff, Respectively)

Record Period		Averages During the Following Periods of Years:													Progressive Averages		Five-Year Average Excess[b]				No. of Stations	
		5	10	15	20	25	30	35	40	50	60	80	100	120			Inches		%			
From:	To:														Years	cfs[a]	P	R	P	R	P	R
1817	1821	599	....	....	....	....	....	....	....	....	....	....	....	....	5	599	+2.2	0.0	+5.2	0.0	1	1
1822	1826	646	622	....	....	....	....	....	....	....	....	....	....	....	10	622	+2.7	+0.56	+6.4	+7.8	1	1
1827	1831	605	....	617	....	....	....	....	....	....	....	....	....	....	15	617	−0.5	+0.07	−1.2	+1.0	3	1
1832	1836	649	627	....	625	....	....	....	....	....	....	....	....	....	20	625	−1.1	+0.60	−2.7	+8.3	6	1
1837	1841	535	....	....	....	607	....	....	....	....	....	....	....	....	25	607	−2.2	−0.77	−5.3	−10.7	14	1
1842	1846	672	604	619	....	....	618	....	....	....	....	....	....	....	30	618	−0.3	+0.88	−0.7	+12.2	15	2
1847	1851	737	....	....	....	....	....	635	....	....	....	....	....	....	35	635	+3.2	+1.66	+7.7	+23.1	15	2
1852	1856	535	636	....	620	....	....	....	622	....	....	....	....	....	40	622	−2.1	−0.77	−5.0	−10.7	21	2
1857	1861	631	....	634	....	....	....	....	....	....	....	....	....	....	45	623	+0.4	+0.38	+1.0	+5.3	24	5
1862	1866	649	640	....	....	645	....	....	....	626	....	....	....	....	50	626	+1.2	+0.60	+3.1	+8.3	21	4
1867	1871	622	....	....	....	....	....	....	....	....	....	....	....	....	55	626	−0.2	+0.28	−0.5	+3.9	27	9
1872	1876	587	605	619	622	....	627	....	....	....	622	....	....	....	60	622	+1.2	−0.14	+3.3	−1.9	45	11
1877	1881	540	....	....	....	....	....	....	....	....	....	....	....	....	65	616	+2.5	−0.71	+7.1	−9.9	45	17
1882	1886	649	595	....	....	....	....	602	....	....	....	....	....	....	70	618	+3.1	+0.60	+9.1	+8.3	44	21
1887	1891	581	....	590	....	596	....	....	....	....	....	....	....	....	75	616	−0.1	−0.22	−0.3	−3.1	42	24
1892	1896	497	539	....	567	....	....	....	595	....	....	609	....	....	80	609	−2.6	−1.24	−7.5	−17.3	50	25
1897	1901	546	....	....	....	....	....	....	....	....	....	....	....	....	85	605	−0.9	−0.64	−2.6	−8.9	50	25
1901	1906	588	567	544	....	....	567	....	....	....	....	....	....	....	90	604	0.0	−0.13	0.0	−1.8	50	26
1907	1911	613	....	....	....	....	....	....	....	....	....	....	....	....	95	604	−0.6	+0.17	−1.7	+2.4	50	26
1912	1916	616	615	....	591	572	....	....	....	584	....	....	599	....	100	605	−0.4	+0.20	−1.2	+2.8	50	26
1917	1921	582	....	604	....	....	....	575	....	....	....	....	....	....	105	604	−0.4	−0.20	−1.2	−2.8	50	26
1922	1926	551	567	....	....	....	....	....	....	....	....	....	....	....	110	602	−1.3	−0.58	−3.7	−8.1	50	26
1927	1931	631	....	....	....	....	....	....	....	....	....	....	....	....	115	603	−0.6	+0.38	−1.7	+5.3	50	28
1932	1936	514	572	565	569	....	585	....	580	....	576	....	....	599	120	599	−2.1	−1.03	−6.0	−14.3	50	29
1937	1938	599	....	....	....	....	....	....	....	....	....	....	....	....	122	599	0.0	0.0	0.0	0.0	50	30
1817	1938	599	599	599	599	599	599	599	599	599	599	599	599	599	122	599	....	....	....	....	....	....

[a] Cfs = cubic feet per second. [b] Minus sign denotes "deficiency."

for the 20-yr period, 1894 to 1913, and adhering quite closely to the cross-sectional areas and dimensions shown in *Professional Paper No. 13.*[16] Furthermore, all available official data, whether published or unpublished, were summarized and plotted for successive periods in Figs. 2 and 3, for six important gaging stations on the lower river. These show not only the meanderings of the rating curves, but also the near approach to constancy for considerable periods, and particularly in the lower segments of such curves.

TABLE 7.—MAXIMUM GAGE HEIGHTS AND DISCHARGES AT MEMPHIS, TENN., AND VICKSBURG, MISS.

No.	Date	Gage readings (ft)	THOUSAND CUBIC FEET PER SECOND[a]		
			Discharge	Storage[b]	Maximum[c]
1	June 23, 1858[d]	35.3	(1,300)	(180)	(1,480)
2	June 26, 1858	46.98[a]			(1,400[a])
3	April 27, 1862	51.10[a]			(1,700[a])
4	March 20–21, 1882	48.75[a]			(1,540[a])
5	March 25, 1884	49.00[a]			(1,550[a])
6	March 16–17, 1890	35.60	1,400	170	1,570
7	April 24–25, 1890	49.05[a]			(1,550[a])
8	April 2–4, 1891	48.1[a]			(1,490[a])
9	May 10, 1892	48.4[a]			(1,500[a])
10	May 22–23, 1893	48.3[a]			(1,495[a])
11	March 20–21, 1897	37.66			(1,600)
12	April 16, 1897	52.48[a]			(1,800[a])
13	April 10–11, 1898	37.22			(1,600)
14	April 24–25, 1898	49.4[a]			(1,580[a])
15	March 20, 1903	40.10			(1,630)
16	March 28, 1903	51.80[a]			(1,800[a])
17	April 11, 1904	39.20	1,620	....	1,620
18	February 3, 1907	40.30			(1,700)
19	February 12–13, 1907	49.65[a]			(1,700[a])
20	March 22, 1909	38.60			(1,600)
21	April 1–2, 1909	48.00[a]			(1,550[a])
22	April 6, 1912	45.23	(1,800)	240	2,040
23	April 12, 1912	51.65[a]	(1,850[a])	(300[a])	(2,150[a])
24	April 9, 1913	46.55	(2,030)	220	2,250
25	April 27–28, 1913	52.20[a]	(1,900[a])	(300[a])	(2,200[a])
26	February 9, 1916	43.4	(1,750)	....	1,750
27	February 15, 1916	53.85[a]			(1,700[a])
28	April 23, 1917	49.98[a]			(1,730[a])
29	April 27, 1920	50.90[a]			(1,825[a])
30	April 1–2, 1922	42.5	(1,560)	....	1,560
31	April 28, 1922	54.85[a]			1,826[a]
32	April 23–25, 1927	45.8	(1,860)	500	2,360
33	May 4, 1927	58.40[a]	(2,145[a])	(350[a])	2,495[a]
34	July 12–16, 1928	49.30[a]			(1,670[a])
35	May 25, 1929	41.5	(1,600)	....	1,600
36	June 6–7, 1929	55.2[a]	1,741[a]		1,741[a]
37	February 19, 1932	38.7	1,308	....	1,308
38	February 28–29, 1932	50.27[a]	(1,410[a])		1,410[a]
39	April 9, 1933	40.1	1,416	....	....
40	June 10, 1933	47.50[a]	1,360[a]		1,360[a]
41	March 19, 1934	30.0	839	....	....
42	April 13–14, 1934	34.58[a]	877[a]		877[a]
43	April 17, 1936	40.3	1,360	....	1,360
44	April 30, 1936	42.54[a]	1,280[a]		1,280[a]
45	February 7, 1937	50.4	2,020	500	2,520
46	February 17–22, 1937	53.2[a]	2,080[a]		2,080[a]

[a] Discharges marked thus: [a]—are Vicksburg readings, and the remainder are Memphis readings. Values in parentheses are unofficial or estimated quantities. [b] Additional floodway and valley storage discharge. [c] Maximum river and floodway discharge. [d] Item 1 represents the record-period maximum for 40 years or more preceding 1858.

Further evidence concerning the progressive minor changes in rating curves, showing the relation of gage reading to river discharge, is offered in Table 7.

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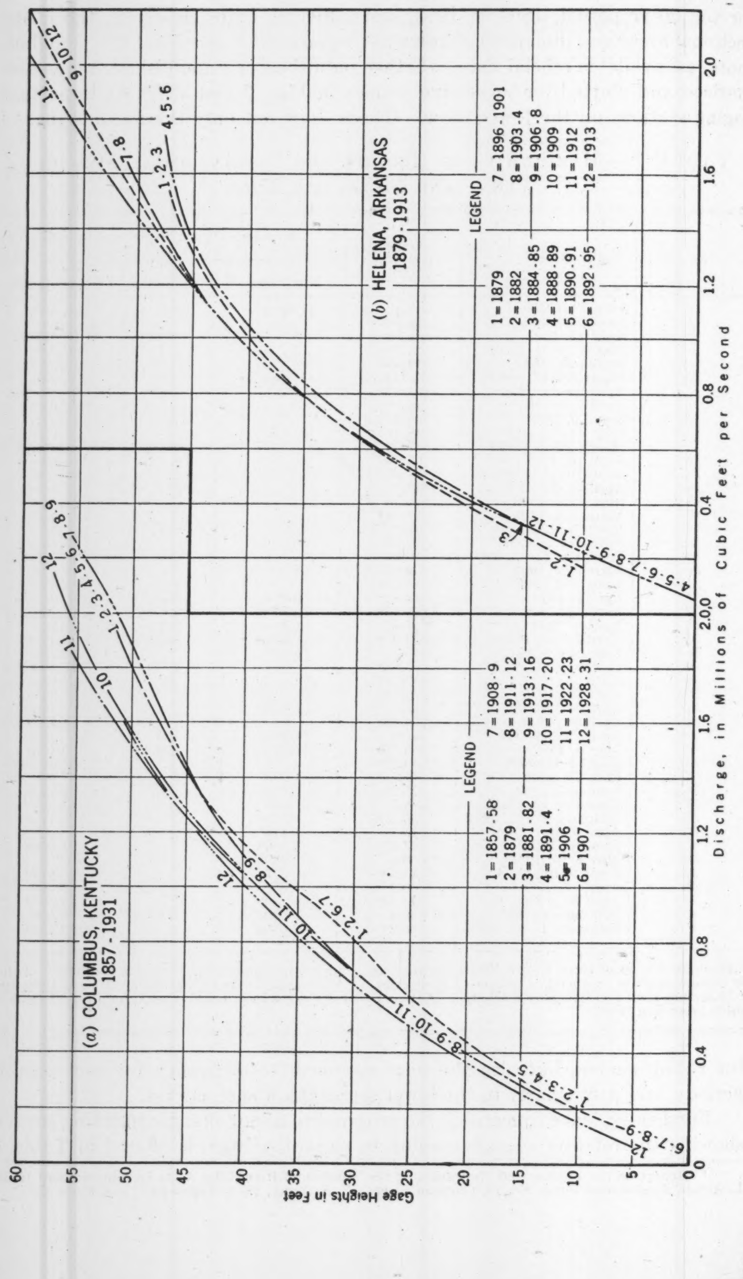
[16] "Report on the Physics and Hydraulics of the Mississippi River," by A. A. Humphreys and Henry L. Abbot, *Professional Paper No. 13*, Corps of Engrs., U. S. Army, 1876, Appendix C and Plate X.


(a) COLUMBUS, KENTUCKY
1857-1931
LEGEND
1 = 1857-58
2 = 1879
3 = 1881-82
4 = 1891-4
5 = 1906
6 = 1907
7 = 1908-9
8 = 1911-12
9 = 1913-16
10 = 1917-20
11 = 1922-23
12 = 1928-31
(b) HELENA, ARKANSAS
1879-1913
LEGEND
1 = 1879
2 = 1882
3 = 1884-85
4 = 1888-89
5 = 1890-91
6 = 1892-95
7 = 1896-1901
8 = 1903-4
9 = 1906-8
10 = 1909
11 = 1912
12 = 1913
6·7·8·9
12
5
1·2·3·4·5
8·9·10·11
10·11
8·9
12
1·2·6·7
10
11
12
1·2·3·4·5·6·7·8·9
4·5·6·7·8·9·10·11·12
1·2
3
4·5·6
1·2·3
7·8
9·10·12
11
Gage Heights in Feet
0
5
10
15
20
25
30
35
40
45
50
55
60
Discharge, in Millions of Cubic Feet per Second
0
0.4
0.8
1.2
1.6
2.0,0
0.4
0.8
1.2
1.6
2.0

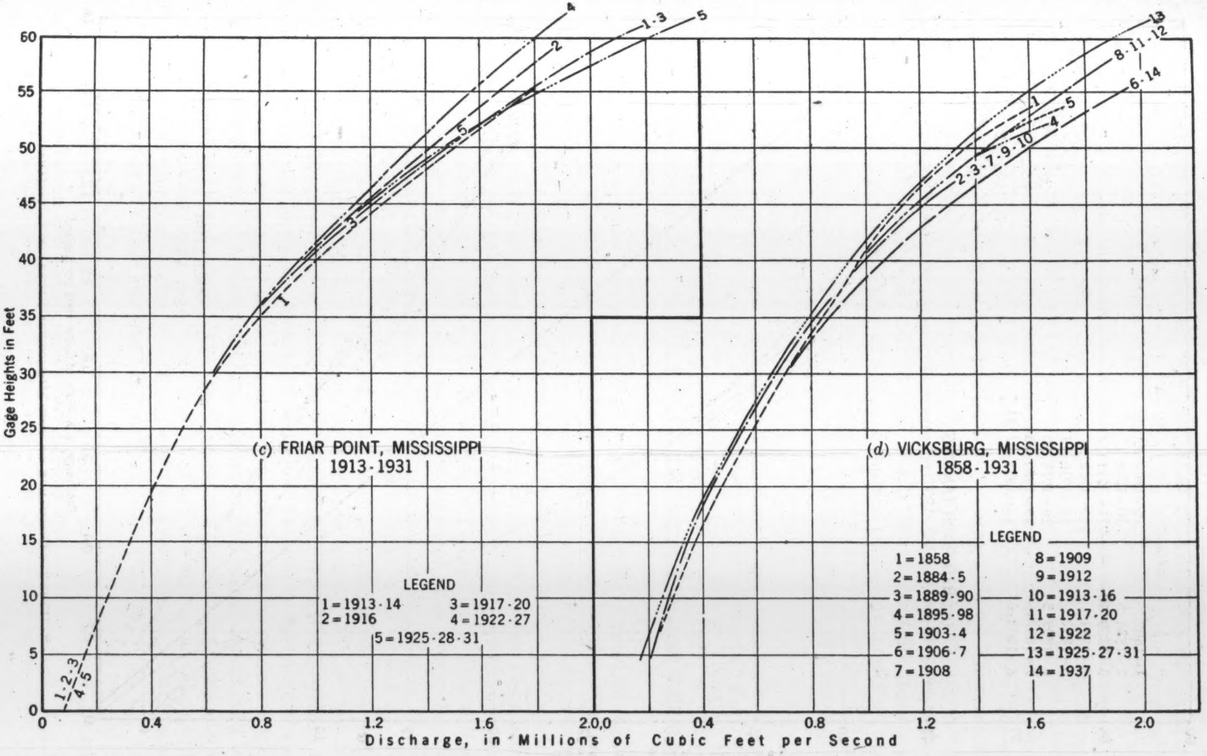


FIG. 2.—DISCHARGE OF THE MISSISSIPPI RIVER COMPILED FROM THE MISSISSIPPI RIVER COMMISSION AND THE U. S. CORPS OF ENGINEERS

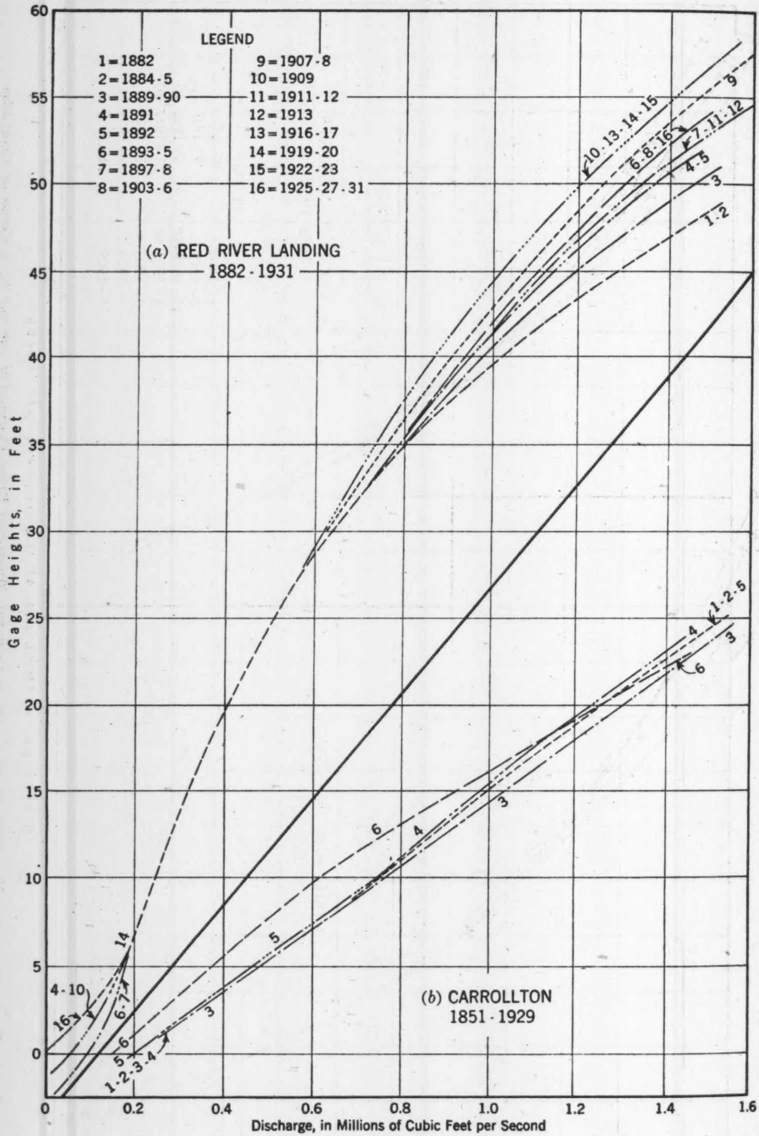


Fig. 3.—Discharge of the Mississippi River, in Louisiana

Table 7 lists the official, derived, and estimated quantities for the principal flood events to date, beginning with the 40-yr (or longer) maxima recorded at both Memphis, Tenn., and Vicksburg. This tabulation indicates that an addition of about 67% to the peak discharge at these two stations produced corresponding increases in maximum gage readings of 15.1 ft and 11.4 ft, respectively. Some undetermined part of these changes must be ascribed to reduced lateral flood-plain storage and total flood-channel section, due to confinement by levees. For Vicksburg, the rise of flood surface due to restrictions by levees was approximately 5 ft, according to data in Fig. 2(*d*). There is shown likewise some evidence of increased channel capacity during early periods, associated with removal of snags or other obstructions to navigation as well as to stream flow; and finally, in the 6-yr period, 1931 to 1937, the influence of cutoffs apparently restored the maximum discharge capacity, as proved by the low position of the rating curve No. 14. This is likewise in accordance with the findings of Brig.-Gen. Harley B. Ferguson,[17] M. Am. Soc. C. E., during his tour as President of the Mississippi River Commission.

Similarly, the writer found:[18]

> "For example, the gage-heights of 46.2 feet corresponding to a discharge of 1,520,000 cfs (Stages and discharge-observations, Lower Valley of the Mississippi River; Mississippi River Commission report, 1937) on March 30, 1891, was exceeded by 4-feet or more for equal discharges during the past 15 years or so."

In Fig. 4 are depicted the yearly, 5-yr, and 10-yr trends of precipitation at Marietta, and the Mississippi River runoff depths in inches over the watershed at Vicksburg, throughout the 123 years of record, including 1939. Apparently a fair correlation is obtained for 92 years, or 75% of the time on a yearly basis, slightly better on the 5-yr basis, and still more uniformly on the 10-yr basis. Likewise, the graphs of cumulative rainfall and runoff departures have several points in common, or nearly so, on the scales adopted; and either the parallelism of trends or their close association and interweaving of lines proves beyond reasonable challenge that this one rainfall station is more of an index to Mississippi River discharge at Vicksburg than would ordinarily be expected. No doubt a part of the explanation lies in the fact that Marietta is near the center of the most productive tributary basin, contributing nearly 50% of the discharge at Vicksburg. The addition of other outstanding station-records as they became available, up to a total of 10, was found to improve the correlation in some parts of the record; but the addition of widely distributed stations to a total of 50 seemed to disturb such correlation of the latter part of the record, as if too much weight was being accorded the areas of low runoff, such as the Great Plains.

For plotting the rainfall and runoff data to afford the most convenient comparison, it was found advisable to utilize a scale for precipitation depths either four or five times the scale adopted for runoff depths, as shown in Fig. 4. For the lower part of the drawing, dealing with cumulative departures, the datum or zero lines do not coincide. In the upper part of the same figure, the mean annual precipitation and runoff depths are slightly separated.

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[17] "Effects of Mississippi River Cut-Offs," by Harley B. Ferguson, *Civil Engineering*, December, 1938, p. 828, Fig. 24.

[18] *Transactions*, Am. Geophysical Union, 1939, p. 165.

TABLE 8.—SUMMARIZED DISCHARGE RECORDS AND DERIVED OR ESTIMATED
(Parentheses Designate Unofficial,

No.	River	Station	YEARS OF RECORD		Drainage area	MEAN DISCHARGE FOR DESIGNATED			
						Water Years, October 1 to September			
			Gagings	Gage Hts.		1933–34	1934–35	1935–36	1936–37
1	Column No.		1	2	3	4	5	6	7
2	Units[a]		(yr)	(yr)	(Sq mile)	cfs	cfs	cfs	cfs
3	Allegheny	Franklin, Pa.	20	54[c]	5,982	6,846	8,148	9,451	13,170
4	Kiskiminetas	Avonmore, Pa.	31	....	1,723	2,260	3,162	3,404	(4,370)
5	Tygart	Fetterman, W. Va.	31	....	1,304	1.887	2,903	2,891	2,777
6	Cheat	Pisgah, W. Va.	25	25[c]	1,360	2,219	3,383	3,159	3,281
7	Youghiogheny	Connellsville, Pa.	29	66[c]	1,326	1,757	2,686	2,688	3,059
8	Monongahela	Charleroi, Pa.	5	53	5,213	6,260	9,494	9,497	10,320
9	Ohio	Sewickley, Pa.	5	67[c]	19,500	20,960	29,840	32,030	39,520
10	Ohio	Wheeling, W. Va.	....	38	24,800	(23,300)	(34,460)	(37,065)	(47,420)
11	Ohio	Parkersburg, W. Va.	....	51	37,914	(29,820)	(48,020)	(51,265)	(68,660)
12	Ohio	Huntington, W. Va.	4	25	55,200	(46,030)	82,920	79,370	95,110
13	Ohio	Cincinnati, Ohio	....	68	76,580	(59,000)	(111,900)	(100,600)	(126,000)
14	Ohio	Louisville, Ky.	4	67	91,200	(68,200)	(131,600)	111,900	149,000
15	Ohio	Evansville, Ind.	....	46	107,000	(84,000)	(167,100)	(127,900)	(180,800)
16	Ohio	Metropolis, Ill.	12 39[c]	80[c]	203,000	(161,000)	(285,200)	226,600	337,100
17	Muskingum	McConnelsville, Ohio	17	26	7,411	2,879	6,607	7,182	11,730
18	Kanawha	Kanawha, W. Va.	61	61	8,367	7,824	16,180	13,840	11,480
19	Scioto	Chillicothe, Ohio	17	31	3,847	883	2,038	2,803	5,167
20	Miami	Hamilton, Ohio	27	29[c]	3,639	1,179	1,397	2,257	5,467
21	Licking	Catawba, Ky.	10	52[c]	3,320	2,240	6,896	3,889	5,394
22	Kentucky	Lockport, Ky.	11	53[c]	6,170	5,346	12,950	8,237	8,893
23	Green	Livermore, Ky.	8	40	7,500	7,791	17,440	7,806	15,060
24	Wabash	Mt. Carmel, Ill.	11	54	28,600	8,043	22,710	14,510	39,250
25	Cumberland	Nashville, Tenn.	50	65	12,860	(16,700)	(23,500)	(16,300)	(24,600)
26	Holston	near Rogersville, Tenn.	36	37	3,035	2,848	5,114	4,523	4,013
27	Little Tennessee	McGhee, Tenn.	33	35	2,443	3,909	5,051	6,080	5,715
28	Clinch	below Norris Dam	11	55[c]	2,913	3,200	5,009	4,264	4,639
29	Hiwassee	Charleston, Tenn.	19	56	2,298	3,217	3,732	5,126	4,365
30	Tennessee	Chattanooga, Tenn.	64	65	21,400	(26,000)	35,700	38,600	36,810
31	Tennessee	Florence, Ala.	44	67	30,810	37,400	49,650	55,740	53,570
32	Tennessee	Johnsonville, Tenn.	49	62	38,520	46,140	62,540	61,740	67,700
33	Mississippi	Elk River, Minn.	23	....	14,500	1,454	2,429	2,889	2,960
34	Mississippi	St. Paul, Minn.	48	61	36,800	1,935	3,688	5,330	5,379
35	Mississippi	La Crosse, Wis.	9	64	62,800	11,300	22,930	23,570	17,580
36	Mississippi	Le Claire, Iowa	65	65	88,600	18,870	41,820	37,390	33,160
37	Mississippi	Keokuk, Iowa	60	71	119,000	21,540	57,260	47,680	49,900
38	Mississippi	Grafton, Ill.	39	59	170,000	(36,800)	(106,000)	(72,200)	(86,000)
39	Mississippi	St. Louis, Mo.	10	81	701,000	67,700	187,600	113,700	146,800
40	Mississippi	Memphis, Tenn.	9	67	932,800	244,285	511,400	349,900	516,700
41	Mississippi	Arkansas City, Ark.	39	59	1,130,700	286,900	651,500	387,900	623,600
42	Mississippi	Natchez, Miss.	35	105	1,149,400	(308,150)	(683,660)	(405,740)	(647,800)
43	Mississippi[b]	...........[b]	45	65	1,242,700	(342,665)	(772,870)	(424,730)	(701,080)
44	Minnesota	Mankato, Minn.	35	....	14,600	136	590	1,084	1,601
45	St. Croix	St. Croix Falls, Wis.	33	....	5,930	1,754	3,356	3,803	3,130
46	Chippewa	Chippewa Falls, Wis.	28	....	5,600	2,563	5,993	5,117	3,445
47	Wisconsin	Muscoda, Wis.	24	66[c]	10,300	4,454	11,040	7,419	6,944
48	Rock	Como, Ill.	24	....	8,700	1,741	4,783	3,190	5,938
49	Iowa	Wapello, Iowa	23	....	12,480	1,210	5,622	4,961	6,654
50	Des Moines	Keosauqua, Iowa	25	32	13,900	607	5,288	3,823	4,216
51	Illinois	Peoria, Ill.	28	67	13,480	10,810	18,590	13,320	16,120
52	Missouri	Fort Benton, Mont.	57	57	24,600	4,952	4,307	4,559	3,618
53	Missouri	Hermann, Mo.	39	65	528,200	29,750	80,010	41,090	59,000
54	Osage	Bagnell, Mo.	13	....	14,000	2,381	14,790	3,950	11,630
55	White	Clarendon, Ark.	11	54	25,750	17,709	43,252	10,246	(35,163)
56	Arkansas	Little Rock, Ark.	39	67	157,900	18,740	65,400	14,010	35,700
57	Yazoo	Greenwood, Miss.	11	54[c]	7,450	5,720	10,230	(7,000)	(14,900)
58	Red	Alexandria, La.	39	67	65,850	(20,085)	42,447	10,885	(27,000)

[a] Cfs = cubic feet per second; csm = cubic feet per second per square mile. [b] Mississippi-Atchafalaya, near

QUANTITIES AT SELECTED STATIONS IN THE MISSISSIPPI RIVER BASIN
Derived, or Estimated Quantities)

PERIODS ENDING SEPTEMBER 30, 1938, AND PERCENTAGES OF LATEST FIVE-YEAR MEANS

30	Years of Record; Observed, (Derived), or (Estimated)									
1937–38	5		10		15		20		25	30
8 cfs	9 cfs	10 csm	11 cfs	12 %	13 cfs	14 %	15 cfs	16 %	17 %	18 %
10,970	9,717	1.62	9,500	97.8	10,036	103.3	9,747	100.3	(103.9)	(106.0)
(2,765)	3,170	1.84	2,789	88.0	3,003	94.7	2,952	93.1	93.0	95.8
2,833	2,658	2.04	2,447	92.1	2,633	99.1	2,594	97.6	97.1	98.2
3,360	3,080	2.26	2,852	92.6	(3,000)	97.4	(2,990)	97.1	(98.5)	(100.6)
2,410	2,520	1.90	2,311	91.7	2,450	97.2	(2,441)	96.9	96.9	99.2
9,341	8,982	1.72	(8,260)	92.0	(8,870)	98.8	(8,860)	98.6	(98.6)	(101.3)
33,120	31,090	1.595	(30,000)	96.5	(31,600)	101.5	(31,500)	101.3	(101.6)	(101.3)
(39,270)	(36,530)	1.473	(35,130)	96.2	(37,120)	101.6	(36,900)	101.0	(101.6)	(101.0)
(55,490)	(50,880)	1.342	(48,370)	95.1	(51,000)	100.2	(50,600)	99.5	(100.2)	(99.3)
85,240	77,730	1.408	(71,680)	92.2	(73,000)	93.9	(74,000)	95.2	(96.8)	(96.8)
108,700	(100,860)	1.317	(94,700)	93.9	(96,000)	95.2	(98,000)	97.2	(99.0)	(98.8)
124,000	116,920	1.282	(111,220)	95.1	(114,000)	97.5	(116,000)	99.2	(100.5)	(99.2)
(144,200)	(140,800)	1.316	(134,000)	95.2	(140,000)	99.4	(143,000)	101.6	(101.3)	(99.4)
269,800	255,940	1.261	253,970	99.2	263,000	102.7	272,000	106.3	104.7	107.4
8,504	7,380	0.996	6,891	93.4	7,200	97.6	(7,250)	98.2	(98.9)	(98.9)
13,170	12,499	1.494	11,561	92.5	11,656	93.3	11,821	94.6	95.3	95.1
3,503	2,879	0.748	3,257	113.1	3,427	119.0	(3,300)	114.6	(110.0)	(109.8)
4,439	2,948	0.810	3,168	107.5	3,387	114.9	3,478	118.0	120.1	(117.0)
3,516	4,387	1.321	3,986	90.9	(4,060)	92.5	(4,150)	94.6	(95.7)	(95.7)
(7,270)	8,539	1.384	8,052	94.3	(8,200)	96.0	(8,400)	98.4	(99.5)	(99.5)
(9,553)	11,530	1.537	(10,874)	94.3	(11,000)	95.4	(11,700)	101.5	(104.1)	(102.3)
35,020	23,907	0.836	25,929	108.5	(27,000)	112.9	(27,000)	112.9	(112.1)	(112.1)
(18,000)	(19,820)	1.541	(20,128)	101.6	20,579	103.8	21,584	108.9	108.4	108.8
4,647	4,229	1.393	4,120	97.4	4,183	98.9	4,257	100.7	100.2	99.8
5,447	5,240	2.145	5,431	103.6	5,280	100.8	5,669	108.2	110.4	112.7
4,303	4,283	1.470	4,316	100.8	(4,350)	101.6	(4,400)	102.7	(102.7)	(103.9)
4,633	4,215	1.834	4,656	110.5	4,477	106.2	4,791	113.7	(112.5)	(113.9)
35,720	34,566	1.615	35,283	102.1	34,762	100.6	36,702	106.2	105.2	105.9
49,600	49,192	1.597	51,506	104.7	50,604	102.9	53,443	108.6	106.8	107.7
58,330	59,290	1.539	63,205	106.6	62,597	105.6	66,258	111.8	108.9	109.6
4,819	2,910	0.201	2,941	101.1	3,197	109.9	3,663	125.9	(146.0)	(140.7)
8,218	4,910	0.133	4,906	99.9	5,346	108.9	6,492	132.2	158.3	157.3
30,390	21,154	0.337	19,947	94.3	(21,150)	100.0	(23,000)	108.7	(114.6)	(114.6)
49,770	36,200	0.409	35,845	99.0	37,734	104.2	39,566	109.3	114.5	114.0
65,520	48,380	0.407	48,218	99.7	50,352	104.1	51,880	107.2	112.3	112.8
(99,000)	(80,000)	0.471	84,844	106.1	93,130	116.4	93,900	117.4	120.9	120.9
156,800	134,520	0.192	145,670	108.3	(165,500)	123.0	(168,200)	125.0	(129.0)	(127.8)
445,500	413,550	0.443	426,000	103.0	(460,000)	111.2	(472.000)	114.1	(114.9)	(115.3)
550,700	500,140	0.442	514,400	102.9	555,800	111.1	574,870	114.9	(116.5)	(117.3)
(573,200)	(523,670)	0.456	(539,600)	103.0	(567,100)	108.3	(579,000)	110.6	(110.8)	(110.7)
(668,000)	(581,870)	0.468	(596,085)	102.4	(629,200)	108.1	(649,800)	111.7	(110.7)	(110.9)
1,939	1,070	0.073	994	92.9	(1,160)	108.4	(1,410)	131.8	(179.0)	(186.9)
4,710	3,351	0.565	3,045	90.9	3,079	91.9	3,154	94.1	98.5	96.7
7,094	4,842	0.865	4,233	87.4	4,396	90.8	4,588	94.8	98.4	(97.4)
11,380	8,247	0.801	7,983	96.8	8,388	101.7	8,487	102.9	104.9	(105.4)
6,855	4,502	0.517	4,675	103.8	5,169	114.8	5,331	118.4	120.1	(120.1)
5,139	4,717	0.378	5,372	113.9	5,648	119.7	5,702	120.9	123.8	(120.8)
3,675	3,522	0.253	4,007	113.8	4,035	114.6	4,400	124.9	134.0	(131.6)
17,410	15,250	1.131	16,405	107.6	17,617	115.5	17,158	112.5	109.5	108.4
6,010	4,608	0.187	5,122	111.2	6,232	135.2	6,496	141.0	155.8	165.3
56,640	53,300	0.101	59,520	111.7	71,257	133.7	73,165	137.3	142.0	138.9
8,303	8,211	0.587	7,527	91.7	9,285	113.1	(9,600)	116.9	(121.8)	(125.4)
(37,000)	(28,674)	1.112	30,250	105.5	(32,000)	111.6	(30,000)	104.6	(106.4)	(104.6)
46,590	36,088	0.229	36,200	100.3	40,260	111.5	41,240	114.3	110.8	103.1
(11,240)	(9,818)	1.318	10,293	104.8	(10,500)	106.9	(10,500)	106.9	(101.8)	(106.9)
(50,000)	(30,060)	0.456	30,120	100.2	29,090	96.8	30,630	101.9	97.0	91.6

Red River Landing, La. [a] Gage heights recorded at a neighboring station.

TABLE 8—(*Continued*)

No.	Mean Discharge and Percentages (*Continued*)						Maximum Discharge			
	Years of Record (*Continued*)						Q (max)		Date	Myers rating
	40	50	60	65	80	100				
	19 %	20 %	21 %	22 %	23 %	24 %	25 cfs	26 csm	27	28 %
3	(107.0)	(108.0)	(107.0)	(106.5)	(106.0)	(105.0)	196,000	32.8	3–17–1865	25.4
4	(95.3)	(97.8)	(99.1)	(98.4)	(97.5)	(96.2)	200,000	116.	3–18–1936	48.3
5	(99.3)	(100.1)	(102.5)	(102.0)	(101.6)	(99.7)	74,300	57.0	7–25–1912	20.3
6	(100.7)	(98.7)	(98.1)	(97.4)	(96.1)	(94.8)	160,000	118.	7–10–1888	43.4
7	(98.8)	(99.2)	(100.0)	(100.0)	(99.2)	(98.0)	92,500	69.8	3–18–1936	25.4
8	(101.6)	(102.0)	(101.6)	(101.3)	(101.0)	(100.0)	156,000	29.9	7–11–1888	21.6
9	(100.3)	(100.7)	(101.6)	(101.6)	(100.0)	(98.4)	574,000	29.4	3–18–1936	41.1
10	(102.1)	(102.9)	(104.0)	(103.7)	(105.0)	(104.0)	507,000	20.4	2 — 1884	32.2
11	(100.4)	(103.2)	(104.2)	(103.8)	(103.9)	(102.6)	(600,000)	15.8	3–30–1913	30.8
12	(99.8)	(103.6)	(104.2)	(103.8)	(102.9)	(101.6)	654,000	11.8	1–28–1937	27.8
13	(101.1)	(104.1)	(104.9)	(104.1)	(103.1)	(102.1)	894,000	11.6	1–26–1937	32.3
14	(101.5)	(104.3)	(105.2)	(104.3)	(103.5)	(102.6)	1,110,000	12.2	1–26–1937	36.4
15	(101.2)	(103.7)	(105.8)	(104.4)	(104.0)	(105.0)	1,410,000	13.2	1–29–1937	43.1
16	105.5	(103.9)	(107.4)	(105.9)	(107.1)	(111.0)	1,850,000	9.1	2– 1–1937	41.0
17	(100.0)	(103.0)	(104.0)	(103.0)	(103.8)	(103.0)	270,000	36.4	3–27–1913	31.4
18	100.1	103.7	104.2	(102.4)	(103.2)	(102.4)	270,000	32.3	9–14–1878	29.5
19	(107.7)	(112.0)	(113.0)	(114.0)	(111.0)	(110.0)	260,000	67.6	3–26–1913	42.0
20	(108.5)	(113.6)	(115.3)	(117.0)	(113.6)	(111.9)	352,000	96.7	3–26–1913	58.4
21	(98.0)	(100.3)	(103.9)	(104.9)			86,200	26.0	1–23–1937	14.9
22	(101.9)	(104.2)	(107.7)	(108.0)			99,000	16.0	1–24–1937	12.6
23	(103.2)	(104.1)	(107.5)	(108.0)			208,000	27.7	1–27–1937	24.0
24	(108.8)	(110.8)	(112.9)	(112.1)	(112.9)	(117.1)	428,000	15.0	3–30–1913	25.3
25	105.9	105.0	(104.6)	(104.4)	(103.9)	(106.0)	203,000	15.8	1– 1–1927	17.9
26	(106.4)	(106.4)	(111.1)	(108.8)	(109.9)	110.8	(200,000)	65.9	3–10–1867	36.3
27	(114.5)	(114.5)	(115.8)	(116.4)			(137,000)	56.1	3 — 1867	27.8
28	(105.1)	(106.2)	(108.6)	(109.7)			70,000	24.0	3–23–1929	13.0
29	(118.6)	(118.6)	(121.0)	(122.2)			70,000	30.5	3–31–1886	14.6
30	107.4	107.9	110.2	110.9	(109.9)	(112.8)	459,000	21.4	3–11–1867	31.4
31	107.3	(107.7)	109.8	110.8	(109.8)	(113.8)	444,000	14.4	3–19–1897	25.3
32	108.0	107.3	(109.6)	(110.5)	(109.6)	(113.0)	460,000	11.9	3–24–1897	23.4
33	(154.5)	(159.6)	(164.9)	(161.5)	(160.0)	(156.0)	31,300	2.2	4– 5–1917	2.6
34	179.4	(181.3)	(190.0)	(185.3)	(183.3)	(177.2)	107,000	2.9	4–29–1881	5.6
35	(129.1)	(127.6)	(130.0)	(129.1)	(129.1)	(127.6)	(160,000)	2.6	6–19–1880	6.4
36	124.3	123.5	130.8	132.1	(132,6)	(129.8)	250,000	2.8	6–25–1880	8.4
37	120.5	118.1	125.6	(126.9)	(128.2)	(126.1)	314,000	2.6	5–18–1888	9.1
38	126.6	(123.8)	(130.0)	(131.3)	(132.5)	(130.0)	(350,000)	2.1	6 — 1844	8.5
39	(131.9)	(128.0)	(133.3)	(133.3)	(131.0)	(128.8)	1,300,000	1.85	6–28–1844	15.5
40	(116.1)	(113.2)	(117.0)	(116.1)	(115.8)	(116.6)	2,520,000	2.70	2– 9–1937	26.1
41	(117.6)	(115.0)	(115.2)	(114.4)	(114.0)	(117.0)	2,160,000	1.91	2–16–1937	20.3
42	(111.2)	(110.1)	(110.4)	(110.8)	(111.9)	(115.7)	2,050,000	1.78	2–19–1937	19.1
43	(111.0)	(109.6)	(110.0)	(110.3)	(111.4)	(114.8)	(2,620,000)	2.11	2–18–1937	23.5
44	(200.0)	(196.3)	(215.0)	(205.6)	(200.0)	(196.0)	65,000	4.45	4 — 1881	5.4
45	(103.3)	(104.4)	(107.4)	(107.4)	(106.0)	(104.0)	35,800	6.04	3–26–1920	4.6
46	(105.0)	(106.0)	(112.0)	(112.0)	(110.0)	(108.0)	78,000	13.9	3–27–1920	10.4
47	(111.0)	(110.0)	(115.0)	(115.2)	(113.0)	(115.0)	80,800	7.84	9–16–1938	8.0
48	(125.0)	(122.8)	(128.8)	(129.9)	(128.0)	(129.0)	51,000	5.86	2–22–1937	5.5
49	(124.0)	(121.9)	(127.2)	(129.3)	(126.0)	(128.0)	67,500	5.41	3–19–1929	8.0
50	(138.1)	(133.4)	(140.8)	(142.0)	(139.1)	(136.3)	97,000	6.98	6– 1–1903	8.2
51	(112.6)	(110.8)	(114.1)	(114.8)	(114.1)	(111.4)	58,300	4.32	10– 9–1926	5.0
52	173.7	179.7	182.6	(180.0)	(175.8)	(169.3)	140,000	5.69	6– 7–1908	8.9
53	140.5	(135.1)	(138.8)	(137.0)	(131.3)	(127.6)	(600,000)	1.14	6–19–1844	8.3
54	(131.3)	(137.6)	(146.1)	(143.7)	(140.0)	(137.0)	150,000	10.7	6 — 1844	12.7
55	(104.6)	(108.1)	(101.1)	(104.6)	(101.1)	(101.1)	440,000	17.1	4–23–1927	27.4
56	104.2	(110.8)	(99.8)	(105.3)	(104.2)	(102.5)	422,000	2.67	6–22–1935	10.6
57	(96.8)	(96.8)	(96.8)	(99.8)	(99.8)	(98.8)	72,900	9.8	1–29–1932	8.4
58	96.1	(99.8)	(96.5)	(96.5)	(99.8)	(103.1)	210,000	3.2	7– 2–1908	8.2

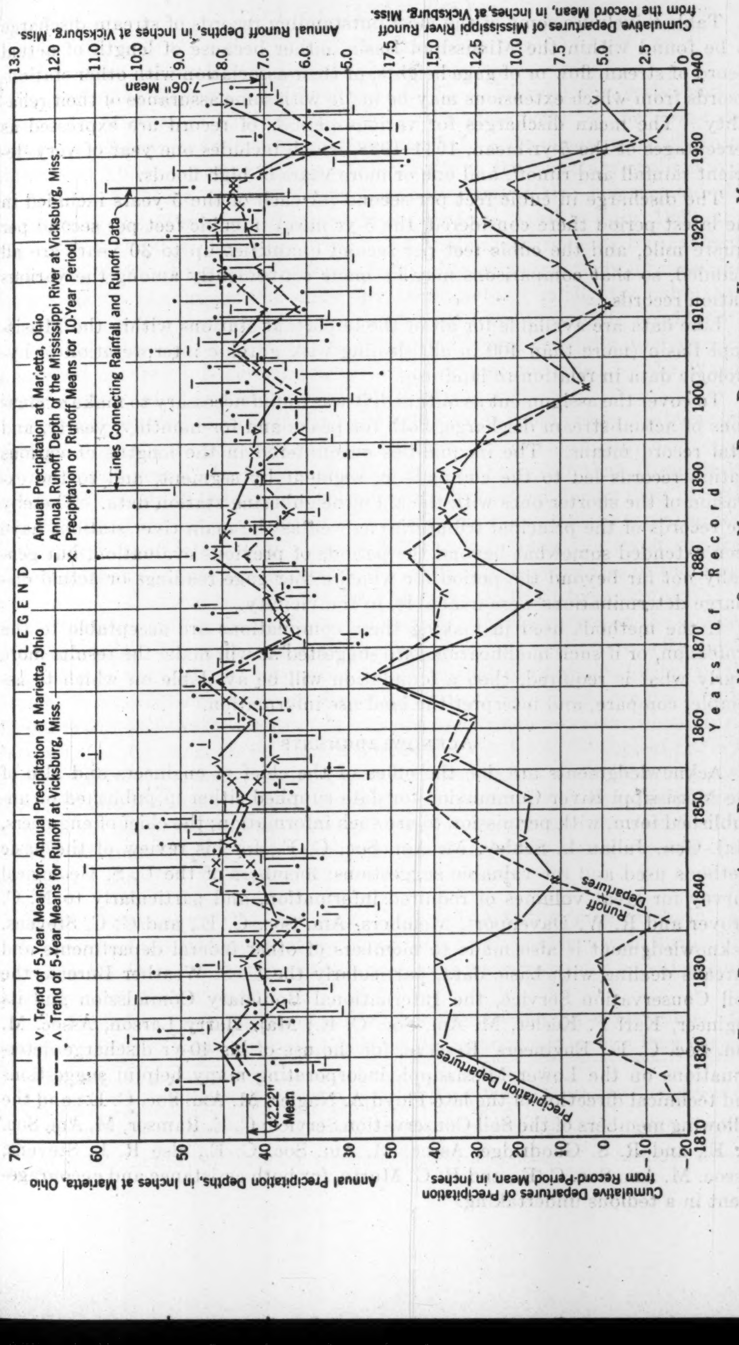

FIG. 4.—RELATIONS OF ANNUAL PRECIPITATION AT MARIETTA, OHIO, TO MISSISSIPPI RIVER DISCHARGE AT VICKSBURG, MISS.

Table 8 includes some of the most outstanding records of stream discharge to be found within the Mississippi Basin, either because of length of actual record of stream flow or of gage heights, or their association with other station-records from which extensions may be made with some assurance of their reliability. The mean discharges for various periods of record are expressed as percentages of the 5-yr mean, 1934–1938, which includes one year of very deficient rainfall and runoff, and one or more years of high floods.

The discharge in cubic feet per second for each of the 5 years included in the latest period there considered, the 5-yr mean in cubic feet per second per square mile, and the cubic feet per second quantities up to 30 years are all included, so that comparisons may be made conveniently among the various station records.

Like data are available for all of the important stations within the Mississippi Basin (more than 400 in all) dealing with graphic interpretation of hydrologic data in relation to land use.

To cover the assignment as outlined, it was found necessary to make evaluations of actual stream discharge, both for peaks and for monthly, yearly, and total record means. The inequalities encountered in the lengths of various station records led to the comparison, segment by segment, and to the extension of the shorter ones with the aid of neighboring station data. Thereby the records of the principal tributaries as well as the main river stations have been extended somewhat beyond the periods of previous evaluation, but generally not far beyond the period for which either gage readings or actual discharge determinations were available, in the vicinity.

If the methods used in making these compilations are acceptable to the profession, or if such modifications are suggested as will make the results more nearly what is required, then a foundation will be available on which to assemble, compare, and interpret the land use information.

ACKNOWLEDGMENTS

Acknowledgments are due the office of the chief of engineers and also of the Mississippi River Commission for data supplied either in published or unpublished form, with permission to use such information; the chief of engineers, Maj.-Gen. Julian L. Schley, M. Am. Soc. C. E., for his review of the basic methods used and for valuable suggestions; members of the U. S. Geological Survey for great volumes of required information, and particularly to N. C. Grover and R. W. Davenport, Members, Am. Soc. C. E., and G. C. Stevens. Acknowledgment is also made to members of other federal departments and bureaus dealing with basic data, particularly the U. S. Weather Bureau, the Soil Conservation Service, the International Boundary Commission and its engineer, Karl F. Keeler, M. Am. Soc. C. E.; Maj. Harry Larson, Assoc. M. Am. Soc. C. E., Engineers' Reserve, for the use of his 40-yr discharge determinations on the Lower Mississippi, incorporating many helpful suggestions and technical direction of the late Floyd A. Nagler, M. Am. Soc. C. E.; and the following members of the Soil Conservation Service: C. E. Ramser, M. Am. Soc. C. E., and R. S. Goodridge, Assoc. M. Am. Soc. C. E., also R. L. Stevens, Assoc. M. Am. Soc. C. E., and H. C. Murto, for both assistance and encouragement in a tedious undertaking.

DISCUSSION

Robert Follansbee,[19] M. Am. Soc. C. E.—Although the early contributions to Mississippi River hydrology were made chiefly by the Army engineers, the Navy made its contribution even before the classic work of Humphreys and Abbot. Acting under instructions from the Secretary of the Navy, Lt. Robert A. Marr, master of the Memphis Navy Yard, investigated the discharge of the Mississippi River at Memphis from April to July, 1849, and from March, 1850, to February, 1851. He made measurements by means of floats, and deduced the mean velocity from surface observations by means of floats sunk to various depths. The reduction factor was slightly more than 10%. He also obtained daily gage heights, and from these gage heights and the results of the discharge measurements he computed the daily discharge for the period covered. A description of this work was presented by the Washington Astronomical and Meteorological Observatory.[20]

The author refers to the earnest approaches made by the early contributors to Mississippi River hydrology toward the more rigorous standards developed during recent years. In that connection it may be noted that Humphreys and Abbot's study of the vertical-velocity curves obtained during their investigations indicated that the velocity at different depths varied as the abscissas of the parabola which had its x-axis parallel to and below the water surface. However, they did not pursue further the study of the properties of the parabolic curve. That was reserved for a British engineer, Maj. Allen Cunningham, who, during the 1870's, made 565 sets of vertical-velocity curves on the Ganges Canal in India. He concluded that, by considering the vertical-velocity curve as a common parabola, its properties indicated that the mean velocity equals $\frac{1}{2}$ (velocity at 0.211-depth + velocity at 0.789-depth).[21] These are practically the 0.2-depth and the 0.8-depth. It was not until 1906, however, when H. K. Barrows, M. Am. Soc. C. E., then a member of the U. S. Geological Survey, made a study of vertical-velocity curves and found that their form was that of a parabola, the mean ordinate of which was closely the average of the ordinates at the 0.2-depth and the 0.8-depth, that the procedure of measuring the velocity of these depths and taking the average as the mean velocity was adopted by the U. S. Geological Survey as its standard.

R. W. Davenport,[22] M. Am. Soc. C. E.—The author's research in the early records of the Mississippi River constitutes a valuable service in furnishing a "panoramic view" of the flow of this dominant waterway for a period of considerably more than a century.

The long-time compilations of runoff presented in the paper are subject to use in a variety of ways, not only for fixing the mean runoff over a long period of years but for studies of trends, and variation in magnitudes, for possible correlation with other factors. The author has included comparisons of deviations of runoff with those of rainfall. He also has in mind that the data may form a foundation on which to assemble, compare, and interpret land-use information.

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[19] Dist. Engr., U. S. Geological Survey, Denver, Colo.

[20] "Observations on the Mississippi River at Memphis, March 1850 to March 1851," by Robert A. Marr, Washington Astronomical and Meteorological Observatory, Vol. 3.

[21] "Recent Hydraulic Experiments," by Allen Cunningham, *Proceedings*, Inst. C. E., Vol. 71, London, 1883.

[22] Hydr. Engr. in Chg, Div. of Water Utilization, Dept. of the Interior, U. S. Geological Survey, Washington, D. C.

It is particularly desirable in such uses of runoff data that the relative refinement of the data be known so that one may consider the extent, if any, to which evidence of relationships, apparent or sought for, may be influenced by fortuitous or systematic errors. It is not uncommon for such errors to obscure or to invalidate the significance of the evidence. It is desirable, as the author indicates, to suggest modifications that will make the results more nearly what is required for such uses. It is also very essential to know the inherent limitations of the results so that they may not be applied unwisely. If the data are satisfactorily free of such errors, the further question should be considered as to whether the significance of the factor investigated, such as land use, may be masked by other factors, particularly those of climatic variation.

The writer has been greatly impressed with the tremendous quantity of data concerning the history and behavior of the Mississippi River and feels quite humble in discussing a subject of which his knowledge encompasses comparatively little. The points raised herein are not intended to be in the way of criticisms, but more in the way of questions on which it is hoped the author may throw light.

The longest record developed is that of the Mississippi River at Natchez from 1817 to 1938, a period of 122 years, the results being shown in Tables 5 and 6. The technique used in the determination of the record in the earlier years may be learned by a perusal of *Professional Paper No. 13*,[23] which presents results for 33 out of the 42 years from 1818 to 1860. These records apparently were based on stages prior to 1848 solely at Natchez, and subsequently mostly at Carrollton and Donaldsonville, La. There seems to be very little information for appraising the accuracy of the basic stage records.

The stage-discharge relation for the Natchez station was determined from discharge measurements in January and February, 1858, at Natchez, and for the remainder of the year it was determined at Vicksburg and transferred to Natchez. In developing the relation a mean curve was apparently drawn through the loops described by the individual measurements of rising and falling stages. The inaccuracy involved in the determination of the discharge for a given day would tend to be compensative in computing the discharge for the year. The mean annual discharge was determined essentially by application of the twelve mean monthly stages of the year to the rating table, adding the results, and dividing by twelve.

The discharge determinations consisted of velocity measurements made by means of timing double floats, the most common form being a surface float of cork to which was attached a paint keg, the depth of submergence of which was fixed by a cord attached to the cork. The kegs were passed at all depths from the bottom to the top, but a uniform depth of 5 ft seems to have been adopted for general use. The mean velocities of the floats were determined for divisions 200 ft in width, and the products of the mean surface velocity and the areas of the cross sections for such divisions were computed and added for the entire cross section to give the "approximate discharge." Soundings were made in the vicinity of Natchez in January and in the vicinity of Vicksburg in February, September, and December during 1858, which apparently gave the basis for the areas of cross sections to be used in the discharge computations.

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[23] "Report on the Physics and Hydraulics of the Mississippi River," by A. A. Humphreys and Henry L. Abbot, *Professional Paper No. 13*, Corps of Engrs., U. S. Army, 1876.

The "approximate discharge" was corrected by a ratio designed to adjust the float velocity to the mean velocity in the section and also to take account of the prevalent condition of the wind. Similarly, the discharges at Carrollton and Donaldsonville were computed from measurements made in 1851 and 1852.

This early river measurement technique, briefly summarized herein but described fully in *Professional Paper No. 13*,[23] was unquestionably the best afforded by the knowledge and facilities of the time and was conducted with supreme care. It may be compared with that used at the present time by the Geological Survey for the gaging station at Vicksburg:[24]

> "Discharge Measurements.—Discharge measurements are made from the downstream side of the highway section of the bridge. The highway section is on the upstream side of the bridge, and the railway section on the downstream side. The metering equipment is lowered through an opening between the floors of the highway and railway. The equipment is operated by means of a reel and boom mounted on a low-wheeled truck frame. The truck frame also carries a four-cylinder automobile engine which supplies power for operating the metering equipment and moving the truck from point to point along the bridge. Depth of water and position of meter are shown by a depth indicator attached to the reel. During medium and high stages the depth of water and velocity are such that the metering equipment is carried downstream and the indicated depth is too large. Conversion to true depth is effected by use of two correction tables, one of which gives the correction to slant length of supporting line above water surface, and the other gives the correction to curved length of line below water surface. The use of each table requires a measurement of the vertical angle in the supporting line above the water surface, and this measurement is accomplished by use of a protractor applied to the line just below the bridge floor. The corrections to indicated depth are lessened by the use of large sounding weights. The maximum sounding weight used at the Vicksburg station is 300 pounds.
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> "Determination of Discharge.—Daily discharge determined by shifting-control method. Changes in stage-discharge relation determined by frequent measurements."

From the establishment of the Vicksburg gaging station in 1930 to the end of the water year 1941, 1,524 discharge measurements have been made at this station, generally two to three days apart, to record the vagaries of the flow of the river. Fig. 5 illustrates the plotting of discharge measurements in 1938 in excess of 700,000 cu ft per sec. The writer is unable to state how typical this period may have been. However, the plotted points suggest the complications involved in the determination of the stage-discharge relation for application to the stage records. Soundings are made for each discharge measurement and the cross sections thus determined indicate considerable variation during short periods. Comparisons of areas of cross sections for gage heights of 28 to 30 ft from soundings from 1933 to 1939 indicate an increase in area of about 10%. This increase may have been associated with the effects of cutoffs below Vicksburg. However, it illustrates another changing aspect of the behavior of the Mississippi River.

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[24] *Water-Supply Paper No. 747*, U. S. Geological Survey, Washington, D. C., 1933, p. 16.

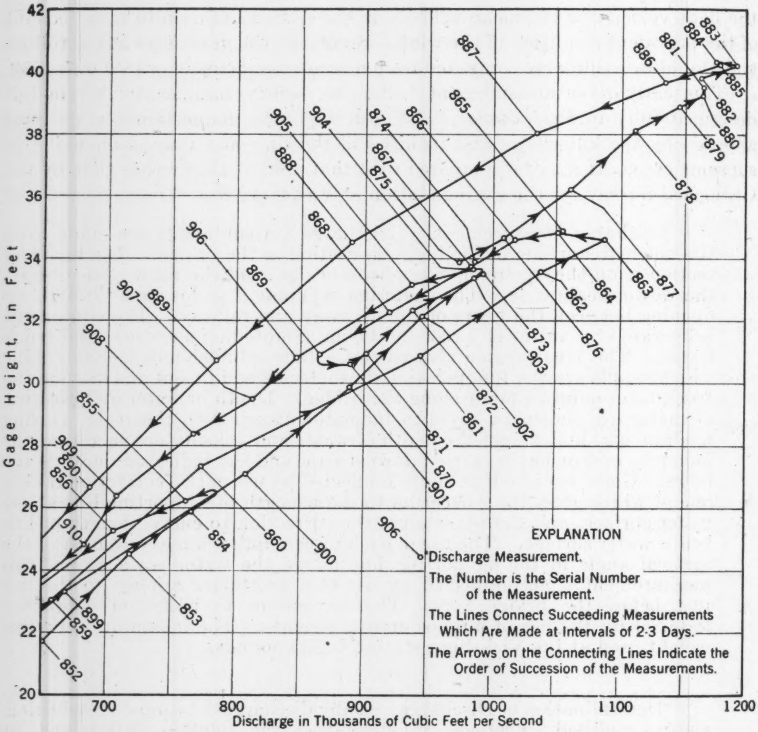

FIG. 5.—DISCHARGE MEASUREMENTS, MISSISSIPPI RIVER AT VICKSBURG, MISS., 1938, U. S. GEOLOGICAL SURVEY

The application of the stage-discharge relation as determined for Natchez in 1858 involves questions of reliability of stage records, the accuracy of the compensative adjustment by use of a mean curve drawn through rising and falling branches of the plotted measurements, uncertainty as to cross-sectional areas applicable at the time of a measurement, and other matters pertinent to a pioneer technique. Much more serious than these questions, however, are those pertinent to the application of the 1858 stage-discharge relation preceding the period in which the determining observations were made as far back as

1818. Illustrations in "The Improvement of the Lower Mississippi River for Flood Control and Navigation"[25] delineate pronounced progress in channel changes in the Mississippi River below Vicksburg and Natchez between 1820 and 1861 and also, apparently, in the construction of levees.

Fig. 2, which shows rating curves of the Mississippi River at Vicksburg from 1858 to 1937, indicates that in that period the mean rating curves that have been defined have varied within limits approximating ±5%. It is apparently not demonstrated, however, that changes influential on the Vicksburg-Natchez section of the river may not have been substantially more operative in the period 1817 to 1858 than in the later period.

The critical feature in respect to the use of the earlier records is the cloud of uncertainty resting over their reliability. The author recognizes this situation in his supposition that modifications may be suggested that will make the results more nearly what is required for the analysis of land-use information. It is notable that the Mississippi River Commission has not included such earlier records in its compilations of past records—for what reason the writer is not informed. In their present status, it is important to caution earnestly that in any applications of the records their inherent nature as respects accuracy be kept in mind.

Charles Senour,[26] M. Am. Soc. C. E.—A well-deserved tribute is paid, in this interesting paper, to the early observers of Mississippi River phenomena, whose painstaking care and understanding went far toward compensating for the shortcomings of their comparatively crude equipment.

Table 1 records annual discharges of the Mississippi River at Natchez for most of the period 1818 to 1859, as obtained from *Professional Paper No. 13.*[23] Captain Humphreys and Lieutenant Abbot measured discharges at Natchez in 1851 and 1858 by subsurface floats. A vertical curve of velocities for the station was prepared and the equation determined for converting velocity at 5-ft depth to mean velocity. From these measurements they developed a stage-discharge relationship curve, and, utilizing this in conjunction with monthly mean stages, deduced the mean annual discharges. Mr. Jarvis has supplemented the Humphreys and Abbot data with later observed or derived data and, in Tables 5 and 6, has extended Table 1 to 1939.

To compare actual measured discharges with those indicated by the stage-discharge curve, the writer plotted all observed discharges at Natchez or at stations between that point and Old River from 1851 to date, in terms of the gage used by Humphreys and Abbot. Fig. 6 shows the zone of extreme variations of those plottings, as compared with the rating curve. The discharges were found to fit the curve very well except for the stages below 15 ft. Of course, there is a fairly wide spread (about 140,000 cu ft per sec in the upper portion of the field) between rising stage and falling stage discharge for the same gage height, so that the mean curve, as a rule, would not give the true discharge for any particular date except (perhaps) at the crest of a rise; but in using it to develop an annual mean or total discharge such discrepancies would tend to compensate. At this particular station, the stage-discharge relationship appears to have been rather constant for the years in which discharge was measured. Measurements have been far from continuous, however. The range has never been considered a highly desirable one, because of very deep water and large boils which give rise to questions regarding the accuracy of both velocity measurements and soundings. Since 1884, measurement of either high water or low water has been made practically every year at both Vicksburg, Miss., and Arkansas City, Ark., and since 1927 the observations have covered a much longer period of each year. At Red River Landing the same process has been in use since 1881. At Natchez, however, following those of Humphreys and Abbot, no measurements were made until 1890–1892. Again there was a hiatus until 1927 and another between 1927 and 1934. In 1911–12–13 measurements were at Point Breeze, about 57 miles below Natchez. In 1928 a range was established at Tarbert Landing, just above Old River, and measurements have been made there annually since that date. Since 1934, annual measurements have also been made at Natchez. What changes the stage-discharge relationship may have undergone during the long periods for which there are no data, and what that relationship may have been between 1818 and the Humphreys and Abbot measurements, are not known. It is known, however that there were major adjustments in the form of Shreves Cutoff and Raccourci Cutoff in 1831 and 1848, respectively.

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[25] "The Improvement of the Lower Mississippi River for Flood Control and Navigation," prepared by D. O. Elliott, under the direction of T. H. Jackson, U. S. Waterways Experiment Station, Vicksburg, Miss., Vol. III, May 1, 1932, Plates VI(f), VI(g), VI(h), and XXXII.

[26] Head Engr., Mississippi River Commission, Vicksburg, Miss.

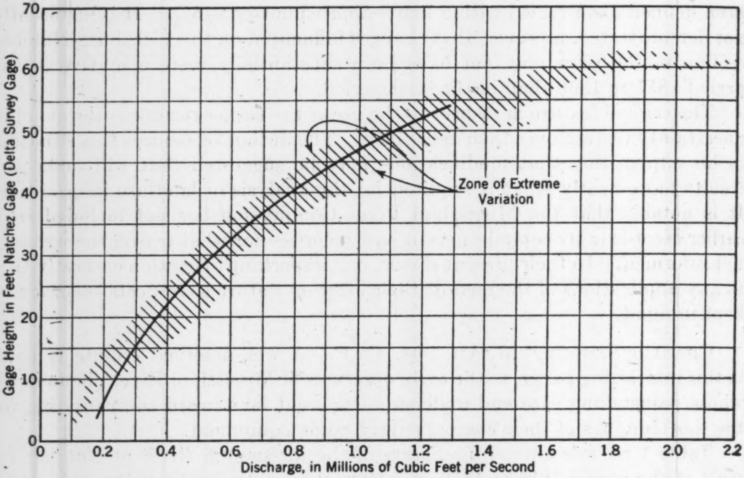

FIG. 6.—DISCHARGE OBSERVATIONS; MISSISSIPPI RIVER, AT NATCHEZ, MISS.

Constancy of stage-discharge relationship is by no means the rule on the lower river. The gradual growth of the levee system to a stature that really permitted confinement of flood flows to the channel, the gradual lengthening of river bends, and their subsequent elimination from time to time by cutoffs, natural or planned, together with numerous other phenomena of regimen, all have combined to produce a changing rather than a fixed relationship. At some stations (Arkansas City and Vicksburg, for example) the changes have been very pronounced, as is evidenced by the rating curves for various years submitted by Mr. Jarvis. Records of actual measurements show that at Vicksburg the stage required to pass a rising stage discharge of 1,000,000 cu ft per sec was about 40 ft in 1890, 38 ft in 1896, 40 ft in 1898, 37 ft in 1906, 40 ft in 1916, 42 ft in 1928, and 31 ft in 1940. Bankfull stage is 42 ft. The increment, in discharge per foot of stage, is about 50,000 cu ft per sec on this part of the curve. Deduction of past flows from present rating curves could introduce errors of a couple of hundred thousand cubic feet per second or more in many cases. During periods of Vicksburg observations the results at that point could be translated to Natchez, probably without very serious discrepancy, but this device is not available for the period anterior to Humphreys and Abbot. This is not intended as a condemnation of the methods employed by the author, who has used the data very ingeniously, indeed. Parenthetically it might be observed that at Vicksburg the 1937 rating curve presented by Mr. Jarvis lies about 100,000 cu ft per sec to the left of that developed by the Mississippi River Commission for the same year below the 35-ft stage. There is a closer agreement above that stage.

On the Mississippi River, the derivation of annual discharge by the use of monthly mean stages should yield fairly accurate figures, provided the rating curve reflects the true relationship between discharge and stage and provided no flow by-passes the station by reason of diversions through levee breaks. In extreme floods of earlier years there were numerous crevasses in the levees, and outflow through those in the west bank levees below the Arkansas River would not reenter the Mississippi above Natchez. The method of monthly mean stages was applied to the Vicksburg record for the years 1930 through 1936, using the appropriate rating curve in each case. Compared with the results obtained by adding together the daily discharges as actually measured from day to day or as scaled from loop curves connecting measurements two to several days apart, the mean annual discharge as developed from monthly mean stage was:

Year	Percentage higher or lower than the sum of daily discharges
1930	0.04% higher
1931	2.85% higher
1932	1.03% lower
1933	1.02% lower
1934	0.96% lower
1935	1.61% lower
1936	0.74% lower

In Table 3 Mr. Jarvis presents maximum stage and discharge at Carrollton (New Orleans) for each of the years 1811–1860. The Carrollton gage readings prior to 1847 were apparently deduced from readings on the Natchez gage. The relationship between gages on the lower Mississippi is not constant and the Mississippi River Commission has not as a rule included in its published pamphlets any stages other than actual gage readings or water marks connected to the gage by levels. Incidentally, crest stages at Carrollton as established by high-water marks and published by the Mississippi River Commission for the years 1832, 1840, and 1844 do not agree with those reported in Table 3. The discharges reported in Table 3 were taken from a Carrollton rating curve developed from discharge measurements at Red River Landing (just below Old River) in 1851 and at New Orleans in 1858. Fig. 7 shows this rating curve about which the writer had plotted all available discharge observations at Carrollton. The zone of extreme variation of these plottings is indicated by the cross-hatched area.

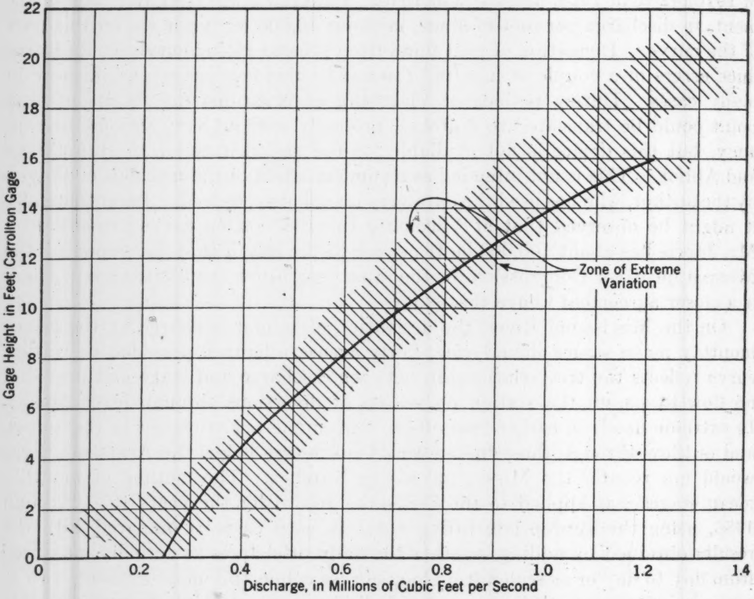

FIG. 7.—DISCHARGE OBSERVATIONS, MISSISSIPPI RIVER, AT CARROLLTON, LA. (1851–1941)

Some clarification of the data presented in Table 7 would be desirable. The table presents maximum gage heights and discharges at Memphis and Vicksburg for the period 1858 to 1937. The Memphis stage for June 23, 1858, should read (with reference to zero of present gage) 34.16 ft instead of 35.3 ft. The stages tabulated for Vicksburg from 1858 to 1931 relate to the U. S. Engineer gage, whereas those subsequent to 1931 apparently relate to the U. S. Weather Bureau bridge gage and are from 0.6 to 2.3 ft lower than the corresponding readings on the Engineer gage. Many of the discharges tabulated are unofficial or estimated quantities. Presumably they are estimated confined flows and so, in the cases of floods not completely confined, do not correspond to the observed gage heights tabulated opposite.

It is noted that for 1909, 1916, 1920, and 1928, estimated discharges are given for Vicksburg, whereas these floods were fully confined at that point and measured confined discharges are available. The discharge given for 1922 was measured after a crevasse had developed 93 miles below Vicksburg and two days after the crest had passed, and was probably influenced by the crevasse. The Vicksburg discharge given for 1927 (namely, 2,495,000 cu ft per sec) appears to be made up of the estimated confined inflow at Arkansas City plus the flow of the Yazoo River. The estimated confined flow at Vicksburg[27] is 2,278,000 cu ft per sec. The actual measured peak flow was 1,806,000 cu ft per sec on May 1, 1927, at a stage of 57.2 ft. The maximum stage was 58.4 ft on May 4, 1927.

The value (2,020,000 cu ft per sec) is the actual measured confined discharge past Memphis in 1937. It is not believed necessary to add the extra 500,000 cu ft per sec.

The data presented in Table 7 should be used with caution. The peak flows of the older great floods were seriously affected by crevasses both above and below the gaging points. The only common basis for comparison is that of confined flow, and it is extremely difficult in many cases to arrive at a satisfactory estimate of it.

In conclusion it might be well to comment on the reference to the blue clay in which the early and many later observers fancied the river bed to lie. Borings disclose that the bed is in sands and gravels. The clays generally exist in old abandoned channels of the river and in old marshes and appear to be the results of deposits of fines from the river itself.

John C. Hoyt,[28] **M. Am. Soc. C. E.**—The pioneer work in the collection of river flow by Humphreys and Abbot[2] and others on the Mississippi River indicated a fair recognition of the fundamental principles of the subject. The principles then recognized have stood the tests of later studies of the subject, and they have been the basis for the further development of the science of river hydrology.

Mr. Jarvis' paper brings together the results of much of the early work on the Mississippi River in a form that makes them readily available for further comparison and study. The comparison of records of river discharge taken at different times depends on three main factors: (1) The methods used in their collection; (2) the equipment and facilities available for collecting the records; and (3) the permanency of conditions along the streams which regulate the flow.

Although the methods of measurement are fundamentally the same, the equipment and facilities for making the measurements have been greatly modified and improved so that the accuracy in recent work has been greatly increased; for example, improvement in sounding equipment has added materially to the accuracy of records. Whereas some streams have fairly permanent channels which govern their regimen, many of the larger ones change and shift their channels to such an extent that measurements of discharge or stage made at one time are not comparable with others made at other times.

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[27] "Basic Data, Mississippi River," *Annex No. 5*, H.R. Doc. No. 798, 71st Cong., 3d Session, 1931, p. 105.

[28] Cons. Hydr. Engr., U. S. Geological Survey, Washington, D. C.

In comparing records of different dates collected along the Mississippi River, the effect of the changes in conditions which affect the flow must be kept in mind, and there will always be an element of uncertainty which requires caution in their use. Although these comparisons may give a general idea of the conditions, they will always be open to question and should be used with great caution.

GEORGE O. GUESMER,[29] ASSOC. M. AM. SOC. C. E.—The source and accuracy of original data which are the basis for a study of any subject determine the value of the conclusions reached. Mr. Jarvis' diligence in "digging out" information buried for years in the records of various offices in the Mississippi River Valley has given him an excellent background.

The writer had an opportunity to observe Mr. Jarvis gathering information used in the preparation of this paper. The comprehensiveness of his research was evident not only from the continuous testing and correlation of each new "lead" but also from the new derivations made in the light of basic data assembled in the course of related studies and research projects. Every effort was made to be certain that each item was thoroughly examined and worthy of consideration. Thus, the paper is based on information carefully weighed by the author during selection at the original source.

H. T. CORY,[30] M. AM. SOC. C. E.—It is most unfortunate that a wholly unjustifiable adoration of the Golden Calf Precision has blinded so surprisingly many to the solid substantiality of what early investigators and constructors did and knew. That Mr. Jarvis should have realized the Humphrey and Abbott early Mississippi records merited thoroughgoing examination by modern critical methods and should have discovered that the endeavor was very much worth while—is reassuring. It is to be hoped that his example will be followed, and followed frequently.

Sometimes the rejection of old-time "crudely observed" data approaches the tragic. An example coming within the writer's personal experience is a case in point. In 1912 he had occasion to meet an old gentleman then 94 years old, who had a remarkably keen memory, supplemented by scattered notes preserved in old notebooks, which were not diaries in any sense.

Among the many impressive and stirring events discussed, the most striking was his being taken, when a youth, by an uncle on a trip to Cincinnati, Ohio. At this time there must have been a notable drought for he vividly recalled having seen the steamboats resting crosswise on the Ohio River bed with the only water in the river running under and not touching the boat bottom.

The writer naturally grilled the old gentleman thoroughly, and was convinced he had described exactly what he had seen at Cincinnati on that trip. Absurd as it may seem to those who have not, more than once, been dumbfounded by the "incredibility" of high and of low extremes of stream discharge and yet lived to see such extremes reached if not exceeded, the writer is convinced that about 1835 the discharge of the Ohio River at Cincinnati fell to, relatively speaking, zero. Several times the incident has been brought to the attention of the federal water resources officials during the thirty years since 1913 but apparently to no purpose. Such "stream gaging" is certainly very crude, and yet —.

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[29] Engr., U. S. Engr. Office, St. Paul, Minn.

[30] Superv. Engr., Reconstruction Finance Corp., Los Angeles, Calif.

After all, there is no doubt that the best way to bring proper attention to such cases of "rejected data" is to follow Mr. Jarvis' example and to present carefully prepared papers to the Society for its information and discussion. The present paper may well be taken as a model.

C. S. JARVIS,[31] M. AM. SOC. C. E.—Mr. Davenport has contributed an interesting, thought-provoking discussion, which discloses specifically why the most earnest attempts at evaluation and interpretation of early discharge records are under such a handicap, in the judgment of those best equipped to carry the project forward or to encourage and guide other investigators. Against all the implications that such a task cannot be accomplished with even a fair degree of reliability, only one answer is ventured:

> Every independent, unbiased, logical approach toward the evaluation and coordination of those early records pays a tangible dividend, either by establishing, confirming, or revising usable estimates and approximations. We are surely nearer to the truth now than we were before any sustained attempts were made at evaluation. Further investigations and revisions are invited. Thus far, many confirmations have resulted from various approaches.

Acting in conformity with basic legislative and administrative authority, Humphreys and Abbot collected available pertinent information concerning flood heights, crevasses, levee construction, and dimensions on the Mississippi River,[32] "and well authenticated changes in the banks of the river, etc. * * *. In each set of cross-sections, the velocity of the current was measured—in some instances, with great elaboration." In view of such items of record, exception must be taken to the concept expressed by Mr. Davenport, implying that practically the sole authority for discharge and cross-sectional data, as well as for the rating curve for the early years of record, was the 1858 series of discharge and gage readings at Vicksburg, Miss., and Natchez, Miss. As opposed to such a theory, note the comparison of methods used for float measurements in 1851 and 1858 and, also, the unsatisfactory tests of Saxton's current meter.[33] Some of the following references prove that fragmentary data were collected prior to 1850 (some as early as 1838), and that the results were available for incorporation in the final report.

Among the many references that establish the broad scope of investigations included in the Humphreys and Abbot report, the following have been selected:[34]

> "The corresponding velocity, taken from a table constructed for the purpose, was considered to be the mean velocity of the division—absolute for the observations of 1851, and relative for those of 1858. For the shore divisions, unless the floats happened to be well distributed through them,

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[31] With Office, Chf. of Engrs., Washington, D. C.

[32] "Report on the Physics and Hydraulics of the Mississippi River," by A. A. Humphreys and Henry L. Abbot, *Professional Paper No. 13*, Corps of Engrs., U. S. Army, 1876, p. 2.

[33] "Report on the Physics and Hydraulics of the Mississippi River," by A. A. Humphreys and Henry L. Abbot, *Professional Paper No. 13*, Corps of Engrs., U. S. Army, 1876, p. 234.

[34] *Ibid.*, p. 235.

the mean velocity was assumed to be eight-tenths of that at the outer edge, a rule deduced from a subdivision and study of the velocity, when thoroughly measured in these divisions [200 ft apart]."

Also in this report, included in George G. Meade's letter,[35] is a brief description of his extensive observations relating to current velocities, directions, and vertical distributions in and around the delta channels during the year 1838; and there is a reference[36] to the 90-ft cord used for connecting the surface float with the submerged float. The location[37] of the maximum velocity in a depth of 100 ft is given as 30 ft from the surface—a fair approach to what is acceptable today.

Furthermore, there are numerous references to records and measurements conducted by Lt. Robert A. Marr, U. S. Navy, Prof. C. G. Forshey, M. Am. Soc. C. E., and Charles Ellet, Jr., during years earlier than 1850 (for example, Lieutenant Marr's record of daily rainfall, gage heights, temperature, and river discharge at Memphis, Tenn., together with weekly evaporation, April 1 to July 15, 1848,[38] river data, March 1, 1850, to March 1, 1851, and related activities[39]).

After describing the discharge and other measurements on the Lower Mississippi River from February, 1851, to February, 1852, and from December, 1857, to December, 1858,[40] Humphreys and Abbot declared, "Besides these continuous series, many measurements of width, depth, area of cross-section, discharge, etc., were made both upon the Mississippi and upon its tributaries and bayous from the Ohio to the Gulf." It seems reasonable to assume that all such data underlay the discharge determinations for the 33 years listed in Table 1. These values are of acceptable quality, as proved repeatedly by various approaches described hereafter.

Even though the scope of the statement concerning other measurements outside the continuous series may apply mainly to the one decade covering the Humphreys and Abbot assignment, the aforementioned and associated references indicate that it is applicable in some degree for the years preceding 1850. Thus, the persistent attempts to evaluate the tributary, main channel, and crevasse discharges, and to correlate these quantities at successive stations from Columbus, Ky., to Carrollton (New Orleans), La.,[41] unmistakably indicate the avidity with which the authors incorporated basic data and extended their use within practicable limits. Unquestionably, some of those segments of gage-height hydrographs for the Natchez, Vicksburg, and New Carthage, Miss., stations (the latter lying between the first two),[42] for 33 years, 1818 to 1860, were associated with series of discharge determinations, by means of floats; either natural—driftwood and uprooted trees—or artificial—weighted rods or multiple floats—as noted in various pages of the text. Otherwise, the separate determinations of discharge by Lieutenant Marr, Professor Forshey, and Mr. Ellet cannot be accounted for nor accredited reasonably.

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[35] "Report on the Physics and Hydraulics of the Mississippi River," by A. A. Humphreys and Henry L. Abbot, *Professional Paper No. 13*, Corps of Engrs., U. S. Army, 1876, p. 507.

[36] *Ibid.*, p. 664.

[37] *Ibid.*, p. 661.

[38] *Ibid.*, p. 201.

[39] *Ibid.*, pp. 129, 202, and 378–394.

[40] *Ibid.*, p. 230.

[41] *Ibid.*, pp. 128–132, 182, and 372.

[42] *Ibid.*, Plates V, VI, and VII.

The rating curve for the earliest biennium of record at Carrollton, 1851 and 1852, depended on more than 100 separate gagings, of which 21 were definitely dated on the chart. There can be no reasonable doubt that the earlier fragmentary or detached data were incorporated in some manner to make the rating curves more nearly representative of conditions during earlier years. Otherwise, it would be difficult to account for the close agreement in the last two columns of Table 1, showing 33-yr means of 614,000 cu ft per sec and 617,000 cu ft per sec, a difference less than 0.5%, even though the means for the last 10 years of the same tabulation differed by nearly 2.0%. It would be gratifying, indeed, if one could be assured that stream-discharge determinations under the most up-to-date, standard practice could stay within such close limits, or that the separate records of earlier periods were within limits five times as great.

Fig. 5 is of unusual interest because it follows the general form of patterns set in earlier years by the Mississippi River gagings at Natchez, Vicksburg, and Warrenton, Miss., beginning in 1858; at Columbus, in 1857; and at Carrollton, as early as 1851. For the last-named station, well-defined segments of rating curves are available for twenty-one separate years or groups of years between 1851 and 1929. They all lie within the limits of those curves selected for inclusion in Fig. 3 for Carrollton and are numbered chronologically from the earliest to curve 6, the rating curve for 1929. With few exceptions, the successive patterns at all gaging stations on the Lower Mississippi River show deviations of more nearly ± 5% as against ± 10% in the positions of rating curves near El. 34 (Fig. 5). The most outstanding exceptions are afforded by the records of river gagings at Columbus, for the years 1857–1858, and 1881–1882, showing somewhat greater variations than ± 10%. This pronounced variability persists in diminishing degree at the same station to include the year 1937, according to the writer's interpretation of official data, published mainly by the Mississippi River Commission. This analysis seems to provide a tentative answer to the query, for which Mr. Davenport admitted he had none, as to how nearly typical was the variation in plotted gagings for the year 1938 at Vicksburg. Fig. 5 typifies the shifting of rating curves for the more erratic periods, but not for the maximum variability.

Recent official records for the Lower Mississippi River stations disclose consistent relationships generally, but occasional departures of 5% or more for gagings by different agencies (or even by the same agency) with the river surface elevation practically constant.

1. Would it seem inconsistent to ascribe some portion, possibly approaching one half of the aforementioned 20% variation for the gagings in Fig. 5, to both the personal equation of the operators and the inherent defects or erratic movements of current-meter vanes due to suspended silt and fine granular material or other debris encountered in all such rivers, but never in the standard rating flumes of laboratories?

2. With all the refinements now applied to the measurements of depths at intervals along the cross section, what uncertainties still persist because the

heavy sounding weight settles into the mud or silt of the stream bed or because it encounters either a ridge or a valley?

3. Is the square foot, or portion thereof, occupied by the current meter for the purpose of determining velocity of flow actually representative of the thousands or tens of thousands of such elementary areas within the cross section for which it serves as a sample?

In view of such unsolved problems as well as the shifting of currents and bars within a cross section and in view of the pulsations of waves or unpredictable movements of eddies in the course of turbulent flow (with all other uncertainties inherent in stream gaging on deep alluvium even where the best practical standards prevail), it seems noteworthy that such consistent relationships have been developed and maintained between river stage and discharge.

No less amazing are the results achieved normally in connection with the early gagings of the Mississippi River system without the conveniences afforded by fixed bridges (being dependent upon floating equipment and shore installations or observations, as well as extra labor and ingenuity) to accomplish the difficult assignments under pioneer conditions. Surely, some compensations must have accrued from the direct measurements of stream velocities by weighted rods, multiple floats, or uprooted trees. This procedure integrated and averaged the movement within much larger samples of cross-sectional area, and furthermore avoided the inherent approximations of rating tables for current meters and the recognized uncertainties involved in their practical use. There is some reassurance in the observation that velocities are determined by the use of floats to this day, when circumstances warrant, either for flood crest or for routine gagings; and results are duly published along with those obtained by the use of current meters, with symbols to indicate whether one or two meters, double floats, rods, or surface floats were employed. Many current-meter gagings have had the advantage of float measurements to test the action of the meter vanes, even though these experiments were not necessarily included in the published reports.

Mr. Follansbee has contributed some very interesting historical notes relating to the development of the art and science of stream gaging.

Mr. Hoyt has written his impressions, based on a much broader experience than comes to most. With deep gratitude the writer acknowledges the encouragement accorded by Mr. Hoyt, particularly in the projects for evaluation and comparison of flood data pertaining to foreign rivers, such as the Nile, as well as to American rivers, especially those with the longer records.

Mr. Senour has contributed a number of important items that would be difficult to obtain outside the central office of the Mississippi River Commission. Fig. 6 provides an authoritative answer to many queries that have been directed toward the writer, as to the stability of stage-discharge relationships at the lower stations. Because the 4,900 sq miles of intervening drainage area between Vicksburg and Natchez represent only 0.43% of the total, although the proportionate runoff was more nearly 1%, the data at each of these stations have served at times for both, with the single percentage differential taken into account wherever circumstances warranted. Inclusion of the complete data for 1937, which were not available when Fig. 2 was prepared, provides an added rating curve, as described by Mr. Senour, to the right of curve 6 for Vicksburg. For 1941 the rating curve is still farther to the right, according to the U. S. Geological Survey, presumably measuring the increase of channel capacity resulting from cutoffs. The Natchez gage readings were about 5 ft higher for equal discharges, traceable to a lower datum with respect to hydraulic slopes for successive stages.

The three crest stages for Carrollton for the years 1832, 1840, and 1844, to which Mr. Senour takes exception, were merely transcribed along with associated data in Table 3 from *Professional Paper No. 13.*[43] If the several curves of Fig. 3 pertaining to Carrollton gagings are plotted or superimposed on Fig. 7, these rating curves will be found to cover approximately one half of the shaded area along the central strip. This shows that the variations among single observations are nearly double the range occupied by rating curves. When it is realized how nearly the extreme variations recorded in Fig. 5 equal those in Figs. 6 and 7 (the latter two covering 90 years instead of the single year 1938), the factors of the personal equation of operators and the practical limitations on the accuracy of current-meter gagings seem to demand consideration.

In response to Mr. Senour's request for clarification of data presented in Table 7, it should be noted that the writer aimed to record the progressively increasing concentrations of flood water associated with peak discharge and aggravated soil wastage, including both the confined channel flow and the well-defined storage or lateral floodway capacities then in use. Examples of this concept were reported in 1929 by C. W. Kutz, M. Am. Soc. C. E.,[44] as 2,662,000 cu ft per sec at the latitude of Arkansas City, Ark., in April, 1927, or 2,350,000 cu ft per sec near Old River, La. At the same time, General Kutz (then Colonel) dealt with possible or probable maximum floods at Cairo, Ill., ranging from 2,250,000 cu ft per sec to 2,400,000 cu ft per sec. According to George R. Clemens,[45] M. Am. Soc. C. E.:

> "Thus the Mississippi River met the first thrust of the 1937 flood. Cairo was the only dry city in the Ohio basin. The New Madrid floodway functioned essentially as planned, and protected not only Cairo but the entire St. Francis basin from a flood that most certainly would have overtopped the levees as they stood in 1927. At the crest, about 2,000,000 cu ft per sec passed Cairo, and about 500,000 cu ft per sec was carried in the floodway."

The writer's concept was that the same 500,000 cu ft per sec could be regarded as in temporary storage or detention between Cairo and Memphis, Tenn. The plan sponsored by the Chief of Engineers to[46] "provide for a flood which without reservoir control would reach 2,600,000 cubic feet per second in the Mississippi between Cairo and the Arkansas" was based on both a knowledge and a thorough understanding of the data including the 1937 flood rise, the well-organized defense of the Cairo levees, the relief afforded by the access to the New Madrid floodway, and the potentialities of future floods.

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[43] "Report on the Physics and Hydraulics of the Mississippi River," by A. A. Humphreys and Henry L. Abbot, *Professional Paper No. 13*, Corps of Engrs., U. S. Army, 1876, p. 443.

[44] "The Work of the Mississippi River Commission," by C. W. Kutz, *Transactions*, Am. Soc. C. E., Vol. 93 (1929), p. 707.

[45] "The Mississippi Meets the 1937 Flood," by George R. Clemens, *Civil Engineering*, June, 1937, p. 381.

[46] "Comprehensive Flood-Control Plan for Ohio and Lower Mississippi Rivers," *Committee Document No. 1*, House Committee on Flood Control, 75th Cong., 1st Session, April 6, 1937, p. 6, paragraph 20.

Another reference which seems to support the concept of total discharge from both confined channel and floodway or lateral storage is to be found in an article by Maj. Raymond G. Moses:[47] "At the crest, of course, the Arkansas-White backwater area was still taking water but the flows from those two streams were sufficient to increase the flow in the main river to about 2,150,000 cfs at the crest at Arkansas City * * *."

The tabulated discharges for the years 1909, 1916, 1920, and 1928 are weighted means of either official or semi-authoritative data. Official quantities including the latest revisions should be preferable.

Mr. Guesmer not only observed some of the initial investigations of Mississippi River discharge data, but also participated to the extent of locating some of the most valuable folios and summaries of unpublished hydrologic data, of which the writer had no previous knowledge. It seemed logical to deal first with the headwaters and then with the successive tributaries and main river stations. The procedure thus developed was found to be very satisfactory and capable of deriving successive discharge quantities generally within 5%, and often within 2%, as illustrated in Table 9 for the Tennessee River at Savannah, Tenn.

TABLE 9.—DISCHARGES, IN CUBIC FEET PER SECOND, FOR THE DESIGNATED YEARS OR OTHER PERIODS

Description	Latest 10 yr	5 yr 1934–1938	1937–1938	1936–1937	1935–1936	1934–1935	1933–1934
271							
Derived	55,163	52,701	53,458	57,312	59,320	53,216	40,197
Departures (numerical)	254	263	−1,500 748[a]	−828	290	−824	417
Departures (%)	0.5	0.5	1.5	−1.5	0.5	−1.6	1.0
272							
Observed	54,909	52,438	51,210	58,140	59,030	54,040	39,780

[a] Initial filling for storage in reservoir was 1,091,000 acre-ft, representing an inflow of 1,500 cu ft per sec for one year.

The initial reckoning disclosed an unusual departure of nearly 4.5% for the water year 1937–1938 until it was discovered that the initial filling of the reservoir accounted for the withdrawal of 1,500 cu ft per sec from the stream flow. In the same manner, it was quite a simple problem to derive the yearly and periodic mean discharges for Cincinnati, Ohio. For this purpose, the observed discharges of the Ohio River at Huntington, W. Va., were correlated with records of the gaged tributaries and estimated yields for the ungaged areas. The latter were either equal to or somewhat less than the observed discharge per square mile of the gaged tributaries. For any of the intermediate stations on the Mississippi River and its major tributaries, or even the minor branches, this procedure seemed to render acceptable values.

Various tests showed that the ungaged portions of valley floors ordinarily yield somewhat less than do equal areas in steeper and more elevated regions.

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[47] "High Water in the Lower Mississippi," by Raymond G. Moses, *The Military Engineer*, May-June, 1937, p. 167.

Furthermore, it is generally known that the larger tributaries head on the higher ground; therefore it is customary to estimate the yield per square mile of ungaged areas, including minor tributaries, as 80% or 90% of the unit yield for the adjacent gaged tributaries. Observed and derived discharges for 56 stations selected from among the 400 or more available are listed in Table 8.

The progressive summations, comparisons, samplings, and adjustments which led to these results for all important stations within the Mississippi River basin involved the following principal phases:

1. Assembly of outstanding and representative precipitation records;
2. Interpolations from neighboring stations to fill all gaps;
3. Comparisons of monthly or yearly means for the group of stations with the total record means to obtain excess and deficient depths for successive months or years;
4. Computations of temperature departures from normal in the same manner, for one or more representative stations;
5. Computations of runoff departures from record mean at the selected lower river station (record-mean annual runoff depth minus successive yearly depths);
6. Comparison of rainfall and runoff departures in the light of temperature departures from normal;
7. Investigation of particular storm patterns in all cases where rainfall and runoff departures are greatly at variance or of opposite signs;
8. Derivation of discharge at the lower river station from summations of tributary discharges and estimates of ungaged areas based thereon;
9. Interpretation of hydrographs in the light of all previously assembled data and their established relationships; and
10. Checking the arrays of data by the use of weighting factors, used in multiplying the different tributary discharges in order to build up the approximation to the lower station discharge, year by year, month by month, or day by day, if time factors, valley storage, and flood routing are all taken into account.

Weighting Factors.—The concept and use of weighting factors may be explained best by a practical example. In the quest for suitable samples of tributary flow that may be used as an index of lower station discharge, the essential criteria for tributary records include not only their length, continuity, and quality, but also their degree of correlation with discharge at the lower station. The Gasconade River at Rich Fountain, Mo., is outstanding for such correlation; the Kanawha River at Kanawha Falls, W. Va., has a record of outstanding length and reliability. Their respective drainage areas of 3,180 sq miles and 8,367 sq miles, totaling 11,547 sq miles, represent nearly 1% of the 1,144,500 sq miles drained above Vicksburg, and the average yield is 2.667% of the total. Evidently, an approximation to the annual discharge at Vicksburg is afforded by taking $\frac{100}{2.667} = 37.5$ times the recorded discharges at Rich Fountain and Kanawha Falls, according to the normal procedure for the "sampling method."[48] The weighting-factor development came in response to the query, "Would it be more satisfactory to accord different weights to the tributary drainages, calling for a greater factor for the Gasconade River and a lesser one for the Kanawha River, or vice versa?" Adopting the water years 1939 and 1940 for the observation or test period, the simultaneous equations become: For 1938–1939,

$$2{,}787\,x + 10{,}090\,y = 572{,}600 \ldots\ldots (1)$$

and for 1939–1940,

$$1{,}611\,x + 10{,}940\,y = 332{,}500 \ldots\ldots (2)$$

$$\frac{2{,}787}{1{,}611} \times \text{Eq. } 2 = 2{,}787\,x + 18{,}926\,y = 575{,}219 \ldots\ldots (3)$$

$$\text{Eq. } 3 - \text{Eq. } 1 = 8{,}836\,y = 2{,}619 \ldots\ldots (4)$$

$$y = 0.296401 \ldots\ldots (5)$$

Substituting the value of y (Eq. 5) in both Eq. 1 and Eq. 3—$2{,}787\,x = 569{,}609$; and $x = 204.3807$.

This disparity in weighting factors results in almost complete dependence on the smaller tributary, outstanding indicator of Vicksburg discharge.

Use of the foregoing weighting factors for the Gasconade River and Kanawha River annual discharges for the 20-yr period, 1921 to 1940, resulted in a maximum departure of + 46.3%, with a mean absolute of ± 21.0%, as recorded for Item 17, Table 10. According to data under Item 91 (Table 10*B*), the substitution of discharge data from the Greenbrier branch of the Kanawha River, draining only 1,357 sq miles, with the Gasconade area, representing only 0.4% of that at Vicksburg, resulted in new weighting factors of 204.9578 and 25.756673, respectively. Under a 20-yr test, 1922 to 1941, these factors proved to be more satisfactory than the first pair, as the maximum and mean-absolute departures were reduced to − 42.4% and 15.1%, respectively, for the Mississippi and Atchafalaya rivers. However, Item 22 indicates what may be attained by such tests with 87.2% gaged, resulting in reduced percentages of + 10.2% and 3.6% over a 22-yr checking period. Such percentages approach so nearly the attainable accuracies in river gaging on alluvial formations that they seem to attest the consistency and reliability of the basic data underlying those derivations or estimates; furthermore, they may constitute a proof or indorsement of the derivation and testing method employed. Similarly, in Item 7 (departures of − 23.9% and 9.6% for 65 years), in Item 32 (departures of + 20.4% and 7.9% for 52 years), and in Item 67 (departures of + 18.2% and 5.7% for 52 years), the fairly close agreement between derived and observed discharge data extends beyond the range of mere coincidence.

The fact that some other series of weighting factors, similarly derived, bring more discordant results would seem to indicate that either the test years or the individual tributary discharges were less representative. For example, the aforementioned Kanawha River discharges showed a larger quantity for the lean year (1939–1940) than for the average year (1938–1939) at Vicksburg, and, therefore, the resulting weighting factor is small, and shifts most of the burden to the smaller Gasconade River.

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[48] "Estimating Annual Yield from Ungaged Drainage-Basins and Effect of Land-Use Treatment on Surface-Runoff," by C. S. Jarvis and H. C. Murto, *Transactions*, Am. Geophysical Union, 1941, pp. 875–893.

Least Squares Solutions.—Items 102 to 107 in Table 10*B* and Items 45 and 46 in Table 10 were solved by the method of least squares. The obvious advantage of this approach is that the observation period may be extended beyond the number of years that corresponds to the number of unknowns. Actually, the practical application of the simplest algebraic approach was found to accomplish the same results. For example, a 20-yr period may be divided into four groups of 5 years each and used effectively for the determination of four unknown weighting factors, as shown in Table 10, Item 54.

After a searching investigation and numerous tests, the most significant of which are summarized in Tables 10, it was found that the method of least squares (with all the extra labor that the procedure involves, particularly in the formation of normal equations) yields either identical results that accrue from the simplest algebraic approach with simultaneous equations, or a generally comparable series of factors. Occasionally, the results from the least squares procedure show up to advantage, but more often the advantage was with the simpler approach. Wherever the constant value is included among the unknowns, the summations of normal equations direct the solution toward the series of factors that will satisfy the arithmetic means of the observation equation coefficients and, therefore, will not likely satisfy any one of them separately.

The main objective of Tables 10 is to illustrate some of the methods used both in extending the records and also in checking the values thus derived as to their consistency, one with another, in the light of such fragmentary or local records and such other information or authoritative interpretations as were available.

These and associated tests have disclosed errors in some of the published basic data, and have indicated the required correction. For example, the mean annual discharge of the Missouri River at Hermann, Mo., for the calendar year 1925, once published as the equivalent of 76,130 cu ft per sec, was found to be more nearly 50,000 cu ft per sec by various tests similar or related to those in Tables 10, and also by reference to the established rainfall-runoff relationships as applied to that year at the stations involved. Likewise, the numerous maximum departure percentages recorded in Col. 25 indicate that the lower station discharges for the year 1931 are probably deficient and should be increased by at least 10%, revising data of that year prior to July, when the present station was established.

The method of least squares brings identical results with those from the simplest algebraic method whenever the unknowns, the tributary stations, and years or other time units in the test period are all equal in number. Values derived in connection with a constant term, which automatically extends the test period by one year, represent mean discharges for the entire test period, and therefore should not normally satisfy any of the individual observation equations.

The aforementioned observations are not intended to disparage the use of the least squares procedure in its proper field, for it has done much to facilitate

TABLE 10.—OUTSTANDING COMBINATIONS OF TRIBUTARIES AND TEST PERIODS DETERMINED EITHER BY THE SIMPLEST SOLUTION OF SIMULTANEOUS
"Weighting Factors" Arranged in Columns Headed by the Names of Rivers and
and Checking Periods Listed in Col. 20

Item No.	Test period	Gaged area (%)	GAGING					
			Huntington, W. Va.	Ft. Benton, Mont.	Le Claire, Iowa	De Valls Bluff, Ark.	Rich Fountain, Mo.	Garland, Ark.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
River			Ohio	Missouri	Mississippi	White	Gasconade	Red
Drainage area, sq miles			55,200[h]	24,600[h]	88,600[h]	23,800[h]	3,180[h]	51,500[h]
								COMBINED WITH
Gaging station			Metropolis,	Hermann,	Keokuk,	Clarendon,	Little Rock,	Alexandria,
State			Ill.	Mo.	Iowa	Ark.	Ark.	La.
River			Ohio	Missouri	Mississippi	White	Arkansas	Red
Drainage area, sq miles			203,000	528,200	119,000	25,750	157,900	65,900
5[b]	1894–1895[g]	11.1	....	....	8.1266	....	....	....
7[b]	1934–1938[g]	15.4	....	−23.7704[h]	3.1095	....	92.9170[h]	....
17[b]	1939–1940[g]	1.0	....	....	....	....	204.3807[h]	....
18[i]	1938–1939[g]	2.2	....	....	....	10.5687	118.7696[h]	....
18[i]	1938–1939[g]	0.4	....	....	....	....	193.8274[h]	....
19	1934–1938	83.2	1.05125	2.76405	0.72409	3.80011	0.68856	....
21	1933–1938[g]	87.2	1.77613	2.04285	−2.34821	−5.23820[h]	0.44926	13.2874[h]
22	1935–1940[g]	87.2[j]	1.01424	3.46250	−0.31917	4.86462[h]	−0.39241	3.31514[h]
22[i]	1934–1939[g]	87.3[j]	0.843654	4.297565	0.223160	4.475945	−1.78116	3.47500[h]
22[i]	1934–1939[g]	88.5[j]	0.878997	4.210438	0.182211	4.03473	−0.87311	1.35515
23	1934–1938[g]	85.3[j]	1.527006	−0.540756	0.534084	....	5.18984	0.37641[h]
23[i]	1934–1938[g]	73.7[j]	1.287651	4.367897	−1.283654	....	....	3.64381
26	1927–1929	74.3	1.582193	0.606386	2.455054	....	....	....
28	1933–1934	28.1	1.637119	....	2.511146	....	....	....
29	1933–1934	63.9	1.545347	2.548392	....	....	....	....
32	1937–1938	28.1	1.422663	....	2.865939	....	....	....
								ITEMS 35 TO 44,
45[k]	1933–1939	87.3	1.214501	3.054400	−0.337125	3.110661	−2.672776	8.667083
46[k]	1933–1938	90.3	1.163258	2.081342	0.105700	3.111428	0.305337	....
47	1933–1939	87.3	1.107809	2.473716	0.664610	3.224007	0.624616	1.395254
48	1933–1938	90.3	1.792211	−5.738581	1.417405	−1.240462	7.911413	....
49	1933–1938	87.3	1.216067	1.495442	0.739574	2.455200	1.342141	2.249353
52	1934–1938	83.2	1.051546	2.761618	0.724266	3.799658	0.690306	....
Using 5-yr Means:								
53[i]	1914–1938	75.8	140.680963	65.24110	−371.38345	−472.67235	....	....
53[i]	1914–1938	69.8	174.631612	80.35262	−458.24473	−586.23519	....	....
54	1919–1938	69.8	2.523752	0.271677	−3.146566	3.054882	....	....
57	1919–1938	73.6	2.622843	1.016706	−6.312512	....	....	....
58	1924–1938	67.8	2.944415	0.910599	−4.210654	....	....	....
60	1924–1938	28.0	2.344919	....	−2.694294	4.353857	....	....
64	1929–1938	25.9	3.789960	....	−7.507301	....	....	....
Yearly Means:								
66	1934–1938	69.8	0.984279	3.492675	0.622489	4.178986	....	....
67	1934–1938	75.8	1.021313	2.637703	0.253073	3.106944	....	....
								ITEMS 68 TO 116,
115[l]	1932–1941[g]	90.2	1.233898	1.233898	1.233898	1.233898[h]	1.233898	....
116[l]	1932–1941[g]	83.0	1.375846	1.375846	1.375846	1.375846[h]	1.375846	....

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[a] Lower river stations. [b] For additional values in Cols. 8 to 17 inclusive, see Tables 10*A* and 10*B*. [c] Weight-
are drought years except 1939 (an average year). [f] With Atchafalaya, Simmesport, La. [g] Discharge data for
[i] Additional related tests. [j] Some revisions of data from *Transactions*, American Geophysical Union, 1941, p.
for the test period were divided by the sum of designated tributary discharges to obtain the ratios, or uniform

for Extending River Discharge Data by the Use of "Weighting Factors," Algebraic Equations or by Application of Least Squares Procedure

Gaging Stations to which They Are Related, for Test Periods Listed in Col. 2
(For Cols. 11 to 17, See Table 10A)

STATIONS:	CHECKING PERIOD		DEPARTURES OF DERIVED DISCHARGES FROM OBSERVED OR OTHER BASIC DATA (%)										Item No.
Vicksburg, Miss.	No. years	Range of years	Checking Period			Selected Years Showing Wide Range of Yield							
			Mean	Absolute[d]	Maximum	1927[e]	1931	1932	1937[e]	1939	1940	1941	
(18)[a,b,c]	(19)	(20)	(21)	(22)	(23)	(24)	(25)	(26)	(27)	(28)	(29)	(30)	(1)
Mississippi 1,144,500[h]	....	....	....	....	....	....	....	....	....	....	....	....	....
GAGING STATIONS AT:													
Angola[f] La. Mississippi 1,242,700	....	....	....	....	....	....	....	....	....	....	....	....	....
000[h]	20	1886–1905[g]	−15.8	±16.7	−39.6	−19.5	+ 8.1	− 6.8	−17.8	+ 1.9	+19.9	+58.8	5[b]
000[h]	65	1875–1939[g]	+2.2	9.6	−23.9	− 6.7	+14.6	−19.4	0.0	+ 0.6	−44.3	+16.5	7[b]
000[h]	20	1922–1941[g]	+3.5	21.0	+46.3	+40.5	+46.3	−44.9	− 4.7	0.0	0.0	+18.5	17[b]
....	20	1921–1940[g]	−0.7	13.5	−41.1	+16.8	+21.0	−33.0	−12.6	0.0	+ 0.5	− 0.8	18[i]
....	20	1921–1940[g]	+2.3	15.5	+50.7	+18.7	+29.3	−37.9	− 3.8	0.0	+ 6.3	+12.5	18[i]
....	50	1889–1938	+3.9	5.1	+19.0	+ 8.7	+ 6.5	− 4.6	+ 0.0	− 1.4	− 7.6	....	19
....	15	1927–1941[g]	−3.2	7.0	−22.3	−18.6	− 8.2	+ 7.7	0.0	−22.3	+ 4.4	+18.1	21
....	22	1920–1941[g]	+2.3	3.6	+10.2	+ 2.3	+ 8.5	− 1.2	0.0	0.0	0.0	+ 5.6	22
....	15	1927–1941[g]	+1.2	2.6	+12.3	− 1.7	+ 8.9	− 2.8	0.0	0.0	− 0.5	+ 3.8	22[i]
....	15	1927–1941[g]	+2.3	2.8	+11.8	+ 4.6	+ 9.0	− 1.5	0.0	0.0	− 1.7	+ 4.9	22[i]
....	15	1927–1941[g]	−3.9	4.7	−15.1	− 5.6	−15.1	− 1.7	0.0	−10.2	−10.1	+ 1.6	23
−25,037	15	1927–1941[g]	+1.5	2.7	+11.6	+ 8.4	− 1.2	+ 0.6	0.0	− 7.3	− 0.8	+11.6	23[i]
000[h]	22 52	1919–1940 1889–1940	+5.7 +6.7	7.9 9.2	+23.6 +33.7	0.0	− 3.5	+ 4.4	+10.9	+ 9.7	+18.7	....	26
000[h]	22	1919–1940	+1.2	7.1	+21.4	− 6.3	−12.9	+ 0.5	+ 8.7	+ 7.5	+16.9	....	28
000[h]	22	1919–1940	+3.1	7.3	+27.9	+ 9.2	− 4.6	+ 5.8	+ 6.8	+ 2.8	+ 8.2	....	29
000[h]	22 52	1919–1940 1889–1940	−6.0 −4.2	8.2 7.9	−19.2 +20.4	−13.4	−17.5	− 7.3	0.0	− 0.2	+ 8.8	....	32
SEE TABLE 10A													
−17,176	52	1889–1940	+1.8	8.3	−29.2	− 9.9	− 5.2	+ 4.3	+ 2.1	− 0.6	+ 7.7	....	45[h]
11,060[h]	52	1889–1940	+3.5	5.3	−29.4	+15.5	+ 1.2	− 0.8	+ 2.0	+ 3.2	+ 5.2	....	46[h]
−2,181	52	1889–1940	+3.0	3.5	+13.7	+ 4.5	+ 3.2	− 2.3	0.0	0.0	− 2.6	....	47
55,066[h]	52	1889–1940	−13.0	15.5	−54.6	−13.8	−29.0	−13.9	0.0	−14.9	+ 2.8	....	48
000	52 10	1889–1940 1923–1932	+0.9 +3.7	5.6 4.0	+20.4 +10.5	− 0.4	− 2.0	− 3.2	0.0	+ 5.6	− 1.9	....	49
000	10	1923–1932	+5.4	9.0	+15.6	+ 8.7	+ 6.8	− 2.8	0.0	+ 5.8	+11.4	....	52
−6,469,256[h]	10	1923–1932	−162.4	(Too many conditions imposed)									53[i]
−8,212.098	10	1923–1932	−179.8	(Too many conditions imposed)									53[i]
000	10	1923–1932	−0.7	6.0	−20.2	− 2.9	−20.2	+ 2.5	+18.8	+10.0	+ 5.6	....	54
117,921[h]	10	1923–1932	−0.5	12.0	−23.6	− 0.6	−12.1	+12.3	+21.9	+ 6.9	+21.0	....	57
000	10	1923–1932	−1.6	10.4	−24.9	− 6.4	−24.9	+ 8.4	+23.1	+ 8.0	+ 6.6	....	58
000	10	1923–1932	−0.3	5.9	−18.2	− 1.5	−18.2	− 0.5	+17.0	+10.8	+ 5.2	....	60
000	10	1923–1932	−3.7	14.4	−37.9	− 7.0	−37.9	+12.9	+34.3	+11.9	+ 9.8	....	64
−3,203	10	1923–1932	+10.8	11.0	+19.3	+10.8	+ 9.5	− 1.8	0.0	+ 3.2	− 3.8	....	66
18,359[h]	10 52	1923–1932 1889–1940	+8.2 +4.2	8.2 5.7	+16.1 +18.2	+16.1	+ 5.2	− 0.7	0.0	+ 2.5	+ 4.8	....	67
SEE TABLE 10B													
000[h]	15	1927–1941[g]	+1.6	2.5	− 5.2	+ 2.6	+ 3.0	+ 0.4	− 2.2	− 3.4	+ 1.0	+ 4.0	115[j]
000[h]	15	1927–1941[g]	−1.2	4.2	+10.0	− 7.2	− 4.0	+ 0.2	− 2.5	− 0.5	− 1.4	− 2.8	116[j]

ing factors constant. [d] Mean absolute (values approximate). [e] Flood year, in the columns, unmarked years water years ending September 30 of each year so listed. [h] Referring to the gaging stations in the main headings. 889. [i] Computed by least squares, and normal equations solved. [j] Mean discharges at the lower river station weighting factors above listed.

scientific analysis and research. However, the criteria for the application of that method are violated mostly in the hydrologic field, especially as to a large number of observations made with equal care and skill, all yielding small deviations from the mean. Items 45 and 46 in Table 10 represent the best of the least squares tests, with maximum departures of some 29% in a 52-yr checking period, in contrast with 13.7% for Item 47.

TABLE 10*A*.—SUPPLEMENT TO TABLE 10

Item No. (1)	Test period[e] (2)	Gaged area (%) (3)	GAGING STA McCon- nelsville, Ohio (10)[a]	 Alder- son, W. Va. (11)	 Chat- tanooga, Tenn. (12)	 Nash- ville, Tenn. (13)	 Lock- port, Ky. (14)	 Hamil- ton, Ohio (15)	 Chilli- cothe, Ohio (16)
River			Mus- kingum	Greenbrier	Tennessee	Cumber- land	Kentucky	Miami	Scioto
Drainage area, sq miles			7,411[h]	1,357[h]	21,400[h]	12,860[h]	6,310[h]	3,639[h]	3,847[h]
									COMBINED WITH GAGING
Gaging station			Mt. Carmel,	Kanawha Falls,	Johnson- ville,	Clarks- ville,	Avon- more,	Fetterman,	Red House[m]
State			Ill.	W. Va.	Tenn.	Tenn.	Pa.	W. Va.	N. Y.
River			Wabash	Kanawha	Tennessee	Cumber- land	Kis- kiminetas	Tygart	Allegheny
Drainage area, sq miles			28,600	8,367	38,520	16,000	1,723	1,304	1,690
5[n]	1894–1895	11.1	....	10.5935	....	....	....	....	....
7[n]	1934–1938	15.4	....	−26.6305	16.3435[h]	....	....	....	....
17[n]	1939–1940	1.0	....	0.2964	....	....	....	....	....
18[i, n]	1938–1939	0.4	....	47.4144[h]	....	....	....	....	....
									ITEMS 19 TO 33,
35	1935–1939	18.3	15.4156[h]	....	....	0.87738[h]	13.0246[h]	21.4527[h]	−20.0958[h]
36[q]	1935–1937	14.6	14.60[o]	....	....	3.32[o]	7.13[o]	....	....
40	1935–1939	38.7	2.88126	18.1615	5.00196[o]	....	....	−166.1619	40.7842
41	1935–1937	37.2	4.96272	7.3593	0.85418[o]	....	....	....	....
44[i]	1937–1938	28.6	4.8195[h]	3.3603	....	....	....	....	....
44[i]	1937–1939	26.7	1.2877[h]	23.1711[h]	....	....	....	....	2.4188[m]
44[i]	1938–1939	15.9	5.5771[h]	18.3464[h]	....	....	....	....	....
44[l]	1937–1938	5.6	....	33.5852[h]	....	....	5.79404	....	....
44[i]	1937–1938	1.5	....	75.7273[h]	....	....	41.13051	....	....
									ITEMS 45 TO 116,
112[l]	1932–1941	18.3	5.437880[h]	....	....	5.437880	5.437880[h]	5.437880[h]	5.437880[h]
113[l]	1932–1941	38.7	2.453260	2.453260	2.453260	....	....	2.453260	2.453260
114[l]	1932–1941	57.0	1.690571	1.690571	1.690571	1.690571	1.690571	1.690571	1.690571

[a] Lower river stations. [c] Weighting factors constant. [d] Mean absolute (values approximate). [e] Flood water years ending September 30 of each year so listed. [h] Referring to the gaging stations in the main headings. sums of designated tributary discharges to obtain the ratios, or uniform weighting factors listed above. [m] In Item
[n] For additional values in Cols. 4 to 9, see Tables 10 and 10*B*. [o] Including the latest revisions of both observed

According to authoritative analyses of probabilities,[49] the standard error of the standard deviation should be as large as the deviation itself whenever the number, *n*, of observation equations and unknowns is equal to 3; and for *n* equals 10, the odds are 3 to 2 that the calculated probable error is within 20% of the correct value; furthermore, the odds are 30 to 1 that the error is within 50%.

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[49] "Handbook of Engineering Fundamentals," by Ovid W. Eschbach, Am. Telephone and Telegraph Co., John Wiley & Sons, Inc., New York, N. Y., 1936, Chapter 2, p. 127.

(ITEMS 35 TO 44 AND ITEMS 112 TO 114)

TIONS:	CHECKING PERIOD		DEPARTURES OF DERIVED DISCHARGES FROM OBSERVED OR OTHER BASIC DATA (%)										Item No.
Huntington,[a] W. Va.	No. years	Range of years[g]	Checking Period			Selected Years, Showing Wide Range of Yield							
			Mean	Absolute[d]	Maximum	1927[e]	1931	1932	1937[e]	1939	1940	1941	
(17)[e]	(19)	(20)	(21)	(22)	(23)	(24)	(25)	(26)	(27)	(28)	(29)	(30)	(1)
Ohio	....	....	....	....	....	....	....	....	....	....	....	....	....
55,200[h]	....	....	....	....	....	....	....	....	....	....	....	....	....
STATIONS AT:													
Metropolis,	....	....	....	....	....	....	....	....	....	....	....	....	....
Ill.	....	....	....	....	....	....	....	....	....	....	....	....	....
Ohio	....	....	....	....	....	....	....	....	....	....	....	....	....
203,000	....	....	....	....	....	....	....	....	....	....	....	....	....
....	20	1886–1905	−15.8	±16.7	−39.6	−19.5	+ 8.1	− 6.8	−17.8	+ 1.9	+19.9	+58.8	5[m]
....	65	1875–1939	+ 2.2	9.6	−23.9	− 6.7	+14.6	−19.4	0.0	+ 0.6	−44.3	+16.5	7[m]
....	20	1922–1941	+ 3.5	21.0	+46.3	+40.5	+46.3	−44.9	− 4.7	0.0	0.0	+18.5	17[m]
....	20	1921–1940	+ 2.3	15.5	+50.7	+18.7	+29.3	−37.9	− 3.8	0.0	+ 6.3	+12.5	18[i,m]
SEE TABLE 10													
000[e]	15	1927–1941	− 6.1	6.6	−21.1	−11.9	−21.1	− 8.4	0.0	0.0	+ 3.8	− 1.9	35
000	15	1927–1941	− 2.8	4.8	−12.3	− 5.5	−12.3	+ 1.6	− 0.1	− 2.4	+10.5	+ 5.7	36[q]
000[e]	{15 43	1927–1941 1899–1941	+12.8 +10.7	17.6	+57.5	− 7.1	+26.7	+35.3	0.0	0.0	−11.5	−29.8	40
000	15	1927–1941	+ 6.0	7.4	+20.7	+11.1	+14.0	+ 8.1	+ 0 3	− 2.8	− 5.7	− 8.9	41
000[h]	10	1931–1940	− 0.5	4.2	−12.7	− 6.1	+ 0.2	+ 0.2	0.0	− 8.4	+ 3.2	− 0.1	44[i]
000[h]	10	1931–1940	+ 9.5	10.1	+19.1	+11.1	+17.2	+19.1	0.0	0.0	+11.9	+ 5.5	44[i]
000[h]	10	1931–1940	+ 5.6	8.6	+16.9	+12.0	− 7.9	+11.0	+ 8.9	0.0	+ 9.6	− 0.4	44[i]
000[h]	15	1927–1941	+14.2	14.9	+43.8	+16.4	+28.6	+26.3	0.0	+ 3.0	+11.8	− 0.4	44[l]
000	15	1927–1941	+ 3.6	19.0	+44.6	− 9.7	+44.6	+ 2.1	0.0	−18.3	+40.6	+35.5	44[i]
SEE TABLES 10 AND 10*B*													
000	15	1927–1941	+ 0.2	4.2	−14.0	+ 1.4	−14.0	+11.5	+ 0.1	− 1.2	+ 1.5	− 6.9	112[l]
000	15	1927–1941	+ 1.8	7.4	+14.0	− 1.5	+12.6	+12.0	− 9.3	− 6.1	−11.9	+ 0.5	113[l]
....[p]	15	1927–1941	+ 1.3	5.0	+11.8	− 0.6	+ 4.3	+11.8	− 6.4	− 4.6	− 7.7	− 1.8	114[l]

year; in these columns unmarked years are drought years except 1939 (an average year). [g] Discharge data for
[i] Additional related tests. [l] Mean discharges at the lower river station for the test period were divided by the
44, Col. 16, the assumption indicated refers to the gaging station at Franklin, Pa. (drainage area, 5,982 sq miles).
and derived in estimated data. [p] All ten stations. [q] Slide-rule calculations only.

The investigations underlying this paper have demonstrated that the suitability or representative character of the observation equations far outweighs the influence of number. Some arrays of data covering only 2 or 3 years have outdone other arrays of 6 or 7 years, or such numbers as 10, 15, or 20. One test of 25 years in groups of five each, yielding a set of five 5-yr means, was found to have imposed so many conditions that the weighting

TABLE 10*B*.—SUPPLEMENT TO TABLE 10

Item No.	Test period[g]	Gaged area %	GAGING STATIONS: Rich Fountain, Mo.	Alderson, W. Va.	Vicksburg, Miss.	CHECKING PERIOD No. years	Range of years[g]
(1)	(2)	(3)	(8)[c]	(11)[c]	(18)[a,e,f]	(19)	(20)
River			Gasconade	Greenbrier	Mississippi	....	....
Drainage area, sq miles			3,180[h]	1,357[h]	1,144,500[h]	....	....
COMBINED WITH GAGING							
Gaging station			Little Rock,	Kanawha Falls,	Angola,[f]	....	....
State			Ark.	W. Va.	La.	....	....
River			Arkansas	Kanawha	Mississippi	....	....
Drainage area, sq miles			157,900	8,367	1,242,700	....	....
ITEMS 5 TO 67, SEE							
1922–1941 Means, and:							
68	1940–1941	1.0	181.34312[h]	14.29019	−115,981[h]	20	1922–1941
69	1940–1941	0.9	201.82239[h]	16.82064	−129,155	20	1922–1941
Yearly Means Only:							
75	1938–1939	0.9	203.08791[h]	5.351215	000	20	1922–1941
1922–1941 Means, and:							
78	1940–1941	0.4	182.89615[h]	65.182267[h]	−88,208[h]	20	1922–1941
83	1940	0.4	207.26092[h]	23.837951[h]	000	20	1922–1941
91	1939–1940	0.4	204.957790[h]	25.756673[h]	000	20	1922–1941
1922–1941 Means, and:							
92	1941	0.4	85.889908[h]	152.810789[h]	000[h]	20	1922–1941
93	1941	0.4	97.566239[h]	172.553078[h]	000	20	1922–1941
Five-Year Means Only:							
94	1927–1941	1.0	97.071751[h]	25.091831	−13,417[h]	20	1922–1941
95	1927–1941	0.9	117.817958[h]	22.326052	24,910	20 53	1922–1941 1889–1941
96	1932–1941	1.0	93.211972[h]	24.712427	000[h]	20	1922–1941
97	1932–1941	0.9	124.986265[h]	23.030062	000	20	1922–1941
98	1927–1941	0.4	106.886839[h]	96.660423[h]	52,504[h]	20 53	1922–1941 1889–1941
99	1927–1941	0.4	126.551046[h]	86.006116[h]	83,564	20 53	1922–1941 1889–1941
100	1932–1941	0.4	123.831488[h]	102.838480[h]	000[h]	20	1922–1941
101	1932–1941	0.4	153.519985[h]	95.839137[h]	000	20 53	1922–1941 1889–1941
One-Year Means:[k]							
102	1927–1941	0.4	85.129169[h]	116.394081[h]	74,377[h]	20	1922–1941
103	1927–1941	0.4	108.391795[h]	117.171683[h]	72,306	20	1922–1941
104	1927–1941	1.0	84.774019[h]	21.541561	62,064[h]	20	1922–1941
105	1927–1941	0.9	108.819457[h]	21.004572	65,498	20	1922–1941
106	1932–1941	1.0	78.531555[h]	32.185600	−60,249[h]	20	1922–1941
107	1932–1941	0.9	92.958831[h]	34.626670	−67,045	20	1922–1941
Ten-Year Means Only:[l,r]							
108	1932–1941	1.0	36.442748[h]	36.442748	000[h]	20	1922–1941
109	1932–1941	0.9	40.635115[h]	40.635115	000	20	1922–1941
110	1932–1941	0.4	114.080564[h]	114.080564[h]	000[h]	20	1922–1941
111	1932–1941	0.4	127.204370[h]	127.204370[h]	000	20 53	1922–1941 1889–1941
113	1932–1941	38.7	....	2.453260	....	15	1927–1941
114	1932–1941	57.0	....	1.690571	....	15	1927–1941
115	1932–1941	90.2	1.233898	....	000[h]	15	1927–1941
116	1932–1941	83.0	1.375846	....	000[h]	15	1927–1941

[a] Lower river stations. [c] Weighting factors constant. [d] Mean absolute (values approximate). [e] Flood Simmesport, La. [g] Discharge data for water years ending September 30 of each year so listed. [h] Referring [l] Mean discharges at the lower river station for the test period were divided by the sums of designated tributary missing columns, see Tables 10 and 10*A*.

(ITEMS 68 TO 111)

DEPARTURES OF DERIVED DISCHARGES FROM OBSERVED AND OTHER BASIC DATA (%)

Checking Period			Selected Years, Showing Wide Range of Yield							Item No.
Mean (21)	Absolute[d] (22)	Maximum (23)	1927[e] (24)	1931 (25)	1932 (26)	1937[e] (27)	1939 (28)	1940 (29)	1941 (30)	(1)
....	....	....	....	....	....	....	....	....	....	....
....	....	....	....	....	....	....	....	....	....	....
STATIONS AT:										
....	....	....	....	....	....	....	....	....	....	....
....	....	....	....	....	....	....	....	....	....	....
....	....	....	....	....	....	....	....	....	....	....
....	....	....	....	....	....	....	....	....	....	....
TABLES 10 AND 10A										
0.0	18.2	−44.6	+34.5	+27.8	−44.6	−8.4	−6.8	0.0	0.0	68
0.0	15.1	−43.3	+22.6	+16.1	−43.3	−5.3	−2.7	0.0	0.0	69
−5.0	15.1	+42.5	+19.7	+28.7	−42.3	−5.1	0.0	+1.5	+13.6	75
0.0	18.0	−42.3	+34.7	+28.8	−42.3	−7.9	−4.8	0.0	0.0	78
0.0	15.7	±42.5	+20.7	+27.6	−42.5	−5.1	+0.4	0.0	+12.5	83
−1.7	15.1	−42.4	+19.9	+27.1	−42.4	−5.5	0.0	0.0	+6.1	91
0.0	12.9	+36.7	+2.4	+36.7	−20.3	−10.7	−7.6	+30.5	0.0	92
0.0	12.4	+29.2	−6.6	+24.0	−19.2	−7.5	−3.4	+29.2	0.0	93
−0.8	13.5	+38.9	+4.4	+38.9	−28.7	−12.2	−10.9	+18.4	+6.2	94
−3.3 +0.04	11.6	−29.9	−4.0	+28.5	−29.9	−9.9	−6.7	+20.8	+3.8	95
−1.0	13.7	+40.5	+2.6	+40.5	−28.1	−12.6	−11.1	+19.3	+7.2	96
−2.8	11.2	−30.9	−1.2	+26.7	−30.9	−9.3	−6.3	+19.3	+2.1	97
−0.6 +2.0	13.9	+43.8	−6.3	+43.8	−26.4	−10.9	−6.8	+16.8	+9.6	98
−3.2 −1.0	11.5	+31.7	−2.7	+31.7	−28.1	−8.9	−3.4	+19.4	+6.4	99
+0.5	13.3	+38.9	+14.1	+38.9	−28.7	−9.3	−5.7	+19.8	+5.4	100
−1.6 +1.2	11.9	−31.4	+7.5	+25.5	−31.4	−6.5	−1.7	+13.9	+0.7	101
−0.2	13.8	+46.4	−0.7	+46.4	−21.0	−11.2	−7.1	+31.3	+10.3	102
−1.6	12.2	+30.3	−6.5	+30.3	−23.8	−9.0	−3.8	+24.6	+8.8	103
−0.5	15.5	+48.6	−1.6	+48.6	−25.6	−12.5	−9.9	+30.6	+14.5	104
−1.8	12.6	+32.5	−7.3	+32.5	−28.0	−10.1	−6.3	+23.8	+13.2	105
−3.7	13.9	+29.5	−2.1	+29.5	−28.3	−15.4	−15.6	+25.8	−3.5	106
−5.1	11.4	−29.3	−9.9	+25.9	−29.3	−13.5	−12.7	+10.9	−4.6	107
−5.2	16.0	+37.6	−18.0	+35.3	−20.9	−17.9	−18.0	+37.6	−0.1	108
−7.8	14.3	+42.6	−25.8	+20.9	−21.1	−16.2	−15.6	+34.2	−6.7	109
−0.8	12.8	+37.6	−1.2	+37.6	−27.1	−10.2	−6.7	+21.6	+3.5	110
−4.2 −0.8	11.7	−27.3	−10.6	+22.9	−27.3	−8.4	−4.0	+18.7	−3.4	111
+1.8	7.4	+14.0	−1.5	+12.6	+12.0	−9.3	−6.1	−11.9	+0.5	113
+1.3	5.0	+11.8	−0.6	+4.3	+11.8	−6.4	−4.6	−7.7	−1.8	114
+1.6	2.5	−5.2	+2.6	+3.0	+0.4	−2.2	−3.4	+1.0	+4.0	115
−1.2	4.2	+10.0	−7.2	−4.0	+0.2	−2.5	−0.5	−1.4	−2.8	116

year; in these columns, unmarked years are drought years except 1939 (an average year). [f] With Atchafalaya, to the gaging stations in the main headings. [h] Computed by least squares, and normal equations solved. discharges to obtain the ratios, or uniform weighting factors above listed. [r] For further information regarding factors were unwieldy and thus useless for the purpose intended, with constants exceeding 6,400,000 cu ft per sec and 8,200,000 cu ft per sec, respectively, for Vicksburg and Angola-Simmesport, Items 53, Table 10.

The only basis thus far disclosed for satisfying the fundamental criterion (that the sum of the squares of deviations shall be a minimum) involves, either directly or indirectly, some multiple of the arithmetic means of the respective coefficients appearing in the observation equations. Comparable results may be had by including the means of all or of a part of the test period, along with any desired number of seemingly representative yearly data, as illustrated by Items 92 and 93, Table 10*B*. Somewhere within the twilight zones of mathematical probability, fantastic legends have grown up, attributing almost magic powers to the least squares procedure, when simpler methods achieve either identical or comparable results, as shown by many items associated with those in Tables 10.

Mr. Cory's comment regarding an extremely low stage of the Ohio River at Cincinnati calls forth a simple test, based on official records as shown in Table 11.[50]

TABLE 11.—RAINFALL DEPTHS IN INCHES AT MARIETTA, OHIO, FOR CRITICAL MONTHS, 1833–1836

Month	1833	1834	1835	1836	Means, 1817–1930
July	6.84	5.08	2.58	3.16	4.63
August	0.25	1.33	6.54	3.16	3.85
September	3.30	0.67	2.75	2.08	3.03
October	3.17	3.75	4.80	2.50	2.91
Yearly depth	40.37	34.66	42.46	36.76	42.25

A casual examination of the foregoing data in the light of the close correlation of Marietta rainfall with runoff as far down the river system as Vicksburg justifies the conclusion that the time of minimum discharge of the Ohio River at Cincinnati was September, 1834, in the second successive month of subnormal rainfall, and in a year of drought following a slightly sub-normal year. As to the actual discharge, it might have been anything from 3,000 to 5,000 cu ft per sec inasmuch as such minima have been observed from Huntington to Louisville, Ky., within recent years. A channel cross section of about 70 ft by 10 ft might have been spanned by those early steamboats. As mean velocities of 4 to 7 ft per sec would account for such minimum discharges, there is no occasion to challenge the veteran's boyhood memories; they correspond with the probable conditions of flow.

It is worthy of note that the same year that marked the establishment of methodical rainfall records at Marietta, Ohio, and of Mississippi River gage heights at Natchez, saw the election of the eminent scientist, Pierre Laplace, to the presidency of the French Academy. Napoleon's complaint upon the occasion of his dismissing Laplace from the office of Minister of the Interior had to do with "administration on the basis of differential calculus," or "of infinitesimals," according to other translations.

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[50] *Bulletin W*, U. S. Weather Bureau, 1930, and later data.

The accustomed praise accorded the Humphreys and Abbot hydrologic data and findings, coupled with the practical disregard and discounting to which they have been subjected entirely too long, may well have its origin in the application of calculus or the Laplace and Gaussian theories of probabilities to data in the rough, where practical application of experience and the simplest mathematical approach would have served better. Intensive studies of those early data pertaining to the Mississippi River system would have been richly repaid if no other result had been achieved than disclosing the fruitless hours devoted to building up the usual normal equations, preparatory to their solution as simultaneous equations, when the basic observation equations are capable of yielding either identical, fairly comparable, or more satisfactory values of the respective unknowns, as described herein. This comment in no wise circumscribes the use of calculus, probabilities, least squares, or other preferred methods within their proper sphere, but is aimed against ill-advised or improper application of such procedure, either inside or outside their respective provinces. The gross errors inherent in either the instrumentation, the sampling, the observations in the field, or their interpretations in the office, or the summation of these defects, would seem to disqualify the ordinary run-of-the-mill hydrologic data for analysis by the method of least squares, where large numbers of small deviations constitute the most suitable arrays. Nevertheless, the normal alternation of plus and minus errors seems to effect such compensations so as to render the best attainable values of rainfall and runoff; and these have consistently withstood the most practical tests as to their suitability and reliability for the purposes intended—evaluation, conservation, and utilization of water resources.

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CLASSIFICATION OF IRRIGABLE LANDS

By W. W. Johnston,[1] Esq.

With Discussion by Messrs. R. Earl Storie, Walter W. Weir, W. L. Powers, and W. W. Johnston.

Synopsis

Land classification as conducted in the Bureau of Reclamation is on the basis of utility for irrigation farming. Three main types are made which are similar in main essentials but differ in detail and refinement. The principal function of all these is to eliminate from the area to be irrigated (arable land) lands which do not have sufficient productive capacity to support a farm family at a reasonable level of living and permit the payment of construction assessments. The "arable" lands are further divided into classes and subclasses on the basis of productive value under irrigation.

Introduction

The term "Land Classification," as used by various agencies in recent years, has been applied to a variety of land maps that differ widely both in purpose and detail. In some cases it means the separation of broad areas on the basis of "type of use," such as industrial areas, recreational areas, and the like, as distinguished from lands that are primarily suited to agricultural uses. More commonly, lands are classified on the basis of their adaptation to different types of agricultural use, such as forest areas, grazing lands, and lands suited to different types of crop production.

Land classification, as conducted in the Bureau of Reclamation and discussed in this paper, is on the basis of utility for irrigation farming, and much of its strength lies in this singleness of purpose. The most important phase of this work is the separation of lands suitable for irrigation (termed "arable") from inferior lands not suited to irrigation development (non-arable). To be suitable for irrigation, land must not only be located under the ditch and of a surface relief even enough to be irrigated, but it also must have sufficient productive capacity to support a family at a reasonable level of living and to permit the payment of irrigation assessments. The arable lands are divided fur-

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[1] War Dog Reception and Training Center, San Carlos, Calif. (formerly Reclamation Economist, Bureau of Reclamation, U. S. Dept. of the Interior, Ephrata, Wash.).

ther into two or three classes on the basis of productive value and the ability to meet irrigation assessments, class 1 representing the highest class arable land and class 3 the lowest. The non-arable lands are placed in class 6 except where special classes are mapped that include portions of the "non-arable" land as here defined, on which surplus waters may be used to advantage on a yearly rental basis for some limited use, such as irrigating pasture. Frequently, subsequent economic investigations will indicate that when intermingled with better land, so that a farm unit will include only a small percentage of this type, such portions of the acreage in the special classes properly may be supplied with a regular water right and be subject to regular irrigation assessments. In reclassification of old projects, class 5, a suspended class, is frequently used to include lands that are temporarily unproductive, pending the construction of needed drainage or other action which will restore productivity.

A study of its history shows that since its inception there has always been some kind of land classification practiced by the Bureau of Reclamation. The earlier land studies were generally not extended to great detail, and the importance of soil conditions was not recognized to the same extent as at present. Little was known about arid soils and their response to irrigation, and in the earlier years there was a tendency on the part of practically every one interested in irrigation to believe that all land below the irrigation ditch was susceptible of producing satisfactory crops if only the water were delivered to it. Experience has demonstrated, however, that good soil is as important as sufficient water, and few irrigation developments have stabilized with as much land under irrigation as was originally contemplated.

The Modern System of Land Classification

The modern system of classifying irrigable lands began with the passage of the so-called "Fact-Finders Act" of December 9, 1924, wherein provision was made for the classification of project lands on the basis of ability to meet construction charges. Experience and study during the intervening years have resulted in the development of methods and standards now in use. Although land classification is used for essentially economic purposes, it is necessary to base the distinction between classes on physical conditions, of which soil, topographic, and drainage characteristics are of primary importance. Climate is likewise an important factor in gaging the productive capacity of an area, but generally it is reasonably uniform within the boundaries of a single project and therefore does not enter into the division of land into classes.

Soil characteristics vary so greatly from place to place that it is difficult to establish a set of soil conditions that are uniformly applicable as a guide in determining irrigability. In the central valleys of California, for instance, medium to light-textured sandy loams and loams of recent alluvial deposition, such as river deposits adjoining the Sacramento River and soils formed from alluvial fan deposits of the smaller streams, are the more successfully irrigated, whereas the older alluvial soils have developed hardpans and in many cases have been leached too extensively to be highly productive or to respond well to irrigation. In eastern Montana, on the other hand, the soils of mature development, which have become well weathered and have developed definite profile characteristics, are usually the most successful irrigated lands; and here the soils of recent alluvial deposition are likely to be either heavy intractable clays that have been formed from materials washed in from "raw" shale hills, or sandy soils derived from sandstone alluvium that has been little altered and is of limited fertility.

Certain soil characteristics are uniformly necessary, however. To be satisfactory for irrigation development, the soil must have a reasonably high water-holding capacity; it must be readily penetrable by water and still have a slow enough infiltration rate so that it can be irrigated without excessive percolation losses; it must be free from "black alkali" or a sodium-saturated condition; and it must be either nonsaline or of a type from which the moderate amounts of saline salts that it may contain can be leached readily. Also, the soil must be sufficiently deep to allow necessary root development and permit soil drainage. Arid and semi-arid soils that meet these requirements are practically always well supplied with mineral plant nutrients. The quantity of nitrogen is generally deficient, but this element is readily maintained by the production of leguminous crops that are the backbone of irrigation agriculture.

Topographic considerations are of two main types: First, the degree of slope; and second, the character of the surface relief—whether it is smooth or uneven. Likewise, the characteristics of the soil in respect to erosion, permeability to water, and susceptibility to "subbing" or lateral movement of water between furrows must be considered, and the methods of irrigation that have proved to be adapted to the region cannot be ignored in fixing the topographic limits for the different classes. In the broad smooth valleys of Arizona and California, the use of large heads of water and flooding in borders, checks, or basins has proved to be the most practicable method of irrigation for all except row crops and possibly orchards. In central Washington, on the other hand, the corrugation method of irrigation has proved to be well adapted to soil and topographic conditions and is used for all types of crops, almost to the exclusion of other methods. Irrigation heads are small and, particularly on the steeper slopes, the use of flumes and irrigation pipe, which permit accurate control of water to the corrugation, is common practice. Under the latter conditions slopes of from 10% to 15% are used successfully under irrigation and are classified as arable where soil conditions are favorable. Under conditions prevailing in the former localities, on the other hand, the permissible limit of slope is generally less than 10%.

Lands of moderate gradient, but of uneven surface, are considered both from the standpoint of the cost of necessary leveling and the probable effect of surface soil removal on fertility. In general, deep soils of recent alluvial deposition can be graded heavily with only temporary curtailment of productive capacity, but older soils that have developed zones of lime accumulation in the subsoil, and also soils of limited depth to rock or gravel, cannot be heavily graded without seriously decreasing crop production in spots where heavy cuts have been made.

Some future drainage requirements are to be anticipated on all new projects, and it is difficult, and in many cases impossible, to determine in advance where seepage development will take place. In land classification, therefore, eliminations from the irrigable area because of adverse drainage conditions are limited to areas without surface outlets and to narrow draws and valleys where early seepage development is to be anticipated and drainage costs will obviously be in excess of benefits. Lands are placed in class 2 or 3 because of adverse drainage conditions where more than average hazards of seepage development exist but conditions are favorable for artificial drainage construction at reasonable cost.

METHODS

In order to achieve the objectives of land classification, as previously discussed, and to assure uniform work where a number of classifiers are employed, it has been found necessary to establish certain physical standards or specifications for each class. This must be done for each project studied, although standards used in different parts of the West include certain similarities, and those applying to two areas having similar soils, climate, and irrigation practices will be correspondingly similar. In so far as conditions can be duplicated, standards are based on actual performance of similar lands under irrigation, and to this end it is customary, when establishing classification standards for an area of new land, to study carefully the results that are being obtained under similar soil and topographic conditions in near-by developed areas.

For instance, in starting the classification of Columbia Basin project lands, it was important to know which sandy soils of the area were suitable for irrigation development. This problem was studied on three near-by irrigation districts where sandy texture has been the principal limiting factor in the success of the area under irrigation. Soils were examined on farms that have been farmed profitably under irrigation for a long period of years, on others that have been less successful, and on farms that have gone out of cultivation after a few years' trial. The history of these farms was obtained from well-informed local people, the record of water use was secured, and soil samples were taken for mechanical analyses. The information obtained, considered in the light of the amounts of irrigation water that can be supplied economically, made it possible to fix the standard of soil texture for each class on the basis of demonstrated experience. The standards with respect to topography, stoniness, depth to rock or gravel, and other factors were determined in the same way, although it was necessary to consider irrigated lands in the Yakima and Kittitas valleys, in addition to projects immediately adjoining the Columbia River, in order to duplicate all of the principal soil and topographic conditions encountered on the Columbia Basin project.

The methods followed in making a land classification vary somewhat, depending on the detail required in the particular study. In all cases, however, the boundaries between classes are drawn on the maps in the field after careful examination of all physical conditions, including subsoil examinations. The reason for placing land in a lower class than "1" is indicated by placing an appropriate letter after the classification number. The letter "S" indicates a soil deficiency; "R" loose rock in the plow zone; "T" a topographic deficiency; and "D" a deficiency in drainage characteristics. Two or more letters are used when more than one factor is involved. Thus, the classification as 2ST indicates that the land is inferior to class 1 in both soil and topographic characteristics, whereas an area marked 2T is equal to class 1 in soil, drainage, and freedom from loose rock in the plow zone, but it is either too rough in surface or too steep in slope to be first class. The boundaries between lands placed in the same class for different reasons are delineated on the land-classification maps, as well as the boundaries between classes. In the ordinary procedure, only the acreage in each class as a whole is measured and reported. The maps include the necessary data, however, to permit the measurement of areas in the several subclasses as well.

Where detailed topographic maps or good aerial photographs are available for use as base maps, it is unnecessary to survey the actual boundaries because with such maps a competent man can sketch the boundaries between classes with no greater error than his judgment in determining where the class boundary should be. Where base maps of this kind are not available, it is necessary to use a plane table. Sufficient test pits or auger borings are made to explore subsoil conditions carefully, these being ordinarily extended to depths of 5 ft. The minimum number for detailed studies is one for each 40 acres; for semi-detailed classification, one for each 160 acres; and enough in reconnaissance studies to meet the less exacting requirements of that kind of study. Alkali determinations of pertinent soils samples are made where the need is indicated, and in some of the larger and more detailed studies field laboratories are maintained where mechanical analyses, percolation studies, and other soil tests can be made.

Types

In recent years the Bureau of Reclamation has attempted to standardize on three main types of land classification that are similar in the main essentials but differ in the detail and degree of accuracy attained. These are "Reconnaissance Land Classification," "Semi-Detailed Land Classification," and "Detailed Land Classification." It has been possible to standardize quite definitely in the case of the last two types, but the degree of accuracy attained in reconnaissance classifications has varied somewhat from one area to another, depending on the particular requirements of the investigation.

Reconnaissance Land Classification

The more general type of reconnaissance is usually made on maps having a scale of 2,000 ft to the inch or on aerial photographs of whatever scale available. This type is applicable to the study of large areas where it is desired to secure general information and to determine the location and extent of areas that are sufficiently promising to warrant more detailed investigations. An example is found in studies made in southwestern Wyoming in connection with the Colorado River investigations. Inspection showed this area to be made up largely of extremely shallow soil but to include also bodies of land of variable extent made up of better soil Consequently, it was decided to make a general reconnaissance land classification for the primary purpose of locating areas of sufficient promise to warrant more detailed study. In the absence of better base maps, township sheets with horizontal scales of 2,000 ft to the inch were made up from township data, and section corners were located and flagged on the exterior boundaries of each township. With these corners for initial control and a plane table for use in maintaining orientation and location, the general boundaries of class 1 and class 2 areas were mapped in each township, this work requiring an average of four days per township for two men. Arable areas having sufficient size to warrant further consideration, as indicated in the reconnaissance, were then covered by a semi-detailed land classification, this being the type that was followed more generally in these investigations. The results were satisfactory and the cost was a fraction of the amount which would have been required had the entire area been covered in the usual detail.

An example of a more accurate reconnaissance is the investigation the Bureau of Reclamation is making in the Sacramento Valley. This work, which eventually will cover several million acres, is for the purpose of obtaining an inventory of lands that may be irrigated from the Shasta reservoir and from storage which may be developed on tributary streams. Since the complete consummation of plans to be developed as a result of the investigations will stretch far into the future, the present requirement is for information on the location of lands of different capabilities and an estimate of acreage that is reliable for good-sized blocks but not necessarily applicable to a single farm or individual 40-acre tract. It is planned that a more detailed land classification will be made of any project resulting from these and other preliminary investigations before starting construction.

In this study, standards in respect to soil conditions are defined in terms of soil types as mapped in the county soil surveys, but all of the area is examined and the features of the classification are delineated on aerial photographs in the field. Where recent soil surveys are available, little additional work is necessary for the better soil types except to map topographic features and give due consideration to alkali. For inferior soil types, however, where only the more favorable phases are suitable for irrigation development, it is necessary to put down a reasonable number of soil pits and to give careful consideration to soil changes. This has been found to be true with all soil types in areas not covered by recent soil surveys.

The good valley lands that are suited to a wide variety of crops are included in classes 1 and 2. Class 1 includes the better lands which have deep, well-drained soils that are free from harmful quantities of alkaline or saline salts and are of favorable topography. Class 2 includes areas which are still "good farm lands" but are distinctly lower in productive capacity than class 1 because of some limitation such as heavy texture, limited soil depth, moderate salinity, or somewhat unfavorable topography. Class 3 is a "special" class made up of lands having restricted utility for irrigation farming. Class 3a includes the deeper and more productive phases of the "hardpan lands," and class 3b comprises the heavy clays, which have proved suitable for growing rice or for irrigated pasture but are not suited to a wide variety of crops. Portions of the rolling foothills that have irrigation possibilities are likewise in "special" classes, being designated as H-1 or H-2, the first representing the best and the second representing the less desirable of the hill lands that are found to be suitable in some capacity for irrigation.

Class 1 and class 2 lands include those which are capable of continued production and which will give a fair return to the operator, thus permitting the payment of substantial irrigation assessments. Lands in the "special" classes are types that have been proved by experience to be capable of some production under irrigation; but their development as irrigated land is more hazardous from the standpoint of the man who may settle on them and also the agency that must collect the construction costs. Areas found to be unsuited for irrigation, even in the limited capacity of the special classes, are classed as non-arable (class 6).

SEMI-DETAILED LAND CLASSIFICATION

The semi-detailed land classification (sometimes called detailed reconnaissance) is generally made on maps having a scale of 1,000 ft = 1 in., and is the type most usually followed in areas under investigation for development as individual irrigation projects. Ordinarily, only two classes of arable land (class 1 and class 2) and one class of non-arable land (class 6) are mapped, but special classes are sometimes included when the requirement is indicated. This type is very similar to the "Detailed Land Classification" except that, as the name implies, it is not carried to the same degree of refinement. With this limitation, the discussion of the latter type that follows in later paragraphs will apply to the semi-detailed type, and a thorough description is omitted from this section in the interest of brevity. A set of classification standards, which was used in the investigation of one project (Kendrick project, Wyoming), follows:

SEMI-DETAILED LAND CLASSIFICATION

Class 1, Arable.—
- Soil.—
  - Depth.—Four feet or more to shale or hard sandstone, 2 ft or more to gravel or loose rock.
  - Texture.—Fine sandy loam to silt loam and not seriously compacted.
  - Alkali.—Free from more than slight indication of neutral salts (white alkali); free of indication of black alkali either present or prospective.
- Topography.—
  - Smooth slopes of 5% or less with reasonably good-sized areas sloping in the same plane.
- Drainage Conditions.—
  - Soil and topographic conditions such that seepage would not be expected to develop soon and artificial drainage would be feasible at moderate cost.

Class 2, Arable.—(Areas placed in class 2 on account of deficiencies in either soil, topography, or drainage with the following minimum standards):
- Soil.—
  - Depth.—Three feet or more to shale, hard sandstone, or impervious clay; $1\frac{1}{2}$ ft or more to gravel or loose rock.
  - Texture.—Sandy loam to clay loam—some compaction permitted.
  - Alkali.—Slight to moderate indication of white alkali but not enough to prevent crop growth; free of indication of black alkali.

Topography.—

Smooth slopes between 5% and 10%, or rougher slopes that may be less than 5% in general gradient.

Drainage Conditions.—

Soil and topographic conditions such that seepage may be expected to develop rather quickly, but with reclamation by artificial drainage at moderate cost appearing feasible.

Class 6, Non-Arable.—(Areas placed in non-arable class because of failure to meet the minimum requirements for class 2 in either soil, topography, or drainage, or because of combinations of such deficiencies, including the following):

Soil.—

Depth.—Shallow soils usually overlying shale.

Texture and Alkali.—Compact clay soils that are generally somewhat alkaline and are only slightly permeable to water; excessively sandy soils that are generally more or less blown up into dunes and depressions.

Topography.—

Areas with rough topography that could be leveled and irrigated only at excessive expense and areas with general slopes greater than 10%.

Drainage Conditions.—

Areas to be classed as non-arable on account of drainage alone only in the case of stream bottoms, where it is obvious that the cost of artificial drainage will be excessive, and in depressed areas without outlets, although adverse drainage conditions as a contributing factor along with adverse soil conditions will result in further exclusions.

DETAILED LAND CLASSIFICATION

In detailed land classification the arable land is typically separated into three classes:

(1) Class 1 includes lands of relatively high utility for irrigation farming and suited to a wide range of crops.

(2) Lands in class 2 are suitable for irrigation farming but are less desirable than those in class 1 either because they will produce less or will be more expensive to farm under irrigation. They may have a lower water-holding capacity, as indicated by light texture or limited soil depth; they may be only slowly permeable to water because of clay layers or slightly penetrable hardpan in the subsoil; or they may be more expensive to irrigate because of steeper slopes, or more expensive to level because of uneven topography. Also, they may be restricted in soil depth, moderately saline, or have less favorable drainage characteristics than the class 1 lands.

(3) Class 3 is made up of land that is likewise suitable for irrigation development but is of restricted utility because of more extreme deficiency in one or more respects. Land placed in this class because of relatively steep slopes, but with first-class soil, may be very desirable for fruit production because of favorable air drainage and can be safely irrigated if a permanent cover crop is

maintained between the trees to prevent washing. On the other hand, class 3 land that is so classed because of inferior soil will be restricted to a smaller percentage of row crops in the rotation or, if of limited depth to impenetrable materials, may need to be kept in crops that shade the ground in order to limit saline accumulations. Class 3, therefore, includes lands on which the hazards of farming and also of collecting construction assessments are more acute than in the case of the higher classes.

The land classification of the Columbia Basin project in Washington, which was completed in the summer of 1941, is a good example of the "Detailed Land Classification." The order of land studies on this project included: First, the retracement of land lines; second, topographic surveys, in which maps were prepared having a contour interval of 2 ft and a horizontal scale of 400 ft to the inch; third, the land classification; and fourth, land appraisal. The results of each phase of the investigation were utilized fully in each successive phase. The topographic maps were used as base maps for the land classification, section sheets being prepared showing the various features of the previous surveys, with spaces provided for recording the results of soil profile examinations, analytical data, and notes. The physical standards that were developed on the basis of actual experience, as described in an earlier paragraph, are listed in Table 1. Although these standards were set up as a general guide, they were adhered to closely, except in cases where the peculiar combination of conditions required slight deviation. The limitations of soil depth, and the like, represent minimum requirements for the class, the average being considerably better than these minimum limits. For example, although the permissible minimum limit of 12 in. of loam above gravel or coarse sand was permitted for class 3, this minimum depth was approached in this class only in the case of small-sized spots in areas of larger size where the average depth was considerably greater, and where the topography was such that little or no leveling of the more shallow phases would be required. Lands classed as nonarable because of limited depth, on the other hand, will include many small-sized bodies of land that will exceed this minimum permissible depth for class 3. In other words, it is impossible to set up a set of classification standards that will cover all combinations of conditions encountered, and the individual land classifier was expected to use his judgment in following the "standards," to the end that class 1 would include the land of highest productive capacity with no specific limitations in utility; class 2 also would include good average arable land with a wide range of utility; and class 3, although of more restricted capability, still would be land that could be utilized successfully under irrigation if handled with reasonable care.

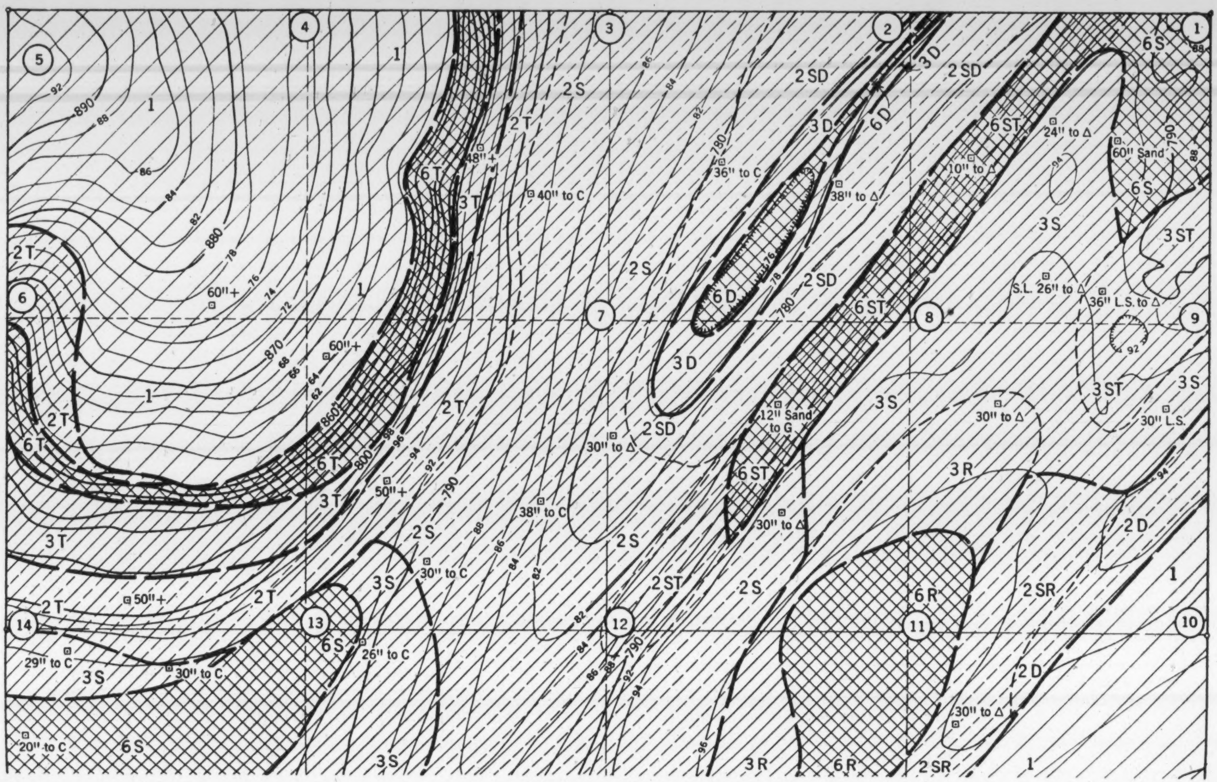
Table 2 contains a hypothetical map which, with Fig. 1, illustrates the principal types of soil and topographic conditions encountered on the Columbia Basin project and the manner in which they were handled in the land classification. Although this example is hypothetical in the sense that no single square mile exists on the project which is the counterpart of this map, it was made up by combining parts of several maps and therefore represents actual field conditions. No attempt has been made to have the acreage in the different

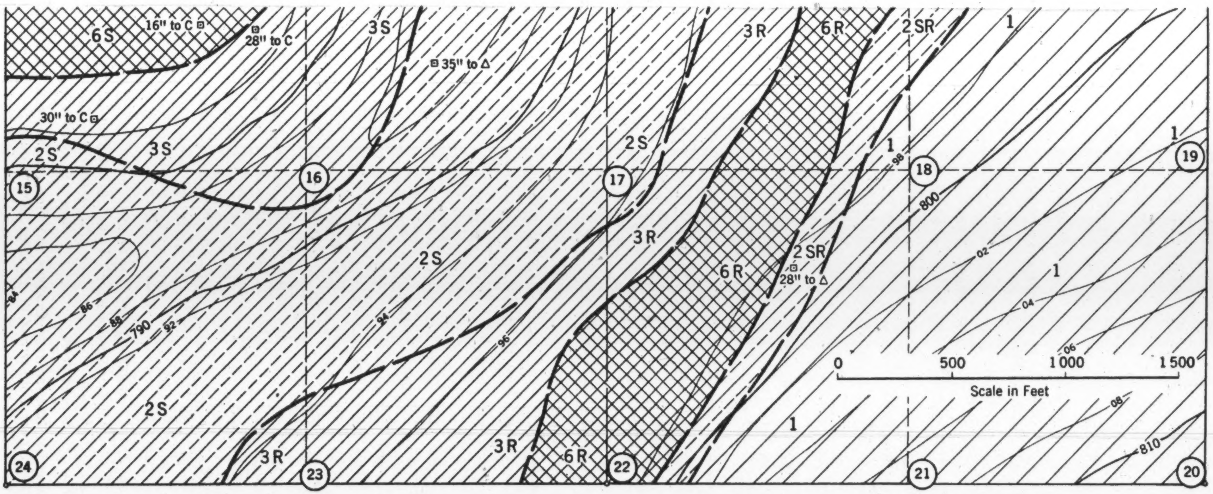
TABLE 1.—LAND CLASSIFICATION STANDARDS.—COLUMBIA BASIN PROJECT

Land characteristics	Class 1—Arable	Class 2—Arable	Class 3—Arable	Class 6—Non-Arable
(a) SOILS				Includes lands which do not meet the minimum requirements for Class 3. Also small areas of arable land lying within larger bodies of non-arable land when these would obviously not make usable fields. Arable areas which are isolated for irrigation by gravity (high) or those where the feasibility of making water delivery is questionable will be classified in the regular way and the matter of eliminating arable land because of cost of serving with irrigation water will be left for future determination.
Texture	Sandy loam to friable clay loam.	May be loamy sand if underlain by finer textured soil.	Loamy sand to friable clay.	
Depth:				
To gravel, sand or cobble	30 in. plus.	20 in. plus.[a]	12 in. plus.[a]	
To solid rock or impervious hardpan	48 in. plus.	36 in. plus.	36 in. plus, where flat or hummocky; 24 in. on smooth slopes.	
To caliche hardpan (slightly penetrable)	30 in. with 48 in. penetrable.	24 in. with 36 in. penetrable.	16 in. with 24–36 in. penetrable.	
To penetrable lime zone	18 in. with 48 in. penetrable.	14 in. with 36 in. penetrable.	8 in. with 24–36 in. penetrable.	
Alkali:				
Black alkali	pH under 9.0 in surface 3 ft; may have higher pH in subsoil if highly calcareous and total salts are low.	pH under 9.0 in surface 2 ft; may have higher pH below 2 ft if highly calcareous and total salts are low.	pH under 9.0 in surface 12 in.; may have higher pH below 12 in. if highly calcareous and total salts are low.	
Total salts	0.2 of 1% or less.	0.30 of 1% or less.	0.5 of 1% or less.	
Rock	None in the plow zone of a size that will interfere with cultivation.	Only scattering loose rock large enough to interfere with cultivation.	Not more than are generally cleared for cultivation.	
(b) TOPOGRAPHY				
Slopes	Up to 5% if reasonably large sized bodies slope in the same plane.	Up to 10% if reasonably large sized bodies slope in the same plane.	Up to 15% if slopes are very favorable.	
Surface	Smooth enough that leveling can be accomplished almost entirely with a float.	Moderate grading may be required.	May require heavy grading in spots but not more than has been general on similar developed projects.	
(c) DRAINAGE				
Soil and topography	Soil and topographic conditions such that no specific drainage requirement is to be anticipated.	Soil and topographic conditions such that drainage will probably be required but with reclamation by artificial means appearing feasible at reasonable cost.	Narrow valleys, etc., where drainage will probably be required. Also areas requiring surface outlets where these can be provided at reasonable cost.	

[a] At the minimum depths the soil should be sandy loam or heavier. Loamy sands underlain by coarse sand or gravel should be 20 to 30 in. deep (depending on the percentage of silt and clay) to qualify for class 3. Although areas may be placed in a lower class than "1" for any of the reasons listed above, they will generally be placed in the next lower class if decidedly inferior in two or more respects. The reasons for placing areas in a lower class than "1" should be indicated by placing the letters "S," "T," or "D" after the classification number, depending on whether the deficiency is in "soil," "topography," or "drainage." The boundary between the different classes should be delineated with a solid line and the boundary between areas placed in the same class for different reasons should be a broken line.

TABLE 2.—HYPOTHETICAL EXAMPLE, LAND CLASSIFICATION; AREAS IN ACRES (SEE FIG. 1)


1
2
3
4
5
6
7
8
9
10
11
12
13
14
6S
3ST
3S
6ST
2SD
3D
6D
2S
2T
6T
3T
2ST
3R
6R
2SR
2D
60" Sand
24" to Δ
10" to Δ
38" to Δ
36" to C
40" to C
48" +
60" +
50" +
30" to C
38" to C
30" to Δ
12" Sand to G
S.L. 26" to Δ
36" L.S. to Δ
30" L.S.
26" to C
29" to C
20" to C
890
880
870
860
800
790
780


6S
16" to C
28" to C
30" to C
2S
3S
3S
35" to Δ
2S
3R
6R
2SR
1
2SR
28" to Δ
6R
3R
2S
3R
3R
6R
1
1
1
15
16
17
18
19
20
21
22
23
24
84
86
88
790
92
94
96
98
800
02
04
06
08
810
0
500
1 000
1 500
Scale in Feet

Class	Southeast Quarter Section				Southwest Quarter Section				Northwest Quarter Section				Northeast Quarter Section				Total
40's	SE	SW	NW	NE	SE	SW	NW	NE	SE	SW	NW	NE	SE	SW	NW	NE	
1	40.0	15.6	0.9	31.7	.....	.....	.....	.....	2.2	14.2	37.0	16.2	3.8	.....	.....	.....	161.6
2		5.9	16.1	6.4	20.1	36.6	3.2	25.1	28.0	11.0	3.0	15.1	11.9	21.2	30.5	6.4	240.5
3		3.1	9.6	.....	16.0	3.4	17.2	13.8	4.7	9.9	.....	2.1	21.5	10.2	4.9	19.2	135.6
6		15.4	13.4	1.9	3.9	.....	19.6	1.1	5.1	4.9	.....	6.6	2.8	8.6	4.6	14.4	102.3
1–3	(40.0)	(24.6)	(26.6)	(38.1)	(36.1)	(40.0)	(20.4)	(38.9)	(34.9)	(35.1)	(40.0)	(33.4)	(37.2)	(31.4)	(35.4)	(25.6)	(537.7)
Total	40.0	40.0	40.0	40.0	40.0	40.0	40.0	40.0	40.0	40.0	40.0	40.0	40.0	40.0	40.0	40.0	640.0

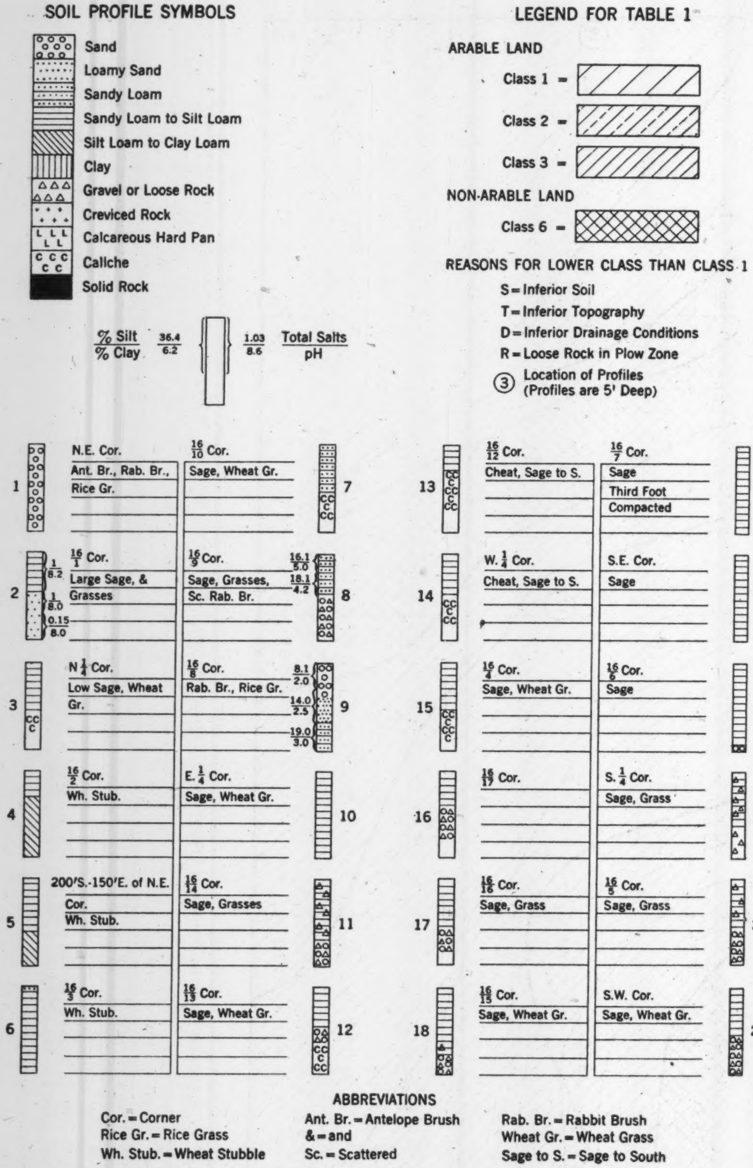

SOIL PROFILE SYMBOLS
Sand
Loamy Sand
Sandy Loam
Sandy Loam to Silt Loam
Silt Loam to Clay Loam
Clay
Gravel or Loose Rock
Creviced Rock
Calcareous Hard Pan
Caliche
Solid Rock
% Silt / % Clay
36.4 / 6.2
1.03 / 8.6
Total Salts / pH
LEGEND FOR TABLE 1
ARABLE LAND
Class 1 =
Class 2 =
Class 3 =
NON-ARABLE LAND
Class 6 =
REASONS FOR LOWER CLASS THAN CLASS 1
S = Inferior Soil
T = Inferior Topography
D = Inferior Drainage Conditions
R = Loose Rock in Plow Zone
③ Location of Profiles (Profiles are 5' Deep)
1 N.E. Cor. Ant. Br., Rab. Br., Rice Gr.
7 16/10 Cor. Sage, Wheat Gr.
2 16/1 Cor. Large Sage, & Grasses
1/8.2
1/8.0
0.15/8.0
8 16/9 Cor. Sage, Grasses, Sc. Rab. Br.
16.1/5.0
18.1/4.2
3 N 1/4 Cor. Low Sage, Wheat Gr.
9 16/8 Cor. Rab. Br., Rice Gr.
8.1/2.0
14.0/2.5
19.0/3.0
4 16/2 Cor. Wh. Stub.
10 E. 1/4 Cor. Sage, Wheat Gr.
5 200'S.-150'E. of N.E. Cor. Wh. Stub.
11 16/14 Cor. Sage, Grasses
6 16/3 Cor. Wh. Stub.
12 16/13 Cor. Sage, Wheat Gr.
13 16/12 Cor. Cheat, Sage to S.
19 16/7 Cor. Sage Third Foot Compacted
14 W. 1/4 Cor. Cheat, Sage to S.
20 S.E. Cor. Sage
15 16/4 Cor. Sage, Wheat Gr.
21 16/6 Cor. Sage
16 16/17 Cor.
22 S. 1/4 Cor. Sage, Grass
17 16/16 Cor. Sage, Grass
23 16/5 Cor. Sage, Grass
18 16/15 Cor. Sage, Wheat Gr.
24 S.W. Cor. Sage, Wheat Gr.
ABBREVIATIONS
Cor. = Corner
Rice Gr. = Rice Grass
Wh. Stub. = Wheat Stubble
Ant. Br. = Antelope Brush
& = and
Sc. = Scattered
Rab. Br. = Rabbit Brush
Wheat Gr. = Wheat Grass
Sage to S. = Sage to South

Fig. 1.—Hypothetical Profiles, Illustrating Classes and Sub-Classes for Principal Soil and Topographic Conditions of Project (See Table 2)

classes on this map proportional to the project areas, the extent of the more inferior types being relatively large on the hypothetical map in order that the maximum number of different conditions encountered in the land classification might be illustrated.

It will be noted that the results of twenty-four soil-profile examinations are recorded in Fig. 1. Conventional symbols, as illustrated in the upper left-hand corner, are used to indicate the soil texture and nature of materials encountered in the 5-ft profile examined. The location of each profile in Fig. 1 is indicated on the map of Table 2 by the number of the profile in a circle located on the map at a point corresponding to the place where the profile examination was made in the field. At twenty-eight other locations the soil depth to various materials is indicated directly on the map as, for instance, in the south center of the NE¼NW¼ where the small "square" appears, representing the location of a place where the depth was indicated as 35 in. to gravel. At such locations less detailed soil-profile examinations are made than at the regularly numbered profiles. These extra examinations are necessary in many cases in order to delineate properly the boundaries between classes.

It will be noted that in the northwest part of the section the land is class 1. Profiles 4 and 5 and the two "spot holes" near the center of the northwest quarter section (Fig. 1) show that this is a deep soil of medium texture, and the topography, although perhaps not ideal, is very good for irrigation purposes. Also, there is no indication of future drainage troubles. In the vicinity of profile 6, the classification is marked "2T," which indicates that, although the soil is good enough for class 1, the land has been placed in class 2 because the slope is somewhat steeper and the direction of fall is not uniform. The strip of land below profile 6, marked "6T," is in excess of 15% in general gradient and is being classified as non-arable on this account, in spite of the fact that the soil is of high quality. The classification of the more moderate slope below is "3T" and where the slope is still more favorable it changes to "2T." Farther south, in the W½ of the SW¼, the topography and drainage conditions are sufficiently favorable for classification as class 1, but there are soil deficiencies, as indicated by the letter "S" following the classification numbers. The classification in the NW¼SW¼ is class 3 and class 6 because of limited depth of soil above caliche (a slightly penetrable calcareous hardpan). The land in class 6 is too shallow for safe utility as irrigated land, and the land in class 3 is deep enough to be arable but has distinct limitations because of shallow soil depth.

Crossing the shallow "draw" and divide, in the S½ of the SW¼, a low terrace is encountered that is underlain by clear sand and gravel. This is classed as "2S" because of the limited depth of good soil above this loose material, which condition is indicated by profiles 17 and 24, Fig. 1. The areas to the east marked class 3R and class 6R, respectively, are made up of soils as desirable as the 2S area except for the presence of cobble and boulders in the plow zone, which has resulted in the least seriously affected part being classed as arable (class 3) and the remainder, where the boulders are too large and too thick to be removed economically to permit cultivation, being classed as non-arable (class 6). The narrow strip of "2SR" was so classed both because of limited soil depth above gravel and because of some boulders in the plow zone which will need to be removed to permit cultivation. An area of very fine soil and excellent topography occurs in the southeastern part of the section, which is in class 1.

Another set of conditions is encountered in the N½ of the NE¼. The class 6S area in the eastern part has topography that would permit irrigation, but this land is classed as non-arable because the soil is a loose sand. The area just to the south, which is marked "3ST," is in the lowest arable class for both "soil" and "topographic" reasons, the soil being sand and loamy sand, with a sandy loam subsoil, and the surface being rough enough to require a rather large amount of leveling. The topography in this area differs from that in the "3T" area, previously described, by having a rough surface of moderate gradient, whereas the first area was smooth but steep. The area of class 3S to the west differs from the class 2 land which it joins on the south only in being of more limited depth above gravel and sand. The rather steep slope marked "6ST" is non-arable both because of shallow soil and unfavorable topography.

The area immediately to the west of the last-described land illustrates the classification of three types of drainage conditions. This entire area meets the soil specifications for class 2. It is anticipated, however, that seepage will develop after the adjoining higher lands are placed under irrigation, which will be more serious in the class 6 and class 3 areas. It certainly would not be economical to build a drainage ditch unless it is necessary for the class 2 area, and in this case it undoubtedly would not be deep enough to protect the class 6 land, and the class 3 land probably would have to be used for pasture. The area of class 2S at the N½ corner has a slope that would facilitate drainage and that is sufficiently smooth and low enough in gradient to be placed in class 1, but it is placed in class 2 because of the limited depth above caliche.

The more general types of land classification, as described, provide the basis for the determination of irrigable acreage and, as far as land is concerned, the basis for project planning prior to the construction stage. The more detailed type, as described for the Columbia Basin, provides information necessary for detailed project planning, for the allocation of construction assessments, and for land appraisal as required by federal laws pertaining to irrigation projects. Much use is being made of the land-classification maps on the Columbia Basin project by land purchasers who are rapidly learning that the information is available. The information provided is expected to become increasingly useful as the different blocks are placed under irrigation.

CONCLUSIONS

The system of land classification described, including, as it does, the elimination of inferior (non-arable) lands from those to be irrigated and the division of the "arable" lands into classes on the basis of physical characteristics which determine utility for farming under irrigation, is unavoidably subject to errors in judgment. Every effort is made, however, to hold such errors to a minimum by basing the standards or specifications for each class on demonstrated experience on operating irrigation projects.

Much of the strength of the method lies in its singleness of purpose but a method including a larger number of classes would provide a better basis for determining suitable crops and farm practices. The work is made more useful in the latter respects than would be the case otherwise by the inclusion of a record of soil profile examinations on each land classification sheet.

The determination of the proper point of division between "arable" and "non-arable" land is not easy and is subject to much difference of opinion. Obviously the limitation of the irrigable area of an irrigation project to "ideal" land would not be to the public interest unless the water supply were only sufficient for such an area. On the other hand, the inclusion of land in the irrigable area which will not produce sufficiently well to maintain a farm family, results only in loss to the settler and to the construction agency. It appears generally safe to include, in the irrigable area, lands of a type that have remained in continued production on older projects and to exclude lands similar to those on developed projects which have been abandoned after years of trial.

DISCUSSION

R. Earl Storie,[2] Esq.—In discussing this paper it can be stated that soil technologists are extremely pleased and happy that engineers and economists are getting down to the fundamentals of land classification—the soil. As Mr. Johnston has stated, the ability to produce depends to a large degree on the quality of the soil.

Many world travelers who have studied the historical development of irrigated agriculture have questioned its permanency. Was this decadence due to wars, the loss of soil fertility, a general lack of capacity of the land to produce, or what? An inadequate inventory of the land resources before development took place may have been responsible for much of this decadence. This should be a real challenge to those working in the land-classification field. There is more arable land in San Diego County, California, than water, and it is extremely important to select areas of land that can be benefited by, and can pay for, the water. Agriculturalists are very happy that Mr. Johnston and his associates are going into the classification of land so thoroughly, because certainly there is no question but that this work should be done in advance of any construction.

Consider the question of the so-called specialized soils—those that may produce a very few specialized crops. Often these specialized soils will not pay the carrying charges, and soil specialists certainly agree with the land classifiers of the Bureau of Reclamation that the soils which will grow many different kinds of crops are the ones that will stand the test of time. It is cause for satisfaction that more rigid standards have been set up and adhered to during the 15 years since about 1927. Had the standard of land classification been as rigid 30 years ago, the Nation certainly would have less irrigation projects that stand as tombstones to the memory of unwise irrigation planning.

To aid this work further, investigators at the University of California, in Berkeley, have been supplying the land classifiers with Natural Land Type Maps covering the various parts of the state. The natural land type is a land unit having more or less uniform, stable conditions of slope, soil profile, and soil texture, and the less stable factors of drainage, salt content, erosion, nutrient level, and micro-relief.

The soil profile is probably the one factor that is least considered by land classifiers, yet it probably is the most important over a long period of time. By all means it should be very carefully considered to insure a permanent irrigated agriculture.

There seems to be some danger in using a too generalized reconnaissance soil map for land-classification purposes. In the future there will be less danger because modern soil surveys are made on a scale sufficient to show factors of soil texture, profile, slope, erosion, alkali content, drainage, etc.

Walter W. Weir,[3] Assoc. M. Am. Soc. C. E.—It was a step in the right direction when the state and federal agencies concerned with the development of new agricultural lands learned that, when the engineering problems of supplying water had been solved, there was still the question of the suitability of the land to be taken into consideration. The sequence of these two actions is of vital importance and when properly timed may avoid many disappointments.

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[2] In charge, Soil Survey, Univ. of California, Berkeley, Calif.

[3] Drainage Engr., Div. of Soils, Univ. of California, Berkeley, Calif.

It is common knowledge that water has been made available to some lands that could not be profitably farmed even though water were free. Although it is not a recent discovery of the Bureau of Reclamation that the inherent quality of the soil is of major importance, the Bureau is not entirely without fault even in some of its later projects. As long ago as 1925 the writer was employed to make a soil classification of a Reclamation Service project as an aid in making adjustments in water charges which had been found to be out of line with the producing power of the soil.

Considerable advancement has been made in the land-classification concept since the passage of the "Fact-Finders Act" in 1924, and of course that is as it should be. Mr. Johnston has presented the results of some 18 years of Bureau of Reclamation experience in attempting to arrive at a sound basis for land classification.

The author intimates that the purpose of land classifications is to determine what lands are arable (using his definition of the word "arable"), and that this information will be used in the design of the irrigation system. Unfortunately, the timing of these land-classification surveys is faulty, and even some of the recent surveys have not been completed until after construction is well under way or completed.

For many years the University of California, in Berkeley, has urged a detailed soil survey and land classification of the East Imperial Mesa in California, an area of some 225,000 acres of desert, for which water might be made available from the All-American Canal. The classification of the soils on this mesa is an example of poor timing, as it was not until after the completion of the All-American Canal and the Coachella Branch (from which branch much of the land is to be irrigated) that the Bureau of Reclamation undertook a land-classification survey. The writer is a member of the Advisory Committee on Land Classification for this area and assisted in setting up the standards of classification. These followed substantially the system described in Mr. Johnston's paper.

The Coachella Branch was constructed with capacity sufficient to irrigate nearly 200,000 acres on this mesa. This was based on a very rough reconnaissance soil survey made in 1920, which would not now qualify as a reconnaissance land classification as described by Mr. Johnston. The detailed classification completed in 1942 places the arable lands on the mesa at a substantially lower amount—in fact, so much lower that, had the information been available sooner, the construction of a canal of substantially different cross section might have resulted. Although the writer does not have the data at hand, it is probable that a somewhat similar situation may be found in the Columbia Basin area in the State of Washington.

With regard to the East Imperial Mesa survey, the Advisory Committee was in full agreement on the standards established, having in mind the particular use that was to be made of the survey, but the Committee strongly advised that a detailed soil survey be made either before or concurrently with the land classification. This suggestion has been followed; the University of California cooperated with the Bureau of Reclamation and the two surveys were made concurrently, to the advantage of both agencies.

One criticism that may be made of the land classifications as outlined in this paper is that the procedure is such that it can be followed by engineers whose training in soils may be limited. The Bureau of Reclamation personnel consists largely of men of engineering training and it has been exceedingly slow in going far afield for men trained in other branches of science. Land classification is not strictly an engineering job and, aside from the reconnaissance type of survey, should not be made by engineers.

The writer frequently has disagreed with the Bureau of Reclamation in its contention that detailed soil survey data are not directly applicable to its needs and that the Storie Index for soil rating cannot be applied generally to soils outside of California because the basic principles used in the index are profile development and texture of the surface soil.

Closer correlation in land classification can be maintained among different areas in the same or different states through the use of the Storie Index and R. Earl Storie's "Natural Land Types" than by any method with which the writer is familiar; but they must be applied by men familiar with soil. Land classification, made for whatever purpose, is only a special adaptation of soil survey information.

Mr. Johnston's statement that soil characteristics vary so greatly that it is difficult to make a uniform application as a guide to irrigability, and his comparison of conditions in the Sacramento Valley and eastern Montana, will hardly support a contention that detailed soil surveys are not the most logical basis for land classification. This contention may have some merit if the classifiers are not soil trained, but it certainly is not true if a little closer coordination could be secured between the engineer and the soil technician.

It has been found in the work of Mr. Storie and the writer at the University of California that, the more detailed the soil survey, the more accurately a land classification can be made. Their most severe criticism has come from attempts to make classifications in the absence of up-to-date soil data. By up-to-date soil data is meant a detailed soil survey which not only delineates and describes the soil profile in terms of its development and texture of the various horizons and strata, but also contains information on alkali, drainage, water-holding capacity, slope, and erosion. Many of the published soil survey reports are admittedly not satisfactory for the purpose, and only the most recent surveys are adequate. This condition is only to be expected, however, as soil surveying as well as land classification has developed rather rapidly, from a modest beginning, into a science.

The writer can agree with Mr. Johnston in his statement that future drainage requirements on new irrigation projects are difficult and often impossible to determine in advance. He has never fully agreed with some persons (not engineers) who have advocated the construction of drainage works at the same time as the construction of irrigation works. On the other hand, a knowledge of the soil, its age, or profile development, mode of formation, parent material, and other features that may be furnished by a soil survey are a material aid in predicting what lands may be subject to seepage, waterlogging, or salt accumulation, and their placement in the proper land-classification group.

W. L. POWERS,[4] ESQ.—The land-classification work in which Mr. Johnston and the writer were so closely associated before and after the enactment of the 1924 amendment to the Federal Reclamation Act has been developed to a high state of precision and usefulness. Members of the five-man board of review of class 3s and class 3d lands had an opportunity to test the accuracy and value of the Columbia Basin project classification in July, 1941. A similar procedure may be used in a soil survey and classification of irrigation-project lands to be undertaken for the Venezuelan Department of Public Works beginning approximately September 15, 1942.

The usable water capacity of the soil is perhaps its most important physical property in land-classification work for irrigation. The water capacity of the soil profile used by plant roots seems to be a more useful criterion than definite limitations as to soil depth. In successful projects with surface irrigation in Northwestern United States, this should be at least 4 acre-in. Some improvement in soil profile water capacity may be effected by the use of deep-rooted, humus-building legumes and farm manures from livestock enterprises.

If irrigation is to be permanent and is to repay those financially concerned, it should be preceded by a rigid land classification. This paper agrees closely with the writer's experience.

W. W. JOHNSTON,[5] ESQ.—According to Mr. Weir, a soil survey is a better basis for the purpose defined by the paper than a land classification as made by the Bureau of Reclamation. There is no rivalry between soil surveys and land classifications as defined by this paper. They are two different things. The soil survey is probably one of the most valuable pieces of work per unit of cost that is made in land studies; but, since it is made for a large variety of uses, it provides reliable data for its direct adaptation to a "land classification" only when there are occasional changes in soils and where topography and drainage conditions do not enter importantly into the problem. In lands remaining for consideration in planning an irrigation project, uniformity of this kind is rare, and in old irrigation projects the spotted occurrence of alkaline and seeped areas likewise requires that each unit of land be classified directly in the field where the importance of seepage and topography can be considered along with differences in soil.

Many attempts have been made by the writer to adapt soil surveys to land-classification uses under irrigation but these have not proved to be reliable except for the requirements of a general reconnaissance. There are several reasons—first, soil-survey maps generally are made on so small a scale that the acreage of lands in each class cannot be computed with a sufficient degree of accuracy; second, topography is considered only in a general way; third, drainage and alkali characteristics are not considered in sufficient detail;

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[4] Soil Scientist in Chg., Oregon Agri. College and Experiment Station, Oregon State College, Corvallis, Ore.

[5] War Dog Reception and Training Center, San Carlos, Calif.

and fourth (perhaps most important), the men who make soil surveys generally are selected because of their technical knowledge of soils and usually do not have sufficient knowledge of or interest in the problems either of the engineer or of irrigation to delineate properly between lands of varying utility for irrigation farming. In other words, to classify land properly for irrigation farming, the classifiers must be "irrigation men" as well as "soils men." As was indicated in the paper, much of the strength of the method discussed lies in its singleness of purpose. The problem is too important to be solved by a secondary manipulation of a general study.

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TRANSACTIONS

Paper No. 2192

MOMENT BALANCE: A SELF-CHECKING ANALYSIS OF RIGIDLY JOINTED FRAMES

BY R. J. CORNISH,[1] ESQ.

WITH DISCUSSION BY MESSRS. D. D. MATTHEWS, WILLIAM A. LARSEN, A. A. EREMIN, L. E. GRINTER, B. J. ALECK, FREDERICK S. MERRITT, RALPH W. STEWART, G. W. STOKES, BRUCE JAMEYSON, R. E. BOWLES, WILLIAM MORRIS, JOHN MASON, AND R. J. CORNISH.

SYNOPSIS

The paper describes a method of analysis that has two main features: (*a*) If the end moments have been estimated previously, the estimate can be adjusted rapidly (a preliminary estimate is not essential, however); and (*b*) the arithmetical work is almost entirely self-checking.

1. NOTATION

The letter symbols in this paper are defined where they first appear and are assembled for convenience of reference in the Appendix.

2. THEORY

The method depends on an easily established relationship between the end moments in the members meeting at a rigid joint. In Fig. 1, four members of uniform moment of inertia are shown meeting at point O. The stiffness K_{ON} is equal to $\frac{I_{ON}}{L_{ON}}$, etc.

The slope-deflection relationships give rise to the following expressions for such members:

$$6\,E\,K_{ON}\,\phi_O = 2\,M_{ON} - M_{NO} + X_{ON} \qquad (1a)$$

$$6\,E\,K_{OP}\,\phi_O = 2\,M_{OP} - M_{PO} + X_{OP} \qquad (1b)$$

$$6\,E\,K_{OQ}\,\phi_O = 2\,M_{OQ} - M_{QO} + X_{OQ} \qquad (1c)$$

and

$$6\,E\,K_{OR}\,\phi_O = 2\,M_{OR} - M_{RO} + X_{OR} \qquad (1d)$$

in which E = modulus of elasticity; ϕ_O = the slope of the members at point O;

NOTE.—Published in May, 1942, *Proceedings*.

[1] Dept. of Municipal Eng., College of Technology, Manchester, England.

M_{ON} = external bending moment at end O of member ON (etc.); and X = a term that usually appears in the slope-deflection equation as

$$X = \frac{6\,A}{L^2}\,(L - \bar{x})\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The subscript *ON* (etc.) denotes, as before, the end O of member ON. When the moment of the transverse load about the joint under consideration is clockwise, X is positive. In Eq. 2, A is the area of a "simply supported" bending moment diagram; $\bar{x}$ is the distance of its center of area from point O; and L = the effective length of the member.

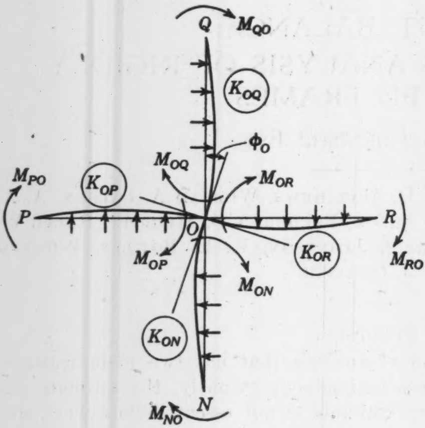

Q
M_{QO}
K_{OQ}
ϕ_O
M_{OQ}
M_{OR}
M_{PO}
K_{OP}
P
R
O
M_{OP}
K_{OR}
M_{RO}
M_{ON}
K_{ON}
N
M_{NO}

FIG. 1

Addition of Eqs. 1 shows that

$$6\,E\,(\Sigma K)\,\phi_O = -\Sigma M' + \Sigma X\,.\,(3)$$

since

$$M_{ON} + M_{OP} + M_{OQ} + M_{OR} = 0\ldots\ldots(4)$$

Then, by substitution for $6\,E\,\phi_O$ from Eq. 3 in Eq. 1*a*,

$$M_{ON} = \frac{1}{2}\left[M_{NO} - X_{ON} + \frac{K_{ON}}{\Sigma K}\,(\Sigma X - \Sigma M')\right]..(5)$$

or, in general,

$$M = \tfrac{1}{2}\,(M' - X + \kappa)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

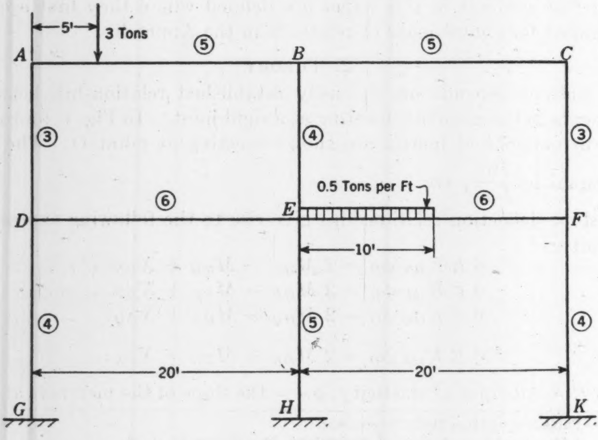

5'
3 Tons
A
B
C
⑤
⑤
③
④
③
0.5 Tons per Ft
⑥
⑥
D
E
F
10'
④
⑤
④
20'
20'
G
H
K

FIG. 2

in which, for simplicity,

$$\kappa = \frac{K}{\Sigma K}(\Sigma X - \Sigma M') \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

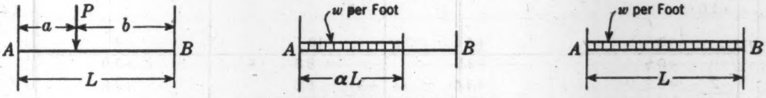
In Eq. 6, M is the external moment acting on the member at O, the joint under consideration, and M' is the moment at the outer end; all external moments are considered positive when clockwise.

The usual assumption is made that the frame can be represented by a line diagram. The prevention of linear displacements means that sidesway and longitudinal strains in the members are neglected, the problem of sidesway being discussed subsequently as a separate refinement.

3. Example

The use of Eq. 6 in frame analysis is best illustrated by an example. Fig. 2 shows a simple frame, carrying a concentrated load of 3 tons on the beam AB and a distributed load of 0.5 ton per ft on the left half of beam EF. The column ends, G, H, and K, are fixed vertically; and the stiffnesses of the various members are shown in the circles.

TABLE 1.—Values of X and the Fixed-End Moment M_F



UNIFORMLY DISTRIBUTED LOAD:
(1) Single Concentrated Load (2) On Part Span; and (3) On Entire Span

No.	A	X_{AB}	X_{BA}	$M_{F(AB)}$	$M_{F(BA)}$
1	$\frac{1}{2} P a b$	$\frac{P a b}{L^2}(L + b)$	$-\frac{P a b}{L^2}(L + a)$	$-\frac{P a b^2}{L^2}$	$\frac{P a^2 b}{L^2}$
2	$\frac{w \alpha^3 L^3}{12} \times (3 - 2\alpha)$	$\frac{w \alpha^2 L^2}{4} \times (4 - 4\alpha + \alpha^2)$	$-\frac{w \alpha^2 L^2}{4} \times (2 - \alpha^2)$	$-\frac{w \alpha^2 L^2}{12} \times (6 - 8\alpha + 3\alpha^2)$	$\frac{w \alpha^3 L^2}{12} \times (4 - 3\alpha)$
3	$\frac{1}{12} w L^3$	$\frac{1}{4} w L^2$	$-\frac{1}{4} w L^2$	$-\frac{1}{12} w L^2$	$\frac{1}{12} w L^2$

Values of X and the Fixed-End Moment M_F.—These values can be obtained at once with the aid of Table 1, and are as follows:

X_{AB} = 19.7 ft-tons; X_{BA} = − 14.1 ft-tons; X_{EF} = 28.1 ft-tons; and

X_{FE} = − 21.9 ft-tons.

$M_{F(AB)}$ = − 8.4 ft-tons; $M_{F(BA)}$ = 2.8 ft-tons; $M_{F(EF)}$ = − 11.5 ft-tons; and

$M_{F(FE)}$ = 5.2 ft-tons.

Values of X_{AB} (Table 1) are used when balancing joint A, and X_{BA} when balancing joint B. As a check on the foregoing, note that, in Table 1,

$$X_{AB} - X_{BA} = \frac{6\,A}{L} \quad \text{. . . . .} \quad (8a)$$

and

$$M_{F(BA)} - M_{F(AB)} = \frac{2\,A}{L} \quad \text{. . . . .} \quad (8b)$$

Moment Balance.—The next step is to estimate the end moments at the joints. A good designer, guided by the calculated fixed-end moments, can make a very close estimate, and in Fig. 3 the initial estimates have been underlined. The moments are balanced in Table 2, successive approximations being entered in Fig. 3 below the initial (underlined) estimates.

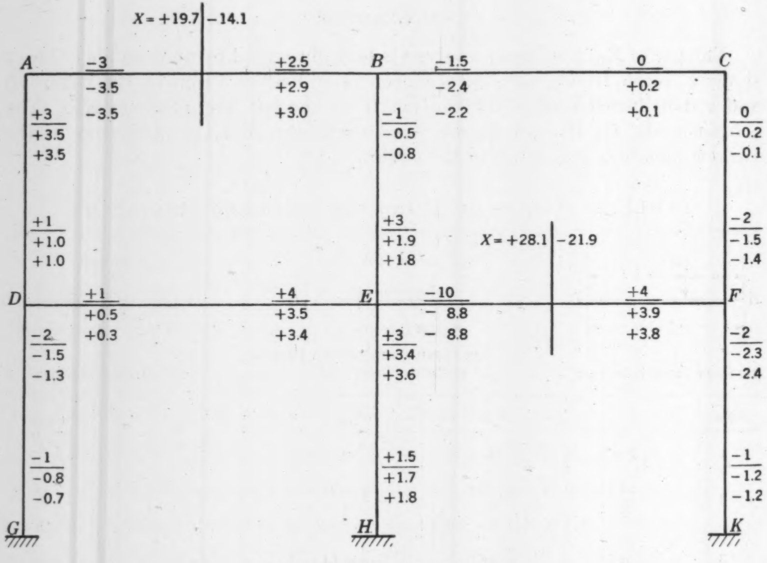

X = +19.7
-14.1
A
-3
-3.5
-3.5
+2.5
+2.9
+3.0
B
-1.5
-2.4
-2.2
0
+0.2
+0.1
C
+3
+3.5
+3.5
-1
-0.5
-0.8
0
-0.2
-0.1
+1
+1.0
+1.0
+3
+1.9
+1.8
X = +28.1
-21.9
-2
-1.5
-1.4
D
+1
+0.5
+0.3
+4
+3.5
+3.4
E
-10
- 8.8
- 8.8
+4
+3.9
+3.8
F
-2
-1.5
-1.3
+3
+3.4
+3.6
-2
-2.3
-2.4
-1
-0.8
-0.7
+1.5
+1.7
+1.8
-1
-1.2
-1.2
G
H
K

FIG. 3 (Initial Estimates Underlined)

In the table there is a column for each member meeting at a joint, and for each joint there is a "Σ" column also. Lines 1 to 5 show the first balance and lines 6 to 10 the second. Joint A has been considered first. In line 1, $-X_{AD}$ is zero, $-X_{AB}$ is -19.7, therefore $-\Sigma X$ is -19.7. The estimated values $(+1, +2.5)$ of M_{DA}, M_{BA} are entered in line 2, giving $\Sigma M' = +3.5$ and $\Sigma X - \Sigma M' = +19.7 - 3.5 = +16.2$. This value is distributed (line 3 and Eq. 7) between the members in proportion to their stiffnesses. (There is no object in computing distribution factors. It is only necessary to set ΣK $(= 8)$ under $\Sigma X - \Sigma M'$ $(= 16.2)$ on the top scale of the slide rule, and

TABLE 2.—EXAMPLE OF COMPUTATION SHEET

No.	Quantity	**Joint A**			**Joint B**				**Joint C**		
		Member AD	Member AB	Σ	Member BA	Member BE	Member BC	Σ	Member CB	Member CF	Σ
..	K	3	5	8	5	4	5	14	5	3	8
(*a*) **First Balance**											
1	$-X$	0	−19.7	−19.7	+14.1	0	0	+14.1	0	0	0
2	M'	+1	+ 2.5	+ 3.5	− 3.5	+3	0	− 0.5	−2.4	−2	−4.4
3	κ	+6.1	+10.1	+16.2	− 4.8	−4.0	−4.8	−13.6	+2.8	+1.6	+4.4
4	$2M$	+7.1	− 7.1	....	+ 5.8	−1.0	−4.8	....	+0.4	−0.4	....
5	M	+3.5	− 3.5	....	+ 2.9	−0.5	−2.4	....	+0.2	−0.2	....
(*b*) **Second Balance**											
6	$-X$	0	−19.7	−19.7	+14.1	0	0	+14.1	0	0	0
7	M'	+1.0	+ 2.9	+ 3.9	− 3.5	+1.9	+0.2	− 1.4	−2.2	−1.5	−3.7
8	κ	+6.0	+ 9.8	+15.8	− 4.6	−3.6	−4.6	−12.7	+2.3	+1.4	+3.7
9	$2M$	+7.0	− 7.0	....	+ 6.0	−1.7	−4.4	....	+0.1	−0.1	....
10	M	+3.5	− 3.5	....	+ 3.0	−0.8	−2.2	....	+0.1	−0.1	....

TABLE 2.—(*Continued*)

No.	Quantity	**Joint D**				**Joint E**					**Joint F**			
		Member DA	Member DG	Member DE	Σ	Member ED	Member EB	Member EH	Member EF	Σ	Member FE	Member FC	Member FK	Σ
..	K	3	4	6	13	6	4	5	6	21	6	3	4	13
(*a*) **First Balance**														
1	$-X$	0	0	0	0	0	0	0	−28.1	−28.1	+21.9	0	0	+21.9
2	M'	+3.5	−1	+4	+6.5	+0.5	−0.5	+1.5	+ 4	+ 5.5	− 8.8	−0.2	−1	−10.0
3	κ	−1.5	−2.0	−3.0	−6.5	+6.5	+4.3	+5.3	+ 6.5	+22.6	− 5.5	−2.8	−3.6	−11.9
4	$2M$	+2.0	−3.0	+1.0	....	+7.0	+3.8	+6.8	−17.6	....	+ 7.8	−3.0	−4.6	....
5	M	+1.0	−1.5	+0.5	....	+3.5	+1.9	+3.4	− 8.8	....	+ 3.9	−1.5	−2.3	....
(*b*) **Second Balance**														
6	$-X$	0	0	0	0	0	0	0	−28.1	−28.1	+21.9	0	0	+21.9
7	M'	+3.5	−0.8	+3.5	+6.2	+0.3	−0.8	+1.7	+ 3.9	+ 5.1	− 8.8	−0.1	−1.2	−10.1
8	κ	−1.4	−1.9	−2.9	−6.2	+6.6	+4.4	+5.5	+ 6.6	+23.0	− 5.5	−2.7	−3.6	−11.8
9	$2M$	+2.1	−2.7	+0.6	....	+6.9	+3.6	+7.2	−17.6	....	+ 7.6	−2.8	−4.8	....
10	M	+1.0	−1.3	+0.3	....	+3.4	+1.8	+3.6	− 8.8	....	+ 3.8	−1.4	−2.4	....

the distributed moments are read over $K = 3$ and $K = 5$.) Lines 1, 2, and 3 are added to give $2M$ (line 4), and the values of M in line 5 are entered in Fig. 3. This completes the first balance of joint A.

For the first balance of joint B, the revised estimate -3.5 is used for M_{AB}; M_{EB} and M_{CB} have been estimated at $+3$ and 0, respectively; and the work proceeds as before. Joints C, D, E, and F are balanced in turn. Since DG, EH, and FK are direction-fixed at G, H, and K, the estimates of M_{GD}, M_{HE}, and M_{KF} are revised by dividing each fresh estimate of M_{DG}, M_{EH}, and M_{FK} by two.

As the original estimates were fairly close to the actual, the first balance gave results precise enough for practical purposes. However, a second balance of all the joints has been made in this example.

A few points may be noted:

(1) The accuracy of the original estimate does not affect the accuracy of the result. In the example, if the moments at all joints are initially assumed to be zero, results of the same order as those of Fig. 3 can be obtained with two balances.

(2) The fact that, to a large extent, the work is self-checking is evident. It is easy to check whether the final values of M (lines 5 and 10) at each joint add up to zero, and even if an error is made it is eliminated in subsequent balances.

(3) Although, for convenience of demonstration, the joints were balanced in the order A, B, C, D, E, F, it is probably best to start the first balance at E, and proceed in the order E, F, A, B, C, D. For the second balance some other order, which can be quickly decided by inspection, may be better.

(4) If $-\frac{1}{2}X$ and $\frac{1}{2}M'$ are used in lines 1 and 2 of Table 2 instead of $-X$ and M', line 3 will also be half as great, and the total of lines 1, 2, and 3 will be M instead of $2M$. Thus the written work for each balance can be reduced by 20% at the expense of a possible source of inaccuracy, in that one must remember each time to divide the values of M', shown in Fig. 3, by 2.

4. SUMMARY OF PROCEDURE

The basic procedure of the Method of Moment Balance may be summarized briefly as follows:

(*a*) The term X and the fixed-end moment M_F are computed for all members carrying transverse loads;

(*b*) Reasonable values are assumed for the end moments and are entered on the design sheet;

(*c*) The joints are balanced successively with the aid of Eq. 6, beginning at a member carrying a transverse load;

(*d*) When one, or more, of the members connected to a joint can be regarded as direction-fixed at the outer end, the moment must be carried over to that end. If, in Fig. 1, ON is fixed vertically at N,

$$M_{NO} = \tfrac{1}{2}(M_{ON} - X_{NO}) \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

In the case of building frames with vertical loads, X for the columns is zero, and half the moment at the top of each of the lowest columns is carried over to the base.

5. SIDESWAY

With an unsymmetrical system of vertical loads in a structure there is linear displacement of the joints as well as rotation. Except in the case of very simple frames such as the single "portal," the moments due to this displacement can be neglected, since they make very little difference to the moment at any point when all dead and live loads are taken into account.

Sidesway moments of some consequence may be caused by horizontal wind loads. The method of moment balance can be applied to such cases in a manner similar to that already described herein.

Referring to Fig. 1, if the transverse loads are omitted and the points N, P, Q, and R are given deflections Δ_N, Δ_P, Δ_Q, and Δ_R in a clockwise direction, it can be shown easily that, in general,

$$M = \frac{1}{2}\left[M' - \frac{6\,E\,K\,\Delta}{L} + \frac{K}{\Sigma K}\left(6\,E\,\Sigma\,\frac{K\,\Delta}{L} - \Sigma M'\right)\right] \;\ldots\ldots(10)$$

The analysis then follows the lines suggested by Hardy Cross and N. D. Morgan,[2] Members, Am. Soc. C. E.: "Assume as many different combinations of horizontal displacements as there are degrees of freedom of horizontal movement, and for each combination determine a set of shears corresponding to a given set of moments." This leads to the formation of as many simultaneous equations as there are floors.

The method is rather cumbersome for more than two stories, and for practical purposes another method given by Messrs. Cross and Morgan[3] provides a more rapid analysis of wind-load moments.

6. ACKNOWLEDGMENTS

The writer wishes to express his gratitude to Denis Matthews, Jun. Am. Soc. C. E., and G. W. Stokes, for their very helpful criticism and advice in the preparation of this paper.

APPENDIX

NOTATION

The following letter symbols, adopted for use in this paper, conform essentially to Standard Letter Symbols for Mechanics, Structural Engineering, and Testing Materials, prepared by a Committee of the American Standards Association with Society representation and approved by the Association in 1932:[4]

A = the area of a "simply supported" bending moment diagram due to transverse loads;

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[2] "Continuous Frames of Re-inforced Concrete," by Hardy Cross and N. D. Morgan, John Wiley & Sons, Inc., New York, N. Y., 1932, p. 228.

[3] *Ibid.*, pp. 228–233.

[4] ASA—Z10a—1932.

a = distance of a load from the left-hand support;
b = distance of a load from the right-hand support;
E = modulus of elasticity;
I = moment of inertia; second moment of the area of a cross section;
K = stiffness ratio $\frac{I}{L}$: ΣK = sum of the stiffness ratios of members meeting at a joint;
L = effective length of a member, as designated by proper subscripts;
M = external end moment of a member, designated by the proper subscripts (considered positive when clockwise):
  M' = external moment at the outer end of a member (the terms "inner" and "outer" ends are used, respectively, for the ends of a member at, and remote from, the joint under consideration);
  M_F = fixed-end moment;
P = a concentrated load;
w = uniformly distributed load per unit length;
X = a term commonly encountered in the slope deflection analysis (see Eq. 3), a subscript—ON, for example—denoting the end O of member ON; X is positive when the moment of the transverse load about the joint under consideration is clockwise;
$\bar{x}$ = distance of the center of area of a "simply supported" bending moment diagram from the "inner" end of a member (the terms "inner" and "outer" ends are defined under M');
α = proportion of the effective span L covered by a uniformly distributed load w;
Δ = deflection of the "outer" end of a member relative to the "inner" end (the terms "inner" and "outer" ends are defined under M');
κ = an arbitrary substitution factor used to simplify Table 2 (line 3); defined by Eq. 7; and
ϕ = slope of a member at any point.

DISCUSSION

D. D. Matthews,[5] **Jun. Am. Soc. C. E.**—The "moment-balance" method of analyzing statically indeterminate frames may deservedly claim attention for two reasons: First, it involves a principle that yields an interesting picture of the structural action involved. Any set of assumed joint moments is adjusted in the analysis to satisfy statics and the geometry of continuity. "Moment balance" is essentially an automatic trial and revision process. Secondly, the practical convenience in applying the method to highly indeterminate frames with negligible sidesway is sufficient to repay the time spent in mastering the method, because of the remarkable self-checking feature of the procedure. Figs. 4, 5, 6, and 7 illustrate an example of a moment-balance analysis of a three-story frame with fifteen joints. The tables in Fig. 7 differ from those in Mr. Cornish's example only in that, for ease of comparison, the columns for the first and second balances have been placed next to each other.

For example, consider joint 3, Fig. 5, as isolated in Fig. 6. The value of X in this case is $+ 81$ (or $- X = - 81$). The balanced moment, $- 19.1$ in Fig. 7, is computed from Eqs. 5 and 6, as follows: $\kappa = \frac{10}{30.5}(81 - 26.8) = + 17.8$; and $M = \frac{1}{2}(- 81 + 25 + 17.8) = - 19.1$. This is the first balance; the result of the second balance (see Fig. 7) is 19.8. The units throughout are in ton-feet, and the analysis neglects sidesway.

For a practical analysis, blanks containing the outlines of Figs. 4, 5, and 7 could be arranged on a single sheet and a new blank filled out separately for each of the load combinations that produce maximum moments.

The writer wishes to record his thanks to Hardy Cross, M. Am. Soc. C. E., for permission to include in Fig. 7 his system for designating members by self-explanatory symbols.

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[5] Lt. Comdr., R. N. V. R., Edinburgh, Scotland.

(*Figures 4 to 7 follow*)

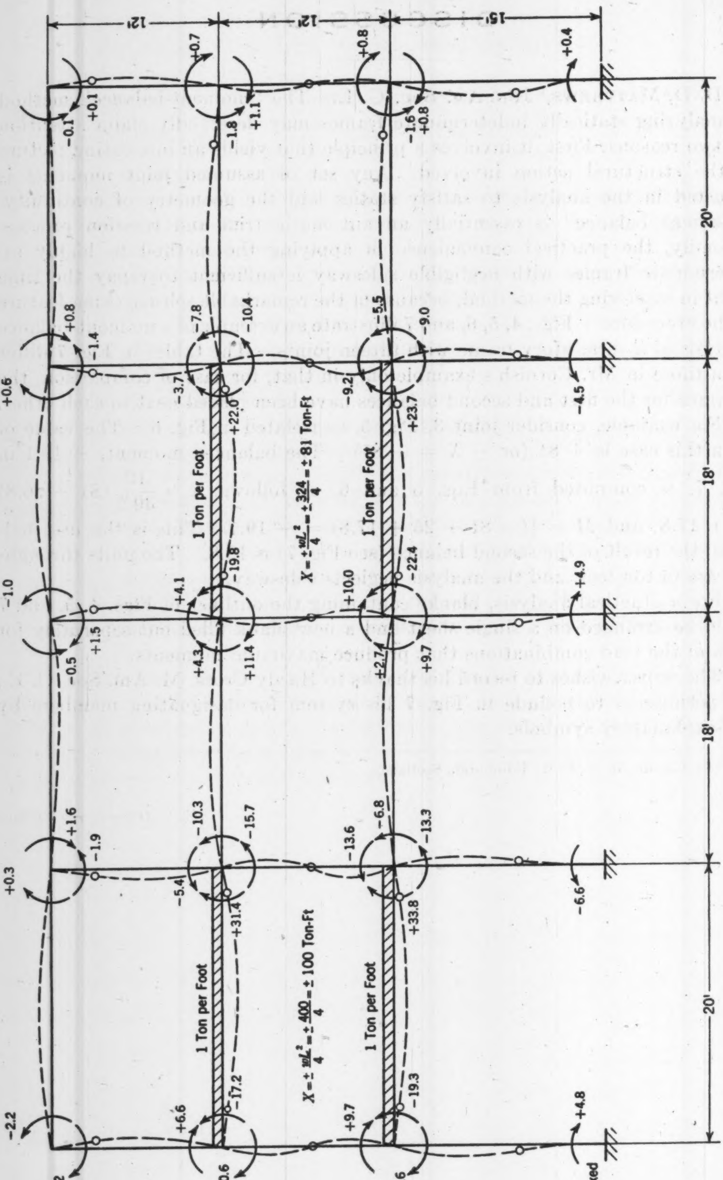

12'
12'
15'
20'
18'
18'
20'
Fixed
1 Ton per Foot
1 Ton per Foot
1 Ton per Foot
1 Ton per Foot
X = ± wL²/4 = ± 324/4 = ± 81 Ton-Ft
X = ± wL²/4 = ± 400/4 = ± 100 Ton-Ft
+0.7
+0.8
+0.4
+0.1
-0.1
+1.1
-1.8
-1.6
+0.8
+0.8
+1.4
+0.6
7.8
10.4
5.5
9.0
-3.7
+22.0
9.2
+23.0
-4.5
-1.0
+4.1
-19.8
+10.0
-22.4
+4.9
+1.5
-0.5
+4.3
+11.4
+2.7
+9.7
+1.6
+0.3
-1.9
-10.3
-15.7
-5.4
+31.4
-13.6
-6.8
-13.3
+33.8
-6.6
-2.2
+2.2
+6.6
-17.2
+10.6
+9.7
-19.3
+9.6
+4.8

FIG. 4.—DIMENSIONS, LOADS, AND FINAL MOMENTS

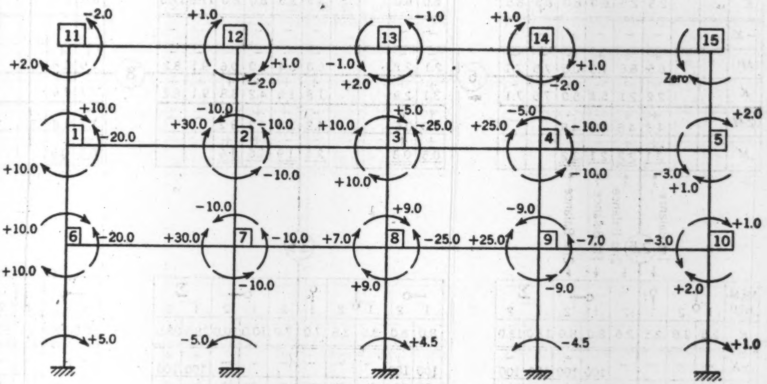

11
12
13
14
15
1
2
3
4
5
6
7
8
9
10
-2.0
+2.0
+1.0
+1.0
-2.0
-1.0
-1.0
+2.0
+1.0
+1.0
-2.0
Zero
+10.0
-20.0
+10.0
-10.0
+30.0
-10.0
-10.0
+5.0
+10.0
-25.0
+10.0
-5.0
+25.0
-10.0
-10.0
+2.0
-3.0
+1.0
+10.0
-20.0
+10.0
-10.0
+30.0
-10.0
-10.0
+9.0
+7.0
-25.0
+9.0
-9.0
+25.0
-7.0
-9.0
+1.0
-3.0
+2.0
+5.0
-5.0
+4.5
-4.5
+1.0

FIG. 5.—ASSUMED MOMENTS, AND ORDER OF BALANCING

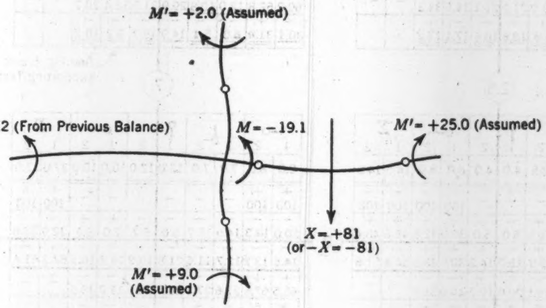
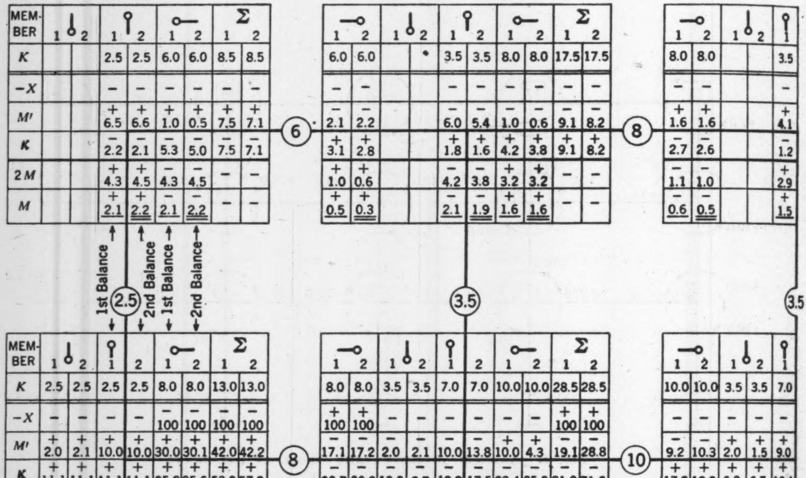

M' = +2.0 (Assumed)
M' = -9.2 (From Previous Balance)
M = -19.1
M' = +25.0 (Assumed)
X = +81
(or -X = -81)
M' = +9.0
(Assumed)

FIG. 6



MEMBER	│○ 1	│○ 2	○│ 1	○│ 2	○— 1	○— 2	Σ 1	Σ 2
K			2.5	2.5	6.0	6.0	8.5	8.5
−X			−	−	−	−	−	−
M′			+6.5	+6.6	+1.0	+0.5	+7.5	+7.1
K			−2.2	−2.1	−5.3	−5.0	−7.5	−7.1
2M			+4.3	+4.5	−4.3	−4.5	−	−
M			+2.1	+2.2	−2.1	−2.2	−	−

—○ 1	—○ 2	│○ 1	│○ 2	○│ 1	○│ 2	○— 1	○— 2	Σ 1	Σ 2
6.0	6.0			3.5	3.5	8.0	8.0	17.5	17.5
−	−			−	−	−	−	−	−
−2.1	−2.2			−6.0	−5.4	−1.0	−0.6	−9.1	−8.2
+3.1	+2.8			+1.8	+1.6	+4.2	+3.8	+9.1	+8.2
+1.0	+0.6			−4.2	−3.8	+3.2	+3.2	−	−
+0.5	+0.3			−2.1	−1.9	+1.6	+1.6	−	−

—○ 1	—○ 2	│○ 1	│○ 2	○│ 1
8.0	8.0			3.5
−	−			−
+1.6	+1.6			+4.1
−2.7	−2.6			−1.2
−1.1	−1.0			+2.9
−0.6	−0.5			+1.5

MEMBER	│○ 1	│○ 2	○│ 1	2	○— 1	○— 2	Σ 1	Σ 2
K	2.5	2.5	2.5	2.5	8.0	8.0	13.0	13.0
−X	−	−	−	−	−100	−100	−100	−100
M′	+2.0	+2.1	+10.0	+10.0	+30.0	+30.1	+42.0	+42.2
K	+11.1	+11.1	+11.1	+11.1	+35.8	+35.6	+58.0	+57.8
2M	+13.1	+13.2	+21.1	+21.1	−34.2	−34.3	−	−
M	+6.5	+6.6	+10.6	+10.6	−17.1	−17.2		

—○ 1	—○ 2	│○ 1	│○ 2	○│ 1	2	○— 1	○— 2	Σ 1	Σ 2
8.0	8.0	3.5	3.5	7.0	7.0	10.0	10.0	28.5	28.5
+100	+100	−	−	−	−	−	−	+100	+100
−17.1	−17.2	−2.0	−2.1	−10.0	−13.8	+10.0	+4.3	−19.1	−28.8
−22.7	−20.0	−10.0	−8.7	−19.9	−17.5	−28.4	−25.0	−81.0	−71.2
+60.2	+62.8	−12.0	−10.8	−29.9	−31.3	−18.4	−20.7	−	−
+30.1	+31.4	−6.0	−5.4	−14.9	−15.7	−9.2	−10.2		

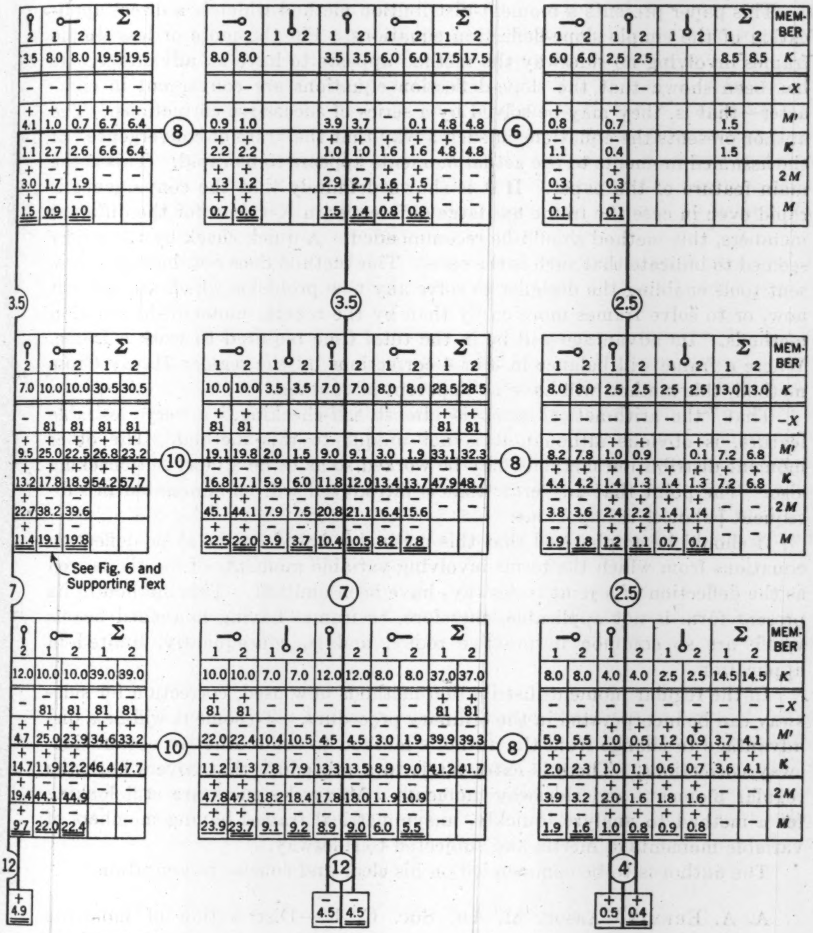
—○ 1	—○ 2	│○ 1	│○ 2	○│ 1
10.0	10.0	3.5	3.5	7.0
−	−	−	−	−
−9.2	−10.3	+2.0	+1.5	+9.0
+17.8	+18.9	+6.2	+6.7	+12.4
+8.6	+8.6	+8.2	+8.2	+21.4
+4.3	+4.3	+4.1	+4.1	+10.7

MEMBER	│○ 1	│○ 2	○│ 1	2	○— 1	○— 2	Σ 1	Σ 2
K	2.5	2.5	4.0	4.0	8.0	8.0	14.5	14.5
−X	−	−	−	−	−100	−100	−100	−100
M′	+10.6	+10.6	+5.0	+5.0	+30.0	+32.8	+45.6	+48.4
K	+9.4	+8.9	+15.0	+14.2	+30.0	+28.5	+54.4	+51.6
2M	+20.0	+19.5	+20.0	+19.2	−40.0	−38.7	−	−
M	+10.0	+9.7	+10.0	+9.6	−20.0	−19.3	−	−

—○ 1	—○ 2	│○ 1	│○ 2	○│ 1	2	○— 1	○— 2	Σ 1	Σ 2
8.0	8.0	7.0	7.0	12.0	12.0	10.0	10.0	37.0	37.0
+100	+100	−	−	−	−	−	−	+100	+100
−20.0	−19.3	−14.9	−15.7	−5.0	−6.7	+7.0	+3.1	−32.9	−38.6
−14.5	−13.2	+12.7	−11.6	−21.8	−19.9	−18.2	−16.6	−67.2	−61.4
+65.5	+67.5	−27.6	−27.3	−26.8	−26.6	−11.2	−13.5	−	−
+32.8	+33.8	−13.8	−13.6	−13.4	−13.3	−5.6	−6.8	−	−

—○ 1	—○ 2	│○ 1	│○ 2	○│ 1
10.0	10.0	7.0	7.0	12.0
−	−	−	−	−
−5.6	−6.8	+10.7	+11.4	+4.5
+11.9	+12.2	+8.3	+8.6	+14.3
+6.3	+5.4	+19.0	+20.0	+18.8
+3.1	+2.7	+9.5	+10.0	+9.4

FIG. 7.—MOMENT-BALANCE DESIGN



⫯ 2	o— 1	o— 2	Σ 1	Σ 2
3.5	8.0	8.0	19.5	19.5
−	−	−	−	−
+4.1	+1.0	+0.7	+6.7	+6.4
−1.1	−2.7	−2.6	−6.6	−6.4
+3.0	−1.7	−1.9	−	−
+1.5	−0.9	−1.0		

—o 1	—o 2	⫰ 1	⫰ 2	⫯ 1	⫯ 2	o— 1	o— 2	Σ 1	Σ 2
8.0	8.0			3.5	3.5	6.0	6.0	17.5	17.5
−	−			−	−	−	−	−	−
−0.9	−1.0			−3.9	−3.7	−	−0.1	−4.8	−4.8
+2.2	+2.2			+1.0	+1.0	+1.6	+1.6	+4.8	−4.8
+1.3	+1.2			−2.9	−2.7	+1.6	+1.5	−	−
+0.7	+0.6			−1.5	−1.4	+0.8	+0.8	−	−

—o 1	—o 2	⫯ 1	⫯ 2	⫰ 1	⫰ 2	Σ 1	Σ 2	MEMBER
6.0	6.0	2.5	2.5			8.5	8.5	*K*
−		−	−			−	−	−*X*
+0.8		+0.7				+1.5		*M′*
−1.1		−0.4				−1.5		*K*
−0.3		+0.3						2*M*
−0.1		+0.1						*M*

⫯ 2	o— 1	o— 2	Σ 1	Σ 2
7.0	10.0	10.0	30.5	30.5
−	−81	−81	−81	−81
+9.5	+25.0	+22.5	+26.8	+23.2
+13.2	+17.8	+18.9	+54.2	+57.7
+22.7	−38.2	−39.6		
+11.4	−19.1	−19.8		

—o 1	—o 2	⫰ 1	⫰ 2	⫯ 1	⫯ 2	o— 1	o— 2	Σ 1	Σ 2
10.0	10.0	3.5	3.5	7.0	7.0	8.0	8.0	28.5	28.5
+81	+81							+81	+81
−19.1	−19.8	−2.0	−1.5	−9.0	−9.1	−3.0	−1.9	−33.1	−32.3
−16.8	−17.1	−5.9	−6.0	−11.8	−12.0	−13.4	−13.7	−47.9	−48.7
+45.1	+44.1	−7.9	−7.5	−20.8	−21.1	−16.4	−15.6	−	−
+22.5	+22.0	−3.9	−3.7	−10.4	−10.5	−8.2	−7.8		

—o 1	—o 2	⫯ 1	⫯ 2	⫰ 1	⫰ 2	Σ 1	Σ 2	MEMBER
8.0	8.0	2.5	2.5	2.5	2.5	13.0	13.0	*K*
−	−	−	−	−	−	−	−	−*X*
−8.2	−7.8	+1.0	+0.9	−	+0.1	−7.2	−6.8	*M′*
+4.4	+4.2	+1.4	+1.3	+1.4	+1.3	+7.2	+6.8	*K*
−3.8	−3.6	+2.4	+2.2	+1.4	+1.4	−	−	2*M*
−1.9	−1.8	+1.2	+1.1	+0.7	+0.7	−	−	*M*

⫯ 2	o— 1	o— 2	Σ 1	Σ 2
12.0	10.0	10.0	39.0	39.0
−	−81	−81	−81	−81
+4.7	+25.0	+23.9	+34.6	+33.2
+14.7	+11.9	+12.2	46.4	47.7
+19.4	−44.1	−44.9	−	−
+9.7	−22.0	−22.4	−	−

—o 1	—o 2	⫰ 1	⫰ 2	⫯ 1	⫯ 2	o— 1	o— 2	Σ 1	Σ 2
10.0	10.0	7.0	7.0	12.0	12.0	8.0	8.0	37.0	37.0
+81	+81	−	−	−	−	−	−	+81	+81
−22.0	−22.4	−10.4	−10.5	−4.5	−4.5	−3.0	−1.9	−39.9	−39.3
−11.2	−11.3	−7.8	−7.9	−13.3	−13.5	−8.9	−9.0	−41.2	−41.7
+47.8	+47.3	−18.2	−18.4	−17.8	−18.0	−11.9	−10.9	−	−
+23.9	+23.7	−9.1	−9.2	−8.9	−9.0	−6.0	−5.5	−	−

—o 1	—o 2	⫯ 1	⫯ 2	⫰ 1	⫰ 2	Σ 1	Σ 2	MEMBER
8.0	8.0	4.0	4.0	2.5	2.5	14.5	14.5	*K*
−	−	−	−	−	−	−	−	−*X*
−5.9	−5.5	+1.0	+0.5	+1.2	+0.9	−3.7	−4.1	*M′*
+2.0	+2.3	+1.0	+1.1	+0.6	+0.7	+3.6	+4.1	*K*
−3.9	−3.2	+2.0	+1.6	+1.8	+1.6	−	−	2*M*
−1.9	−1.6	+1.0	+0.8	+0.9	+0.8	−	−	*M*

+4.9

−4.5 −4.5

+0.5 +0.4

OF A THREE-STORY FRAME

William A. Larsen,[6] Esq.—Much has been written the past few years on methods of obtaining moments in frames by a series of successive corrections. For any new presentation to be of value, a definitely more desirable feature than is found in the present methods should be given.

This paper presents a moment-distribution method which is a direct application of the simple slope-deflection equations. For the more or less simple frames involving no sidesway the method appears to have an advantage. It has been shown that the slope-deflection equations are convergent in character—that is, they may be solved by a series of successive corrections. The author presents the equations in such a way that the "rate of convergence" of the assumed moments to the actual moments appears to be rapid. This is the main feature of the paper. If it is shown definitely that the convergence is rapid even in case the frame has large differences in *K*-values for the different members, this method should be recommended. A quick check by the writer seemed to indicate that such is the case. This method does not, however, present tools enabling the designer to solve any new problems which are difficult now, or to solve frames more easily than by the regular moment-distribution methods. Its advantage will be in the total time required to work a frame. Where a frame will balance in 3 or 4 corrections by the regular Hardy Cross method, this method will have no apparent advantage.

That "the arithmetical work is almost self-checking," a very desirable feature, is obvious although it is not unique to this method. The other moment-distribution methods may be worked so as to be almost self-checking also. The point that any error that is introduced will be eliminated in subsequent balances is important.

It should be emphasized that this method is based on the slope-deflection equations from which the terms involving variable moments of inertia as well as the deflection of a joint (sidesway) have been omitted. This method in its present form is not applicable, therefore, to frames having haunched beams which are so common in practice today, and is, consequently, limited in application.

In the regular moment-distribution methods now used, correction for sidesway can be incorporated in the ordinary procedure. To some it will be a disadvantage to change from the author's method to some other to obtain sidesway corrections. There is extra merit to a procedure that covers both the regular moments and sidesway moments. Many designers are still looking for a method to analyze, quickly, unsymmetrical frames having members of variable moments of inertia and subjected to sidesway.

The author is to be commended on his clear and concise presentation.

A. A. Eremin,[7] Assoc. M. Am. Soc. C. E.—Distribution of moments in rigid frames or continuous beams is an interesting problem in the analysis of statically indeterminate structures. In the past few decades numerous analytical and graphical methods of distribution of moments have been developed. The methods are based on the same well-known principle of the deformation of an elastic member loaded with the moment area from the given loading. Therefore, the proximity of the methods is usually the same, and the merit of the methods varies with their simplicity and the saving of time and effort in their application to practice.

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[6] Associate Engr., Bureau of Reclamation, U. S. Dept. of the Interior, Denver, Colo.

[7] Associate Bridge Engr., Bridge Dept., Div. of Highways, State Dept. of Public Works, Sacramento, Calif.

The slope-deflection equations for the distribution of moments converge rapidly. Therefore, various attempts have been made to develop algebraic equations that will reduce the number of steps in the computations.

To reduce the balancing operations the author, in Eq. 7, assumes the value of M'. To select a suitable approximation for M' is not always a simple matter. Often it requires preliminary analytical computations or graphical constructions. The assumed value of M' may vary from its exact value not only in magnitude but also in sign. The operations with the assumed value of M' may take more time than the additional steps required to balance the fixed-end moments in the successive approximations.

Considerable difficulty is involved in computing the moment area and its center in Eq. 2. This is especially true in relation to the haunched beams carrying irregular loading.

The graphical method of distribution of moments developed by L. H. Nishkian and D. B. Steinman,[8] Members, Am. Soc. C. E., has similar difficulty in establishing the centers in the moment areas. Complexity in finding the center of moment area in the graphical constructions was an important reason why the graphical method could not find wide application, and preference is generally given to the algebraic methods based on the fixed-end moment distribution. It is quite possible to prepare the tables and diagrams for computing the moment areas and their centers for various shapes of members and type of loading. However, these tables and diagrams will be more complicated than those prepared for computing fixed-end moments.

Distribution of moments in rigid frames or continuous beams has an important economic value in the theory of design. Therefore, the author is to be commended for his contribution toward the improvement of designing methods.

L. E. GRINTER,[9] M. AM. SOC. C. E.—This interesting and useful paper presents a method of successive approximations for analyzing rigid frames for fixed loadings. It should be noted distinctly that the procedure is one of successive approximations rather than a method of successive corrections. In this feature, it is different from the Cross method of moment distribution which has been emphasized by the writer as a convenient tool of analysis. Nevertheless, there are several points in common between these procedures.

In the author's paper the working formula (Eq. 6) is $M = \frac{M'}{2} - \frac{X}{2} + \frac{\kappa}{2}$, which is derived from the slope-deflection equations. This may lead some readers to misunderstand the background of the method, which is not the use of slopes and deflections as redundants. This will be evident if the working formula is written out without the need for mathematical derivation by a study of the physical action of the structure in Fig. 8. The moment at the end O of the member OR may be obtained as the summation of the carry-over effects

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[8] "Moments in Restrained and Continuous Beams by the Method of Conjugate Points," by L. H. Nishkian and D. B. Steinman, *Transactions*, Am. Soc. C. E., Vol. 90 (1927), p. 1.

[9] Vice-Pres. and Dean, Graduate School, Illinois Inst. of Technology, Chicago, Ill.

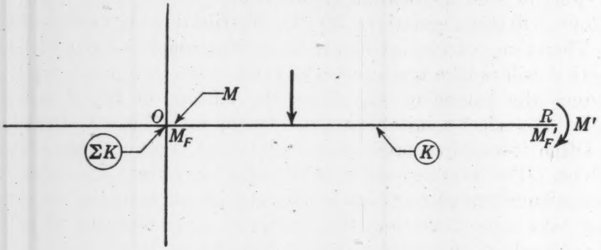

FIG. 8.—DERIVATION OF BASIC FORMULA

at that point of the moments applied at R $\left(\text{that is, } \frac{M'}{2} \text{ and } \frac{M'_F}{2}\right)$ plus the fixed-end moment at the end O plus the balancing moment at O for the member OR when the joint is allowed to rotate. Thus, without reference to sign, one may write

$$M = \left[\frac{M'}{2}\right] + \left[M_F + \frac{M'_F}{2}\right] + \left[\frac{K}{\Sigma K}\left(\frac{\Sigma M'}{2} + \Sigma M_F + \frac{\Sigma M'_F}{2}\right)\right] \quad (11)$$

In Eq. 11, the three terms enclosed in brackets will be found to correspond exactly to the author's definitions of the three terms of his Eq. 1. Signs, of course, follow automatically if some such rule is adopted as "a positive moment tends to rotate the adjacent joint clockwise."

After Eq. 6, or Eq. 11, is established, the author's procedure is simple and convenient since it permits the operator to shorten the convergence by exercising his judgment. In many problems, it is true that the final moments can be "guessed" with reasonable accuracy, and these become the first M'-values in Eq. 11. The values of M_F and M'_F, being fixed-end moments, are known exactly. Evidently, all values of M can then be computed by repeated use of Eq. 11. It must be realized, of course, that a given moment is an M-value for that point but an M'-value when the moment at the far end of the member is being computed.

This method of analysis has many points in its favor, particularly if the procedure is brought clearly into view by eliminating the unnecessary terms X and κ. On the other hand, it lacks the close association with the deformations of the structure that is so evident in the Cross procedure. For example, it is convenient and almost necessary to follow the author's procedure of placing the calculations in tabular form, rather than upon the diagram of the structure, since a term such as the author's X or κ is not an individual moment but a combination of a number of moments that are best selected from a well-organized table. The writer tried the other procedure of writing all moment terms upon an outline of the structure but found it not entirely practicable. Also, there were a great many more terms and many more calculations than are involved in an analysis by the usual method of balancing moments.

To be more useful than the regular procedures of balancing moments, this method should be preceded by a careful estimate of the moments. If the moments can be estimated so accurately that only one revision becomes necessary, the method might be faster than any other in common use; but, if the method is for use by inexperienced designers, several revisions of moments will be necessary and the procedure will be rather lengthy.

The self-correcting feature of the method deserves comment. Since this is a method of successive approximations, an error in recording a moment will be corrected in the succeeding revision. In this direction the method has merit. The check of $\Sigma M = 0$ at each joint, of course, is not an independent check because this tool is used as one part of the procedure of analysis. An error in computing a fixed-end moment, in computing K or ΣK, or in sign may not be discovered. In these computations, care must be exercised, as always.

The writer suggests that this method should be given the author's name because the title "Moment Balance" or "Balancing Moments" has been used for years as an alternate to "Moment Distribution" to designate the Cross method, or its variations, based upon successive corrections.

B. J. ALECK,[10] JUN. AM. SOC. C. E.—"Moment balance" may be extended to apply to members of variable moment of inertia and to some other special cases. Let all moments and joint rotations be considered positive when they are counterclockwise at the ends of the member. Let: $(K_f)_{AB}$ = stiffness at A when B is free to rotate; $(M_f)_{AB}$ = fixed-end moment at A when B is free to rotate; and C_{AB} = carry-over factor from A to B. Then

$$M_{AB} = (M_f)_{AB} + C_{BA}\, M_{BA} + (K_f)_{AB}\, \phi_A \ldots\ldots\ldots\ldots (12a)$$

and

$$\Sigma M_{AX} = \Sigma[(M_f)_{AX} + C_{XA} M_{XA}] + \Sigma(K_f)_{AX}\, \phi_A = 0 \ldots\ldots (12b)$$

at joint A; and

$$\phi_A = -\frac{\Sigma[(M_f)_{AX} + C_{XA} M_{XA}]}{\Sigma(K_f)_{AX}} \ldots\ldots\ldots\ldots (12c)$$

The procedure from this stage follows the outline set up by Mr. Cornish: $(M_f)_{AX}$ is computed; $C_{XA}\, M_{XA}$ is approximated and used in Eq. 12c to find ϕ_A; and ϕ_A is then used in Eq. 12a to find the moments at joint A.

It should be emphasized that members with known conditions at the far end can be represented with "modified" stiffness and fixed-end moment, in a manner identical with that used in "moment distribution." For such members,

$$M_{AB} = (M_{MF})_{AB} + (K_M)_{AB}\, \phi_A \ldots\ldots\ldots\ldots\ldots (13)$$

when $(M_{MF})_{AB}$ = modified fixed-end moment and $(K_M)_{AB}$ = modified stiffness.

Since Eq. 13 is of the same form as Eq. 12a, it may be used whenever convenient as a substitute, and relieves one of estimating M_{BA} before M_{AB}

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[10] Research Engr. (Materials and Structures), Curtis Wright, Plant 5, Buffalo, N. Y.

is known. The moments at the far end of such members are found, using the modified carry-over factors.

For common members, data compiled for use with the moment-distribution method can be modified to evaluate all the terms used in Eqs. 12 and 13. For uncommon members, they can be evaluated from the properties of the proper $\frac{M}{E\,I}$-diagram.

Mr. Cornish's method provides a check in one cycle on answers obtained by any method. To the writer, this feature alone seems sufficient reason for engineers to learn the principles of "moment balance."

FREDERICK S. MERRITT,[11] JUN. AM. SOC. C. E.—The method of analyzing continuous structures by moment balance has two main features to recommend it. First, by enabling designers to estimate final moments, considerable work may be saved. Second, its self-checking character may also save time, in that errors may be either immediately located or neutralized by further balancing. Accounting for sidesway moments is as much of a handicap in other methods as it is in this.

The method of moment balance may be applied in a somewhat different form with the advantage of ease of generalization and saving in computation. Let: α = six times the rotation at one end of a simply supported beam of unit length and least $K = \frac{I}{L} = 1$ due to a unit moment applied at that end; β = six times the rotation at one end of this beam due to a unit moment applied at the other end; K = the ratio of least moment of inertia to span length; M = actual end moment minus fixed-end moment; and ϕ = the rotation at the end of any beam. Clockwise moments will be considered positive and counterclockwise moments negative. Then, referring to Fig. 1, in general, neglecting deflection at beam ends:

$$6\,E\,K_{ON}\,\phi_O = \alpha_{ON}\,M_{ON} - \beta_{ON}\,M_{NO} \qquad (14a)$$

$$6\,E\,K_{OP}\,\phi_O = \alpha_{OP}\,M_{OP} - \beta_{OP}\,M_{PO} \qquad (14b)$$

$$6\,E\,K_{OQ}\,\phi_O = \alpha_{OQ}\,M_{OQ} - \beta_{OQ}\,M_{QO} \qquad (14c)$$

and

$$6\,E\,K_{OR}\,\phi_O = \alpha_{OR}\,M_{OR} - \beta_{OR}\,M_{RO} \qquad (14d)$$

Divide each equation by its value of α_O and let $S = \frac{K}{\alpha}$ and $C = \frac{\beta}{\alpha}$. Note that C is the carry-over factor toward O for the beam, considering it as fixed-ended. Summing Eqs. 14 yields:

$$6\,E\,\phi_O\,\Sigma S_O = \Sigma M_O - \Sigma(C\,M') \qquad (15)$$

in which M' is the change in the fixed-end moments at the far ends. Since $-\Sigma M_O = \Sigma M^F{}_O$ (the sum of the fixed-end moments at O due to loads)—

$$6\,E\,\phi_O\,\Sigma S_O = -\left[\Sigma M^F{}_O + \Sigma(C\,M')\right] \qquad (16)$$

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[11] 1st Lt., Coast Artillery Corps, U. S. A., Camp Davis, N. C.

TABLE 3.—COMPUTATION SHEET

No.	Quantity	JOINT A			JOINT B				JOINT C		
		Member AD	Member AB	Σ	Member BA	Member BE	Member BC	Σ	Member CB	Member CF	Σ
....	K	3	5	8	5	4	5	14	5	3	8
....	M^F	....	−8.4	−8.4	+2.4	....	....	+2.4	....	....	....
					(a) FIRST BALANCE						
1	$\frac{1}{2} M'$	+0.5	+0.1	+0.6	+2.5	+1.5	0	+4.0	−1.2	−1.0	−2.2
2	m	+2.9	+4.9	−7.8	−2.3	−1.8	−2.3	+6.4	+1.4	+0.8	−2.2
3	Change	+3.4	+5.0	....	+0.2	−0.3	−2.3	....	+0.2	−0.2	....
Final moment		+3.4	−3.4	....	+2.6	−0.3	−2.3	....	+0.2	−0.2	....

TABLE 3.—(*Continued*)

No.	Quantity	JOINT D				JOINT E					JOINT F			
		Member DA	Member DG	Member DE	Σ	Member ED	Member EB	Member EH	Member EF	Σ	Member FE	Member FC	Member FK	Σ
....	K	3	4	6	13	6	4	5	6	21	6	3	4	13
....	M^F	....	....	....	....	....	....	....	−11.5	−11.5	+5.2	....	....	+5.2
						(a) FIRST BALANCE								
1	$\frac{1}{2} M'$	+1.7	−0.5	+2.0	+3.2	+0.3	−0.2	+0.8	−0.6	+ 0.3	+1.3	−0.1	−0.5	+0.7
2	m	−0.7	−1.0	−1.5	+3.2	+3.2	+2.1	+2.7	+3.2	−11.2	−2.7	−1.4	−1.8	+5.9
3	Change	+1.0	−1.5	+0.5	....	+3.5	+1.9	+3.5	+2.6	....	−1.4	−1.5	−2.3	....
Final moment		+1.0	−1.5	+0.5	....	+3.5	+1.9	+3.5	−8.9	....	+3.8	−1.5	−2.3	....

Substituting Eq. 16 in Eq. 14*a*:

$$M_{ON} = C\, M_{NO} - \frac{S_{ON}}{\Sigma S_O} \left[\Sigma M^F{}_O + \Sigma (C\, M') \right] \ldots\ldots\ldots\ldots (17)$$

Using Eq. 17, the method of moment balance may be applied to continuous frames containing beams of variable moment of inertia. The final moments are obtained by adding the results of the last balance to the fixed-end moments. For beams of constant moment of inertia, Eq. 17 becomes:

$$M_{ON} = \tfrac{1}{2}\, M_{NO} - \frac{K_{ON}}{\Sigma K_O} \left(\Sigma M^F{}_O + \tfrac{1}{2}\, \Sigma M' \right) = \tfrac{1}{2}\, M_{NO} - m \ldots\ldots (18)$$

in which

$$m = \frac{K_{ON}}{\Sigma K_O} \left(\Sigma M^F{}_O + \tfrac{1}{2}\, \Sigma M' \right) \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

Using Eq. 18, the frame in Fig. 2 was analyzed for a comparison of results (see Table 3). The same changes in fixed-end moments as assumed by the author were also taken as a preliminary estimate. Joint A was balanced first. One half the change at the far ends was recorded for AD and AB and the sum placed under the sigma. This sum was then added to the fixed-end moment (in this case zero), and the result, entered in Table 3, was distributed, with sign changed, to AD and AB in proportion to their *K*-values. The change in the fixed-end moment was obtained by adding the two values entered for each member. If the final moment were desired at this stage, the addition would include the fixed-end moment, and the change in moment would not be entered. It will be noticed that at the end of the first balance results agreeing with those of the author were obtained.

The method of moment balance has the same disadvantage in practical application as other methods of analysis by converging increments—that is, in locating loads for maximum moment. Because of the unknown end conditions in interior spans, it is difficult to apply these methods to problems of moving loads.

Ralph W. Stewart,[12] M. Am. Soc. C. E.—The "Synopsis" of this paper states that the trial and revision procedure which it describes is rapid and that the arithmetical work is almost entirely self-checking. These claims are attractive and led the writer to experiment some with the author's demonstration problem shown in Fig. 2.

TABLE 4.—Comparison of Four Independent Solutions (see Fig. 2)

Description	Joint A		Joint B			Joint C	
	AD	AB	BA	BE	BC	CB	CF
Effect of fixed-end moment, AB.	−3.23	+3.23	−1.74	+0.81	+0.94	+0.19	−0.19
Effect of fixed-end moment, BA.	−0.20	+0.20	−1.83	+0.87	+0.96	+0.20	−0.20
Effect of fixed-end moment, EF.	+0.01	−0.01	+0.38	−0.77	+0.38	−0.01	+0.01
Effect of fixed-end moment, FE.	−0.03	+0.03	+0.11	−0.02	−0.09	−0.33	+0.33
Combined moment...........	−3.45	+3.45	−3.08	+0.89	+2.19	+0.05	−0.05
R. J. Cornish................	3.5	3.5	3.0	0.8	2.2	0.1	0.1
H. O. Springer...............	3.40	3.40	3.09	0.88	2.21	0.01	0.01
L. Blume....................	3.4	3.4	3.0	0.8	2.2	0.1	0.1

TABLE 4.—(*Continued*)

Description	Joint D			Joint E				Joint F		
	DA	DG	DE	ED	EB	EH	EF	FE	FC	FK
Effect of fixed-end moment, AB.	−1.29	+0.57	+0.72	+0.16	+0.27	−0.22	−0.21	−0.02	−0.05	+0.07
Effect of fixed-end moment, BA.	−0.07	+0.06	+0.01	−0.11	+0.36	−0.13	−0.11	+0.01	−0.07	+0.06
Effect of fixed-end moment, EF.	+0.34	+0.58	−0.92	−3.16	−2.18	−2.99	+8.33	−0.92	+0.34	+0.58
Effect of fixed-end moment, FE.	+0.03	+0.07	−0.10	−0.35	−0.21	−0.33	+0.89	−2.85	+1.15	+1.70
Combined moment...........	−0.99	+1.28	−0.29	−3.46	−1.76	−3.67	+8.90	−3.78	+1.37	+2.41
R. J. Cornish................	1.0	1.3	0.3	3.4	1.8	3.6	8.8	3.8	1.4	2.4
H. O. Springer...............	0.99	1.26	0.29	3.45	1.76	3.68	8.89	3.79	1.35	2.42
L. Blume....................	1.0	1.3	0.3	3.4	1.7	3.6	8.9	3.7	1.4	2.3

Table 4 shows the results of three independent solutions of this problem, as well as the final moments computed by the author. For the practical design of a building frame of this type it is necessary to consider various combinations of loading, including wind or earthquake loads. This requires that the frame be solved for the effect of a fixed-end moment at each of its joints. The first solution in Table 4 gives the effect of each individual fixed-end moment and is carried to an accuracy of 0.01 ft-ton. This first solution was performed without the use of simultaneous equations and without the use of any kind of successive approximations.

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[12] Engr. of Bridge and Structural Design, City of Los Angeles, Los Angeles, Calif.

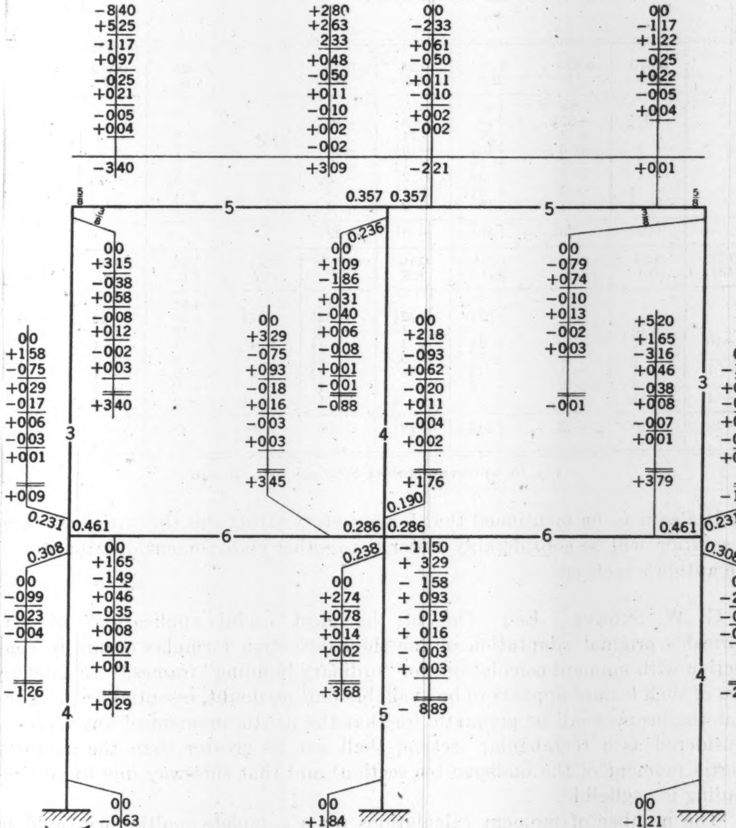
-8|40
+5|25
-1|17
+0|97
-0|25
+0|21
-0|05
+0|04
-3|40
+2|80
+2|63
2|33
+0|48
-0|50
+0|11
-0|10
+0|02
-0|02
+3|09
0|0
-2|33
+0|61
-0|50
+0|11
-0|10
+0|02
-0|02
-2|21
0|0
-1|17
+1|22
-0|25
+0|22
-0|05
+0|04
+0|01
5/8
3/8
5
0.357
0.357
5
3/8
5/8
0.236
0|0
+3|15
-0|38
+0|58
-0|08
+0|12
-0|02
+0|03
+3|40
0|0
+1|09
-1|86
+0|31
-0|40
+0|06
-0|08
+0|01
-0|01
-0|88
0|0
-0|79
+0|74
-0|10
+0|13
-0|02
+0|03
-0|01
0|0
+1|58
-0|75
+0|29
-0|17
+0|06
-0|03
+0|01
+0|09
3
0|0
+3|29
-0|75
+0|93
-0|18
+0|16
-0|03
+0|03
+3|45
4
0|0
+2|18
-0|93
+0|62
-0|20
+0|11
-0|04
+0|02
+1|76
+5|20
+1|65
-3|16
+0|46
-0|38
+0|08
-0|07
+0|01
+3|79
3
0|0
-1|58
+0|37
-0|19
+0|07
-0|03
+0|01
-1|35
0.190
0.231
0.461
6
0.286
0.286
6
0.461
0.231
0.308
0.238
0.308
0|0
-0|99
-0|23
-0|04
-1|26
0|0
+1|65
-1|49
+0|46
-0|35
+0|08
-0|07
+0|01
+0|29
0|0
+2|74
+0|78
+0|14
+0|02
+3|68
-11|50
+ 3|29
- 1|58
+ 0|93
- 0|19
+ 0|16
- 0|03
+ 0|03
- 8|89
0|0
-2|11
-0|26
-0|05
-2|42
4
5
4
0|0
-0|63
-0.63
0|0
+1|84
0|0
-1|21

FIG. 9.—FIFTY-MINUTE SOLUTION BY H. O. SPRINGER

The solution by H. O. Springer (see Fig. 9) was performed by conventional end-moment distribution in 50 min, including the time required to make the sketch and write on it the stiffness factors and fixed-end moments. The solution by L. Blume (see Fig. 10), which is of the same order of accuracy as the author's two-place solution, required only 15 min after the sketch showing stiffnesses and fixed-end moments was prepared.

The author deserves credit for having made a scholarly investigation of the analysis of rigidly jointed frames. However, between the superior speed of the conventional end-moment distribution solution and the superior quality of the first solution in Table 4, it appears to the writer that the author's method is encircled so that it has no useful field.

	0.375 *AD*	0.625 *AB*	0.36 *BA*	0.28 *BE*	0.36 *BC*		0.625 *CB*	0.375 *CF*	
	....	−84	+28	....	....		....	....	
	+31½	+52½	−10	− 8	−10		....	....	
	....	− 5	+26	+11	....		− 5	− 6	
	+ 2	+ 3	−13½	−10½	−13		+ 7	+ 4	
	− 4	− 7	+ 1½	+ 1½	+ 3½		− 7	− 2	
	+ 4	+ 7	− 2¼	− 2	− 2¼		+ 5½	+ 3½	
	− ½	− 1	+ 3½	+ 1½	+ 3		− 1	− ½	
	+ ½	+ 1	− 3	− 2	− 3		+ 1	+ ½	
	+33½	−33½	+30½	− 8½	−22		+ ½	− ½	
0.31 *DG*	0.23 *DA*	0.46 *DE*	0.285 *ED*	0.19 *EB*	0.285 *EF*	0.24 *EH*	0.46 *FE*	0.23 *FC*	0.31 *FK*
....	....	....	....	....	−115	....	+52	....	....
....	....	....	+32½	+22½	+ 32½	+27½	−24	−12	−16
....	+16	+16	....	− 4	− 12	....	+16	....	....
−10	− 7	−15	+ 4½	+ 3	+ 4½	+ 4	− 7	− 4	− 5
....	+ 1	+ 2	− 7½	− 5	− 3½	....	+ 2	+ 2	....
− 1	− 1	− 1	+ 4½	+ 3	+ 4½	+ 4	− 2	− 1	− 1
....	+ 2	+ 2	− ½	− 1	− 1	....	+ 2	+ 2	....
− 1	− 1	− 2	+ 1	+ ¼	+ 1	+ ¼	− 2	− 1	− 1
−12	+10	+ 2	+34½	+18½	− 89	+36	+37	−14	−23

FIG. 10.—FIFTEEN-MINUTE SOLUTION BY L. BLUME

It also may be mentioned that for one-story structures the "quality" type of solution will be considerably faster than either end-moment distribution or the author's method.

G. W. STOKES,[13] ESQ.—One of the most useful applications of Mr. Cornish's original adaptation of the slope-deflection formulas occurs in connection with moment calculations for "ordinary building" frames.[14] No definition of such frames appears to be available but, no doubt, essential features are that the beams shall be prismatic (or that the inertia moment of any haunch, considered as a rectangular section, shall not be greater than the assumed inertia moment of the midspan tee section) and that sidesway due to vertical loading is negligible.

The number of moment calculations for a complete multi-story building frame is usually large and is magnified by the alternative arrangements of superload which give maximum moments. Since the relative K-values are in the first place unknown, it is common practice to proceed by means of intelligent approximation of these values, based in the first place on consideration of fixed-

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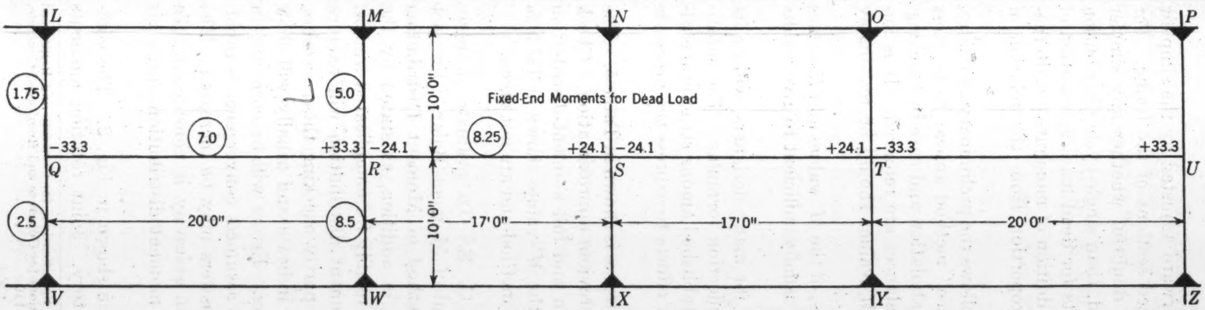
[13] Cheadle Hulme, Cheshire, England.

[14] "Recommended Practice and Standard Specifications for Concrete and Reinforced Concrete," Rept. of the Joint Committee on Standard Specifications for Concrete and Reinforced Concrete, *Proceedings*, Am. Soc. C. E., June, 1940, Pt. 2, paragraph 805(c), p. 46.

TABLE 5.—SOLUTION OF A SIMPLE BUILDING FRAME



Quantity	Joint Q				Joint R					Joint S					Joint T					Joint U			
	Member QL	Member QR	Member QV	Σ	Member RQ	Member RM	Member RS	Member RW	Σ	Member SR	Member SN	Member ST	Member SX	Σ	Member TS	Member TO	Member TU	Member TY	Σ	Member UT	Member UP	Member UZ	Σ
(a) Span Q-R Loaded																							
K	1.75	7.0	2.5	11.25	7.0	5.0	8.25	8.5	28.75	8.25	5.0	8.25	8.5	30.0	8.25	5.0	7.0	8.5	28.75				
M_F	....	−3.33	....	....	+3.33	....	....	....	....	....	....	....	....	....	....	....	....	....	....				
$-X$	0	−100.0	0	−100.0	+100.0	0	0	0	+100.0	0	0	0	0	0	0	0	0	0	0				
M'	+3.0	+34.0	+4.2	+41.2	−14.8	−3.9	−4.7	−6.6	−30.0	−12.4	+0.5	0.6	+0.9	−10.4	+1.7	−0.1	0	−0.1	+1.5				
κ	+9.2	+36.5	+13.1	+58.8	−17.0	−12.2	−20.1	−20.7	−70.0	+2.9	+1.7	+2.8	+3.0	+10.4	−0.5	−0.1	−0.4	−0.5	−1.5				
$2M$	+12.2	−29.5	+17.3	....	+68.2	−16.1	−24.8	−27.3	....	−9.5	+2.2	+3.4	+3.9	....	+1.2	−0.2	−0.4	−0.6	....				
M	+6.1	−14.8	+8.7	....	+34.1	−8.0	−12.4	−13.7	....	−4.8	+1.1	+1.7	+2.0	....	+0.6	−0.1	−0.2	−0.3	....				
(b) Span R-S Loaded																							
K	1.75	7.0	2.5	11.25	7.0	5.0	8.25	8.5	28.75	8.25	5.0	8.25	8.5	30.0	8.25	5.0	7.0	8.5	28.75	7.0	1.75	2.5	11.25
M_F	....	....	....	....	....	....	−24.1	....	....	+24.1	....	....	....	....	....	....	....	....	....	....	....	....	....
$-X$	0	0	0	0	0	0	−72.3	0	−72.3	+72.3	0	0	0	+72.3	0	0	0	0	0	0	0	0	0
M'	−0.2	+6.1	−0.5	+5.4	+1.3	+2.4	+20.0	+4.0	+27.7	−19.8	−2.3	−2.7	−4.0	−28.8	−7.4	+0.1	+0.2	+0.1	−7.0	+0.9	−0.1	−0.1	+0.7
κ	−0.8	−3.4	−1.2	−5.4	+10.8	+7.8	+12.8	+13.2	+44.6	−12.0	−7.2	−12.0	−12.3	−43.5	+2.0	+1.2	+1.7	+2.1	+7.0	−0.4	−0.1	−0.2	−0.7
$2M$	−1.0	+2.7	−1.7	....	+12.1	+10.2	−39.5	+17.2	....	+40.5	−9.5	−14.7	−16.3	....	−5.4	+1.3	+1.9	+2.2	....	+0.5	−0.2	−0.3	....
M	−0.5	+1.4	−0.9	....	+6.1	+5.1	−19.8	+8.6	....	+20.3	−4.6	−7.4	−8.1	....	−2.7	+0.7	+0.9	+1.1	....	+0.3	−0.1	−0.2	....

K = stiffness ratio I/L
M_F = fixed-end moment
X = a slope-deflection term
M' = external moment at outer end
κ = a substitution factor
M = final moment

end beam moments and statical reactions, afterward adjusted by the simplest possible calculations of end moments for limited sections of the frame. For these preliminary estimates of moments it is doubtful whether any simpler method has been found, or is likely to be found, than single-cycle distribution and carry-over of individual fixed-end moments for dead load, summation of such moments and further summation by the addition of moments for alternative superload arrangements, obtained by proportion from the relation of superload to dead load.

It is in the final stage of calculation, which follows the preliminary estimates, that the characteristics of the "moment-balance" method appear to be most valuable—namely, rapid convergence, simple tabulation and ease in checking. The method is somewhat lengthy if several balances are required. It is important, therefore, that the estimates of the M'-values should be reasonably close.

By the insertion, in their appropriate columns, of the M'-values and K-values already obtained, one "balancing" operation is usually sufficient to give results accurate enough to form the basis for design.

A point in favor of Mr. Cornish's method, for many designers, will be its clear derivation from the well-known slope-deflection formulas. The substitution of values in formulas is a practice well established among structural engineers, a fact which no doubt explains the many efforts to express the processes of moment distribution in terms of symbols.[15]

An example of a section of a simple building frame is shown in Table 5. The M'-values and K-values are those obtained by previous approximation. Check calculations made by the usual slope-deflection and full moment-distribution methods give results practically identical with the M'-values shown in Table 5. The necessary summations are common to any method and are not given.

Bruce Jameyson,[16] **Assoc. M. Am. Soc. C. E.**—To evaluate properly the relative merits of the new Cornish Method of Moment Balance the fact should not be overlooked that the Cross Method of Moment Distribution will accomplish nearly the same results and, in addition, as stated by the author, will include the sidesway feature more readily for complicated frames. When starting with the correct answers, moment distribution will balance most types of problems in one cycle, thereby partly checking these answers. If the answers are incorrect, one cycle will so indicate and usually will show the approximate location of the error or errors. Errors will be corrected in one or more cycles. If fair starting values are assumed, convergence is rapid. The accuracy (decimal places) of the trial answers may be increased. The foregoing statements apply equally well when sidesway is considered. To illustrate, the author's example is worked by moment distribution, using his starting trial answers.

Example.—Consider the example and frame shown in Fig. 2. The computations in Table 6 are almost self-explanatory. Joint rotation moments

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[15] For example, see "Recommended Practice and Standard Specifications for Concrete and Reinforced Concrete," Rept. of the Joint Committee on Standard Specifications for Concrete and Reinforced Concrete, *Proceedings*, Am. Soc. C. E., June, 1940, Pt. 2, Appendix 2, p. 110.

[16] Associate Prof., Civ. Eng., Univ. of California, Berkeley, Calif.

(M_R-values) were computed for the six upper joints, using the author's assumed starting answers in the girders and fixed-end moments (F-values). For example, for joints A and B:

$$M_{AB} + \tfrac{1}{2} M_{BA} = \text{answer} - F = 3.0 - 8.44$$

and

$$\tfrac{1}{2} M_{AB} + M_{BA} = \text{answer} - F = 2.5 - (-2.81)$$

Solving, joint rotation moments (M_R-values) are:

$$M_{AB} = -7.46$$

and

$$M_{BA} = 4.04$$

These two moments are the controlling moments for rotation of joints A and B, respectively. Then

$$M_{AD} = \frac{3}{5} \times (-7.46) = -4.48$$

$$M_{BE} = \frac{4}{5} \times 4.04 = 3.23$$

and

$$M_{BC} = \frac{5}{5} \times 4.04 = 4.04$$

In actual practice (except when checking answers, say, from another method) these joint rotations would not be computed, as in Table 6, but would be guessed directly instead of guessing the trial answers. Some work thus would be saved. Also, little would be gained by attempting to guess joint rotations at joints C and D, which are far away from any load. This, then, becomes simply the general application of the arbitrary joint-rotation, short-cut method treated by Professor Cross and applied by him to a very special case.[17] The rotation moments may be applied to any or all joints, but, when the control rotation moment for one end of one member has been computed or assumed, the M_R-values for all other members framing into the joint must be in accordance with relative stiffnesses (K-values). This follows from the fundamental definition of stiffness—that is: "The moment at one end of a member necessary to produce unit rotation of the same end, opposite end fixed (or free) and with no relative end deflection." Carry-over factors from these rotation moments (C_R) must be applied to opposite ends of members thus rotated. Note that, for member BA, $F + M_R + C_R$ = assumed estimated starting answer. Table 6 shows "Close answers at end of four cycles" (work of third and fourth cycles not shown). These answers are correct to a maximum possible error of 0.02.

The following facts may be noted:

(*a*) At the end of the first cycle results by moment balance are more accurate than by moment distribution.

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[17] "Continuous Frames of Re-inforced Concrete," by Hardy Cross and N. D. Morgan, John Wiley & Sons, Inc., New York, N. Y., 1932, pp. 228, 229, and 230.

TABLE 6.—COM-

Description	Joint A			Joint B				Joint C		
Member	....	AD	AB	BA	BE	....	BC	CB	CF	....
Stiffness factors, K	....	3	5	5	4	....	5	5	3	....
Distribution	....	0.375	0.625	0.357	0.286	....	0.357	0.625	0.375	....
Carry-over factors, C	....	↓ 0.5	0.5←	→0.5	↓ 0.5	....	0.5←	→0.5	↓ 0.5	....
Estimated answers, author's values	....	−3.0	+3.0	−2.5	+1.0	....	+1.5	0	0	....
(*a*) First Cycle										
Fixed-end moments, F	....	0	+8.44	−2.81	0	....	0	0	0	....
Computed joint rotations, M_R	....	−4.48	−7.46[a]	+4.04[a]	+3.23	....	+4.04	−1.00[a]	−0.60	....
Carry-over factors, C_R	....	+0.33	+2.02	−3.73	−0.92	....	−0.50	+2.02	+0.65	....
Sidesway corrections, f, neglected	....	....	....	....	....	....	....	....	....	....
Balancing moments, B	....	+0.43	+0.72	−1.20	−0.96	....	−1.19	−0.67	−0.40	....
(*b*) Second Cycle										
Carry-over factors, C	....	+0.08	−0.60	+0.36	−0.52	....	−0.33	−0.58	+0.14	....
Sidesway corrections, f, neglected	....	....	....	....	....	....	....	....	....	....
Balancing moments, B	....	+0.20	+0.32	+0.18	+0.14	....	+0.17	+0.28	+0.16	....
(*c*) Totals										
Answers at end of two cycles	....	−3.4	+3.4	−3.2	+1.0	....	+2.2	+0.05	−0.05	....
Close answers at end of four cycles	....	−3.42	+3.42	−3.10	+0.88	....	+2.22	0.00	0.00	....
Author's answers at end of two cycles	....	−3.5	+3.5	−3.0	+0.8	....	+2.2	−0.1	+0.1	....

(*b*) At the end of the second cycle both methods yield results of about the same accuracy (nine differences of 0.1 in each case).

(*c*) Original starting answers (moments) have been roughly checked, and errors and mistakes in these starting answers have been corrected.

(*d*) The computations (except the original answers) were carried by slide rule to two decimal places, thereby increasing the accuracy of the four-cycle answers. Some of the computations by moment balance could not have been obtained to this accuracy by slide rule.

(*e*) One cycle of moment distribution, other than the first, involves smaller numerals and less work than for moment balance. The first cycle of both methods involves roughly the same amount of work when M_R-values are guessed, in moment distribution, instead of the answers.

(*f*) In this problem, sidesway corrections (fixed-end moments *f*), if desired, would be added directly after the carry-over values and would be based upon the subtotals at this stage.[18] In either case, if the starting answers were correct, all balancing moments would be zero, and the first and only cycle would partly check the assumed answers. The *f*-values, for this problem,

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[18] "Lateral Loads and Members of Variable Section," by G. E. Lange and C. T. Morris, *Bulletin No. 66*, Ohio State Univ. Studies, Engineering Series, 1931.

PUTATION SHEET

Description	Joint D			Joint E				Joint F		
Member	DG	DA	DE	ED	EB	EH	EF	FE	FC	FK
Stiffness factors, *K*	4	3	6	6	4	5	6	6	3	4
Distribution	0.308	0.231	0.461	0.286	0.190	0.238	0.286	0.461	0.231	0.308
Carry-over factors, *C*	↓ 0.5	↑ 0.5	0.5←	→0.5	↑ 0.5	↓ 0.5	0.5←	→0.5	↑ 0.5	↓ 0.5
Estimated answers, author's values	+2.0	−1.0	−1.0	−4.0	−3.0	−3.0	+10.0	−4.0	+2.0	+2.0
(*a*) First Cycle										
Fixed-end moments, *F*	0	0	0	0	0	0	+11.46	−5.21	0	0
Computed joint rotations, M_R	+0.89	+0.67	+1.33[a]	−2.76	−1.84	−2.30	−2.76[a]	+2.59[a]	+1.30	+1.73
Carry-over factors, C_R	0	−2.24	−1.38	+0.67	+1.62	0	+1.30	−1.38	−0.30	0
Sideway corrections, *f*, neglected	....	....	....	....	....	....	....	....	....	....
Balancing moments, *B*	+0.22	+0.17	+0.34	−1.54	−1.03	−1.28	−1.54	+0.59	+0.29	+0.39
(*b*) Second Cycle										
Carry-over factors, *C*	0	+0.22	−0.77	+0.17	−0.48	0	+0.30	−0.77	−0.20	0
Sidesway corrections, *f*, neglected	....	....	....	....	....	....	....	....	....	....
Balancing moments, *B*	+0.17	+0.13	+0.25	0	0	0	+0.01	+0.45	+0.22	+0.30
(*c*) Totals										
Answers at end of two cycles	+1.3	−1.05	−0.2	−3.5	−1.7	−3.6	+8.8	−3.7	+1.3	+2.4
Close answers at end of four cycles	+1.28	−0.98	−0.30	−3.46	−1.76	−3.66	+8.88	−3.78	+1.35	+2.43
Author's answers at end of two cycles	+1.3	−1.0	−0.3	−3.4	−1.8	−3.6	+8.8	−3.8	+1.4	+2.4

[a] Controlling moment for joint.

are so easily computed that there usually would be little excuse for neglecting sidesway.

Conclusion.—It is the writer's opinion that the advantage is with moment distribution, at least in so far as the solution of the author's particular example is concerned. The writer believes, however, that the author has made a valuable contribution and that his method will find a permanent place in the field of structural analysis.

R. E. Bowles,[19] **Esq.**—The theory upon which this paper is based is a simple one and one that has now probably become familiar to those interested. It does not involve any intermediate assumptions that are not in accordance with the actual structure. Thus it avoids the complication of locking and releasing the joints. The paper is a valuable contribution to the subject for the following reasons:

1. The process solves, directly, for the quantity required—that is, the moments, and not the slopes.

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[19] Dept. of Building, College of Technology, Manchester, England.

2. It is simple in operation, largely self-checking, and error-eliminating.

3. It is considered to be an advantage that the resulting calculations are not written on the frame diagram, but are written in a table considerably easier to arrange than that of moment distribution for a complicated structure. It avoids the addition of the results of successive stages, which in moment distribution involves the summation of numerous quantities of opposite sign.

4. As the order of the "balances" is unimportant it is possible to concentrate on those parts of the structure where given loading produces maximum effect.

5. It permits the experienced designer to minimize work by inserting intelligent forecasts of the desired results, whereas less experienced designers may obtain equally accurate results with slightly more labor.

6. A few simple examples that have been solved by this method suggest that even without experience the designer finds it less laborious and quicker than other methods.

7. The practical user will probably appreciate that at each successive stage the required moment is seen clearly without resort to summation.

8. Although the method, as published, does not directly accommodate varying moments of inertia or sidesway, it may still be applied to many practical problems where the effects of these factors are frequently neglected.

The foregoing conclusions have been confirmed mostly by actual computations for a simple frame, which the writer had previously solved, for comparison, by other methods.

William Morris,[20] Esq.—Mr. Cornish appears to be justified in his claim to have established a method which is self-checking, and which gives quite a rapid solution of rigid frames. The writer tested the method on several frames, using assumed end moments which were obviously a bad guess, and even used moments of the wrong sign. It was found that three balances gave results which were sufficiently accurate in the worst case of assumption.

It does not seem necessary, however, to calculate the values of X by use of the formulas given in Table 1, as the fixing moments can be used directly for this purpose. To avoid the excessive amount of arithmetic which appears to be involved, the writer suggests the following form of solution which will, no doubt, commend "moment balance" to those engineers who are familiar with the moment distribution form of solution. For example, Eq. 5 may be written in the form

$$M_{ON} = \frac{1}{2} M_{NO} - \frac{K_{ON}}{\Sigma K}\left[\sum\left(-\frac{X}{2}\right) + \sum \frac{M'}{2}\right] + \frac{1}{2}\left(-X_{ON}\right) \ldots (20)$$

This indicates a process similar to moment distribution as follows:

(*a*) Assume moments at the ends of the members remote from the joint under consideration;

(*b*) Carry half these moments over to the joint;

(*c*) Sum the latter together with $\sum\left(-\frac{X}{2}\right)$ and distribute to each member in proportion to their K-values; and

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With Matthews and Mumby, Manchester, England.

(*d*) Add the results obtained (not forgetting to include $-\frac{X}{2}$ for those members which are loaded) to give the final moments. The process is continued around the frame as shown in the paper. It is to be noted that the values of $-\frac{X}{2}$ for each end of a loaded member can be obtained from the fixing moments by carrying over to the far end half the moment of the near end and changing sign. Sum the results to give $-\frac{X}{2}$ for each end.

It seems preferable to determine the values of $-\frac{X}{2}$ for all loaded spans first, although, for a building frame designed by loading only one span at a time to give maximum moments, two values only are required. This means that the values of $-\frac{X}{2}$ are only included in the balancing twice, as Eq. 20 reduces to the form

$$M_{ON} = \frac{1}{2} M_{NO} - \sum \frac{M'}{2} \quad \text{.................... (21)}$$

for the remaining joints.

The process is illustrated in Fig. 11 on which values of $-\frac{X}{2}$ have been kept clear of the calculations and the assumed moments have been written at

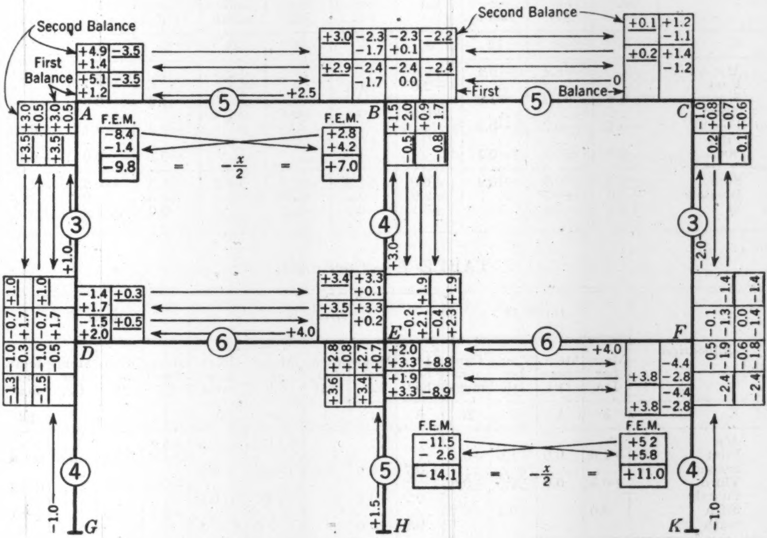
Second Balance
First Balance
Second Balance
First
Balance
A
B
C
D
E
F
G
H
K
F.E.M.
−8.4
−1.4
−9.8
F.E.M.
+2.8
+4.2
+7.0
$= -\frac{x}{2} =$
F.E.M.
−11.5
− 2.6
−14.1
F.E.M.
+5.2
+5.8
+11.0
$= -\frac{x}{2} =$
+2.5
+4.0
+4.0
0
+1.0
+3.0
−2.0
−1.0
+1.5
−1.0
+4.9
+1.4
−3.5
+5.1
+1.2
−3.5
+3.0
−2.3
−1.7
−2.3
+0.1
−2.2
+2.9
−2.4
−1.7
−2.4
0.0
−2.4
+0.1
+1.2
−1.1
+0.2
+1.4
−1.2
−1.4
+1.7
+0.3
−1.5
+2.0
+0.5
+3.4
+3.3
+0.1
+3.5
+3.3
+0.2
+2.0
+3.3
−8.8
+1.9
+3.3
−8.9
+3.8
−4.4
−2.8
+3.8
−4.4
−2.8

FIG. 11

the side of the table. The example chosen is that used by Mr. Cornish and the results at each joint are naturally identical with those of the author, who shows that they "self-check." The method is easily adaptable to portal frames without sidesway and continuous beams and, of course, can be used for haunched beams when the fixed-end moments and stiffness values have been ascertained.

The practical usefulness of the method for problems involving sidesway is not apparent.

JOHN MASON,[21] ESQ.—There are two points suggested by this most interesting paper, upon which the writer would like to comment:

(1) Why introduce a new concept "*X*" in addition to the fixed-end moment M_F? The fixed-end moment must be found as a guide to the estimate of the end moment, and it will be shown that it can be re-used just as easily in the moment balance as the new expression.

(2) Any saving in labor by this method depends on a reasonable initial estimate of the end moments which, it is submitted, is more than offset by the certainty of working methodically to the result by methods that do not depend on any such estimate.

TABLE 7.—COMPUTATIONS FOR METHOD OF ALLOCATED MOMENTS

Allocation	JOINT A			JOINT B				JOINT C		
	Member AD	Member AB	Joint moment	Member BA	Member BE	Member BC	Joint moment	Member CB	Member CF	Joint moment
K	3	5	16	5	4	5	28	5	3	16
M_F	....	−8.4	−8.4	+2.8	....	....	+2.8	....	....	0
First	....	....	....	+2.6	+1.1	0	+3.7	....	....	....
Second	−0.4	−1.2	−1.6	....	....	....	....	−1.2	−0.8	−2.0
Third	....	....	....	+0.5	+0.3	+0.6	+1.4	....	....	....
Fourth	−0.1	−0.2	−0.3	....	....	....	....	−0.2	−0.1	−0.3
Fifth	....	....	....	+0.1	+0.1	+0.1	+0.3	....	....	....
Sixth	−0.0	−0.1	−0.1	....	....	....	....	−0.1	−0.0	−0.1
Sum	−0.5	−9.9	−10.4	+6.0	+1.5	+0.7	+8.2	−1.5	−0.9	−2.4
Balance	+3.9	+6.5	....	−2.9	−2.4	−2.9	....	+1.5	+0.9	....
M	+3.4	−3.4	....	+3.1	−0.9	−2.2	....	0.0	0.0	....

TABLE 7.—(*Continued*)

Allocation	JOINT D				JOINT E					JOINT F			
	Member DA	Member DG	Member DE	Joint moment	Member ED	Member EB	Member EH	Member EF	Joint moment	Member FE	Member FC	Member FK	Joint moment
K	3	4	6	26	6	4	5	6	42	6	3	4	26
M_F	...	...	...	...	...	...	...	−11.5	−11.5	+5.2	...	...	+5.2
First	+1.6	0.0	+1.6	+3.2	...	...	...	....	....	+1.6	0.0	0.0	+1.6
Second	...	...	...	...	−0.7	−0.9	0.0	−1.6	−3.2	...	...	...	...
Third	+0.3	0.0	+0.5	+0.8	...	...	...	....	....	+0.5	+0.4	...	+0.9
Fourth	...	...	...	...	−0.2	−0.2	...	−0.2	−0.6	...	...	...	...
Fifth	0.0	...	+0.1	+0.1	...	...	...	....	....	+0.1	0.0	...	+0.1
Sixth	...	...	...	...	0.0	0.0	...	0.0	....	...	...	...	...
Sum	1.9	0.0	+2.2	+4.1	−0.9	−1.1	0.0	−13.3	−15.3	+7.4	+0.4	...	+7.8
Balance	−0.9	−1.3	−1.9	...	+4.4	+2.9	+3.6	+4.4	....	−3.6	−1.8	−2.4	...
M	+1.0	−1.3	+0.3	...	+3.5	+1.8	+3.6	−8.9	....	+3.8	−1.4	−2.4	...

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[21] Cons. Engr., London, England.

Using the method of "allocated moments" presented by the writer[22] in 1940, an alternate solution of the example given in Table 2 (using the same sign convention and notation) is given in Table 7.

The essential feature of the method of allocated moments consists in finding the angular rotation of the joint or, for convenience, a value called the "joint moment," which is a measure of the angular rotation of the joints, being related to it by the equation:

$$\text{Joint rotation} = \frac{\text{Joint moment}}{4 \times \text{joint stiffness} \times \text{E}} = \frac{\text{Joint moment}}{4\,E\,\Sigma K} \quad \ldots\ldots (22)$$

in which the "stiffness of the joint" equals the sum of the stiffness of all members radiating from that joint.

The joint moment is found by taking the sum of the fixing moments and adding algebraically to this value a series of allocated moments, each allocated moment being equal to $-\frac{1}{2}$ the adjacent joint moment distributed in proportion to the stiffness of the members at that joint. The allocated moments are the same as the "carry-over" moments in "moment-distribution" methods. In the concise method on which Table 7 is built, however, the moments totaling +3.7 (allocated to joint B in the first allocation) are first added to the fixed-end moment, +2.8, and the sum, +6.5, is allocated in bulk to the adjacent joints.

Once the rotation of the joint has been established, the end moments follow from the slope-deflection relationships:

$$\begin{aligned} M_{ON} &= M_{F(ON)} + 4\,E\,K_{ON}\,\phi_O + 2\,E\,K_{NO}\,\phi_N \\ &= M_{F(ON)} + K_{ON} \times \frac{\text{Joint moment}_O}{\Sigma K_{ON}} + \frac{K_{NO}}{2} \times \frac{\text{Joint moment}_N}{\Sigma K_{NO}} \quad \ldots (23a) \end{aligned}$$

$$\begin{aligned} M_{NO} &= M_{F(NO)} + 4\,E\,K_{NO}\,\phi_N + 2\,E\,K_{ON}\,\phi_O \\ &= M_{F(NO)} + K_{NO} \times \frac{\text{Joint moment}_N}{\Sigma K_{NO}} + \frac{K_{ON}}{2} \times \frac{\text{Joint moment}_O}{\Sigma K_{ON}} \quad \ldots (23b) \end{aligned}$$

or, in general,

$$M = M_F - \mu - \frac{\mu'}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$$

Now the sum of the allocated moments is equal to

$$\frac{\mu'}{2} = \frac{K_{NO}}{2} \times \frac{\text{Joint moment}_N}{\Sigma K_{NO}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

The method is therefore self-checking by comparison of

$$\frac{\mu'}{2} = \frac{K_{NO}}{2} \times \frac{\text{Joint moment}_N}{\Sigma K_{NO}}$$

from Eq. 23*a* with half the complimentary term $\mu = K_{NO} \times \frac{\text{Joint moment}_N}{\Sigma K_{NO}}$ from Eq. 23*b*.

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[22] *Journal*, The Inst. of Structural Engrs., April, 1940, p. 577; for discussion, *ibid.*, March, 1941, p. 43.

It should be noted that, since the rotation at a fixed end is zero, there is no allocated moment from a fixed end; hence, the entries in Cols. DG, EH, and FK are all zero.

Applying Eq. 24 to the moment balance system, the allocated moments $\left(\frac{\mu'}{2} = 2\,E\,K_{NO}\,\phi_N\right)$ are first estimated and added algebraically to the fixing moments to find the joint moment, which is then balanced. If, on comparison of $\mu = 4\,E\,K_{NO}\,\phi_N$ with $\frac{\mu'}{2} = 2\,E\,K_{NO}\,\phi_N$, the solution is not sufficiently accurate, the process is repeated, using one half of the last-found value of μ instead of $\frac{\mu'}{2}$ as the allocated moment in the next balance.

The computation is now as shown in Table 8, the order of balancing being joints B, D, F, A, C, and E and the initial estimates for $\frac{\mu'}{2}$ being obtained by balancing the fixed moment directly. Thus, +8.4 from joint A balances into AD = + 3.2 and AD = + 5.2. The initial estimates are underlined in Table 8. Values from the revised estimates have been used as soon as found.

TABLE 8.—COMPUTATION FOR METHOD OF MOMENT BALANCE USING FIXING MOMENTS

Quantity	JOINT A			JOINT B				JOINT C		
	Member AD	Member AB	Σ	Member BA	Member BE	Member BC	Σ	Member CB	Member CF	Σ
K	3	5	8	5	4	5	14	5	3	8
					(*a*) FIRST BALANCE					
M_F	0	−8.4	−8.4	+2.8	0	0	+2.8	0	0	0
$\frac{\mu'}{2}$	−0.4	−1.2	−1.6	$\underline{+2.6}$	$\underline{+1.1}$	0	+3.7	−1.2	−0.8	−2.0
μ	+3.8	+6.2	+10.0	−2.3	−1.9	−2.3	−6.5	+1.2	+0.8	+2.0
M	+3.4	−3.4	....	+3.1	−0.8	−2.3	....	0.0	0.0	....
					(*b*) SECOND BALANCE					
M_F	0	−8.4	−8.4	+2.8	0	0	+2.8	0	0	0
$\frac{\mu'}{2}$	−0.5	−1.4	−1.9	+3.1	+1.4	+0.6	+5.1	−1.4	−0.9	−2.3
μ	+3.9	+6.4	+10.3	−2.8	−2.3	−2.8	−7.9	+1.4	+0.9	+2.3
M	+3.4	−3.4	....	+3.1	−0.9	−2.2	....	0	0	....
					(*c*) THIRD BALANCE					
M_F	0	−8.4	−8.4	+2.8	0	0	+2.8	0	0	0
$\frac{\mu'}{2}$	−0.5	−1.5	−2.0	+3.2	+1.5	+0.7	+5.4	−1.5	−0.9	−2.4
μ	+3.9	+6.5	+10.4	−2.9	−2.3	−2.9	−8.2	+1.5	+0.9	+2.4
M	+3.4	−3.4	....	+3.1	−0.8	−2.2	....	0	0	....

TABLE 8.—(*Continued*)

Quantity	Joint D				Joint E					Joint F			
	Member DA	Member DG	Member DE	Σ	Member ED	Member EB	Member EH	Member EF	Σ	Member FE	Member FC	Member FK	Σ
K	3	4	6	13	6	4	5	6	21	6	3	4	13
(a) First Balance													
M_F	0	0	0	0	0	0	0	−11.5	−11.5	+5.2	0	0	+5.2
$\frac{\mu'}{2}$	+1.6	0	+1.6	+3.2	−0.7	−0.9	0	− 1.6	− 3.2	+1.6	0	0	+1.6
μ	−0.7	−1.0	− 1.5	−3.2	+4.2	+2.8	+3.5	+ 4.2	+14.7	−3.1	−1.6	−2.1	−6.8
M	+0.9	−1.0	+0.1	...	+3.5	+1.9	+3.5	− 8.9	....	+3.7	−1.6	−2.1	...
(b) Second Balance													
M_F	0	0	0	0	0	0	0	−11.5	−11.5	+5.2	0	0	+5.2
$\frac{\mu'}{2}$	+1.9	0	+2.1	+4.0	−0.9	−1.1	0	− 1.8	− 3.8	+2.1	+0.4	0	+2.5
μ	−0.9	−1.2	−1.9	−4.0	+4.4	+2.9	+3.6	+ 4.4	+15.3	−3.5	−1.8	−2.4	−7.7
M	+1.0	−1.2	+0.2	...	+3.5	+1.8	+3.6	− 8.9	....	+3.8	−1.4	−2.4	...
(c) Third Balance													
M_F	0	0	0	0	0	0	0	−11.5	−11.5	+5.2	0	0	+5.2
$\frac{\mu'}{2}$	+1.9	0	+2.2	+4.1	−0.9	−1.2	0	− 1.8	− 3.9	+2.2	+0.4	...	+2.6
μ	−0.9	−1.3	−1.9	−4.1	+4.4	+2.9	+3.7	+ 4.4	+15.4	−3.6	−1.8	−2.4	−7.8
M	+1.0	−1.3	+0.3	...	+3.5	+1.7	+3.7	− 8.9	....	+3.8	−1.4	−2.4	...

It will be seen that, although the second balance is quite close enough for practical purposes, the calculation has been taken to the third balance as a check. It also will be noted that the values in the moment balance (Table 8) for $\frac{\mu'}{2}$ and Σ are in all cases the same as the sum of the figures at the corresponding stages in the method of allocated moments (Table 7).

It should be pointed out that the example taken is a very simple one, the incidence of the loading being such that the allocated moments in any one column are always of the same sign and that the joint moment, therefore, is always increasing in absolute value. This would not necessarily be the case if all the bays were loaded.

The outstanding merit of the method suggested by Mr. Cornish is that it enables a very rapid first approximation to be made of the moments at a joint in terms of the members immediately adjacent to that joint. It is of great value in the initial stages of a design, and perhaps at a time when the other members have yet to be designed, to be able to arrive at this first approximation. Any such approximation can also be checked and corrected, as the design of the members proceeds, by taking out subsequent balances.

It is submitted, however, that the introduction of the concept "X" is unnecessary.

Although it is admitted that members remote from the joint under consideration only affect the moments at that joint in a secondary degree, clearly any method which achieves quickness by discarding certain factors must be limited in that respect.

R. J. CORNISH,[23] ESQ.—The paper originated in an attempt to adapt, to structural frameworks, methods of successive approximation which had proved successful in the study of flow in networks of pipes.[24] In the latter case, there are two basic facts: At a given junction the head is the same in all pipes; and the total quantity entering equals the total quantity leaving. In a rigid frame, at any joint the angle of slope is the same for all members, and the sum of the positive moments equals the sum of the negative moments. Here the analogy appears to end. There seems to be nothing in a structural member which corresponds to the flow of water through a pipe, and it is difficult to visualize a moment "entering" a frame as water enters a pipe network. The quantity X was an attempt to represent the application of moments to the frame. This approach probably explains why the writer failed to notice that X can be simply expressed in terms of the fixed-end moments $X = -2\,M_F + M'_F$, and he is indebted to Professor Grinter and Mr. Morris for making this clear. Table 3, by Mr. Merritt, in which the term X does not appear, might well supersede Table 2. It is preferable to that suggested by Mr. Mason, although it is useful to have his variation on record.

Messrs. Aleck and Merritt have adapted the method to members of varying section, thus answering an objection raised by Messrs. Larsen and Eremin. Messrs. Matthews and Morris have shown that the calculations can be tabulated "on the frame." The writer considers a separate tabulation to be clearer, an opinion with which Mr. Bowles expresses agreement. The time factor, mentioned by Messrs. Larsen and Stewart, is admittedly an important one, but the time occupied in checking work, and in locating suspected mistakes, should be taken into account, although it is hardly possible to estimate it. Any one who has had the task of checking a frame computation knows the labor involved and the difficulty of deciding whether discrepancies are serious. Messrs. Aleck and Stokes both refer to the value of moment balance in this connection. It has the merit of indicating the magnitude of errors in addition to locating them.

The self-checking feature of moment balance is an advantage to those engineers whose arithmetical accuracy is not above reproach, and for any one there is less mental strain when this feature is present. For the same reason the additional written work on which Professor Grinter comments may not be entirely a disadvantage. As he remarks, it is partly because this is a method of successive approximation rather than of successive correction, and partly because operations which would be performed mentally in other methods are included in the table.

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[23] Dept. of Municipal Eng., College of Technology, Manchester, England.

[24] "Analysis of Flow in Networks of Conduits or Conductors," by Hardy Cross, Univ. of Illinois Eng. Experiment Station, *Bulletin No. 286*, 1936; and "Analysis of Flow in Networks of Pipes," by R. J. Cornish, *Journal*, Institution of Civ. Engrs., December, 1939, p. 147.

The writer cannot agree with Mr. Stewart that the first method of Table 4 gives results of superior "quality." Any method of successive approximations or successive corrections can give results to any number of decimal places provided that enough balances are made; but few designers think it worth while to get meticulously "accurate" answers, considering that the appropriate values of E, I, and L can be known only approximately. The important thing is to find a method that will quickly give an answer that is accurate enough for practical purposes.

Mr. Jameyson's adaptation of the Cross method of analysis to the correction of initial estimates is interesting and valuable; it may be that the most useful service the paper has rendered was to provoke this contribution.

A defect of moment balance, as presented in the paper, was that correction for sidesway was awkward; this difficulty has now been overcome.

Section 5 of the paper outlines a method of computing sidesway moments, involving the solution of as many simultaneous equations as there are floors. This makes the method unsuitable for general use. This limitation was suggested by Messrs. Larsen and Bowles. In the following procedure, this disadvantage is eliminated. Each story is given an arbitrary initial deflection and the resulting moments are balanced with the help of Eq. 10; the deflections then are adjusted to make the sums of the column-end moments equal to the sidesway moments. One repetition of this process usually gives sufficient accuracy for practical purposes.

As a simple example, the frame of Fig. 2 will be analyzed for the horizontal forces shown in Fig. 12. With 1.5 tons at A and 2.0 tons at D, the shears in the upper and lower stories are 1.5 tons and 3.5 tons, respectively. If AD is 15 ft and DG is 20 ft, the corresponding sidesway moments are 22.5 ft-tons and 70.0 ft-tons.

It can be assumed as a first approximation that, in its deflected form, Col. ADG remains a straight line. (This is close to the truth for tall, regular frames,[25] although in this example it is far from true; however, the error is eliminated rapidly.) Then, if Δ is the relative displacement of the top and bottom of a column of length, L, $\frac{\Delta}{L}$ has the same value for each story, and the initial values of $\frac{6\,E\,K\,\Delta}{L}$ must be proportional to K, the stiffness ratio. Since at first one is concerned only with relative deflections, it is convenient to begin by making $\frac{6\,E\,K\,\Delta}{L}$ equal to K. The slope deflection equations show that, when the ends of a member are deflected without rotation, the moment induced at each end is $\frac{-\,6\,E\,K\,\Delta}{L}$; the initial column moments are therefore simply the stiffness ratios K. These are entered in Fig. 12 beside each column, and are thus the moments resulting from deflection of the frame without rotation of the joints. Since the frame is symmetrical, only one half of it need be considered, and moments needed in the right-hand half can be inserted by symmetry.

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[25] "Theory of Modern Steel Structures," by L. E. Grinter, Macmillan, New York, N. Y., 1937, Vol. 2, p. 168.

In the next step the moments are balanced; that is, rotation without deflection takes place. The balance could be started from the K-values already entered, but time is saved by a rough preliminary adjustment, reducing the moments of the outer columns by rather more than those of the inner columns. Very little care is needed except where a member is direction-fixed at one end, in which case the moment at that end must be altered by half as much as the moment at the other end. For instance, when M_{DG} is reduced from -4.0 to -3.0, M_{GD} must be reduced to -3.5.

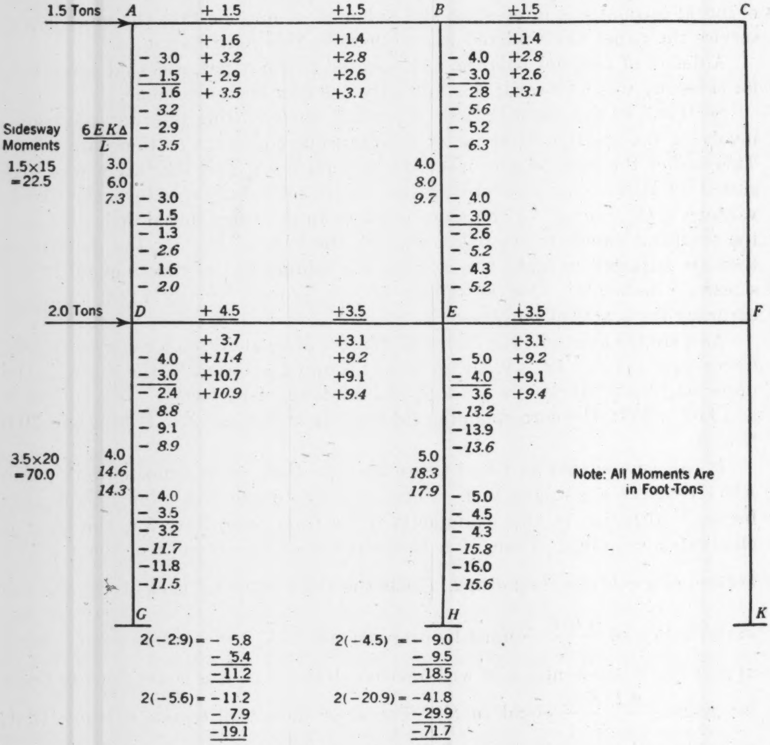
1.5 Tons
A
+ 1.5
+1.5
B
+1.5
C
+ 1.6
+ 3.2
+ 2.9
+ 3.5
+1.4
+2.8
+2.6
+3.1
+1.4
+2.8
+2.6
+3.1
− 3.0
− 1.5
− 1.6
− 3.2
− 2.9
− 3.5
− 4.0
− 3.0
− 2.8
− 5.6
− 5.2
− 6.3
Sidesway Moments
6EKΔ/L
1.5×15 =22.5
3.0
6.0
7.3
4.0
8.0
9.7
− 3.0
− 1.5
− 1.3
− 2.6
− 1.6
− 2.0
− 4.0
− 3.0
− 2.6
− 5.2
− 4.3
− 5.2
2.0 Tons
D
+ 4.5
+3.5
E
+3.5
F
+ 3.7
+11.4
+10.7
+10.9
+3.1
+9.2
+9.1
+9.4
+3.1
+9.2
+9.1
+9.4
− 4.0
− 3.0
− 2.4
− 8.8
− 9.1
− 8.9
− 5.0
− 4.0
− 3.6
−13.2
−13.9
−13.6
3.5×20 =70.0
4.0
14.6
14.3
5.0
18.3
17.9
Note: All Moments Are in Foot-Tons
− 4.0
− 3.5
− 3.2
−11.7
−11.8
−11.5
− 5.0
− 4.5
− 4.3
−15.8
−16.0
−15.6
G
H
K
2(−2.9) = − 5.8
− 5.4
−11.2
2(−5.6) = −11.2
− 7.9
−19.1
2(−4.5) = − 9.0
− 9.5
−18.5
2(−20.9) = −41.8
−29.9
−71.7

FIG. 12

After the preliminary column moments have been selected, the corresponding beam moments are entered in Fig. 12. These rough estimates have been underlined to show that they form the start of the balance.

The balance closely resembles that given in the paper, except that Eq. 10 is used instead of Eq. 6, and, thus, in Table 9, $\frac{-6EK\Delta}{L}$ takes the place of $-X$ in Table 2.

TABLE 9.—COMPUTATION SHEET FOR SIDESWAY

No.	Quantity	Joint A				Joint B				
		Member AD	Member AB		Σ	Member BA	Member BE	Member BC		Σ
..	K	3	5		8	5	4	5		14
	(a) FIRST BALANCE									
1	$\frac{-6EK\Delta}{L}$	−3.0	0		−3.0	0	−4.0	0		−4.0
2	M'	−1.3	+1.5		+0.2	+1.6	−2.6	+1.6		+0.6
3	$\frac{K}{\Sigma K}\left[\Sigma\left(\frac{6EK\Delta}{L}\right)-\Sigma M'\right]$	+1.1	+1.7		+2.8	+1.2	+1.0	+1.2		+3.4
4	$2M$	−3.2	+3.2		....	+2.8	−5.6	+2.8		....
5	M	−1.6	+1.6		....	+1.4	−2.8	+1.4		....
	(b) SECOND BALANCE									
1	$\frac{-6EK\Delta}{L}$	−6.0	0		−6.0	0	−8.0	0		−8.0
2	M'	−1.6	+2.8		+1.2	+2.9	−4.3	+2.9		+1.5
3	$\frac{K}{\Sigma K}\left[\Sigma\left(\frac{6EK\Delta}{L}\right)-\Sigma M'\right]$	+1.8	+3.0		+4.8	+2.3	+1.9	+2.3		+6.5
4	$2M$	−5.8	+5.8		....	+5.2	−10.4	+5.2		....
5	M	−2.9	+2.9		....	+2.6	−5.2	+2.6		....

TABLE 9.—(*Continued*)

No.	Quantity	Joint D				Joint E				
		Member DA	Member DG	Member DE	Σ	Member ED	Member EB	Member EH	Member EF	Σ
..	K	3	4	6	13	6	4	5	6	21
	(a) FIRST BALANCE									
1	$\frac{-6EK\Delta}{L}$	−3.0	−4.0	0	−7.0	0	−4.0	−5.0	0	−9.0
2	M'	−1.5	−3.5	+3.5	−1.5	+3.7	−3.0	−4.5	+3.7	−0.1
3	$\frac{K}{\Sigma K}\left[\Sigma\left(\frac{6EK\Delta}{L}\right)-\Sigma M'\right]$	+1.9	+2.7	+3.9	+8.5	+2.6	+1.7	+2.2	+2.6	+9.1
4	$2M$	−2.6	−4.8	+7.4	....	+6.3	−5.3	−7.3	+6.3	....
5	M	−1.3	−2.4	+3.7	....	+3.1	−2.6	−3.6	+3.1	....
	(b) SECOND BALANCE									
1	$\frac{-6EK\Delta}{L}$	−6.0	−14.6	0	−20.6	0	−8.0	−18.3	0	−26.3
2	M'	−3.2	−11.7	+9.2	−5.7	+10.7	−5.6	−15.8	+10.7	0
3	$\frac{K}{\Sigma K}\left[\Sigma\left(\frac{6EK\Delta}{L}\right)-\Sigma M'\right]$	+6.1	+8.1	+12.1	+26.3	+7.5	+5.0	+6.3	+7.5	+26.3
4	$2M$	−3.1	−18.2	+21.3	....	+18.2	−8.6	−27.8	+18.2	....
5	M	−1.6	−9.1	+10.7	....	+9.1	−4.3	−13.9	+9.1	....

TABLE 10.—COMPUTATION FOR

No.	Quantity	JOINT E				JOINT F		
		Member EA	Member EK	Member EF	Σ	Member FE	Member FB	Member FL
..	K	29.4	35.4	19.5	84.3	19.5	30.4	35.5
								(*a*) FIRST
1	$\frac{-6\,E\,K\,\Delta}{L}$	−29.4	−35.4	0	−64.8	0	−30.4	−35.5
2	M'	−8.0	−7.6	+14.0	−1.6	+14.7	−15.0	−15.4
3	$\frac{K}{\Sigma K}\left[\Sigma\left(\frac{6\,E\,K\,\Delta}{L}\right) - \Sigma M'\right]$	+23.2	+27.8	+15.4	+66.4	+11.7	+18.2	+21.1
4	$2\,M$	−14.2	−15.2	+29.4	....	+26.4	−27.2	−29.8
5	M	−7.1	−7.6	+14.7	....	+13.2	−13.6	−14.9
								(*b*) SECOND
1	$\frac{-6\,E\,K\,\Delta}{L}$	−29.1	−34.2	0	−63.3	0	−30.1	−34.3
2	M'	−7.1	−7.4	+13.0	−1.4	+14.0	−13.4	−14.4
3	$\frac{K}{\Sigma K}\left[\Sigma\left(\frac{6\,E\,K\,\Delta}{L}\right) - \Sigma M'\right]$	+22.6	+27.2	+15.0	+64.7	+11.0	+17.3	+20.1
4	$2\,M$	−13.6	−14.4	+28.0	....	+25.0	−26.2	−28.6
5	M	−6.8	−7.2	+14.0	....	+12.5	−13.1	−14.3

TABLE 10.—

No.	Quantity	JOINT O				JOINT P		
		Member OK	Member OS	Member OP	Σ	Member PO	Member PL	Member PT
..	K	35.4	35.6	21.4	92.4	21.4	35.5	35.6
								(*a*) FIRST
1	$\frac{-6\,E\,K\,\Delta}{L}$	−35.4	−35.6	0	−71.0	0	−35.5	−35.6
2	M'	−10.0	−9.2	+17.0	−2.2	+17.0	−17.0	−19.8
3	$\frac{K}{\Sigma K}\left[\Sigma\left(\frac{6\,E\,K\,\Delta}{L}\right) - \Sigma M'\right]$	+28.0	+28.2	+17.0	+73.2	+12.4	+20.7	+20.8
4	$2\,M$	−17.4	−16.6	+34.0	....	+29.4	−31.8	−34.6
5	M	−8.7	−8.3	+17.0	....	+14.7	−15.9	−17.3
								(*b*) SECOND
1	$\frac{-6\,E\,K\,\Delta}{L}$	−34.6	−37.2	0	−71.8	0	−34.7	−37.2
2	M'	−7.7	−7.8	+15.0	−0.5	+15.9	−15.0	−18.6
3	$\frac{K}{\Sigma K}\left[\Sigma\left(\frac{6\,E\,K\,\Delta}{L}\right) - \Sigma M'\right]$	+27.7	+27.8	+16.8	+72.3	+12.5	+20.7	+20.8
4	$2\,M$	−14.6	−17.2	+31.8	....	+28.4	−29.0	−35.0
5	M	−7.3	−8.6	+15.9	....	+14.2	−14.5	−17.5

SIDESWAY, WILSON-MANEY BENT

		Joint K				Joint L					No.
Member FG	Σ	Member KE	Member KO	Member KL	Σ	Member LK	Member LF	Member LP	Member LM	Σ	
26.2	111.6	35.4	35.4	21.4	92.2	21.4	35.5	35.5	26.2	118.6	..
Balance											
0	− 65.9	−35.4	−35.4	0	−70.8	0	−35.5	−35.5	0	−71.0	1
+15.0	−0.7	−8.0	−8.7	+14.0	−2.7	+15.5	−16.0	−15.9	+18.0	+1.6	2
+15.6	+66.6	+28.2	+28.2	+17.1	+73.5	+12.5	+20.8	+20.8	+15.3	+69.4	3
+30.6	....	−15.2	−15.9	+31.1	....	+28.0	−30.7	−30.6	+33.3	....	4
+15.3	....	−7.6	−7.9	+15.5	....	+14.0	−15.4	−15.3	+16.7	....	5
Balance											
0	−64.4	−34.2	−34.6	0	−68.8	0	−34.3	−34.7	0	−69.0	1
+14.9	+1.1	−7.3	−7.3	+13.5	−1.1	+14.9	−14.4	−14.5	+16.4	+2.4	2
+14.9	+63.3	+26.8	+26.8	+16.3	+69.9	+12.0	+19.9	+20.0	+14.7	+66.6	3
+29.8	....	−14.7	−15.1	+29.8	....	+26.9	−28.8	−29.2	+31.1	....	4
+14.9	....	−7.4	−7.5	+14.9	....	+13.5	−14.4	−14.6	+15.5	....	5

(Continued)

		Joint S				Joint T					No.
Member PQ	Σ	Member SO	Member SW	Member ST	Σ	Member TS	Member TP	Member TX	Member TU	Σ	
29.2	121.7	35.6	25.8	30.5	91.9	30.5	35.6	25.8	37.3	129.2	..
Balance											
0	−71.1	−35.6	−25.8	0	−61.4	0	−35.6	−25.8	0	−61.4	1
+20.0	+0.2	−10.0	−18.9	+20.0	−8.9	+21.7	−20.0	−22.9	+25.0	+ 3.8	2
+17.0	+70.9	+27.2	+19.7	+23.4	+70.3	+13.6	+15.9	+11.5	+16.6	+57.6	3
+37.0	....	−18.4	−25.0	+43.4	....	+35.3	−39.7	−37.2	+41.6	....	4
+18.5	....	−9.2	−12.5	+21.7	....	+17.6	−19.8	−18.6	+20.8	....	5
Balance											
0	−71.9	−37.2	−30.2	0	−67.4	0	−37.2	−30.2	0	−67.4	1
+18.7	+1.0	−8.7	−22.4	+20.0	−11.1	+23.1	−18.1	−26.0	+22.5	+1.5	2
+16.9	+70.9	+30.4	+22.0	+26.1	+78.5	+15.5	+18.1	+13.2	+19.1	+65.9	3
+35.6	....	−15.5	−30.6	+46.1	....	+38.6	−37.2	−43.0	+41.6	....	4
+17.8	....	−7.8	−15.3	+23.1	....	+19.3	−18.6	−21.5	+20.8	....	5

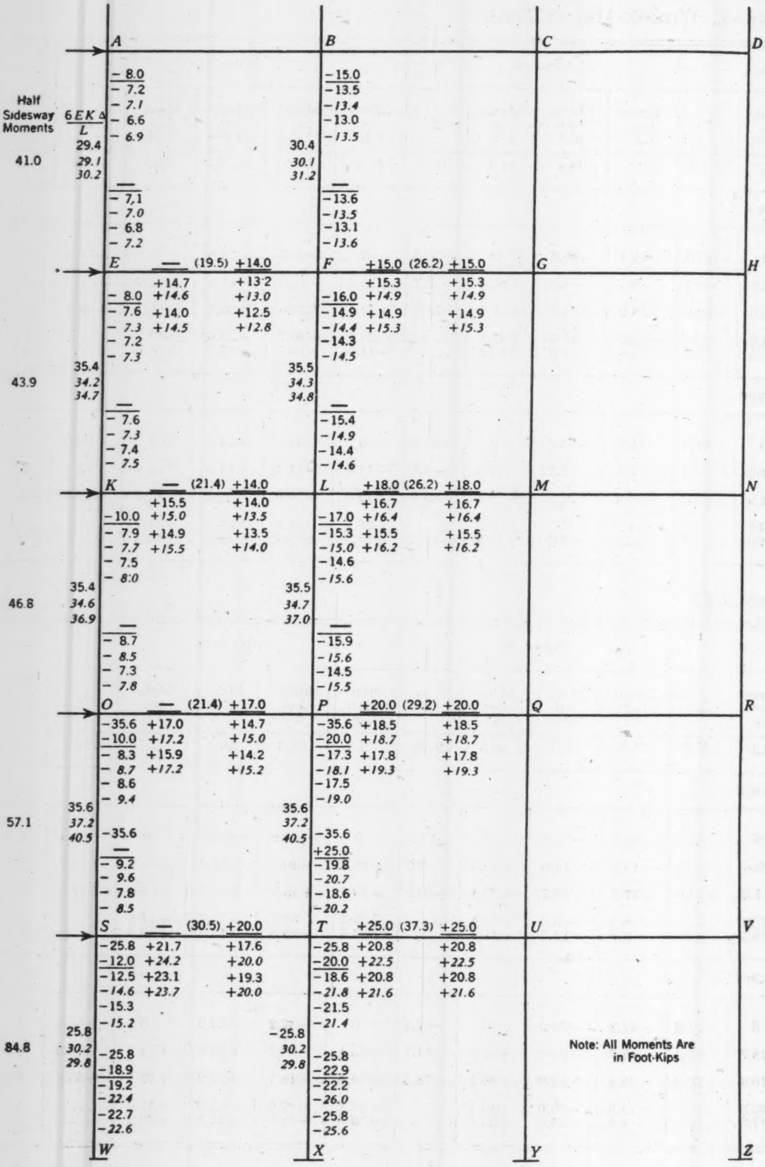
A
B
C
D
Half Sidesway Moments
6EKΔ/L
41.0
29.4
29.1
30.2
− 8.0
− 7.2
− 7.1
− 6.6
− 6.9
—
− 7.1
− 7.0
− 6.8
− 7.2
30.4
30.1
31.2
−15.0
−13.5
−13.4
−13.0
−13.5
—
−13.6
−13.5
−13.1
−13.6
E
— (19.5) +14.0
+14.7 +13.2
+14.6 +13.0
+14.0 +12.5
+14.5 +12.8
− 8.0
− 7.6
− 7.3
− 7.2
− 7.3
43.9
35.4
34.2
34.7
—
− 7.6
− 7.3
− 7.4
− 7.5
F
+15.0 (26.2) +15.0
+15.3 +15.3
+14.9 +14.9
+14.9 +14.9
+15.3 +15.3
−16.0
−14.9
−14.4
−14.3
−14.5
35.5
34.3
34.8
—
−15.4
−14.9
−14.4
−14.6
G
H
K
— (21.4) +14.0
+15.5 +14.0
+15.0 +13.5
+14.9 +13.5
+15.5 +14.0
−10.0
− 7.9
− 7.7
− 7.5
− 8.0
46.8
35.4
34.6
36.9
—
− 8.7
− 8.5
− 7.3
− 7.8
L
+18.0 (26.2) +18.0
+16.7 +16.7
+16.4 +16.4
+15.5 +15.5
+16.2 +16.2
−17.0
−15.3
−15.0
−14.6
−15.6
35.5
34.7
37.0
—
−15.9
−15.6
−14.5
−15.5
M
N
O
— (21.4) +17.0
+17.0 +14.7
+17.2 +15.0
+15.9 +14.2
+17.2 +15.2
−35.6
−10.0
− 8.3
− 8.7
− 8.6
− 9.4
57.1
35.6
37.2
40.5
−35.6
—
− 9.2
− 9.6
− 7.8
− 8.5
P
+20.0 (29.2) +20.0
+18.5 +18.5
+18.7 +18.7
+17.8 +17.8
+19.3 +19.3
−35.6
−20.0
−17.3
−18.1
−17.5
−19.0
35.6
37.2
40.5
−35.6
+25.0
−19.8
−20.7
−18.6
−20.2
Q
R
S
— (30.5) +20.0
+21.7 +17.6
+24.2 +20.0
+23.1 +19.3
+23.7 +20.0
−25.8
−12.0
−12.5
−14.6
−15.3
−15.2
84.8
25.8
30.2
29.8
−25.8
−18.9
−19.2
−22.4
−22.7
−22.6
T
+25.0 (37.3) +25.0
+20.8 +20.8
+22.5 +22.5
+20.8 +20.8
+21.6 +21.6
−25.8
−20.0
−18.6
−21.8
−21.5
−21.4
−25.8
30.2
29.8
−25.8
−22.9
−22.2
−26.0
−25.8
−25.6
U
V
Note: All Moments Are in Foot-Kips
W
X
Y
Z

Fig. 13

Joint D was first balanced. As in the paper, M' refers to moments at ends of members remote from D—that is, M_{AD}, M_{ED}, and M_{GD}. The items in lines 1 and 2 of the Σ-column are added with the sign changed, and the result (+8.5) is distributed to the members in proportion to their stiffnesses. The values in lines 1, 2, and 3 are added to give line 4, which is divided by 2 to give M_{DA}, M_{DE}, and M_{DG} (line 5). The values are entered in Fig. 12 at D, and M_{GD} is altered by half as much as M_{DG}. The remainder of the first balance, in the order, *E*, *A*, *B*, is done in the same way.

The sidesway moment which is actually being taken by each story is now found by adding the column-end moments. Since those in CF and AD are the same, the sum for the upper story is equal to $2(-1.6 - 1.3) - 2.8 - 2.6$, which equals -11.2 (an abbreviated form of this computation is shown at the bottom of Fig. 12). All the upper-story column moments therefore must be increased in the ratio $\frac{22.5}{11.2}$. Similarly, the column moments in the lower story must be increased in the ratio $\frac{70.0}{19.1}$, and corresponding alterations must be made in the beam moments. The new values are entered in italics in Fig. 12. The values of $\frac{6\,E\,K\,\Delta}{L}$ are increased in the same ratios and are entered in Fig. 12 on the left of each column.

One more balance and adjustment of sidesway moments has been shown. The process may be repeated to give any order of accuracy desired, but wind loading is so indefinite that great precision is not necessary. For most design purposes a single balance gives enough information provided that the preliminary estimates were reasonably close; the second balance provides a general check.

The method is especially useful with regular frames of several stories. If the balance is started at the bottom, at first only rough estimates for the two lowest floors need be inserted. The results of the first balance of the bottom floor moments guide one in the next estimates, and so on. The moments at the higher floors can be written down almost before the balance is made. The adjustment of beam moments after allowing sidesway is not so obvious in a frame of several bays as in the simple example of Fig. 12, but any common sense basis is satisfactory.

Fig. 13 and Table 10 show the calculations for the lower floors of the Wilson-Maney truss.[26] These are given here because this truss has been used as a typical example for other methods,[27] with which moment balance can be compared.

In an actual computation it is better to arrange the table with the stories beneath one another, instead of as in Table 10, where convenience of printing has been considered.

The order of balance adopted is S, T, O, P, K, L, E, and F. The *K*-values for the two lowest floors are first inserted as column-end moments in Fig. 13, and rough estimates, which have been underlined, are made where necessary. (A dash has been placed where an estimate is not required.) Rough values for M_{TS} and M_{TU} are found by distributing $M_{TP} + M_{TX}$ approximately in the ratio of the stiffness (shown in brackets) of TS and TU. Joints S and T are then balanced (Table 10). The results of this balance help in estimating the moments necessary for the balance of joints O and P, remembering that values tend to fall off in upper floors. For example, since the first balance gave M_{TS} as +17.6, M_{TO} is taken as +17.0.

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[26] *Bulletin No. 80*, Univ. of Illinois Eng. Experiment Station, p. 20.

[27] "Continuous Frames of Re-inforced Concrete," by Hardy Cross and N. D. Morgan, John Wiley & Sons, Inc., New York, N. Y., 1932, p. 230; and "Theory of Modern Steel Structures," by Linton E. Grinter, Macmillan, New York, N. Y., 1937, Vol. 2, p. 169.

Thus rough estimates and the first balance are made for each floor successively. Table 10 shows the balance for the four lowest stories and includes the values of M_{AB} and M_{BF}, which were obtained by balancing joints A and B.

The sum of the column-end moments for each story is now found, and, although the additions can be recorded in Fig. 13, they have been omitted to save printing. The column moments are altered to make their sum equal to the appropriate sidesway moments for half the truss, shown on the left of Fig. 13. Since, for the bottom floor, the sum of the moments is −72.5, they are increased in the ratio $\frac{84.8}{72.5}$. The values of $\frac{6\,E\,K\,\Delta}{L}$ for SW and TX are increased in the same ratio.

The new values of M_{TP} and M_{TX} are now −20.7 and −21.8. Their sum, −42.5, must be balanced by the beam moments M_{TS} and M_{TU}. Thus, the closest approximation presumably would be obtained by dividing 42.5 in the ratio of the last found values (as, for example, +17.6 and +20.8) of M_{TS} and M_{TU}, but great accuracy is not required, and a "round figure" adjustment to +20.0 and +22.5 is sufficient at this stage. All sidesway adjustments are shown in italics in Fig. 13.

A second balance and sidesway adjustment are now made with the result that values for moments and deflections are within about 5% of the "exact" values given by Messrs. Wilson and Maney.

It may be concluded that the method offered herein gives satisfactory results in the analysis of moments due to wind loading. Although the time taken may be a little longer than by some other methods, the reduction in arithmetical checking more than compensates for this. There is a fair amount of written work, because nearly all the calculations are recorded. However, this is an advantage rather than otherwise, since it facilitates the checking of the data. As the method involves the calculation of $\frac{6\,E\,K\,\Delta}{L}$ for each column, the final floor deflections can readily be computed if required.

Finally, the writer thanks all the contributors to the discussion and expresses his appreciation of the points raised. Some do not call for a detailed reply, and some were answered by other contributors. Time will show whether "moment balance" has any permanent value; if it has, future developments will be greatly helped by the discussion.

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TRANSACTIONS

Paper No. 2193

PERMEABILITY OF MUD MOUNTAIN DAM CORE MATERIAL

By Allen S. Cary,[1] Assoc. M. Am. Soc. C. E., and Boyd H. Walter,[2] and Howard T. Harstad,[3] Juniors, Am. Soc. C. E.

With Discussion by Messrs. F. H. Kellogg, J. E. Christiansen and Milton Fireman, William P. Creager, and Allen S. Cary, Boyd H. Walter, and Howard T. Harstad.

Synopsis

During the early investigation of the material contemplated for use in the impervious core of the Mud Mountain Dam (in a northwestern state), it was proved that the moisture content of the material when compacted had a much greater influence on the permeability than did the density of the material, within the range of densities at which it would be compacted during construction. Since the voids in the soil act as conduits for the water, it is postulated that the water content of the soil at the time of compaction governed the arrangement of the voids. At a high water content, the voids were apparently all of uniformly small size and the permeability was low. At a low water content, but at the same numerical void ratio or density, the voids were apparently of a wide range of sizes and the permeability was high. Thus, a wide or narrow range in void sizes may be likened to a wide or narrow range in particle sizes.

That the wide range or narrow range of void sizes may be imparted to the soil at will, with absolutely no variation in density being necessary, is proved by the extreme range in permeability among groups of samples compacted at very nearly identical densities but at different moisture contents.

This paper reports the results of an investigation inaugurated to determine the permeability-void ratio relationships of the proposed borrow-pit material as a basis for the distribution of the material in the various zones in the Mud Mountain Dam. So pronounced were the effects of slight moisture differences on the permeability of the material that moisture control during construction

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Note.—Published in September, 1942, *Proceedings*.

[1] Associate Geologist, Chf. Seattle Dist., Soils Laboratory, U. S. Engr. Dept., Seattle, Wash.

[2] Aviation Cadet, U. S. Army, Los Angeles, Calif.

[3] Ensign, U. S. Coast Guard, Norfolk, Va.

became of paramount importance, not so much for the usual density control but primarily for control of the permeability of the core material.

Throughout the paper reference is made to "optimum water content" and "void ratio." In order to accommodate those who may not be conversant with this terminology, the following definitions are given:

> Optimum water content—"The water content at which the maximum density is produced in a soil by a specific amount of compaction."[4]
>
> Void ratio—"The ratio of the volume of intergranular space to the volume of solid particles in a given soil mass without regard to the proportions of liquid, air, or gas which may occupy the space."[4]

This paper will discuss in turn: (1) The material tested, (2) method and equipment, (3) test results, (4) interpretations, and (5) further investigation.

MATERIALS TESTED

Test results in this paper apply only to the materials used in the impervious core of the Mud Mountain Dam. The soil is a mixture of stream gravel and glacial till. The average mechanical analysis of the material after removing all boulders larger than 6 in. is shown in Fig. 1. Curves 1 and 2 represent 20% and 40% till admixture, respectively. The curves reveal an extreme range of particle sizes from colloids to 6 in.

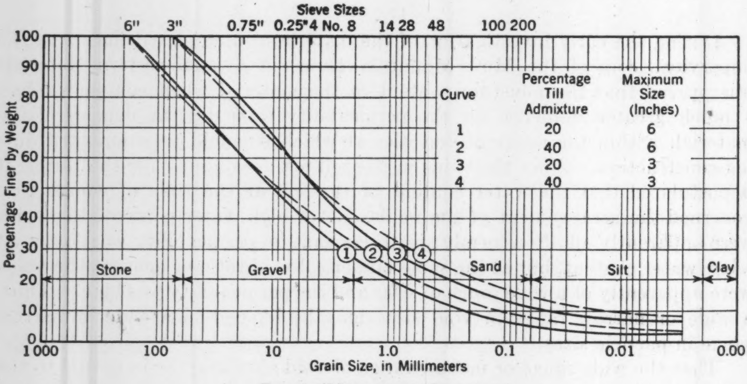

FIG. 1.—MECHANICAL ANALYSIS

The colloids are found primarily in the stream gravel and occur naturally as a coating around the particles in the deposit. They have been identified by the National Bureau of Standards by X ray and petrographic analysis as montmorillonite, an alteration product of volcanic glass and ash. Because of its extreme flatness, montmorillonite has been described as a "one-dimensional colloid." The extreme sensitivity of the mineral to excess moisture results in a marked loss of stability in the gravel when it is slightly above the optimum water content.

---

[4] "Soil Mechanics Nomenclature," *Manual of Engineering Practice No. 22*, Am. Soc. C. E., 1941.

For all permeability tests, samples containing only those particles passing a 3-in. screen were used. Mechanical analyses of this material are shown in Fig. 1, curves 3 and 4. The range of void ratios at which the material was tested was limited to those expected during ordinary construction on the fill.

EQUIPMENT AND METHODS

Apparatus.—Cylinders 13.54 in. in diameter and 12 in. high (1 cu ft) were used throughout the testing program for compaction and permeability. The permeameters were arranged with screens top and bottom to permit the free passage of water to and from the sample, and they were operated either as variable-head or constant-head upflow permeameters, as shown diagrammatically in Fig. 2. Except for very pervious materials, all tests were run with the variable-head hookup.

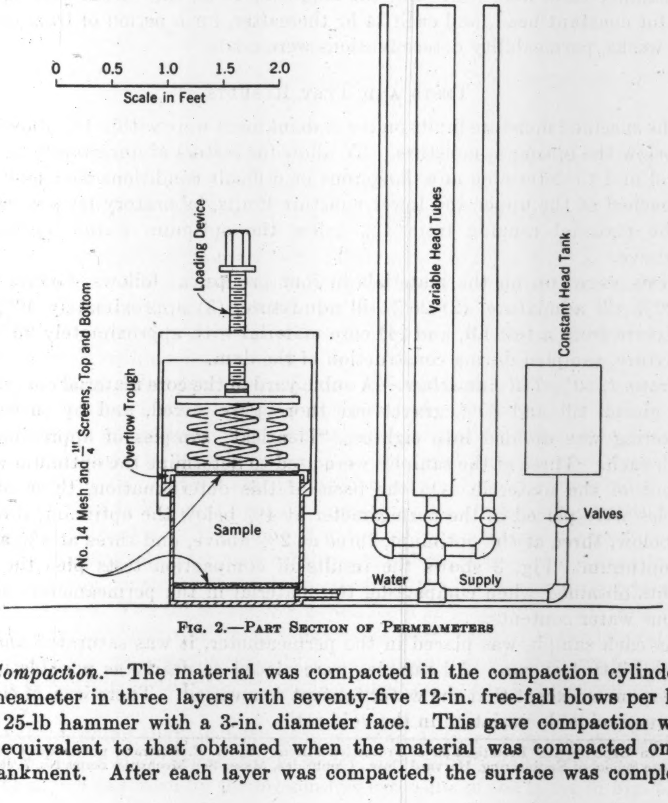
0
0.5
1.0
1.5
2.0
Scale in Feet
Loading Device
Variable Head Tubes
Constant Head Tank
No. 14 Mesh and $\frac{1}{4}$'' Screens, Top and Bottom
Overflow Trough
Sample
Valves
Water
Supply

FIG. 2.—PART SECTION OF PERMEAMETERS

Compaction.—The material was compacted in the compaction cylinder or permeameter in three layers with seventy-five 12-in. free-fall blows per layer of a 25-lb hammer with a 3-in. diameter face. This gave compaction which was equivalent to that obtained when the material was compacted on the embankment. After each layer was compacted, the surface was completely scarified to break up any surface skin that might tend to reduce the permeability.

Water.—The water was distilled and preheated to above room temperature so that there was a falling temperature gradient through the sample during the entire run. This assisted in the removal of trapped air in the sample and hence in the prevention of air-locking.

An apparatus for de-airing water was assembled similar to that described elsewhere by G. E. Bertram.[5] It was impossible to obtain enough water to run all of the tests and, since preheating the water seemed as effective in this material, the de-airing was discontinued entirely.

Test Procedure.—A sample was compacted in the cylinder at the desired moisture content and loaded with 1 ton per sq ft. Water was started flowing up through the sample with a constant-head hookup as shown in Fig. 2. When the sample had become saturated, or rather when water appeared at the upper surface of the sample, the valves were turned for variable head, and duplicate permeability tests were run. For the long-time tests, the valves were turned back for constant head, and each 24 hr thereafter, for a period of from one to eight weeks, permeability determinations were made.

Tests and Test Results

The specified moisture limits on the embankment were within 1% above and 2% below the optimum moisture. To allow for factors of uncertainty in field control and to determine any dangerous or difficult conditions that might be approached at the upper and lower moisture limits, laboratory tests were run on the material ranging from 4% below the optimum water content to 4% above.

Tests were run on the materials in four groups, as follows: Gravel with (1) 20% till admixture, (2) 25% till admixture, (3) approximately 40% till admixture from a test fill, and (4) core material with approximately 35% till admixture, sampled during construction of the dam.

Group 1, 20% Till Admixture.—A cubic yard of the core material containing 20% glacial till and 80% gravel was thoroughly mixed, and by successive quartering was divided into eighteen "identical" samples of approximately 160 lb each. Three of the samples were used to determine the optimum water content of the material. On the basis of this determination, three of the samples were placed in the permeameter at 4% below the optimum, three at 2% below, three at the optimum, three at 2% above, and three at 4% above the optimum. Fig. 3 shows the results of compaction tests plus the unit weights obtained when compacting the material in the permeameters at the various water contents.

As each sample was placed in the permeameter, it was saturated and the permeability test run. All samples were placed as nearly as possible in the same manner except that the water content was varied. There was, of course, some unavoidable variation in the void ratio.

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[5] **"An Experimental Investigation of Protective Filters," by G. E. Bertram, publications from the Graduate School of Engineering, Harvard Univ., Cambridge, Mass., Soil Mechanics Series No. 7, January, 1940.**

When permeability is plotted against void ratio, on semilogarithmic paper, a straight-line relationship is usually apparent for most soils. In group 1, however, there is none of the usual straight-line correlation between the permeability and the void ratio of the entire series, as shown in Fig. 4(*a*). However, there is a good correlation between each set of three points obtained from those samples placed at the same water content. Fig. 5(*a*) shows a definite correlation between the permeability and the water content at which the material was placed.

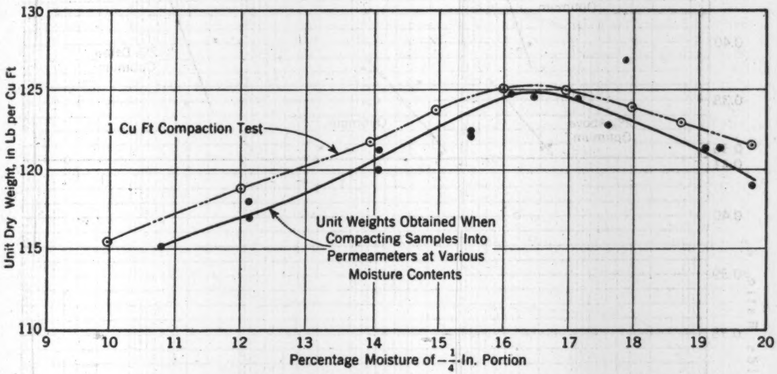

FIG. 3.—UNIT WEIGHT VERSUS MOISTURE CONTENT; 20% TILL ADMIXTURE

Group 2, 25% Till Admixture.—A similar series of tests was run for a 25% till admixture with the same permeability-moisture correlation as obtained in group 1. These tests are plotted, permeability versus water content, in Fig. 5(*b*). Permeability-void ratios are plotted in Fig. 4(*b*) and show none of the usual straight-line relationships.

Group 3, 40% Till Admixture.—A third series of tests was run using samples selected at random from a test fill constructed of core material with approximately 40% till admixture. The permeability-moisture correlation in Fig. 5(*c*) is better than might be expected when it is considered that this material varied considerably in till content and in mechanical analysis. Permeability-void ratios are plotted in Fig. 4(*c*) with the same complete absence of correlation.

Group 4, Embankment Material.—Fig. 6 shows the results of observations on 256 of the samples taken for permeability control of the core of the dam during the early part of construction, and directly from the borrow pit during excavation operations. The till admixture varied from 25% to 40%. All tests were run at the water content at which the samples were received in the laboratory. Here again, the permeability-moisture correlation was surprisingly favorable when it is considered that the mechanical analyses of the samples varied considerably.

It will be noted that, in both groups 3 and 4, there is a limited range of water contents. Had the range been wider, there is little doubt that the curves could be drawn as shown by the broken-line extension of the curve in Fig. 6.

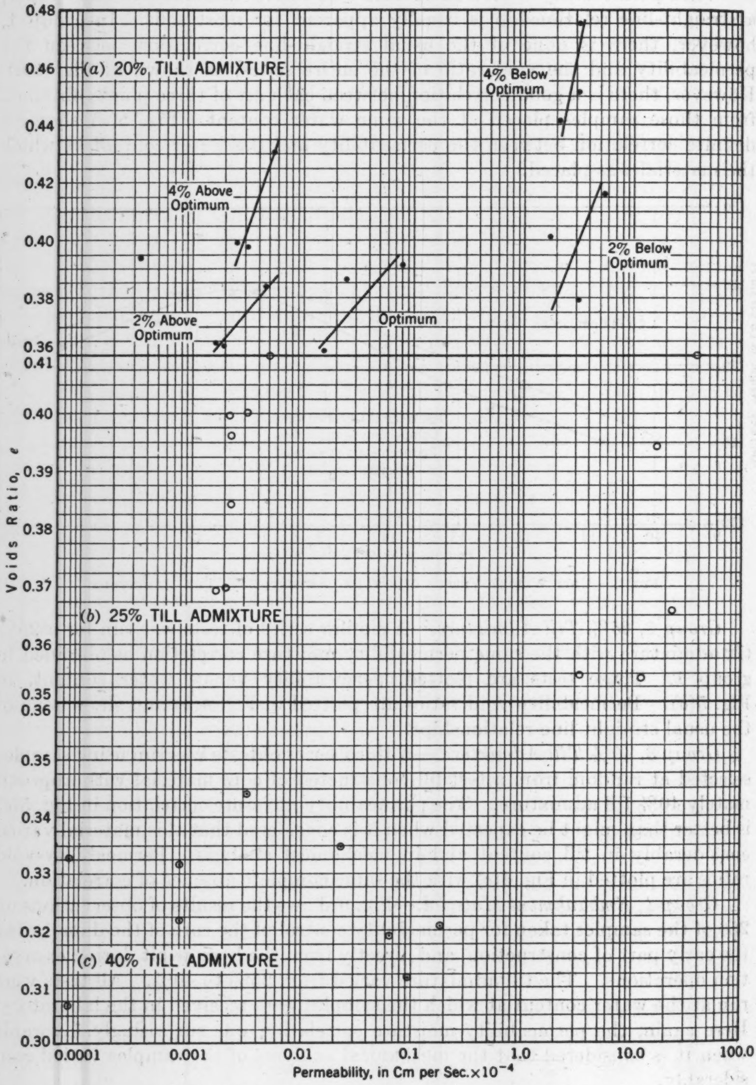

FIG. 4.—PERMEABILITY VERSUS VOID RATIO

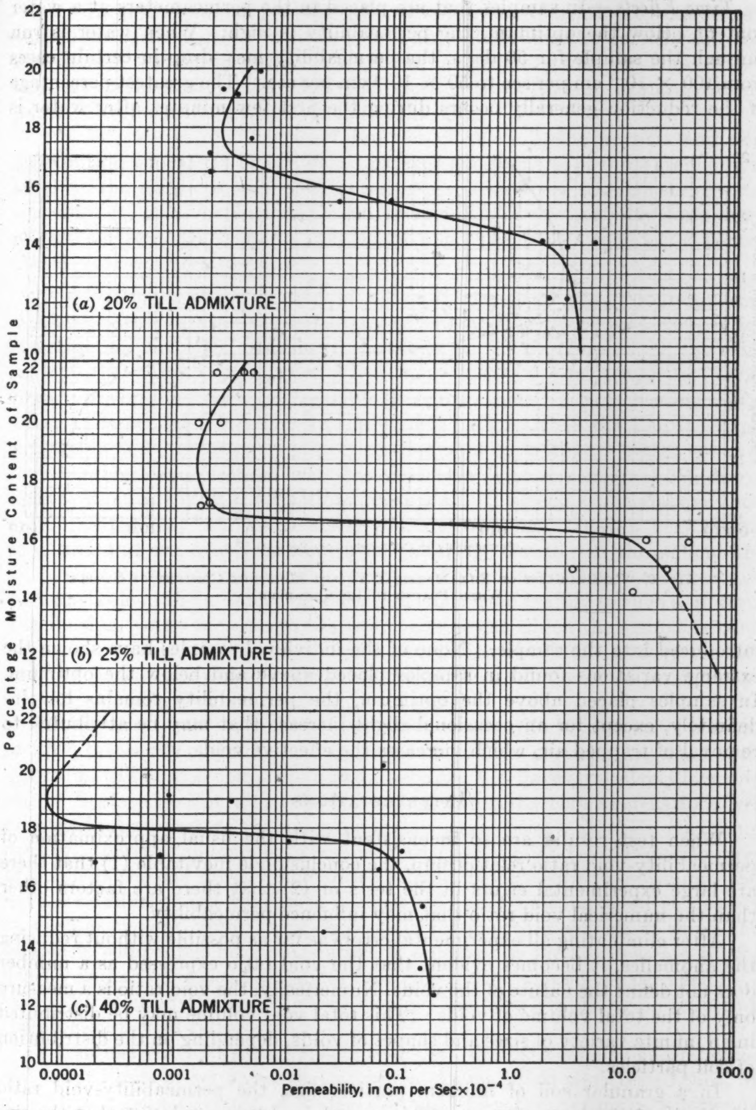

Fig. 5.—Permeability Versus Moisture Content of Sample When Placed in Permeameter

Time Effects.—In samples that are placed in the permeameters at a water content below the optimum, the permeability is high. When water is run through the sample for 56 days, the permeability may drop in certain cases from 100 × 10^{-4} cm per sec to 50 × 10^{-4} cm per sec. The greatest percentage of the reduction generally occurs during the first few minutes after water is introduced into the sample. None of the individual samples have shown the extreme variations found in samples placed above and below the optimum. In samples placed above the optimum, the permeability remains low indefinitely, except for an occasional slight increase that may be attributed to removal of trapped air, which increases the effective voids.

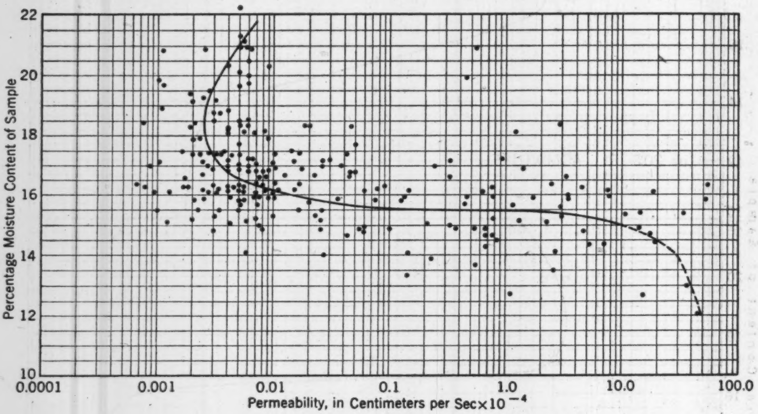

FIG. 6.—PERMEABILITY OF FILL MATERIAL VERSUS MOISTURE CONTENT OF SAMPLE WHEN PLACED IN PERMEAMETER

INTERPRETATIONS

When test results are so inconsistent with the usual approximation of permeability-void ratio relationship, the conclusion is inevitable (1) that there are large experimental errors in the tests or (2) that there are factors other than the numerical void ratio that may influence permeability.

After eliminating all experimental errors as far as possible without reducing the anomalies, it becomes evident that the void ratio expressed as a number does not define the nature of the voids. Numerically, the void ratio is a measure only of the total volume of voids. This total void volume may be distributed in an infinite variety of sizes and shapes of voids, depending on the distribution of soil particles.

In a granular soil of uniform particle size, the permeability-void ratio correlation is fairly consistent. This is explained by postulating that the size and distribution of the voids are more or less constant. In a well-graded soil with a small amount of clay included, consistent permeability-void ratio correlation can be obtained only when, and if, the particle and void distribution can be controlled. Control may be exercised under two conditions:

(1) In any given sample, the void ratio may be reduced progressively by consolidating the sample without appreciably disturbing the particle and void distribution. If permeability tests are run concurrently with the consolidation and the permeability-void ratio relationship is plotted on semilogarithmic paper, the plot will approximate a straight line.

(2) In a series of "identical" samples of the soil, the particle and void distribution appears to be controlled by exercising control of the water content at which the sample is compacted into the permeameter. At a constant water content but with varying void ratio, the results of permeability tests when plotted on semilogarithmic paper again approximate a straight-line permeability-void ratio relationship.

The optimum water content of a 1-cu-ft sample of Mud Mountain core material, with a maximum particle size of 3 in. as determined by a compaction investigation, is 11.5% of the dry weight of the entire sample. In the same sample, the water content of the fractions passing a ¼-in. screen is 16.5%. Now, if the optimum of a small sample using only those fractions passing a ¼-in. screen is determined, it is found to be 15.0%. Thus, when the 1-cu-ft sample containing the 3-in. maximum particle size is at its optimum water content, small aggregations of those fractions passing a ¼-in. screen are 1.5% above their optimum. With the fractions passing a ¼-in. screen slightly above their optimum, the clay, and particularly colloidal fractions, are in a very plastic condition and are relatively free to be uniformly distributed throughout the soil system during the compaction process. The montmorillonite clay becomes very unstable in the presence of water and lends itself readily to such a uniform distribution. Any increase in the water content at which the sample is compacted above this stage is reflected in a general increase in all voids in the soil, and the permeability-void ratio relationship then approaches a straight line (see Fig. 4), reflecting the numerical void ratio.

If this same sample were to be compacted into a cylinder at a water content of 1.5% below the optimum, the aggregations of finer particles will be at their optimum, will be well compacted, and, therefore, will have a relatively low void ratio. Surface tension and cohesion acting within the clay aggregations will be contributing factors in maintaining the aggregations at their maximum density, thereby enhancing a nonuniform particle and void distribution throughout the soil mass. The entire sample, being below its optimum, will have a relatively high void ratio because it cannot be compacted as densely as when at the optimum. Any decrease in the moisture content at which the sample is compacted below this stage is reflected in further decrease in the density and an increase in certain of the voids in the soil, and the permeability-void ratio relationship approaches a straight line (see Fig. 4), again reflecting the numerical void ratio.

The numerical void ratio appears to govern the permeability at the upper and lower limits of water content at which the materials were tested, but, at a relatively narrow range of water contents near the optimum, enormous changes of permeability take place with no change in void ratio necessary. This range of water content marks the transition from a condition of more or less uniform distribution of the fines on the wet side to a condition of concentration of fines into aggregations held together by cohesion and surface tension on the dry side. It is interesting to note that this range of moisture on either side of the optimum is almost the exact range originally specified for field control on Mud Mountain Dam.

Possible Application

In determining the permeability of certain pervious foundation materials, it is sometimes quite impossible to obtain undisturbed samples or to test the material in place, in which case it is common practice to determine the permeability of remolded samples over a range of void ratios. If this is done in conjunction with a single consolidation test, it has been suggested that the permeability-void ratio relationship should approach a straight line. However, if a series of tests is run on samples of the material compacted at various water contents, the results will indicate the degree of confidence that may be placed in the predictions of seepage through the foundation materials. If wide variations in permeability are obtained that cannot be attributed to void-ratio change, some other means of determining the permeability of the foundation materials is indicated.

The early practice of puddling cores in earth dams accomplished the same end as is attained by compacting this material at some moisture content slightly above the optimum. Puddling results in a uniform soil particle distribution, in a uniform void-distribution, and hence in a low permeability for that particular soil. The fact that, in the Mud Mountain core material, there is only 1.5% to 3% of clay, and in a puddled core there may be nearly 100% of clay, does not alter the fact that a high water content assists in a uniform particle distribution.

Further Investigations

The investigation has been limited thus far to a single material that is rather distinctive in its particle-size distribution. In this material, the effects of moisture are so extremely pronounced that they completely mask, over a certain range, all effects of numerical void-ratio change. Investigations of materials from four other sources, with a narrower range of particle sizes and no montmorillonite clay, are being conducted. It is indicated that the results will not reach the almost impossible extremes of the results obtained by using Mud Mountain borrow-pit material, but that herein may lie the answer to certain minor variations from the expected straight-line permeability versus void-ratio relationships.

Authority and Acknowledgment

The laboratory investigation that forms the subject of this paper was conducted in the U. S. Engineer Soils Laboratory, at Mud Mountain Dam, under the direction of Capt. J. D. Lang, Frank R. Preble, William E. Hoy, and Albert H. Rodes, successive resident engineers on the Mud Mountain Dam project. The writers wish to express their appreciation to the entire laboratory personnel for their critical and painstaking work in developing the testing methods and equipment, and for the interpretations as herein set forth. The constant interest and friendly criticism of A. E. Jaskar, Jun. Am. Soc. C. E., and H. E. Johnson deserve special mention. H. R. Cedergren, Jun. Am. Soc. C. E., of the Portland (Ore.) Division, U. S. Engineers, and Profs. R. G. Hennes, Assoc. M. Am. Soc. C. E., and J. Hoover Mackin of the University of Washington, at Seattle, have read the manuscript and offered a number of valuable suggestions.

DISCUSSION

F. H. Kellogg,[6] **M. Am. Soc. C. E.**—The fact that distribution of voids in a soil, rather than total voids volume, may be a diagnostic criterion of soil behavior has been excellently and strikingly demonstrated by the authors. The results of a number of permeability tests on several soil types are shown, in Fig. 7, to give results similar to those shown in Figs. 5 and 6. Such behavior is of practical significance in several ways, particularly in considering the swelling and softening of a compacted soil due to wetting.

Experience with fine soils compacted to form very hard, resistant masses, at moistures lower than optimum, has indicated that subsequent wetting of these masses results in a striking loss in strength. The wetting frequently causes considerable swelling. The same soils, compacted at moistures higher than optimum, show little or no swelling and softening after subsequent wetting. This has led to the control of moisture for such purposes as minimizing swelling of a highway subgrade under a concrete slab.[7] These effects of compacting moisture on swelling and softening, as actually observed, could be predicted, qualitatively at least, by test results such as those shown in Figs. 8 and 9. Fig. 8 shows, as points I, II, III, etc., the moistures and densities of a soil as compacted, and, as points 1, 2, 3, etc., the corresponding moistures and densities after wetting subsequent to compaction. Fig. 9(*a*) shows, as ordinates, the coefficients of internal friction of a soil that was submerged and saturated after being compacted at the moistures shown as abscissas; and Fig. 9(*b*) shows corresponding data for cohesion. These data, together with those presented by the authors, indicate the disadvantages of compacting fine soils at less than optimum moisture, due to the comparative ease with which water can find its way between hard lumps in the soil, softening them and weakening the entire mass. This was graphically illustrated by experiences with subgrades of A-5-7-type soils,[8] where time requirements and availability of materials prevented selection or special treatment of subgrade materials. Subgrades were either opened to traffic compaction or compacted at controlled moisture just before placing the 6-in. mechanically stabilized base and the asphaltic seal courses. After 2 to 4 months of predominately rainy weather, subgrade failures occurred at several places. At the points of failure, the subgrade had become wet and highly plastic. After checking for the effects of coarse aggregate and chemical changes in the clay, the following conclusions were made:

(1) Smooth, hard subgrades compacted at moistures lower than optimum were softened during wet weather to a plastic clay incapable of supporting the required wheel loads. This was true in spite of measures taken to promote drainage.

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[6] Senior Materials Engr., TVA, Knoxville, Tenn.

[7] *Engineering News-Record*, March 18, 1937, pp. 405–409.

[8] *Public Roads* 12, July, 1931, pp. 136–137.

(2) Subgrades compacted at moistures higher than optimum and immediately covered by the slab would not soften appreciably.

(3) Compacted clay subgrade could not be successfully plowed, moistened, and recompacted just prior to paving, as insufficient time for uniform distribution of moisture was available. Such a procedure merely served to wet the surface of the clay lumps, without affecting the compacted mass as a whole.

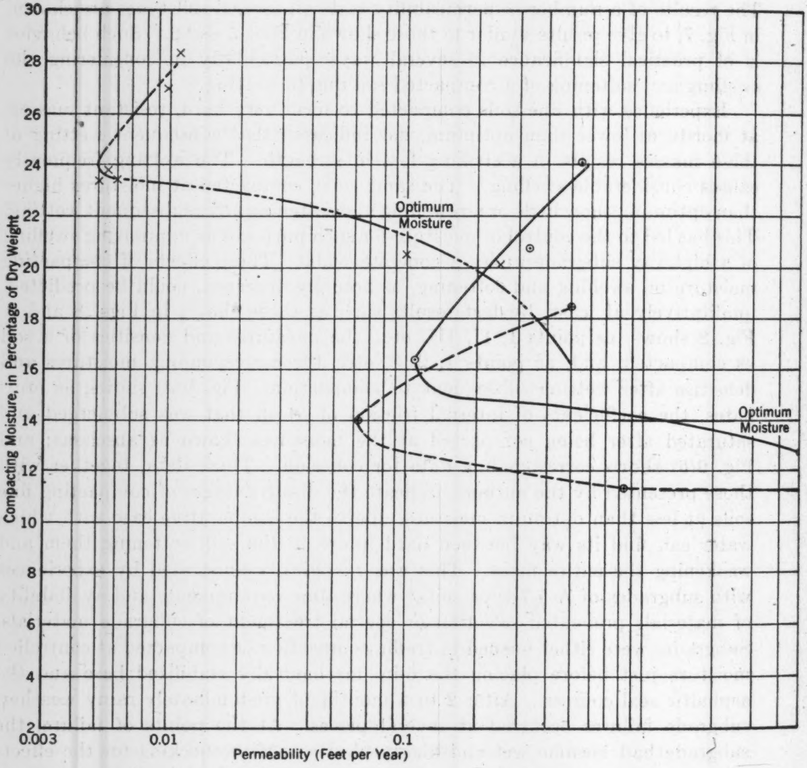
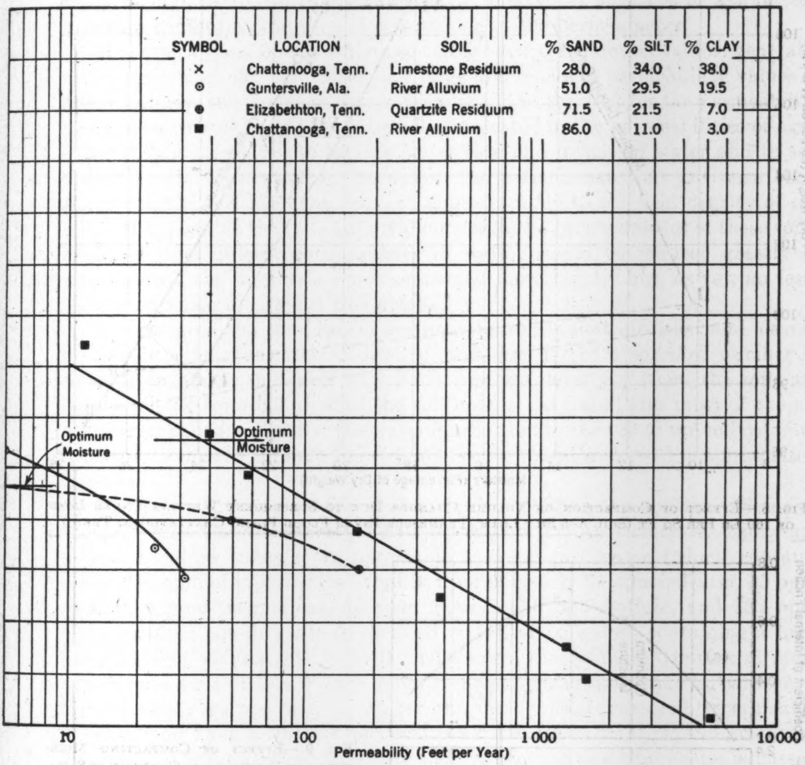


FIG. 7.—EFFECT, ON PERMEABILITY,

In the construction of certain impounding dikes, the relationships demonstrated by the authors have been used to advantage where only one type of material was available. At the upstream face of such a dike, high compacting moistures were used, giving a fill of low permeability. Downstream, lower compacting moistures gave a more pervious fill, increasing the effectiveness of the drains and partly offsetting the effects of anisotropy. Such a procedure, of course, must be consistent with the stability requirements of the dike, and the upstream fill must not be compacted when it is so wet as to cause cracking under wheel loads.

In the writer's experience, any soil that can be compacted at a given moisture to within 4 lb of the density which would permit no air voids at that moisture, and that has a soil mortar with an appreciable plasticity index, will deviate from the "normal" permeability-void ratio relationships. In other words, materials containing sufficient clay to form lumps at low moistures, and to lubricate coarse grains at high moistures, should fall into this class. Since montmorillonite is particularly sensitive to physicochemical changes, materials containing appreciable quantities of that mineral would require a lower percentage of fines to bring them into this class than would other materials.



(Figures 8 and 9 follow on next page)

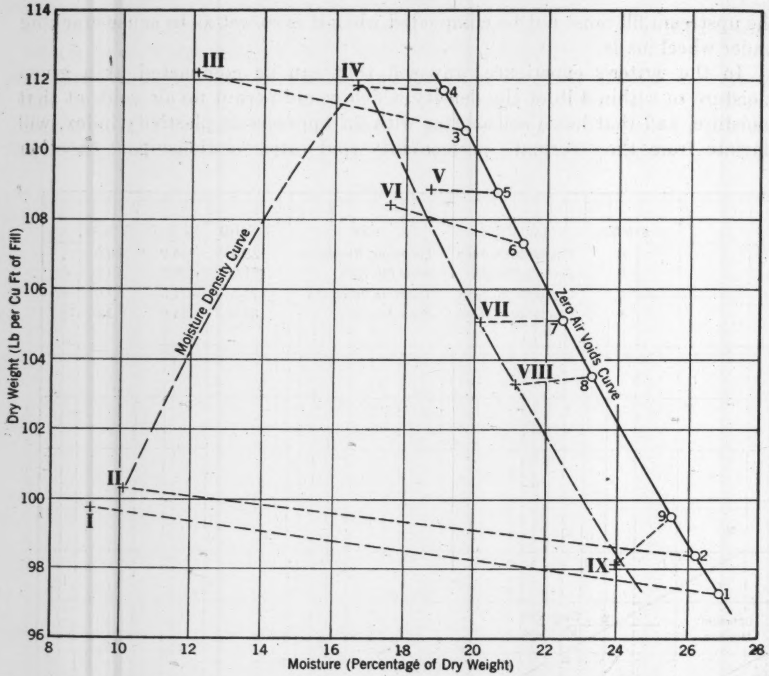


FIG. 8.—EFFECT OF COMPACTION ON VOLUME CHANGES DUE TO SUBSEQUENT WETTING UNDER LOAD OF 160 LB PER SQ FT (SOIL A-5 SILT LOAM, TENNESSEE RIVER FLOOD PLAIN, CHATTANOOGA, TENN.)

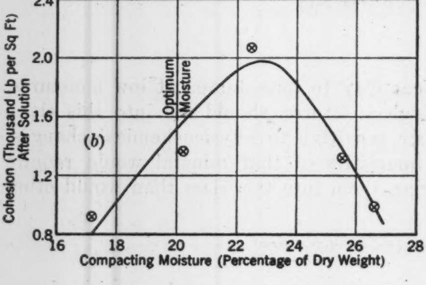


FIG. 9.—EFFECT OF COMPACTING MOISTURES ON FRICTION AND COHESION OF SATURATED SUBMERGED SOIL (SOIL A-5-7 IS A CLAY LOAM, WITH DOLOMITE RESIDUUM, FROM DANDRIDGE, TENN.)

J. E. Christiansen,[9] **Assoc. M. Am. Soc. C. E., and Milton Fireman,**[10] **Esq.**—Differences in permeability that can be obtained merely by varying the soil-moisture content at which the material is compacted are emphasized in this interesting paper. They reveal a phenomenon that probably has not received sufficient attention and about which very little has been reported in the literature on this subject. The tests reported show variations in permeability as high as 10,000 to 1, caused by differences of only 1% or 2% in soil-moisture content at the time of compaction in the permeameter.

It is the purpose of the writers to call attention to another factor, generally overlooked, that also can cause very large differences in permeability values of soils—namely, the chemical composition of the water used for the permeability test. The writers have been primarily interested in the permeabilities of agricultural soils as related to infiltration of rain and irrigation water and to soil drainage. In these studies, the soil in the permeameter has not been compacted highly, but has been brought approximately to the field density of the soil. It is possible that the large differences in the permeabilities of these soils, resulting from differences in quality of water, would be reduced greatly in magnitude if the materials were compacted more highly, but, as yet, no tests have been made to support this possibility.

In preparing the samples for testing, the soil is first air-dried and sieved, poured into the permeameter, leveled, and then jarred a specified number of times to bring the materials to a fairly uniform density. Next, the soils are slowly wetted from below until the soil mass is saturated and most of the air has been expelled. Finally, the water is admitted to the top of the soil column; it passes downward through the soil and is caught and measured in a graduated cylinder.

Distilled water was used in the studies reported by the authors. Some of the soils tested at the Regional Salinity Laboratory in Riverside, Calif., are several hundred times more permeable to local tap water (which contains about 280 ppm of mixed salts—that is, approximately 34 ppm calcium, 35 ppm sodium, 6 ppm magnesium, 16 ppm chloride, 36 ppm sulfate, and 153 ppm bicarbonate) than they are to distilled water, or to waters containing a high percentage of sodium salts. As is well known, the colloidal complex of most agricultural soils of the West is mainly calcium saturated; that is, calcium is the principal absorbed cation. In general, these soils are reasonably permeable to water and have a suitable physical condition for cultivation and crop production. When water containing a high percentage of sodium salts is passed through these soils, base exchange takes place. The sodium ions replace a large part of the absorbed calcium ions, and the soils may become deflocculated, sticky, jellylike, and much less permeable to water. When leached with high sodium water, some soils remain moderately permeable as long as the salt content is fairly high, because the presence of salt tends to keep the soil flocculated. If the salt solution is replaced by a water of considerably lower salt concentration, the soil will disperse immediately and become relatively impermeable.

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[9] Irrig. and Drainage Engr., U. S. Regional Salinity Laboratory, Bureau of Plant Industry, Soils, and Agricultural Eng., Agricultural Research Administration, U. S. Dept. of Agriculture, Riverside, Calif.

[10] Junior Soil Scientist, U. S. Regional Salinity Laboratory, Bureau of Plant Industry, Soils, and Agricultural Eng., Agricultural Research Administration, U. S. Dept. of Agriculture, Riverside, Calif.

Characteristic permeability curves for two agricultural soils are shown in Fig. 10. The curves show the changes in permeability that occurred with the amount of water which passed through the soil. The Hesperia sandy loam is a surface soil from near Bakersfield, Calif., and the Imperial clay loam is from near El Centro, Calif. These soils are typical of many others tested in that they show marked differences in permeability to distilled water as compared with ordinary tap water, or with waters containing only a few hundred parts per million of calcium salts. It will be noted that the Hesperia soil is more permeable to a solution of sodium chloride than it is to distilled water. However, had this soil been subsequently leached with distilled water, it would have been practically impermeable. One soil tested was 35,000 times as permeable to a water containing 800 ppm of calcium chloride as to distilled water.

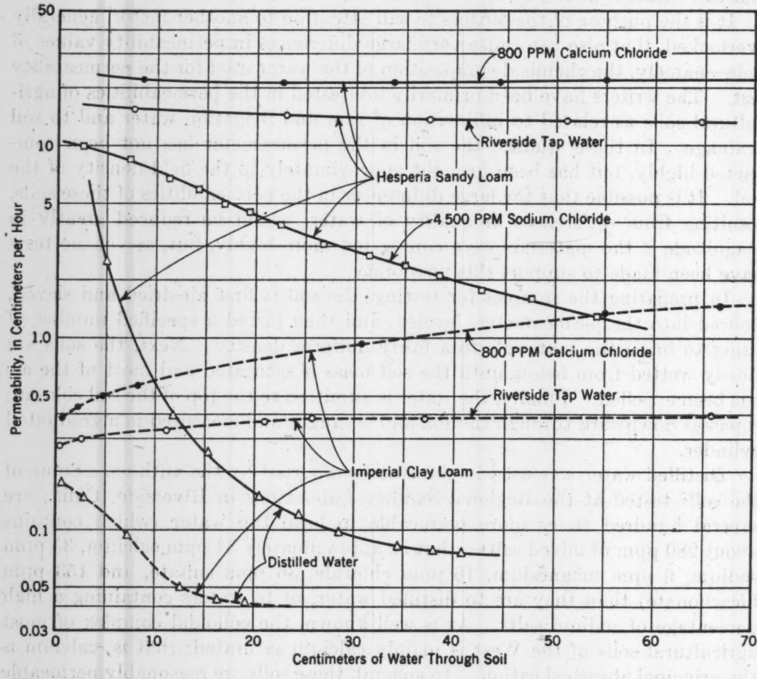

Fig. 10.—Permeabilities of Two Agricultural Soils as Influenced by Water Composition

Therefore, there may be a danger in using distilled water for testing the permeability of soils used for structural purposes. In place in a dam or other embankment, soils are subjected to the percolation of natural soil waters containing dissolved salts. They may be many times more permeable to these soil solutions than to distilled water.

William P. Creager,[11] **M. Am. Soc. C. E.**—For impervious materials an amount of moisture should be provided which is slightly greater, rather than slightly less, than optimum, particularly when natural moisture content is deficient and more must be added. This fact is indicated strikingly in this paper.

It is understood that the laboratory tests of the Mud Mountain Dam core material showed no tendency to swell when saturated subsequent to loading. Although the montmorillonitic fraction of the soil is subject to swelling, it constituted only about 1% of the total volume.

However, the following description of the expected behavior of a soil subject to swelling is so similar to that observed at Mud Mountain that it is felt that, although the Mud Mountain soil did not swell, there must have been some as yet undiscovered action allied to that produced by swelling. The following discussion applies to a soil subject to swelling.

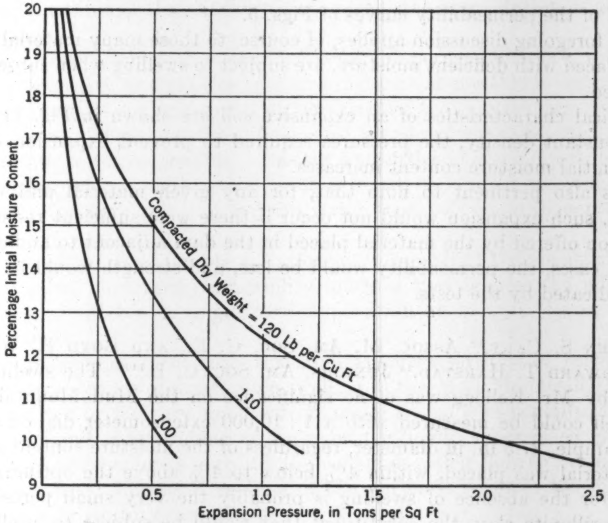

Fig. 11

It is interesting to trace the behavior of a soil swelling having different amounts of moisture when applied on the dam or in the laboratory: For a small amount of applied moisture, the applied moisture is not well distributed throughout the soil, the material is lumpy, and no amount of churning or compaction can distribute the moisture uniformly. Thus, a fraction of the material has too little, and another fraction too great, an amount of moisture. The fraction that has too little moisture may be subject to swelling when subsequently saturated, with resulting increase in voids and loss of strength.

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[11] Cons. Engr., Buffalo, N. Y.

For instance, consider Fig. 3 with 16.5% of applied moisture. If one fraction of the material contains only 13.7% moisture and the other fraction 19.6% moisture and the total dry weight is 120 lb, the fraction containing only 13.7% moisture is subject to swelling, increased voids, greater permeability, and loss of strength.

However, if 19.6% of moisture were applied to the total mass, the dry weight still would be 120 lb, but the material would not be subject to swelling with consequent loss of desirable attributes. As the amount of moisture is increased, the distribution will be much more uniform; swelling and permeability have decreased and strength has increased.

When the moisture is increased to slightly greater than optimum, swelling has been practically eliminated, and the material has reached its optimum of density and strength after subsequent saturation, as shown by Fig. 5(*a*) for a 20% till admixture which has an optimum moisture of 16.5%. The use of an abundance of moisture, however, is obviously deleterious, as shown by the reversal of the permeability curves of Figs. 5.

The foregoing discussion applies, of course, to those many materials which, when placed with deficient moisture, are subject to swelling when subsequently saturated.

Typical characteristics of an expansive soil are shown in Fig. 11. Even for a constant density, the pressures required to prevent expansion decrease as the initial moisture content increases.

It is also pertinent to note that, for any given material and moisture content, such expansion would not occur if there were sufficient resistance to expansion offered by the material placed in the dam adjacent to and above it. In such cases, the permeability would be less, and strength would be greater, than indicated by the tests.

ALLEN S. CARY,[12] ASSOC. M. AM. SOC. C. E., AND BOYD H. WALTER,[13] AND HOWARD T. HARSTAD,[14] JUNIORS, AM. SOC. C. E.[14a]—The swelling mentioned by Mr. Kellogg was of no significance on the Mud Mountain Dam. No swell could be measured with a 1 : 10,000 extensometer dial on a 12-in.-thick sample, 27.5 in. in diameter, regardless of the moisture content at which the material was placed, within 4% below to 4% above the optimum. One reason for the absence of swelling is probably the very small percentage of montmorillonite clay, the constituent that would be subject to swell. Also, as stated in the paper (see heading, "Interpretations"), when the entire sample is at its optimum moisture content for maximum compaction, the finer fractions are considerably above their optimum. It may be surmised, then, that, with the entire sample 4% below its optimum, the fine fractions are near their optimum, at least nearly enough so that any swelling that does occur is so minute that it cannot be measured.

If the Mud Mountain core material were placed at some moisture content lower than optimum, there can be little doubt that subsequent wetting would lower the shearing resistance of the material appreciably. It could not be compacted to the maximum density at a low moisture, and thus the full shearing resistance of the material would not be mobilized. The slightly lower than maximum shearing resistance of the material when placed at a moisture above the optimum is an absolute minimum for the structure as a whole and will tend to increase as settlement and consolidation within the mass take place. However, the core serves no structural purpose in this particular dam; it merely acts as an impervious member held in place by the dumped rock shells.

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[12] Associate Geologist, Chf. Seattle Dist., Soils Laboratory, U. S. Engr. Dept., Seattle, Wash.

[13] Aviation Cadet, U. S. Army, Los Angeles, Calif.

[14] Ensign, U. S. Coast Guard, Norfolk, Va.

The interesting discussion by Messrs. Christiansen and Fireman reveals another point that complicates further the determination of the permeability of a soil. As stated in the paper (see heading, "Equipment and Methods—Water"), distilled and de-aired water was used in the Mud Mountain Dam laboratory for the early part of the work but was discarded later in favor of heated distilled water. The heating was required to insure a falling temperature gradient through the soil sample to prevent the precipitation of air within the sample.

There are so many factors that influence the permeability of a soil that it is almost necessary to eliminate as many variables as possible in order to have a common basis of comparison for those variables that are inherent within the soil structure. Therefore, the influence of air, dissolved salts, etc., must be eliminated in so far as possible. Present knowledge of the subject forbids an unvarying procedure; rather, all variables must be studied individually and collectively for each soil and for the water that will come in contact with the soil.

In view of the considerable variation in permeability that is attributed to dissolved salts, no future permeability analysis that ignores the effects will be entirely applicable to a practical application as in dam, levee, or filter construction.

Mr. Creager has mentioned the swelling characteristics of certain soils. Probably the absence of measurable swelling in the Mud Mountain Dam core material is due to the very small percentage of the subject constituent. It is entirely probable that swelling of the montmorillonite takes place, partly, at least, in such a direction as to reduce the void sizes rather than to cause swelling of the soil mass. The total swelling of the soil mass becomes too minute to measure in a laboratory sample.

Apparently it was not clearly stated in the paper that the moisture content of the tested samples was reduced as little as possible below that at which the samples were to be tested. This precaution was taken to reduce the nonuniform moisture distribution in the soil mass.

AMERICAN SOCIETY OF CIVIL ENGINEERS
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Paper No. 2194

RESISTANCE OF SOFT FILL TO STATIC WHEEL LOADS

By W. Watters Pagon,[1] M. Am. Soc. C. E.

With Discussion by Messrs. Jacob Feld, and W. Watters Pagon.

Synopsis

The mathematics of stress in an elastic body, under a load system applied at the surface of the body, is outlined in this paper. The principles are applied to the construction of airplane runways upon a plastic mud fill, to resist airplane wheel loads and to control the placing of granular fill upon the mud. Finally, the paper shows the practical application of the mathematics to the construction of the landing field of the Baltimore (Md.) Municipal Airport.

Historic Background of the Problem

A general history of the Municipal Airport in Baltimore was presented by the writer in 1940, giving in some detail the character of the mud that was pumped into the site as a base for the construction of a landing field, and how this mud fill was allowed to stand for a long period of years before it could be used.[2] Tests indicative of the character of the mud are given in this reference.

At the beginning of the work described herein, about 12,000,000 cu yd of mud, and about 4,000,000 cu yd of granular fill, had been placed by contract. About 400,000 cu yd of the total fill remained to be placed. The granular fill had been hauled by large capacity trucks, either from a distant borrow pit or from barges loaded with dredged material which were moored to the airport bulkheads. It had been placed in the manner most convenient to trucking and, therefore, much of it was in the form of long "fingers" extending into mud areas, the fingers having a depth such that they had been able to support the weight of the heavy trucks.

When it was decided to abandon this method of construction, post-hole auger borings were made on 5-ft centers each way throughout the area of the four landing strips to determine how much granular fill had been placed on the

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Note.—Published in February, 1942, *Proceedings*.

[1] Cons. Engr., Baltimore, Md.

[2] *Transactions*, Am. Soc. C. E., Vol. 106 (1941), p. 1340.

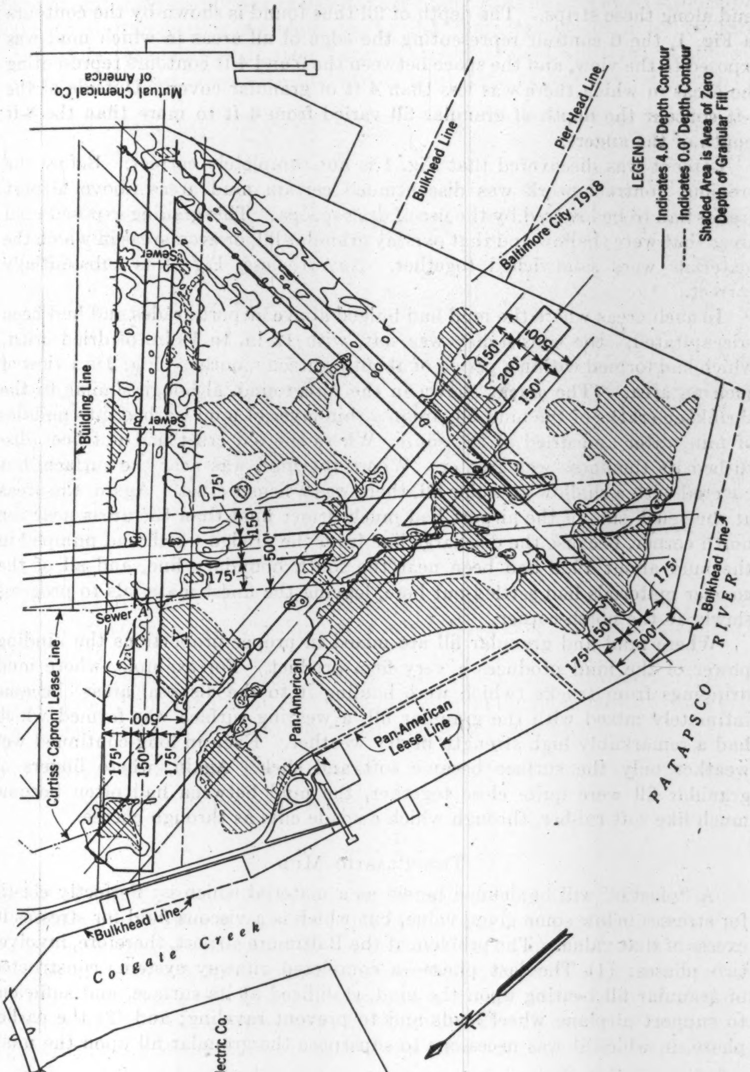
Mutual Chemical Co. of America
Bulkhead Line
Pier Head Line
Baltimore City 1918
LEGEND
Indicates 4.0' Depth Contour
Indicates 0.0' Depth Contour
Shaded Area is Area of Zero Depth of Granular Fill
Sewer
Building Line
Sewer B
150'
200'
150'
175'
150'
500'
175'
Sewer A
Curtiss-Caproni Lease Line
175'
150'
175'
500'
Pan-American
Pan-American Lease Line
PATAPSCO RIVER
Bulkhead Line
175'
150'
500'
175'
Bulkhead Line
Colgate Creek
Western Electric Co.

FIG. 1.—DEPTH OF GRANULAR FILL AT THE BEGINNING OF THE WORK

mud along these strips. The depth of fill thus found is shown by the contours in Fig. 1, the 0 contour representing the edge of all areas in which mud was exposed to the view, and the space between the 0 and 4-ft contours representing the areas in which there was less than 4 ft of granular cover. Outside of the 4-ft contour the depth of granular fill varied from 4 ft to more than the 8-ft length of the augers.

Later it was discovered that Fig. 1 is not completely correct. Before the previous contract work was discontinued certain mud areas above airport grades had to be lowered by the use of drag scoops. This grading exposed mud areas that were shallow and that overlay granular fill, or even areas in which the materials were sandwiched together. Nevertheless, Fig. 1 is substantially correct.

In such areas where the mud had bulged above airport grades and had been "decapitated," the consistency was soft, with 12 in. to 18 in. of dried crust, which had formed within a period of about eighteen months. Fig. 2 is a view of such an area. (The cracks shown in the foreground, although similar to the shrinkage cracks of the mud fill, are in a thin layer of mud washed into puddles of rain water and dried in the sun.) Where the old crust had not been disturbed its thickness was greater. Where the mud was near the surface, but covered with a shallow granular fill, there was a firmer crust. Again, the areas at the south side of the airport had much firmer mud than the areas near the north corner, because the discharge line from the dredge which had pumped in the mud about 1928 had been near the south boundary line, and all of the coarser materials had settled out here, leaving the finest materials to progress slowly to the north corner.

Where mud and granular fill are mixed in proper proportions the binding power of the mud produces a very firm product. For example, where mud drippings from trucks (which were hauling it to the disposal areas) became intimately mixed with the granular fill, a wearing surface was formed which had a remarkably high strength in dry weather. Even in long-continued wet weather only the surface became soft and slick. Again, where fingers of granular fill were quite close together, the mud between had often become much like soft rubber, through which a spade cuts as through cheese.

THE PLASTIC MUD

A "plastic" will be defined herein as a material which is: Perfectly elastic for stresses below some given value, but which is a viscous fluid for stresses in excess of that value. The problem at the Baltimore airport, therefore, involves two phases: (1) The last phase—a completed runway system, constructed of granular fill bearing upon the mud, stabilized at its surface, and sufficient to support airplane wheel loads and to prevent raveling; and (2) the earlier phase, in which it was necessary to superpose the granular fill upon the mud.

THE PROBLEM PRESENTED

The construction problem consisted merely of removing the least quantity of mud, and placing a minimum quantity of granular fill—that is, securing a maximum of uniformity in the depth of fill. The depth of fill must be sufficient to prevent weaving of the runway surface under the airplane loads specified by the Civil Aeronautics Administration; the character of fill must be satisfactory for asphalt stabilization because the entire field will slowly settle with time as the water leaves the underlying mud, causing any overlying concrete pavement to break. As far as possible, the material used for fill must be sharp and gritty so as to have a maximum of interlocking capacity and a minimum of capillarity.

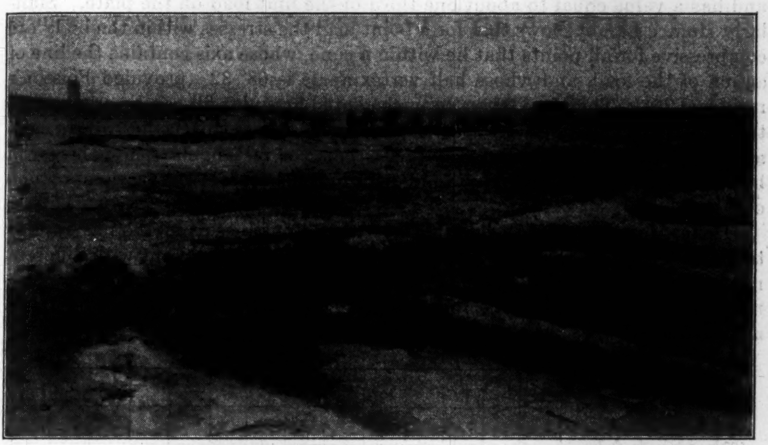

FIG. 2.—CHARACTERISTIC VIEW OF EXCAVATED MUD AREA

The purpose of this paper is to describe the method of determining how great a depth of granular fill should be placed upon the mud, how this fill should be placed, and how the mud should be, and was, removed.

STRESS IN A BODY PRODUCED BY A POINT SOURCE AT ITS SURFACE

The curves of Boussinesq's formula

$$s_z = \frac{3\,P\,z^3}{2\,\pi\,R^5} \qquad (1)$$

are shown in Fig. 3, in which s_z is the stress at various depths z below the ground surface and distances R from the point of application of the load P (r in Fig. 3 is the horizontal component of R). For a depth of 24 in. the stress is about one fourth of that for a 12-in. depth, but for a 36-in. depth it is again more than halved. Of course this theory presupposes a strictly elastic body throughout, whereas the airport consists of a granular and earthy fill overlying elastic mud. Experience has shown that the theory is a good approximation for earth materials, but there is no theory and no experience for the case in which the earth materials are superposed upon a mass of mud. It is fair to assume that so long as the mud is not overstressed, it can be considered an elastic support under a stratum of material that will be pseudo-elastic, or at least capable of transmitting load through a cone of material. That earth materials can so transmit load was demonstrated at The Pennsylvania State College at State College, Pa.[3]

It can be shown that for a load applied uniformly over a circular plate the maximum shear occurs at a depth about equal to one third of the plate diameter, and has a value equal to about one third of the unit load on the plate. Similarly Horace Lamb[4] shows that for a point load the stresses within the body are compressive for all points that lie within a cone, whose axis contains the line of action of the load, and whose half vertex angle is 68° 32′ (provided Poisson's ratio is 0.25). Therefore, even an uncemented granular fill can be assumed to transmit vertical load to any area, at any depth, whose surface lies within such a cone. For a load distributed over a circular or rectangular plate there will be a frustum of a cone or pyramid within which there will be compression only.

Hence, for want of a theory that expresses Boussinesq's method as applied to a composite body having two layers, each perfectly elastic but with different moduli of elasticity, it becomes necessary to use the cone-of-compressed-material concept, and in the following the half vertex angle has been taken as 60°.

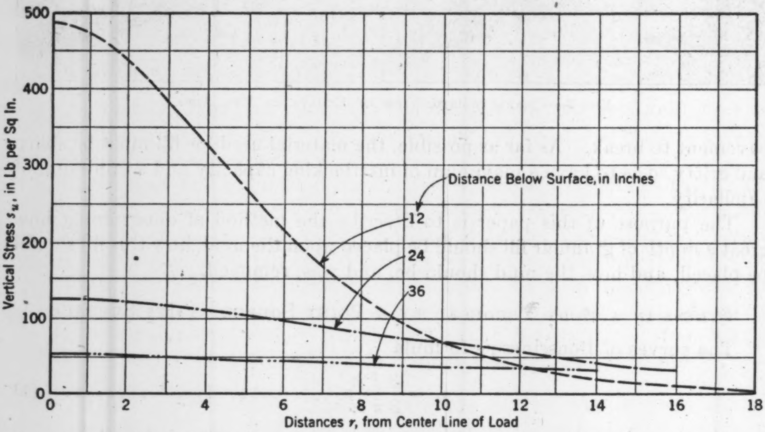

FIG. 3.—CURVES OF THE BOUSSINESQ FORMULA FOR A THOUSAND-POUND LOAD

If the computation is based upon an airplane wheel concentration of P = 50,000 lb, there immediately arises the question of contact area between tire and paving. At present it is true that the maximum tire pressure is about 65 lb per sq in., and to allow for further development 75 lb per sq in. will be used. This means that such a wheel concentration would require 667 sq in. of contact surface—that is, a circle of 29-in. diameter, or an ellipse of 24 by 36 in., etc. Bomber B-19 is reported to have a wheel load of 82,000 lb on a tire-imprint area of 1,000 sq in.[5] In order not to exceed a safe basis of computation, a diameter of 36 in. will be used which, for a depth of granular fill of 3 ft, and a half angle of 60°, would load an area of 13.4-ft diameter, or 140 sq ft; and, for a depth of 4 ft, an area of 225 sq ft. The respective uniform unit loads on the mud surface, for $P = 50{,}000$ lb, would then be 360 lb per sq ft, and 225 lb per sq ft.

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[3] "Experiments on the Distribution of Vertical Pressure in Earth," by R. B. Fehr, *Bulletin No. 11*, The Pennsylvania State College, State College, Pa., 1919, p. 118.

[4] "Hydrodynamics," by Horace Lamb, University Press, Cambridge, 1925, pp. 188–189; also "A Treatise on the Mathematical Theory of Elasticity," by Augustus Edward Hough Love, University Press, Cambridge, 2d Ed., 1906, p. 189.

Using the load of 225 lb per sq ft on the surface of the mud crust, and neglecting any value in the mud crust, the maximum shear, from the foregoing statement, would be about 70 lb per sq ft, and would occur beneath the center of the wheel concentration and at a depth of one or more feet below the mud crust. This might place the highest shear stress in the semiliquid mud; hence it becomes necessary to determine whether the mud can carry a shear of 70 lb.

If the shear exceeds the shearing resistance of the mud, movement will occur until the material flows sufficiently to reduce the maximum stress.

VISCOSITY TEST

To determine roughly what shearing strength the mud possessed, a test was made by screwing a duckpin ball to a galvanized pipe and loading the affair.

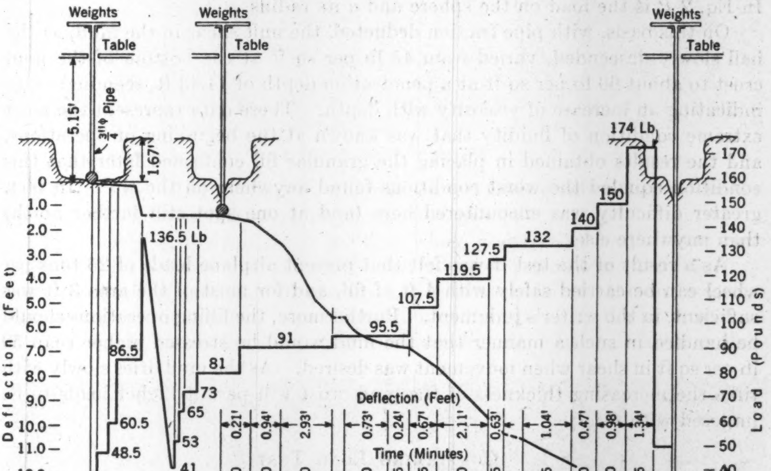

FIG. 4.—MUD VISCOSITY TEST, BALTIMORE MUNICIPAL AIRPORT, OCTOBER 20, 1939

Fig. 4 shows the curve of this test, with deflections, loading, time, and average velocities of descent. The test could not be entirely continuous, because twice it was necessary to add lengths of pipe, which involved removing the load. The ball had a diameter of $4\frac{1}{2}$ in. The site chosen at point X, Fig. 1, was an area in which the mud had been decapitated with a drag scoop about 18 months previously, and had then dried in the sun as shown in Fig. 2. The test was started at a depth of about 1.67 ft below the surface, which was where the mud was close to the plastic limit. It was impossible to know what friction was developed on the pipe, but the mud tended to develop a water film on the pipe as it closed in behind the ball, whose frictional resistance would be very small. Therefore, a friction force of 20 lb per sq ft of pipe surface was assumed.

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[5] *Construction Series No. 59*, The Asphalt Inst., p. 5.

In the deflection range from 2.67 to 5.60 ft, the coefficient of viscosity is about 3,400; from 6.98 to 10.39, about 6,300; from 10.60 to 13.09, 9,700; and from 13.09 to 14.43, 5,900 $\left(\text{units are } \frac{\text{pounds} \times \text{seconds}}{\text{feet}^2}\right)$. If a greater value is given to the pipe friction, these results would differ accordingly. The maximum unit shearing stress on a sphere moving through a viscous liquid is 0.25 p, in which

$$p = \frac{P}{\pi a^2} \qquad (2)$$

In Eq. 2 P is the load on the sphere and a its radius.

On this basis, with pipe friction deducted, the unit shear in the mud, as the ball slowly descended, varied from 45 lb per sq ft at the bottom of the mud crust to about 60 lb per sq ft at a penetration depth of 14.43 ft, seemingly also indicating an increase of viscosity with depth. These data represent the most extreme condition of fluidity that was known at the beginning of operations, and the results obtained in placing the granular fill confirmed later that this condition equaled the worst conditions found anywhere on the site. In fact, greater difficulty was encountered here (and at one spot still farther north) than anywhere else.

As a result of the test it was felt that present airplane loads of 25 tons per wheel can be carried safely with 4 ft of fill, and for most of the area 3 ft was sufficient, in the writer's judgment. Furthermore, the filling operations should be handled in such a manner that the mud would be stressed higher than 50 lb per sq ft in shear when movement was desired. As the mud dries slowly with time the increasing thickness of the mud crust will permit higher loads to be imposed with safety.

Confirming Load Test

As a practical check on the mathematics, a load test was instituted. A truck was obtained of the low-body type used for hauling large equipment. Its weight was 11,275 lb on two rear axles, each axle being equipped with a pair of dual wheels, and a load of 5,500 lb on the front axle, equipped with two wheels. The wheel base of the rear end is 4 ft 3 in. between axles, and the front axle is 15 ft 10 in. from the forward rear axle. The truck body is 8 ft wide. The rear dual wheels are 12 in. apart, with 5 ft between centers of inner wheels. All wheels are equipped with 9.75 by 20 pneumatic tires. This truck was loaded with 17 tons of pig lead (see Fig. 5), making a total rear-end load of 22.6 tons. Allowing for tire compression the 22.6 tons were applied on two areas, each about 2 ft by 5 ft or somewhat more, one surrounding each set of four wheels. On actual airplanes having loads comparable to these, the load wheels are from 18 to 22 ft apart, whereas the two areas combined approximate, roughly, the load on one single airplane wheel group.

FIG. 5.—TEST TRUCK LOADED WITH SEVENTEEN TONS OF PIG LEAD

This truck was driven about the field where granular fill had already been placed. The most severe condition occurred at one spot where borings indicated a depth of granular fill of only 1.9 ft measured from the surface, and actually several inches less from the bottom of the rut which the tires made in the soft, loose, untreated airport surface (see Fig. 6). In some areas a bulldozer was required to increase the tractive effort. The truck was left in such a position for 24 hr, and showed a settlement of about $\frac{3}{8}$ in. Again at various spots a 47-ton caterpillar crane on a wood mat was operated continuously upon fresh fills of 4-ft depth. When in operation the crane would jar the mud, and

FIG. 6.—RUTS FORMED IN THE SOFT SURFACE OF THE GRANULAR FILL

slowly settle, so that every hour or so it was necessary to back it off and place more fill, and then to move it forward for a further reach. The mud clearly is "thixotropic"—that is, it becomes less viscous and more fluid when jarred, but regains its viscosity when allowed to be quiet. The truck-hauling roads also "held up" with a depth of about 4 ft, although in the softest areas there was slow and continuous settlement.

STRESSES DURING CONSTRUCTION OF THE GRANULAR BLANKET

In 1912 S. D. Carothers[6] derived a method of computing stresses within an elastic body when subjected, not to point loads, but to uniform, or uniformly varying, loads, having application only to two-dimensional problems. By assuming certain plane harmonic functions, or combinations of functions, he derived the stress and strain, for example, for a load having a triangular cross section, beginning with zero load along a straight line, and increasing linearly at right angles to this line. If, at a distance a, a reverse load of equal amount is then applied, there results a load condition which represents a uniform fill bounded on one edge by a sloping berm. For instance, this would represent the condition of the airport under a uniform fill advancing across the field with a straight front edge and having a slope along the front edge (see Fig. 7). By other modifications he obtained the stress condition for an infinitely long prismatic load—as, for example, a highway or railroad fill.

The writer converted this principle to derive the stress condition when a trench of infinite length is opened through the dried mud crust (see Fig. 8). He then amplified this theory by superposing a triangular prism along one or both edges of the trench as shown in Fig. 9. Since shear stress is the condition sought, the shear for different types of loading was computed, and is shown in Fig. 9 in a section at right angles to such loadings, with a sloping berm, having a horizontal width, a, the depth of fill being h, and the weight of fill $w\,h$, or p, per square foot. The drawings are dimensionless; that is, they are proportional throughout to the length a. It will be seen that the shear is maximum, for the blanket, under the middle of the slope, and at a depth of about 3.5 a, with a value of 0.318 $w\,h$. For other relations between the width of side slopes and the width of the trench, the numerical results would change slightly. The maximum shear occurs at a depth approximately equal to a and has a value varying from 0.31 p to 0.35 p. Fig. 10 represents half of the width of a runway, having a crown equal to h, bordered by a trench as shown.

Fig. 11 represents the program that was established for the construction of the runway fills through the soft mud areas. The initial condition is shown in Fig. 11(a), and the final condition in Fig. 11(g). The runway is crowned, and at each edge there is a perforated drain. The depth of granular fill is 4 ft, as is the minimum depth of each lateral drain. Fig. 11(b) indicates the mud area that is subject to motion for the condition shown. The intent was to construct, first, a hauling road down the center of the runway, noted as Fill 2, with (if possible) a hand-placed fill, marked Fill 1, placed previously. As Fill 2 settled under the vibration of the trucks hauling in the material, and dumping

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[6] "Plane Strain: The Direct Determination of Stress," by S. D. Carothers, *Proceedings*, Royal Soc. of London, Vol. 97, 1920, Series A, p. 110.

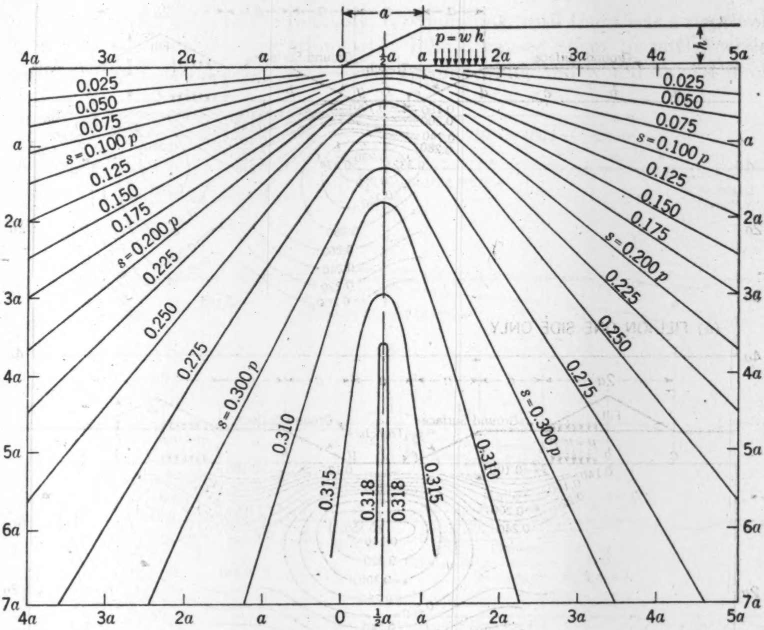

FIG. 7.—SHEAR AT EDGE OF GRANULAR BLANKET

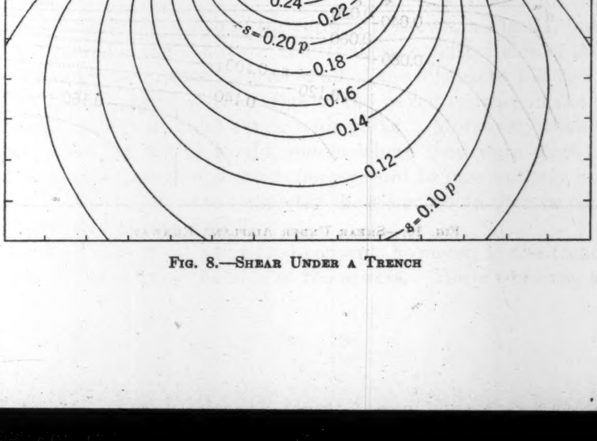

FIG. 8.—SHEAR UNDER A TRENCH

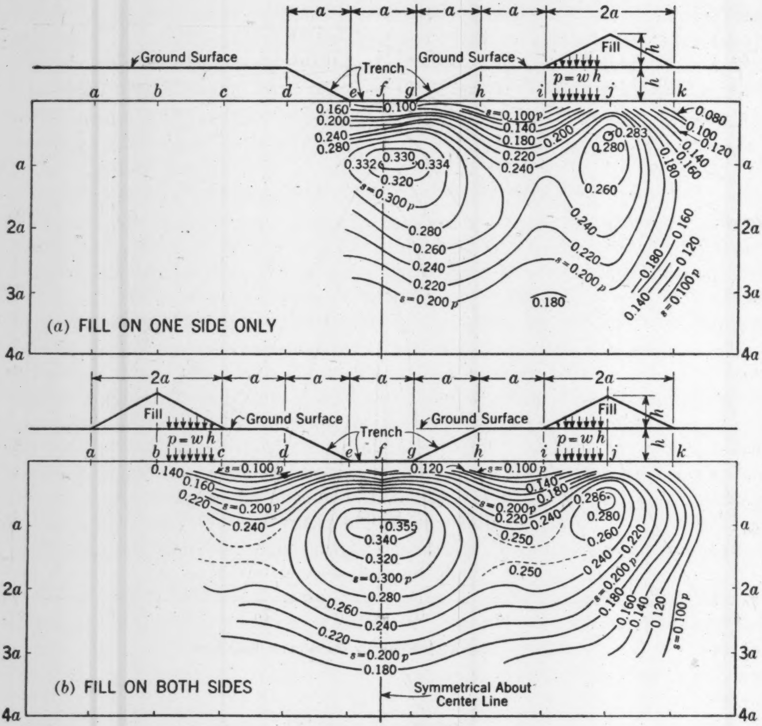
a
2a
h
Ground Surface
Trench
Ground Surface
Fill
$p=wh$
a
b
c
d
e
f
g
h
i
j
k
2a
3a
4a
0.160
0.200
0.240
0.280
0.100
$s=0.100p$
0.140
0.180
0.220
0.240
0.200
0.283
0.280
0.080
0.100
0.120
0.140
0.160
0.180
0.330
0.332
0.334
0.320
$s=0.300p$
0.260
0.280
0.260
0.240
0.220
$s=0.200p$
0.240
0.220
$s=0.200p$
0.180
0.160
0.180
0.120
0.140
$s=0.100p$
(a) FILL ON ONE SIDE ONLY
0.140
0.160
0.220
$s=0.200p$
0.240
0.120
0.355
0.340
0.320
$s=0.300p$
0.280
0.260
0.240
0.220
$s=0.200p$
0.180
0.250
0.286
0.280
0.260
0.240
0.220
$s=0.200p$
0.180
0.160
0.140
0.120
$s=0.100p$
(b) FILL ON BOTH SIDES
Symmetrical About Center Line

FIG. 9.—SHEAR UNDER A TRENCH, WITH ADJACENT FILL

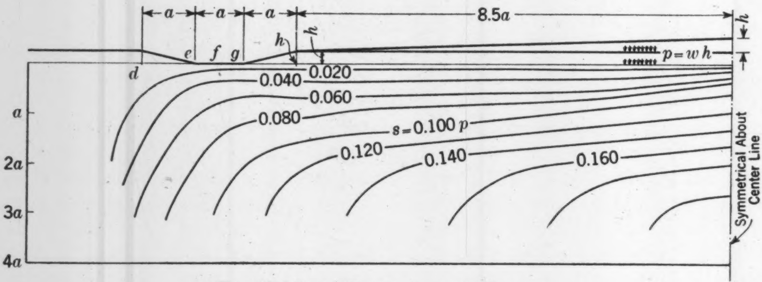
a
8.5a
h
$p=wh$
d
e
f
g
h
0.020
0.040
0.060
0.080
$s=0.100p$
0.120
0.140
0.160
2a
3a
4a
Symmetrical About Center Line

FIG. 10.—SHEAR UNDER AIRPLANE RUNWAY

it along the two sides of the roadway, it would sink until there was a depth of 4 ft. As the trucking operations over Fills 3 caused them to settle, while placing Fills 4, the hauling road would become wider and would have 4 ft of

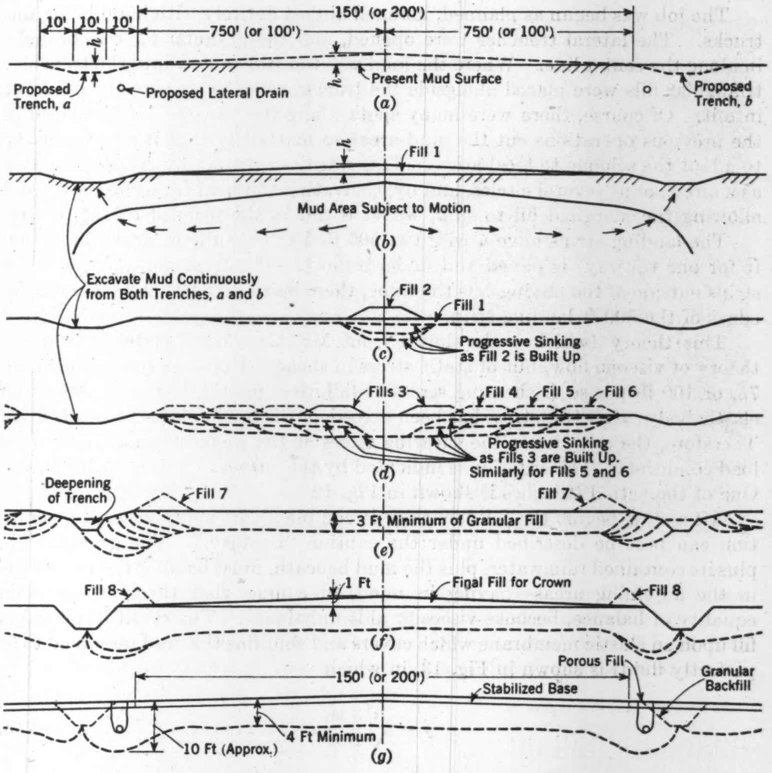
10'
10'
10'
150' (or 200')
750' (or 100')
750' (or 100')
h
h
Proposed Trench, a
Proposed Lateral Drain
Present Mud Surface
Proposed Trench, b
(a)
h
Fill 1
Mud Area Subject to Motion
(b)
Excavate Mud Continuously from Both Trenches, a and b
Fill 2
Fill 1
Progressive Sinking as Fill 2 is Built Up
(c)
Fills 3
Fill 4
Fill 5
Fill 6
Progressive Sinking as Fills 3 are Built Up. Similarly for Fills 5 and 6
(d)
Deepening of Trench
Fill 7
Fill 7
3 Ft Minimum of Granular Fill
(e)
Fill 8
1 Ft
Final Fill for Crown
Fill 8
(f)
Porous Fill
150' (or 200')
Granular Backfill
Stabilized Base
4 Ft Minimum
10 Ft (Approx.)
(g)

FIG. 11.—PROCEDURE FOR BLEEDING SOFT MUD FROM BENEATH 500-FT LANDING STRIPS

fill beneath it. Only the width of the runway is shown in Fig. 11, but the procedure is the same for the 175-ft (or 150-ft) strips on either side of the runway width. The same description applies to all of the fills up to Fill 7.

To place lateral drains at each edge of the paved runway in the soft mud was impossible, except at great expense for piling. Moreover, there would be no certainty that the drains would remain where they were placed. Hence Fills 7 were designed to extend much deeper, and to pass entirely by the site of the sewer. When they were completed the sewer would then be constructed in an open trench dug into Fills 7.

There was an important additional concept, however, in the trench shown on each side of the runway, outside of the sewers. These trenches, known as "bleeding trenches," were planned as the line along which mud would be excavated. As the crust would be cut near the same line for the sewers anyhow, two purposes were served at the same time, and the strength of a considerable area of crust thus conserved.

The job was begun as planned, manned almost entirely with hand labor and trucks. The lateral trenches were opened, and the granular fill was brought in along the center line. Where the mud was too stiff to flow into the trenches, triangular fills were placed alongside the trench, so as to squeeze the mud out into it. Of course, there were many spots where the "fingers" of fill placed in the previous operations cut the mud areas so materially that it was necessary to adapt the scheme to local areas. This was done merely by treating an area as a circle, or as several circles, and by excavating the mud from the center and allowing the marginal fill to sink, which it did in the form of a large crater.

The landing strips have a width of 500 ft, but only the central 150 ft (200 ft for one runway) is paved and underdrained. The treatment of the 175-ft strips outside of the paving was the same, there being an outer trench along the edges of the 500-ft landing strip.

This theory (which is developed from Mr. Carothers' system[6]) is not a theory of viscous flow, but of static stress in shear. Until the test value of 50, 75, or 100 lb per sq ft shearing stress at failure is passed, the mud acts as an elastic body; after this stress has been exceeded, the mud acts as a viscous fluid. Therefore, the criterion is the shearing stress at the weakest point under each load condition. The criterion is indicated by the curves of unit shearing stress. One of the actual trenches is shown in Fig. 12.

After flow begins the mathematics is complex. In many cases the condition can best be described under the caption "isostasy." The granular fill, plus its contained rain water, plus the mud beneath, must balance the mud alone in the adjoining areas—having in mind, of course, that there is not exact equality of balance, because viscosity aids unbalance. The result of placing a fill upon an elastic membrane which covers and contains the mud (assumed to be perfectly fluid) is shown in Fig. 13, in which

$$j = \sqrt[4]{\frac{3\,w_2}{E\,h^3}} \qquad \text{(3)}$$

In Eq. 3, E = modulus of elasticity, w_2 = unit weight of mud, and h = depth of granular fill. When w_2 = 80 lb per cu ft and E = 300,000 lb per sq ft, x_0 for depths of 1, 2, 3, and 4 ft, respectively, is:

h (ft)	x_0 (ft)
1	14
2	24
3	32
4	40

This form of mud wave is quite similar to the waves that actually have developed on the field. It is clear that the shear is a maximum at the edge of the fill. Of course, the dried mud crust cannot be considered a perfect membrane.

However, it seldom happened that the crust sheared, so that in effect the crust did form a membrane. At the north corner of the airport the runway grade is 3 ft above the field, and, because of this inclination of the crust as the fill advanced, there were some places where the crust was forced forward until it reached solid fill, when it would at times be forced up over this fill like ice on a lake shore. This was corrected by filling uphill instead of downhill.

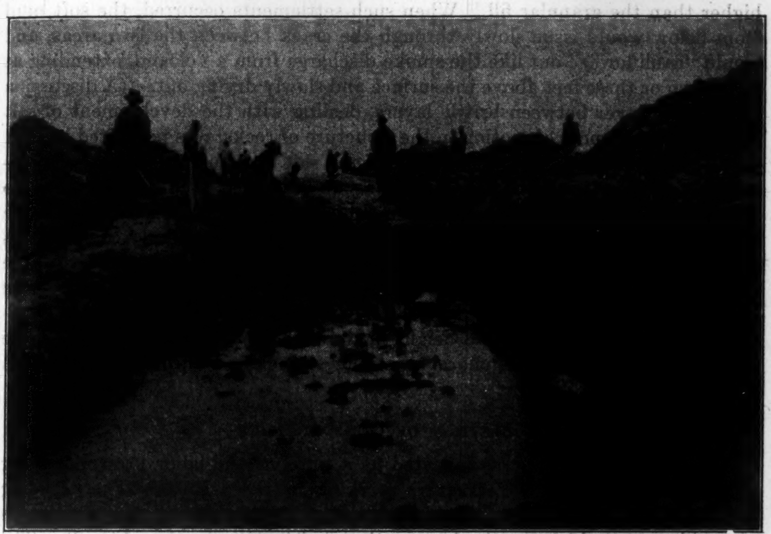

FIG. 12.—ONE OF THE BLEEDING TRENCHES AFTER A PERIOD OF RAINY WEATHER

In the area near where the ball test was made the mud did not behave as well as elsewhere. Through lack of foresight, and due to the fact that the fill occurred only along one side of the runway, the hauling road was held too long at one site, with several fills made upon it in order to maintain grade. The result was that after a heavy rain storm a considerable waterlogged area sheared

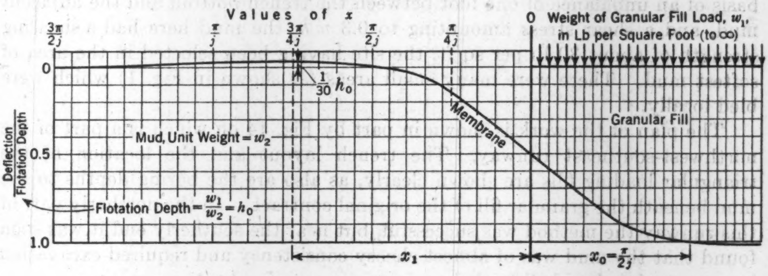
Values of x
$\frac{3\pi}{2j}$
$\frac{\pi}{j}$
$\frac{3\pi}{4j}$
$\frac{\pi}{2j}$
$\frac{\pi}{4j}$
0
Weight of Granular Fill Load, w_1, in Lb per Square Foot (to ∞)
0
0.5
1.0
$\frac{\text{Deflection}}{\text{Flotation Depth}}$
$\frac{1}{30}h_0$
Membrane
Granular Fill
Mud, Unit Weight = w_2
Flotation Depth = $\frac{w_1}{w_2} = h_0$
x_1
$x_0 = \frac{\pi}{2j}$

FIG. 13.—FORM OF MUD WAVE, ASSUMING CRUST TO BE AN ELASTIC MEMBRANE OVER A FRICTIONLESS FLUID

down past the near-by fills and sank a foot or more. It later became stable. At another point, very close to the south side of the field (where the mud in general was stiffest), there was an area of somewhat similar character in which the granular fill subsided, and the mud near-by rose until it stood perhaps 5 ft higher than the granular fill. When such settlements occurred, the soft mud from below would issue slowly through the crack between the two areas, and would "cauliflower" out like the smoke discharge from a volcano, extending at times two or three feet above the surface and slowly drying out. (A discussion of a plastic layer between brittle layers, dealing with the development of salt domes and evidences of motion in the structure of rocks, was presented in 1931 by A. Nadai.[7] For example, the flow of the salt layer under the superincumbent layer of rock is described, and its accumulation at points of weakness to form domes. Photographs of models demonstrate the flow of lava through the throat of a volcano; and they also show slip lines where the plastic materials have sheared on either side of the throat as well as flow lines indicating the character of flow into and up through the throat. At the lower end of the throat the slip lines converge, and the flow lines indicate a sort of "vena contracta," which disappears at a height in the throat of roughly three times the throat diameter, such as is characteristic of viscous flow through a tube, with slip at the walls.)

METHOD OF CONSTRUCTION

As stated, the work was initiated as a purely hand-labor project, having about 1,000 laborers and truckers working in three shifts. The bleeding trenches were first opened up through the mud crust, using runway boards alongside upon which to wheel away the mud in barrows (see Fig. 2). The mud was sticky, which made it difficult to disengage it from the shovel unless a pail of water was handy. The men had to stand on planking. Each day the trench would be excavated to a depth of from 12 in. to 24 in.; each night it would fill up again to within about 6 in. to 12 in. of the surface, the cross section of the surface of the mud as it oozed up being approximately a parabola. On the basis of an unbalance of one foot between the trench bottom and the adjacent mud, and a shear stress amounting to 0.3 $w\,h$, the mud here had a shearing strength of about 30 lb per sq ft, the site having been selected in the area of softest mud. There were many small areas (as shown in Fig. 1) which were bled locally.

The plan of the work is shown in part by Fig. 14 on which is a part of the northwest-southeast runway. The trench layout and the location of the triangular loading fills are shown clearly, as also are the boring depths to the mud beneath the granular fill of the original contract. In the northerly part of this runway the method was successful, but near the southerly end it was soon found that the mud was of almost cheesy consistency and required excavation by means of a clamshell bucket.

The hand-labor method, however, proved to be too slow, and progress was delayed. This was aggravated by the fact that the work was begun in De-

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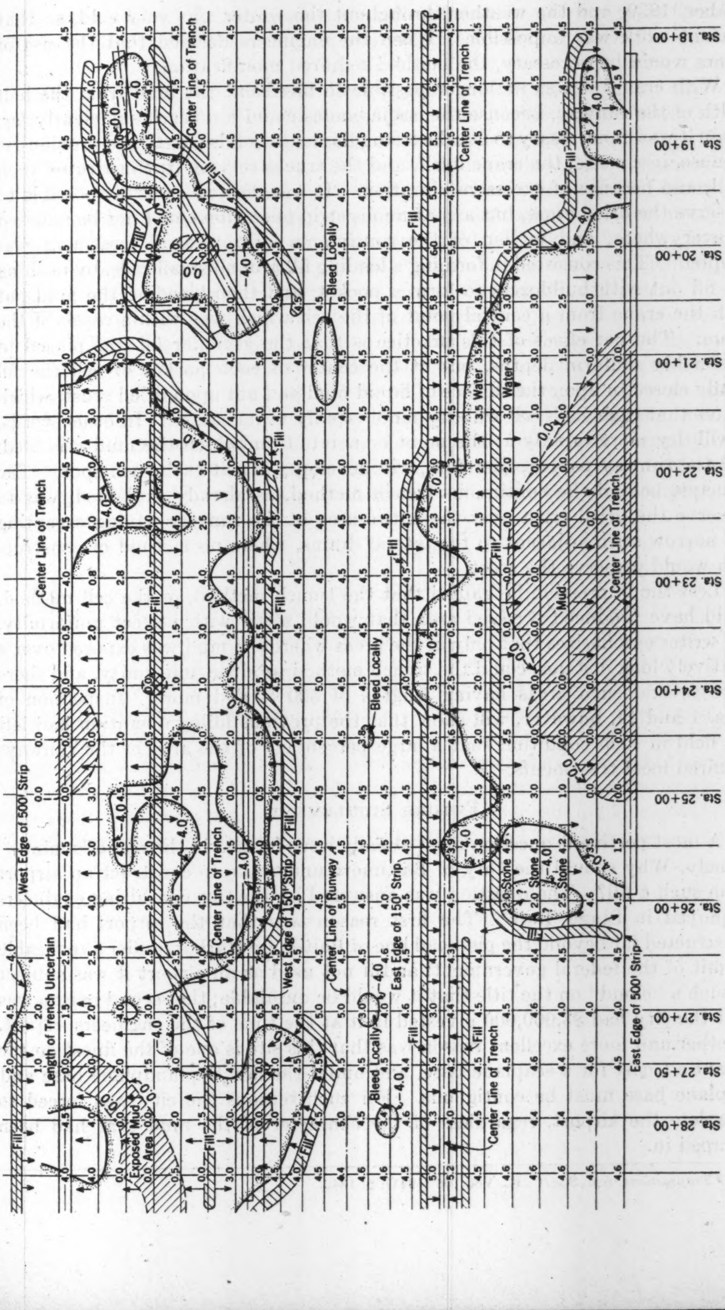
[7] "Plasticity; A Mechanics of the Plastic State of Matter," by A. Nadai, McGraw-Hill Book Co., Inc., 1931, Chapters 43–45.


Sta. 18+00
Sta. 19+00
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Sta. 23+00
Sta. 24+00
Sta. 25+00
Sta. 26+00
Sta. 27+00
Sta. 27+50
Sta. 28+00
West Edge of 500' Strip
Length of Trench Uncertain
Exposed Mud Area
Center Line of Trench
Center Line of Runway
West Edge of 150' Strip
East Edge of 150' Strip
East Edge of 500' Strip
Bleed Locally
Fill
Water
Mud
Stone

FIG. 14.—TYPICAL LAYOUT OF WORK FOR HAND-LABOR OPERATION

cember, 1939, and the weather throughout the winter was very cold, so that efficient work was impossible. When the engineers decided that the use of cranes would be necessary, the method required modification.

With cranes it was no longer possible to bleed out the mud from the half width of the runway, because the crane booms could not reach sufficiently far. Again it was unnecessary to use the triangular loading fills mentioned previously, because, in a sense, the crane itself, and the trucks serving it, constituted such a fill, and because of excessive vibration. It was desired as far as possible to preserve the dried crust, but a continuous strip operation could not be adhered to everywhere; and in lieu of the continuous strip, a local treatment was adopted. This consisted in forming a loading fill around small cells, by pushing the fill out with bulldozers to form a pocket, and then bleeding the mud out with the crane from a central point of the cell which was within reach of the boom. The net effect of this practice is that the granular fill was placed to reasonably uniform depths, but at the center of each pocket or cell the fill finally closed in upon the surface of liquid mud and not upon dried crust, which means that each such cell will dry more rapidly at its center. In other words, it will dry more rapidly at the point or points from which the mud was bled, and the runway paving, therefore, will develop a sag at each such spot. The principle behind the continuous trench method, as already explained, was to preserve the dried crust over the entire area of the runway paving excepting the narrow strip adjacent to the lateral drains, where no method of construction would preserve the crust.

Lest the impression be gained that the trench method, or the cell method, could have produced a paved area that would settle with perfect uniformity, the writer emphasizes that only in the areas where the mud was exposed over a relatively long distance could the trench method produce uniformity, and there were several such areas having lengths of 500 ft and more. Inspection of Figs. 1 and 14, however, will show that the previous filling operation had left the field in such condition that a large percentage of the area of the runways required local treatments.

FUTURE SETTLEMENT

A most pertinent question must already have formed in the reader's mind—namely, Why should the City of Baltimore undertake to construct an airport upon such a fill? The question was answered by the two consulting engineers employed in Stage Two.[8] The first reason was that the airport had been constructed far beyond the pierhead line of Baltimore Harbor, under a revocable permit of the federal government, and if not used as an airport it was subject to such a "cloud" on the title that it would be unsalable; the second reason was that the city had $3,000,000 invested in it at the time of the engineers' report. Another and more excellent reason was that this site is one of the finest on the Atlantic Coast for a seaplane base, and under the law the landplane base and seaplane base must be contiguous. For such reasons the city was forced to complete the airport, regardless of the character of the mud that had been pumped in.

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[8] *Transactions*, Am. Soc. C. E., Vol. 106 (1941), p. 1342.

The consulting engineers stated that the fill would settle for years to come. The mud surface has performed as was predicted, because there was a settlement of the surface of from 2 to 3 ft (in a depth of about 25 ft) in ten years. However, the rate of settlement slows rapidly as the upper part of the mud dries, because the rate of percolation of the water to the surface where it can dry out in the sun is extremely slow, due to the fineness of the mud particles.

Where the dried crust was undisturbed in the operation, the settlement will be slow, for this reason, although it will continue for a long period; where the dried crust was removed, and the granular fill placed upon fresh soft mud, the settlement will be greater, because the upper part of the mud will dry rapidly by absorption of water into the dry fill. This was one of the major reasons why the writer endeavored in every way not to remove the dry crust.

Where the stabilized surface of the runways has been completed for six to ten months the settlement has been sufficiently uniform so that few water pockets have formed on the surface, despite the fact that the crown of the runways is only 9 in. in a width of 150 ft. However, where one runway crosses three main sewers that were built previously upon a pile foundation, and over an old buried concrete bulkhead, there is an evident settlement of nearly 3 in. No other measurements can be made because the surface of the stabilization was not sufficiently exact to permit measurement of the small amount of settlement.

ACKNOWLEDGMENTS

A part of the funds expended, and most of the labor required, were provided by the Works Progress Administration, with Lt.-Col. Harry D. Williar, Jr., M. Am. Soc. C. E., as state administrator, and C. F. Bornefeld, M. Am. Soc. C. E., as state director, Division of Operations. The work was done under the direction of George Cobb, chief engineer, and handled directly by Charles F. H. Foard, associate engineer. Sewer construction was under the direction of George E. Finck, M. Am. Soc. C. E., sewerage engineer. Paving and grading were under George A. Carter, highways engineer, and were handled in the field by Charles N. Berigtold, assistant engineer. The writer is airport consulting engineer, and was directly responsible for all mud removal and placing of granular fill, as well as for the buildings and field lighting.

DISCUSSION

Jacob Feld,[9] **M. Am. Soc. C. E.**—The correlation of the elastic analysis and field tests with observations of actual construction procedure and of results forms a valuable contribution to the subject of control and use of unstable ground areas. The superposing of a granular sand as a blanket over plastic or semifluid soils for the preparation of roadways is often part of heavy construction procedure in outlying areas. Mr. Pagon's analysis of the necessary thicknesses of granular materials will be of great aid. As reported by the writer[10] in 1940, similar results were found by trial experience in the construction of temporary roads in salt marshes. A blanket of sand 3 ft deep, placed by dumping from trucks, provided a roadbed which sustained heavily loaded construction trucks even though the adjacent ground could not be walked upon.

In very soft spots a layer of metal mesh or chain fabric pressed into the top surface of the sand will provide additional carrying capacity for the period in which the natural consolidation of the material makes it somewhat softer than the adjacent consolidated area. Such a method would aid in reducing the upkeep costs of runways placed upon plastic mud fills.

The action of the diaphragm formed in the contact zone, where the granular material and the finer grains of the original plastic mud intermix, was studied very carefully by Gen. M. C. Barberis, of the Italian Navy, from 1910 until quite recently. One of the reports easily available was presented at the 16th International Congress of Navigation, Brussels, Belgium, 1935, under the title, "Recent Examples of the Foundations of Quay Walls Resting on Poor Subsoil." This is the only report written by General Barberis in English, and makes reference to his various other more lengthy papers in Italian and French.

A summary of all of the studies made by General Barberis as listed by him for the writer in 1935, after a lengthy discussion following an inspection of the work in 1931, is as follows:

(1) Docks built in Spezia, Italy, on a foundation of fluid subaqueous mud showed movements in accordance with the hypothesis and theories of Vierendeel and Rankine.

(2) The interposition of a layer of clean sand hydraulically placed between subaqueous mud and rock-fill foundation of the docks shows a resistance three times as great as would be expected from the Rankine theory in the work performed from 1912 to 1915, and six times the theoretical resistance was found in the work performed from 1930 to 1934.

(3) The great additional resistance resulting from the interposition of a clean, fine, sand layer is due to the large cohesion between that material and the underlying mud.

(4) When the sand layer is sufficiently extensive in area (say, 200 ft measured at right angles to the face of the dock) a thickness of 6 to 8 ft is sufficient to imprison the underlying clayey mud. After this condition had existed

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[9] Cons. Engr., New York, N. Y.

[10] *Proceedings*, Conference on Soil Mechanics, S. P. E. E., Purdue Univ., Lafayette, Ind., 1940, pp. 101–108.

eighteen years, borings taken through the material indicated that the mud had not penetrated very far into the sand and that below the contact zone the mud had not been changed physically from the samples of uncovered or natural mud. Settlements of rock fill placed on the sand in 1912 had practically ceased in 1927.

(5) The artificial sand layer was dropped through 30 ft of water and spread very rapidly, and in a short time covered a large area of mud without disturbing the structure of the ancient mud layer.

(6) When the sand layer is 15 ft or more and extends for widths of 300 ft, the settlement of superimposed structure can be computed from the consolidation determined for the sand alone and without considering the effect on the underlying mud strata. Analyses indicate that a 15-ft depth of sand eliminated any possible excessive horizontal forces in the mud layer.

(7) In the work done between 1930 and 1934, sand placed 9 ft thick upon the mud supported loads of 2.71 kg per sq cm, or approximately 3 tons per sq ft. This was seven times the theoretical bearing capacity of the mud layer.

(8) In a section of the quay wall at the pierhead of one of the breakwaters built in 1934, with loads of 2.5 kg per sq cm placed on a sand layer 15 ft thick, there was no displacement of the foot of the quay wall, and no formation of a mud wave beyond the limits of the load. This indicated definitely that the sand layer takes the full load and distributes it to the mud without any local overstresses.

(9) In the breakwater built at Spezia, the layer of sand was 6 ft high and supported 2 km (1.24 miles) of breakwater. Similar construction of breakwaters was used at Kobe, Japan, except that the mud was disturbed by dumping rock from the surface into a trench dredged for the sand layer.

(10) Tidal changes of the range of 6 ft have not affected the stability of structures built on such subaqueous sand layers.

As Mr. Pagon states, the general idea of covering mud fill with granular materials is economically possible, but it must be done uniformly if satisfactory results are to be expected. Such uniformity is more easily obtained by placing the granular blankets by subaqueous methods, as was practiced by General Barberis. Developments of these studies to practical application for increasing the bearing values of mud fills not only for airport runways, but for the support of structures and roads, will be of great value to the profession, and as a contribution along this line, Mr. Pagon's paper is most welcome.

W. WATTERS PAGON,[11] M. AM. SOC. C. E.—The citation, by Mr. Feld, of work done in Italy is very much to the point in any discussion of construction upon foundations which are of so yielding a character as those of the Baltimore Municipal Airport and of many docks, quays, etc., which necessarily must be located in soft bottom. A recognized method of handling such locations heretofore has been to dredge away the softer material and backfill with sand or other suitable material.

The engineering profession would have profited more if Mr. Feld had given a discussion of the theories of Vierendeel and Rankine mentioned in Items 1 and 2 of his discussion.[12] The observation that some of the work showed a resistance three times, and other work six times, the resistance determined by the Rankine theory then would offer some idea of the shears involved. Undoubtedly, the cohesion between sand fill and mud is of considerable importance; but such a contact surface does not in any way diminish the liquidity of the mud until a long time after construction is completed. This is true because a long time is required for the mud to exude its water against the tenacious force which causes it to adhere to the mud particles and against the high resistance to percolation through the minute interstices of the mud. If the area covered is great, and, if the shear stress in the mud is not exceeded locally, there will be no displacements, but only a slow settlement as the water leaves it.

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[11] Cons. Engr., Baltimore, Md.

The experience at the Baltimore Municipal Airport has been that at two areas there has been localized settlement, amounting to about 15 in., which was repaired by filling in and replacing the bituminous surfacing. One of these areas had previously given trouble (as described in the paper), certainly not so much due to a yielding of the mud, as to the fact that the fill was being placed during a very wet March and was thoroughly soaked, so that the trucks hauling fill wallowed in the soupy fill. This clayey material, therefore, became thoroughly water-soaked.

Supplementing the data of the paper: A recent test was made, for the writer, of some of the undisturbed mud crust, which, so far as can be known, was not disturbed for thirteen or fourteen years but lay drying in the sun. The sample was taken from a depth of about 12 in. within this crust, where there still was considerable plasticity. The compressive strength was 40 lb per sq ft, or a shear strength of 20 lb per sq ft, which is less than the results found in the ball test described in the paper. The moisture content was 126% dry weight. The initial modulus of elasticity or deformation was 400 lb per sq ft, as compared with one sand test in which the initial modulus (stress difference ÷ axial strain) was 400,000 lb per sq ft, the minor principal stress being 2,000 lb per sq ft. The ratio of maximum stress to corresponding strain (16% reduction in height) was 250 lb per sq ft.

One point might here be emphasized; namely, that where such strains are involved, failure does not develop when some small part of the subgrade is overstressed, because merely local yielding occurs. Failure develops only when yielding occurs over a complete surface which extends to the top of the subgrade, such that the material can be thrust aside, allowing the load to penetrate. For a circular area on a level surface, a uniform pressure of 3.1 times the shearing strength will cause local overstress, whereas 5.6 times the shearing stress is required on a circular die to reach the maximum bearing capacity.

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[12] "History of the Development of Lateral Earth Pressure Theories," *Proceedings*, Brooklyn Engrs. Club, January, 1928.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2195

VISCOSITY AND SURFACE TENSION EFFECTS ON V-NOTCH WEIR COEFFICIENTS

By Arno T. Lenz,[1] Assoc. M. Am. Soc. C. E.

With Discussion by Messrs. A. A. Kalinske, Glen N. Cox and F. J. Germano, Emery H. Willes, Ralph W. Powell, G. H. Hickox, and Arno T. Lenz.

Synopsis

Results of 652 test runs in the Hydraulic and Sanitary Laboratory at the University of Wisconsin, in Madison, are reported in this paper. Square-edged, brass, V-notch weirs, with six different angles, were tested, with heads up to 0.88 ft, using water and two oils at several temperatures.

Viscosity varied from 1 to 150 times that of water; surface tension from 1 to 0.41 times that of water; and density from 1 to 0.85 times that of water. Also reported are 45 tests made at the University of California, in Berkeley, with two oils, at heads up to 0.20 ft, and 92 tests made at Cornell University, in Ithaca, N. Y., with water, at heads up to 3.43 ft.

The purpose of the Wisconsin tests was to obtain experimental data with regard to the effect of viscosity and surface tension on a weir coefficient *C*. It was found that the weir coefficient increases with both viscosity and surface tension, and that it decreases with increase in head and angle of V-notch.

A general equation expressing these relations mathematically was derived with the aid of dimensional analysis and is given in the paper, along with the limiting conditions beyond which the equation is not valid. As it should be good for any oil, its validity was checked with previously unpublished data from tests made at the University of California with two different oils. The coefficients computed by the general equation were within 1% of the test values.

Comparison of published coefficients, from similar tests with water, with the values of *C* computed from the general equation shows agreement within about 1% for all weir angles between 28° and 90°. Head-discharge equations are given which express the results of these water tests.

Notation

The letter symbols used in this paper are defined where they first appear and are assembled for convenience of reference in the Appendix.

Note.—Published in March, 1942, *Proceedings*.

[1] Madison, Wis.

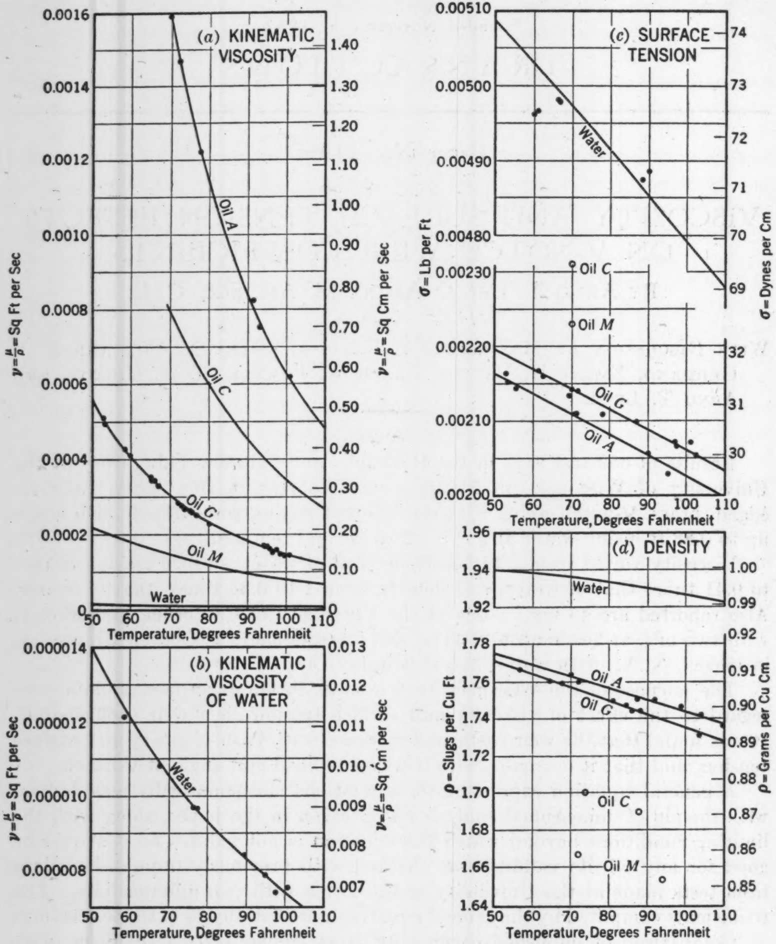

FIG. 1.—PHYSICAL PROPERTIES OF LIQUIDS TESTED

INTRODUCTION

Since the first work on V-notch (triangular) weirs by Prof. James Thomson (13)[2] in 1858, many experimenters starting with the equation

$$Q = c\,H^{2.5} \dots\dots\dots\dots\dots\dots\dots\dots\dots (1)$$

have determined the value of the coefficient c for various values of head, angle of notch, type of weir crest, size of approach channel, and temperature of flowing liquid. The effect of the angle of notch is largely eliminated by the use of the formula,

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[2] Numerals in parentheses, thus: (13), refer to corresponding items in the Bibliography (see Appendix).

$$Q = C\frac{8}{15}\sqrt{2g}\tan\frac{\theta}{2}H^{2.5} \qquad (2)$$

and since this is the most general fundamental equation available it is used as the starting point in this paper. In Eqs. 1 and 2, Q, of course, is the flow; c is an empirical coefficient; H is the hydraulic head; g is the gravity constant; and θ is the angle of the notch. The value C in Eq. 2 is really the dimensionless ratio of the actual discharge to the discharge as computed by calculus methods, usually termed the theoretical discharge. Therefore, C should express the effect of the various physical properties of the flowing liquid, other factors being constant. The methods of dimensional analysis have been applied to this problem by Hubert Mawson (8), B. M. Thornton, Ed S. Smith, Jr. (11), and H. N. Eaton (5), but all the experimental work by which they prove their conclusions was done with only one liquid—water. Since the physical properties of water change little with temperature, it was necessary to experiment with liquids having widely different properties in order to show definitely how the value of C depends on viscosity, surface tension, and density.

Three groups of new test data are reported. They are: (*a*) Tests made by the writer and assistants at the University of Wisconsin, (*b*) tests made at the University of California under the direction of Morrough P. O'Brien, M. Am. Soc. C. E., and (*c*) tests made by Chitty Ho, Assoc. M. Am. Soc. C. E., and Sze-Ling Wu at Cornell University (7). The trade names of the various oils used have been replaced by arbitrary letter designations, serving the same purpose as far as discussion is concerned. Their physical characteristics are given in Fig. 1.

EXPERIMENTAL METHODS

Wisconsin Tests.—The apparatus used in all tests with water and oil G is shown in Fig. 2. The weir tank had for its front end a frame of 2-in. steel angles. Two steel end plates, one above the other, closed the opening in the end frame. A small opening through which the liquid could flow was cut in the upper plate to which were bolted two brass weir plates as shown in Figs. 3 and 4. Two designs of upper end plate were used—one for the 10°, 20°, 28°, and 45°, and one for the 60° and 90°, V-notches. It was thus possible to use the same two weir plates for tests at all angles.

The weir plates were $\frac{1}{4}$-in. brass plates, $8\frac{1}{2}$ in. by 13 in., beveled to $\frac{1}{32}$-in. thickness at the weir crest, with square upstream corners and straight edges. A filler plate with the correct angle was in place below the weir plates, as shown in Fig. 3, throughout each series of runs, in order to provide a smooth upstream surface of at least $8\frac{1}{2}$ in. from the crest in all directions.

The point of the notch was 3 ft above the bottom of the tank. This gave a ratio of approach-channel depth to head of 4.4 for the maximum head of 0.88 ft with the 10° notch and 7.0 for the maximum head of 0.50 ft with the 90° notch. The width of the channel of approach (3.5 ft) was 7.0 times the head for the maximum head on the 90° notch. The work of James Barr (2) would

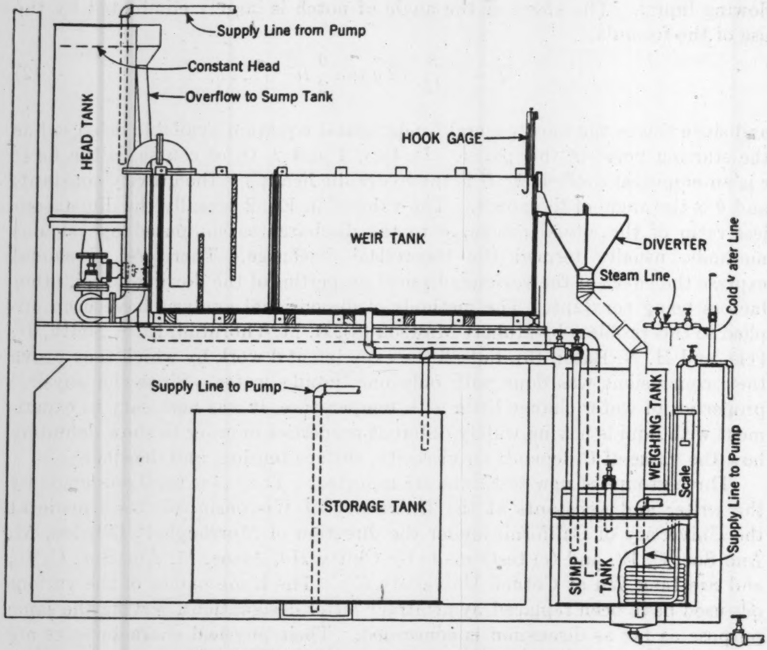
Supply Line from Pump
Constant Head
Overflow to Sump Tank
HEAD TANK
HOOK GAGE
WEIR TANK
DIVERTER
Steam Line
Cold Water Line
Supply Line to Pump
WEIGHING TANK
Scale
Supply Line to Pump
STORAGE TANK
SUMP TANK

FIG. 2.—WEIR APPARATUS (SECTIONAL ELEVATION)

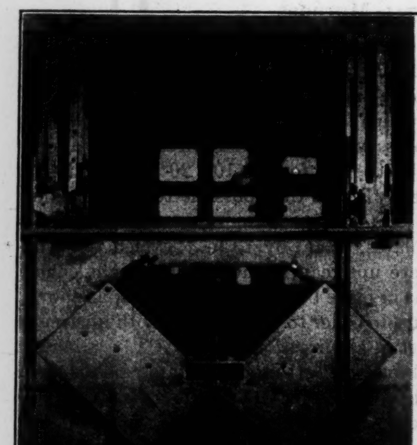

FIG. 3.—APPARATUS FOR DETERMINING ZERO READING

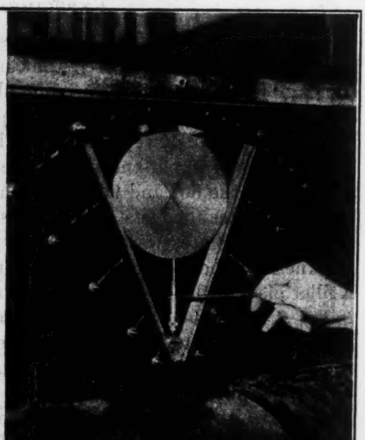

FIG. 4.—MEASUREMENT OF ANGLE OF NOTCH

indicate that there should be no influence due to side-walls or floor with ratios of this magnitude.

The liquid flowed over the notch into a funnel-shaped diverter (Fig. 2) from which it could be discharged into the sump tank or into either one of two weighing tanks. A coil in the sump tank was provided with steam and cold-water connections and was used to maintain a constant temperature in the liquid. From the sump tank the liquid was pumped to a constant-head tank from which it flowed to the weir tank. The liquid entered the weir tank behind a series of three baffles, the last of which was 7 ft 9 in. from the weir plates. The flow from the last baffle to the weir plates was exceedingly smooth, and the head was maintained constant throughout each run.

The apparatus used in the tests with oil A was essentially the same as with oil G, with one exception. Each V-notch was cut in a single brass plate 12 in. by 24 in. by $\frac{3}{32}$ in. thick. The edges were beveled to the same profile as in the other tests. The principal objection to this type of notch was the difficulty in fixing the exact vertex of the V-notch at the geometric intersection of the sides, particularly for the narrow angles.

Measurements.—The head was measured by two hook gages reading to 0.0001 ft, one on each side of the notch and rigidly fastened to the headframe as shown in Fig. 3. A third gage 5 ft upstream from the weir was also read until 300 runs proved beyond doubt that the gages fastened to the headframe were beyond any drawdown influence from the notch. This method of reading heads was much more satisfactory than the use of a hook gage in a stilling well, particularly at high temperatures, for it was found that the level of the quiet liquid in the well was not the same as in the weir tank, the difference in level being in proportion to the difference in liquid density due to the temperature variation between stilling well and weir tank.

Zero readings on the hook gages were obtained by measuring with a small auxiliary micrometer hook gage from a shelf resting on a small disk placed in the V-notch, to the level of the quiet liquid in the tank (Fig. 3), and computing the difference in level between the liquid and geometrical point of the notch. The angle of notch was obtained by measuring with a micrometer caliper between two disks of known diameter and computing sin $\theta/2$ (Fig. 4). For the 90° notch a level bar was used instead of the upper disk.

The actual discharge over the weir was determined by weighing the liquid caught in a measured time. For small rates of flow a small scale, accurately calibrated, measured the weight to 0.01 lb. For a net weight of more than 300 lb a large tank with a maximum capacity of 1,500 lb was used. The scale was sensitive to 0.2 lb. Runs of 5-min duration were made for low rates of flow, the time decreasing to about 1 min for high rates. Time was measured by an accurate stop-watch, graduated to 0.2 sec, with the reading estimated to 0.1 sec.

Temperature was read on a thermometer graduated to half degrees Fahrenheit, and readings were estimated to the nearest 0.1°. The correct unit weight for the actual temperature of the liquid was used in computing the volumetric rate of discharge. The value of g was taken as 32.169 ft per sec^2 (980.5 cm per sec^2). It is believed that all measurements taken are accurate within 0.2%, most of them being accurate to within 0.1%.

Physical Properties of Liquids.—Oil A was a fuel oil, No. 5 grade, the type used in large oil burners. Oil G was a dustproofing oil, the type used to spray on coal to reduce dust trouble before delivery. Both were selected because of their viscosity range and also because the 2,200 gal required in the circulating system could be resold after the tests were completed. Lake water was used in all the water tests. The appearance of the liquid jet for oil G may be seen in Fig. 5.

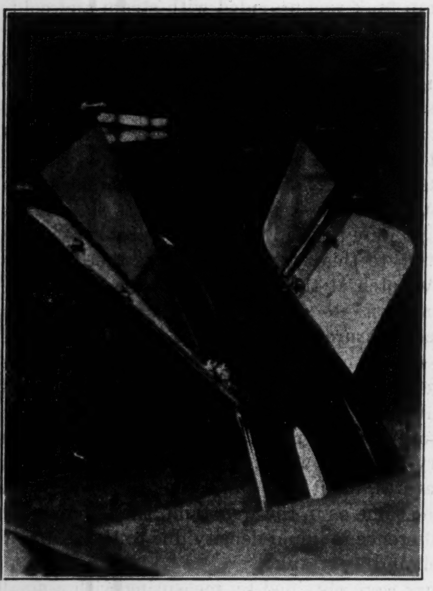
Fig. 5.—Oil G Flowing Over a 60° V-Notch Weir

The density (ρ) of the liquids used in the tests is shown in Fig. 1(*d*). The values for oils A and G were obtained by tests with the hydrometer and Westphal balance, and by weight determinations, using flasks of known volume. Plotted points in Fig. 1(*d*) indicate test results.

Kinematic viscosity (ν) was measured by means of an Ostwald viscosimeter. The instrument was calibrated with distilled water as recommended by the American Society for Testing Materials (A.S.T.M.) Standards (1). The curve in Fig. 1(*b*) is from the International Critical Tables (9), whereas the plotted points were obtained from water measurements using the instrument constant obtained by calibration. Test data for oils and a comparison with the water curve is shown in Fig. 1(*a*). Computations for Reynolds' number **R** were made using values taken from the smooth curves.

Surface tension (σ) was determined by measuring the rise in the capillary tube of the apparatus shown in Fig. 6. The difference in level between the liquid in the two legs of the U-tube is directly proportional to the surface tension of the liquid and inversely proportional to ρ, g, and an instrument constant. This capillary rise was measured with a micrometer comparator telescope reading to 0.005 mm. Comparison of the test data obtained with distilled water with the values published in the International Critical Tables (9) is given in Fig. 1(*c*). Test data from oil measurements and the mean curves used in computations are also shown. The water bath shown in Fig. 6 was used for both the surface tension and viscosity measurements to hold the temperature constant.

California Tests.—The two oils M and C were used in the California tests, which were conducted at low heads (all less than 0.20 ft) and with 90° weirs. Tests were made with each oil at two temperatures, approximately 70° F and

90° F. The experimental data are summarized in Table 1(*a*). The tank used consisted of a weir box 14½ in. high, 10⅜ in. wide, and 40¾ in. long. It contained two baffles, the last one 24¼ in. upstream from the 3¼-in. by 6-in. brass weir plate in which was cut a 90° V-notch. A head gage in a stilling well 3½ in. in diameter was connected to the weir tank with a ⅜-in. tube, the connection being 15¾ in. upstream from the weir plate. Maximum head in any of the test runs was 0.20 ft; minimum head selected from the data for use in this paper was 0.10 ft. Head-gage readings were recorded to 0.001 ft; time to 0.1 sec, with a minimum of 15 sec at 0.20-ft head; and weights to 0.1 lb, with a minimum net weight of 30 lb.

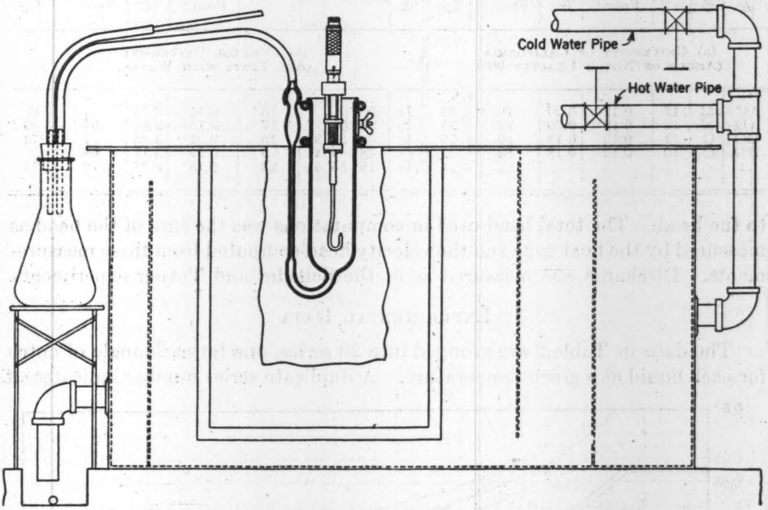

Fig. 6.—Arrangement of Surface Tension U-Tube in a Constant-Temperature Bath

Cornell Tests.—All the Cornell tests were made with water only, but with five angles of notch, 90°, 60°, 37°, 28°, and 19°, at heads up to 3.43 ft (see Table 1(*b*). They were made with "brass plates ⅛ in. thick and 6 in. wide with sharp square edges in order to make the water sheet discharge freely." These plates were attached to a bulkhead near the end of the same concrete channel—6 ft wide and 11.6 ft deep—that was used in the weir experiments by E. W. Schoder, M. Am. Soc. C. E., and the late K. B. Turner (10), first published in 1927. The vertex of each notch was set 8 ft above the floor level. The closest baffle was 24.6 ft from the weir. Heads were measured with float gages connected 11.74 ft upstream from the notch. One float gage was equipped with a vernier, and readings were recorded to 0.0001 ft; the other was read to 0.01 ft and was used as a check only. Velocity measurements upstream from the weir were made either before or after the runs at relatively high heads by means of wooden floats, the submerged depth of the float being adjusted equal

TABLE 1.—SUMMARY OF EXPERIMENTAL DATA

Series	Oil	Runs (45)	RANGE OF HEAD (FEET)		RANGE OF TEMPERATURE (DEGREES F)		Series	Exact angle of notch	Runs (92)	RANGE OF HEAD (FEET)		RANGE OF TEMPERATURE (DEGREES F)	
			From:	To:	From:	To:				From:	To:	From:	To:
(*a*) UNIVERSITY OF CALIFORNIA (ANGLE OF NOTCH EXACTLY 90°)							(*b*) CORNELL UNIVERSITY (ALL TESTS WITH WATER)						
AC	M	12	0.12	0.20	70	73	D	90° 0′	15	0.15	2.19	..	67
AD	M	12	0.12	0.20	92	95	E	60° 0′	17	0.15	2.98	68	70
DC	C	11	0.10	0.18	69	75	F	36° 52′.2	18	0.32	3.43	71	73
DD	C	10	0.12	0.18	88	..	G	28° 4′.4	19	0.29	3.17	68	..
..	..	..	....	...	..	..	H	18° 55′.5	23	0.18	3.37	72	73

to the head. The total head used in computations was the sum of the head as measured by the float gage and the velocity head computed from these measurements. Discharge was measured as in the Schoder and Turner experiments.

EXPERIMENTAL DATA

The data in Table 2 are grouped into 20 series, one for each angle of notch for each liquid at a given temperature. A duplicate series number indicates an entirely independent set of runs. The original data are on file at the University of Wisconsin, and a summary of data for each run is on file in the Engineering Societies Library.[3]

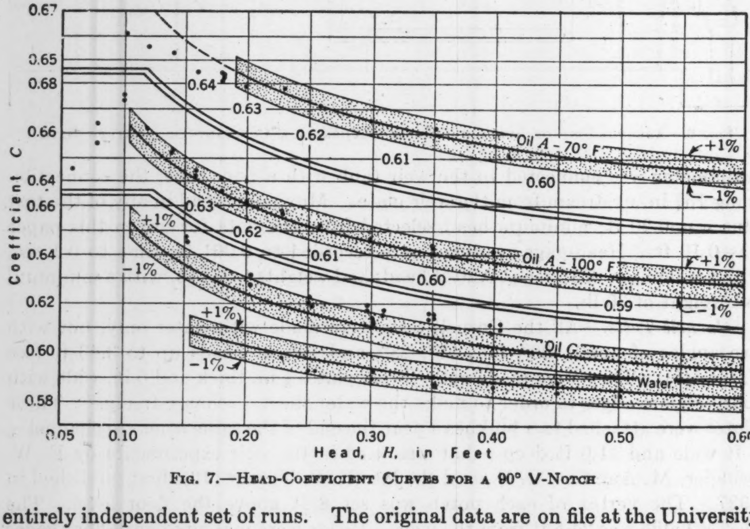

FIG. 7.—HEAD-COEFFICIENT CURVES FOR A 90° V-NOTCH

Head-Coefficient Data and Curves.—The experimental coefficients for the 90° V-notch for oils A and G are shown in Fig. 7. The curves in Fig. 7 are computed from the formula

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[3] 29 West 39th Street, New York, N. Y.

$$C = 0.56 + \frac{B}{\mathbf{R}^n \mathbf{W}^m} \quad \text{.......................(3)}$$

in which B is a known constant; m and n are constant exponents; and $\mathbf{W}$ = Weber's number. All test points within the shaded band are within 1% of the corresponding computed value. The broken line at low heads (oil A at 70° F) indicates the extension of Eq. 3 beyond its range of usefulness. At high heads the coefficient tends to become constant. Similar curves plotted for all the series in Tables 1 and 2 show that nearly all test points are within 1% of the corresponding values computed by Eq. 3, and that within the useful range the variation of the test points from the equation seems to be unrelated to head, viscosity, surface tension, or angle of notch.

TABLE 2.—SUMMARY OF EXPERIMENTAL DATA, WISCONSIN TESTS

Series[a]	Angle of notch	Runs[b]	Range of Head. (Feet) From:	Range of Head. (Feet) To:	Range of Temperature (Degrees F) From:	Range of Temperature (Degrees F) To:
90A 70	89° 53′.9	13	0.10	0.35	69.8	70.2
90A 100	89° 53′.9	14	0.06	0.35	99.8	100.2
90A 100	89° 53′.9	13	0.06	0.40	99.9	100.3
90G	89° 49′.6	23	0.15	0.40	64	68
90D	89° 53′.2	14	0.23	0.50	61	63
60G	59° 56′.9	23	0.15	0.55	62	67
60D	59° 51′.0	33	0.16	0.60	54	62
45A 70	44° 52′.0	16	0.10	0.50	69.7	70.4
45A 70	44° 52′.0	27	0.18	0.50	69.6	72.2
45A 100	44° 52′.0	25	0.10	0.55	99.2	100.1
45A 100	44° 52′.0	32	0.10	0.55	99.5	102.0
45G	44° 40′.0	22	0.18	0.60	58	65
45D	44° 40′.0	18	0.20	0.55	56	60
28A 70	28° 13′.1	28	0.16	0.55	70.1	70.4
28A 70	28° 13′.1	35	0.16	0.60	71.5	74.0
28A 100	28° 13′.1	24	0.10	0.55	99.6	100.0
28A 100	28° 13′.1	36	0.10	0.60	98.9	100.6
28G	27° 53′.0	25	0.18	0.70	64	69
28D	27° 55′.0	18	0.26	0.70	51	59
20A 70	19° 54′.7	27	0.14	0.60	70.1	70.3
20A 70	19° 54′.7	9	0.18	0.55	70.0	71.7
20A 100	19° 54′.7	33	0.10	0.60	99.3	101.0
20A 100	19° 54′.7	14	0.10	0.55	99.7	100.1
20G	19° 56′.3	26	0.18	0.70	64	69
20D	19° 55′.0	24	0.18	0.70	54	67
20D	20° 2′.5	19	0.24	0.70	54	65
10G	10° 7′.8	28	0.20	0.80	58	64
10D	10° 20′.1	33	0.24	0.88	52	54

[a] "A" denotes "oil A"; "G" denotes "oil G"; and "D" denotes "water." [b] Total, 652.

THEORY

The derivation of a general equation that should be applicable to any liquid flowing over a V-notch weir is aided by a dimensional analysis of the problem. Although the equation derived (Eq. 3) is an empirical one, the fact that it is dimensionally correct should remove some objections to its use for liquids other than those actually tested.

The discharge Q over a V-notch weir is a function of the measurements shown in Table 3. As L and A are functions of the angle θ, there remain, for a notch of given angle, six independent variables. From dimensional analysis it is known that

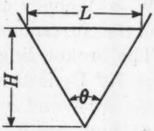
$$\phi\,(V, H, \rho, \mu, \sigma, g) = 0 \quad \text{.......................(4a)}$$

According to the π-theorem (3a), a complete equation of the form of Eq. 4a has the solution:

$$\phi'\,(\pi_1, \pi_2, \cdots) = 0 \quad \text{.......................(4b)}$$

in which π_1, π_2, π_3, etc., are the independent products of the arguments V, H, ρ, $\cdots$, and are dimensionless in the fundamental dimensional units: Force (F), length (L), and time (t). In this case there are three terms in the π-theorem, each a dimensionless product of powers of the six independent variables.

TABLE 3.—Variable Quantities in the Measurement of Flow

No.	Measurement	Symbol	Unit	Dimensions[a]	Weir section
1	Head	H	Feet	L	
2	Crest length (projected)	L	Feet	....	
3	Angle of notch (projected)	θ	Degrees	....	
4	Area of notch (projected)	A	Square feet	....	
5	Mean velocity $= Q/A$	V	Feet per second	L/t	
6	Density	ρ	Slugs per cubic feet	$F\,t^2/L^4$	
7	Viscosity	μ	Pound-seconds per square foot	$F\,t/L^2$	
8	Surface tension	σ	Pounds per foot	F/L	
9	Acceleration of gravity	g	Feet per second per second	L/t^2	

[a] Six independent variables.

Since $\frac{\mu}{\rho} = \nu$, Eq. 4*b* thus becomes

$$\phi''\left(\frac{\nu}{V\,H}, \frac{\sigma}{V^2\,\rho\,H}, \frac{g\,H}{V^2}\right) = 0 \quad \text{(5)}$$

For a given angle of V-notch it is then possible to write:

$$V = \sqrt{g\,H}\,f\left(\frac{\nu}{V\,H}, \frac{\sigma}{V^2\,\rho\,H}\right) \quad \text{(6a)}$$

or

$$Q = A\,V = \sqrt{g}\tan\frac{\theta}{2}H^{2.5}f\left(\frac{\nu}{V\,H}, \frac{\sigma}{V^2\,\rho\,H}\right) \quad \text{(6b)}$$

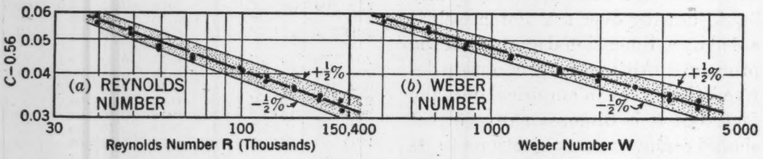


Fig. 8.—Coefficient Curves for Water (56° to 62° F) Flowing Through a 45° V-Notch Weir

When Eq. 6*b* is compared with Eq. 2, a solution is obtained for the weir coefficient C:

$$C = f'\left(\frac{\nu}{V\,H}, \frac{\sigma}{V^2\,\rho\,H}\right) \quad \text{(7)}$$

The value of C for a given angle of notch, therefore, is a function of the kinematic viscosity and surface tension of the flowing liquid. Since C is to be used to compute the mean velocity of flow and the actual discharge, the presence of the factor V in each of the foregoing dimensionless numbers is a serious disadvantage even if the functional relationship between C and the dimensionless products were known. It is proposed, therefore, to replace V in each of the products by a dimensional equivalent, $\sqrt{g\,H}$. Then

$$C = f''\left(\frac{\nu}{g^{0.5}\,H^{1.5}}, \frac{\sigma}{\rho\,g\,H^2}\right) \quad \text{(8)}$$

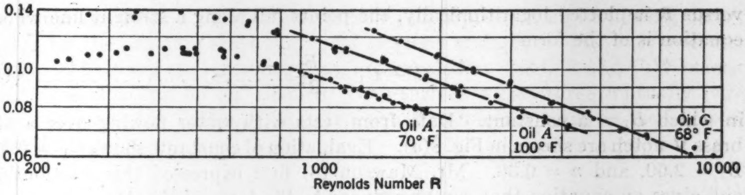


FIG. 9.—COEFFICIENT CURVES FOR OIL FLOWING THROUGH A 28° V-NOTCH WEIR

Taking the reciprocal value in each case to obtain whole numbers, let Reynolds' number

$$\mathbf{R} = \frac{g^{0.5}\,H^{1.5}}{\nu} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9a)$$

and Weber's number

$$\mathbf{W} = \frac{\rho\,g\,H^2}{\sigma} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9b)$$

because they are dimensionless numbers expressing the effect of viscosity and surface tension under the given flow conditions. However, the exact relationship between V and H is not expressed, and therefore the usual critical values of these numbers for pipe flow cannot be applied to V-notch weir flows. Then

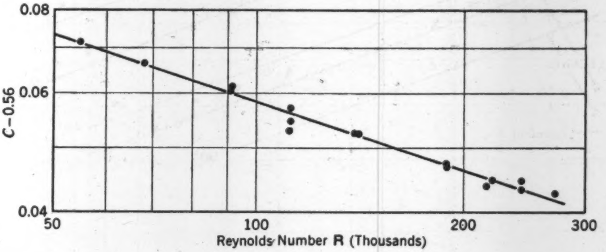


FIG. 10.—COEFFICIENT CURVE FOR WATER AT 56° F, FLOWING THROUGH A 28° V-NOTCH WEIR

$$C = f''\left(\frac{1}{\mathbf{R}}, \frac{1}{\mathbf{W}}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

ANALYSIS OF DATA

Development of General Equation.—The functional relationship, f'', between C, $\mathbf{R}$, and $\mathbf{W}$ for a given weir must be obtained from experimental data. This relationship may be one of many types, but certain well-known characteristics of C are an aid in determining its character. First, C is approximately a constant. Furthermore, it is generally supposed that viscosity plays a much more important rôle than surface tension in determining C.

Assume that C is a function of $\mathbf{R}$ alone and independent of Weber's number, and that for large values of Reynolds' number (high heads and low viscosities) $1/\mathbf{R}$ approaches zero and C approaches a constant value C_1. When $C - C_1$

versus **R** is plotted logarithmically, the points fall along a straight line whose equation is of the form

$$C - C_1 = \frac{B_1}{\mathbf{R}^n} \quad \text{(11)}$$

in which B_1 = a constant. Data from tests with water flowing over a 45° brass V-notch are shown in Fig. 8(*a*). Evaluation of constants shows $C_1 = 0.56$, $B_1 = 2.60$, and $n = 0.36$. Mr. Mawson (8) first expressed this relationship and gives an equation that can be reduced to the form of Eq. 11.

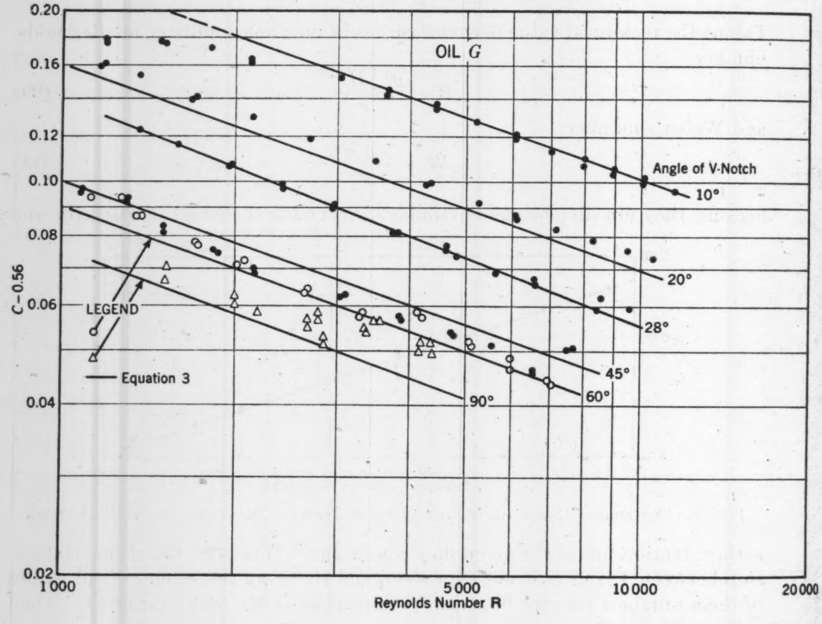


FIG. 11.—COEFFICIENT CURVES;

Data from tests on a 28° V-notch for water and two oils are shown in Figs. 9 and 10. Again the straight-line relationship previously noted holds true for Reynolds' number greater than 850. Evidently the form of Eq. 11 is generally correct, but the assumption that C is independent of Weber's number is not correct.

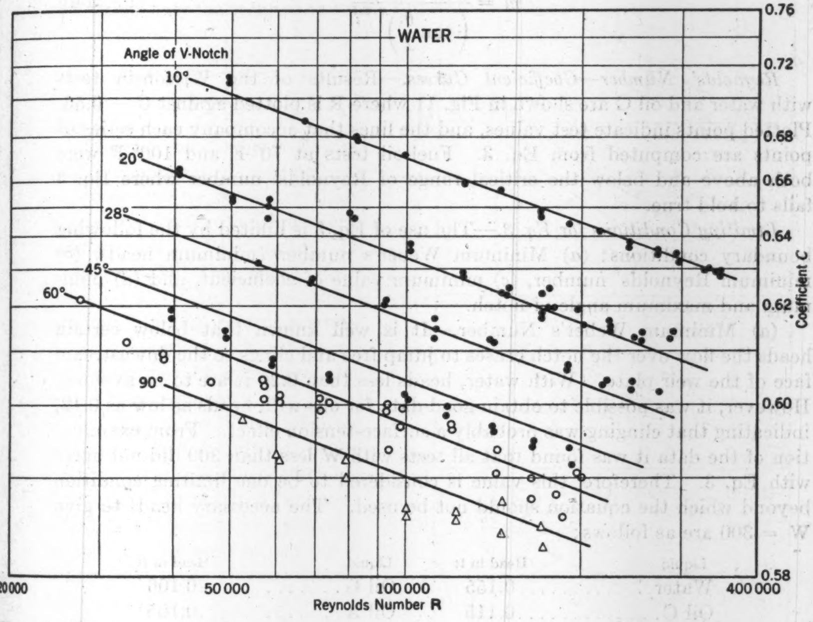
Plotting $C - C_1$ against **W** logarithmically again shows a straight-line relationship, the equation for the line being of the form:

$$C - C_2 = \frac{B_2}{\mathbf{W}^m} \quad \text{(12)}$$

in which C approaches a constant value C_2, and B_2 = a constant. Experimentally it has been found that $C_2 = C_1$. Coefficients from tests shown in Fig. 8(*a*) are plotted against **W** in Fig. 8(*b*). Evaluation of the constants shows that $C_2 = 0.56$, $B_2 = 0.329$, and $m = 0.28$.

Both Eqs. 11 and 12 are those of a line on a plane, the plane being determined by the ordinate scale C and the abscissa scale **R** or **W**. It is evident, therefore, that the straight lines that are revealed on the C-**R** and C-**W** planes are traces of a line on an inclined plane surface whose three mutually perpendicular axes are C, **R**, and **W**. The equation for such a line on the inclined plane surface is

$$C - C_x = \frac{B_x}{\mathbf{R}^n\,\mathbf{W}^m} \qquad (13)$$



Wisconsin Tests

The experimental data plot as straight lines on such inclined planes, one line for each liquid in each plane, and one plane for each angle of notch. Since it is difficult to show such three-dimension figures, the traces of these lines in the C-**R** plane are shown in Fig. 11. However, the position of any point depends on **W** as well as **R**. For a given angle of V-notch, there is only one value of coefficient for each set of **R** and **W** values.

Evaluation of the constant C_x in Eq. 13 has shown it to be equal to 0.56 for all angles of notch, at least through 90°. The general equation for the coefficient of discharge (see Eq. 3) is $C = 0.56 + \dfrac{B}{\mathbf{R}^n\,\mathbf{W}^m}$.

The values B, n, and m are functions of the angle of notch and may be computed from the following equations or read from curves of these equations:

$$B = 0.475 + \frac{0.225}{\left(\tan\frac{\theta}{2}\right)^{0.80}} \quad \text{.................}(14a)$$

$$n = 0.165\left(\tan\frac{\theta}{2}\right)^{0.09} \quad \text{.................}(14b)$$

and

$$m = \frac{0.170}{\left(\tan\frac{\theta}{2}\right)^{0.035}} \quad \text{.................}(14c)$$

Reynolds' Number—Coefficient Curves.—Results of the Wisconsin tests with water and oil G are shown in Fig. 11 where **R** is plotted against $C - 0.56$. Plotted points indicate test values, and the lines that accompany each series of points are computed from Eq. 3. Fuel-oil tests at 70° F and 100° F were both above and below the critical range of Reynolds' number where Eq. 3 fails to hold true.

Limiting Conditions for Eq. 3.—The use of Eq. 3 is limited by the following boundary conditions: (*a*) Minimum Weber's number (minimum head), (*b*) minimum Reynolds' number, (*c*) minimum value of coefficient, and (*d*) minimum and maximum angle of notch.

(*a*) Minimum Weber's Number.—It is well known that below certain heads the flow over the notch ceases to jump free and clings to the downstream face of the weir plate. With water, heads less than 0.20 ft are to be avoided. However, it was possible to obtain good data for oils with heads as low as 0.12, indicating that clinging was probably a surface-tension effect. From examination of the data it was found that all tests with **W** less than 300 did not agree with Eq. 3. Therefore, this value is considered to be one limiting condition beyond which the equation should not be used. The necessary heads to give **W** = 300 are as follows:

Liquid	Head in ft	Liquid	Head in ft
Water	0.155	Oil G	0.106
Oil C	0.115	Oil A	0.105
Oil M	0.112		

The foregoing values are for 70° F, but change very little within the range of temperatures tested. The curves in Figs. 7 and 12 were stopped at these heads. It should be noted that if the jet clings to the face of the weir plate, Eq. 3 will not give correct results even if **W** is greater than the minimum value stated here. The minimum value of **W** appears to be unrelated to the angle of notch.

(*b*) Minimum Reynolds' Number.—Study of Fig. 13 (and similar curves for several values of θ) indicates that, below a certain minimum critical value of **R**, viscosity affects the flow, and consequently the coefficient, in a different manner than above that value of **R**. This change in coefficient is not to be confused with that which has been discussed under the heading "Minimum Weber's Number." For example, comparison of data in Fig. 13(*a*) for the 28° V-notch shows that below **R** = 845, $C - 0.56$ definitely falls away from the line of Eq. 3. Although the break occurs at the same value of **R** for series at both 70° and 100° F, corresponding heads in the two series are 0.363 ft and 0.207 ft, respectively, and the corresponding values of **W** are 2,840 and 1,180, respectively. Similar results are evident in all the tests of oil A, although the value of **R** below which the viscosity effect changes is not the same for all angles. This critical value may be expressed by the equation

$$\mathbf{R} = \frac{300}{\left(\tan\frac{\theta}{2}\right)^{0.75}} \ldots (15)$$

Clearly, the change in flow is due to changes in viscosity effect as expressed by **R** rather than changes in surface tension effect as expressed by **W**, or head. It is also evident that the critical value of **R** is much lower than can be obtained with water under any practical test and therefore need not be considered in water measurement.

(*c*) Minimum Value of Coefficient.—For infinitely large values of **R** and **W** the coefficient C computed from Eq. 3 becomes equal to 0.56; yet no data obtained in these tests or any published have ever shown values that low.

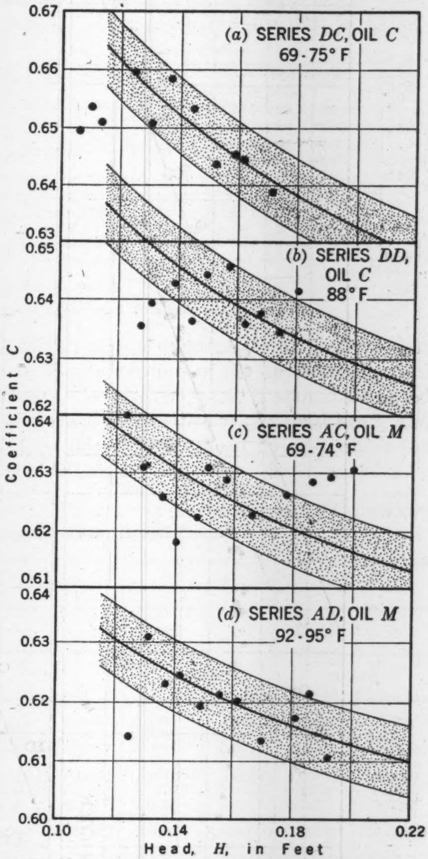


FIG. 12.—HEAD-COEFFICIENT CURVES; CALIFORNIA TESTS WITH A 90° V-NOTCH WEIR

The newest tests under conditions that give large values of **R** and **W** are the Cornell tests (7). These test values of $C - 0.56$ are plotted against **R** in Fig. 14. In all cases the coefficient was found to decrease to a minimum value and to remain approximately constant at that value for all greater values of **R**, independent of the angle of the V-notch. This constant value of C was 0.5886. The minimum C obtained by D. R. Yarnall (14) with his 90° notch was 0.5808.

Although it appears from the foregoing data that there is a limit to the applicability of Eq. 3, the reason for that limit is not obvious. It has been

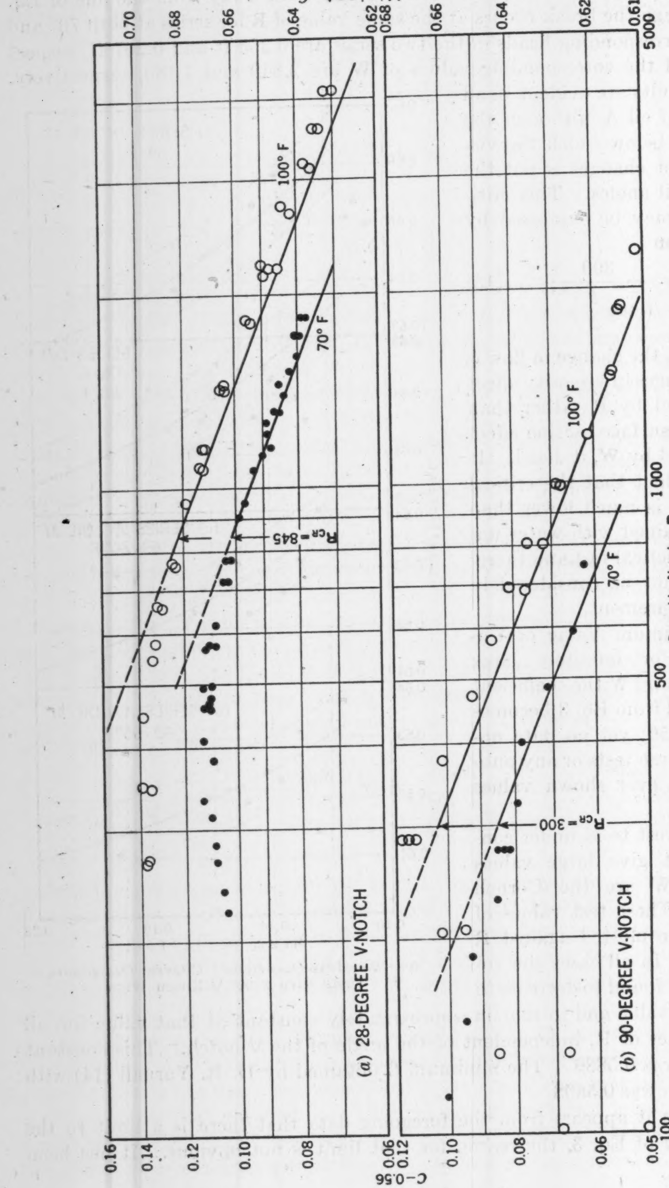


FIG. 13.—REYNOLDS' NUMBER—COEFFICIENT CURVES; WISCONSIN TESTS, OIL A

suggested by Messrs. Eaton (5) and Smith (11) that the ratios $\frac{H}{Z}$, $\frac{H}{b_w}$, and $\frac{L}{b_w}$ as given in Table 4 influence the velocity of approach in such a manner as to counteract the decrease in C as expressed by Eq. 3 (Z = height of the vertex of the notch and b_w = width of weir box). It appears from the data in Table 4 that the experimental range of the three ratios is so great for constant C that there must be some factor other than the dimensions of the approach channel responsible for the constant coefficient. As further proof, it should be noted that in all cases the velocity head is a small proportion of the total head, and that the velocity itself, as measured by floats, at the head beyond which the coefficient is constant, is less than 0.08 ft per sec.

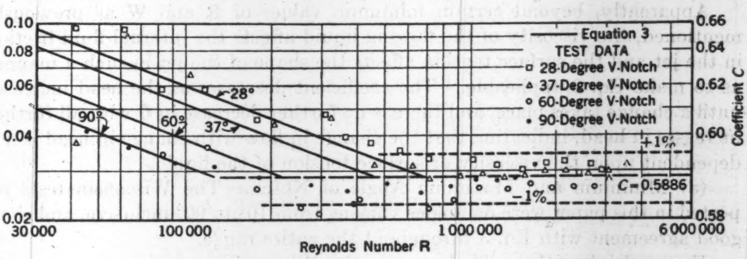


FIG. 14.—REYNOLDS' NUMBER—COEFFICIENT CURVES; CORNELL TEST

TABLE 4.—RANGE OF DATA (CORNELL TESTS) FOR APPROXIMATELY CONSTANT VALUES OF COEFFICIENT C

No.	Description	$\theta = 90°$		$\theta = 60°$		$\theta = 37°$		$\theta = 28°$	
		From:	To:	From:	To:	From:	To:	From:	To:
Average coefficient		0.5872		0.5856		0.5889		0.5915	
1	Head H (ft)	0.420	2.190	0.680	2.976	0.862	3.431	0.982	3.172
2	Discharge Q	0.287	17.95	0.552	22.12	0.579	18.39	0.604	11.31
3	L/b_w	0.1400	0.7300	0.1309	0.5727	0.0958	0.3812	0.0817	0.2640
4	Velocity head	0.0001(−)	0.0026	0.0001(−)	0.0050	0.0001(−)	0.0034	0.0001(−)	0.0009
5	H/Z	0.0525	0.2738	0.0850	0.3725	0.1078	0.4288	0.1228	0.3963
6	H/b_w	0.0700	0.3650	0.1133	0.4967	0.1437	0.5717	0.1637	0.5283
7	Reynolds' number	142,000	1,669,000	294,000	2,714,000	450,000	3,498,000	535,000	2,949,000

Mr. Eaton shows also a dimensionless ratio H/w, in which w is the thickness of the crest. If a minimum value of this ratio determines the head beyond which the weir coefficient is constant, this limiting value should be the same for both the Cornell and Yarnall tests with 90° weirs. It is not the same, being 40.3 and 246, respectively. Apparently H/w has no important bearing on the limit to Eq. 3, but w alone may have some bearing, as the minimum coefficient for $w = \frac{1}{8}$ in. (Cornell tests) was 0.5856; and for $w = \frac{1}{32}$ in. (Yarnall tests), 0.5808. Wisconsin tests, with $w = \frac{1}{32}$ in., approach the latter value.

In summary it should be stated that the coefficients decrease with increasing head in reasonably good agreement with Eq. 3 until a certain minimum coefficient is obtained. For all greater heads the coefficient is nearly constant for $w = \frac{1}{32}$ in. at approximately 0.581, which value is unrelated to θ, H, **R**, **W**, $\frac{H}{Z}$, $\frac{H}{b_w}$, $\frac{H}{w}$, or $\frac{L}{b_w}$.

Apparently, beyond certain minimum values of **R** and **W** as previously mentioned, the viscosity of the flowing liquid affects the internal fluid friction in the jet and the surface tension affects the shape of the jet in such a manner as to make Eq. 3 applicable. The coefficient decreases as the head increases until a change takes place, and there is no further decrease in C with all further increases in head, indicating that the change in flow with change in head is not dependent upon the viscosity or surface tension of the liquid.

(*d*) Minimum and Maximum Angle of Notch.—The Wisconsin tests reported in this paper were on angles varying from 10° to 90° inclusive, and show good agreement with Eq. 3 throughout the entire range.

For angles less than 28°, however, the Wisconsin tests do not agree with other published test data as follows—Eq. 3 is: (*a*) 3.2% below the Yarnall data for a 13° V-notch, (*b*) about the same amount below the Cone formula (4)

$$Q = (0.025 + 2.462\, s)\, H^{(2.5-\alpha)} \qquad (16)$$

which is not recommended for angles less than 28°, and (*c*) 3½% below the Cornell data for a 19° V-notch. In Eq. 16, $s = \tan\frac{\theta}{2}$ = the slope of the side, and

$$\alpha = \frac{0.0195}{s^{0.75}} \qquad (17)$$

Eq. 16 has been reduced to the form

$$C = \frac{0.5758}{H^{\alpha}} + \frac{0.005845}{s\, H^{\alpha}} \qquad (18)$$

F. W. Greve (6) did not publish his data for a 10° notch because it was "not in harmony" with his other tests. When such disagreement exists among careful experimenters whose work agrees for the larger angles of notch, it would seem advisable to restrict the use of the V-notch to angles between 28° and 90° if good results are to be expected without an actual calibration of the weir in place.

One difficulty encountered in the use of narrow angles is the determination of the correct zero reading, because the geometrical point of the vertex is not likely to be at the actual metal point of the V. It may be noted in Fig. 15 that Eq. 3, the Cone formula, and Yarnall test data for the 13° V-notch are in closer agreement at high heads where an error in the determination of the zero reading is of less importance than at low heads.

HEAD-DISCHARGE EQUATION FOR WATER

When Eq. 3 is substituted for C in Eq. 2, the water discharge for a given angle of V-notch may be computed directly from the head because the water temperature has very little effect on the coefficient. The combination of the two equations reduces to

$$Q = \left(2.395 + \frac{N}{H^{e}} \right) \tan \frac{\theta}{2} H^{2.5} \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

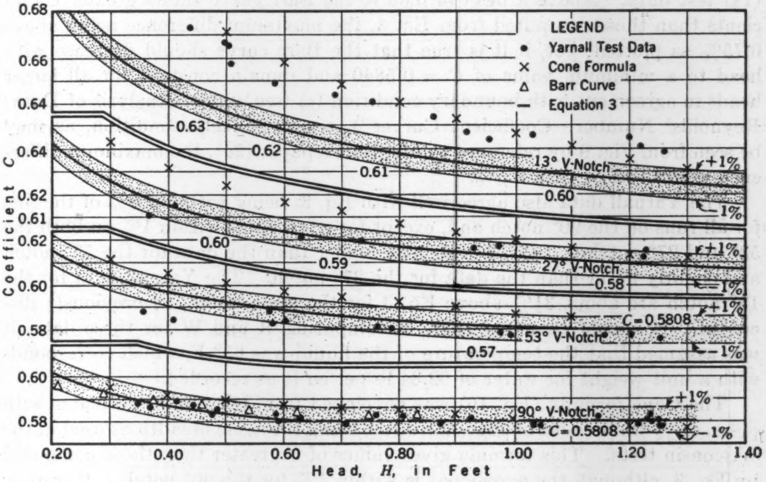


FIG. 15.—HEAD-COEFFICIENT CURVES; YARNALL, CONE AND BARR TESTS

Values of the constants N and e are given in Table 5 for water at 70° F. The discharge, Q, is in cubic feet per second, and H is in feet. The minimum value of the ratio $\frac{N}{H^{e}}$ is 0.090. For accuracy the head must be more than 0.155 ft, and the jet must not cling to the downstream face of the weir plate.

TABLE 5.—VALUES OF THE CONSTANTS N AND e (EQ. 19)

Constant	ANGLE θ:					
	90°	60°	45°	28° 04′	20°	10°
N	0.068	0.087	0.102	0.135[a]	0.167	0.267
e	0.588	0.582	0.579	0.575[a]	0.573	0.569

[a] Side slope 4 on 1 equals 28° 04′.

COMPARISON WITH PUBLISHED DATA

Water Tests.—Few liquids other than water have ever been measured over V-notches so that comparisons made with published data can reveal little about the effect of viscosity and surface tension on the coefficients; yet, if Eq. 3 is applicable for all liquids of reasonable viscosity, it must be applicable for water. Head-coefficient curves for water tests are shown in Fig. 15, in comparison with curves computed from Eq. 3, and demonstrate reasonable agreement for angles of 28° and larger.

The best known water tests on V-notches are those made by James Barr at Glasgow University in Scotland (2). He ran tests (2*a*) using a 90° V cut in a 27-in. by 11-in. brass plate $\frac{3}{16}$ in. thick but "filed up" so that the breadth of the edge was reduced to about $\frac{1}{16}$ in. This group of his tests conforms most closely with the Wisconsin tests of Series 90D, and points from his test curve are plotted in Fig. 15 in comparison with Eq. 3, the Cone formula and Yarnall (14) test data. Above a head of 0.33 ft the Barr curve shows greater coefficients than those computed from Eq. 3, the maximum difference being about 0.75% as plotted; yet, if it is true that the Barr curve should decrease with head to a minimum value of $C = 0.5840$ and remain constant for all larger heads in agreement with boundary condition (*c*) (see under "Analysis of Data: Reynolds' Number—Coefficient Curves"), a not unlikely condition, as may be seen from Fig. 9 by reference to Mr. Barr's paper (2*a*), the maximum difference would be less than 0.5%.

The Yarnall data also agree well with Eq. 3, being within 0.5% of the line for all runs on the 90° notch and, except for a few runs, within 1% on both the 53° and 27° notches. Eq. 3 is slightly higher than the data for the 53° notch and slightly lower than the data for the 27° notch. The Yarnall data for the 13° notch are about $3\frac{1}{3}$% above Eq. 3 for the same notch, as previously discussed. It should be noted that, in computing **R** and **W** for these data, it was assumed that the temperature of the liquid was 65° F, which corresponds with a unit weight for water of 62.34 lb per cu ft as recorded.

The Cone formula (Eq. 16) was developed from tests on brass plates with $\frac{1}{16}$-in. or $\frac{1}{32}$-in. breadth of edge, the latter being the same breadth as used in the Wisconsin tests. This formula gives values of C greater than those computed by Eq. 3, although the agreement is within 1% for the 90° notch. However, except for the 13° notch, the Yarnall data are in closer agreement with Eq. 3 than with the Cone formula.

Within the range of applicability, Eq. 3 checks head-coefficient data for the Cornell tests fairly well, particularly for both the 28° and 90° notches. The data for the 60° notch were lower than computed, and for the 37° notch higher than computed. With the exception of a few runs, all the data were within 1% of Eq. 3 where it is applicable.

Data by Professor Greve on tests at Purdue University caused him to conclude (6*a*): "Apparently the coefficient is dependent upon head and independent of size [of notch]." The latter conclusion is contrary to the conclusions of this paper. His average coefficients for angles of V-notch between 20° and 120° are plotted in Fig. 16 in comparison with curves computed from Eq. 3 for V-notches of the Wisconsin tests (Series D). Professor Greve used soft steel plates as compared with the bronze used by Mr. Yarnall and brass as used by Messrs. Barr and Cone, and the writer.

Oil Tests.—The California tests on oil are the only ones that have come to the attention of the writer, on liquids other than water, and hence may be used as a check of Eq. 3 in so far as major viscosity and surface tension effects are concerned. Results of the California tests are plotted in Figs. 12 and 17. In each case the plotted points are from test data, and for comparison the line represents Eq. 3. It is of interest to note that, although they are rather variable (as would be expected at low heads), the coefficients are nearly all within 1% of Eq. 3; yet the variation in viscosity and surface tension in the group of tests caused a 3.6% change in coefficient at a head of 0.18 ft, and the tests with the most viscous oil showed an increase in coefficient of 5.5% over that for water at the same head (0.18 ft). Since the surface tension of the oils tested varied as well as the viscosity, and since the ratio of surface tension to viscosity varied also, the agreement is considered satisfactory.

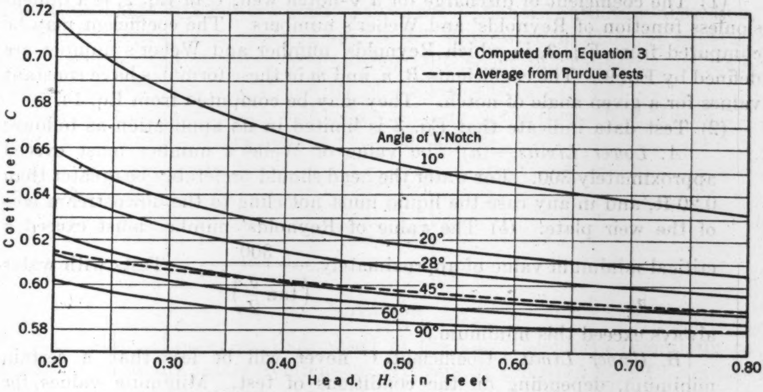


FIG. 16.—HEAD-COEFFICIENT CURVES FOR WATER

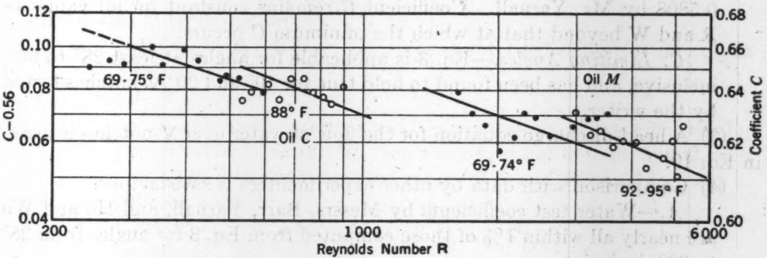


FIG. 17.—COEFFICIENT CURVES FOR CALIFORNIA TESTS ($\theta = 90°$)

Temperature Effect.—The effect of change of temperature on the flow of water over a V-notch is included in Eq. 3. Computations made for a head of 0.50 ft on a 90° weir give values of C as follows:

C	Temperature
0.5860	40° F
0.5840	70° F
0.5803	165° F

The coefficient decreases 0.34% between 40° and 70° F.

F. G. Switzer, M. Am. Soc. C. E. (12*a*), concludes that "the effect of water temperature is inappreciable compared with other factors present" after tests on 90° and 54° V-notches with water at 39°, 83°, 119°, and 165° F; yet he also states (12*a*) that "no point is more than 2 per cent from a mean and the great majority of the points are within 1 per cent of the mean." Since the total change in C by Eq. 3 for this temperature range is but 1%, it is possible that the temperature effect may be hidden by other variables.

CONCLUSIONS

(1) The coefficient of discharge for a V-notch weir, C in Eq. 2, is a dimensionless function of Reynolds' and Weber's numbers. The coefficient may be computed from Eq. 3, in which Reynolds' number and Weber's number are defined by Eq. 9. The coefficients B, n, and m in these formulas have constant values for a given angle of notch. They may be computed from Eq. 14.

(2) Test data indicate that Eq. 3 is limited in its application as follows:

A. Lower Limits.—(*a*) The value of Weber's number must exceed approximately 300. For water the head should preferably be greater than 0.20 ft, and in any case the liquid must not cling to the downstream face of the weir plate. (*b*) The value of Reynolds' number must exceed a critical minimum value of approximately $\dfrac{300}{\left(\tan\dfrac{\theta}{2}\right)^{0.75}}$. Tests with water always exceed this minimum.

B. Upper Limit.—Coefficient C never will be less than a certain minimum, depending on the conditions of test. Minimum values for published test data are within 1% of 0.585. The lowest obtained was 0.5808 by Mr. Yarnall. Coefficient C remains constant for all values of $\mathbf{R}$ and $\mathbf{W}$ beyond that at which the minimum C occurs.

C. Limiting Angles.—Eq. 3 is applicable for angles at least 28° to 90°, inclusive, and has been found to hold true for 10° and 20° V-notches tested by the writer.

(3) A head-discharge equation for the flow of water over V-notches is given in Eq. 19.

(4) Comparison with data by other experimenters is satisfactory.

A.—Water test coefficients by Messrs. Barr, Yarnall, and Ho and Wu are nearly all within 1% of those computed from Eq. 3 for angles from 28° to 90°, inclusive.

B.—Oil-test coefficients obtained at the University of California are within 1% of those computed from Eq. 3, although in extreme cases the values are 5.5% greater than those for water at the same head.

(5) The change in viscosity and surface tension when the temperature of water is increased from 40° F to 70° F is sufficient to cause a decrease in the computed discharge coefficient of 0.34%. Increasing the temperature from 40° F to 165° F decreases the computed C about 1%.

ACKNOWLEDGMENTS

The writer is indebted to F. M. Dawson, M. Am. Soc. C. E., formerly at the University of Wisconsin, who suggested the problem as a thesis subject[4] and

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[4] "The Effect of Viscosity and Surface Tension Upon V-Notch Weir Coefficients," by Arno T. Lenz; thesis presented to the University of Wisconsin in 1940, in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

gave many helpful suggestions. The work was completed under the direction of J. G. Woodburn, M. Am. Soc. C. E. S. E. Kotz built the apparatus for the runs of Series A and assisted in obtaining data for those runs. Two thesis students, J. R. Frost and A. J. Gollnick, assisted with the construction of the apparatus for the runs of Series D and G. Several students assisted with the collection of data under appropriations from the National Youth Administration.

The California data, applying only to 90° weirs, were obtained by students working under the direction of Professor O'Brien at the University of California, and were loaned to the writer by Professor O'Brien.

The description of the Cornell tests was based on the research of Messrs. Ho and Wu (7) under the direction of Professor Schoder.

APPENDIX

NOTATION

The following letter symbols, used in the paper, conform essentially to American Standard Letter Symbols for Hydraulics, prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1942:[5]

A = area of cross section;

B = a coefficient: B_1, B_2, $\cdots$ B_x = constant coefficients for any given weir angle;

b_w = width of channel;

C = V-notch weir coefficient: C_1 and C_2 = constant weir coefficients for large values of $\mathbf{R}$ and $\mathbf{W}$, respectively, equal to 0.56; and C_x = a constant weir coefficient for values of $\mathbf{R}$ and $\mathbf{W}$ equal to 0.56;

c = coefficient = $Q/h^{2.5}$;

e = a constant exponent for any given weir angle;

f = a function;

F = a force;

g = gravity constant;

H = hydraulic head, in feet;

L = a length;

m = a constant coefficient;

N = a constant coefficient for any given weir angle;

n = a constant coefficient;

Q = flow, in cubic feet per second;

$\mathbf{R}$ = Reynolds' number;

s = slope of the side of a weir;

t = time;

V = velocity = Q/A;

$\mathbf{W}$ = Weber's number;

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[5] ASA-Z10.2-1942.

w = thickness of the crest of a weir;
Z = height of the vertex of a notch;
θ = notch angle;
μ = viscosity;
ν = kinetic viscosity;
π = π_1, π_2, etc. = independent products of the arguments V, H, ρ, etc.;
ρ = density of liquids; and
σ = surface tension.

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(1) **"Tentative Method for Test of Kinematic Viscosity,"** A. S. T. M. Standard D445–39T, 1937; revised, 1938 and 1939.

(2) **"Experiments Upon the Flow of Water Over Triangular Notches,"** by James Barr, *Engineering*, Vol. 89, April 8, 1910, pp. 435–438, and April 15, 1910, pp. 470–473. (*a*) See Fig. 9.

(3) **"Dimensional Analysis,"** by P. V. Bridgman, Revised Ed., Yale Univ. Press, New Haven, Conn., 1931. (*a*) p. 40.

(4) **"Flow Through Weir Notches with Thin Edges and Full Contractions,"** by V. M. Cone, *Journal of Agricultural Research*, U. S. Dept. of Agriculture, Vol. 23, March, 1916, p. 1051.

(5) **"The V-Notch Weir for Hot Water,"** by Ed S. Smith, Jr., discussion by H. N. Eaton, *Transactions*, A. S. M. E., Vol. 57, 1935, pp. 249–250.

(6) **"Flow of Water Through Circular, Parabolic, and Triangular Vertical Notch-Weirs,"** by F. W. Greve, *Engineering Bulletin*, Purdue Univ., Lafayette, Ind., Vol. 40, March, 1932, pp. 1–84. (*a*) p. 33.

(7) **"The Flow of Water Over Sharp Crested Weir Notches, Rectangular, Trapezoidal, and Triangular,"** by Chitty Ho and Sze-Ling Wu; thesis presented to Cornell Univ. in November, 1931, in partial fulfilment of the requirements for the degree of Master of Civil Engineering.

(8) **"Applications of the Principles of Dimensional and Dynamical Similarity to the Flow of Liquids through Orifices, Notches and Weirs,"** by Hubert Mawson, *Proceedings*, The Institution of Mechanical Engrs., Vol. 1, 1927, p. 1033.

(9) **"International Critical Tables,"** National Research Council of the United States, 1st Ed., New York, N. Y., 1926.

(10) **"Precise Weir Measurements,"** by E. W. Schoder and the late K. B. Turner, *Transactions*, Am. Soc. C. E., Vol. 93 (1929), p. 999.

(11) **"The V-Notch Weir for Hot Water,"** by Ed S. Smith, Jr., *Transactions*, A. S. M. E., Vol. 56, 1934, pp. 787–789.

(12) **"Tests of the Effect of Temperature on Weir Coefficients,"** by F. G. Switzer, *Engineering News*, Vol. 73, April 1, 1915, pp. 636–637. (*a*) p. 636.

(13) **"On Experiments on the Measurement of Water by Triangular Notches in Weir Boards,"** by J. Thomson, Annual Rept., British Assn. for the Advancement of Science, 1858, pp. 181–185; 1861, pp. 151–158.

(14) **"Accuracy of the V-Notch Weir Method of Measurement,"** by D. R. Yarnall, *Transactions*, A. S. M. E., Vol. 48, 1926, pp. 939–964.

DISCUSSION

A. A. Kalinske,[6] **Assoc. M. Am. Soc. C. E.**—Any engineer concerned with the measurement of discharge of various liquids by the use of the V-notch weir should welcome this most noteworthy addition of experimental data. The precision of the author's measurements and the wide variation of viscosity and angle of notch are the outstanding features of this paper. The writer is particularly interested in the analyses of the data and therefore will confine this discussion to that phase.

The writer cannot agree that a general equation (Eq. 3) has been derived relating all the pertinent variables. The equation holds only for a certain range of Reynolds' and Weber's numbers since it was derived only for a range of data where the quantity ($C - 0.56$) versus **R** and **W** plots as a straight line on log-paper. There is, of course, no physical reason why this should be true for all ranges of **R** and **W**. The situation is comparable to the pipe friction factor, f, versus **R** plotting for pipe friction for which various power functions can represent the relation between f and **R** for different ranges of **R**-values. It should be kept in mind that an empirical equation derived by fitting a curve to a certain range of data can scarcely be expected to be general, or physically sound. The fact that the equation is dimensionally consistent does not necessarily mean that it is generally applicable.

So far as the writer can determine, Eq. 3 was derived by the analysis of the data in Fig. 8. By themselves these data do not really prove that C is a function of **R** and **W** as indicated, since the viscosity and surface tension were not varied—only the head. The writer does not wish to imply that C may not be a function of **R** and **W**; he wishes only to state that viscosity and surface tension should be independently varied to demonstrate experimentally that **R** and **W** affect C in the manner indicated. The author has other data, of course, which prove that C is definitely a function of **R** and that it is related to **W**, although the surface tension was not really varied over a very great range. However, it should be re-emphasized that the data in Fig. 8 are not in themselves sufficient to warrant the extensive use to which they are put.

Eqs. 11 and 12, which are derived from Fig. 8, are not in any sense different equations. Since **R** varies as $H^{1.5}$ and **W** as H^2, and since τ and σ were constant for the data in Figs. 8, the data plotted in these curves are actually a plotting of ($C - 0.56$) against $H^{1.5}$ and H^2. Naturally, then, the exponent of **R** in Eq. 11 and that of **W** in Eq. 12 should be as the ratio $2.0/1.5 = 1.33$, which is approximately what the author determines, since $0.36/0.28 = 1.29$. The value of B_2 in Eq. 12 could be determined from B_1 in Eq. 11 and the expressions for **R** and **W**, since all terms in these expressions were constant except H. In other words, Eqs. 11 and 12 are not actually independent equations.

In Section (*c*) of the discussion on the limiting conditions of Eq. 3, the author states that for infinitely large values of **R** and **W** the coefficient C

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[6] Asst. Prof. and Research Engr., Univ. of Iowa, Iowa Inst. for Hydr. Research, Iowa City, Iowa.

becomes 0.56; however, he states further that no data bear this out. This is very easily explained: Eq. 3 is applicable for a limited range of **R**-values and **W**-values and the fact that it does not hold for all values is no more surprising than the fact that the Blasius relation for f and **R** for pipe friction holds only for a limited range of **R**-values.

The foregoing remarks do not, of course, invalidate the use of Eq. 3, since the author is careful to point out the limits of its applicability. The reasons for the limitations are quite obvious since a simplified empirical equation cannot be expected to describe completely a complex flow phenomenon when viscosity and surface tension are both influencing factors, as is the case in this problem. Although entirely empirical equations with clearly defined limitations are useful, a curve or curves indicating the values of C for all practical ranges of **R** and **W** probably would be the best so far as engineering use is concerned.

GLEN N. COX,[7] M. AM. SOC. C. E., AND F. J. GERMANO,[8] ASSOC. M. AM. SOC. C. E.—It is indeed refreshing to read a paper such as that presented by Professor Lenz. With this paper, the explored realm of fluid mechanics and dimensional analysis has been widened, and the author is to be congratulated on the capable manner in which this has been accomplished.

Since the publication of Reynolds' outstanding paper,[9] a number of papers have been presented on the flow in pipes in which an attempt has been made to obtain a wider correlation and to make the results of more general application. Although further advance has been, and will be, made in this connection, the paper by Kemler[10] stands out as the pioneering attempt to make the results of general use to the engineering profession. The solution of the practical pipe problems is now commonly made by means of these generalized pipe friction coefficients.

As the generalized method became more commonly accepted as desirable for the solution of practical pipe-flow problems, attention turned to obtaining generalized coefficients for the various measuring devices. Excellent generalizations of the coefficients for venturi tubes[11] and orifice meters[12] have been presented. The current paper extends the same type of reasoning to the coefficients of the V-notch weir and covers the widest range of conditions contained in any of the papers on weirs which have been published to date. Although certain refinements and additions may follow, the paper must be considered to be outstanding in assisting in the extension of the generalized reasoning.

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[7] Prof., Mechanics and Hydraulics, Louisiana State Univ., Baton Rouge, La.

[8] Asst. Prof., Eng. Mechanics, Louisiana State Univ., Baton Rouge, La.

[9] "An Experimental Investigation of the Circumstances Which Determine Whether the Motion of Water Shall be Direct or Sinuous, and of the Law of Resistance in Parallel Channels," by Osborne Reynolds, *Philosophical Transactions*, Royal Soc. of London, Vol. 174, Pt. III, 1883.

[10] "A Study of the Data on the Flow of Fluids in Pipes," by Emory Kemler, *Transactions*, A. S. M. E., Vol. 55, 1933, HYD-55-2, p. 7.

[11] "Fluid Meters—Their Theory and Applications," *Research Publication*, A. S. M. E., 4th Ed., 1937.

[12] "Orifice Discharge Coefficients for Viscous Liquids," by G. L. Tuve and R. E. Sprenkle, *Instruments*, Vol. 6, 1933, pp. 201–206; and Vol. 8, 1935, pp. 202–205, 225, and 232–234.

Eq. 13 does not follow as obviously from Eqs. 11 and 12 as might be expected from a cursory reading of the explanatory paragraph. Eqs. 11 and 12 are those of the projections on the *C*–**R** and *C*–**W** planes of a line in the plane represented by Eq. 13. Such a plane is illustrated as *G H I* in Fig. 18. Line *A B* is a particular line in this plane. For a given angle of notch, the experimental data plot as a series of parallel lines in the same plane, one line for each fluid.

Since the exponents, *m* and *n*, of Eqs. 11 and 12 do not refer to the same things as those in Eq. 13, confusion might have been avoided by using letters in the equations of the projections of the lines differing from those relating to the plane.

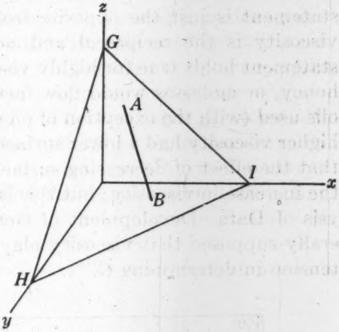

Fig. 18

It would increase the value of the paper if Professor Lenz would prepare a three-dimensional figure showing the plane on which the line (Eq. 13) appears and showing the projections of this line that give Eqs. 11 and 12. The values of *m* and *n* from Eqs. 11, 12, and 13 should be indicated, as well as the three values of *B* for these equations. This could be done for one angle of the notch—say 45°. To do so effectively, at least two experimental curves must be drawn.

It would appear that undue emphasis has been placed upon the failure of Eq. 3 to apply for the full range in the values of **R** and **W**. Experience has taught engineers that certain factors predominate over a certain range, whereas others may govern over another. Thus, there are three definite ranges for the coefficient of drag for the flow past a cylinder, or for the value of *f* for the flow through pipes. There are also three ranges for the value of *C* for the flow through a pipe orifice, but there are only two ranges for the value of *C* for the flow through a venturi tube. There are three ranges for the coefficient for the V-notch weir—namely, that for which the flow is primarily governed by the surface tension of the liquid, that for which it is primarily governed by the viscosity of the liquid, and then that for which it is independent of the surface tension and the viscous effects. It seems rather unreasonable to expect one empirical equation (even though derived rationally) to cover the various ranges without loss in accuracy. It should suffice simply to state the applicable range without the need for a feeling of apology.

Emery H. Willes,[13] **Jun. Am. Soc. C. E.**—In the third paragraph of the "Synopsis" the author makes the statement: "It was found that the weir coefficient increases with both viscosity and surface tension, and that it decreases with increase in head and angle of V-notch." If an increase of viscosity and surface tension does increase the weir coefficient, it naturally would follow that an increase in head and angle of V-notch would decrease the weir coefficient, for as the head and angle of V-notch increases the effect of viscosity and surface tension decreases, and hence the weir coefficient decreases. Is it true that the weir coefficient will increase with viscosity and surface tension? This statement is just the opposite from that which the writer would expect, for viscosity is the reciprocal and not a measure of fluidity. If the author's statement holds true for highly viscous liquids, then, for a given head, glucose, honey, or molasses would flow faster than water. The writer noticed for the oils used (with the exception of oil C) that at a given temperature an oil with a higher viscosity had a lower surface tension (see Fig. 19). This might indicate that the effect of decreasing surface tension was to overcome the effect due to the increase in viscosity; but this is contrary to the author's statement ("Analysis of Data: Development of General Equation"): "Furthermore, it is generally supposed that viscosity plays a much more important rôle than surface tension in determining *C*."

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[13] Hydrographer, Utah Copper Co., Salt Lake City, Utah.

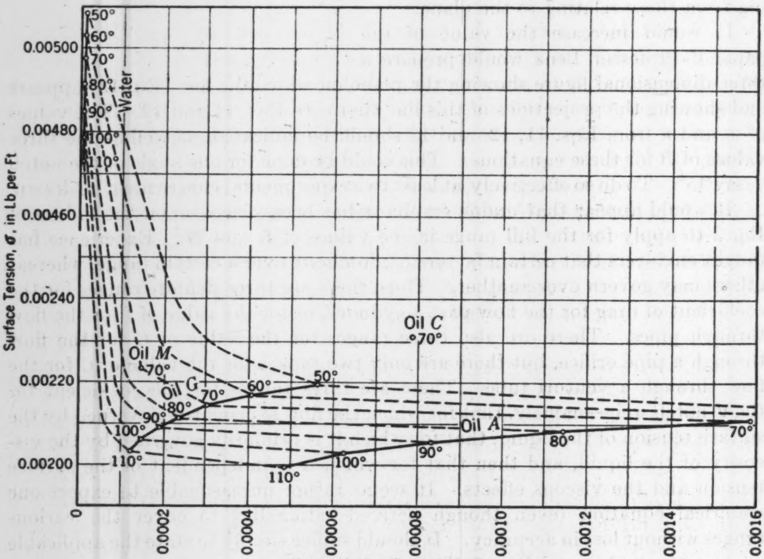

FIG. 19.—RELATIONSHIP BETWEEN VISCOSITY AND SURFACE TENSION FOR THE GIVEN OILS

If the same oil is used and the temperature varies (Fig. 12), the surface tension increases with the viscosity. This relationship is just the reverse of that noticed for different oils at the same temperature. These two opposite relationships, together with the author's experimental results, bear out the author's statement that "viscosity plays a much more important rôle than surface tension," and prove the falsity of the writer's suggestion herein.

What, then, is the answer to the apparent reversal of natural laws? In Fig. 13, for values of Reynolds' number greater than a certain critical value (for example, $\mathbf{R}_{cr} = 845$ for a 28° V-notch) the plotted points fall on the author's straight line; but for values of Reynolds' number less than this critical value there is a rapid drawing away from the straight line, apparently forming a parabolic curve. Thus, for the lower values of Reynolds' number—or, in other words, as the viscosity increases, or becomes more important—the coefficient (C) will decrease instead of increase. At some value of Reynolds' number the tangent to the true curve becomes zero ($\mathbf{R}$ = approximately 400 for a 28° V-notch). For values of Reynolds' number less than this the slope of the tangent is positive and for values of Reynolds' number greater than this the slope is negative. Thus, for the highly viscous liquids the coefficient will decrease as the viscosity increases, which would be expected.

In 1643 Torricelli in his *De Motu Gravium Naturaliter Accelerato* enunciated the law that is the base of all weir formulas—namely:

> "The velocity of a fluid passing through an orifice in the side of a reservoir is the same as that which would be acquired by a heavy body falling freely through the vertical height measured from the surface of the fluid in the reservoir to the center of the orifice."

By the use of Torricelli's law and calculus, Eqs. 1 and 2 are easily derived. This derivation is simple and may be found in most hydraulic textbooks. The author arrives at the same formula using a method developed by Edgar Buckingham[14] in 1915—namely, the "π-theorem." This method of dimensional analysis, because its value has been proved in model experiments in hydraulic laboratories, is now often used. Because of this increase in importance, a detailed investigation of the development of this theorem may be in order.

If the physical quantities, $G_1, G_2, \ldots G_n$, such as a length, a force, a velocity, a density, etc., suffice to describe any physical phenomenon completely, then they are mutually dependent in some known or unknown manner. This mutual dependence may be stated mathematically as a function of n independent variables:

$$f(G_1, G_2, \ldots G_n) = 0 \ldots\ldots\ldots\ldots (20)$$

In other words, the mutual dependence of the n quantities can be expressed as a polynomial in n independent variables. Any polynomial of these n variables may be written:

$$K_1 (G_1)^{a_1} (G_2)^{b_1} \ldots (G_n)^{m_1} + K_2 (G_1)^{a_2} (G_2)^{b_2} \ldots (G_n)^{m_2} + \ldots K_n (G_1)^{a_n} (G_2)^{b_n} \ldots (G_n)^{m_n} = 0 \ldots\ldots\ldots\ldots (21)$$

or

$$\Sigma[K (G_1)^a (G_2)^b \ldots (G_n)^m] = 0 \ldots\ldots\ldots\ldots (22)$$

in which K is a dimensionless coefficient and $a, b, \ldots m$ are exponents.

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[14] "Model Experiments and the Forms of Empirical Equations," by E. Buckingham, *Transactions*, A. S. M. E., Vol. 37, 1915, p. 289.

If Eq. 22 is divided by one of the terms of Eq. 21—$K_i\ (G_1)^{a_i}\ (G_2)^{b_i}\ \ldots\ (G_n)^{m_i}$:

$$\Sigma[J\ (G_1)^{\alpha}\ (G_2)^{\beta}\ \ldots\ (G_n)^{\mu}] = 0 \ldots\ldots\ldots\ldots (23a)$$

in which $J = \frac{K}{K_i}$, $\alpha = \frac{a}{a_i}$, $\beta = \frac{b}{b_i}$, $\ldots\ \mu = \frac{m}{m_i}$; or, since one of the terms is equal to unity:

$$1 + \Sigma[J\ (G_1)^{\alpha}\ (G_2)^{\beta}\ \ldots\ (G_n)^{\mu}] = 0 \ldots\ldots\ldots\ldots (23b)$$

In his "Deduction of the π-Theorem"[14] Buckingham makes the statement:

> "Now all the terms of any physical equation must have the same dimensions and the coefficients [*J*] have no dimensions because they are merely ratios of the dimensionless coefficients [*K*]."

This is merely another way of expressing one of the first principles of physical mathematics—namely, dogs, cats, and pigs cannot physically be added together; they must be restated in terms of some common attribute. In other words, each term of Eq. 21 or Eq. 22 must have the same dimensions. Upon the validity of this statement depends the proof of the theorem and hence the usefulness of dimensional analysis. Since each term in Eq. 21 or Eq. 22 must have the same dimensions, each term in Eq. 23 will be dimensionless.

Every physical quantity is composed of two distinct characteristics—magnitude and dimension. Magnitude is dimensionless and may be represented by a numerical constant *K*, whereas the dimension of the quantity is determined by certain fundamental units such as mass, length, time, etc., to some power which is known or can be determined as soon as the quantity itself is known. In mechanics the number of these fundamental units is three—mass, length, and time. In problems dealing with thermal and electromagnetic as well as mechanical quantities, the number of fundamental units will not exceed five. The dimension of a quantity may be represented, therefore, by M^x, L^y, T^z, $\ldots\ (D_k)^n$, in which *k* represents the number of fundamental units and *M*, *L*, *T*, and *D* are symbols for these units. Thus,

$$G = K\ M^x\ L^y\ T^z\ \ldots\ (D_k)^n \ldots\ldots\ldots\ldots (24)$$

Assuming for simplicity that mechanical quantities only are involved, Eq. 23 becomes,

$$1 + \Sigma\,[J\ (K_{G1})^{\alpha}\ (K_{G2})^{\beta}\ \ldots\ (K_{Gn})^{\mu}\ (M^{x_1}\ L^{y_1}\ T^{z_1})^{\alpha}\ (M^{x_2}\ L^{y_2}\ T^{z_2})^{\beta}$$
$$\ldots\ (M^{x_n}\ L^{y_n}\ T^{z_n})^{\mu}] = 0 \ldots\ldots\ldots\ldots (25)$$

As noted previously, Eq. 23 is dimensionless; hence,

$$(M^{x_1}\ L^{y_1}\ T^{z_1})^{\alpha}\ (M^{x_2}\ L^{y_2}\ T^{z_2})^{\beta}\ \ldots\ (M^{x_n}\ L^{y_n}\ T^{z_n})^{\mu} = [1] \ldots (26)$$

in which [1] represents the expression which has the same dimension as unity. Eq. 25, therefore, becomes,

$$1 + \Sigma[J\ (K_{G1})^{\alpha}\ (K_{G2})^{\beta}\ \ldots\ (K_{Gn})^{\mu}] = 0 \ldots\ldots\ldots\ldots (27)$$

and from Eq. 26:

$$\left.\begin{aligned} \alpha\, x_1 + \beta\, x_2 + \ldots\ \mu\, x_n &= 0 \\ \alpha\, y_1 + \beta\, y_2 + \ldots\ \mu\, y_n &= 0 \\ \alpha\, z_1 + \beta\, z_2 + \ldots\ \mu\, z_n &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (28)$$

The unknowns in Eqs. 28 are $\alpha, \beta, \dots \mu$. The remainder will be known in any particular problem.

Again referring to Buckingham's paper:[15]

> "If k is the number of fundamental units needed, there is always, among all the n [quantities], at least one set of k which are independent of one another and might themselves be used as fundamental units if we paid no attention to the question of representing and preserving the fundamental units by satisfactory primary standards."

Letting $G_1, G_2, \dots G_k$ represent the chosen quantities in Eq. 20 that may be used as dimensions and $P_1, P_2, \dots P_{n-k}$ be the quantities that are left, then from the foregoing statement:

$$[P_1] = [(G_1)^a (G_2)^b \dots (G_k)^i] \dots\dots\dots\dots\dots (29a)$$

in which the brackets are used as a symbol of "the dimension of." Substituting the dimensions of Eq. 24 in Eq. 29*a*, then:

$$M^{x_p} L^{y_p} T^{z_p} \dots (D_k)^{n_p} = [M^{x_1} L^{y_1} T^{z_1} \dots (D_k)^{n_1}]^a [M^{x_2} L^{y_2} T^{z_2}$$
$$\dots (D_k)^{n_2}]^b \dots [M^{x_k} L^{y_k} T^{z_k} \dots (D_k)^{n_k}]^i \dots\dots\dots (29b)$$

Letting π equal the ratio of the terms on both sides of the equal sign in Eqs. 29:

$$\pi = (G_1)^A (G_2)^B \dots (G_k)^I P \dots\dots\dots\dots\dots\dots (30a)$$

in which $A = -a$, $B = -b$, etc. Also,

$$\pi = [M^{x_1} L^{y_1} T^{z_1} \dots (D_k)^{n_1}]^A [M^{x_2} L^{y_2} T^{z_2} \dots (D_k)^{n_2}]^B$$
$$\dots [M^{x_k} L^{y_k} T^{z_k} \dots (D_k)^{n_k}]^I M^{x_p} L^{y_p} T^{z_p} \dots (D_k)^{n_p} = [1] \,..\, (30b)$$

in which the brackets are again used as a symbol of "the dimension of." As demonstrated in Eqs. 28, it is possible to write from Eq. 30*b*:

$$\left.\begin{array}{l} A_{x_1} + B_{x_2} + \dots I_{x_k} + x_p = 0 \\ A_{y_1} + B_{y_2} + \dots I_{y_k} + y_p = 0 \\ A_{z_1} + B_{z_2} + \dots I_{z_k} + z_p = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ A_{n_1} + B_{n_2} + \dots I_{n_k} + n_p = 0 \end{array}\right\} \dots\dots\dots\dots\dots (31)$$

As in Eqs. 28, the unknowns are $A, B, \dots I$ and, since there are as many equations as there are unknowns, Eqs. 31 can be solved. Hence, Eqs. 30 are completely solved and Eq. 27 becomes:

$$1 + \Sigma[J (\pi_1)^{r_1} (\pi_2)^{r_2} (\pi_3)^{r_3} \dots (\pi_{n-k})^{r_{n-k}}] = 0 \dots\dots (32a)$$

The only unknowns now are the value of the exponents "r" for each π-value and each term. Since the "1" in Eq. 32*a* is also one of the terms of Eq. 22 divided by itself, it also may be included in the summation, and therefore

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[15] "Model Experiments and the Forms of Empirical Equations," by E. Buckingham, *Transactions*, A. S. M. E., Vol. 37, 1915, p. 290.

Eq. 32*a* can be written:

$$\Sigma[J\,(\pi_1)^{r_1}\,(\pi_2)^{r_2}\,(\pi_3)^{r_3}\ \ldots\ (\pi_{n-k})^{r_{n-k}}] = 0 \tag{32b}$$

or

$$f(\pi_1,\ \pi_2\ \ldots\ \pi_{n-k}) = 0 \tag{32c}$$

which is the same as Eq. 4*b* of the paper.

By choosing *V*, *H*, and ρ as the independent quantities to be used as fundamental units:

$$\left.\begin{aligned}
&\pi_1 = V^{A_1}\,H^{B_1}\,\rho^{C_1}\,\mu \\
&= (K_V)^{A_1}\,(K_H)^{B_1}\,(K_\rho)^{C_1}\,K_\mu\left[\left(\frac{L}{T}\right)^{A_1} L^{B_1}\left(\frac{F\,T^2}{L^4}\right)^{C_1}\left(\frac{F\,T}{L^2}\right)\right] \\
&\pi_2 = V^{A_2}\,H^{B_2}\,\rho^{C_2}\,\sigma \\
&= (K_V)^{A_2}\,(K_H)^{B_2}\,(K_\rho)^{C_2}\,K_\sigma\left[\left(\frac{L}{T}\right)^{A_2} L^{B_2}\left(\frac{F\,T^2}{L^4}\right)^{C_2}\left(\frac{F}{L}\right)\right] \\
&\pi_3 = V^{A_3}\,H^{B_3}\,\rho^{C_3}\,g \\
&= (K_V)^{A_3}\,(K_H)^{B_3}\,(K_\rho)^{C_3}\,K_g\left[\left(\frac{L}{T}\right)^{A_3} L^{B_3}\left(\frac{F\,T^2}{L^4}\right)^{C_3}\left(\frac{L}{T^2}\right)\right]
\end{aligned}\right\} \tag{33}$$

and from Eq. 6

$$\left.\begin{aligned}
&\left(\frac{L}{T}\right)^{A_1} L^{B_1}\left(\frac{F\,T^2}{L^4}\right)^{C_1}\left(\frac{F\,T}{L^2}\right) = [1] \\
&\left(\frac{L}{T}\right)^{A_2} L^{B_2}\left(\frac{F\,T^2}{L^4}\right)^{C_2}\left(\frac{F}{L}\right) = [1] \\
&\left(\frac{L}{T}\right)^{A_3} L^{B_3}\left(\frac{F\,T^2}{L^4}\right)^{C_3}\left(\frac{L}{T^2}\right) = [1]
\end{aligned}\right\} \tag{34}$$

When the exponents in Eqs. 34 are determined:

$$\left.\begin{aligned}
\pi_1 &= \frac{\mu}{V\,H\,\rho} = \frac{\nu}{V\,H} \\
\pi_2 &= \frac{\sigma}{V^2\,H\,\rho} \\
\pi_3 &= \frac{g\,H}{V^2}
\end{aligned}\right\} \tag{35}$$

When these values of π are substituted in Eq. 32*c*, Eq. 5 is the result, from which the author develops Eq. 7.

To get Eq. 8 the author replaces "*V* in each of the products by a dimensional equivalent, $\sqrt{g\,H}$." If the particular function of Eq. 7 were known, this substitution would not be entirely correct because the true relationship between *V* and *g H* is Eq. 6*a*; but, since the particular function is not known, this replacement is permissible. At least it is dimensionally correct. A more direct approach to Eq. 8 would be to choose *H*, *g*, and ρ as the three units to be used as dimensions instead of V, H, and ρ. Thus,

$$\left.\begin{aligned}
\pi_1 &= H^{A_1} g^{B_1} \rho^{C_1} V \\
&= (K_H)^{A_1} (K_g)^{B_1} (K_\rho)^{C_1} K_V \left[L^{A_1} \left(\frac{L}{T^2}\right)^{B_1} \left(\frac{F\,T^2}{L^4}\right)^{C_1} \left(\frac{L}{T}\right)\right] \\
\pi_2 &= H^{A_2} g^{B_2} \rho^{C_2} \mu \\
&= (K_H)^{A_2} (K_g)^{B_2} (K_\rho)^{C_2} K_\mu \left[L^{A_2} \left(\frac{L}{T^2}\right)^{B_2} \left(\frac{F\,T^2}{L^4}\right)^{C_2} \left(\frac{F\,T}{L^2}\right)\right] \\
\pi_3 &= H^{A_3} g^{B_3} \rho^{C_3} \sigma \\
&= (K_H)^{A_3} (K_g)^{B_3} (K_\rho)^{C_3} K_\sigma \left[L^{A_3} \left(\frac{L}{T^2}\right)^{B_3} \left(\frac{F\,T^2}{L^4}\right)^{C_3} \left(\frac{F}{L}\right)\right]
\end{aligned}\right\} \ldots (36)$$

and solving:

$$\left.\begin{aligned}
\pi_1 &= \frac{V}{\sqrt{g\,H}} \\
\pi_2 &= \frac{\mu}{H^{1.5}\,g^{0.5}\,\rho} = \frac{\nu}{H^{1.5}\,g^{0.5}} \\
\pi_3 &= \frac{\sigma}{H^2\,g\,\rho}
\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (37)$$

Substituting these values of π in Eq. 32c and solving for C, as demonstrated by the author, will yield Eq. 10.

The writer takes exception to the author's statement ("Analysis of Data: Development of General Equation"): "The functional relationship, f'', between C, R, and W for a given weir must be obtained from experimental data." This statement should be: "The functional relationship * * * at present must be obtained from experimental data."

There is a theoretical solution to every phenomenon in nature. The trouble lies with Man. Either the relationship is so simple that Man has not looked in that direction yet, or it is so complex that at present it exceeds the capacity of the mind to solve it. The fact still remains, however, that there is a theoretical solution to the problem. Eq. 10 will be of the form

$$N_1 (\pi_1)^{n_1} (\pi_2)^{m_1} + N_2 (\pi_1)^{n_2} (\pi_2)^{m_2} + \ldots N_n (\pi_1)^{n_n} (\pi_2)^{m_n} = 0 \ldots (38)$$

Substituting $\pi_1 = \frac{1}{\mathrm{R}}$, $\pi_2 = \frac{1}{\mathrm{W}}$, and $N = B_x$, Eq. 38 becomes:

$$\frac{B_{x_1}}{\mathrm{R}^{n_1}\,\mathrm{W}^{m_1}} + \frac{B_{x_2}}{\mathrm{R}^{n_2}\,\mathrm{W}^{m_2}} + \ldots \frac{B_{x_n}}{\mathrm{R}^{n_n}\,\mathrm{W}^{m_n}} = 0 \ldots\ldots\ldots (39)$$

or letting C_x be the value of the function as π_1 and π_2 approach zero,

$$C - C_x = \frac{B_{x_1}}{\mathrm{R}^{n_1}\,\mathrm{W}^{m_1}} + \frac{B_{x_2}}{\mathrm{R}^{n_2}\,\mathrm{W}^{m_2}} + \ldots \frac{B_{x_n}}{\mathrm{R}^{n_n}\,\mathrm{W}^{m_n}} \ldots\ldots (40)$$

or

$$C - C_x = \sum \frac{B_x}{\mathrm{R}^n\,\mathrm{W}^m} \ldots\ldots\ldots\ldots\ldots\ldots (41)$$

If for simplicity all terms but one are assumed either zero or unimportant: $C - C_x = \frac{B_x}{\mathbf{R}^n \, \mathbf{W}^m}$ (see Eq. 13). If the effect of Weber's number (**W**) is insignificant, Eq. 13 becomes $C - C_x = \frac{B_x}{\mathbf{R}^n}$ (compare Eq. 12). As noted previously, these equations can be used only between certain limits. Beyond these limits Eq. 41 must be used, for the experimental data when plotted on logarithmic paper seem to pull away from the author's straight line (see Fig. 13) and form a curve resembling a parabola. Just what the particular function of Eq. 41 is probably will be revealed or approximated by further experimental investigation.

RALPH W. POWELL,[16] M. AM. SOC. C. E.—This paper has been long desired, as it not only lays a basis for the measurement of viscous liquids by means of notch weirs, but also indicates the way in which the flow of less viscous fluids, like water, may be affected by changes of temperature. For many years the writer has felt that, instead of trying to study the effect of temperature upon the coefficient of notched weirs which measure water, by great refinement of measurement, it would be more satisfactory to find the influence of viscosity by the use of more viscous liquids where the effect would be greatly increased and therefore more readily measured. Mr. Lenz now shows that surface tension also has an important effect and gives quantitative results.

However, the writer feels constrained to mention several things that he personally wishes had been done differently. One minor point is that he prefers to define the Reynolds number of a weir as $\frac{V\,H}{\nu}$, in which V is $\sqrt{2\,g\,H}$, the ideal velocity due to the head H. This gives $\mathbf{R} = \frac{\sqrt{2\,g}\,H^{1.5}}{\nu}$, or $\sqrt{2}$ times as much as Eq. 9*a*. Similarly, he would follow the definition of the Weber number given by Hunter Rouse,[17] Assoc. M. Am. Soc. C. E.—namely, $\frac{V}{\sqrt{\frac{\sigma}{\rho\,H}}}$, which reduces to $\mathbf{W} = \sqrt{\frac{2\,g\,\rho}{\sigma}}\,H = H\sqrt{\frac{2\,\gamma}{\sigma}}$. This is the square root of twice the value given by Eq. 9*b*. However, in the following comments, the author's definitions have been used. The writer also very much wishes that the actual numerical data had been included in the paper. Desiring to test the possibility of representing the author's experimental results by a formula of different form, he obtained a copy of the original thesis[4] from the University of Wisconsin, and was surprised to find that even this did not give the original measurements.

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[16] Associate Prof. of Mechanics, Ohio State Univ., Columbus, Ohio.

[17] "Fluid Mechanics for Hydraulic Engineers," by Hunter Rouse, McGraw-Hill Book Co., Inc., New York, N. Y., 1938, p. 23.

In 1910 T. P. Strickland[18] showed that the coefficient for water flowing through a V-notch weir could be very accurately represented over a wide range of heads by a formula of the form

$$C = a + \frac{b}{\sqrt{H}} \quad \text{(42)}$$

Scaling values from the curves of the paper and plotting C against $\frac{1}{H}$ indicated that all the author's values for a 90° notch above the minimum heads shown in Fig. 7 fit this formula with a maximum discrepancy of about 0.5%. This is using $a = 0.552$ for all four liquids and the value of b for each liquid that gives the best fit.

The writer wished to investigate a formula of the form

$$C = a + \frac{B_1}{\mathbf{R}^n} + \frac{B_2}{\mathbf{W}^m} \quad \text{(43)}$$

as an alternative to Eq. 3. Since **R** contains $H^{1.5}$ and **W** contains H^2, Eq. 42 would require that, in Eq. 43, $n = \frac{1}{3}$ and $m = \frac{1}{4}$. It was found that the best fit for a formula of this type and the author's data on 90° notch weirs was given by

$$C = 0.552 + \frac{0.435}{\mathbf{R}^{1/3}} + \frac{0.201}{\mathbf{W}^{1/4}} \quad \text{(44)}$$

All the points shown in Fig. 7 (except those below the minimum head indicated) were computed by this formula and the discrepancies tabulated. The algebraic mean of the discrepancies was 0.0003 and the arithmetic mean, 0.0045. This is about the same mean discrepancy as the author's formula gives, and it seems to the writer that Eq. 44 is somewhat simpler than Eq. 3. At least it has the advantage of separating the viscosity and surface tension "corrections." It is interesting to note that, according to Eq. 44 for water, the effect of surface tension is nearly three and one-half times as great as that of viscosity. As is to be expected, for the oils the viscosity had more effect than the surface tension —nearly twice as much in the case of oil A at 70° F.

It should also be noted that, whereas the range of viscosities studied is quite large, the variation in surface tension is relatively small. The property that is effective here is what might be called the "kinematic surface tension," $\frac{\sigma}{\rho g}$ or $\frac{\sigma}{\gamma}$, where γ is the specific weight. The expression $\frac{\sigma}{\gamma}$ is an area, say feet squared, and

$$\mathbf{W} = \frac{H^2}{\text{kinematic surface tension}} \quad \text{(45)}$$

When one consults a handbook for possible liquids that would give a wider range of kinematic surface tensions than those used by the author, the search is not very rewarding. Mercury has a surface tension some seven times as great as water, but it weighs 13.6 times as much, so that its kinematic surface tension is less—about the same as that of the oils used by the author. Water seems to stand at the top of the list (except that some water solutions of substances like $CaCl_2$ are somewhat higher). Carbon tetrachloride has a kinematic surface tension of about one fourth that of water and would probably be the best liquid to compare with water to get a more accurate measure of the effect of surface tension. Another liquid that might be used is water to which a small amount of saponin and a still smaller amount of oleic acid have been added.[19] This will not give nearly as wide a range as carbon tetrachloride (in fact a little less than the oil used by the author), but it would have the advantage of having the same viscosity as water alone, and therefore of varying only one factor at a time.

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[18] *Engineering* (London), Vol. 90, October 28, 1910, p. 598.

The writer has not taken time to develop formulas similar to Eq. 44 for other notch angles, and to study the variation of the coefficients with θ. It would be only after such a study that the relative merits of a formula of the form of Eq. 44 could be compared with the author's Eqs. 3 and 14, and of course there may be some other form that is better than any of these.

The author has given little attention to the interesting question of the variation of C with **R** and **W** when either or both of the latter are below what he calls the minimum values. Water is the best liquid with which to study low Weber numbers, and Barr's experiments ran down to a Weber number of about 54. These results, put in the units of this paper, and with the temperature assumed to be 46.4° F (the mean of the range given), are plotted in Fig. 20. (Some of these data were used by the writer[20] elsewhere; but there the coefficient unfortunately was plotted against the Reynolds number, a procedure which this paper shows to have been quite beside the point.) Curve A of Fig. 20 is the most striking, and it is interesting that in this semilogarithmic plotting it

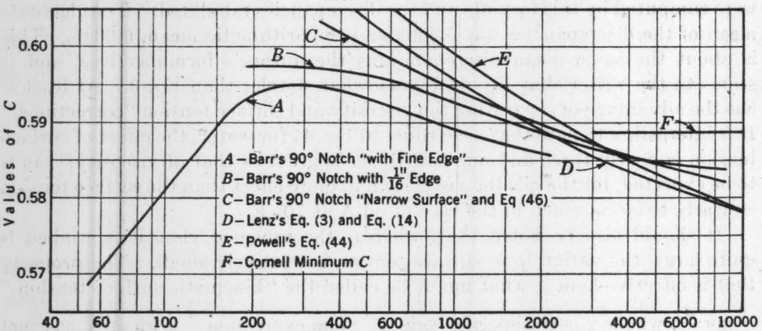

FIG. 20

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[19] "A Study of Foam Formation in Aqueous Solutions," by Virgil Leland Hansley, an unpublished doctor's thesis, Ohio State Univ., Columbus, Ohio, 1928, p. 110.

[20] "Mechanics of Liquids," by Ralph W. Powell, Macmillan Co., 1940, Fig. 109, p. 199.

is perfectly straight from **W** = 54 to **W** = 166. The similarity between this curve and that of orifice coefficients[21] plotted against **R** is also worthy of note. The fact that curves *A*, *B*, and *C* are so different may be partly due to the fact that the writer's assumption of the same temperature for the three cases introduces an appreciable error, but it is more probable that the slight difference in the three weirs was responsible. If so, before the V-notch weir can ever become an instrument for absolute measurement (dependable without calibration), the size of plate, thickness of edge, and angle of bevel must be standardized more definitely. (Barr's original data included a record of the temperature, as shown in the sample that was printed. Is it possible that these records are still in existence, so that **R** and **W** could even yet be found for each of his runs?)

For water at the temperature assumed, **R** and **W** were computed for a number of different heads, and from these *C* was computed by Eqs. 3 and 14, and 44, and the results plotted in Fig. 20. As the author's edge was $\frac{1}{32}$ in., it would be expected that his results at this temperature would lie between curves *A* and *B*, but nearer curve *B*. (Curve *B* is the one from which Barr's values, plotted in Fig. 15, are taken.) To make the discrepancy no more than 1% would require changing the minimum **W** from 300 to about 600. As a matter of fact, any formula of the kind proposed by the author (or the writer) could not be expected to fit beyond the point of inflection of the curve, and for curve *B* this seems to be at about **W** = 900. For curve *C*, however, it is at least as low as the lowest **W** on the curve—that is, 340. This is the curve fitted by Mr. Strickland,[18] and changing his formula into one similar to Eq. 44 gives

$$C = 0.565 + \frac{0.435}{\mathbf{R}^{1/3}} + \frac{0.102}{\mathbf{W}^{1/4}} \qquad (46)$$

This fits curve *C* with a maximum discrepancy of about 0.04%. There are other pairs of numerical coefficients, of course, which would fit equally well—any that would reduce to

$$C = 0.565 + \frac{0.0157}{\sqrt{H}} \qquad (47)$$

Undoubtedly similar formulas could be deduced to fit curves *A* and *B* up to the point of inflection with not much greater discrepancies.

That the author found, for very large values of **R** and **W**, that *C* becomes independent of either is not surprising in view of the analogous facts regarding the coefficients for orifices, nozzles, and pipes (and, the writer believes, for channels as well); but it is somewhat surprising that the limit is reached for both **R** and **W** at precisely the same value of *C*. Also, the fact that at high heads all of Barr's weirs gave values of *C* less than 0.5886 would seem to indicate that this limiting value of *C* depends on the weir thickness (as well as its sharpness and the smoothness and size of the upstream face).

Finally, physical chemists have discovered that certain solutions, such as 500 ppm of peptone in water or 100 ppm of saponin in water, have what is called "surface viscosity."[22] This is quite different from ordinary bulk viscosity and seems to be due to the formation of a plastic surface film. It is not likely that the liquids used by the author had this property, but it would seem probable that, if a liquid had it, its flow over a notched weir would be affected thereby. Therefore, if experiments were to be made on a saponin and oleic acid solution as suggested herein, this matter should be investigated.

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[21] "Mechanics of Liquids," by Ralph W. Powell, Macmillan Co., 1940, p. 190.

G. H. HICKOX,[23] M. AM. SOC. C. E.—This paper is a genuine contribution to knowledge in the field of flow measurement, and the author is to be congratulated on his effort in bringing this information before the profession. A number of years ago, the writer became interested in the effect of viscosity on weir discharge because of an inquiry as to the proper weir discharge coefficient to be used in the measurement of printer's ink. He assisted in the preparation of the experimental apparatus on which the University of California results, referred to by the author, were obtained. It is gratifying to see that the ensuing experiments have been put to good use.

In the past, practically all information on weir discharge coefficients has been obtained from experiments using water. The effects of viscosity, surface tension, and inertia have been combined into a single coefficient that also includes the geometry of the approach channel and the roughness of the weir surface, as well as any other variables which might affect the discharge. Attempts have been made to evaluate the effect of the geometry of the approach channel by including the velocity of approach with the measured head. The results have not been entirely satisfactory. The most consistent results have been obtained when the dimensions of the approach channel have been such that the velocity of approach was negligible. A notable exception is the use of the ratio of the head on the weir to the height of weir crest above the channel bottom in the case of a rectangular weir. The effects of viscosity and surface tension have been almost totally neglected.

It should be recognized that, after the author's demonstration of the use of **R** and **W** as suitable parameters, the treatment is empirical. It is quite possible that the form of the function indicated by Eq. 10 might be considerably different from that developed in Eq. 13. It is conceivable that several different functions might exist which would satisfy the experimental data equally well. Extrapolation of these functions outside the experimental range would doubtless give varying results. For this reason, extrapolation of the results of the experiments beyond the range of the experimental data is unwarranted and should be made with caution or not at all.

The use of V-notch weirs as measuring devices need not be confined solely to fluids of low viscosity, such as water or lubricating oils. These weirs may find application in the handling of heavy inks, paints, and sugar solutions, including molasses, to mention only a few examples. Additional experiments with fluids of high viscosity would determine the effects of viscosity, and possibly surface tension, over a much wider range. This would be advantageous in the application to fluids of lower viscosity since it would remove the uncertainties existing at the end of the present useful range.

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[22] "A Study of Foam Formation in Aqueous Solutions," by Virgil Leland Hansley, an unpublished doctor's thesis, Ohio State Univ., Columbus, Ohio, 1928, pp. 41–55.

[23] Senior Hydr. Engr., TVA, Hydraulic Laboratory, Norris, Tenn.

The effects of viscosity and surface tension on V-notch weirs are paralleled more or less closely by the effects on small ogee weirs. Tests on small-scale models of ogee spillways are being made in large numbers, and the results are being transferred to the much larger prototype structures by means of Froude's law, neglecting the effects of viscosity and surface tension. Such a transference of results assumes that the coefficient for the prototype is the same as for the model. Experimental verification of this assumption is almost entirely lacking, and the evidence available does not point to any definite conclusions. From experiments on three different scale models of San Gabriel Dam No. 1, Paul Baumann,[24] M. Am. Soc. C. E., found that the discharge coefficient increased with the size of the model. A comparison[25] of measured discharges over an eastern dam with model tests made by the Bureau of Reclamation on a 1 : 72 scale model showed that the prototype coefficients were smaller than those of the model. A. H. Gibson[26] and D. Thoma[27] have presented similar results. No definite information is available to explain the differences that have been found. Although it is possible that prototype discharge measurements may be in error, it seems much more likely that the neglected effects of viscosity and surface tension in small models would be found to be responsible.

In spite of the writer's own caution against extrapolation of results, he would like to inquire whether any work done by the author has indicated a possible answer to this problem. The question is not academic. The matter of predicting spillway discharges from model tests becomes important on those rivers that are completely canalized and that also develop power at the navigation dams. Under such conditions, an accurate measurement of discharge is useful for reservoir regulation and power production. The usual rating stations are drowned out by downstream dams, and the discharges can be determined only by calibration of the turbines and spillway gates. The easiest way of making the spillway calibration is by means of model tests, if it can be established that they are reliable. It is possible that studies similar to those reported in the paper would make it possible to evaluate the effects of viscosity and surface tension on the results of small-scale model tests.

ARNO T. LENZ,[28] ASSOC. M. AM. SOC. C. E.—The writer wishes to express his sincere thanks to those who discussed his paper. The discussions called attention to some points not made clear originally and other points not considered by the writer but which might well be noted. The discussions will be answered in order of their presentation.

Professor Kalinske objects to the expression "general equation" for Eq. 3. The writer used the term in the sense that the equation was applicable to liquids other than water as well as to water, and held over a wide range of notch angles. He did recognize that the equation applied only within certain limits, as evidenced by the fact that those limits are stated carefully. He agrees with Professors Cox and Germano that it would be too much to expect one empirical equation to cover all possible conditions.

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[24] "Design and Construction of San Gabriel Dam No. 1," by Paul Baumann, *Transactions*, Am. Soc. C. E., Vol. 107 (1942), p. 1595.

[25] "Conformity between Model and Prototype, A Symposium: Spillway Coefficients at an Eastern Dam," by G. H. Hickox, *Proceedings*, Am. Soc. C. E., October, 1942, p. 1334.

[26] "The Use of Models in Hydraulic Engineering," by A. H. Gibson, *Water and Water Engineering*, Winter 1934, pp. 697–704.

[27] *Transactions*, Hydraulic Lab. of The Technical University of Munich, Germany, No. 2, 1928.

[28] Madison, Wis.

Eq. 3 was not derived from the data in Fig. 8 alone. In the section "Development of General Equation," an attempt was made to rationalize an equation which might appear rather complicated if picked from thin air. Fig. 8(*a*) was used only as an example to illustrate the plotting of the data. These data were used to evaluate constants for Eq. 11, but it was considered obvious that the data in Fig. 11 for oil G and water, and, in Fig. 13, for oil A at two different temperatures also gave equations of the form of Eq. 11. Data on Weber's number for all series could have been shown in figures similar to Fig. 8(*b*). Eqs. 3 and 13 were derived from study of the complete data.

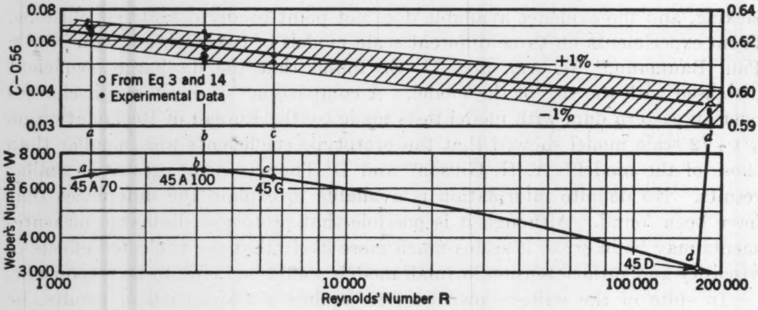


FIG. 21.—COEFFICIENT CURVE FOR 45° V-NOTCH WEIR AT 0.50-FT HEAD

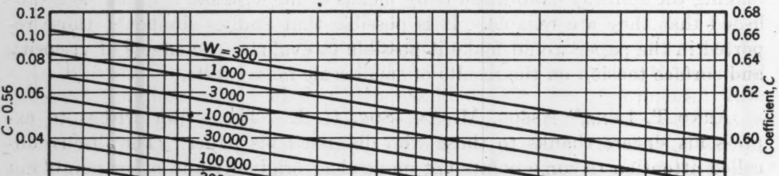


FIG. 22.—COEFFICIENT CURVES FOR 90° V-NOTCH WEIR

The writer agrees heartily with Professor Kalinske in the conclusion that C is a function of **R** and **W** could not be proved by changing the head alone. Fig. 21 is presented as proof that C is a function of **R** and **W** even though the head remains constant. The curved line abcd is determined by **R** and **W** for a constant head $H = 0.50$ ft. The experimental coefficients are plotted on a logarithmic scale perpendicular to this line. Similar data for other heads and other angles could be drawn, within the limits of Eq. 3.

The statement that Eqs. 11 and 12 are not independent equations is true, of course, in case of a single liquid at a given temperature but would not be true for different liquids unless ν and σ were related. Daniel Silverman and W. E. Roseveare[29] have shown that viscosity and surface tension are related, and hence the problem is somewhat simpler than it might have been otherwise.

If a family of curves is to be preferred to Eq. 3, the reader is referred to Fig. 22. Of course, it would be necessary to plot nine such curves to cover

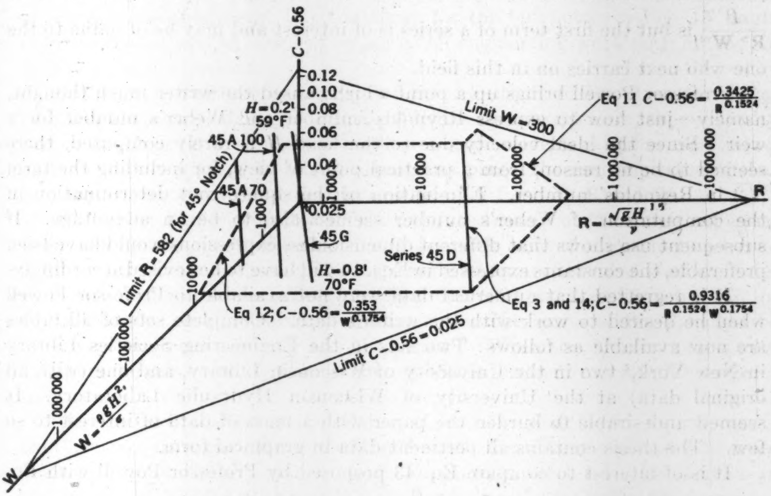

C−0.56
0.12
0.10
0.08
0.06
0.04
0.025
H=0.2'
59°F
45 A 100
45 A 70
45 G
Series 45 D
H=0.8'
70°F
Limit W=300
Eq 11 $C-0.56=\frac{0.3425}{R^{0.1524}}$
1 000
10 000
100 000
1 000 000
10 000 000
R
$R=\frac{\sqrt{g}H^{1.5}}{\nu}$
Eq 3 and 14; $C-0.56=\frac{0.9316}{R^{0.1524}W^{0.1754}}$
Eq 12; $C-0.56=\frac{0.353}{W^{0.1754}}$
Limit C−0.56=0.025
Limit R=582 (for 45° Notch)
10 000
100 000
1 000 000
$W=\frac{\rho g H^2}{\sigma}$
W

FIG. 23.—THREE-DIMENSIONAL PLOTTING OF EQ. 3 FOR 45° V-NOTCH WEIR

the angles 90°, 60°, 45°, 37°, 28°, 27°, 20°, 13°, and 10° which are compared with test data from the University of Wisconsin, Cornell University, University of California, and tests by Messrs. Yarnall, Cone, and Barr.

The writer appreciates the kind comments of Professors Cox and Germano. At their suggestion he submits the three-dimensional diagram, Fig. 23. The four lines represent Eq. 3 on the plane for the 45° V-notch for the four series of Wisconsin data. The line for oil G is not parallel to the other three because the temperature varies for the experimental runs from about 59° F at $H = 0.20$ ft to 70° F at $H = 0.80$ ft. The line remains on the plane as long as the angle of notch remains constant for all four series. The traces of the line for the water tests (Series 45D) are shown on the ($C-0.56$, **R**)-plane and the ($C-0.56$, **W**)-plane. These are similar to the traces shown in Figs. 8(*a*) and 8(*b*) except that they are for **R** = 582 and **W** = 300 instead of **R** = 1 and **W** = 1 and are drawn for Eqs. 3 and 14 instead of for the best line that can be drawn through the data for one liquid and one angle as in Figs. 8(*a*) and 8(*b*).

---

[29] "An Equation Relating Viscosity and Surface Tension," by Daniel Silverman and W. E. Roseveare, *Journal*, Am. Chemical Soc., Vol. 54, 1932, p. 4460.

It was refreshing to have Mr. Willes analyze, carefully, and, from a different point of view, the author's broad conclusions as to the effect of viscosity and surface tension. It is true that when a liquid of great viscosity is used, $\mathbf{R}$ is less than $\mathbf{R}_{cr}$, even at fairly high head, and hence the coefficient C will decrease with greater viscosity at a given head.

The application by Mr. Willes of the π-theorem to the derivation of the terms in Eq. 5 is fundamentally the same as that used by the writer and reported in his thesis[4] which was the source of this paper. The deduction that $\frac{B}{\mathbf{R}^n\,\mathbf{W}^m}$ is but the first term of a series is of interest and may be of value to the one who next carries on in this field.

Professor Powell brings up a point which caused the writer much thought, namely—just how to express Reynolds' number and Weber's number for a weir. Since the ideal velocity due to the head H is rarely computed, there seemed to be no reason, from a practical point of view, for including the term $\sqrt{2}$ in Reynolds' number. Elimination of the square root determination in the computation of Weber's number seemed also to be an advantage. If subsequent use shows that different dimensionless expressions would have been preferable, the constants expressed in Eqs. 14 will have to be revised accordingly.

It is regretted that numerical data were not available to Professor Powell when he desired to work with the writer's data. Complete sets of all tables are now available as follows: Two sets in the Engineering Societies Library in New York,[3] two in the University of Wisconsin Library, and one (with all original data) at the University of Wisconsin Hydraulic Laboratory. It seemed undesirable to burden the paper with a mass of data of interest to so few. The thesis contains all pertinent data in graphical form.

It is of interest to compare Eq. 43 proposed by Professor Powell with Eq. 40 proposed by Mr. Willes. If $\frac{B}{\mathbf{R}^n\,\mathbf{W}^m}$ is but the first term of series, then Eq. 43 is not of the correct type. On the other hand, it is very possible that Eq. 43 may fit the data better than Eq. 3. However, evaluation of constants would be exceedingly difficult. The numerical value of $C-a$ would have to be divided between the terms $\frac{B_1}{\mathbf{R}^n}$ and $\frac{B_2}{\mathbf{W}^m}$. How would one determine what proportion should be considered a viscosity effect and what proportion a surface tension effect? Would these proportions be the same for all angles of notch? Although it might be simpler in use to separate viscosity and surface tension corrections, the writer is inclined to believe that nature considers them together. Considering the "general" equation, if that term may be used to include a considerable range of angle as well as viscosity and surface tension, it is much simpler to determine constants for Eq. 3 than for Eq. 43 because Eq. 3 has one less constant to determine.

The range of surface tension was less than might have been desired because the experimental work was nearly completed before it was realized that surface tension had any effect, and the apparatus was dismantled to make room for another project before the analysis of the data was finished. It would have been interesting to try saponin, peptone, and oleic acid solutions which have considerable effect on surface tension. The brass weir plates were saved. Perhaps some day such tests can be run on these identical plates.

Professor Powell's analysis of conditions for low values of Weber's number is of interest. The tabular data contain some runs with heads as low as 0.03 ft. These tests are far beyond the limits of Eq. 3 and hence of little interest here, but it is hoped they will be helpful at some time.

It is agreed that plate measurements—thickness, sharpness, etc.—must be standardized more definitely before the V-notch weir becomes an instrument of absolute measurement; but the curve for the 90° notch in Fig. 15 would indicate that it can be relied upon for much engineering work with all the accuracy normally desired.

Professor Powell expresses surprise that the limit of Eq. 3 is reached at precisely the same value of C. If Eq. 43 were the correct form, then viscosity and surface tension "corrections" might not become constant at the same values of head; but, since Eq. 3 plots as a line on a plane, the limiting value of that line must of necessity be at the same value of H for both **R** and **W**.

The writer agrees with Mr. Hickox that extension of the empirical equation, beyond the range of tests, should be made with caution, but wishes to point out that, by increasing the viscosity 150 times, the weir coefficient for the same head was increased 3% to 7% for the 90° notch and 5% to 11% for the 20° notch. A reasonable extension, within the limits of Eq. 3, might be in order if too great accuracy was not desired. When this type of weir is used for other liquids, undoubtedly other experimenters will extend the range of tests.

Mr. Hickox' question on the relation of model to prototype is entirely pertinent because, from the geometry of the V-notch, every head on a notch of given angle is a geometrically similar model of some larger head. Coefficients in the formula $Q = C' L H^{1.5}$ may be obtained directly from Eq. 19 for water tests. Then, starting with a 0.155-ft head on a 90° V-notch "model" weir, it is evident that this is a 1:2 scale model of a similar "prototype" weir with $H = 0.310$ ft. Coefficients would be 2.599 and 2.530, a decrease of 2.65% for the prototype. However, if one started with a "model" head of 0.310 ft and

TABLE 6.—EFFECT OF SCALE ON MODEL V-NOTCH WEIRS

Description	SCALE FACTORS, MODEL: PROTOTYPE							
	1 : 1		1 : 2		1 : 10		1 : 50	
Model heads, in feet	0.155	0.310	0.155	0.310	0.155	0.310	0.155	0.310
Head on prototype, H_p	0.155	0.310	0.310	0.620	1.55	3.10	7.75	15.50
Coefficient for prototype, C_p	2.599	2.530	2.530	2.485	2.485	2.485	2.485	2.485
Correction[a] for prototype (%)	0	0	−2.65	−1.78	−4.39	−1.78	−4.39	−1.78

[a] Assuming Eqs. 3 and 14 to be correct with limit c (see heading: "Analysis of Data: Limiting Conditions for Eq. 3").

took a "prototype" head of 0.620 ft, the decrease in coefficient would be only 1.78% (see Table 6). Since the coefficient becomes constant for larger heads $\left(\frac{N}{H^{\epsilon}}\right.$ should never be less than 0.090$\left.\right)$, there would be no decrease in coefficient for larger prototypes.

Examination of Table 6 would lead one to conclude that for a V-notch weir: (1) The prototype coefficient is less than the model coefficient unless the model head is 0.62 ft or more (for the 90° weir); (2) the reduction in coefficient from model to prototype depends on the head on the model; and (3) if the prototype head is 0.62 ft or more, the decrease in coefficient is independent of the scale of the model, depending only on the head on the model (there being no reduction in coefficient if both model and prototype have heads greater than 0.62 ft). It is hoped that these conclusions will not be misinterpreted. They are presented in answer to a question. External friction on an ogee weir very likely modifies them to such an extent that their use for an ogee weir would be indeed an unwarranted extension of the experimental data.

In conclusion the writer hopes that he has explained points not previously made clear.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2196

MODEL TESTS ON STRUCTURES FOR HYDROELECTRIC DEVELOPMENTS

BY L. M. DAVIS,[1] M. AM. SOC. C. E.

WITH DISCUSSION BY MESSRS. EMIL P. SCHULEEN, C. M. ALLEN, ADOLF A. MEYER, AND L. M. DAVIS.

SYNOPSIS

Two series of model tests on hydraulic structures are described in this paper. These are the Holtwood and Safe Harbor hydroelectric developments on the Susquehanna River in southern Pennsylvania near the Maryland border. Data are presented on tests made to determine: (*a*) The discharge coefficients of various spillways; (*b*) the effect of depth of approach on the discharge coefficient; (*c*) the effect of submerged obstructions, such as cofferdam cribs, on discharge; (*d*) the water pressures acting on the spillway face for various conditions; (*e*) the effectiveness of various pier-nose designs; (*f*) the results of erosion in connection with apron design; and (*g*) the proper location of a deflecting wall below the Holtwood Dam.

INTRODUCTION

Hydraulics has been the subject of experimentation for many centuries. During the past century human knowledge concerning the science of hydraulics has increased at a rapid rate, largely due to work done in laboratories throughout the world. Nevertheless, many problems confront the hydroelectric designer, for which a satisfactory and economical solution can be reached only by means of laboratory experiments. Few hydroelectric developments have been built in recent years that were not preceded by a comprehensive testing program.

A number of problems arose in connection with the design of the hydraulic structures for the Safe Harbor development that required model tests in order to insure a satisfactory and economical structure. Consequently, a test program was conducted during the summer of 1930 that included tests on spillway sections, numerous designs of pier noses, and tests to determine a suitable apron to be installed below the dam to prevent the flood waters from causing erosion which would eventually endanger the dam.

NOTE.—Published in April, 1942, *Proceedings*.

[1] Hydr. Test Engr., Pennsylvania Water & Power Co., Holtwood, Pa.

Necessity for model tests on the Holtwood development arose in 1936. The spring flood of that year on the Susquehanna River reached a peak of 860,000 cu ft per sec, which was 20% greater than the value used for the design of the Holtwood development. Following this flood it was decided to make alterations to the hydraulic structures at Holtwood so that they would withstand safely a flood of 1,200,000 cu ft per sec, the value which was used in the design of the Safe Harbor development. It was recognized that under this flow certain parts of the Holtwood Dam would be exposed to negative pressures. To determine the extent and magnitude of these negative pressures, it was decided to conduct model tests. It was also desired to obtain, by means of model tests, a satisfactory and economical design of deflection wall to separate the main river channel from the tailrace and to replace the wall which failed during the flood. An apron had never been provided below the Holtwood Dam. On three occasions since the dam was completed in 1910, paving has been necessary in several areas along the toe of the dam to repair the effects of erosion. It was decided to design a suitable apron that could be installed if repairs should become necessary in the future.

Description of Models

All spillway calibrations made in 1930 for the Safe Harbor series of tests were done in a 3-ft by 35-ft glass-sided wooden flume (see Fig. 1) in the basement of the main laboratory building. The model installed in this flume was a 1 : 40 scale ratio that provided one full spillway opening and two half spillway openings.

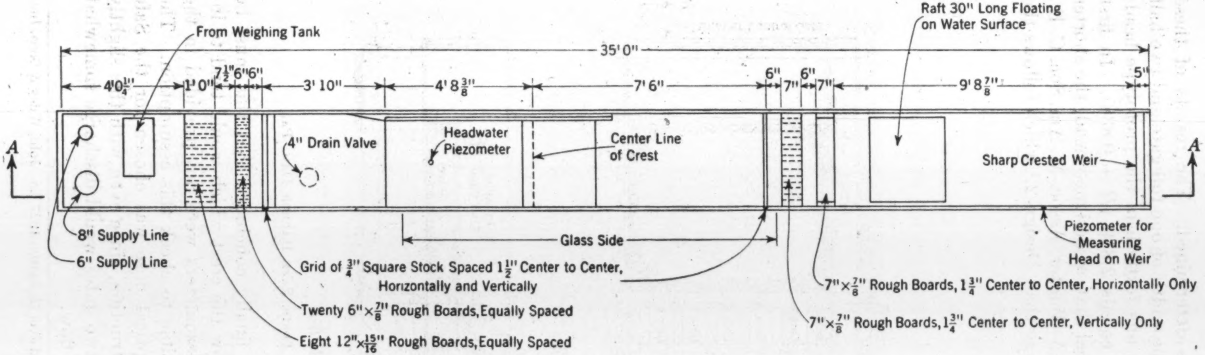
For the erosion tests a 1 : 40 ratio model was installed in the main creek below the laboratory as shown in Figs. 2 and 3. This model was provided with five full spillway openings and two half openings so that the effect of various combinations of "gates open" could be determined. The erosion bed was composed of $1\frac{1}{4}$-in. to 2-in. gravel.

For the 1936 series of tests for the Holtwood development, it was decided to take advantage of a 5-ft concrete flume, located parallel to the creek just below the main laboratory, for the 1 : 30 scale model that was used for calibration, measurement of pressures on the surface of the dam, and for erosion tests. The setup is shown in Fig. 4. There was sufficient room between this 5-ft flume and the creek for the 1 : 50 scale model that was used for the deflection-wall tests. This was a true model of 300 ft of the east end of the dam, east abutment, and deflection wall below the dam. Water for both of these model setups was obtained through a 30-in. pipe from the main laboratory building. This pipe started at a point next to the 50,000-lb weighing tank so that all flows could be checked accurately.

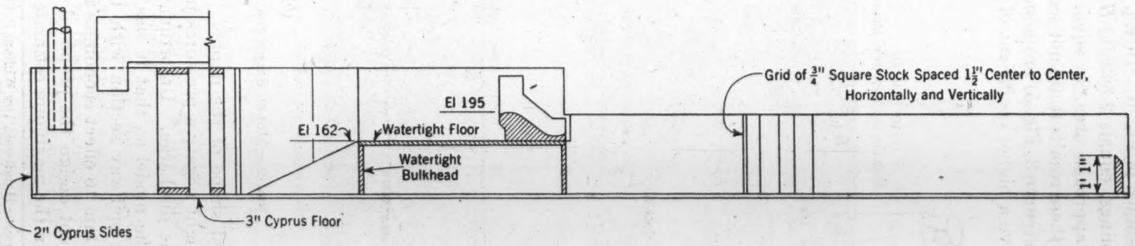
Coefficients of Discharge for Various Spillway Sections Tested

In connection with the tests for the Safe Harbor Dam the coefficients of discharge

$$C = \frac{Q}{L\,H^{1.5}} \qquad (1)$$


From Weighing Tank
35' 0"
4' 0¼"
1' 0"
7½"
6"
6"
3' 10"
4' 8⅜"
7' 6"
6"
7"
6"
7"
9' 8⅞"
5"
Raft 30" Long Floating on Water Surface
A
A
4" Drain Valve
Headwater Piezometer
Center Line of Crest
Sharp Crested Weir
8" Supply Line
6" Supply Line
Glass Side
Piezometer for Measuring Head on Weir
Grid of ¾" Square Stock Spaced 1½" Center to Center, Horizontally and Vertically
Twenty 6"×⅞" Rough Boards, Equally Spaced
Eight 12"×15/16" Rough Boards, Equally Spaced
7"×⅞" Rough Boards, 1¾" Center to Center, Horizontally Only
7"×⅞" Rough Boards, 1¾" Center to Center, Vertically Only

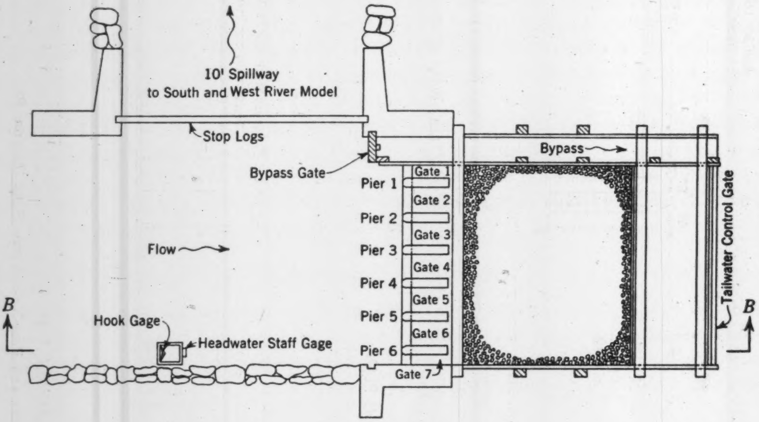
(a) PLAN


El 195
El 162
Watertight Floor
Watertight Bulkhead
Grid of ¾" Square Stock Spaced 1½" Center to Center, Horizontally and Vertically
1' 1"
2" Cyprus Sides
3" Cyprus Floor

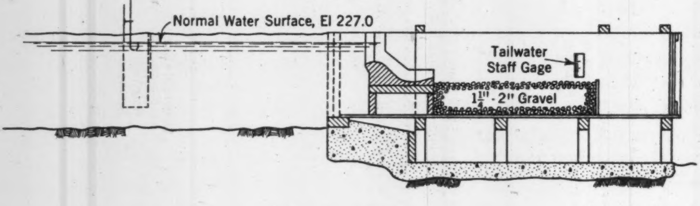
(b) SECTION *A-A*

FIG. 1.—MODEL SETUP FOR CALIBRATION AND PIER-NOSE TESTS, SAFE HARBOR, PA.

of four proposed spillway sections were determined. The results of these tests are shown in Fig. 5. In Eq. 1, L, the width of one full gate and two half gate openings, equals 2.392 ft; H equals one fortieth of the prototype head; and the approach area, in square feet, equals 2.819 ($H + 0.838$). In first studying these results it did not seem logical that, at designed head, the shorter dam nose (curve 3, Fig. 5) proposed by I. A. Winter, Assoc. M. Am. Soc. C. E., should give a higher coefficient of discharge than Design 2,[2] which follows the theoretical shape of the nappe. It was finally concluded that this must be due to slight negative pressures just below the crest which would tend to increase the discharge. Unfortunately, piezometers were not provided in the face of the model so that it was impossible to check this assumption. The parabolic spillway section with the elliptical nose was chosen for the Safe Harbor Dam to effect economies in construction, since even with this slightly less efficient design the proposed thirty-two gates would discharge somewhat more than the maximum estimated river flow.


10' Spillway
to South and West River Model
Stop Logs
Bypass Gate
Bypass
Pier 1
Pier 2
Pier 3
Pier 4
Pier 5
Pier 6
Gate 1
Gate 2
Gate 3
Gate 4
Gate 5
Gate 6
Gate 7
Flow
B
B
Hook Gage
Headwater Staff Gage
Tailwater Control Gate

(a) PLAN


Normal Water Surface, El 227.0
Tailwater
Staff Gage
1½"-2" Gravel

(b) SECTION B-B

FIG. 2.—MODEL SETUP FOR EROSION TESTS, SAFE HARBOR DAM, PA.

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[2] "Hydraulic Handbook," by William P. Creager and Joel D. Justin, 1927 Ed., John Wiley & Sons, Inc., New York, N. Y., Table 32, p. 209.

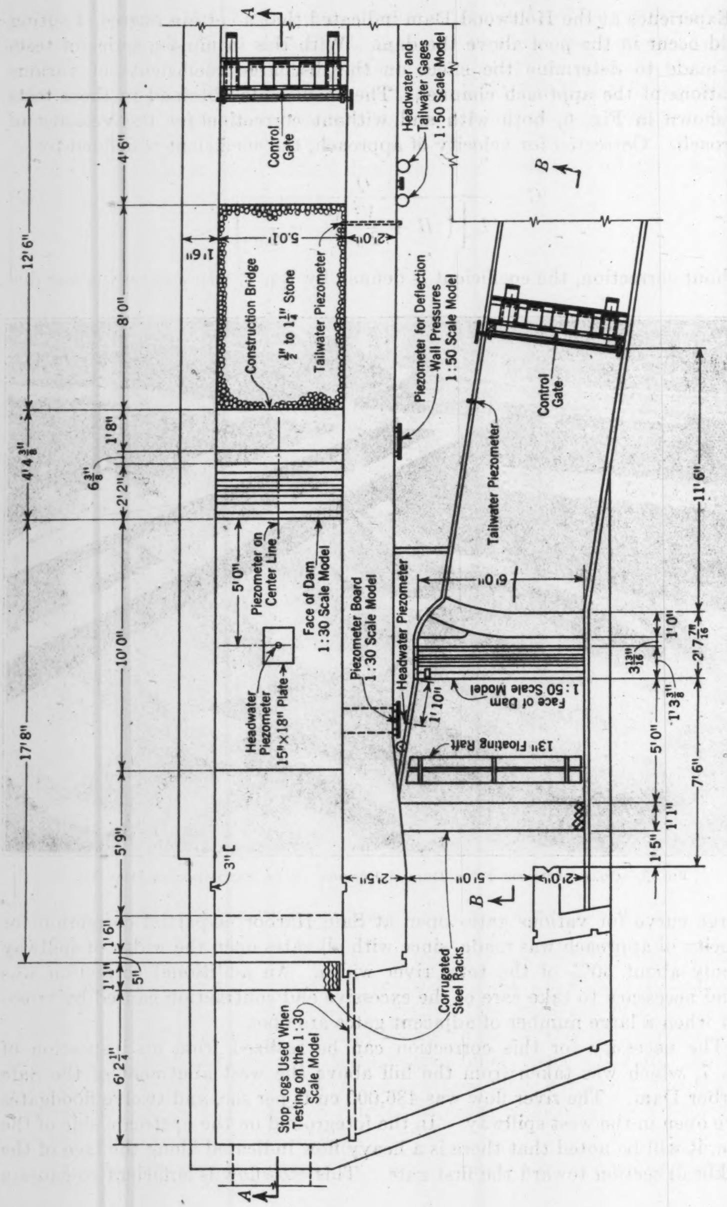
Experience at the Holtwood Dam indicated that a certain degree of silting would occur in the pool above the dam. With this in mind a series of tests was made to determine the effect on the discharge coefficients of various elevations of the approach channel. The coefficients obtained in these tests are shown in Fig. 6, both with and without correction for the velocity of approach. Corrected for velocity of approach, the coefficient is defined by

$$C = \frac{Q}{L\left[\left(H + \frac{V^2}{2g}\right)^{1.5} - \frac{V^2}{2g}\right]} \qquad (2)$$

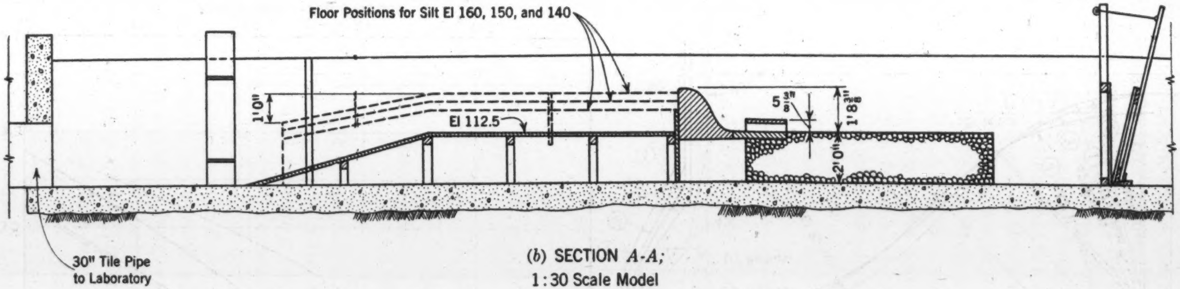
Without correction, the coefficient is defined by Eq. 1. In computing the discharge curve for various gates open at Safe Harbor, a partial correction for velocity of approach was made, since with all gates open the width of spillway is only about 50% of the total river width. An additional correction was found necessary to take care of the excessive end contraction caused by crossflow when a large number of adjacent gates are open.

FIG. 3.—CHARACTERISTIC FLOW DURING EROSION TESTS, SAFE HARBOR DAM, PA.

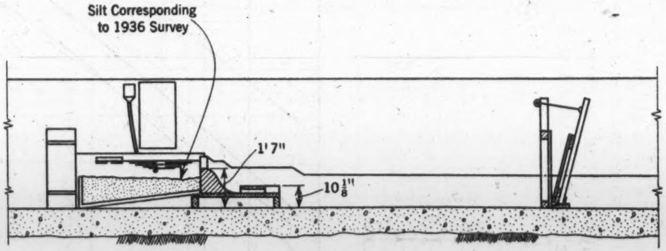
The necessity for this correction can be realized from an inspection of Fig. 7, which was taken from the hill above the west abutment of the Safe Harbor Dam. The river flow was 486,000 cu ft per sec, and twelve floodgates were open in the west spillway. In the foreground on the upstream side of the dam, it will be noted that there is a heavy flow indicated along the face of the bulkhead section toward the first gate. This crossflow is sufficient to cause a


A
A
B
B
Control Gate
Headwater and Tailwater Gages 1:50 Scale Model
12'6"
4'6"
8'0"
1'6"
5.0'
2'0"
Construction Bridge
½" to 1¼" Stone
Tailwater Piezometer
Piezometer for Deflection Wall Pressures 1:50 Scale Model
Control Gate
1'8"
4'4⅜"
6⅝"
2'2"
Tailwater Piezometer
11'6"
17'8"
10'0"
5'0"
Piezometer on Center Line
Face of Dam 1:30 Scale Model
Piezometer Board 1:30 Scale Model
Headwater Piezometer
Headwater Piezometer 15"×18" Plate
6'0"
1'0"
3 13/16"
2'7 7/16"
1'3⅜"
Face of Dam 1:50 Scale Model
1'10"
13" Floating Raft
5'0"
7'6"
1'1"
1'5"
5'9"
3" ⊏
2'5"
5'0"
2'0"
Corrugated Steel Racks
1'6"
1'1"
5"
6'2¼"
Stop Logs Used When Testing on the 1:30 Scale Model

(a) PLAN OF MODELS


Floor Positions for Silt El 160, 150, and 140
1'0"
El 112.5
5 3/8"
1'8 3/8"
2'0"
30" Tile Pipe to Laboratory

(*b*) SECTION *A-A*;
1:30 Scale Model


Silt Corresponding to 1936 Survey
1'7"
10 1/8"

(*c*) SECTION *B-B*;
1:50 Scale Model

FIG. 4.—LAYOUT OF MODELS FOR SPILLWAY AND DEFLECTION-WALL TESTS, HOLTWOOD DAM, PA.

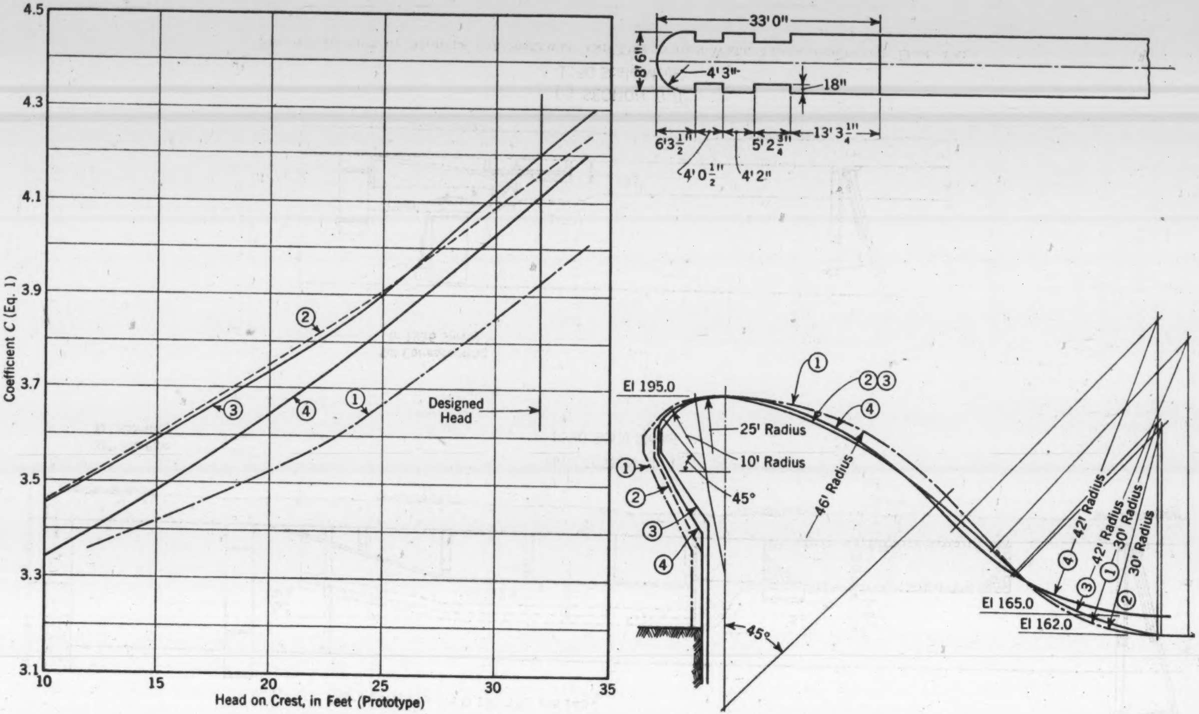


FIG. 5.—COMPARISON OF DISCHARGE COEFFICIENTS FOR VARIOUS SPILLWAY SECTIONS

considerable drawdown on the near side of the first gate and some piling up of the water on the far side. The water is deflected from the pier on the far side of this opening, so that by the time it reaches the bottom of the bucket it has been thrown to the near side of the opening and forms a huge wave where it hits the pier extension. Judging from the wave pattern of the water leaving the apron, the crossflow has some slight effect even on the fourth gate from the ends. It is evident that this crossflow must reduce the discharge of the end-gates appreciably when a considerable number of adjacent gates are open.

At first glance it is difficult to understand why the coefficients of discharge with the velocity-of-approach correction improve with increase in velocity of approach in the case of the Safe Harbor Dam (Fig. 6). The only reasonable explanation seems to be that the recessed face must cause unfavorable flow conditions that increase the losses with increase in depth of approach. It is known that the discharge over a sharp-crested weir is reduced when the face of the weir is inclined upstream.

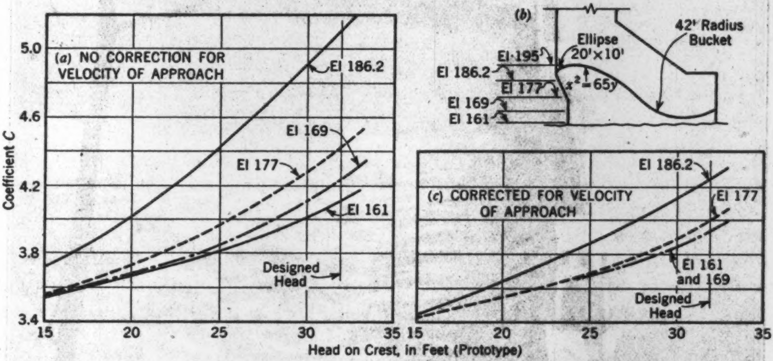


FIG. 6.—DISCHARGE COEFFICIENTS FOR VARIOUS SILT ELEVATIONS IN THE APPROACH CHANNEL, SAFE HARBOR DAM

The results of the calibration tests made on the Holtwood model dam, with various elevations of silt in the approach channel, are shown in Fig. 8. The coefficients of discharge are given with no correction (Eq. 1) and also with full correction for velocity of approach (Eq. 2). In computing the discharge curve for the Holtwood Dam, no correction for velocity of approach was applied, since there are no piers or floodgates on the dam and the approach channel is approximately the same width as the overflow section.

In arriving at the discharge curves for the Safe Harbor and Holtwood spillways, the coefficients of discharge as obtained from the model tests were used without any "step-up." (The term "step-up" is in common use in hydraulic turbine work. Experience with hydraulic turbines has shown that in a homologous series the percentage of losses varies inversely with the size. Since model tests are made on a small size turbine the term "step-up" is applied to the correction necessary to predict the characteristics of the prototype accurately.)

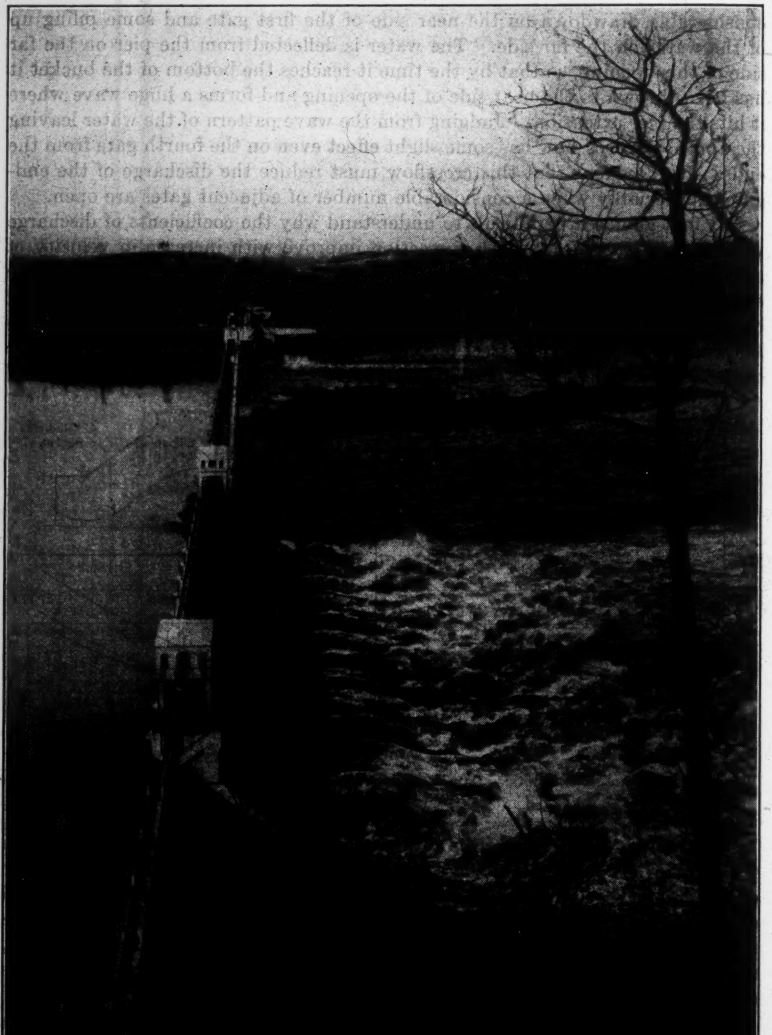


FIG. 7.—VIEW OF SAFE HARBOR DAM FROM THE WEST ABUTMENT

It seems probable that, if the same ideal flow conditions existed on the prototype that existed on the model, some step-up would be obtained. However, ideal flow conditions do not usually exist at flood stages so that it would seem unwise to claim any step-up in preparing prototype discharge curves.

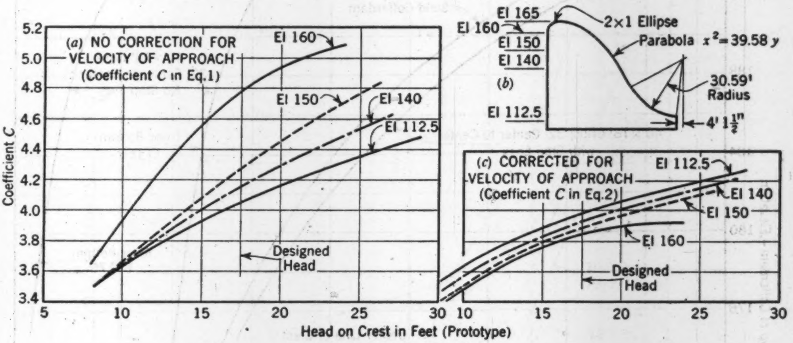


FIG. 8.—DISCHARGE COEFFICIENTS FOR VARIOUS SILT ELEVATIONS IN THE APPROACH CHANNEL, HOLTWOOD DAM

During the 1936 flood there was an excellent opportunity to obtain a comparison between the Susquehanna River discharge as obtained at the Safe Harbor, the Holtwood, and the Conowingo, Md. (14 miles downstream from Holtwood), hydro plants. The flood peak measured at Holtwood was about 1.5% less than the Conowingo measurement and 3.0% less than the Safe Harbor measurement. The Safe Harbor measurement was based on curves that did not take into consideration the excessive end contraction shown in Fig. 7, so it is not surprising that Safe Harbor indicated more water than the other two plants. It is felt that this comparison shows remarkable agreement, considering that the flow was 860,000 cu ft per sec.

EFFECT OF COFFERDAM CRIBS ON SPILLWAY DISCHARGE

Following the completion of the Safe Harbor Dam, it was found that it would be a rather expensive operation to remove the cribs of the upstream cofferdam entirely. To determine the effect that these cribs might have on spillway discharge, a series of tests was made, the results of which are shown in Fig. 9. With the river bottom at El. 165, tests were made with the cribs and stop logs in place, with the cribs in place and the stop logs removed, and with a solid cofferdam, making tests for each case at various stages of removal. Tests were repeated with a river bottom at El. 175 for the cribs without stop logs. It will be noted that the loss in discharge with the cribs and stop logs in place to El. 193 is 18%, whereas the loss with the cribs in place to El. 193 with stop logs removed is only 4.1%. Since it was relatively easy, the stop logs were completely removed from the cofferdam and the cribs were removed to El. 184. This did not involve any underwater work, and the loss in discharge for the cribs in place to this elevation is only 1.7%. Considerable silt has been deposited in the pond but apparently the cribs have set up a disturbance that has kept the river bed clear between the cribs and the dam.

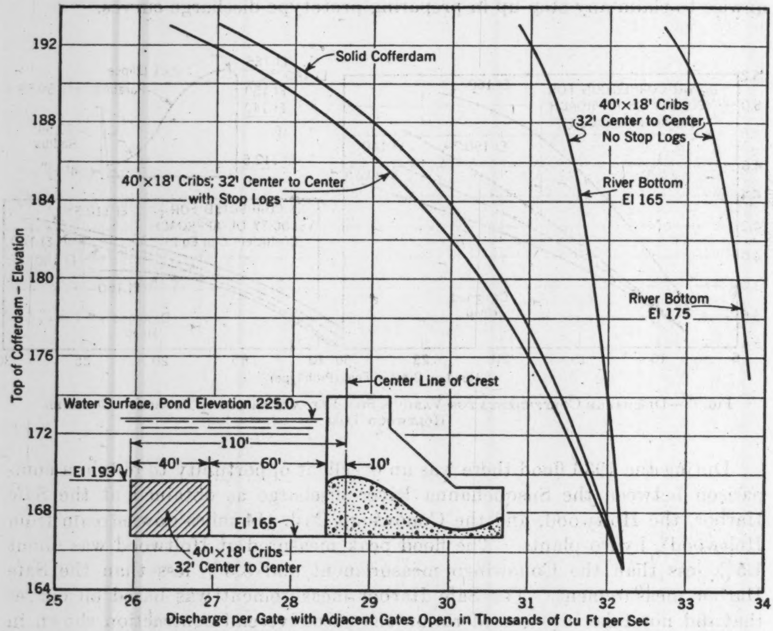


FIG. 9.—EFFECT ON DISCHARGE BY COFFERDAM CRIBS LEFT IN PLACE TO VARIOUS ELEVATIONS, SAFE HARBOR DAM

PRESSURE MEASUREMENT ON THE FACE OF THE DAM

Before accepting the parabolic section for the face of the Safe Harbor Dam, it was deemed advisable to determine the water pressures acting on the face of the dam under designed head condition. From an inspection of Fig. 5 it will be noted that the slope of the parabolic face as it approaches the bucket is somewhat greater than the slope of the theoretical lower nappe. There was some concern that this might result in negative pressures; but the results of the measurement of water pressures on the model (see Figs. 10 and 11) show that this fear was unfounded. It will be noted that the back pressure, resulting from the bucket deflecting the jet of water, extends some distance upstream from the beginning of the bucket. No doubt this back pressure, combined with the friction loss between the water and the face of the dam, is responsible for the rather high positive pressures at a point where negative pressures might be expected.

One spillway section of the Safe Harbor Dam was provided with five piezometers along the center line. It will be noted that the pressures obtained from the prototype agree reasonably well with the pressures obtained on the model at and near the crest of the dam, but diverge widely as the bucket is approached. Of course, part of this divergence is due to the fact that the elevation at the bottom of the bucket of the model was 165, whereas the elevation of the bottom of the bucket of the prototype was 168. The elevation of the water below the dam for the prototype tests was about 5 ft higher than for the model tests, which also might account for some of this discrepancy.

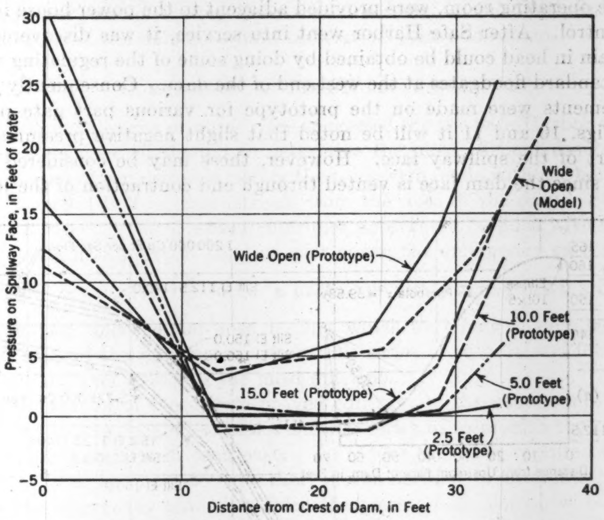


FIG. 10.—PRESSURE ON THE FACE OF THE SPILLWAY, SAFE HARBOR DAM, FOR VARIOUS GATE POSITIONS

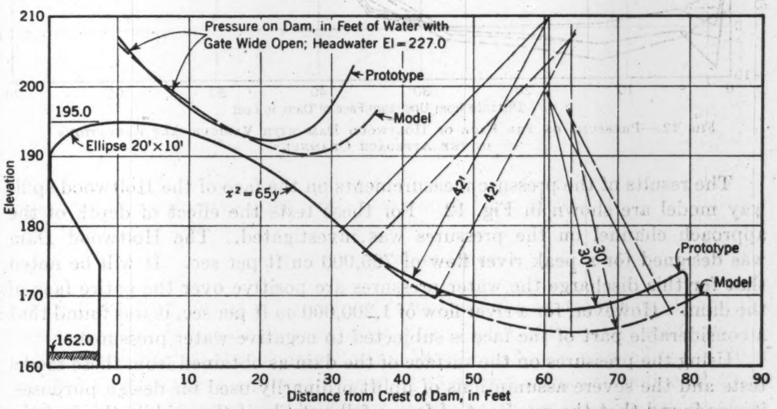


FIG. 11.—COMPARISON OF WATER PRESSURES ON THE SPILLWAY FACE OF MODEL AND PROTOTYPE, SAFE HARBOR DAM

When the model tests were conducted for Safe Harbor, it was felt that none of the standard floodgates would be operated at part gate opening, since four double-leaf regulating gates, which can be operated by remote control from the operating room, were provided adjacent to the power house for pond-level control. After Safe Harbor went into service, it was discovered that a slight gain in head could be obtained by doing some of the regulating with one of the standard floodgates at the west end of the dam. Consequently pressure measurements were made on the prototype for various part gate positions. From Figs. 10 and 11 it will be noted that slight negative pressures existed over part of the spillway face. However, these may be considered of little concern since the dam face is vented through end contraction of the jet.

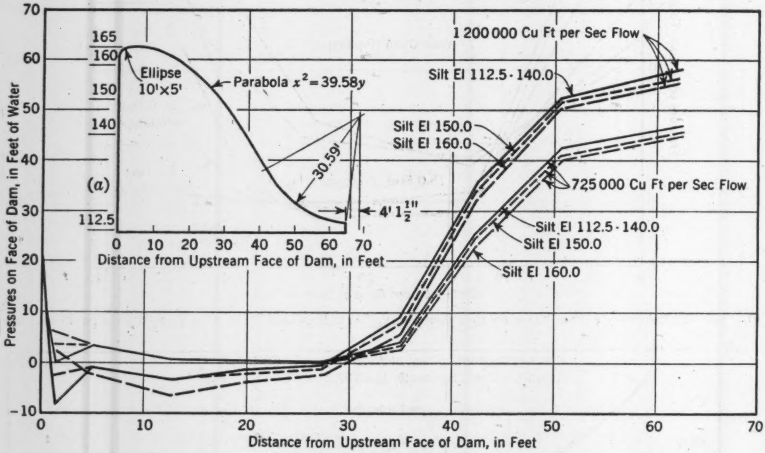


FIG. 12.—PRESSURE ON THE FACE OF HOLTWOOD DAM WITH VARIOUS SILT ELEVATIONS IN THE APPROACH CHANNEL

The results of the pressure measurements on the face of the Holtwood spillway model are shown in Fig. 12. For these tests the effect of depth of the approach channel on the pressures was investigated. The Holtwood Dam was designed for a peak river flow of 725,000 cu ft per sec. It will be noted that for this discharge the water pressures are positive over the entire face of the dam. However, for a river flow of 1,200,000 cu ft per sec, it was found that a considerable part of the face is subjected to negative water pressures.

Using the pressures on the surface of the dam as obtained from these model tests and the severe assumptions of uplift ordinarily used for design purposes, it was found that the resultant of forces fell outside of the middle third of the base, which would produce tension at the heel. Field measurements of uplift pressures actually existing on the dam were made.[3] Using these data in place of the usual uplift assumption, it was determined that the structure as a whole was stable. The dam was poured in 5-ft lifts and 40-ft sections. Large stones were used as keys between the pours. In the event of extreme high flow, which would result in negative pressures of considerable magnitude acting on the top of the dam, weakness in any of the sections at this top-pour joint might cause the loss of some of the top slabs. Such a remote coincident would cause little damage, and the damage would be relatively easy to repair.

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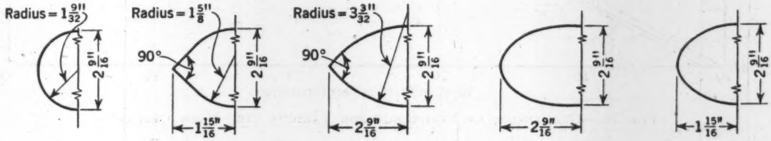
[3] "Uplift Measurements at Holtwood Dam," by Paul E. Gisiger, *Civil Engineering*, July, 1938, p. 447.

COMPARISON OF PIER-NOSE DESIGN

Five different designs of noses were suggested for the Safe Harbor spillway piers, and tests were conducted to determine the most efficient design. The piers had already been designed before the tests were made with the emergency gate slot placed 4 ft 3 in. downstream from the nose of the dam. Since the piers were 8 ft 6 in. wide, a semicircular pier nose would be flush with the dam nose. For the first series of tests on pier noses, the emergency gate slot was fixed a model distance of $1\frac{9}{32}$ in., or a prototype distance of 4 ft 3 in., from the nose of the dam, and the pier-nose designs, which were longer, were permitted to extend beyond the dam nose. From this series of tests it was found that the semicircular pier nose was the most efficient.

Since the results of these tests proved contrary to the usual conception of streamlining, it was decided to make another series of tests with the pier nose flush with the dam nose in each case. To reduce the labor required to make the changes from one nose to the other, it was decided to eliminate both the gate and the emergency gate slots from the piers. The pier noses tested are shown in Fig. 13. As in Fig. 5, the coefficient of discharge was determined by Eq. 1, L equals 2.392 ft; H equals one fortieth of the prototype head; and the approach area, in square feet, equals 2.819 (H + 0.838). The spillway section was the final design of Safe Harbor Dam—that is, a nose ellipse of 20 ft by 10 ft; a face curve expressed by $x^2 = 65\,y$; and a bucket radius of 42 ft. The results of this second series of tests are shown in Fig. 14. It will be noted that for this series of tests the semicircular design was the least efficient of the five designs tested. In the case of Design 2, Fig. 13, the pier nose was 2.3% less efficient at designed head when it extended $\frac{21}{32}$ in. beyond the dam nose than when it was flush with the dam nose. Design 4, Fig. 13, was 2.6% less



① SEMI-CIRCULAR ($1\frac{9}{32}$" Radius)

② 90° SHARP NOSE ($1\frac{5}{8}$" Radius)

③ 90° SHARP NOSE ($3\frac{3}{32}$" Radius)

④ ELLIPSE (2×1)

⑤ ELLIPSE ($2\frac{9}{16}$"×$3\frac{7}{8}$")

FIG. 13.—PIER NOSES TESTED

efficient when it extended $1\frac{9}{32}$ in. beyond the dam nose than when it was flush with the dam nose.

Since it was undesirable to move the gate slots farther downstream, and since there was ample discharge capacity provided without taking advantage of the 2.4% increase that would have been obtained if Design 3 had been used, it was decided to adopt the semicircular pier nose for Safe Harbor.

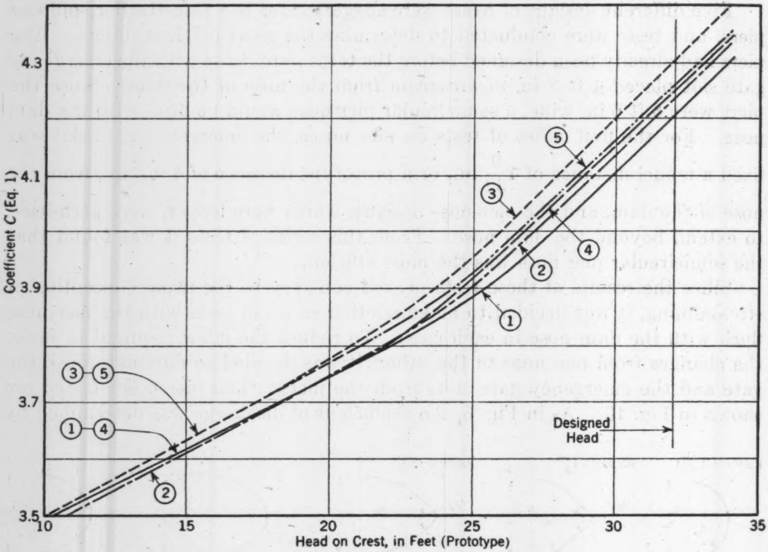


FIG. 14.—COMPARISON OF EFFICIENCY FOR VARIOUS PIER NOSE DESIGNS

The effect, on the various designs of pier noses, of flow approaching the piers from an angle was not investigated. It seems probable that the more streamlined pier noses would lose their advantage over the semicircular pier nose at relatively small angles of flow from the normal.

EROSION TESTS

When the erosion tests were made for the Safe Harbor development, there was no previous experience on a dam comparable to Safe Harbor that could be used as a starting point for this investigation. The normal head acting on the crest of Safe Harbor Dam is 32 ft, and the total drop from normal pond level to the bottom of the bucket is only 62 ft. With only one floodgate discharging, there is very little depth of water below the dam to act as a cushion pool.

Forty-four different designs of apron were tested, including stepped aprons, dentated sills, and aprons of various lengths and degrees of angular upturn.

Early in the program of tests it was found that the floodgate piers had to be extended at their full width to the downstream end of the apron to prevent erosion immediately below the structure where jets from adjacent gates met and seemed to be deflected downward with tremendous energy.

Of all of the types of aprons tested, only three gave adequate protection. One consisted of a 20-ft apron with a 42-ft radius starting from the bottom of the bucket, with a tangent extension of 12° from the horizontal to a step on the downstream end, which was 1 ft 8 in. high and 2 ft 6 in. wide. Erosion for this design started at a point 16 ft from the end of the apron and had a maximum depth of 32 ft at a distance of 128 ft. This design of apron was considered impractical since the Susquehanna River is full of debris at times of high flow, and it was felt that the step at the end of the apron would be damaged easily and would require frequent repairs. A 32-ft apron with a 42-ft radius starting from the bottom of the bucket, with a tangent extension at 16° from the horizontal and without a step, gave protection almost identical with the 20-ft apron with a step. Further tests were made with shorter aprons and steeper angular upturns. It was found that an 18-ft apron, with a 30-ft radius starting from the bottom of the bucket, with a tangent extension of 26° from the horizontal, resulted in erosion starting 20 ft from the end of the apron with a maximum depth of 47 ft at a distance of 140 ft from the end of the apron (see Fig. 15). This apron was adopted for Safe Harbor.

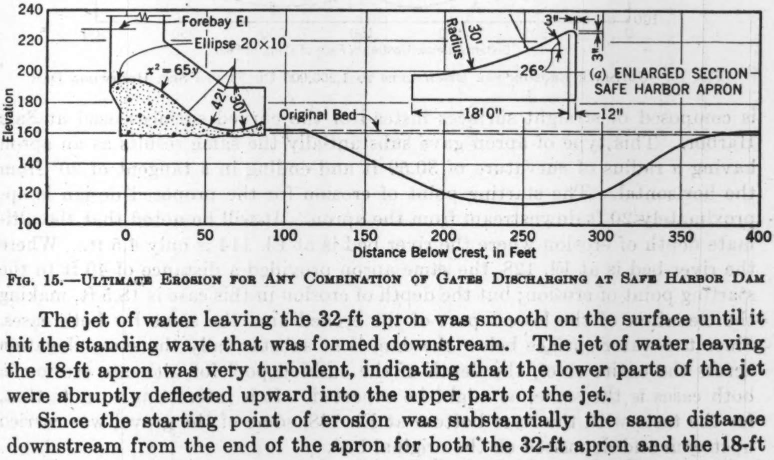


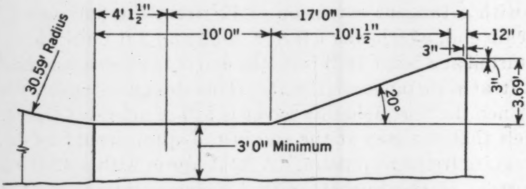
FIG. 15.—ULTIMATE EROSION FOR ANY COMBINATION OF GATES DISCHARGING AT SAFE HARBOR DAM

The jet of water leaving the 32-ft apron was smooth on the surface until it hit the standing wave that was formed downstream. The jet of water leaving the 18-ft apron was very turbulent, indicating that the lower parts of the jet were abruptly deflected upward into the upper part of the jet.

Since the starting point of erosion was substantially the same distance downstream from the end of the apron for both the 32-ft apron and the 18-ft apron, it is evident that the angle with which the jet leaves both aprons must be approximately the same. Undoubtedly this same angle could be obtained with a variety of combinations of length and upturn. However, tests on aprons even slightly shorter than 18 ft, with steeper angles of upturn, resulted in an unstable jet.

After the experience obtained by the erosion tests on the Safe Harbor model, it was a simple matter to arrive at an appropriate design of apron for the

Holtwood Dam. The erosion conditions at Holtwood are far less severe than at Safe Harbor, since there are no gates on this dam and, as the head increases on the dam, the tailwater level builds up, forming a cushion pool that absorbs most of the energy. The proposed design of apron for Holtwood that was developed from these tests is shown in Fig. 16. It will be noted that this apron is composed of straight surfaces instead of the curved surfaces used at Safe Harbor. This type of apron gave substantially the same results as an apron having a radius of curvature of 30.59 ft and ending in a tangent of 20° from the horizontal. The starting point of erosion for the proposed design is approximately 20 ft downstream from the apron. It will be noted that the ultimate depth of erosion where the river bed is at El. 114 is only 4.5 ft. Where the river bed is at El. 128, the same apron provided a distance of 40 ft to the starting point of erosion; but the depth of erosion in this case is 18.5 ft, making the elevation of the lowest part of the eroded area the same for both cases. Since the water surface below the dam is at substantially the same elevation across the entire river, the depth of the cushion pool for ultimate erosion in both cases is the same, as might be expected. It is interesting to note that, for the tests with the river bottom at El. 128, some of the gravel was carried upstream and deposited on the original bed.



(a) PROPOSED DESIGN OF APRON FOR HOLTWOOD

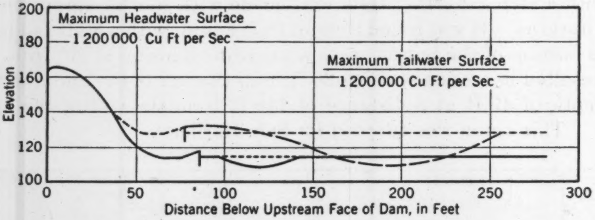


FIG. 16.—ULTIMATE EROSION FOR DISCHARGES TO 1,200,000 CU FT PER SEC, HOLTWOOD DAM

DEFLECTION-WALL TESTS

The Holtwood development was built at the upstream end of an island. This island with one immediately downstream divides the river into two channels for a distance of one mile. There is considerable drop in water-surface levels in this distance. The channel on the east side of these islands is now used as the power plant tailrace, and the main river channel is on the

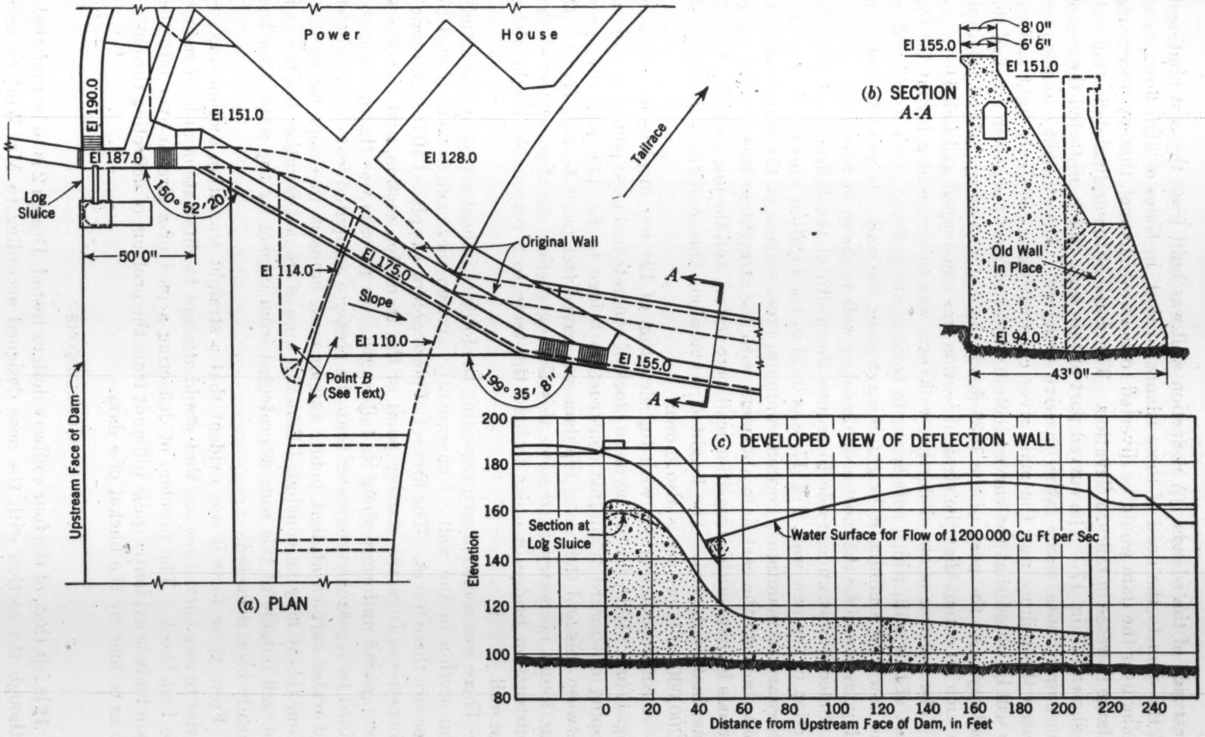

Power House
El 190.0
El 187.0
Log Sluice
El 151.0
El 128.0
Tailrace
150° 52′ 20″
50′ 0″
Original Wall
El 114.0
El 175.0
Slope
El 110.0
Point *B* (See Text)
199° 35′ 8″
El 155.0
A
A
Upstream Face of Dam
(*a*) PLAN
(*b*) SECTION *A-A*
8′ 0″
6′ 6″
El 155.0
El 151.0
Old Wall In Place
El 94.0
43′ 0″
(*c*) DEVELOPED VIEW OF DEFLECTION WALL
Section at Log Sluice
Water Surface for Flow of 1 200 000 Cu Ft per Sec
Elevation
200
180
160
140
120
100
80
0
20
40
60
80
100
120
140
160
180
200
220
240
Distance from Upstream Face of Dam, in Feet

FIG. 17.—TAILRACE DEFLECTION WALL, HOLTWOOD DAM

west side of the islands. A deflection wall was built from the east abutment of the dam to the head of these islands so that, in times of high flow, water going over the dam would be diverted to the west channel, thereby preventing a large increase in tailrace elevation. The shape of the original deflection wall is shown in Fig. 17. The curved part of this wall, which protected the end of the power house, was a rather massive structure. The straight extension between the tailrace and the main river channel was a relatively light section. It will be noted that the curved section of the wall cuts across in front of the spillway. At the peak of the 1936 flood, the water piled up behind this wall to such an extent that the straight section was overtopped and failed.

The purpose of the deflection-wall tests was to arrive at a layout for the part of this wall, which cuts across in front of the spillway, that would deflect the flow with a minimum height of wave along the wall. It was felt that such a layout would be the most economical as well as the most effective.

Since it was undesirable to reduce the width of the tailrace, the straight section of the new wall, Fig. 17, was fixed in the location shown. It was also necessary to maintain a certain minimum cross section of the transition wall that connects the east dam abutment with the straight section of wall. Numerous layouts of this transition wall were tested, and the one shown in Fig. 17 was finally adopted. The height of wave along this wall for a discharge of 1,200,000 cu ft per sec is also shown.

A construction bridge with top elevation of 128 was in place as shown in Fig. 17(*a*). With the old layout a block of concrete had been poured as shown, sloping downward from this construction bridge to El. 114, and the space between this and the toe of the dam was paved (see point *B*, Fig. 17(*a*)). It was found necessary to remove this sloping section and two arches of the construction bridge in order to hold the wave to a reasonable height along the wall.

There was some concern regarding the force of impact acting on the transition section of the wall. Consequently, sixteen piezometers were placed to measure these forces. The first set of piezometers was placed 10 ft (prototype) downstream from the starting point of the transition section, and the others were spaced uniformly along its full length. In no case was the pressure obtained by a piezometer greater than the pressure computed from the height of the water surface at that point, and in most cases it was appreciably less. From this it may be concluded that the zone of impact is rather limited and the wall outside of this zone is protected from impact by the water that has already been deflected.

From these tests it was evident that a straight transition section was superior to any curved section that would attempt to eliminate or at least reduce the first shock. The problem of deflecting a part of a stream over into the main body is evidently quite different from the problem of deflecting the entire jet as is done by the bucket of a dam.

CONCLUSIONS

It is felt that, of the four spillway designs tested, Design 2 is to be preferred. Although the section with the nose designed according to Mr. Winter's test data is about 1% more efficient at designed head, it is probable that negative pressures exist somewhere on the spillway face that might cause damage to the concrete surface.

It was a bit disconcerting to have such a marked difference in the characteristics of the discharge coefficient curves with the velocity of approach correction for the two series of tests. The projecting nose of the Safe Harbor Dam seems to offer the only logical explanation.

The results of the tests with the upstream cofferdam cribs in place are valuable in that there are, no doubt, many instances where there would be a considerable saving of expense by leaving these cribs in place.

The measurements of pressure on the spillway face agreed with what had been anticipated. The check that was obtained on the Safe Harbor spillway with pressures observed on the model agreed within reasonable limits.

The results of the comparative tests on pier noses were most enlightening. It is evident that it is very important from an efficiency point of view to have the pier noses and the dam nose in the same plane.

It is felt that the apron designs that were developed are the most economical possible under the existing conditions in each case. It seems logical that the energy of the water should be utilized to make the starting point of erosion downstream as far as possible from the end of the apron, rather than to attempt to dissipate the energy on the apron.

The tests made to find a suitable design and location for the Holtwood deflection wall were not unusual in any way. Many such problems have been solved in a similar manner and, until the knowledge of hydraulics increases, future problems of a like nature may be solved satisfactorily by means of model tests. It was evident from these tests that a straight transition section was preferable to any other design and that the angle across the flow should be kept as small as possible. The tests showed that the zone of impact was confined to a distance of less than 10 ft from the starting point of the transition wall.

ACKNOWLEDGMENTS

Both series of tests reported in this paper were made at the Alden Hydraulic Laboratory of the Worcester Polytechnic Institute, at Worcester, Mass. The testing was under the supervision of C. M. Allen, M. Am. Soc. C. E. The writer wishes to express his appreciation to all members of the laboratory staff and in particular to Professors Allen, L. J. Hooper, and C. W. Hubbard. Credit is also due J. Marcus Mousson, 2d, M. Am. Soc. C. E., of the Safe Harbor Water Power Corporation for his assistance in this work.

DISCUSSION

Emil P. Schuleen,[4] **Assoc. M. Am. Soc. C. E.**—Holtwood Dam (as described by Mr. Davis) was originally designed for a maximum flood of 725,000 cu ft per sec. During the flood of 1936 this flow was exceeded by about 20% when a discharge of some 860,000 cu ft per sec occurred. It is interesting to note that the subsequent model tests showed the design of the spillway to be safe, with only slight negative pressures on the spillway face, even for the new design requirements for a flood of 1,200,000 cu ft per sec, an increase of about 66% over the original assumptions for maximum flow. It is fortunate that the original design was sufficiently liberal to provide this factor of safety.

The erosion tests conducted during the investigations deserve comment. The reproduction in a model of the resistance of the stream-bed material to scour cannot be made with absolute fidelity by any of the usual laboratory methods. Such tests give qualitative rather than quantitative results, indicating the points of maximum and minimum erosion, and to a limited extent the general location of the deposition of bed materials. The use of a movable bed in the model, even if it consists of gravel or stone sizes closely simulating the size of material found in the prototype, can be considered to indicate only the relative scouring potentialities for the various designs tested. Caution must be exercised, therefore, in the interpretation of the results of erosion tests, which should not be considered to indicate the exact amount of scour to be expected. In this connection, it is also advisable in most instances to obtain data of a negative nature, whereby the investigation would show the behavior of the structure on the assumption that the resistance of the bed material to erosion and movement is much greater than anticipated and that a negligible amount of scour would occur in the prototype. This can be accomplished by the use of a fixed bed in the model. Since the degree of erosion is more or less unpredictable, no possibility should be overlooked in investigations of this type.

The tests to determine the crest profile for the Safe Harbor spillway emphasize the delicate relation between the shape of the spillway at the crest and its effect on discharge efficiency. In general, the coefficient of spillway discharge is dependent on the shape of the spillway crest and the head. If the spillway crest is designed properly, the pressure distribution in the vicinity of the crest will be zero only at the design head. Under this condition the crest profile would be that of the lower surface of the nappe of an equivalent free-flowing jet, and a coefficient of discharge of approximately 4.0 would result. For heads lower than the design head, positive pressures would be produced and the coefficient decreased. Conversely, the reduction in pressure on or in the region of the spillway crest, due to heads in excess of the design head, would result in an increase in the discharge coefficient. It is interesting to note that slight revisions in the shape of the crest profiles tested (some seemingly insignificant) produced appreciable changes in the coefficient of discharge.

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[4] Senior Engr., U. S. Engr. Office, Pittsburgh, Pa.

The reason for the increase in the discharge coefficient for the Safe Harbor spillway, with decrease in the depth of approach due to silting, is not apparent. The opposite is true of the Holtwood spillway, which is in conformity with results obtained by others in model research and in analyses of actual structures. Inspection of both designs reveals that the Safe Harbor spillway differs from Holtwood only in two essential respects—the use of the overhanging lip upstream from the crest, and the presence of gate piers on the spillway section. This would suggest that the increase in the discharge coefficient with decrease in the approach depth may be due either to the effect of the pier noses and the gate recesses, or to a redistribution of flow at higher velocities at a point immediately upstream from the crest. The latter seems to be the more logical answer to the problem. The discharge coefficient, under normal conditions, is practically the same for a spillway crest with an overhanging lip as it is for a crest with a vertical upstream face. The effect of a reversal in flow lines produced by the overhanging lip apparently is compensated for by an increased curvature of the filaments adjacent to the spillway face. However, with increased velocity upstream due to a decrease in the approach depth, it is possible that this compensation does not occur, resulting in a different condition of flow than before and producing an alteration in the shape of the flow profile over the crest of the spillway. However, this is merely a conjectural solution to a problem which can be solved only by intensive research study.

The apron designs for both the Safe Harbor and Holtwood dams are unusual in that no direct dissipation of energy is intended on the apron proper. The purpose of each apron is to deflect the high velocity jet away from the structure and let it dissipate itself in impact as it strikes the tailwater and stream bed below the end of the apron. The model investigations were for the purpose of determining the relative scouring effects and of producing a design whereby the resulting scour would be limited to areas a safe distance from the toe of the dam.

The hydraulic performance of an apron is a function of the head, the velocity and thickness of the issuing jet, and the tailwater conditions available. Since these factors differ widely for each case under consideration, the design details must be such as to fit the physical and hydraulic characteristics of each particular case. The performance must be stable and of such character as to produce no serious damage to the structure or its foundation. These requirements apparently have been met in the case of the Safe Harbor and Holtwood spillways, since each has weathered floods of great magnitude without serious consequence. There is no question, therefore, with respect to the adequacy of the design in these two cases.

There is a tendency among engineers to avoid the expense and time required for model tests by applying the results of tests of other structures to their problem. The practice of making use of all available information on the subject should not be discouraged, but the engineer should be cautioned to make certain that the test results definitely are applicable in his case. In this respect, the general type of apron used for the Safe Harbor and Holtwood dams has certain operation characteristics that would be difficult, if not impossible, to predict except by hydraulic experiments on a model of the particular structure to which the design is to be applied. Studies made on aprons of this general type in the course of the model tests for the Tygart River Dam (in West Virginia) revealed that two distinctly different types of flow can exist, depending on the depth of tailwater available. When the tailwater is sufficiently low, the jet will be deflected upward by the end of the apron, and dissipation of energy will occur by impact of the jet with the tailwater and the stream bed, as indicated by Fig. 18. This type of action is accompanied by a ground roller, which may not be of sufficient magnitude to produce serious scour conditions near the end of the apron, and in fact may even deposit material in that region as in the case of the Holtwood Dam with the river bed at El. 128. If the tailwater is raised sufficiently, the jet, rather than being deflected upward, is submerged and deflected downward, producing

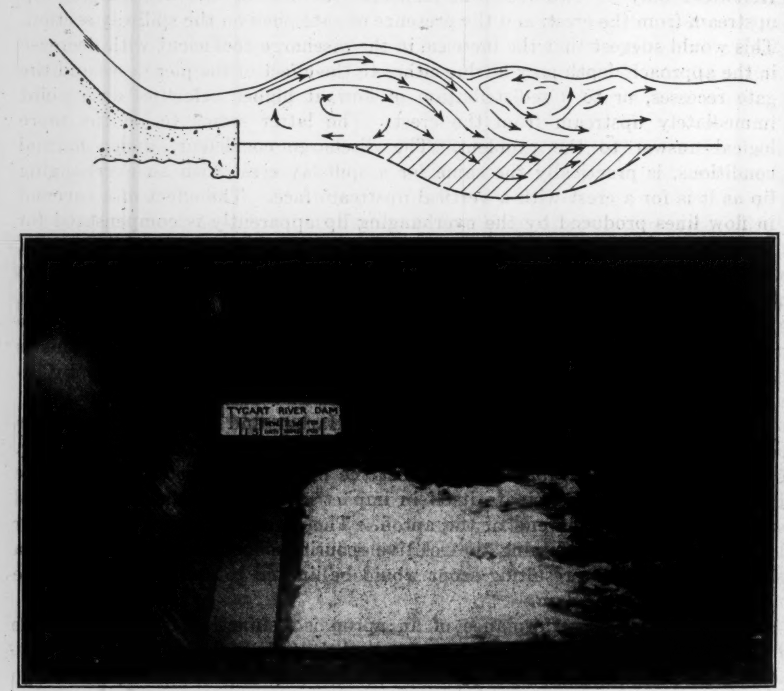

FIG. 18.—FLOW WITH GROUND ROLLER

instead a surface roller, high velocities along the stream bed, and back eddies and turbulence at the end of the apron, as indicated by Fig. 19. The resulting scour is heavy and more general in nature, and energy dissipation occurs through diffusion of the jet into the tailwater, and excessive turbulence. The latter type of flow is undesirable from the standpoint of erosion at the toe of the dam. At some point between these two tailwater elevations, the action will become unstable and the jet will alternately produce ground and surface

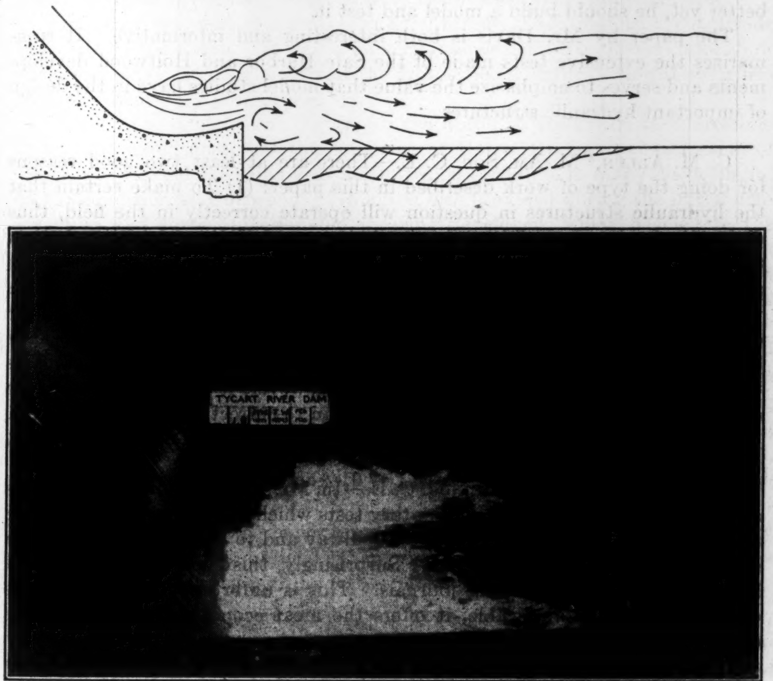

FIG. 19.—FLOW WITH SURFACE ROLLER

rollers, the performance being first of the one type and then of the other. The unstable action is particularly undesirable because, in addition to the scouring effect of the water, loose bed material is swept alternately upstream and downstream with each change in action, exercising an abrasive effect on such bed material as might otherwise be undisturbed.

Because of this uncertainty in operation characteristics, the adoption of the Safe Harbor or Holtwood type of spillway apron in the design of other dams without careful and exhaustive check by model experiments might well lead to undesirable results, especially if the apron adopted is relatively short,

with a comparatively flat angle to the horizontal. The need for intensive and careful study of this type of apron by hydraulic model experiments prior to its adoption as a final design for a proposed structure cannot be overemphasized.

The foregoing comments are in no way a reflection on the designs adopted for the Safe Harbor and Holtwood dams, and are presented only as a warning to engineers who might otherwise be tempted to adopt those designs for other structures without adequate analysis. The engineer should be cautioned to be absolutely certain that the test results are applicable to his problem; or, better yet, he should build a model and test it.

The paper by Mr. Davis is both interesting and informative. It summarizes the extensive tests made of the Safe Harbor and Holtwood developments and serves to emphasize the value that model studies have in the design of important hydraulic structures.

C. M. ALLEN,[5] M. AM. SOC. C. E.—There are at least two good reasons for doing the type of work described in this paper: (1) To make certain that the hydraulic structures in question will operate correctly in the field, thus giving the engineers confidence in the designs; and (2) to reduce the costs of these structures without interfering with the operation and efficiency. In one afternoon of laboratory work, in the writer's experience, the investigators were able to save $100,000 in concrete and excavation, and in addition they obtained better operating conditions. On the other hand, model tests have been made that called for considerable changes in design, and in some few cases the final cost was more than anticipated. However, it is always much more economical to make changes in the model than to correct mistakes in the field.

Mr. Davis is to be complimented on his performance of a fine piece of work.

ADOLF A. MEYER,[6] M. AM. SOC. C. E.—In Mr. Davis' interesting paper is a description of the hydraulic laboratory tests which led to the adoption of the type of apron used at the Safe Harbor spillway and to the decision to install a similar one at the Holtwood Dam. Surprisingly, this type has received very little attention in engineering journals. This is unfortunate because, where conditions make it applicable, it offers the most economical means for safe energy dissipation below spillways.

This form of apron creates two typical conditions: It dissipates the energy in the natural river bed below the apron, and it separates the zone of scouring from the structure by lifting the body of the shooting water at the end of the upturned apron bucket.

The change from shooting to streaming flow is obtained by hydraulic jump in the natural river bed. When free surface rollers only occur, the scour is kept to a minimum. However, no analytical method allows prediction of the type of rollers for all discharge and tailwater conditions. When ground rollers occur, the scour may be very large. Only hydraulic model tests can give positive assurance about the behavior of the discharging water in the scouring zone.

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[5] Prof. of Hydr. Eng., Worcester Polytechnic Inst., Worcester, Mass.

[6] Head Civ. Design Engr., TVA, Knoxville, Tenn.

There are a few specific conditions which prevent the use of this economical apron on certain dams. One is the character of the foundation rock. Obviously it would not be advisable to use this type where the foundation rock is bedded in horizontal layers with weaknesses in the bedding joints. Where expensive shore protection is necessary along the full length of the scouring zone, the over-all economy of this type may become questionable. As high wave action results from this type of energy dissipation, it will be objectionable where navigation channels come close to the downstream end of the dam.

During its ten years of operation at Safe Harbor, floods of unusual magnitude have passed the apron. It would be very interesting if Mr. Davis could show the present-day scouring picture at the prototype and indicate how it compares with the laboratory predictions.

The simple straight-line design for the apron at Holtwood is interesting. It may be noted here that the condition at Hales Bar Dam, construction of which was started the same year as Holtwood (1905), is in many ways similar to the condition at Holtwood. No apron was provided during the initial construction and some objectionable scouring developed, which has led to the decision to install an apron in the near future. However, due to the fact that a navigation channel is adjacent to the downstream end of the dam, an apron will have to be provided which dissipates most of the energy immediately below the dam in order not to disturb navigation by excessive wave action.

The TVA adopted the jump-type apron recently for the Apalachia and Ocoee No. 3 projects. At Apalachia, the energy per linear foot of apron is approximately the same as at Safe Harbor. Laboratory tests showed an apron with a 20°-bucket angle to give unusually satisfactory service. In the model, the sand bed immediately downstream from the apron remained undisturbed, with the scour never exceeding 10 ft in depth anywhere downstream from the apron.

Mr. Davis' interesting paper gives a clear illustration of highly satisfactory results which can be obtained when the design for hydraulic structures is extensively coordinated with hydraulic laboratory model tests.

L. M. DAVIS,[7] M. AM. SOC. C. E.—Two points which are extremely important have been emphasized in the discussion by Mr. Schuleen. First, it is impossible to obtain accurate quantitative results by means of erosion tests on models. However, it is felt that the distance to the starting point of erosion below the toe of the dam can be determined with reasonable accuracy by proper model tests since the path of the jet leaving the apron should have the same relative shape in both model and prototype for similar conditions. As Mr. Schuleen states, it is very important to simulate, in the model, all possible conditions of flow to guard against a design of apron which will lend itself to some of the destructive regimens of flow under certain tailwater conditions as evidenced by the Tygart Dam tests. In connection with the Safe Harbor tests it was found that the most destructive erosion occurred when the jet was unstable and would change periodically from the condition of the depressed jet to the condition of the upward flowing jet. Second, Mr. Schuleen states that it would be dangerous to use the results of these erosion tests in the design of an apron for some other dam—except as a guide in making a preliminary design to be checked by model tests.

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[7] Hydr. Test Engr., Pennsylvania Water & Power Co., Holtwood, Pa.

Professor Allen has given the reasons why model tests are justified. At the present stage of knowledge in hydraulics, model tests should be considered essential in the design of important hydraulic structures.

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STRESS CONCENTRATIONS IN PLATES LOADED OVER SMALL AREAS

By H. M. Westergaard,[1] M. Am. Soc. C. E.

With Discussion by Messrs. George R. Rich, Dana Young, A. Nádai, L. A. MacColl, L. W. Teller and Earl C. Sutherland, D. L. Holl, C. A. P. Turner, and H. M. Westergaard.

Synopsis

If a considerable pressure is applied within a small area as a transverse load on a plate or slab, a concentration of stress will occur within and around that area. This stress concentration differs in type from that observed at fillets, re-entrant corners, or holes in solids; it is caused by concentration of the load. If the load on the plate or slab is not too close to any lines or points of concentrated support or to any edge and is neither too concentrated nor too dispersed, and if the material of the plate or slab is homogeneous, isotropic, and elastic, the important features of the stress concentration produced by the load can be expressed conveniently and with good approximation by means of six coefficients, all of which are pure numbers. Three of these—*B*, *C*, and *D*—are called "place coefficients"; they depend on the place of the load, but not on its manner of distribution over the small loaded area. The remaining three—*K*, *S*, and *T*—are called "area coefficients"; they depend on the size and shape of the loaded area and on the manner of distribution of the load over that area, but are independent of the place of the load. When the place coefficients have been determined in *m* cases and the area coefficients in *n* cases, solutions are thereby made available in *m* times *n* combined cases.

Six examples demonstrate how the place coefficients may be obtained from solutions already available for loads concentrated at a point. Area coefficients are derived in twelve cases.

When the load is concentrated within a very small area, special corrections are needed. The most important of these can be made by replacing the place coefficient *K* by a "substitute coefficient" *K'*.

A final, numerical example shows the use of place coefficients and area coefficients in determining stresses in a concrete pavement on elastic subgrade

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Note.—Published in April, 1942, *Proceedings*.

[1] Dean, Graduate School of Eng., and Gordon McKay Prof. of Civ. Eng., Harvard Univ., Cambridge, Mass.

under a wheel load which is distributed through dual tires over the combined area of two ellipses.

NOTATION

For convenience the plate or slab is assumed horizontal, with the load acting as a downward pressure on the top. The following notation is used:

P = total load;
p = load per unit of area;
A = loaded area;
h = thickness of the plate or slab;
μ = Poisson's ratio;
N = measure of stiffness of the plate or slab, defined by Eq. 2;
x, y = horizontal rectangular coordinates;
r, θ = corresponding polar coordinates;
u, v = particular values of x, y;
R, α = relative polar coordinates, defined by Eqs. 7, and corresponding to the relative rectangular coordinates $x - u$ and $y - v$;
ζ = deflection at point x, y, positive downward;
m_x, m_y, m_{xy} = bending moments and twisting moment at point x, y in the directions of x and y, per unit of width of section, positive when the corresponding stresses at the bottom of the plate or slab are positive; measurable in inch-pounds per inch, or in pounds;
$\sigma_x, \sigma_y, \tau_{xy}$ = horizontal normal stresses and shearing stress at the bottom of the plate or slab in the directions of x and y;
c = distance chosen arbitrarily;
B, C, D = place coefficients, defined through Eqs. 1, 4, 8, and 9, and used by means of Eqs. 13 to 20; pure numbers;
K, S, T = area coefficients, defined by Eqs. 11 and 12, and used by means of Eqs. 13 to 20; pure numbers;
Δ = Laplace's operator, Eq. 3.

Logarithms are natural logarithms unless otherwise stated.

THE THREE PLACE COEFFICIENTS

The deflection ζ of a plate or slab at the point x,y due to a load P concentrated at the point u,v at the distance R from the point x,y may be written as a sum of terms, as follows: First, a term containing $R^2 \log R$, which is needed to account for the singularity at the point of the load;[2] second, a polynomial of second degree in x, y, u, and v; and third, a supplementing series of terms of higher than second degree in x, y, u, and v. When x, y, u, and v are kept small, the first term, containing $\log R$, will be a dominating one in the intended applications; the terms of second degree will be needed; but the supplementing series of terms of higher degree may be ignored. The number of different coefficients in the polynomial of second degree is restricted by Maxwell's law of reciprocal deflections, according to which the deflection at x,y due to P at u,v is equal to the deflection at u,v due to P at x,y. Thus one arrives at the following formula which still requires verification:

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[2] See, for example, "Die elastischen Platten," by A. Nádai (Julius Springer, Berlin), 1925, p. 202.

$$\zeta = \frac{P}{8\,\pi\,N}\Bigg[\,R^2\left(\log\frac{R}{c} - 1 - B\right) + c_0 + c_x\,(u + x) + c_y\,(v + y) + c_u\,u\,x + c_v\,v\,y + c'\,(u\,y + v\,x) - \frac{C}{2}\,(u^2 - v^2 + x^2 - y^2) - D\,(u\,v + x\,y)\,\Bigg] \qquad (1)$$

The constant N in Eq. 1 is the "measure of stiffness" of the plate or slab in bending; this constant is defined by the formula

$$N = \frac{E\,h^3}{12\,(1 - \mu^2)} \qquad (2)$$

in which E is the modulus of elasticity, μ is Poisson's ratio, and h is the thickness of the plate or slab.

By applying Laplace's operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial R^2} + \frac{1}{R}\,\frac{\partial}{\partial R} + \frac{1}{R^2}\,\frac{\partial^2}{\partial \alpha^2} \qquad (3)$$

to Eq. 1 one finds

$$-\,N\,\Delta\,\zeta = \frac{P}{2\,\pi}\left(B - \log\frac{R}{c}\right) \qquad (4)$$

Transverse shears in plates or slabs per unit of length of section are first derivatives[3] of $-\,N\,\Delta\,\zeta$. By differentiating Eq. 4 with respect to R one finds the shear on a circular section with constant R. The value found is $-\,P/(2\,\pi\,R)$, which shows that the terms included in Eq. 1 account properly for the concentrated load P and for no other load within the region under consideration. This completes the verification of the form of Eq. 1.

The bending moments and the twisting moment in a plate or slab can be expressed by the formulas

$$\left.\begin{matrix} m_x \\ m_y \end{matrix}\right\} = -\,\frac{1 + \mu}{2}\,N\,\Delta\,\zeta \mp \frac{1 - \mu}{2}\,N\left(\frac{\partial^2\zeta}{\partial x^2} - \frac{\partial^2\zeta}{\partial y^2}\right) \qquad (5)$$

and

$$m_{xy} = -\,(1 - \mu)\,N\,\frac{\partial^2\zeta}{\partial x\,\partial y} \qquad (6)$$

By taking ζ from Eq. 1 and noting that

$$x - u = R\cos\alpha \qquad \text{and} \qquad y - v = R\sin\alpha \qquad (7)$$

one finds

$$\left.\begin{matrix} m_x \\ m_y \end{matrix}\right\} = \frac{P}{4\,\pi}\left[\,(1 + \mu)\left(B - \log\frac{R}{c}\right) \pm \frac{1 - \mu}{2}\,(C - \cos 2\,\alpha)\,\right] \qquad (8)$$

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[3] "Die elastischen Platten," by A. Nádai (Julius Springer, Berlin), 1925, p. 21.

and

$$m_{xy} = \frac{(1-\mu)P}{8\pi}(D - \sin 2\alpha) \qquad (9)$$

The three coefficients B, C, and D, which occur in Eqs. 1, 4, 8, and 9, will be considered as constants characteristic of the small region that is under consideration. The distance c can be chosen arbitrarily, but the choice of c will affect the value of B; the coefficient B attains its definite meaning after c has been chosen.

In the subsequent applications of Eqs. 8 and 9 it will not be necessary, unless it is specifically stated, to assume that x, y, u, and v are small; it is sufficient that the points x,y and u,v be contained within a small area; the distance R must remain small.

THE THREE AREA COEFFICIENTS

Assume now that the load P is distributed over a small area A, so that the pressure is p per unit of area at u,v and

$$P = \int_A p\, dA \qquad (10)$$

Then Eqs. 8 and 9 may be used with P replaced by $p\,dA$ and with integrations carried out over the area A.

It can now be seen that it will be expedient to introduce the following three "area coefficients," which are functions of x and y obtainable by integrations with respect to u and v over the loaded area:

$$K = -\frac{1}{P}\int_A p\, dA \log \frac{R}{c} \qquad (11)$$

$$S = -\frac{1}{P}\int_A p\, dA \cos 2\alpha \quad \text{and} \quad T = -\frac{1}{P}\int_A p\, dA \sin 2\alpha \qquad (12)$$

The coefficients K, S, and T are pure numbers. In terms of K, S, and T the integrals obtained from Eqs. 8 and 9 are expressed in the following formulas for the moments at point x,y due to the load distributed over the small area:

$$\left.\begin{matrix} m_x \\ m_y \end{matrix}\right\} = \frac{P}{4\pi}\left[(1+\mu)(B+K) \pm \frac{1-\mu}{2}(C+S)\right] \qquad (13)$$

and

$$m_{xy} = \frac{(1-\mu)P}{8\pi}(D+T) \qquad (14)$$

The stresses at the bottom of the plate or slab corresponding to the moments in Eqs. 13 and 14 are

$$\left.\begin{matrix} \sigma_x \\ \sigma_y \end{matrix}\right\} = \frac{3P}{2\pi h^2}\left[(1+\mu)(B+K) \pm \frac{1-\mu}{2}(C+S)\right] \qquad (15)$$

and

$$\tau_{xy} = \frac{3(1-\mu)P}{4\pi h^2}(D+T) \qquad (16)$$

The curvatures and twist corresponding to the moments in Eqs. 13 and 14 are

$$\left.\begin{matrix} -\dfrac{\partial^2\zeta}{\partial x^2} \\ -\dfrac{\partial^2\zeta}{\partial y^2} \end{matrix}\right\} = \frac{P}{4\,\pi\,N}\left[B + K \pm \frac{C+S}{2}\right] \ldots\ldots\ldots (17)$$

and

$$-\frac{\partial^2\zeta}{\partial x\,\partial y} = \frac{P}{8\,\pi\,N}(D+T) \ldots\ldots\ldots (18)$$

Finally, the strains and detrusion at the bottom of the plate or slab corresponding to the stresses in Eqs. 15 and 16 are

$$\left.\begin{matrix} \epsilon_x \\ \epsilon_y \end{matrix}\right\} = \frac{P\,h}{8\,\pi\,N}\left[B + K \pm \frac{C+S}{2}\right] \ldots\ldots\ldots (19)$$

$$\gamma_{xy} = \frac{P\,h}{8\,\pi\,N}(D+T) \ldots\ldots\ldots (20)$$

When the load P is distributed uniformly over the area A, the area coefficients in Eqs. 11 and 12 can be stated in the slightly simpler form

$$K = -\frac{1}{A}\int_A dA \log\frac{R}{c} \ldots\ldots\ldots (21)$$

$$S = -\frac{1}{A}\int_A dA\cos 2\,\alpha \quad \text{and} \quad T = -\frac{1}{A}\int_A dA\sin 2\,\alpha \ldots (22)$$

The coefficient K defined by Eq. 11 or 21 is a logarithmic potential.[4] If p is constant within the loaded area, K may be produced as the gravitational potential created by a long uniform rod with the cross section shaped as the loaded area.

DIFFERENTIAL EQUATIONS FOR THE AREA COEFFICIENTS

The deflections of a plate or slab are governed by Lagrange's equation[3]

$$\Delta^2\zeta = p/N \ldots\ldots\ldots (23)$$

When $\Delta\zeta$ is taken from Eq. 17, Eq. 23 becomes a differential equation for K; namely,

$$\Delta K = -\frac{2\,\pi\,p}{P} \ldots\ldots\ldots (24)$$

By comparing the values of the third derivatives that may be obtained from Eqs. 17 and 18 one finds two more equations for K, S, and T; they are equations of compatibility, expressing requirements of consistent deformations. The equations are

$$\frac{\partial}{\partial x}(2\,K - S) = \frac{\partial T}{\partial y}, \quad \frac{\partial}{\partial y}(2\,K + S) = \frac{\partial T}{\partial x} \ldots\ldots\ldots (25)$$

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[4] See, for example, "Foundations of Potential Theory," by Oliver Dimon Kellogg (Julius Springer, Berlin), 1929, pp. 62, 338, and 377.

The differential Eqs. 24 and 25 have two uses: One is the checking of results found by direct use of the definitions in Eqs. 11 and 12, or 21 and 22; the other use—which is actually more important—is in deriving formulas for K, S, and T. Examples of such derivations will be shown in a later section.

In the special group of cases in which the load is symmetrical around the origin of the coordinates, so that $p = p(r)$, K will be a function of r only, and S and T may be written in terms of a function $f(r)$, of r only, in the form

$$S = f(r) \cos 2\,\theta \qquad \text{and} \qquad T = f(r) \sin 2\,\theta \ldots\ldots\ldots\ldots (26)$$

Then Eq. 24 may be written in the special form

$$\frac{d^2K}{dr^2} + \frac{1}{r}\frac{dK}{dr} = -\frac{2\,\pi\,p}{P} \ldots\ldots\ldots\ldots (27)$$

and it is found that the two Eqs. 25 can be replaced by the single equation

$$\frac{df}{dr} + \frac{2\,f}{r} = 2\,\frac{dK}{dr} \ldots\ldots\ldots\ldots (28)$$

Note on the Special Significance of the Center of the Load

If the terms of third degree had been included in Eq. 1, linear terms in x, y, u, and v would have to be added in the expressions for the second derivatives of the deflection. Accordingly the moments in Eqs. 8 and 9 at x,y due to a load P at u,v would be supplemented by terms that can be written in the following form, the constants $c_1, c_2, \cdots c_{10}$ being of the dimension distance to the power -1:

$$\left.\begin{matrix} m'_x \\ m'_y \end{matrix}\right\} = P\,[(1+\mu)\,(c_1\,x + c_2\,y) \pm (1-\mu)\,(c_3\,x + c_4\,y)$$
$$+ (1+\mu)\,(c_5\,u + c_6\,v) \pm (1-\mu)\,(c_7\,u + c_8\,v)] \ldots\ldots\ldots (29)$$

and

$$m'_{xy} = (1-\mu)\,P\,[(c_2 + c_4)\,x + (c_1 - c_3)\,y + c_9\,u + c_{10}\,v] \ldots\ldots (30)$$

It is observed that the constants c_1, c_2, c_3, and c_4 have been written in Eq. 30 in such a way that the two equations of compatibility corresponding to Eqs. 25 will be satisfied. If Eqs. 29 and 30 are to be used directly, the origin of coordinates should be chosen so that x, y, u, and v can all be kept small.

When the load P is distributed over the area A, the term P in Eqs. 29 and 30 can be replaced by $p\,dA$, and thereafter the expressions on the right side can be integrated with respect to u and v over the area A. The resulting supplementary moments at point x,y will still be expressed by Eqs. 29 and 30 provided that u and v are now re-interpreted as the coordinates of the "center of the load," by which is meant the point of application of the resultant of the distributed load. It follows that there is an advantage in choosing the origin of the coordinates at the center of the load; namely, that the terms in Eqs. 29 and 30 containing u and v and the six constants c_5 to c_{10} thereby drop out. If, besides, the moments or stresses are computed at the center of the load, so that $x = y = 0$, the supplementary moments corresponding to the terms of third degree in the expression for the deflection disappear altogether. At other points the supplementary moments or stresses will be sums of terms proportional to first moments of the load. Inclusion of terms of fourth degree in the expression for the deflection would produce further supplementary terms for the moments and stresses. These further terms would contain moments of inertia and products of inertia of the load.

A useful conclusion can be drawn from these deliberations: It will be desirable generally to determine the place coefficients B, C, and D so that they will be exact at or near the center of the load. When this is done, the origin of coordinates may again be chosen freely according to convenience; in the region of interest the supplementary terms that might otherwise be considered will remain small; and the formulas in Eqs. 13 to 20 may be used without supplementary terms.

Corrections needed when the loaded area is too large to be rated as small will be considered in a later section.

DETERMINATION OF THE PLACE COEFFICIENTS IN SIX EXAMPLES

Example a. Center of a Circular Plate or Slab with a Fixed Edge at r = c.—

When a load P is concentrated at the center, the deflection at any point can be written in the form of Eq. 1 as follows:[5, 6]

$$\zeta = \frac{P}{8\pi N}\left[r^2\left(\log\frac{r}{c} - \frac{1}{2}\right) + \frac{1}{2}c^2\right] \ldots\ldots\ldots\ldots (31)$$

A comparison with Eq. 1 shows that with c chosen equal to the radius

$$B = -\frac{1}{2} \quad \text{and} \quad C = D = 0 \ldots\ldots\ldots\ldots (32)$$

Example b. Center of a Circular Plate or Slab with a Simply Supported Edge at r = c.—

With the load P at the center the deflection at any point can again be written in the form of Eq. 1 with $R = r$ and $C = D = 0$. It follows that Eq. 8 for the moments applies over the whole area. Then at $y = 0$, m_x becomes

$$m_r = -\frac{(1+\mu)P}{4\pi}\log\frac{r}{c} \ldots\ldots\ldots\ldots (33)$$

which after further comparison with Eq. 8 gives

$$B = \frac{1-\mu}{2(1+\mu)} \quad \text{and} \quad C = D = 0 \ldots\ldots\ldots\ldots (34)$$

The values in Eq. 34 can also be obtained like those in Eq. 32 by inspection of the formula for the deflection, which is in this case[6, 7]

$$\zeta = \frac{P}{8\pi N}\left[r^2\log\frac{r}{c} + \frac{3+\mu}{2(1+\mu)}(c^2 - r^2)\right] \ldots\ldots\ldots (35)$$

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[5] See, for example, "Die elastischen Platten," by A. Nádai (Julius Springer, Berlin), 1925, p. 61; or footnote 6.

[6] "Theory of Plates and Shells," by S. Timoshenko (McGraw-Hill Book Company, Inc., New York, N. Y.), 1940, p. 74.

[7] "Die elastischen Platten," by A. Nádai (Julius Springer, Berlin), 1925, p. 60.

Example c. Long Rectangular Plate or Slab Extending Far in the Directions of $\pm y$*, with Simply Supported Edges at* $x = \pm \frac{1}{2} l$*, and with a Load P at* $x = u$, $y = 0$.—

A particularly usable solution of this problem was given by A. Nádai[8,9] in 1921. According to this solution one may write

$$-N\,\Delta\zeta = \frac{P}{2\pi}\,\mathrm{Re}\left[\log\cos\frac{\pi(z+u)}{2l} - \log\sin\frac{\pi(z-u)}{2l}\right]\ldots(36)$$

in which z denotes the complex variable $x + i\,y$ and the symbol Re stands for "real part of the function." Furthermore

$$m_x = m_y \quad \text{at} \quad y = 0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(37)$$

When R, which is the same as the numerical value of $z - u$, is small, Eq. 36 may be written in the simpler form

$$-N\,\Delta\zeta = \frac{P}{2\pi}\left[\log\left(\frac{2}{\pi}\cos\frac{\pi u}{l}\right) - \log\frac{R}{l}\right]\ldots\ldots\ldots\ldots(38)$$

It is convenient to choose $c = l$. A comparison of Eqs. 38 and 37 with Eqs. 4 and 8 then leads directly to the results

$$c = l, \qquad B = \log\left(\frac{2}{\pi}\cos\frac{\pi u}{l}\right) = -\,0.4516 + 2.3026\,\log_{10}\cos\frac{\pi u}{l},$$

$$C = 1 \quad \text{and} \quad D = 0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(39)$$

Example d. Special Case of Example c: The Load Is at Midspan, at $x = y = 0$.—

Eqs. 39 give

$$c = l, \qquad B = -\,0.4516, \qquad C = 1, \quad \text{and} \quad D = 0\ldots\ldots\ldots(40)$$

Example e. Again a Long Rectangular Plate or Slab Extending Far in the Directions of $\pm y$*, but with Fixed Edges at* $x = \pm \frac{1}{2} l$*, with the Load P at Midspan at* $x = y = 0$.—

A supplement[10] to Nádai's solution for the simply supported plate or slab gives results that can be stated as follows:

$$c = l, \qquad B = -\,1.044, \qquad C = 0.538, \qquad \text{and} \qquad D = 0\ldots\ldots(41)$$

Example f. Large Slab on Elastic Subgrade Offering a Reaction per Unit of Area Equal to the Deflection Times a Constant "Modulus of Subgrade Reaction" k; the Place of the Load Being Far from Any Edge.—

This problem was solved in 1884 by Heinrich Hertz[11] who considered a

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[8] "Über die Spannungsverteilung in einer durch eine Einzelkraft belasteten rechteckigen Platte," by A. Nádai, *Der Bauingenieur*, Vol. 2, 1921, pp. 11–16; or "Die elastischen Platten," by A. Nádai (Julius Springer, Berlin), 1925, pp. 86 and 93.

[9] "Computation of Stresses in Bridge Slabs Due to Wheel Loads," by H. M. Westergaard, *Public Roads*, Vol. 11, March, 1930, pp. 1–23, especially p. 6.

[10] *Ibid.*, especially p. 20, formulas 104 and 105.

[11] "Über das Gleichgewicht schwimmender elastischer Platten," by Heinrich Hertz, *Wiedemanns Annalen der Physik und Chemie*, Vol. 22, 1884, pp. 449–455; reprinted in his "Gesammelte Werke," Vol. 1, pp. 288–294.

plate floating on a liquid of weight k per unit of volume, and loaded by a concentrated force P. The solution can be applied to problems of concrete pavements if it is recognized in advance that the assumption of the existence of a constant modulus k of subgrade reaction is at best a workable compromise in the interest of simplicity.[12, 13, 14]

A simple complete derivation will be shown.

The general equation of flexure of plates or slabs, Lagrange's equation

$$N \Delta^2 \zeta = p \qquad \text{(42)}$$

becomes in this case

$$N \Delta^2 \zeta + k \zeta = 0 \qquad \text{(43)}$$

When a distance l, known as the "radius of relative stiffness," and defined by the equation

$$k l^4 = N \qquad \text{(44)}$$

is introduced, Eq. 43 becomes

$$l^4 \Delta^2 \zeta + \zeta = 0 \qquad \text{(45)}$$

If the function

$$Z = \zeta + i l^2 \Delta \zeta \qquad \text{(46)}$$

(in which $i = \sqrt{-1}$) satisfies the equation

$$l^2 \Delta Z + i Z = 0 \qquad \text{(47)}$$

ζ will satisfy Eq. 45. Eq. 47 is satisfied[15] when Z is taken as a constant times any Bessel function of order zero with the argument $\frac{r\sqrt{i}}{l}$. Hankel's Bessel function $H^{(1)}{}_0$ is selected.[16] Then it will be in agreement with Eqs. 45 to 47 to write

$$\zeta = \frac{P l^2}{4 N} \operatorname{Re} H^{(1)}{}_0 \left(\frac{r\sqrt{i}}{l} \right) \qquad \text{(48)}$$

and

$$\Delta \zeta = \frac{P}{4 N} \operatorname{Im} H^{(1)}{}_0 \left(\frac{r\sqrt{i}}{l} \right) \qquad \text{(49)}$$

in which the symbol Re stands for the real part and the symbol Im for the imaginary part of the function. Eq. 48 is a statement of Hertz's solution in a particularly simple form.

To show that Eqs. 48 and 49 apply (without need for adding other particular solutions containing integration constants) in the case of a large slab with the load P at $r = 0$ three further observations will be sufficient: First, both ζ and $\Delta \zeta$ vanish when r goes to infinity; second, $d\zeta/dr = 0$ at $r = 0$; and third, when

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[12] "Stresses in Concrete Pavements Computed by Theoretical Analysis," by H. M. Westergaard, *Public Roads*, Vol. 7, April, 1926, pp. 25–35.

[13] "Analytical Tools for Judging Results of Structural Tests of Concrete Pavements," by H. M. Westergaard, *ibid.*, Vol. 14, December, 1933, pp. 185–188.

[14] "Stresses in Concrete Runways of Airports," by H. M. Westergaard, *Proceedings*, Nineteenth Annual Meeting of the Highway Research Bd., December, 1939, published in 1940, pp. 197–202.

[15] "Funktionentafeln mit Formeln und Kurven," by E. Jahnke and F. Emde (B. G. Teubner, Leipzig), 1909, p. 166.

[16] *Ibid.*, pp. 97 and 139.

r is small, Eq. 49 may be written in the form[17]

$$N \Delta \zeta = \frac{P}{2\pi}\left(\log\frac{r}{l} - 0.1159\right) \qquad (50)$$

Eq. 50 is of the form of Eq. 4 and shows that the load P is accounted for.

A comparison of Eq. 50 with Eq. 4 and a consideration of the symmetry lead to the following statement of results for the slab on elastic subgrade:

$$c = l, \qquad B = 0.1159, \qquad \text{and} \qquad C = D = 0 \qquad (51)$$

with l defined by Eq. 44.

STATEMENT OF THE AREA COEFFICIENTS IN TWELVE CASES

The area coefficients K, S, and T which appear in Eqs. 13 to 20 are defined by Eqs. 21 and 22 when the load is distributed uniformly over the loaded area; otherwise they are defined by Eqs. 11 and 12. Some formulas will now be stated. Notes on their derivation follow later.

Case 1.—

Circle $r = a$ with the load distributed uniformly over the area.

Within the circle:

$$K = \frac{1}{2} + \log\frac{c}{a} - \frac{r^2}{2a^2}, \qquad S = -\frac{r^2}{2a^2}\cos 2\theta, \qquad T = -\frac{r^2}{2a^2}\sin 2\theta \qquad (52)$$

Outside the circle:

$$K = \log\frac{c}{r}, \qquad S = \left(\frac{a^2}{2r^2} - 1\right)\cos 2\theta, \qquad T = \left(\frac{a^2}{2r^2} - 1\right)\sin 2\theta \qquad (53)$$

Case 2.—

Narrow strip along the whole periphery of the circle $r = a$, with the load distributed uniformly over the length.

Inside the circle:

$$K = \log\frac{c}{a}, \qquad S = T = 0 \qquad (54)$$

Outside the circle:

$$K = \log\frac{c}{r}, \qquad \frac{S}{\cos 2\theta} = \frac{T}{\sin 2\theta} = \frac{a^2}{r^2} - 1 \qquad (55)$$

Case 3.—

Circle $r = a$ loaded over the area according to the formula

$$p = \frac{(n+2)\,P\,r^n}{2\pi\,a^{n+2}} \qquad (56)$$

in which $n > -2$.

Within the circle:

$$K = \frac{1}{n+2} + \log\frac{c}{a} - \frac{1}{n+2}\left(\frac{r}{a}\right)^{n+2} \qquad (57)$$

$$\frac{S}{\cos 2\theta} = \frac{T}{\sin 2\theta} = -\frac{2}{n+4}\left(\frac{r}{a}\right)^{n+2} \qquad (58)$$

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[17] "Funktionentafeln mit Formeln und Kurven," by E. Jahnke and F. Emde (B. G. Teubner, Leipzig), 1909, pp. 97, 93, and 141.

Outside the circle:

$$K = \log \frac{c}{r}, \qquad \frac{S}{\cos 2\,\theta} = \frac{T}{\sin 2\,\theta} = \frac{n+2}{n+4}\left(\frac{a}{r}\right)^2 - 1 \tag{59}$$

It is observed that $n = 0$ reproduces Case 1 and $n = \infty$ Case 2.

Case 4.—

Circle $r = a$ loaded over the area according to the formula

$$p = \frac{(n+2)\,P}{n\,\pi\,a^2}\left[1 - \left(\frac{r}{a}\right)^n\right] \tag{60}$$

in which $n > -2$ and $n \neq 0$.

Within the circle:

$$K = \frac{n+4}{2\,(n+2)} + \log \frac{c}{a} - \frac{n+2}{2\,n}\left(\frac{r}{a}\right)^2 + \frac{2}{n\,(n+2)}\left(\frac{r}{a}\right)^{n+2} \tag{61}$$

$$\frac{S}{\cos 2\,\theta} = \frac{T}{\sin 2\,\theta} = -\frac{n+2}{2\,n}\left(\frac{r}{a}\right)^2 + \frac{4}{n\,(n+4)}\left(\frac{r}{a}\right)^{n+2} \tag{62}$$

Outside the circle:

$$K = \log \frac{c}{r}, \qquad \frac{S}{\cos 2\,\theta} = \frac{T}{\sin 2\,\theta} = \frac{n+2}{2\,(n+4)}\left(\frac{a}{r}\right)^2 - 1 \tag{63}$$

It is observed that $n = \infty$ reproduces Case 1.

Case 5.—

Ellipse

$$x^2/a^2 + y^2/b^2 = 1 \tag{64}$$

with the load distributed uniformly over the area.

Within the area of the ellipse:

$$K = \frac{1}{2} + \log \frac{2\,c}{a+b} - \frac{x^2}{a\,(a+b)} - \frac{y^2}{b\,(a+b)} \tag{65}$$

$$S = -\frac{a-b}{a+b} - \frac{2\,a\,b}{(a+b)^2}\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \tag{66}$$

$$T = -\frac{4\,x\,y}{(a+b)^2} \tag{67}$$

Outside the area of the ellipse it is expedient to use a function of the complex variable $z = x + i\,y$; namely,

$$W = \frac{z - \sqrt{z^2 - a^2 + b^2}}{a^2 - b^2} = \frac{1}{z + \sqrt{z^2 - a^2 + b^2}} \tag{68}$$

which at the elliptic boundary has the value

$$W_b = \frac{x}{a\,(a+b)} - \frac{i\,y}{b\,(a+b)} \tag{69}$$

and which may be written for great values of z in the form

$$W = \frac{1}{2\,z} + \frac{a^2 - b^2}{8\,z^3} + \cdots \quad\text{(70)}$$

The following values, which are stated in terms of W in Eq. 68, apply outside the elliptic area and coincide with the values in Eqs. 65 to 67 at the boundary:

$$K = \frac{1}{2} + \mathrm{Re}\,[\log(2\,c\,W) - z\,W] \quad\text{(71)}$$

$$S = -\frac{a-b}{a+b} - \frac{4}{a+b}(b\,x\,\mathrm{Re} + a\,y\,\mathrm{Im})\,W + 2\,a\,b\,\mathrm{Re}\,(W^2) \quad\text{(72)}$$

$$T = \frac{4}{a+b}(b\,x\,\mathrm{Im} - a\,y\,\mathrm{Re})\,W - 2\,a\,b\,\mathrm{Im}\,(W^2) \quad\text{(73)}$$

in which Re stands for "the real part of" and Im for "the imaginary part of."

At $y = 0$, $x \geqq a$, Eqs. 71 to 73 give

$$K = \frac{1}{2} + \log\frac{2\,c}{x + \sqrt{x^2 - a^2 + b^2}} - \frac{x}{x + \sqrt{x^2 - a^2 + b^2}} \quad\text{(74)}$$

$$S = -\frac{a^2 + b^2}{a^2 - b^2} + \frac{4\,b^2\,x}{(a^2 - b^2)\,(x + \sqrt{x^2 - a^2 + b^2})} \quad\text{(75)}$$

$$T = 0 \quad\text{(76)}$$

At $x = 0$, $y \geqq b$ one finds similarly

$$K = \frac{1}{2} + \log\frac{2\,c}{y + \sqrt{y^2 + a^2 - b^2}} - \frac{y}{y + \sqrt{y^2 + a^2 - b^2}} \quad\text{(77)}$$

$$S = -\frac{a^2 + b^2}{a^2 - b^2} + \frac{4\,a^2\,y}{(a^2 - b^2)\,(y + \sqrt{y^2 + a^2 - b^2})} \quad\text{(78)}$$

$$T = 0 \quad\text{(79)}$$

When either y is great or b is fairly close to a, S in Eq. 78 will be close to the value

$$S' = 1 - \frac{a^2}{2\,y^2} + \frac{a^2\,(a^2 - b^2)}{4\,y^4} \quad\text{(80)}$$

For example, with $a = 1.05\,b$, at $x = 0$, $y = b$, Eq. 80 gives $S' = 0.477$, whereas Eq. 78 or Eq. 66 gives $S = 0.475$. The first two terms in the expression in Eq. 80 are suggested by the formula for S in Eqs. 53, which apply when $a = b$.

Case 6.—

Narrow strip along the axis of x with the load $q = q(x)$ per unit of length; q being different from zero only over a short stretch, and

$$\int_{-\infty}^{\infty} q\,dx = P \quad\text{(81)}$$

The solution may be written in terms of a function Z of the complex variable $z = x + i\,y$ in the form

$$K = \operatorname{Re} Z, \qquad S = 2\,y \operatorname{Im} Z' - 1, \qquad T = 2\,y \operatorname{Re} Z' \ldots\ldots (82)$$

in which Re stands for "the real part of" and Im for "the imaginary part of," and Z' denotes the first derivative with respect to z; provided that Z and Z' are determined so that they satisfy the following three conditions, the first two of which define Z':

First,

$$\operatorname{Im} Z' = \pm \frac{\pi\,q}{P} \quad \text{at} \quad y = 0 \ldots\ldots\ldots\ldots\ldots\ldots (83)$$

with the plus and minus signs applying when y approaches zero from the positive and negative side respectively.

Second, $z\,Z'$ must converge toward -1 when z goes to infinity.

Third, in obtaining Z from Z' the integration constant must be chosen so that Re Z converges toward $\log (c/r)$ when z goes to infinity.

Case 7.—

Narrow strip along the axis of x from $x = -a$ to $x = a$, with the load per unit of length equal to

$$q = \frac{2\,P}{\pi\,a^2}\sqrt{a^2 - x^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (84)$$

The solution may be stated either as an application of Case 5 or as an illustration of Case 6.

As application of Case 5: Eqs. 65 to 79 apply with $b = 0$.

As illustration of Case 6: Eqs. 82 apply with

$$Z' = -\frac{2}{a^2}\left(z - \sqrt{z^2 - a^2}\right) \ldots\ldots\ldots\ldots\ldots\ldots (85)$$

and

$$Z = \frac{1}{2} + \log\frac{2\,c}{z + \sqrt{z^2 - a^2}} - \frac{z}{z + \sqrt{z^2 - a^2}} \ldots\ldots\ldots (86)$$

Case 8.—

Narrow strip along the axis of x from $x = -a$ to $x = a$, loaded uniformly over the length.

Eqs. 82 for Case 6 apply with

$$Z' = \frac{1}{2\,a}\log\frac{z - a}{z + a} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (87)$$

$$Z = 1 + \frac{z - a}{2\,a}\log\frac{z - a}{c} - \frac{z + a}{2\,a}\log\frac{z + a}{c} \ldots\ldots\ldots (88)$$

Case 9.—

Narrow strip 1–3 in Fig. 1, loaded uniformly over the length.

Values at point 0:

$$K = 1 + \frac{s}{b}\log\frac{c}{R_1} + \frac{t}{b}\log\frac{c}{R_3} - \frac{R_2\,\beta}{b} \quad\ldots\ldots\ldots\ldots\ldots(89)$$

$$S = \left[1 - \frac{2\,R_2\,\beta}{b}\right]\cos 2\,\alpha + \frac{2\,R_2}{b}\left(\log\frac{R_3}{R_1}\right)\sin 2\,\alpha \quad\ldots\ldots\ldots(90)$$

$$T = \left[1 - \frac{2\,R_2\,\beta}{b}\right]\sin 2\,\alpha - \frac{2\,R_2}{b}\left(\log\frac{R_3}{R_1}\right)\cos 2\,\alpha \quad\ldots\ldots\ldots(91)$$

Case 10.—

Triangle 0–1–3 in Fig. 1, with the load distributed uniformly over the area A.

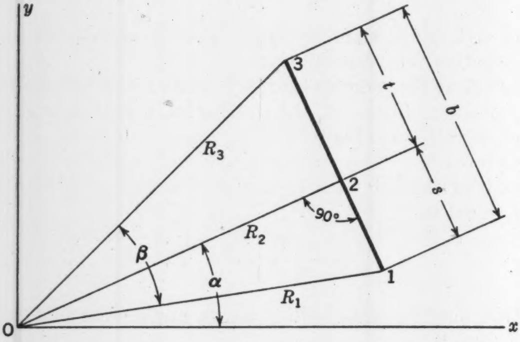

Fig. 1.—Strip 1–3 in Case 9 and Triangle 0–1–3 in Case 10

Values at point 0: For K: add $\frac{1}{2}$ to the value in Eq. 89. The coefficients S and T are the same as in Eqs. 90 and 91.

Case 11.—

Polygon composed of triangles such as 0-($2\,n - 1$)-($2\,n + 1$) in Fig. 2, with $n = 1, 2, 3 \ldots$; the load being distributed uniformly over the whole area A.

Values at the common vertex 0:

$$K = \frac{3}{2} + \frac{1}{2\,A}\sum^{n} R_{2n}\left(s_n \log\frac{c}{R_{2n-1}} + t_n \log\frac{c}{R_{2n+1}} - R_{2n}\,\beta_n\right) \ldots(92)$$

$$S = \frac{1}{A}\sum^{n} R_{2n}\left[\left(\frac{1}{2}b_n - R_{2n}\,\beta_n\right)\cos 2\,\alpha_n + R_{2n}\left(\log\frac{R_{2n+1}}{R_{2n-1}}\right)\sin 2\,\alpha_n\right] \quad\ldots\ldots\ldots\ldots\ldots(93)$$

$$T = \frac{1}{A}\sum^{n} R_{2n}\left[\left(\frac{1}{2}b_n - R_{2n}\,\beta_n\right)\sin 2\,\alpha_n - R_{2n}\left(\log\frac{R_{2n+1}}{R_{2n-1}}\right)\cos 2\,\alpha_n\right] \quad\ldots\ldots\ldots\ldots\ldots(94)$$

Case 12.—
Rectangle with sides a and b in the directions of x and y, loaded uniformly over the area.

At the center:

$$K = \frac{1}{2}\left[3 + \log\frac{4c^2}{a^2+b^2} - \frac{a}{b}\tan^{-1}\left(\frac{b}{a}\right) - \frac{b}{a}\tan^{-1}\left(\frac{a}{b}\right)\right] \quad \text{(95)}$$

$$S = -\frac{a}{b}\tan^{-1}\left(\frac{b}{a}\right) + \frac{b}{a}\tan^{-1}\left(\frac{a}{b}\right) \quad \text{(96)}$$

$$T = 0 \quad \text{(97)}$$

At the two corners at which both x and y have either their smallest values or their greatest values:

$$K = K_{\text{Eq. 95}} - \log 2 \quad \text{(98)}$$

Coefficient S is the same as in Eq. 96.

$$T = \frac{a}{2b}\log\frac{a^2}{a^2+b^2} + \frac{b}{2a}\log\frac{b^2}{a^2+b^2} \quad \text{(99)}$$

At the center of either side of length b: K, S, and T are the same as in Eqs. 95 to 97 except that a is replaced by $2a$.

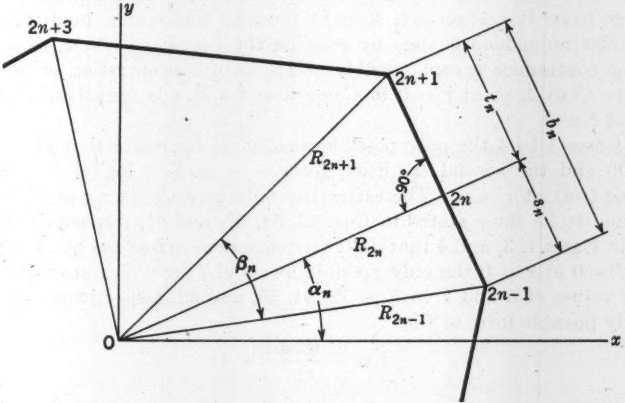

FIG. 2.—POLYGON IN CASE 11

NOTES ON THE DERIVATION OF THE PRECEDING FORMULAS FOR AREA COEFFICIENTS

Some of the formulas for the area coefficients can be derived conveniently by direct use of the definitions in Eqs. 11 and 12, or 21 and 22. For example, in this way the formulas for the uniformly loaded circular area in Case 1, Eqs. 52 and 53, can be obtained by easy integrations; and the formulas for the interior of the uniformly loaded elliptic area in Case 5, Eqs. 65 to 67, can be

obtained by less simple, yet manageable integrations. (Indebtedness is expressed to Louis A. Pipes, Faculty Instructor in Electrical Engineering, Harvard University, who derived Eq. 65 independently and commented on electrical analogies that exist because K is a logarithmic potential.)

In most of the cases, however, including that of the elliptic area in Case 5, if values are desired at all points, it is more convenient to use the governing differential equations.

In Cases 1 to 4 the load is confined by the circle $r = a$; p is a function of r only, and $p = 0$ when $r > a$. The governing differential equations are Eq. 27,

$$\frac{d^2K}{dr^2} + \frac{1}{r}\frac{dK}{dr} = -\frac{2\pi p}{P} \qquad (100)$$

and Eq. 28,

$$\frac{df}{dr} + \frac{2f}{r} = 2\frac{dK}{dr} \qquad (101)$$

in which, by Eqs. 26,

$$f = \frac{S}{\cos 2\theta} = \frac{T}{\sin 2\theta} \qquad (102)$$

If these equations are to be used, it is necessary to note some conditions that apply at special places. The following conditions of this kind are consequences of the definitions of K, S, and T in Eqs. 11 and 12: First, when r goes to infinity, K must converge toward or be equal to $\log(c/r)$ and f must converge toward -1; second, K and f must be continuous functions, so that they suffer no sudden change by crossing the boundary $r = a$; third, dK/dr must be continuous except possibly at places of concentration of load, as at $r = a$ in Case 2, or at $r = 0$ in Cases 3 and 4 if n is negative; and fourth, $f = 0$ at $r = 0$.

In Cases 1 to 4 the only possible form of K for $r \geqq a$ that satisfies both Eq. 100 and the special condition for $r = \infty$ is $K = \log(c/r)$. This gives $K = \log(c/a)$ at $r = a$. Thereafter the only possible forms of K for $r \leqq a$ are found to be those stated in Eqs. 52, 54, 57, and 61, respectively; it being noted in Cases 1, 3, and 4 that dK/dr is continuous across the boundary $r = a$. With $f = 0$ at $r = 0$ the only possible forms of f for $r \leqq a$ are those defined by the values of S and T in Eqs. 52, 54, 58, and 62, respectively. For $r > a$ the only possible form of f is

$$f = \frac{\text{constant}}{r^2} - 1 \qquad (103)$$

in which the constant can be determined by the value of f at $r = a$; this leads to the formulas for S and T in Eqs. 53, 55, 59, and 63. For supplementary checking it may be observed that df/dr is continuous across the boundary at $r = a$ except in Case 2, in which the load is concentrated at the boundary.

In the less simple cases in which p is not a function of r only the differential equations governing K, S, and T are Eq. 24,

$$\Delta K = -\frac{2\pi p}{P} \qquad (104)$$

and Eqs. 25,

$$\frac{\partial}{\partial x}(2K - S) = \frac{\partial T}{\partial y}, \qquad \frac{\partial}{\partial y}(2K + S) = \frac{\partial T}{\partial x} \text{........} (105)$$

Three conditions which apply at special places, and which are consequences of the definitions of K, S, and T in Eqs. 11 and 12, will make the solutions unique. The first is that when r goes to infinity, K, S, and T must converge toward the values

$$K_\infty = \log \frac{c}{r}, \qquad S_\infty = -\cos 2\theta, \qquad T_\infty = -\sin 2\theta \text{......} (106)$$

The second and third conditions will be stated under the assumption that p^2 has a finite maximum value except on lines where load may be concentrated; on such lines the load q per unit of length must nowhere be infinite; and no finite part of the load may be concentrated at a point. Then the second condition is that K, S, and T must be continuous functions with single values, finite or zero, at all points of the xy-plane except at $r = \infty$. The third condition is that $\partial K/\partial x$ and $\partial K/\partial y$ must nowhere be infinite and must be single-valued at all points except at lines where the load is concentrated as a load q per unit of length of the line; in crossing such a line in the direction of a rectangular coordinate n along the normal, the value of $\partial K/\partial n$ is required to drop by the amount $2\pi q/P$, giving the surface x,y,K a corresponding edge or sharp ridge.

When the functions K, S, and T satisfy the second and third special conditions and Eqs. 105, the first derivatives of S and T with respect to x and y will automatically be single-valued at all points except at lines where load is concentrated. The surfaces x,y,S and x,y,T can have no sharp edge except where the surface x,y,K also has one. This conclusion may be used for supplementary verification of results, but the three conditions that were stated and Eqs. 104 and 105 are sufficient to define the single solution that exists in each case.

Three additional, general observations are needed. The first is that a general solution of Eqs. 104 and 105 for the unloaded region, in which $p = 0$, may be written in the following form:

$$K = \text{Re}\, Z \text{.......................} (107)$$

$$S = 2jx\, \text{Re}\, Z' + 2(1-j)\, y\, \text{Im}\, Z' + \text{Re}\, Z_1 \text{.......} (108)$$

$$T = -2jx\, \text{Im}\, Z' + 2(1-j)\, y\, \text{Re}\, Z' - \text{Im}\, Z_1 \text{.......} (109)$$

in which Re stands for "the real part of" and Im for "the imaginary part of," and Z and Z_1 are any analytic functions of the complex variable z, and j is a convenient constant.

The second general observation is that if the loaded area is turned about the point u,v through an angle α_0, so that the original angles α are changed to $\alpha + \alpha_0$, the value of K at point u,v will remain unchanged; but the values S_0 and T_0 of S and T at point u,v will be changed, according to the definitions in Eqs. 12, to

$$S = -\frac{1}{P}\int dP \cos 2\,(\alpha + \alpha_0) = S_0 \cos 2\,\alpha_0 - T_0 \sin 2\,\alpha_0 \ldots\ldots (110)$$

$$T = -\frac{1}{P}\int dP \sin 2\,(\alpha + \alpha_0) = S_0 \sin 2\,\alpha_0 + T_0 \cos 2\,\alpha_0 \ldots\ldots (111)$$

The third general observation deals with the effects of an expansion of the loaded area. Consider three areas A_1, A_2, and A_3 over each of which the load P is distributed uniformly in succession. The area A_1 has the boundary $r = r_1\,(\theta)$; A_2 has the boundary $r = (1 + j)\,r_1\,(\theta)$, in which j is a positive constant; and A_3 is the part of A_2 lying outside A_1. At $r = 0$, S and T will have the same values for the three areas; but K will have three values K_1, K_2, and K_3 at $r = 0$ for the three areas. The following relations apply:

$$K_2 = K_1 - \log\,(1 + j) \ldots\ldots (112)$$

$$A_3\,K_3 = A_2\,K_2 - A_1\,K_1 \ldots\ldots (113)$$

$$A_2 = (1 + j)^2\,A_1, \qquad A_3 = j\,(2 + j)\,A_1 \ldots\ldots (114)$$

Substitutions from Eqs. 112 and 114 in Eq. 113 give

$$K_3 = K_1 - \frac{(1 + j)^2 \log\,(1 + j)}{j\,(2 + j)} \ldots\ldots (115)$$

When j is very small, so that A_3 is reduced to a rim around A_1, K_3 in Eq. 115 becomes

$$K_4 = K_1 - \frac{1}{2} \ldots\ldots (116)$$

The preparations have now been made for the discussion of the remaining individual cases.

In Case 5 an elliptic area is loaded uniformly. The solution in Eqs. 65 to 67 and 71 to 73 is verified by the following observations: First, Eqs. 72 and 73 fit the pattern of Eqs. 108 and 109 when W is any analytic function of the complex variable z; but when W is chosen as in Eq. 68, all three Eqs. 71 to 73 fit the pattern of Eqs. 107 to 109. Second, with W as in Eq. 68, the values in Eqs. 71 to 73 converge toward the values in Eqs. 106 when r goes to infinity, and assume the values defined by Eqs. 65 to 67 on the elliptic boundary. Third, the values of K, S, and T in Eqs. 65 to 67 satisfy Eqs. 104 and 105. Fourth and last, the first derivatives of K with respect to x or y are the same at the elliptic boundary whether they are determined from Eq. 65 or Eq. 71. The solution satisfies all the requirements and is correct. For an additional check it can be observed that the first derivatives of S and T with respect to x or y are the same at the elliptic boundary whether they are determined from Eqs. 66 and 67 or from Eqs. 72 and 73.

In Case 6 the load is at $y = 0$ only. Eqs. 82 fit the pattern of Eqs. 107 to 109. The requirements in Eqs. 106 for $r = \infty$ are satisfied; and Eq. 83 assures that in crossing the axis of x from the side of $-y$ to the side of y the value of

$\partial K/\partial y$, which is equal to $-\text{Im}\,Z'$, drops suddenly by the amount $2\pi q/P$, thereby accounting for the load q per unit of length. All requirements are satisfied. The solution is of the type introduced by Nádai[8] in his paper of 1921 on concentrated loads.

Case 7 is an obvious application of Case 5, obtained by letting one principal radius b of the ellipse approach zero; but Case 7 is much simpler than Case 5 and is solved conveniently by direct application of Case 6. The solution of Case 7 serves therefore as an additional check on the solution of Case 5.

Case 8, in which the load is distributed uniformly over the length of a piece of line, is a direct application of Case 6.

In Case 9 the load is distributed uniformly over the length of the line 1–3 in Fig. 1. The area coefficients are determined only at the origin of the coordinates. The value of K may be obtained either through Case 8 or by integration according to the definition in Eq. 11. The values of S and T are derived conveniently by determining first their values $-S_0$ and $-T_0$ when α in Fig. 1 is $-\pi/2$; this can be done either through Case 8 or by integrations according to the definitions in Eqs. 12. Thereafter Eqs. 110 and 111 are applied with α_0 equal to α in Fig. 1.

In Case 10 the load is distributed uniformly over the area of the triangle 0–1–3 in Fig. 1. The area coefficients are determined at 0 only. The coefficient K is obtained from Case 9 through Eq. 116; S and T are necessarily the same as in Case 9.

Case 11 of a polygon and Case 12 of a rectangle are direct applications of Case 10.

THREE EXAMPLES OF CORRECTIONS NEEDED WHEN THE LOADED AREA IS TOO LARGE TO BE RATED AS SMALL

Example 1.—Consider a circular plate or slab with a fixed edge at $r = c$ and with the load uniformly distributed over the area of the small circle $r = a$. Combination of the place coefficients in Eqs. 32, Example a, with the area coefficients in Eqs. 52, Case 1, gives at the center

$$B + K = \log\frac{c}{a}, \qquad C + S = D + T = 0 \qquad (117)$$

These sums define the bending moment and the greatest stress at the center through Eqs. 13 and 15. The values are proportional to $B + K$.

When the radius a of the loaded area is increased so that it ceases to be small, a correction is needed. The correction can be made by replacing $B + K$ by a corrected combined coefficient $(B + K)'$, which in this example may be determined as follows:

According to Eqs. 4 and 32 a load $P = 1$ at the distance r from the center produces a value of $N\,\Delta\,\zeta$ at the center equal to

$$N\,(\Delta\,\zeta)_0 = \frac{1}{2\pi}\left[\log\frac{r}{c} + \frac{1}{2} + F(r)\right] \qquad (118)$$

in which the function $F(r)$ is zero at $r = 0$ and contributes no singularities.

The surface $x, y, N(\Delta\zeta)_0$ is an influence surface and may therefore be produced as a deflected surface; it must obey the equation $\Delta^2 (N\Delta\zeta)_0 = 0$ for $0 < r \leqq c$ and satisfy the conditions of the fixed edge at $r = c$. The only form of Eq. 118 that meets all these requirements is

$$N(\Delta\zeta)_0 = \frac{1}{2\pi}\left(\log\frac{r}{c} + \frac{1}{2} - \frac{r^2}{2c^2}\right) \ldots\ldots\ldots\ldots (119)$$

When the load P is distributed uniformly over the area of the circle $r = a$ the value of $N\Delta\zeta$ at the center becomes, by use of Eq. 119,

$$N\Delta\zeta = \frac{P}{\pi a^2}\int_0^a (2\pi r\,dr)\,N(\Delta\zeta)_0 = -\frac{P}{2\pi}\left(\log\frac{c}{a} + \frac{a^2}{4c^2}\right) \ldots (120)$$

According to Eq. 17 the uncorrected value of $N\Delta\zeta$ is $-\frac{P}{2\pi}(B + K)$. A comparison with Eq. 120 shows that the corrected value of $B + K$, which will replace the value in Eqs. 117, is

$$(B + K)' = \log\frac{c}{a} + \frac{a^2}{4c^2} \ldots\ldots\ldots\ldots\ldots\ldots (121)$$

The last term in Eq. 121 represents the correction. When $a = c/4$, the correction is 1.1%. When $a = c$, the correction accounts for the whole value of the bending moment at the center.

Example 2.—The same plate or slab and the same load are considered as in Example 1 except that the edge at $r = c$ is now simply supported. Using Eqs. 34 instead of Eqs. 32 one finds

$$B + K = \log\frac{c}{a} + \frac{1}{1+\mu}, \qquad C + S = D + T = 0 \ldots\ldots (122)$$

A corrected value $(B + K)'$ applicable when the radius a of the loaded circle ceases to be small can be obtained in the same way as in Example 1. Instead of Eqs. 119 and 121 one finds

$$N(\Delta\zeta)_0 = \frac{1}{2\pi}\left[\log\frac{r}{c} - \frac{1-\mu}{2(1+\mu)}\left(1 - \frac{r^2}{c^2}\right)\right] \ldots\ldots (123)$$

and

$$(B + K)' = \log\frac{c}{a} + \frac{1}{1+\mu} - \frac{1-\mu}{4(1+\mu)}\frac{a^2}{c^2} \ldots\ldots\ldots (124)$$

The last term in Eq. 124 represents the correction. When $a = c/4$ and $\mu = 0.3$, the correction is $-$ 0.4%.

Example 3.—Consider a large slab on elastic subgrade under a load distributed uniformly over the area of a small circle $r = a$. Combination of the place coefficients in Example *f*, Eqs. 51, with the area coefficients in Case 1, Eqs. 52, gives at the center of the circle the sums

$$B + K = \log\frac{l}{a} + 0.6159, \qquad C + S = D + T = 0 \ldots\ldots (125)$$

in which l is the radius of relative stiffness defined by Eq. 44. Eqs. 125 define the moments and stresses at the center of the circle through Eqs. 13 and 15. The values are proportional to $B + K$, as in Examples 1 and 2.

An analysis[18] applicable when the radius a has ceased to be small leads to a simple formula which will include the needed correction with good accuracy as long as $a < l$. As in Examples 1 and 2, the formula defines a corrected combined coefficient $(B + K)'$ which will replace $B + K$. The formula is

$$(B + K)' = \log \frac{l}{a} + 0.6159 + \frac{\pi}{32}\left(\frac{a}{l}\right)^2 \qquad (126)$$

The last term represents the correction. With $a = l/2$ the correction is 1.9%, which is still small. With $a = l/4$ the correction is 0.3%, which can well be ignored in practical applications.

CORRECTIONS NEEDED WHEN THE LOADED AREA IS VERY SMALL

When the load is concentrated within an area of very small extension, the usual assumption that horizontal normal stresses are proportional to the distance from a neutral plane will not apply in the immediate vicinity of the load. Results found by the ordinary theory of bending of plates or slabs should then be corrected in accordance with the theory of elasticity in three dimensions. When the corrections can be evaluated, and when the stresses at the bottom of the plate or slab are of primary interest, it will be suitable to state corrected results in terms of "substitute coefficients" K', S', and T'. These substitute coefficients should be determined so that they will define the correct stresses at the bottom of the plate or slab when they are used in Eqs. 15 and 16 instead of the true area coefficients K, S, and T.

When the load is concentrated at a point, the function K has an infinitely high peak at that point, while K' will have a rounded peak of finite height; this suggests the importance of replacing K by K' when the loaded area is reduced nearly to a point. On the other hand, S and T have the extreme values ± 1; which suggests that the replacements of S and T by S' and T' will usually be relatively unimportant, and may be ignored in most practical problems. The discussion that follows deals with the replacement of K only.

In Case 1, in which the load is distributed uniformly over the area of the circle $r = a$, the value of K at the center, according to Eqs. 52, is

$$K = \frac{1}{2} + \log \frac{c}{a} \qquad (127)$$

The value of the substitute coefficient K', which will define the correct stress at the bottom of the plate or slab under the center of the circle, may be written in the form

$$K' = \frac{1}{2} + \log \frac{c}{a'} \qquad (128)$$

in which a' is a substitute radius. Computations based on a theory given by

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[18] "Stresses in Concrete Runways of Airports," by H. M. Westergaard, *Proceedings*, Nineteenth Annual Meeting of the Highway Research Bd., December, 1939, published in 1940, especially p. 202.

Nádai[19] led to the numerical values in Table 1 and to the following formulas which fit the numerical values with good approximation:[20]

$$a' = \sqrt{1.6\,a^2 + h^2} - 0.675\,h \quad \text{when} \quad a < 1.724\,h \ldots\ldots\ldots (129)$$

$$a' = a \quad \text{when} \quad a > 1.724\,h \ldots\ldots\ldots (130)$$

in which h is the thickness of the plate or slab.

If the load is distributed uniformly over a non-circular area, or if the area is circular but a point other than the center is considered, Eqs. 128 to 130, or Eq. 128 and Table 1, may still be used to obtain a proper substitute coefficient K'. If the circular area that was considered is replaced by the sector $\theta_1 \leqq \theta \leqq \theta_2$, $r \leqq a$, the value of K' at $r = 0$ will be the same as for the full circle; it follows that if the constant radius a is replaced by a single-valued function

$$a = a\,(\theta) \quad \text{for} \quad 0 \leqq \theta < 2\,\pi \,.\,(131)$$

defining the boundary of a new area in polar coordinates a, θ, the new value of K' at $r = 0$ will be

$$K' = \frac{1}{2\,A}\int_0^{2\pi} a^2\,d\theta\left(\frac{1}{2} + \log\frac{c}{a'}\right).\,(132)$$

in which a' is the same function of a as before.

TABLE 1.—VALUES OF THE RATIO OF THE SUBSTITUTE RADIUS a' OF A UNIFORMLY LOADED CIRCULAR AREA TO THE THICKNESS h, FOR DIFFERENT VALUES OF THE RATIO OF THE TRUE RADIUS a TO THE THICKNESS h

a/h	a'/h
0	0.3254
0.25	0.3709
0.5	0.504
0.75	0.705
1.0	0.944
1.5	1.456
2.0	1.967

Assume next that the range $a_{\max} - a_{\min}$ of a in Eq. 131 is fairly small compared with the thickness h—for example, not greater than $h/4$—then within that range a' may be stated with good approximation by the formula

$$a' = a'_0\,[1 + \lambda\,(a^2 - a^2_0)] \ldots\ldots\ldots (133)$$

in which a_0 is chosen so that

$$a^2_0 \int_0^{2\pi} a^2\,d\theta = \int_0^{2\pi} a^4\,d\theta \ldots\ldots\ldots (134)$$

and a'_0 is the value of a' for $a = a_0$, while λ is a suitable constant. According to Eq. 134 a_0 may be defined as $\sqrt{2}$ times the polar radius of gyration of the area A about $r = 0$, or as the radius of a circular area having the same polar radius of gyration about the center that A has about $r = 0$. The use of Eq. 133 means that a piece of the curve $a' = a'(a)$ is replaced by a suitable parabola. Eq. 133 may be restated approximately in the form

$$\log a' = \log a'_0 + \lambda\,(a^2 - a^2_0) \ldots\ldots\ldots (135)$$

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[19] "Die Biegungsbeanspruchung von Platten durch Einzelkräfte," by A. Nádai, *Schweizerische Bauzeitung*, Vol. 76, 1920, p. 257; or "Die elastischen Platten," by A. Nádai (Julius Springer, Berlin), 1925, pp. 315–322.

[20] "Stresses in Concrete Pavements Computed by Theoretical Analysis," by H. M. Westergaard, *Public Roads*, Vol. 7, April, 1926, especially pp. 32, 26, and 27.

Substitution from Eqs. 135 and 134 in Eq. 132 then leads to the approximate formula

$$K' = \frac{1}{2} + \log \frac{c}{a'_0} \qquad (136)$$

which has the same form as Eq. 128 for the circular area. It is concluded that the value of K' can be found by first changing the loaded area into a circular area with the same polar radius of gyration, provided that the range $a_{\max} - a_{\min}$ is not too great.

When the load is distributed uniformly over the area of an ellipse with the principal radii a and b, as in Case 5, the value of K at the center is according to Eq. 65

$$K = \frac{1}{2} + \log \frac{2c}{a + b} \qquad (137)$$

The polar radius of gyration is proportional to $\sqrt{a^2 + b^2}$. Consequently, when the ellipse is either small enough or sufficiently close to a circle to permit the use of Eq. 136, the rule for changing K into K' may be stated as follows: The value of $\frac{1}{2}(a + b)$ in Eq. 137 is replaced according to Eq. 129 by the substitute radius

$$a'_0 = \sqrt{0.8\,(a^2 + b^2) + h^2} - 0.675\,h \qquad (138)$$

which can be used through Eq. 136. Eq. 129 should not be applied when $|a - b| > h/2$, nor when $a + b > 3\,h$. When either $|a - b| > h/2$ or $a + b > 2\,h$, it is suggested that instead of using Eq. 138, K' at the center be determined as if the ellipse were a circle with the radius $\frac{1}{2}(a + b)$; this means that a in Eqs. 129 and 130 should be replaced by $\frac{1}{2}(a + b)$. Such a computation gives stresses that are slightly too high for some of the combinations of a and b; but the errors are on the side of safety and disappear when a and b become large enough to permit the assumption that $K' = K$.

NUMERICAL EXAMPLE: WHEEL WITH DUAL TIRES ON CONCRETE PAVEMENT

The numerical data defining this example are stated at the head of Table 2. Information kindly supplied by E. F. Kelley, Chief of the Division of Tests, Public Roads Administration, Federal Works Agency, in a letter, indicates that the assumptions stated at the head of Table 2 about the area of contact between tires and pavement represent with fair realism a case of a not excessive overload, such as may occur in a dynamic action of the rear wheels of a truck. The actual area of contact between a tire and the pavement is likely to be something between an ellipse and a circumscribed rectangle, but the difference between the assumed ellipses and a corresponding true area is not important from the present point of view.

With the numerical data stated at the head of Table 2, Eqs. 2, 44, and 51 give

$$N = 89{,}320{,}000 \text{ lb-in.}, \qquad l = c = 36.56 \text{ in.} \qquad (139)$$

$$B = 0.1159, \qquad C = 0 \qquad (140)$$

It is desirable to investigate first the effects of the load $P = 5{,}000$ lb transmitted through one of the tires. With this value of the load Eq. 15 for the normal stresses becomes

$$\left.\begin{matrix}\sigma_x \\ \sigma_y\end{matrix}\right\} = [58.465\,(B + K) \pm 19.488\,S]\ \text{lb-in.}^{-2} \qquad (141)$$

but the possible necessity of replacing K by K' should be investigated. With $a = 3.25$ in. and $b = 6.5$ in. one finds at the center of the single loaded ellipse

TABLE 2.—NUMERICAL EXAMPLE OF STRESSES CREATED AT THE BOTTOM OF A SEVEN-INCH CONCRETE PAVEMENT BY A WHEEL LOAD OF 5,000 LB TRANSMITTED THROUGH A SINGLE TIRE OR BY A WHEEL LOAD OF 10,000 LB TRANSMITTED THROUGH DUAL TIRES, WHEN THE LOAD IS AT A CONSIDERABLE DISTANCE FROM ANY EDGE OR JOINT

Assumed modulus of elasticity of concrete, $E = 3{,}000{,}000$ lb-in.$^{-2}$ Poisson's ratio of the concrete, $\mu = 0.2$.
Modulus of subgrade reaction, $k = 50$ lb-in.$^{-3}$
The load is assumed uniformly distributed over the area of contact between tires and pavement.
In the case of the single tire the area of contact is bounded by the ellipse

$$\frac{x^2}{(3.25\ \text{in.})^2} + \frac{y^2}{(6.5\ \text{in.})^2} = 1 \qquad \text{(Ellipse I)}$$

In the case of the dual tires a second area is added; namely, the area bounded by a second ellipse

$$\frac{(x - 11.5\ \text{in.})^2}{(3.25\ \text{in.})^2} + \frac{y^2}{(6.5\ \text{in.})^2} = 1 \qquad \text{(Ellipse II)}$$

The values are computed from Eqs. 141, 145, 146, and 147.
The decimals are included to show the variations clearly.

AT DISTANCE x FROM CENTER OF ELLIPSE I		STRESSES (LB-IN.$^{-2}$) AT $y = 0$			
		Single Tire		Dual Tires	
x (inches)	Location	σ_x	σ_y	σ_x	σ_y
−3.25	Edge, ellipse I	132.15	136.47	173.25	213.01
0	Center, ellipse I	160.30	147.30	216.34	236.74
1.625	In ellipse I	153.26	144.59	218.52	241.53
3.25	Edge, ellipse I	132.15	136.47	208.35	241.75
4.50	Between ellipses	113.17	128.11	199.43	240.46
5.75	Midpoint	98.31	119.98	196.62	239.96
7.00	Between ellipses	86.26	112.35	199.43	240.46
8.25	Edge, ellipse II	76.20	105.28	208.35	241.75
9.875	In ellipse II	65.26	96.94	218.52	241.53
11.50	Center, ellipse II	56.04	89.44	216.34	236.74
14.75	Edge, ellipse II	41.10	76.54	173.25	213.01

by Eq. 138: $a'_0 = 4.827$ in.; by Eqs. 136, 139, and 140: $B + K' = 2.6406$; by Eq. 66: $S = \frac{1}{3}$; and finally by Eqs. 141 with K replaced by K' the stresses at the center:

$$\sigma'_x = 160.9\ \text{lb-in.}^{-2}, \qquad \sigma'_y = 147.9\ \text{lb-in.}^{-2} \qquad (142)$$

When K is not replaced by K', one finds at the center by Eqs. 65, 139, and 140

$$B + K = 2.6307 \qquad (143)$$

and thereafter by Eq. 141 the stresses at the center

$$\sigma_x = 160.30\ \text{lb-in.}^{-2}, \qquad \sigma_y = 147.30\ \text{lb-in.}^{-2} \qquad (144)$$

which differ only insignificantly from the stresses in Eqs. 142. In the remaining computations K will not be replaced by K'.

When only ellipse I—defined at the head of Table 2—is loaded, the following values apply at points on the axis of x:

Within the ellipse, by Eqs. 143, 65, and 66:

$$B + K = 2.6307 - \frac{1}{3}\left(\frac{x}{a}\right)^2, \quad S = \frac{1}{3} - \frac{4}{9}\left(\frac{x}{a}\right)^2 \ldots\ldots (145)$$

Outside the ellipse, by Eqs. 139, 140, 74, and 75, and with the notation

$$X = x + \sqrt{x^2 + 31.688 \text{ in.}^2} \ldots\ldots (146)$$

the corresponding values are

$$B + K = 0.6159 + \log \frac{73.12 \text{ in.}}{X} - \frac{x}{X}, \quad S = \frac{5}{3} - \frac{16\,x}{3\,X} \ldots (147)$$

The stresses stated in the third and fourth columns of Table 2 were computed from Eqs. 141 and 145 to 147.

When the load $2\,P = 10{,}000$ lb is transmitted through two tires and is distributed uniformly over the area of the two ellipses defined at the head of Table 2, values of the stresses may be obtained by superposition. Since the distance between the centers of the two ellipses is 11.5 in., the stress σ_x or σ_y at the point $x,0$ due to the combined load $2\,P$ is the sum of two stresses produced by the load P over the area of ellipse I; namely, the stress at the point $x, 0$ plus the stress at the point $(11.5 \text{ in.} - x),0$. The last two columns in Table 2 were obtained in this way by simple additions.

CONCLUSION

A pressure concentrated within a small area on the surface of a plate or slab produces a concentration of stress within and around that area. If the loaded area is not too close to any line or point of concentrated support nor to any edge, and if the main dimensions of the loaded area are not too small compared with the thickness of the plate or slab, and if the material of the plate or slab can be considered as homogeneous, isotropic, and elastic, the stresses and deformations in the region of concentration can be expressed by simple formulas in terms of a set of three "place coefficients" and a set of three "area coefficients." Each problem of this kind of stress concentration can be considered as solved when the needed place coëfficients and area coefficients have been determined.

The place coefficients depend on the size and shape of the plate or slab, on the way in which the plate or slab is supported, and on the locality of the load, but do not depend on the size and shape of the loaded area, nor on the distribution of the load over that area. The area coefficients, on the other hand, depend only on the size and shape of the loaded area and on the manner of distribution of the load over that area. Of the three place coefficients or area coefficients in a set one is frequently zero for reasons of symmetry.

The number of problems for which solutions will have been made available at a given time by the method of place coefficients and area coefficients is the product of two numbers; namely, the number of available sets of place coefficients times the number of available sets of area coefficients.

Sets of place coefficients were derived in six examples by inspection of known solutions of particular problems in which the load is concentrated at a single point. The number of such examples can be increased without difficulty, because solutions for a load concentrated at a point are known in more cases than the six that were considered; and new problems of this kind can be solved by the methods of the theory of plates or slabs. Such new solutions will have an extended usefulness because place coefficients can be derived from them, and each new set of place coefficients can be combined with any set of area coefficients.

Area coefficients were derived in twelve cases. The areas confining the load are a circle in Cases 1 to 4, an ellipse in Case 5, narrow strips in Cases 6 to 9, a triangle in Case 10, any polygon in Case 11, and a rectangle in Case 12. While the range of these cases is representative, it will be desirable to have further cases added to the list.

Special devices of analysis are needed in the particular class of problems in which the main dimensions of the loaded area are small compared with the thickness of the plate or slab. Certain "substitute coefficients" were shown to be useful devices serving this purpose.

The use of place coefficients and area coefficients was illustrated by a numerical example. In this example the slab is a pavement, and the load was assumed to be distributed uniformly over the area of two ellipses which represent the area of contact between dual tires and the pavement. The stresses were computed by superposition of the effects identified with each of the two ellipses.

It has been shown that place coefficients and area coefficients can be used with advantage in solving many problems of stress concentrations in plates.

DISCUSSION

GEORGE R. RICH,[21] M. AM. SOC. C. E.—The author has prepared a scholarly paper of great practical interest, in which functions of a complex variable are used to advantage. Since the analytical methods adopted will be valuable for general reference, it is appropriate to indicate that much of the qualifying material preceding Eqs. 107, 108, and 109 follows directly from the specification (from pure mathematics) of conditions that are necessary and sufficient for a function of a complex variable to be analytic.[22, 23] Included in this specification are the familiar Cauchy-Riemann equations which are necessary to verify that Eqs. 107, 108, and 109 satisfy basic Eqs. 104 and 105 when $p = 0$.

In connection with Eqs. 48 and 49, the writer was pleasantly surprised to find that the 1938 edition of the excellent tables of Jahnke and Emde is currently available.[24] However, it is believed that tables of the so-called "ker" and "kei" type of Bessel functions are more commonly used in the United States than those giving Hankel functions of the order of zero for complex values of the argument. Consequently, it may be useful as a reference to give the requisite transformation relation which is developed in the standard treatises:[25, 26, 27]

$$\frac{1}{2}\pi i\, H^{(1)}{}_0\left(\frac{r}{l}\sqrt{i}\right) = \mathrm{ker}\left(\frac{r}{l}\right) - i\,\mathrm{kei}\left(\frac{r}{l}\right) \dots\dots\dots (148)$$

In "ker" and "kei" functions, Eqs. 48 and 49 become:

$$\zeta = \frac{P\,l^2}{4\,N}\,\mathrm{Re}\,H^{(1)}{}_0\left(\frac{r}{l}\sqrt{i}\right) = \frac{P\,l^2}{4\,N}\left(-\frac{2\,\mathrm{kei}\left(\frac{r}{l}\right)}{\pi}\right) \dots\dots (149a)$$

and

$$\Delta\zeta = \frac{P}{4\,N}\,\mathrm{Im}\,H^{(1)}{}_0\left(\frac{r}{l}\sqrt{i}\right) = \frac{P}{4\,N}\left(-\frac{2\,\mathrm{ker}\left(\frac{r}{l}\right)}{\pi}\right) \dots\dots (149b)$$

Modern tables of these functions,[25] with convenient asymptotic expansions[25, 26] for the larger values of the argument, are readily available. A thoroughgoing coordinate treatment of the general theory of Bessel functions is best obtained through the medium of complex variable theory and is preferably introduced by way of the Laplacian equation.[27]

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[21] Chf. Design Engr., TVA, Knoxville, Tenn.

[22] "Theory of Functions of a Complex Variable," by E. T. Copson, Oxford Univ. Press, 1935, p. 40.

[23] "The Theory of Functions," by E. C. Titchmarsh, Oxford Univ. Press, 1939, p. 64.

[24] "Tables of Functions with Formulæ and Curves," by Eugen Jahnke and Fritz Emde (B. G. Teubner, Leipzig and Berlin, and G. E. Stechert and Co., New York, N. Y.), 1938, p. 252.

[25] "Bessel Functions for Engineers," by N. W. McLachlan, Oxford Univ. Press, 1934, p. 170, and Table 10, p. 179.

[26] "A Treatise on Bessel Functions," by Andrew Gray, G. B. Mathews, and T. M. Macrobert, Macmillan and Co., Ltd., London, 1931, pp. 26, 58, and 59.

[27] "Theory of Functions of a Complex Variable," by E. T. Copson, Oxford Univ. Press, 1935, p. 313.

The conversion of the Laplacian operator to cylindrical coordinates, Eq. 3, can be obtained, of course, by elementary methods; but, since the engineer assigned to analytical work has frequent occasion to effect a transformation of many similar types of differential invariants from Cartesian to orthogonal curvilinear coordinates, it may be stated in passing that such conversions are obtained with striking ease and directness by means of the methods of tensor analysis. For example, in the case of Eq. 3: The line element is—

$$\overline{ds^2} = \overline{dR^2} + R^2\,\overline{d\alpha^2} \dots\dots\dots\dots\dots\dots (150a)$$

and the non-zero components of the metric tensor are—

$$g_{11} = 1; \qquad g_{22} = R^2; \qquad g^{11} = 1; \qquad \text{and} \qquad g^{22} = \frac{1}{R^2} \dots (150b)$$

Using Eddington's[28] notation, the tensor expression for the Laplacian operator, valid in any system of coordinates, in space of any number of dimensions, is:[29]

$$g^{\mu\nu}\left(\frac{\partial^2\phi}{\partial x_\mu\,\partial x_\nu} - \{\mu\,\nu,\alpha\}\,\frac{\partial\phi}{\partial x_\alpha}\right),$$

The only non-zero value of the Christoffel three-index symbol in the foregoing case is $\{22, 1\} = -R$, and the conversion is immediate:

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial \alpha^2} \dots\dots\dots\dots (151)$$

The author states that the coefficient K defined by Eqs. 11 and 21 is a logarithmic potential, and mentions electrical potential analogies that exist in connection with the derivation of Eq. 65. A familiar and perhaps closer analogy occurs in the theory of elasticity in which both direct and logarithmic potentials are used in calculating the displacements produced at any point in a solid body by the application of pressure distributed over an area on the bounding plane.[30]

Dana Young,[31] M. Am. Soc. C. E.—The determination of the stresses in a plate in the vicinity of a transverse concentrated load is one of the most troublesome problems in plate analysis. The ordinary theory of bending of plates is not valid for stresses at a point load, and it is necessary to correct the results in accordance with more exact theory or to take into consideration the fact that in most practical applications the load is actually distributed over a small area. A variety of special methods have been used in the past to obtain solutions for a limited number of cases. The author has succeeded in developing an original and powerful treatment of the problem which possesses great generality.

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[28] "The Mathematical Theory of Relativity," by A. S. Eddington, Cambridge Univ. Press, 1937, 2d Ed., pp. 55–64.

[29] "The Theory of Spherical and Ellipsoidal Harmonics," by E. W. Hobson, Cambridge Univ. Press, 1931, pp. 2–8.

[30] "The Mathematical Theory of Elasticity," by A. E. H. Love, Cambridge Univ. Press, 1934, 4th Ed., p. 192.

[31] Prof., Applied Mechanics, The Univ. of Texas, Austin, Tex.

The sets of place coefficients and area coefficients given in the paper provide solutions for many useful cases. Data for simply supported and clamped rectangular plates of finite length were not given in the paper, and it may be of interest to derive the place coefficients for these cases.

For a rectangular plate with simply supported edges at $x = 0$, $x = l$, and $y = \pm \frac{1}{2} l_1$, and with a load P uniformly distributed over a small circular area of radius a, whose center is at $x = v$, $y = 0$, the moments under the center of the load as given by S. Timoshenko[32] are:

$$m_x = \frac{P}{4\pi}\left[(1+\mu) \log \frac{2 l \sin \frac{\pi v}{l}}{\pi a} + 1 + \gamma_1 \right] \ldots\ldots\ldots (152a)$$

and

$$m_y = \frac{P}{4\pi}\left[(1+\mu) \log \frac{2 l \sin \frac{\pi v}{l}}{\pi a} + 1 + \gamma_2 - (1-\mu) \right] \ldots (152b)$$

in which

$$\left.\begin{matrix}\gamma_1\\ \gamma_2\end{matrix}\right\} = 2 \sum_{m=1}^{\infty} \frac{1}{m} \sin^2\left(\frac{m \pi v}{l}\right)\left[(1+\mu)(\tanh \alpha_m - 1) \mp \frac{(1-\mu)\alpha_m}{\cosh^2 \alpha_m} \right] \ldots (153)$$

and $\alpha_m = \dfrac{m \pi l_1}{2 l}\cdot$ Substituting Eq. 153 in Eqs. 152 and rearranging:

$$\left.\begin{matrix}m_x\\ m_y\end{matrix}\right\} = \frac{P}{4\pi}\left\{ (1+\mu)\left[\log\left(\frac{2}{\pi}\sin\frac{\pi v}{l}\right) + \frac{1}{2} + \log\frac{l}{a} + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\sin^2\frac{m\pi v}{l}\right)\right.\right.$$

$$\left.\left.(\tanh \alpha_m - 1)\right] \pm \frac{1-\mu}{2}\left[1 - 4\sum_{m=1}^{\infty}\frac{1}{m}\left(\sin^2\frac{m\pi v}{l}\right)\left(\frac{\alpha_m}{\cosh^2\alpha_m}\right)\right]\right\} \ldots (154)$$

From Eq. 52, with $c = l$, it is noted that $K = \dfrac{1}{2} + \log\dfrac{l}{a}$ and $S = 0$. Then, comparing Eq. 154 with Eq. 13:

$$B = \log\left(\frac{2}{\pi}\sin\frac{\pi v}{l}\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\sin^2\frac{m\pi v}{l}\right)(\tanh \alpha_m - 1) \,.\,. (155a)$$

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[32] "Über die Biegung der allseitig unterstützten rechteckigen Platte unter Wirkung einer Einzellast," by S. Timoshenko, *Der Bauingenieur*, Vol. 3, 1922, p. 51; or "Theory of Plates and Shells," by S. Timoshenko, McGraw-Hill Book Co., Inc., New York, N. Y., 1940, pp. 156–167.

and

$$C = 1 - 4 \sum_{m=1}^{\infty} \frac{1}{m} \left(\sin^2 \frac{m \pi v}{l} \right) \left(\frac{\alpha_m}{\cosh^2 \alpha_m} \right) \ldots\ldots\ldots (155b)$$

When l_1 is very large, $\tanh \alpha_m - 1 \cong 0$, $\frac{\alpha_m}{\cosh^2 \alpha_m} \cong 0$, and Eqs. 155 reduce to Eq. 39 $\left(\text{note that } v = \frac{l}{2} + u \text{ and } \sin \frac{\pi v}{l} = \cos \frac{\pi u}{l} \right)$. The series in the foregoing expressions for B and C converge rapidly. Numerical values for B and C for the case of a central concentrated load $\left(v = \frac{l}{2} \right)$ and several ratios $\frac{l_1}{l}$ are given in Table 3.

TABLE 3.—PLACE COEFFICIENTS FOR SIMPLY SUPPORTED AND CLAMPED RECTANGULAR PLATES WITH A CENTRAL CONCENTRATED LOAD

$\frac{l_1}{l}$	SIMPLY SUPPORTED		CLAMPED	
	B	C	B	C
1.0	−0.6174	0	−1.137	0
1.4	−0.5002	0.5776	−1.049	0.3968
2	−0.4591	0.9065	−1.044	0.5294
∞	−0.4516	1.0000	−1.044	0.538

Notes: $\frac{l_1}{l}$ = ratio of length of plate in y-direction to length in x-direction. Values for B are based on $c = l$. Values for $\frac{l_1}{l_1} = \infty$ are taken from author's paper.

For a similarly loaded rectangular plate, with all four edges clamped, the moments at any point are found by superposing upon the moments for a simply supported plate the moments due to the distributed edge couples of the clamped plate. The solution for a clamped rectangular plate with a central concentrated load has been given,[33] and from this solution it may be shown that the corrections to be added to the place coefficients in Eqs. 155 to obtain the place coefficients for a clamped rectangular plate with a central concentrated load are:

$$B \text{ (correction)} = \frac{4\pi}{P} \sum_{m=1,3,5\ldots} \frac{(-1)^{\frac{m-1}{2}}}{2} \left(\frac{A_m}{\cosh \alpha_m} + \frac{B_m}{\cosh \beta_m} \right) \ldots (156a)$$

and

$$C \text{ (correction)} = \frac{4\pi}{P} \sum_{m=1,3,5\ldots} (-1)^{\frac{m-1}{2}} \left(A_m \frac{\alpha_m \tanh \alpha_m - 1}{\cosh \alpha_m} - B_m \frac{\beta_m \tanh \beta_m - 1}{\cosh \beta_m} \right) \ldots\ldots\ldots\ldots\ldots\ldots (156b)$$

in which $\alpha_m = \frac{m \pi l_1}{2 l}$, $\beta_m = \frac{m \pi l}{2 l_1}$, and A_m and B_m are numerical coefficients with values as given in the reference cited.[33]

Numerical values for the place coefficients for the clamped plate are also given in Table 3.

A. NÁDAI,[34] ESQ.—The paper by Dean Westergaard on concentrated stresses in plates loaded under transversal forces which are distributed over comparatively small areas deserves to be studied most carefully by engineers.

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[33] "Clamped Rectangular Plate with a Central Concentrated Load," by Dana Young, *Journal of Applied Mechanics*, Vol. 6, No. 3, September, 1939, p. A-114; also Vol. 7, No. 1, March, 1940, p. A-43.

[34] Cons. Mechanical Engr., Westinghouse Research Laboratories, East Pittsburgh, Pa.

Those interested in the practical applications of the mathematical theory of bending of elastic plates must welcome this concise study of the effects of concentrated loads. It is an established fact that the foundations for the handling of the problems of the theory of flat plates have long been developed. For example, mathematicians have developed means for determining distributions of certain potential functions in space or over areas, the theory of which can furnish the basis for solving many plate problems. Despite these early developments (and also valuable recent advances made by engineers) for solving such problems, the practical applicability of the bending theory of elastic plates is still somewhat limited by certain facts. It is the intention of this discussion to point out at least a few of these facts, some of which were revealed in the paper.

The methods available for computing stresses in flat plates loaded by concentrated forces belong to this group of questions. It must be noted that certain difficulties that are encountered when the maximum bending stresses in such plates must be evaluated are not based on some inherent weaknesses of the plate theory. Rather, they are due to the fact that engineers dealing with these problems perhaps have not yet applied the special methods which are needed for solving such cases. The contrary is the truth concerning the aforementioned difficulties—namely, that the theory is able to furnish the required and entirely satisfactory means for finding the peak values of the stresses in plates loaded by concentrated forces.

It is well known that in the section in which a single transverse load is applied in a beam the bending moment has a discontinuity. In the bending theory of beams this discontinuity does not introduce any difficulties, and the bending stresses in the vicinity of the area in which a "concentrated load" is applied are computed from the same formula as the stresses in more distant points. The assumption (never mentioned) is that in the practical applications such a load necessarily should be distributed over a sufficiently large contact area, with the adjoining member of the construction transmitting the load, so that no further "local effects" around the pressure area need to be considered in most cases.

When the corresponding case of a concentrated load occurs in a plate, engineers can, and should, take somewhat the same attitude toward the plate theory. The extreme cases of a very strong load concentration in a small area can be excluded from consideration since they also introduce practical difficulties in construction. As Dean Westergaard states clearly, one should distinguish as to whether the concentrated load acts in the interior of the surface of the plate or near or along one of the edge sections. These two cases need somewhat different methods of attack. The present discussion has been confined only to the first case.

Few instances are known to the writer in which authors who have dealt with similar cases have gone to the trouble of establishing the workable expressions needed for expressing the distribution of the bending stresses in the vicinity and within the area through which the concentrated load acts. Although the form of a solution of Lagrange's equation (Eq. 23) for a load which is concentrated in a mathematical point is known in the vicinity of this point, other important properties of the solution rarely have been visualized or discussed by those who make use of these developments in practical applications.

The static feeling of an engineer tells him that it should not, and does not, matter much how a load is actually distributed over an area, which is sufficiently small (not "too small" when compared in its dimensions with the thickness h of the plate and not "too large" when compared with the smallest lateral dimension of the plate, measured parallel to its plane), if one wants to determine the stresses only in points of the plate, which are farther away from the loaded area. Practically very little or no use has been made of this good instinctive feeling in the mathematical applications of the plate theory. In other words, it is useful and helps in the discussion of the case if the load is replaced first by a single force concentrated in a mathematical point. Clearly this is a mathematical abstraction, but a very useful one, among the many other ones used, provided one confines it to its proper limits.

Engineers probably have not been conscious of the fact that finding the proper integral of Lagrange's equation of the plate theory under the given boundary conditions which corresponds to the case of the single force concentrated in a point is the first necessary step toward establishing the expressions required for a computation of the peak stresses also near the finite loaded area. Experience has shown that the deflections and the form of the elastic surface can be computed at any point, including the area loaded, by using this solution.

In the mathematical theory of the distribution of potentials around given bodies or areas, use is made of similar ideas when potential functions are introduced which (1) satisfy the differential equation of the potential, (2) become finite in a single point in a required manner, and (3) satisfy the prescribed boundary conditions along the surface of the body or along curves in the case of two-dimensional problems. These functions are there known as "Green's functions," which solve special cases of boundary conditions. The bending theory of flat plates is closely related to the theory of logarithmic potentials. Evidently, finding the solution for a given contour line of a plate under given boundary conditions and carrying a single force in an arbitrary point is identical with this problem of finding the Green function of Lagrange's plate equation of the fourth order, satisfying given boundary conditions. The computation of the stresses near or within the area in which a concentrated force acts in a plate thus must consist first of a determination of the Green function corresponding to the load concentrated in the center of the loaded area.

After the solution for a single concentrated force in the center of the loaded area has been found or constructed by the methods which are available for such cases the three "place coefficients," B, C, and D, introduced by the author, can be computed readily by comparing it with Eq. 1. It is known, however, that "the logarithmic tube" which expresses the bending stresses in a plate near the point of application of the load increases to infinity near this point, and expressions based on the load concentrated in a point fail at and near the point of application. The author is to be commended for showing clearly that in the most general case three additional functions, K, S, and T, must be constructed as defined in Eqs. 11 and 12. These functions are needed if the bending stresses are to be computed near or within the loaded area. They need to be computed only once for a given pressure distribution and shape of the loaded area and are not dependent on the shape of the contour line of the plate. After these three functions have been constructed, the peak stresses, which are usually found within the loaded area of the plate, can be computed. It is interesting to note that in a few cases in the plate literature evidently these integrals have been constructed implicitly when the complete solutions were found by the ordinary methods of integration.

It is noted that the author has made frequent use of expressions containing functions of the complex variable $z = x + i\,y$. The theories of plane stress and of the bending of plates are both based on the biharmonic equation and closely related to the theory of functions. In plane hydrodynamics when certain patterns of potential flow are discussed the same methods are used.

In this connection perhaps the attempts of M. Bergstraesser,[35] in his doctor's thesis of 1928, may deserve mention. He used the methods of conformal mapping for the numerical construction of the bending moments in parallel strips and in rectangular plates loaded by a single force and satisfying the Navier boundary conditions. His efforts were directed toward the numerical determination of Green's function for the moment sum. The results of his elaborate computations, which were presented in curves plotted for the bending moments m_x and m_y and for the twisting moment m_{xy}, should become even more valuable today if the three place coefficients, *B*, *C*, and *D*, introduced by Dean Westergaard, were computed for the various cases which were discussed in this thesis.

L. A. MacColl,[36] Esq.—The paper by Dean Westergaard makes a valuable contribution to the practical solution of certain problems which are of great importance. It undoubtedly will be welcomed by all engineers who have occasion to work with the subject.

Briefly, the idea is that of expressing approximately such quantities as bending moments, twisting moments, etc., resulting from the application of a moderately concentrated load to an elastic plate, as sums of small numbers of terms, which severally have simple interpretations in terms of the location and distribution of the load.

To the writer the names "place coefficients" and "area coefficients" seem to be unfortunately chosen, for an inspection of the basic formulas shows that they are not coefficients in any very usual sense of the word. By Eq. 13, *B*, *K*, *C*, and *S*—or, more exactly, $\frac{(1+\mu)\,B}{4\,\pi}$, $\frac{(1+\mu)\,K}{4\,\pi}$, $\frac{(1-\mu)\,C}{8\,\pi}$, and $\frac{(1-\mu)\,S}{8\,\pi}$—are bending moments per unit load. Likewise, by Eq. 14, $\frac{(1-\mu)\,D}{8\,\pi}$ and $\frac{(1-\mu)\,T}{8\,\pi}$ are twisting moments per unit load. Hence, the writer would prefer some such names as "place moments" and "area moments," or "location moments" and "distribution moments."

It is suggested in the paper that the so-called area coefficients, *K*, *S*, and *T*, may be determined more easily from the partial differential equations 24 and 25, than from the explicit formulas, Eqs. 11 and 12. The writer doubts that this is true under any very general conditions. Whatever simplification the partial differential equations have afforded in the particular cases considered in the paper has resulted, he thinks, from the fact that those cases are very simple and special, and are not at all typical of the general situation. Certainly, a set of explicit formulas, comparable to Eqs. 11 and 12, is the goal of any method for solving differential equations; and it should be considered fortunate that in the present case the formulas are so simple and so easily interpreted.

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[35] "Versuche mit freiaufliegenden rechteckigen Platten unter Einzelkraftbelastung," by M. Bergstraesser, V.D.I. *Forschungsarbeiten*, Heft 302, 1928.

[36] Bell Telephone Laboratories, Inc., New York, N. Y.

L. W. TELLER,[37] ASSOC. M. AM. SOC. C. E., AND EARL C. SUTHERLAND,[38] ESQ.—That portion of this very useful paper which deals with the action of a large slab on an elastic subgrade is of particular interest to the highway engineer and it is to this particular part of the paper that this discussion will be limited.

In his original analysis of the load-stress relations in a concrete pavement slab,[39] Dean Westergaard assumed that the load was applied to the surface of the pavement over a uniformly loaded small circular area in the cases of (1) a corner loading and (2) a load applied at an interior point some distance from the slab edges and over a uniformly loaded semicircular area for the case of a load applied at an edge at some distance from a corner. For these conditions of loading, formulas for the calculation of the critical load stresses were presented.

In the paper under discussion, the early analysis is extended to permit the computation not only of the critical load stresses but also of other load stresses in the vicinity of the loaded area both for areas that are circular in shape and for areas other than circular in shape. Of the latter, the elliptical area, such as that used by Dean Westergaard in the numerical example (Table 2), is of particular interest since it is a shape which approximates quite closely that of the area of contact of a large pneumatic tire at normal inflation and capacity load.

A number of years ago the Public Roads Administration began a study of the structural behavior of concrete pavement slabs at the Arlington Experiment Farm, Virginia. A part of this investigation was an experimental study of the load-stress relations in pavement slabs of uniform thickness, being, in effect, a comparison of the behavior as predicted by the Westergaard theory with the behavior as actually observed on pavement slabs of ordinary size. Four reports of this work have been published[40] and the fifth (and final) report, which concerns the Westergaard analysis most directly, is nearing completion (August, 1942). The reports already published describe the scope of the investigation and the general procedures and apparatus used.

One of the questions left unanswered in the original analysis by the author was that of the effect of the shape of the loaded area on the magnitude of the critical stress from a given wheel load. So long as it was unanswered it has been the subject of considerable discussion. In an effort to obtain some data on this particular question, a limited program of load tests on areas simulating tire contact areas was made in connection with the aforementioned general investigation.[40] Since these tests concern one of the problems analyzed in the present paper, it is thought that the experimental data may be of interest.

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[37] Prin. Engr. of Tests, Public Roads Administration, Federal Works Agency, Washington, D. C.

[38] Highway Engr., Public Roads Administration, Federal Works Agency, Washington, D. C.

[39] "Computation of Stresses in Concrete Roads," by H. M. Westergaard, *Proceedings*, Fifth Annual Meeting of the Highway Research Board, 1926; also *Public Roads*, Vol. 7, No. 2, April, 1926.

[40] "The Structural Design of Concrete Pavements," reported by L. W. Teller and Earl C. Sutherland.

Part 1—"A Description of the Investigation," *Public Roads*, Vol. 16, No. 8, October, 1935.
Part 2—"Observed Effects of Variations in Temperature and Moisture on the Size, Shape and Stress Resistance of Concrete Pavement Slabs," *ibid.*, No. 9, November, 1935.
Part 3—"A Study of Concrete Pavement Cross Sections," *ibid.*, No. 10, December, 1935.
Part 4—"A Study of the Structural Action of Several Types of Transverse and Longitudinal Joint Designs," *ibid.*, Vol. 17, Nos. 7 and 8, September and October, 1936.

The tests were made on a concrete pavement slab of normal size and of 8-in. uniform thickness. This pavement was laid on a stabilized subgrade formed by incorporating sand with the original subgrade material and compacting to maximum density at the optimum moisture content. Loads were applied on bearing plates cut to a shape which simulated the area of contact of low pressure pneumatic tires. This shape is shown in Fig. 3(*a*). It is a conventionalized average of a fairly large number of actual tire impressions and, even when extended to the larger sizes, seems to conform rather well to available tire imprints. In these comparisons the impressions of airplane tires at recommended capacity load were used.

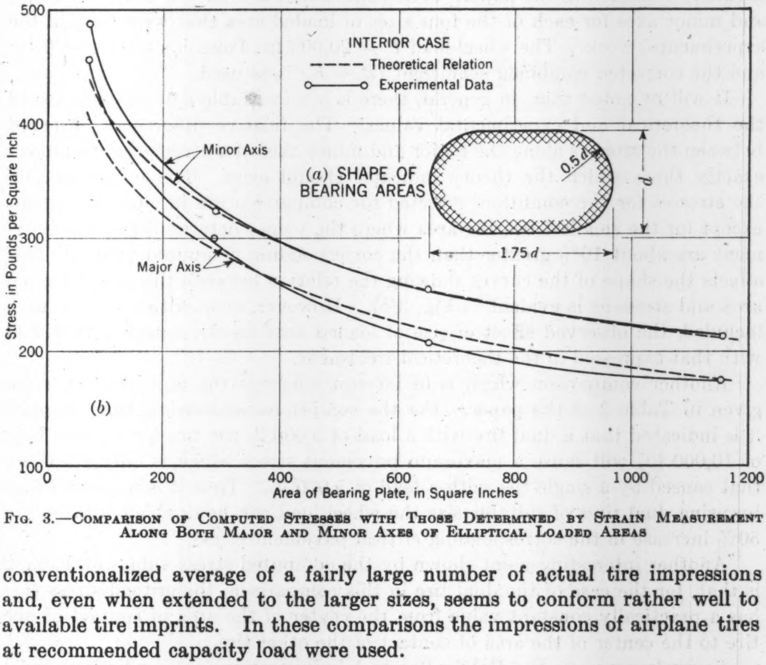

FIG. 3.—COMPARISON OF COMPUTED STRESSES WITH THOSE DETERMINED BY STRAIN MEASUREMENT ALONG BOTH MAJOR AND MINOR AXES OF ELLIPTICAL LOADED AREAS

In the loading tests four bearing plates of this conventionalized shape and of the following sizes were used:

Width (minor axis, *d*) in inches	Length (major axis) in inches	Area, in square inches
6.86	12.00	72
13.71	24.00	289
20.57	36.00	649
27.43	48.00	1,155

The loading apparatus was arranged so as to distribute the applied load uniformly over the area of contact of the bearing plate. Strains were measured in the upper surface of the pavement at the center of the bearing area along both the major and minor axes of the area. The gage used was the recording gage described in the first of the published reports mentioned previously.[40]

The modulus of elasticity of the concrete E equals 4,800,000 lb per sq in., and the modulus of subgrade reaction k equals 250 lb per cu in., as determined by deflection tests on the pavement slab in the manner suggested by Dean Westergaard in one of his earlier papers.[41]

Fig. 3(*b*) shows a comparison between the observed stresses in the aforementioned 8-in. pavement slab and corresponding stresses computed by the methods described in the paper. The comparison is made for both the major and minor axes for each of the four sizes of loaded area that were used in the experimental work. The wheel-load $P = 20{,}000$ lb; Poisson's ratio $\mu = 0.15$; and the corrected combined coefficient $(B + K)'$ was used.

It will be noted that, in general, there is a remarkable agreement between the theoretical and experimental values. The relative differences observed between the stresses along the major and minor axes of a given area are almost exactly those which the theory indicates should exist. The magnitudes of the stresses for the conditions selected for comparison are in close agreement except for the smallest bearing area where the values determined by measurement are about 10% greater than the corresponding computed values. This affects the shape of the curves showing the relation between the size of loaded area and stress as is evident in Fig. 3(*b*). However, even with this exception included, the observed effect of size of loaded area on stress agrees very well with that expressed in the theoretical treatment.

Another comparison which is of interest concerns the numerical example given in Table 2 of the paper. For the conditions assumed in this example, it is indicated that a dual tire with a load of 5,000 lb per tire (or a wheel load of 10,000 lb) will cause a maximum pavement stress which is only 1.5 times that caused by a single tire with a load of 5,000 lb. Thus it is indicated that by using dual tires of suitable size the wheel load can be doubled with only a 50% increase in the corresponding critical pavement stress.

Another interesting point shown by the computed stress values of Table 2 is that, for the case of the dual tire in this comparison, the critical stress (σ_y) has a practically constant value from the center of the area of contact of one tire to the center of the area of contact of the other tire.

Several years ago the Public Roads Administration made a limited series of load tests in which an effort was made to compare the stress produced by a given load when the load was applied through a single circular bearing plate with the stress developed when twice that load was applied through two plates of the same size, separated by a distance comparable to the dual tire spacings of large tires. Although the program was quite limited and the data apply only to circular areas of contact, the tests afford an interesting comparison with the indications of Table 2.

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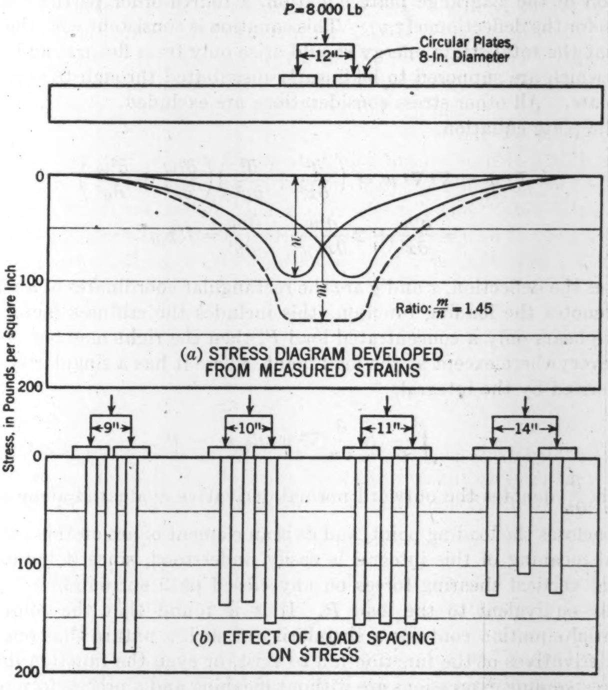
[41] "Analytical Tools for Judging Results of Structural Tests of Concrete Pavements," by H. M. Westergaard, *Public Roads*, Vol. 14, No. 10, December, 1933.

In the tests referred to, loads were applied on an 8-in. circular plate in the interior region of a slab of uniform thickness. Strains were measured in the direction of the longitudinal axis of the pavement in the center of the loaded area and at various distances from the center on a line perpendicular to the longitudinal axis of the pavement through the center of the loaded area. From these strain data a curve was developed showing how the magnitude of the longitudinal stress varied across the pavement slab as the distance from the loaded area was changed.

With this stress-variation curve as a basis, the effect of two loads was examined by combining two such curves displaced horizontally 12 in. with respect to each other as shown in Fig. 4(*a*) and obtaining combined stress values by addition. This study indicated that a load of 8,000 lb, applied through dual 8-in. diameter areas, would cause a stress 1.45 times as great as would be caused by 4,000 lb applied through a single area of the same size.



(*a*) STRESS DIAGRAM DEVELOPED FROM MEASURED STRAINS

(*b*) EFFECT OF LOAD SPACING ON STRESS

Fig. 4.—Stress Conditions Under Dual Loads; Interior of Slab

Fig. 4(*b*) shows some additional data obtained in the same experiments by tests in which strains were measured under and between the loaded areas.

It will be noted that both the combined stress curve of Fig. 4(*a*) and the data for the 11-in. spacing of Fig. 4(*b*) indicate that the stress between the loaded areas is practically equal to that within the loaded areas.

Although the data presented in this discussion are quite limited in scope and apply to but one part of the paper under discussion, the close agreement between the behavior predicted by the theory and that observed during the loading tests tends to inspire confidence in the general soundness of the theoretical treatment.

D. L. HOLL,[42] ESQ.—Any attempt to analyze the stresses that arise in areas of a plate which are subjected to high concentration of normal loading is subject to the limitations of the theory used. The author carefully has presented the hypothesis of the Poisson-Kirchhoff theory, which is used in the derivation of the Lagrange plate equation, a fourth-order partial differential equation for the deflection $w(x,y)$. This equation is consistent with the requirement that the total strain energy should arise only from flexural and torsional stresses which are supposed to be linearly distributed through the thickness h of the plate. All other stress considerations are excluded.

In the plate equation,

$$\nabla^4 w = \nabla^2 \nabla^2 w = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$
$$= \frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^4 w}{\partial x^2\, y^2} + \frac{\partial^4 w}{\partial y^4} = f(x,y) \ldots\ldots\ldots\ldots\ldots (157)$$

w denotes the deflection, x and y are the rectangular coordinates of a point, and $f(x,y)$ denotes the loading function (this includes the stiffness factor N). If the plate bears only a concentrated load P, then the right member of Eq. 157 is zero everywhere except at the load point, where it has a singularity which is characterized by the integral,

$$\int - N \frac{\partial}{\partial n}(\nabla^2 w)\, ds = - P \ldots\ldots\ldots\ldots\ldots (158)$$

in which $\frac{\partial}{\partial n}$ denotes the outward normal derivative evaluated along any path which encloses the loading point, and ds is an element of arc on this path. The physical meaning of this integral is easily understood, since it indicates that the total vertical shearing forces on any closed path surrounding the load is statically equivalent to the load P. If it is found that the solution of a differential equation contains a singularity of such a nature that one or more of the derivatives of the function fail to exist, or even the function itself, then the corresponding expressions are without meaning and a precise formulation of the result is not possible. This is not a weakness of the theory but a consequence of the mathematical abstraction—namely, the representation of a point load by some type of function which shows a sudden discontinuity.

Not infrequently a point load is considered as the limiting case of allowing a uniform pressure distribution over a finite area to increase indefinitely in magnitude as the area becomes infinitesimal in such a manner that the product of pressure times area remains finite and equal to the load P. As an example, a pressure p_o is uniform over a circle of radius c; then a concentrated load is

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[42] Prof., Dept. of Mathematics, Iowa State College, Ames, Iowa.

$$P = \left[\begin{matrix} \text{limit} \\ p_o \rightarrow \infty \\ c \rightarrow 0 \end{matrix} \right] \pi p_o c^2 \quad \text{(159)}$$

The theory of elasticity is based upon the hypothesis that the induced stresses and strains are limited to those which are admissible when the linear relationship is given by Hooke's law. If the area over which a load is distributed is decreased indefinitely, and at the same time the resultant load is to remain constant, then the basic assumption of this law is violated. This does not imply that such concepts as concentrated forces or moments find no place in the theory, but rather that by introducing the fiction of a point load one can obtain more general results at distances somewhat removed from the zone of high stress concentration. It is known that the latter values are valid by Saint Venant's principle. Another important use of the concept of a concentrated load is made when one knows that the principle of reciprocity holds for both flexural and twisting moments as well as for deflections.

The specific contribution of this paper is the fact that the stresses arising from the bending and twisting moments which are induced in a thin plate by normal loads can be characterized to a fair degree of accuracy by two sets of dimensionless numbers called "place coefficients" and "area coefficients." The latter are independent of the former and depend only on the size and shape of the loaded area and on the distribution of load over that area. These area coefficients are defined by

$$K = \frac{-1}{P} \int_A p \log \frac{R}{c} \, dA \quad \text{(160)}$$

and

$$S, T = -\frac{1}{P} \int_A p \, (\cos 2\alpha, \sin 2\alpha) \, dA \quad \text{(161)}$$

in which dA is an element of the loaded area A over which the total load P has a local pressure p, and R is the distance from the element at (u,v) to the point x,y where the stress intensity is desired. The angle α is given by

$$R \sin \alpha = y - v, \qquad R \cos \alpha = x - u \quad \text{(162)}$$

The area coefficient K in Eq. 160 may be defined in terms of a potential problem. It gives the logarithmic potential at (x,y) of an infinitely long rod of section A over which the mass distribution is of density p. One should note that the quantity c in Eq. 160 can be chosen arbitrarily and is generally some representative length either of the plate or of the area of loading, and thus the value of K is not an absolute number, but a relative one. This coefficient also satisfies the differential equations

$$\nabla^2 K = 0 \text{ (in regions of no loading)} \quad \text{(163}a\text{)}$$

and

$$\nabla^2 K = -\frac{2\pi p}{P} \text{ (within regions of loading)} \quad \text{(163}b\text{)}$$

In general, the determination of the value of K at an arbitrary point is not an elementary problem. Its existence is conditioned by the nature of the loading, but more specifically by the corners and sharp angular points along the boundary, and by line distributions. That these difficulties are recognized by the author is shown by his listing of three conditions for the radially symmetric case (paragraph following Eq. 102), three conditions for the general case (Eq. 106), and three additional observations (Eqs. 107 to 116) when analytic functions of complex variables are used. It is the writer's opinion that a greater use of the mathematical tool of analytic function theory does not make the problem more difficult, but rather tends to clarify it.

The author uses three "place coefficients," B, C, and D, which depend upon the size and shape of the plate, upon the conditions of support at the edge, and upon the position of the load relative to the edges; but these coefficients do not depend upon the load distribution. The reason for limiting them to three such coefficients is that it is assumed that one is interested in stress concentrations near small loading areas in which the deflection function can be suitably represented by a term containing an appropriate singularity and by polynomial terms of not more than the second degree. These place coefficients must be determined by examining known solutions of plate problems under given edge conditions. Since these place coefficients arise as coefficients of polynomial solutions of $\nabla^4 w = 0$ (that is, of biharmonic polynomials which are symmetric in the coordinate pairs (x,y) and (u,v)), the number of coefficients will usually be greater than three. The number depends upon the edge conditions, the shape of the plate, and the position of the load. If terms of third order are required to express the deflection of the plate, the author gives Eqs. 29 and 30 for the supplementary moments. These equations show the need of ten additional constants. In the six examples cited by the author, symmetry conditions yield a zero value for the place coefficient D and, of the remaining two, B is usually the more important one.

As stated at the beginning of this discussion, the stress concentration under a point load is infinite. To avoid this difficulty Dean Westergaard invokes the theory of elasticity in three dimensions, thereby considering all the stresses in a plate. Hence, when the loaded area is very small, "substitute coefficients," K', S', and T', for K, S, and T, respectively, are determined in such a way that the new coefficients define the correct stresses on the bottom surface of the plate and directly under the center of the small area of loading. This amounts to the determination of an equivalent radius of loading such that, by using this new radius in the definitions of K, S, and T, the calculated stresses agree with the actual stresses as evaluated from the three-dimensional or thick-plate theory. It is the writer's opinion that Eqs. 129 and 130 are valid when there is no support on the lower surface of the plate near to the area of loading. One raises the question as to the validity of using this equivalent radius in a plate on an elastic subgrade. It would seem rational that such an equivalent radius should be a function not only of the actual loading radius and of the plate thickness, but also of the stiffness of the subgrade or of the ratio of relative stiffness of the plate to the subgrade. Until such an equivalent or substitute radius is found, one may accept the author's substitute coefficient K' for K, although he states that the corrected values S and T may be ignored in most practical problems.

The author also gives three examples in which the combination of area and place coefficient $(B + K)$ is corrected to $(B + K)'$ when it seems that the area of loading is too large to be rated as small. This determination of a new value for $(B + K)$ becomes rather difficult at any arbitrary point, but can be obtained at the center of the loading area if it is known that the reciprocal theorem for moments is valid. In view of the conditions limiting the use of these coefficients, it appears that the designing engineer will need to have a reasonably thorough understanding of their properties before he can utilize them in a new situation.

The writer wishes to supplement this discussion with some additional examples. The first two examples are to illustrate cases where the deflection of the plate contains polynomial terms far beyond the second degree.

Example 1. Clamped Square Plate with Central Point Load.—Let the side of the square be of length a, and the origin of the coordinate axes be at the geometrical center with axes directed parallel to the edges of the plate. The deflection is given by

$$w(x,y) = \frac{P}{8\,\pi\,N}\left[\,R^2 \log\frac{R}{a} + 0.5625 - 0.5562\,R^2 - \frac{0.0757}{a^2}\,(x^4 - 6\,x^2\,y^2 + y^4) + \frac{0.0654}{a^4}\,(x^6 - 5\,x^4\,y^2 - 5\,x^2\,y^4 + y^6) - \frac{0.00035}{a^6}\,(x^8 - 28\,x^6\,y^2 + 70\,x^4\,y^4 - 28\,x^2\,y^6 + y^8) + \cdots\right] \qquad (164)$$

The place coefficients are $B = -\,0.4438$ and $C = D = 0$. All the polynomial terms of higher degree are solutions of $\nabla^4\,w = 0$. One may readily compute the correction to the moments arising out of the supplemental terms.

Example 2. Centrally Loaded Square Plate Having Two Opposite Edges Clamped and the Other Edges Free.—The deflection is:

$$w(x,y) = \frac{P}{8\,\pi\,N}\left[\,R^2 \log\frac{R}{a} + 0.7783 + 0.0436\,R^2 - 0.3843\,(x^2 - y^2) + \frac{0.0183}{a^2}\,(x^4 - 6\,x^2\,y^2 + y^4) + \frac{0.2211}{a^2}\,(x^4 - y^4) + \cdots\right] \qquad (165)$$

The place coefficients are $B = -\,1.0436$, $C = 0.7886$, and $D = 0$. Comparing these coefficients with a clamped infinite strip (Example *e* of the paper, in which $B = -\,1.044$, $C = 0.538$, and $D = 0$), one notes an apparent correlation in one case and a considerable divergence in the other. These coefficients also may be compared with the values $B = -\,0.4516$, $C = 1$, and $D = 0$ in Example *d* for a simply supported infinite plate strip symmetrically loaded by a single concentrated load. It is to be noted that, in the term $\log\dfrac{R}{a}$, the distance a can be chosen arbitrarily, but the choice of this distance will affect the value of B; the coefficient B attains its definite value after a has been chosen. In the foregoing examples, a is the width of the plate strip.

Example 3. Simply Supported Equiangular Plate; Concentrated Load at the Centroid.—Let the origin of a rectangular coordinate system be at the middle of an edge of the plate and the x-axis directed along an altitude of the triangular plate. Let the load be applied at $\left(\frac{a}{3}, 0\right)$, in which a is the altitude. By a summation process for the cumulative effects of an infinite number of alternating image loads, S. Woinowsky-Krieger[43] finds that the moments at points on the x-axis may be expressed with sufficient accuracy by

$$M_x, M_y = \frac{(1+\nu) P}{4\pi}\left[\log\frac{\sqrt{3}}{\pi} - 0.379 - \log\frac{R}{a}\right] \mp \frac{(1-\nu) P}{8\pi} \quad (166)$$

This yields a value of $B = -1.277$ and $C = D = 0$. If, in the term $\log\frac{R}{a}$, one uses the radius $c = \frac{a}{3}$ of the inscribed circle, then the expression in brackets in Eq. 166 is $\left(-0.179 - \log\frac{R}{c}\right)$ and B is -0.179. This indicates the change in the place coefficient B when the length c in the term $\log\frac{R}{c}$ is varied.

One may compare the moments in Eq. 166 with the moments M_r and M_t in a simply supported circular plate of radius a_o. These moments are:

$$M_r = -\frac{(1+\nu) P}{4\pi}\log\frac{R}{a_o} \quad (167a)$$

and

$$M_t = -\frac{(1+\nu) P}{4\pi}\log\frac{R}{a_o} + \frac{(1-\nu) P}{4\pi} \quad (167b)$$

Hence, the stresses in the equiangular plate near the load are smaller by $\frac{(1-\nu) P}{8\pi}$ than those in a circular plate of radius,

$$a_o = \frac{a\sqrt{3}}{\pi} e^{-0.379} = 0.279\, a \quad (168)$$

If the load P is uniformly distributed over a circle of radius c centered at the centroid of the equiangular plate, then it is known[44] that at the center of loading the moments $M_x = M_y$ are greater by $\frac{P}{4\pi}$ than M_r at the boundary of the circle c. This is true for any symmetrical figure. Hence, in the triangular plate these maximum moments are:

$$M_x = M_y = \frac{(1+\nu) P}{4\pi}\left(\log\frac{a\sqrt{3}}{\pi c} + 0.121\right) = \frac{1+\nu}{4\pi}\left(\log\frac{a}{c} - 0.474\right) \quad (169)$$

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[43] "Berechnung der ringsum frei aufliegenden gleichseitigen Dreiecksplatte," by S. Woinowsky-Krieger, *Ing-Archiv*, Vol. IV, pp. 254–262.

[44] "Die elastischen Platten," by A. Nádai (Julius Springer, Berlin), 1925, p. 62.

S. Timoshenko[45] utilizes this superposition principle in finding the maximum moments under a circular loading centered upon a rectangular plate which has all its edges simply supported. These moments are:

$$M_x = \frac{P}{4\,\pi}\left[(1+\nu)\log\frac{a}{2\,c} + (1+\gamma_1)\right] \ldots\ldots\ldots\ldots (170a)$$

and

$$M_y = \frac{P}{4\,\pi}\left[(1+\nu)\log\frac{a}{2\,c} + (\nu+\gamma_2)\right] \ldots\ldots\ldots\ldots (170b)$$

in which a is the width of the plate, c is the radius of loading, and γ_1 and γ_2 are coefficients which depend upon the plate ratio $\frac{b}{a}$. For varying ratios, $\frac{b}{a}$, these coefficients are listed in Table 4.

TABLE 4.—VALUES OF PLACE COEFFICIENTS AND COEFFICIENTS γ, IN EQS. 170

Coefficient	PLATE RATIOS, $\frac{b}{a}$						
	1.0	1.2	1.4	1.6	1.8	2.0	∞
γ_1	−0.250	−0.035	0.103	0.190	0.242	0.273	0.314
γ_2	0.450	0.431	0.400	0.369	0.348	0.337	0.314
B	−0.616	−0.541	−0.500	−0.478	−0.466	−0.459	−0.452
C	0	0.34	0.58	0.74	0.85	0.91	1

The area coefficients K, S, and T, evaluated at the center of an area loading of radius c, are $K = \frac{1}{2} + \log\frac{a}{c}$ and $S = T = 0$. Comparing Eq. 170b with Eq. 13:

$$M_x, M_y = \frac{P}{4\,\pi}(1+\nu)(B+K) \pm \frac{P}{8\,\pi}(1-\nu)(C+S) \ldots\ldots (171)$$

The B and C coefficients are listed in Table 4.

In the determination of these values of the place coefficients B and C, the writer has assumed that a value $\nu = 0.3$ was used by Professor Timoshenko in obtaining the γ_1 and γ_2 values. This calculation yields the correct values of B and C for the infinite strip $\frac{b}{a} = \infty$ and for C when $\frac{b}{a} = 1$. The values of B and C in Table 4 are particular values of these constants for a centrally placed load, for rectangular plates with "pimeed" edges, and for a length c which is chosen as the radius of the circle over which the load is distributed. Furthermore one should use these coefficients only in Eq. 171 when determining the moments at the center of the loading area.

Application of Analytic Function Theory.—Let $\phi(z)$ and $\psi(z)$ denote analytic functions of the complex variable $z = x + i\,y$. Then $\bar{z} = x - i\,y$ is the conjugate variable and $\overline{\phi(z)}$ and $\overline{\psi(z)}$ denote the conjugate functions of $\bar{z}$ obtained from ϕ and ψ, respectively, by replacing all terms containing i by $-i$. The following results are readily obtained:

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[45] "Über die Biegung der allseitig unterstützten Rechteckigen Platte unter Wirkung einer Einzellast," by S. Timoshenko, *Bauingenieur*, Vol. 3, 1922, pp. 51–54.

$$x = \tfrac{1}{2}(z + \bar{z}), \quad y = \frac{z - \bar{z}}{2\,i} \qquad \text{(172}a\text{)}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \quad \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right) \qquad \text{(172}b\text{)}$$

and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right) \qquad \text{(172}c\text{)}$$

Since ϕ and ψ are analytic, both their real and imaginary parts satisfy $\nabla^2 = 0$, and a solution of a thin-plate problem having a single concentrated load can be put in the form:

$$w = \frac{P}{16\,\pi\,N}\left[z\,\bar{z}\log z\,\bar{z} + \phi(z) + \bar{z}\,\psi(z) + z\,\overline{\psi(z)} + \overline{\phi(z)}\right] \qquad (173)$$

Eq. 173 satisfies Eq. 157 when the right member of the latter is zero (that is, when $\nabla^4 w = 0$). The first term of Eq. 173 is the necessary singularity which satisfies Eq. 158. The entire expression for w in Eq. 173 is real. The functions ϕ and ψ can contain no singularities and one seeks to determine them by satisfying two boundary conditions along each arc of the boundary of the plate.

To meet various boundary conditions, the following reference formulas will be needed:

$$2\,\frac{\partial w}{\partial \bar{z}} = \frac{\partial w}{\partial x} + i\,\frac{\partial w}{\partial y} = \frac{P}{8\,\pi\,N}\left[z\,(1 + \log z\,\bar{z}) + \psi(z) + z\,\overline{\psi'(z)} + \overline{\phi'(z)}\right] \qquad (174)$$

$$\begin{aligned}\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial z^2} + 2\,\frac{\partial^2 w}{\partial z\,\partial \bar{z}} + \frac{\partial^2 w}{\partial \bar{z}^2} = \frac{P}{16\,\pi\,N}\biggl\{&\frac{z^2 + z^2}{z\,\bar{z}} + 4 + 2\log z\,\bar{z}\\ &+ 2\,\mathrm{Re}\left[\phi''(z) + \bar{z}\,\psi''(z) + 2\,\psi'(z)\right]\biggr\}\end{aligned} \qquad (175)$$

$$\begin{aligned}\frac{\partial^2 w}{\partial y^2} = -\left(\frac{\partial^2 w}{\partial z^2} - 2\,\frac{\partial^2 w}{\partial z\,\partial \bar{z}} + \frac{\partial^2 w}{\partial \bar{z}^2}\right) = \frac{-P}{16\,\pi\,N}\biggl\{&\frac{z^2 + z^2}{z\,\bar{z}} - (4 + 2\log z\,\bar{z})\\ &+ 2\,\mathrm{Re}\left[\phi''(z) + \bar{z}\,\psi''(z) - 2\,\psi'(z)\right]\biggr\}\end{aligned} \qquad (176)$$

$$\nabla^2 w = \frac{4\,\partial^2 w}{\partial z\,\partial \bar{z}} = \frac{P}{4\,\pi\,N}\left[2 + \log z\,\bar{z} + \psi'(z) + \overline{\psi'(z)}\right] \qquad (177)$$

$$\begin{aligned}\frac{\partial^2 w}{\partial x\,\partial y} &= i\left(\frac{\partial^2 w}{\partial z^2} - \frac{\partial^2 w}{\partial \bar{z}^2}\right)\\ &= \frac{i\,P}{16\,\pi\,N}\left\{\frac{z^2 - \bar{z}^2}{z\,\bar{z}} + \mathrm{Im}\left[\phi''(z) + \bar{z}\,\psi''(z)\right]\right\}\end{aligned} \qquad (178)$$

$$\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} = 2\left(\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial \bar{z}^2}\right) = \frac{P}{16\,\pi\,N}\left[8 + 4\log z\,\bar{z} + 8\,\mathrm{Re}\,\psi'(z)\right] \qquad (179)$$

$$M_x = -N\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = -N\nabla^2 w + N(1-\nu)\frac{\partial^2 w}{\partial y^2} \ldots (180)$$

$$M_y = -N\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) = -N\nabla^2 w + N(1-\nu)\frac{\partial^2 w}{\partial x^2} \ldots (181)$$

$$V_x = -N\frac{\partial}{\partial x}\nabla^2 w = \frac{-P}{4\pi}\left[\frac{z+\bar z}{z\bar z} + \psi''(z) + \overline{\psi''(z)}\right] \ldots\ldots (182)$$

$$V_y = -N\frac{\partial}{\partial y}\nabla^2 w = \frac{-iP}{4\pi}\left[\frac{\bar z - z}{z\bar z} + \psi''(z) - \overline{\psi''(z)}\right] \ldots\ldots (183)$$

$$M' = \frac{-N}{2}(1+\nu)\nabla^2 w = -N(1+\nu)\,2\frac{\partial^2 w}{\partial z\,\partial\bar z} \ldots\ldots\ldots (184)$$

$$M'' = \frac{-N}{2}(1-\nu)\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}\right) = -N(1-\nu)\left(\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial \bar z^2}\right) \ldots (185)$$

$$M_x = M' + M'' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (186)$$

$$M_y = M' - M'' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (187)$$

In Eqs. 174 to 187, $\psi'(z)$ and $\overline{\phi'(z)}$ denote $\frac{\partial\psi}{\partial z}$, $\frac{\partial\phi}{\partial\bar z}$, etc.; Re means "real part" and Im means "imaginary part." All the right members of these equations are real.

Example 4. Clamped Circular Plate with Central Point Load.—The point load P is located at the center of a clamped plate of radius a. If the edge of the plate is clamped horizontally at $x^2 + y^2 = a^2$, then the right member of Eq. 174 vanishes when $z\bar z = a^2$, and

$$z(1 + \log z\bar z) + \psi(z) + z\psi'(z) + \overline{\phi'(z)} = 0 \ldots\ldots\ldots (188)$$

with $z = a e^{i\theta}$. Since $\phi(z)$ and $\psi(z)$ are analytic within $|z| \leqq a$, they may be represented as simple power series:

$$\phi(z) = b_o + b_1 z + b_2 z^2 + \cdots \ldots\ldots\ldots\ldots\ldots (189a)$$

and

$$\psi(z) = c_o + c_1 z + c_2 z^2 + \cdots \ldots\ldots\ldots\ldots\ldots (189b)$$

Inserting Eqs. 189 in Eq. 188 for boundary values of z, one readily determines:

$$c_1 = -\tfrac{1}{2}(1 + 2\log a), \qquad b_1 = -c_o \ldots\ldots\ldots\ldots (190a)$$

$$c_n = b_n = 0, \quad \text{for} \quad n \geqq 2 \ldots\ldots\ldots\ldots\ldots (190b)$$

and

$$w(r) = \frac{P}{8\pi N}\left(r^2\log\frac{r}{a} + b_o - \frac{r^2}{2}\right) \ldots\ldots\ldots\ldots (190c)$$

Eq. 190c is obtained from Eq. 173 and the constant b_o will be $\frac{a^2}{2}$ if the deflection is measured from the plane of the clamped edge—that is, $w = 0$ at $r = a$. With this value of b_o, Eq. 190c is the required solution, and the place coefficients are $B = -\frac{1}{2}$ and $C = D = 0$.

Example 5. Circular Plate Centrally Loaded and Naviér Édge Conditions.—This edge condition requires that $\nabla^2 w = 0$ at $z\bar{z} = a^2$. The right member of Eq. 177 vanishes where $z = a\,e^{i\theta}$; that is,

$$2 + \log z\bar{z} + \psi'(z) + \overline{\psi'(z)} = 0 \qquad (191)$$

The coefficients in Eq. 189b become $c_1 = -(1 + \log a)$ and $c_n = 0$ for $n \geqq 2$. The property of radial symmetry requires that $c_o = 0$ and $b_n = 0$ for $n \geqq 1$, and from the deflection at the edge, $b_o = a^2$; hence,

$$w = \frac{P}{8\pi N}\left(r^2 \log\frac{r}{a} + a^2 - r^2\right) \qquad (192)$$

The place coefficients are $B = C = D = 0$. The physical meaning of the imposed boundary condition is that the moment sum vanishes at $r = a$; that is,

$$M_x + M_y = M_r + M_\theta = 0 \qquad (193)$$

This can be realized in a circular plate by clamping the edge so that the slope is given by

$$\frac{dw}{dr} = -\frac{P\,a}{4\pi N} \qquad (194)$$

at the boundary. In a polygonal plate where the edges suffer no deflection this condition is equivalent to simply supported edges.

Example 6. Simply Supported Circular Plate Centrally Loaded.—The edge condition requires that

$$M_r = -N\left[\nabla^2 w - (1-\nu)\frac{dw}{r\,dr}\right]$$
$$= -N\left[\nabla^2 w - (1-\nu)\left(\frac{1}{\bar{z}}\frac{\partial w}{\partial z} + \frac{1}{z}\frac{\partial w}{\partial \bar{z}}\right)\right] = 0 \qquad (195)$$

at $z\bar{z} = a^2$. One may use this operator in Eq. 173 and evaluate it at the boundary. Symmetry conditions and the vanishing of the boundary deflection lead to

$$w = \frac{P}{8\pi N}\left[r^2\log\frac{r}{a} + \frac{3+\nu}{2(1+\nu)}(a^2 - r^2)\right] \qquad (196)$$

It is natural for the reader to inquire as to what has been gained by using two unknown analytic functions. The answer is not immediately given. The simplicity of Examples 4, 5, and 6 is such that the method of this analysis seems unnecessarily cumbersome, since other more direct methods are available. In the following examples, in which the radial symmetry is not present, the writer uses the method of mapping, conformally, the area of the given plate. This plate is considered to occupy a region of the complex z-plane which is mapped into a region of the complex ζ-plane, which is entirely within a circle of unit radius. The boundary of the plate in the z-plane becomes the boundary of the unit circle $|\zeta| = 1$ in the new plane.

Examples of Mapping to a Unit Circle.—Let $z = z(\zeta)$ be a mapping transformation by which a 1 : 1 correspondence is established between the boundary points of the edge of the plate and the point on the periphery of the unit circle in the ζ $(= \xi + i\,\eta)$-plane. Furthermore, let the transformation take the load point $z = z_o$ into the origin $\zeta = 0$ of the circle.

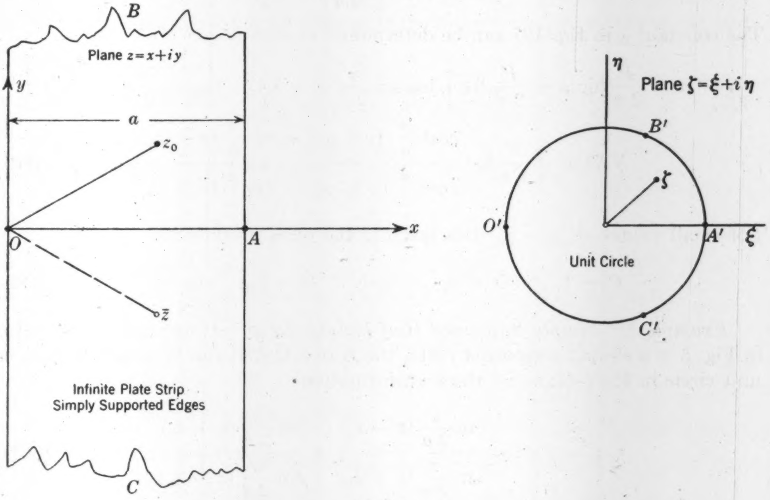

FIG. 5

First, one may treat the case of a polygonal plate with rectilinear edges which are simply supported so that the boundary condition is $\nabla^2 w = 0$. The moment sum $M' = \dfrac{M_x + M_y}{2}$ is proportional to $\nabla^2 w$ and hence satisfies $\nabla^2 M' = 0$ at all points in the plate except at the point where the concentrated load is located. The right member of Eq. 177 shows that M' has a singularity of the type $\log z\,\bar{z}$. It is known from the theory of logarithmic potential[46] that the function M' can be determined from

$$M' = \kappa \log |\zeta| \ldots\ldots\ldots\ldots\ldots\ldots\ldots (197)$$

in which $\zeta = \zeta(z)$ is the aforementioned mapping function and κ is a constant to be determined to meet the singularity condition of a point load given by Eq. 158. Some examples of this method follow.

Example 7. Simply Supported Infinite Plate Strip; Point Load.—This is Example *c* of the paper.[47] The simply supported infinite plate strip of width

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[46] "Foundations of Potential Theory," by Oliver Dimon Kellogg (Julius Springer, Berlin), 1929, p. 365.

[47] See also, "Die elastischen Platten," by A. Nádai (Julius Springer, Berlin), 1925.

a has the coordinate axis and load point z_o shown in Fig. 5. This figure also shows the unit circle in the ζ-plane and the correspondence of points O, C, A, and B on the boundary of the infinite plate with the points O′, C′, A′, and B′, respectively. The transformation which transacts this mapping in such a manner that z_o goes into the origin $\zeta = 0$ is:

$$\zeta = \rho\, e^{i\lambda} = \frac{\sin \frac{\pi}{2\,a}(z - z_o)}{\sin \frac{\pi}{2\,a}(z + \bar{z}_o)} \quad \dots\dots\dots\dots\dots\dots (198)$$

The constant κ in Eq. 197 can be determined so that

$$N\,\nabla^2 w = \frac{P}{2\,\pi}\log \rho = \frac{P}{2\,\pi}\,\mathrm{Re}\left[\log \sin \frac{\pi}{2\,a}(z - z_o) - \log \sin \frac{\pi}{2\,a}(z + \bar{z}_o)\right]$$

$$N\,\nabla^2 w = \frac{P}{4\,\pi}\log \frac{\cosh \frac{\pi}{a}(y - y_o) - \cos \frac{\pi}{a}(x - x_o)}{\cosh \frac{\pi}{a}(y - y_o) - \cos \frac{\pi}{a}(x + x_o)} \quad \dots\dots\dots (199)$$

For small values of $|z - z_o|$ this leads to the place coefficients

$$C = 1, \qquad D = 0, \qquad \text{and} \qquad B = \log \frac{2}{\pi}\sin \frac{\pi\,x_o}{a} \quad \dots\dots\dots (200)$$

Example 8. Simply Supported Half Infinite Strip.—If one half of the strip in Fig. 5 is a simply supported plate, the region O A B can be mapped upon a unit circle in the ζ-plane by the transformation:

$$\zeta = \rho\, e^{i\lambda} = \frac{\sin \frac{\pi}{2\,a}(z - z_o)}{\sin \frac{\pi}{2\,a}(z + \bar{z}_o)} \times \frac{\sin \frac{\pi}{2\,a}(z + z_o)}{\sin \frac{\pi}{2\,a}(z - z_o)} \quad \dots\dots\dots (201)$$

and

$$N\,\nabla^2 w = \frac{P}{2\,\pi}\log \rho \quad \dots\dots\dots\dots\dots\dots\dots (202)$$

in which $\log \rho$ in Eq. 202 will contain two expressions similar to the one obtained from Eq. 198. The physical interpretation of the result is that the moment sum at any point in the simply supported half strip is the combination of two equal and oppositely directed loads applied to an infinite plate strip at the two points z_o and $\bar{z}_o$. In a small neighborhood of the load point $z_o = x_o + i\,y_o$,

$$N\,\nabla^2 w = \frac{P}{2\,\pi}\left\{\log \frac{|z - z_o|}{a} - \left[\log\left(\frac{2}{\pi}\sin \frac{\pi\,x_o}{a}\right) + 2\log \frac{\cosh \frac{2\,\pi\,y_o}{a} - 1}{\cosh \frac{2\,\pi\,y_o}{a} - \cos \frac{2\,\pi\,x_o}{a}}\right]\right\} \quad \dots\dots\dots (203)$$

The place coefficient B is the expression enclosed in the brackets.

Example 9. Sector of a Circle.—Consider a plate sector of a circle of angular opening $\alpha = \frac{\pi}{k}$, in which $k \geqslant 1$, simply supported along two radial edges which extend indefinitely. Let $z_o = r_o e^{i\theta_o}$ be the point of loading. The transformation,

$$\zeta = \frac{z^k{}_o - z^k}{z^k - \bar{z}^k{}_o} = \rho\, e^{i\lambda} \dotfill (204)$$

maps the region of the plate sector within the unit circle with z_o going into $\zeta = 0$. If $R = \frac{r}{r_o}$, in which r and r_o are distances from the vertex of the plate sector, then

$$N\,\nabla^2 w = \frac{P}{4\,\pi} \log \frac{R^k + R^{-k} - 2 \cos k\,(\theta_1 - \theta)}{R^k + R^{-k} - 2 \cos k\,(\theta_1 + \theta)} \dotfill (205)$$

If the load point is on the axis of symmetry and $\theta_1 = \frac{\alpha}{2}$, then

$$N\,\nabla^2 w = \frac{P}{4\,\pi} \log \frac{R^k + R^{-k} - 2 \sin k\,\theta}{R^k + R^{-k} + 2 \sin k\,\theta} \dotfill (206)$$

If $\theta = \frac{\alpha}{2}$ and $\alpha = \frac{\pi}{2}$, then $k = 2$, and along the bisector of the plate,

$$N\,\nabla^2 w = \frac{P}{4\,\pi} \log \frac{r^4 + r^4{}_o - 2\,r^2\,r^2{}_o}{r^4 + r^4{}_o + 2\,r^2\,r^2{}_o} = \frac{P}{2\,\pi} \log \frac{|r^2 - r^2{}_o|}{r^2 + r^2{}_o} \dotfill (207)$$

For $|r - r_o|$, which is a small value:

$$N\,\nabla^2 w = \frac{P}{2\,\pi} \log \frac{|r - r_o|}{r_o} \dotfill (208)$$

This yields the place coefficient $B = 0.0$ when the sector is an infinite quadrant $\alpha = \frac{\pi}{2}$ and the load is at a distance r_o along the bisector. With $M_x = M_y$, $C = 1$ and $D = 0$. If the log term is put in the form, $\log \frac{R}{c}$, in which $c = 2\,r_o$ (that is, twice its distance to the nearest supporting edge), then $B = -\log \sqrt{2}$ $= -\,0.347$. If $k = 1$, and $\theta = \theta_1 = \frac{\pi}{2}$, then Eq. 205 yields:

$$N\,\nabla^2 w = \frac{P}{2\,\pi} \left(\log \frac{|r - r_o|}{r_o} - \log 2 \right) \dotfill (209)$$

and $B = 0.693$ is the place coefficient for a point load at a distance r_o from a long, simply supported, rectilinear edge. It is to be noted that the values of B in these sector plates were derived by choosing a particular value of θ—namely, that along the bisector. In the special case $\alpha = \pi$, a simple particular transformation,

$$\zeta = \frac{i\,r_o - z}{i\,r_o + z} \dotfill (210)$$

transforms the half plane with the point $z = i\, r_o$ going into the center of the ζ-circle. Hence,

$$N\,\nabla^2 w = \frac{P}{2\,\pi}\log\frac{\rho_1}{\rho_2} = \frac{P}{2\,\pi}\left(\log\frac{\rho_1}{r_o} - \log\frac{\rho_2}{r_o}\right)\ldots\ldots\ldots\ldots(211)$$

holds for any point in the plate where ρ_1 and ρ_2 are distances from the load point and its image, respectively. In this general case the place coefficient is $B = \log\frac{\rho_2}{r_o}$.

Example 10. Clamped Circular Plate with Eccentric Point Load.—A circle of radius a is mapped within the circle $|\zeta| = 1$ by the transformation,

$$\zeta = \frac{a\,(z - z_o)}{a^2 - z\,\bar{z}_o}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(212)$$

with the load point, z_o, becoming the origin, $\zeta = 0$. No loss of generality is incurred if z_o is taken at a distance c along the real axis x, the value of c being the distance the load is from the center of the circular plate of radius a. Then $\log|\zeta|$ is $\log\frac{a\,\rho_1}{c\,\rho_2}$, in which ρ_1 and ρ_2 are the respective distances from any point of the plate to the load points and to its circular image, which is located on the real axis at a distance $\frac{a^2}{c}$ from the origin. Before considering the clamped edge, it may be pointed out that the latter logarithmic term determines the place coefficient at any point in an eccentrically loaded circular plate having a Naviér edge condition.

In the case of a clamped edge along which the deflection and slope vanish, a suitable deflection function is:

$$w = \frac{P}{8\,\pi\,N}\left[\rho^2_1\log\frac{a\,\rho_1}{c\,\rho_2} + \psi(\rho_1) + \phi(\rho_2)\right]\ldots\ldots\ldots\ldots(213)$$

since it is known that $\log\frac{a\,\rho_1}{c\,\rho_2}$ vanishes for points along the boundary. The vanishing of the slope normal to the boundary edge is equivalent to the vanishing $\frac{\partial w}{\partial\rho_1}$ and $\frac{\partial w}{\partial\rho_2}$. If $\psi(\rho_1)$ and $\phi(\rho_2)$ are simple polynomial expressions,

$$\left.\begin{aligned}\psi(\rho_1) &= b_o + b_1\,\rho_1 + b_2\,\rho^2_2 + \cdots\\ \phi(\rho_2) &= c_o + c_1\,\rho_1 + c_2\,\rho^2_2 + \cdots\end{aligned}\right\}\ldots\ldots\ldots\ldots(214)$$

Then the complete solution of Michell's clamped circular plate eccentrically loaded is:

$$w = \frac{P}{8\,\pi\,N}\left[\rho^2_1\log\frac{a\,\rho_1}{c\,\rho_2} + \frac{1}{2}\left(\frac{c^2}{a^2}\rho^2_2 - \rho^2_1\right)\right]\ldots\ldots\ldots(215)$$

and

$$N\,\nabla^2 w = \frac{P}{8\,\pi}\left[4\log\frac{a\,\rho_1}{c\,\rho_2} + 4 - \frac{4\,\rho_1}{\rho_2}\cos\beta + 2\left(\frac{c^2}{a^2} - 1\right)\right]$$

$$= \frac{P}{2\,\pi}\left[\log\frac{a\,\rho_1}{c\,\rho_2} + 1 - \frac{\rho_1}{\rho_2}\cos\beta + \frac{1}{2}\left(\frac{c^2}{a^2} - 1\right)\right]\ldots\ldots\ldots(216)$$

From Eq. 216 one may find a place coefficient B. As a special case, if $c \to 0$ with $c\,\rho_2 = a^2$, and $\rho_2 \to \infty$, then $N\,\nabla^2 w = \frac{P}{2\,\pi} \log\left(\frac{r}{a} + \frac{1}{2}\right)$, which agrees with the result obtained from Eq. 190*c*. Another special case arises if both c and a become infinite with $(a - c)$ remaining finite. Then,

$$N\,\nabla^2 w = \frac{P}{2\,\pi}\left(\log\frac{\rho_1}{a_o} - \log\frac{\rho_2}{a_o} + 1 - \frac{\rho_1}{\rho_2}\cos\beta\right)\ldots\ldots\ldots(217)$$

in which a_o now becomes the distance that the load is from a long, clamped, straight edge, and ρ_1 and ρ_2 are the distances from a point x,y of the plate to the load point and its image. The angle β is the angle between the radii ρ_1 and ρ_2.

C. A. P. Turner,[48] **M. Am. Soc. C. E.**—In a circular plate, supported at its edge, the twisting shears may be clockwise or counterclockwise—as a matter of chance. The contours developed will be the same, but the spiral of displacement is right-handed in one case and left-handed in the other.

With the square or column-supported slab, the sign, or the change of sign, of the twisting shears must be considered because it controls the travel of the load to the support. Its neglect makes the mathematical plate theories unsatisfactory in general, as stated by Sir William Thomson and P. G. Tait[49] with reference to their own work: "Mathematicians have not hitherto succeeded in solving this problem with complete generality, for any other form of plate than the circular ring (or circular disc with concentric circular aperture)." This criticism is applicable to the paper under discussion.

Scientific literature and technical discussions are encumbered with a mass of mathematical development, tending toward confusion instead of simplification of the subject matter in a practical manner. This results from academic efforts to develop unknown properties from a hypothetical premise, inconsistent with observed and known phenomena. The utility of mathematical theory depends upon whether the theory enables the engineer to form a more exact conception of the resistance of materials and of the science of design.

In 1906 the writer constructed the concrete frame and floors of the Somerset Block in Winnipeg, Manitoba, Canada, under Garson, Hodgins and Thompson, contractors. Ten or twelve weeks after casting the first floor, the writer met the contractors at the building and was told that an accident had furnished a great test of the strength of the floor construction. The slab was 5½ in. thick, 18½ ft square, and reinforced with ⅜-in. round rods, 6 in. on centers through the middle third and 8 in. on centers for the remainder. A stone cornice block weighing approximately three fourths of a ton had slipped from the derrick boom and dropped 60 ft on to the center slab.

The writer was asked what damage he supposed had been done. He replied that an engineer would expect the stone to be broken and shattered no matter how it hit the slab. If a corner had struck the slab, there probably would have been a crease or indentation of ⅛ in. where the corner struck, with no cracks or other damage to the slab. The astonished contractors believed that the writer must have been told previously of the details of the accident until assured otherwise. The writer questions whether the mathematics in the Westergaard paper would have assisted in predicting this result, because the author's mathematical theory is that of imaginary substances as explained by William J. Ibbetson.[50]

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[48] Cons. Engr., Columbus, Ohio.

[49] "Natural Philosophy," by William Thomson and P. G. Tait, 1867, Article 655, p. 504.

Although recognizing that resistance to elastic change in form was that of intermolecular forces, early elasticians assumed that these forces, although simple in action, were too complicated for mathematical treatment; so they adopted the Boscovitch infinitesimal point action of imaginary substances. Its major error lies in the assumption that the internal shift of energy equals the external work of the applied force, whereas the actual kinetic shift of internal energy is 100% greater than that assumed in mathematical theory.

After a lifetime of endeavor, Prof. Karl Pearson found himself so dissatisfied with the hypothetical character of mathematical elasticity[51] that, in evaluating the work of Kelvin, he expressed the belief that Kelvin's work in coordinating the temperature change under unit force with the coefficient of expansion, weight, mechanical equivalent, and specific heat would bring him recognition as the greatest elastician of his age. Notwithstanding this brilliant achievement, Kelvin so underestimated the mechanical equivalent of the temperature change that he attributed the increased elongation, due to heat radiation, to viscosity instead of to the natural result of energy shift.

Another fundamental error lies in the generalization[52] of Hooke's Latin expression:

> "Ut tensio sic vis: That is, The Power of any Spring is in the same proportion with the tension thereof. If one power stretch or bend it one space, two will bend it two, and three will bend it three, etc. Now as the Theory is very short, so the way of trying it is very easie."[53]

In the thin metal of a spring, the increase of temperature in the compression side is instantly absorbed, through conduction, by the decrease in temperature in the tension side; and Hooke's law of the power of a spring is a law of nature quite different from that of axial force, which endows the specimen with a large increase in strength and resistance before the radiation is complete.

Thus, H. H. Campbell[54] found that the yield point of mild steel—39,000 lb per sq in., under a pulling speed of 0.1 in. per min—was increased to 45,000 lb per sq in. under a high speed of testing. Such differences in the yield point were actual and in no sense hypothetical.

How should elastic resistance be formulated if the textbook translation of Hooke's Latin does not fit the case? An axial force produces a change in volume proportional to the magnitude of the force within the natural limit of elasticity. The same force causes the same change in volume in a given metal, independent of its physical state—whether it is soft ingot, or is in a rolled or drawn condition—thereby coordinating the variations in Young's modulus, E, and Poisson's ratio, μ, which have puzzled philosophers for generations.

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[50] "An Elementary Treatise on the Mathematical Theory of Perfectly Elastic Solids, with a Short Account of Viscous Fluids," by William J. Ibbetson, Macmillan and Co., London, England, and New York, N. Y., 1887, Chapter I.

[51] "A History of the Theory of Elasticity and of the Strength of Materials, from Galilei to the Present Time," by Isaac Toddhunter (edited by Karl Pearson), Cambridge Univ. Press, 1886–1893, Vol. IV.

[52] "A Treatise on the Mathematical Theory of Elasticity," by A. E. H. Love, Cambridge Univ. Press, 1927, p. 95.

[53] "Lectures de Potentia," by Robert Hooke, 1676.

[54] "Manufacture and Properties of Structural Steel," by H. H. Campbell, Ed. 1, Hill Publishing Co., New York, N. Y., and London, England, 1896.

Twenty-one independent coefficients of elasticity have been developed in the mathematical elasticity of hypothetical substances, as opposed to the fact that there are no independent coefficients or properties in actual substances. All are interdependent in endless ways (see discussion on time, space, and energy by Arvid Reuterdahl[55]).

Quoting Messrs. Thomson and Tait:[56]

> "Although the problem of fulfilling arbitrary boundary conditions has not yet been solved for rectangular plates, there is one remarkable case of it which deserves particular notice; not only as interesting in itself, and important in practical application, but as curiously illustrating one of the most difficult points of the general theory."

This appears in the bending of a thin rectangular plate, with a thickness 150th of the span, supported at diagonally opposite corners and loaded at the other two. The inscribed square remains flat and the end triangles bend only in cylindrical curvature. A mathematical theory that deserves recognition should account for this phenomenon, but the mathematics presented by Dean Westergaard is incapable of explaining it.

From the days of Stahl, scholars have entertained fantastic notions of energy and matter, based upon figments of the mind unrelated to mechanical fact and observation. It took fifty years for the textbook writers to recognize the transformation of mechanical energy into heat, although demonstrated conclusively by Benjamin Thompson in 1799. By boring copper with a dull tool under water, he brought the water to the boiling point. A similar backwardness is exhibited in the textbooks of today in failing to recognize the elementary relation of energy to mass. Thus, the cubic energy of elastic resistance is mistreated as linear energy in unsatisfactory mathematical treatments such as those presented by current writers on the subject.

In a current text,[57] conservation of momentum is treated as if elastic resistance were linear energy instead of cubic, presenting the erroneous conclusion that the sum of the momenta of two elastic spheres is the same after impact as before; and the same metaphysical notion appears in the Germanic dream that the intra-atomic energy of a quarter gram of matter is equal to that which the practical engineer develops in burning 5,000 tons of coal.

Since it took old-time scholars fifty years to discard the German phlogistic hypothesis after the demonstration of the relationship of mechanical to heat energy by Thompson, the practical student wonders if present-day scholars will need more than fifty years to discard the complex, artificial, mathematical elasticity of imaginary substances in favor of the natural elasticity of actual substances—Young's modulus coordinated by arithmetic with the atomic weight and number, strength, specific heat, melting point, and the calories of chemical combination.

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[55] "Kinematics," by Arvid Reuterdahl.

[56] "Natural Philosophy," by William Thomson and P. G. Tait, 1867, Article 656, p. 505.

[57] "Handbook of Chemistry and Physics"—Charles D. Hodgman, Editor—21st Ed., Chemical Rubber Co., Cleveland, Ohio, p. 1675.

The development of the exact stress mechanism of plates is a much simpler procedure than the development of the exact stress mechanism of simple and continuous beams. This is true despite the fact that beam action seems simple as treated by the common theory of flexure until it is observed that the shear strain so computed may be from 30% to 130% greater than the actual when determined by precise methods.[58]

H. M. WESTERGAARD,[59] M. AM. SOC. C. E.—All the discussions that have been presented are contributions to the subject of the paper, except the discussion by Mr. Turner, which is irrelevant.

Mr. Rich is right in calling attention to the fact that Bessel functions of the "ker" and "kei" type can be used instead of Hankel's Bessel function in stating the solution of the problem of a slab on elastic subgrade (Eqs. 48 and 49); and that ker and kei functions have the advantage that information about them may be more readily available in the United States than about Hankel's Bessel functions.

It would not be desirable to accept the suggestion offered by Mr. MacColl to substitute the names "place moments" and "area moments" for the names "place coefficients" and "area coefficients" that were introduced in the paper. The latter names were chosen after careful consideration and with knowledge of the fact that it does not pay to take the business of nomenclature lightly. The substitution of "moment" for "coefficient" would be unfortunate because in the analysis of the bending of plates it is already necessary to deal both with moments per unit of width of section, denoted in the paper by the small letter m and measurable in pounds, and with moments on a section of finite length, properly denoted by the capital letter M and measurable in inch-pounds. A very minor disagreement with Mr. MacColl applies to his statement that the cases for which area coefficients were derived in the paper are "very simple and special." Case 6, of a load distributed in any manner over a piece of line, and Case 11, of a load distributed uniformly over the area of a polygon of any shape, are at least fairly general. These cases seem sufficiently general to warrant the belief that the more important further work that need be done is to evaluate place coefficients in more examples rather than area coefficients in more cases.

Mr. Nádai's approval of the paper is gratifying because he was one of the pioneers in bridging the gap between the mathematical aspects of the theory of plates and the practical aspects which are interesting to engineers; his own original contributions in this field are noteworthy and important. Mr. Nádai's comments, in his discussion, on the nature of the singularities and the rôle of Green's functions are useful and enlightening.

The discussion by Professor Young is especially welcome because it contributes new dependable sets of place coefficients, some of which he derived from his own previous original work on rectangular plates with fixed (or clamped) edges. His discussion has the merit of attractive form, clarity, and adherence to the rule of economy of thought. A simple analytical test of the dependability of Professor Young's results may be performed by examining the place coefficient $B = -1.137$ that he states at the top of Table 3 for the case of a square plate with fixed (or clamped) edges, with the load at the center, and with the distance c chosen equal to the side l of the square. Assume tentatively that this value of B is right. The question is now asked, what is the diameter L of a circular plate with fixed edge, with load at the center, and with the same stress concentration under the same load; that is, with the same value of B for $c = l$? Or, what is the diameter L of a circular plate equivalent in this sense? According to Eqs. 32, $c = \frac{L}{2}$ would give $B = -\frac{1}{2}$ for the circular plate; then $c = l$ gives

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[58] "Elasticity, Structure and Strength of Materials," by C. A. P. Turner, Pt. I, Minneapolis, Minn., 1939, pp. 260–292.

[59] Dean, Graduate School of Eng., and Gordon McKay Prof. of Civ. Eng., Harvard Univ., Cambridge, Mass.

$$B = -\tfrac{1}{2} + \log \frac{L}{2\,l} = -1.193 + \log \frac{L}{l} \ldots\ldots\ldots\ldots (218)$$

When the value $B = -1.137$, assumed for the square plate, is required for the circular plate, Eq. 218 gives

$$L = l\,e^{0.056} = 1.058\,l \ldots\ldots\ldots\ldots\ldots\ldots\ldots (219)$$

as the diameter of the equivalent circular plate. This result is within reasonable expectations. Professor Young's table stands this test well.

It is regrettable that undivided praise cannot be given to the discussion by Professor Holl, in which he invested a considerable amount of work. It is a matter of minor criticism that Professor Holl chose to shift from the notation μ, ζ, m, and Δ in the paper to another notation γ, w, M, and ∇^2, which, although it also conforms to good standards, is not preferable, is a disadvantage to the reader unless he reads the discussion only, and acts against the rule of economy of thought by making it necessary to restate a number of equations. It is subject to serious criticism that Professor Holl neglected in a number of cases to state his choice of the distance c which must be chosen and stated if the place coefficient B and the area coefficient K are to attain definite meaning; furthermore he uses the symbol c with varying meanings, so that, for example, immediately before Eq. 171 the reader will have to translate from $\frac{a}{c}$ to $\frac{c}{a}$.

Without quoting any source Professor Holl writes Eq. 164 as a formula for the deflections of a square plate with fixed edges, side a, and the load at the center. From this formula he derives the place coefficient $B = -0.4438$ which is consistent with Eq. 164 if c, the choice of which he does not state, is chosen equal to a; but this result is in serious conflict with the value $B = -1.137$ given by Professor Young, and does not stand up under the simple test that was shown for the latter value and led to Eq. 219; the required correction is close to $-\log 2$. The place coefficients in Professor Holl's Table 4 agree fairly well with Professor Young's Table 3 provided that the distance c is assumed equal to the short span of the rectangular plate. Professor Holl devotes a part of his discussion to advocating the use of the conjugate complex variable $\bar{z} = x - i\,y$; this use has possibilities of serving to simplify some of the analytical processes of the theory of plates.

The discussion by Messrs. Teller and Sutherland deserves high praise. The experimental verifications that they report are as satisfactory as can be expected when the structure tested is a concrete slab on a subgrade. In the theory of elasticity, because of the simplicity of the assumed physical laws, dependable results can be obtained by mathematical means, and much of the work must be by mathematical means. Still the theory of elasticity is a physical science, and physical experiments are needed. The careful and extensive tests conducted by Messrs. Teller and Sutherland and their analysis of the results are a significant, important, and most welcome contribution to the subject of stress concentrations in plates.

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HYDRAULIC DESIGN OF DROP STRUCTURES FOR GULLY CONTROL

By B. T. Morris,[1] Jun. Am. Soc. C. E., and D. C. Johnson,[2] Assoc. M. Am. Soc. C. E.

With Discussion by Messrs. John Hedberg, L. Standish Hall, J. E. Christiansen, Walter T. Wilson, N. A. Christensen and Dwight Gunder, Boris A. Bakhmeteff and Nicholas V. Feodoroff, G. H. Hickox, and B. T. Morris and D. C. Johnson.

Synopsis

In the stabilization of gullies, small overflow dams are used to retain silt and to control the stream grade. These dams are simple drop structures similar to those used in irrigation canals. In this paper the development of rules for the proportioning of such dams is described in terms of the hydraulic requirements for structure performance. The formulas included in the design rules are presented graphically for convenience in application. These rules are based on the accumulated experience of engineers in irrigation and soil conservation work and on the results of a series of laboratory test programs.

Design Problem

Small overflow dams, called drop structures, are installed in a gully to establish permanent control elevations below which an eroding stream cannot lower the channel floor. These dams control the stream grade, not only at the spillway crest itself, but also through the ponded reach upstream from the dam. Thus drop structures, placed at intervals along a gully, can stabilize it by changing its profile from a continuous steep gradient to a series of more gently sloping reaches separated by artificial spillways.

Although the construction of large numbers of drop structures for the stabilization of gullies in many different localities began with the operations of the Soil Conservation Service (SCS), U. S. Department of Agriculture, the development of designs for dams of this type started many years earlier with their application to the control of grades in irrigation canals. There is a

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Note.—Published in January, 1942, *Proceedings*.

[1] Research Engr., C. F. Braun & Co., Alhambra, Calif.; formerly Hydr. Engr., SCS, Pasadena, Calif

[2] Sacramento, Calif.; formerly Asst. Regional Engr., SCS, Berkeley, Calif.

marked similarity between dams that have been built in large gullies in the Pacific Southwest and canal drops (see Figs. 1 and 2).

Records of the design and construction of canal drops are readily available in the literature of irrigation engineering in the United States and the British Empire. Engineers of the Soil Conservation Service have adapted design rules taken from this source to gully-control drop structures. The design rules presented in this paper have been developed from a typical canal-drop design through the analysis of the differences in the operating requirements of the two types of drop and through hydraulic laboratory tests of experimental drop structures.

Notation.—The following letter symbols, introduced in the paper, conform essentially to American Standard Letter Symbols for Hydraulics, prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1942:[3]

a = offset for weir notch ventilation (Fig. 3);
b = width:
    b_n = width of notch (Fig. 3);
    b_w = width of water surface upstream from notch;
C = coefficient:
    C_L = coefficient of apron length $= \frac{L}{\sqrt{h\,d_c}}$ (Eq. 2);
    C_x = coefficient of longitudinal-sill spacing $= \frac{x}{b_n}$ (Eq. 13*a*);
d = depth:
    d_c = critical depth of cross section;
    d_s = depth in upstream or downstream channel;
g = acceleration of gravity;
h = height; height of fall; h' = height of transverse end sill or depth of stilling pool;
L = length of the apron;
Q = discharge;
V = velocity:
    V_c = critical velocity of cross section;
    V_s = mean velocity in upstream or downstream channel;
x = spacing distance of longitudinal sills (Design B, Fig. 3);
x' = spacing distance of longitudinal sills (Design A, Fig. 3).

Gully-Control Problem vs. Canal-Drop Problem.—Both the fundamental purpose of drop-structure installation and the basic problem to be met in drop-structure design are common to canal and gully applications. The purpose is the control of stream grade; and the problem is the design of the spillway part of the structure. In each type of drop installation the permanence and the efficiency of the structure are controlled by the performance of the energy-dissipating and scour-preventing devices installed downstream from the dam proper. The efforts of the writers in the development of design rules for drop structures have been concentrated on the spillway problem. The type of energy-dissipating and scour-preventing device adopted by them, after

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[3] ASA-Z10.2-1942.

FIG. 1.—A NEWLY CONSTRUCTED DROP ON THE LAS POSAS PROJECT, VENTURA COUNTY, CALIF. (BEFORE THE BACKFILL HAS BEEN PLACED BETWEEN THE STRUCTURE AND THE BANKS)

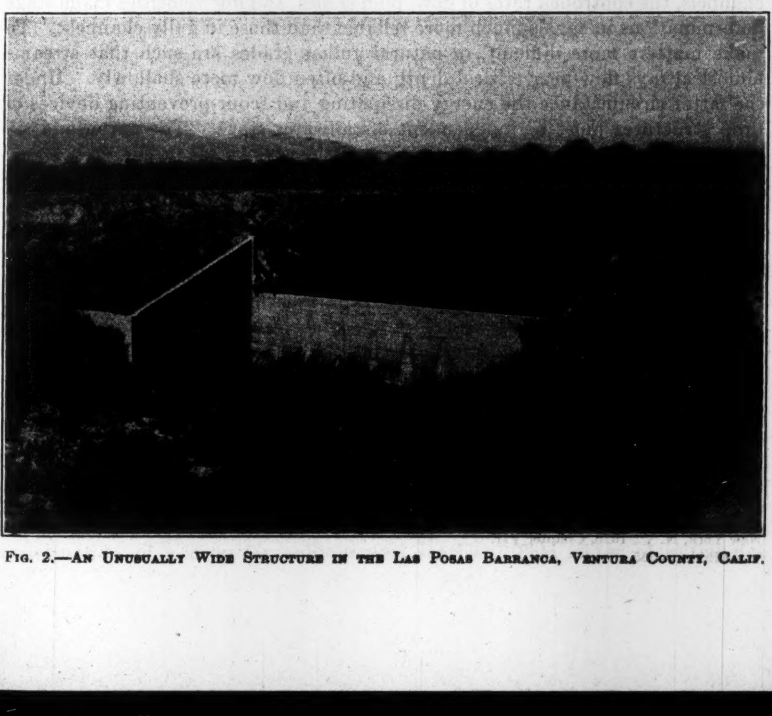

FIG. 2.—AN UNUSUALLY WIDE STRUCTURE IN THE LAS POSAS BARRANCA, VENTURA COUNTY, CALIF.

consideration of various types previously applied to small drop structures by others, is the rectangular apron with transverse end sill that is made an integral part of the dam and supporting walls. (This part of an overflow dam and spillway has been given several names descriptive of its effect on the flow—stilling pool, stilling basin, water cushion, spillway bucket, tumble bay, and cistern. Photographs of flow through the structure, as well as the analysis in this paper under "Criteria for Satisfactory Drop-Structure Performance" will demonstrate the inadequacy of each of these terms taken literally. Needing a term to describe the function of this part of the structure and being unable to provide a satisfactory name, the writers, on occasion, will use the term "stilling pool." Suggestions of better names will be welcomed.) The simplicity of layout and construction of this type of drop structure, as well as the available knowledge of its performance, was recognized in its selection for standard design. Descriptions of its application to irrigation canals have been presented elsewhere by B. A. Etcheverry,[4] M. Am. Soc. C. E.

Differences between the characteristics of flow in irrigation canals and in natural gullies led to the selection of the apron with end sill in preference to the simple apron or any other device which depends on the tailwater stage for the establishment of a hydraulic jump immediately below the dam. The steep grades, irregular runoff rates, varying silt loads, and uncertain roughnesses of gully channels make the elevations of tailwater surface and stream bottom very difficult to predict and, at best, too unreliable for use in controlling the performance of the structure. Although there are many uncertainties involved in the prediction of equilibrium slopes and stable grades in earthen irrigation channels, the controlled rates of flow, mild slopes, and low velocities make stage determinations in canals much more reliable than those in gully channels. To make matters more difficult, in natural gullies grades are such that streams almost always flow near critical depth and often flow more shallowly. Under the latter circumstance the energy-dissipating and scour-preventing devices of drop structures must be independent of tailwater stage. The apron and sill combination is well suited to this last requirement.

The drop structure resulting from the combination of a straight breast wall dam and a rectangular apron with end sill is shown in Fig. 3. The important variables in determining the proportions of the stilling pool are: The height of fall, h; the length of the apron, L; the height of the end sill (or depth of the pool), h'; and the discharge Q, indicated by the critical depth for the weir notch

$$d_c = \left(\frac{Q}{b_n \sqrt{g}}\right)^{\frac{2}{3}} \dots\dots\dots\dots\dots\dots\dots\dots (1)$$

in which b_n is the width of the notch. (In Fig. 3, Design A provides nappe ventilation through the offset of the side-walls from the edge of the notch as well as through the lateral contraction of the flow at the crest; Design B provides nappe ventilation through flow contraction alone.) Professor Etcheverry has presented a rational formula,[5] relating these variables, that may be written, in terms of critical depth, d_c:

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[4] "Irrigation Practice and Engineering," by B. A. Etcheverry, Vol. III, McGraw-Hill Book Co., Inc., New York, N. Y., 1916, Chapter VII.

[5] Ibid., p. 235.

$$L = C_L \sqrt{h\, d_c} \qquad (2)$$

in which C_L, the coefficient of apron length, is variously determined at values from 3.1 to 4.5. The assumptions upon which Eq. 2 is based are: (1) That the nappe trajectory is a parabola with its vertex at the crest of the dam; and (2) that the falling water strikes the apron at a constant fraction of its length (from the crest).

In applying Eq. 2 to the determination of apron lengths in gully-control drop design, it was believed that a smaller value of the length coefficient might be used. Whereas, in the design of drop structures for canal service, it might be well to include a factor of safety in the value of the coefficient, C_L, in gully-control service, the factor of safety for the entire system of structures is included in the determination of the design discharge rate, so that the inclusion of a factor of safety in C_L would be a compounding of safety factors. The compounding of two or more safety factors does not lead to economical construction. Unless the cost of constructing gully-control drops is kept as low as possible, it is very difficult to justify their widespread use on farm land.

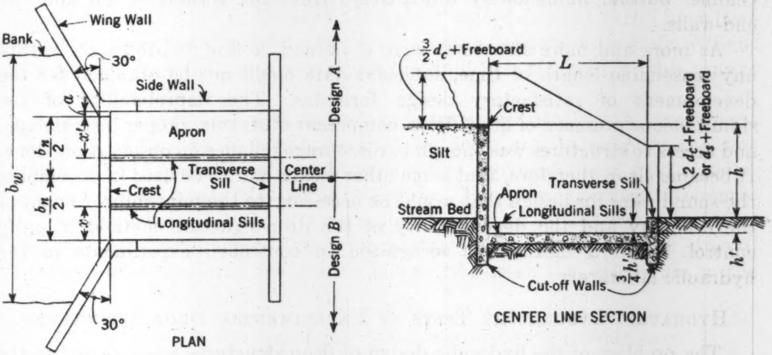
Wing Wall
Bank
30°
Side Wall
α
Apron
b_w
$\frac{b_n}{2}$
x'
Transverse Sill
Center Line
Crest
$\frac{b_n}{2}$
x
Longitudinal Sills
30°
PLAN
Design A
Design B
$\frac{3}{2}d_c$+Freeboard
L
Crest
Silt
$2\,d_c$+Freeboard or d_s+Freeboard
h
Transverse Sill
Apron
Stream Bed
Longitudinal Sills
h'
$\frac{3}{4}h'$
Cut-off Walls
CENTER LINE SECTION

FIG. 3.—GULLY-CONTROL DROP STRUCTURE

The use of gully-control drops on agricultural land imposes an additional consideration over those taken into account in the design of the canal drop. Irrigation organizations and maintenance arrangements are such that preventive measures may be taken immediately wherever excess scour is noted at a drop structure. This factor is taken into account in the recommendation by many designers that riprap stream protection be placed downstream from drops and that this riprap be repaired and extended to suit conditions encountered. The drop design should not rely on maintenance by the farm operator for the safety of the structure. In gully-control service, the drop structure will receive maintenance that is no more likely to be regular than is its frequency of operation. For this reason, in the interest of the safety of the structure, there must be an increased emphasis on the dependability of the original design and construction of a drop for gully-control service. This factor of initial dependability is an economic factor fully as important as initial cost because it represents decrease in maintenance cost and decrease in the probability of loss through damage, failure, and the necessity of replacement.

Early Gully-Control Drop Construction.—The first drop structures built to designs prepared by the Pacific Southwest Region, SCS, had aprons designed with a length coefficient C_L of 2.5 and a sill height h' of $\frac{h}{8}$. The original intent of the designer was that the operation of these structures be observed and the design rules altered to suit observed deficiencies, if any. Some of these structures have been in place since 1936 and none have failed from stilling-pool inadequacy. However, observation of dams that have passed floods near design capacity has indicated an urgent need for improved control of the flow as it leaves the structure, as evidenced by serious scouring of the banks and channel bottom immediately downstream from the transverse sill and the end-walls.

As more and more structures were examined, it was realized that, within any reasonable length of time, sufficient data could not be obtained for the development of satisfactory design formulas. The improbability of the simultaneous presence of flood flows, competent observers, proper instruments, and access to structures was enough to discourage reliance on observation alone. It became clear, therefore, that some other means had to be used in developing the sound rules for design that would be necessary to the joint improvement of the economy and the dependability of the drop-structure method of gully control. Such a means was recognized in controlled experiments in the hydraulic laboratory.

HYDRAULIC LABORATORY TESTS OF EXPERIMENTAL DROP STRUCTURES

The problem of the hydraulic design of drop structures was referred to the hydraulic laboratory of the Soil Conservation Service at the California Institute of Technology, Pasadena, Calif., where a series of experiments was planned for the development of satisfactory designs and rational design rules.

The first experiments dealt with a structure of typical proportions of height of fall, width of crest, and depth of flow. Although it was not essential to the interpretation of the tests that a scale model be used, the experimental drop structure was designed as a one-eighth scale model of a laterally-contracted drop structure (see Fig. 3, Design A) having the dimensions:

Crest width, b_n, in feet	9.6
Fall height, h, in feet	8.5
Discharge, Q, in cubic feet per second	350
Critical depth for notch cross section, d_c, in feet	3.45
Channel width, upstream, b_w, in feet	14.7
Apron width, $b_n + 2a$, in feet	10.0

Expressed in appropriate dimensionless ratios, these drop-structure proportions are: $\frac{\text{fall height}}{\text{critical depth at the notch}} = \frac{h}{d_c} = 2.5$; $\frac{\text{critical depth}}{\text{notch width}} = \frac{d_c}{b_n} = 0.36$; and contraction ratio $= \frac{\text{stream width}}{\text{notch width}} = \frac{b_w}{b_n} = 1.5$.

Stilling pools equivalent to the following combinations of prototype dimensions (in feet) were tested in conjunction with this experimental drop at flow rates from 28% to 200% of the hypothetical design capacity:

Length	Depth
16	2
20	2
24	2
24	3

The downstream channel of the drop installation was represented by a model that had a trapezoidal cross section with 1 on 1 side slopes and a bottom width corresponding to 10.0 ft in the prototype.

The experimental drop-structure installation differed from the prototype it represented in that it consisted of only one half of the symmetrical drop and channel system. A sheet of heavy plate glass, placed at what would have been the center line of the complete structure, made observation possible without distorting the flow sufficiently to impair the similitude required for this type of hydraulic experiment.

Another compromise with nature made interpretation of the experiments difficult. To shorten the time required for tests and to simplify the test technique, solid channel walls and bottom were provided where field installations would furnish natural erodible materials. Thus, the experimental channel had fixed boundaries instead of a movable bed. In an installation of this kind there was no direct measure of the scouring power of the effluent stream. Therefore, other performance criteria had to be chosen that would aid in the selection of the best structure proportions.

CRITERIA FOR SATISFACTORY DROP-STRUCTURE PERFORMANCE

In order to judge the excellence of drop-structure performance, the necessary qualities of a good drop structure must be established in terms of drop-structure use.

First, the structure must drop the water within its own confines and discharge it downstream in such a way as to cause a minimum of locally intensified erosion; and

Second, in order that the structure may perform its first function continuously, it must discharge the stream in such a manner that the flow will not undermine the structure itself.

These two requirements will be recognized by hydraulic engineers as those governing all spillway construction.[6] One difference is important: In soil conservation work, the order of importance of the requirements is the reverse of that found in most spillway work because the reduction of erosion is the original purpose for which the structure is erected. In other types of dam construction, the first rule is generally of secondary importance and there are even some cases in which this consideration has been neglected entirely, although the writers find it difficult to conceive of an installation where there is not some value in saving the channel from gross enlargement.

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[6] "Low Dams" (handbook), National Resources Committee, Washington, D. C., 1938, p. 108.

Before these rules can be applied directly to the interpretation of the results of experiments, they must be redefined and restated in terms of the behavior of the flow through the structure. In order that this flow may conform to the rules for structure performance, the kinetic energy of the falling water must be "dissipated" through its conversion to turbulence energy in the eddy motion of the "stilling pool" and this turbulence energy must be so distributed in the flow (prior to its complete decay through conversion to heat energy by viscous forces) that it will have a minimum of sediment-transporting power and thus a minimum of scouring power. Furthermore, the flow over the end sill must produce a movement of sediment along the channel toward the end sill, rather than away from it, so that undermining may be prevented.

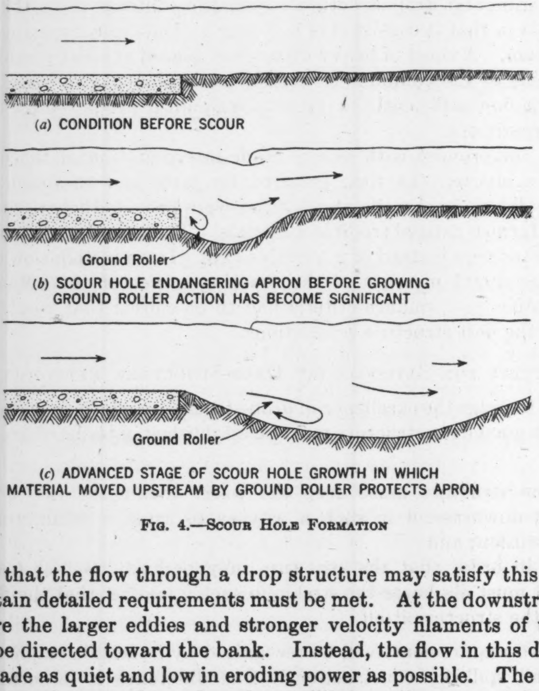
(a) CONDITION BEFORE SCOUR
Ground Roller
(b) SCOUR HOLE ENDANGERING APRON BEFORE GROWING GROUND ROLLER ACTION HAS BECOME SIGNIFICANT
Ground Roller
(c) ADVANCED STAGE OF SCOUR HOLE GROWTH IN WHICH MATERIAL MOVED UPSTREAM BY GROUND ROLLER PROTECTS APRON

FIG. 4.—SCOUR HOLE FORMATION

In order that the flow through a drop structure may satisfy this second set of rules, certain detailed requirements must be met. At the downstream end of the structure the larger eddies and stronger velocity filaments of the stream should not be directed toward the bank. Instead, the flow in this danger zone should be made as quiet and low in eroding power as possible. The reasons for these statements become clear when the equilibrium of scour and deposition at a gully bank is considered. The equilibrium of scour and deposition in the various parts of a natural gully cross section has been treated by N. A. Christensen,[7] Assoc. M. Am. Soc. C. E.

So far as erosion by the stream may be concerned, such equilibrium is often imaginary, for the stream itself cannot deposit material on a bank whose slope is equal to, or greater than, the angle of repose of the inundated material. The bank outline may remain fixed only if sufficient material is added through sliding or flow down the bank to match that withdrawn by the stream. On the basis of these considerations it may be seen that the banks below the structure will be eroded by the stream until this equilibrium can be reached. Therefore:

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[7] "Some Aspects of Gully Development, Classification and Control," by N. A. Christensen, thesis presented to California Institute of Technology in 1938 in partial fulfilment of the requirement for the degree of Doctor of Philosophy.

> Every effort that can be made to reduce the lateral attacking power of the stream will reduce the extent of channel widening.

Just as the components of high transporting power must be kept away from the stream banks at the exit from the stilling pool, so must they be kept away from the stream bottom. The designer and constructor of a drop structure are tempted, at first, to try to decrease the danger of scour by making an "easy" transition from the end sill or apron to the gully floor. Unfortunately, a smooth plane extension of the natural sediment bottom from the masonry of the structure (see Fig. 4) is not always stable under the flow of the stream. If such a surface were maintained by the equilibrium of scour and deposition of the stream, the stability of any local part of the surface (say, at the downstream edge of the structure) would be sensitive to random or accidental fluctuations in the transporting power of the stream. Such a fluctuation would produce a small disturbance in the bottom composed of a pit and a dune of excavated or disturbed material. This disturbance in the bottom will itself give rise to further local increase in the transporting power of the stream. Thus, the requisites for instability are present in the phenomenon, and a scour hole is certain to develop. The rate of growth of this scour hole is controlled, at first, by the balance of the strength of the initial disturbance and the stability of the sediment particles and, later after the scour hole has attained considerable depth, by the relative depths of the scour hole and the stream itself.

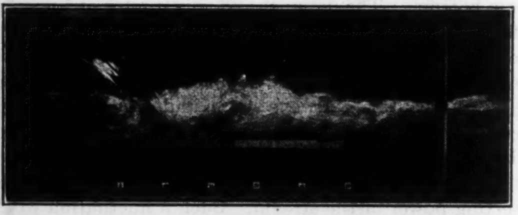
Recognizing the futility of scour hole prevention as a means of protecting the structure from caving, the designer must see to it that the hole that is formed does not endanger the structure. This he may do by forcing it to be developed far enough downstream from the end sill of the structure that the part adjacent to the structure will be too shallow to be dangerous with normal cutoff wall provision. After the first development of a scour hole downstream from the structure, the scour process itself tends to place the deepest part of the hole farther and farther downstream. The discontinuity of flow lines formed at the downstream end of the structure, when scour lowers the stream-bed level, encourages separation in the flow.

This separation is evident in the form of a "ground roller" whose top elements move downstream and whose bottom elements move upstream. Until this ground roller is formed, sediment immediately downstream from the structure can be removed by the stream but cannot be replaced because the motion of the water and the entrained sediment is downstream. After the ground roller has been formed, the upstream flow adjacent to the bottom can bring sediment from the downstream parts of the scour hole to replace that removed through the roller. From this understanding of the behavior of the scour hole in the vicinity of the drop structure can be deduced a rule for the determination of desirable flow conditions at the end of the structure:

> The flow must be such as to develop a protective ground roller before the end of the structure has been laid bare to a dangerous depth.

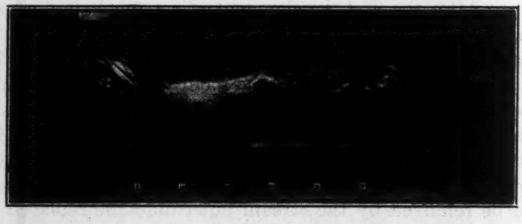
(The importance of the "ground roller" to scour control has been emphasized many times in the literature of spillway design.)[8]

In order that the effluent stream shall have the minimum sediment-transporting power economically obtainable, the over-all dissipation effectiveness of the stilling pool should be a maximum and the excess energy in the effluent stream, a minimum.



(*a*) Tailwater Depth Less than Critical

(*b*) Jump Forming Immediately Downstream



(*c*) Stilling Pool Operating Submerged

FIG. 5.—PERFORMANCE OF AN EXPERIMENTAL DROP STRUCTURE AT THREE TAILWATER STAGES

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[8] "Stauraumverlandung und Kolkabwehr," by A. Schoklitsch, Julius Springer, Vienna, 1935, p. 85; "Low Dams" (handbook), National Resources Committee, Washington, D. C., 1938, p. 106; "The Causes and Prevention of Bed Erosion," by Arthur Douglas Deane Butcher and John Dekeyne Atkinson, *Minutes of Proceedings*, Inst. C. E., Vol. 235 (1932–33), pp. 175–222 and discussion; and "Dissipation of Energy below Falls," by C. C. Inglis and D. V. Joglekar, Bombay, P.W.D., *Technical Paper No. 44*, Bombay, 1933.

For the selection of the best structure from the laboratory experiments, the following forms of the drop-structure performance criteria were used:

(1) The kinetic energy of the effluent stream, as measured by the excess of local velocity over the velocity for absolute minimum specific energy, shall be a minimum (absolute minimum energy is obtained in flow at critical depth);

(2) The flow in the vicinity of the banks of the gully shall be as nearly parallel to the banks as possible and shall have a minimum of eddy motion; and

(3) A large ground roller shall be produced in the flow over the end sill of the stilling pool before any scour has occurred.

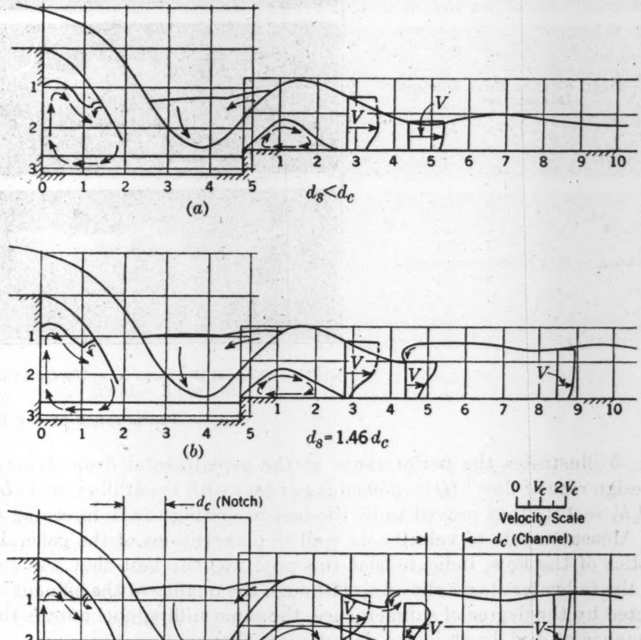
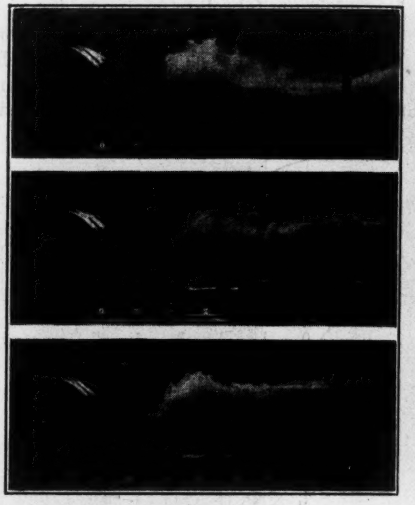

1
2
3
0 1 2 3 4 5
1 2 3 4 5 6 7 8 9 10
V
V
$d_s<d_c$
(a)
1
2
3
0 1 2 3 4 5
1 2 3 4 5 6 7 8 9 10
V
V
V
$d_s = 1.46\,d_c$
(b)
d_c (Notch)
0 V_c $2V_c$
Velocity Scale
d_c (Channel)
1
2
3
0 1 2 3 4 5
1 2 3 4 5 6 7 8 9 10
V
V
V
$d_s = 1.92\,d_c$
(c)

FIG. 6.—DIMENSIONLESS PLOTS OF FLOW PATTERNS AT THE CENTER LINE OF THE EXPERIMENTAL DROP STRUCTURE

RESULTS OF THE FIRST SERIES OF EXPERIMENTS

The first series of experiments with drop structures of varying stilling-pool design furnished considerable information in regard to the selection of apron lengths and sill heights, and led to the development of a new device for the reduction of bank scour.

Although photographs and flow and velocity measurement records were made of more than one hundred combinations of apron length and sill height, stream discharge and tailwater depth, it will be necessary to present only a few of them to demonstrate the type of drop-structure performance obtained.


A.—Tailwater Depth Less than Critical

B.—Jump Forming Short Distance Downstream

C.—Stilling Pool Slightly Submerged

(*a*) Twice Normal Discharge ($Q = 700$ Cu Ft per Sec; and $L = 16$ Ft)

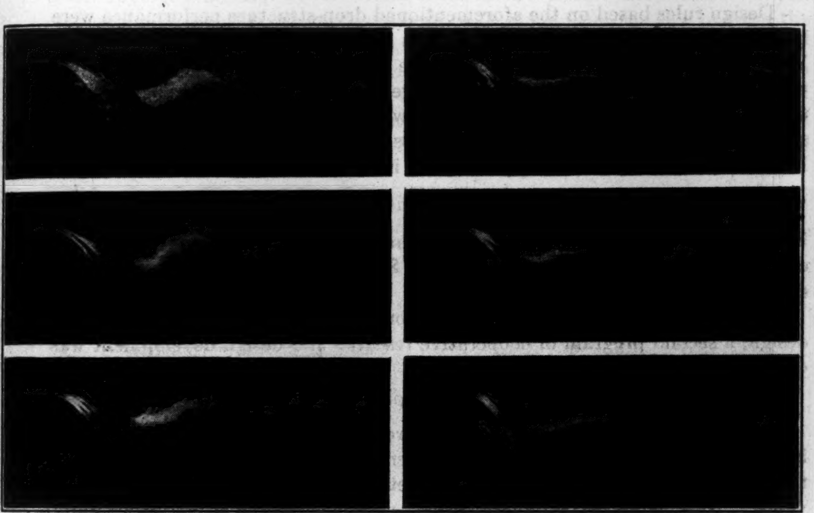
FIG. 7.—PERFORMANCE OF THE EXPERI-

Fig. 5 illustrates the performance of the experimental drop structure at the "design rate of flow" ($Q = 350$ cu ft per sec) with the stilling pool ($L = 16$ ft, and $h' = 2$ ft) that proved to be the best according to the foregoing standards. Measurements of velocity, as well as observations of the general characteristics of the flow, indicate that this pool was the best that was tried at any of the tailwater stages shown. Although the quality of the effluent stream is affected by the degree of submergence, the same stilling pool affords the best performance at any depth in the downstream channel.

Measurements in the downstream channel gave maximum values of the ratio of local velocity to critical velocity of 1.6, 1.2, and 1.1 for the conditions shown in Figs. 5(*a*), 5(*b*), and 5(*c*), respectively. These maximum local velocities occurred well above the stream bottom, away from the sides, and about one pool-length downstream from the end of the structure (see velocity profiles in Fig. 6).

The ground roller required for the protection of the structure from undermining is present even before the excavation of a scour hole might begin. The end sill of the stilling pool has performed a double task in improving the energy dissipation in the pool and in deflecting the departing stream upward sufficiently to insure the development of a ground roller.

When the first tests were made with this experimental drop structure, a large part of the flow leaving the stilling pool was directed against the bank with a high velocity. To reduce the bank scour that would accompany such conditions, flow-straightening longitudinal sills (see Fig. 3) were placed on the apron at the third points of its width. Only one of these sills may be seen in Fig. 5 because only one is needed in the half model.



(*b*) Twice Normal Discharge (Q = 700 Cu Ft per Sec) with Stilling Pool Lengthened (L = 20 Ft)

(*c*) Original Discharge (Q = 350 Cu Ft per Sec) with Stilling Pool Lengthened (L = 20 Ft)

MENTAL DROP STRUCTURE (h' = 2 FT)

To emphasize the fact that material has not been wasted in constructing stilling pools of the size shown in Fig. 5, the photographs of Fig. 7(*a*), which indicate the performance of the same structure under a prototype flow of 700 cu ft per sec, are shown. It is immediately apparent that the apron is too short for either satisfactory energy dissipation or proper development of the ground roller. Unless the stream bed were otherwise protected, such a flow would excavate a scour hole so close to the end sill and so deep as to endanger the stability of the structure itself.

The photographs in Fig. 7(*b*) indicate the size of the stilling pool necessary to accommodate the flow of 700 cu ft per sec just described. Velocity measurements have shown that the energy dissipation performance of this structure corresponds exactly to that of the first structure at 350 cu ft per sec. The fact that both pools have the same depth may be taken as an indication that the pool designed for the 350 cu ft per sec flow might be made a little shallower.

The photographs of Fig. 7(*c*) are presented to indicate the nature of flow in stilling pools that are wastefully long. Here the flow of 350 cu ft per sec has been handled in the stilling pool designed for 700 cu ft per sec. The slight excess of depth is not at once apparent, but the section of parallel flow between the impingement zone and the curving flow at the end sill is a direct indication of the wasted length. Here again velocity measurements have shown energy dissipation performance equivalent to that shown in Fig. 5.

Design rules based on the aforementioned drop-structure performance were introduced in the engineering standards of the Pacific Southwest Region. Within a year and a half, sufficient value had been attached to them by experience in their use, so that requests were made to the regional engineer and to the laboratory for data that would allow the extension of the design rules to structures of other proportions than those used in this first series of tests. Examination of Fig. 5 and Fig. 7(*b*) will indicate that the experimental data were restricted to fairly narrow falls, with height-to-flow-depth ratios, $\frac{h}{d_c}$, between 2.5 and 1.5. Such drop structures were certainly typical of those encountered in the field practice of the Service, but there were also many drops of more extreme proportions.

As a means of extending the application of this type of laboratory data to design, a second program of cooperative research and design development was outlined.

THE SECOND SERIES OF EXPERIMENTS

Because experience in the first test program had indicated techniques to be used and the possibilities to be encountered in further testing, it was possible to plan the new test program in some detail and to design apparatus in advance. New, larger experimental drops were constructed in a testing flume of larger capacity than had been available before.

The new experimental drop (Fig. 8) was designed for operation at discharge rates from 0.5 cu ft per sec to 5.0 cu ft per sec for fall heights of 1.74 and 0.87 ft. Since the "half-model" technique was again used to facilitate examination of the flow at the center line of the structure, the 1.5-ft width of the supply flume corresponded to an approach channel width of 3.0 ft in a complete structure. Although the first arrangement of the experimental drop produced a condition geometrically similar to that used in the previous series of tests, this drop was not considered to be a model of any particular prototype structure. To emphasize the general applicability of the test results, all measurements were expressed as dimensionless ratios.

Rapidity of measurement and adjustment of the experimental variables and ease of visual and photographic observation were given consideration in the design of the experimental drop installation. The flume in which the new installation was made is of the closed circuit type and is equipped with a remote-control variable-speed pump and a pair of venturi meters. With this equipment the time required to change and redetermine the rate of flow through the experimental drop is very short. Timber and plywood construction made the alteration of the proportions of the drop itself a simple task.

The entire working section of the experimental drop installation was placed high enough above the ground to permit horizontal photography at convenient tripod and camera heights. The several sheets of heavy plate glass making up the window side of the installation were butt-joined with a transparent plastic cement that acted both as a water stop and as a structural "cushion" between the imperfectly cut edges of the individual panes. The top edge of the glass was supported laterally by removable crossties or by removable outside braces, depending on whether photographs were to be made from the side or from the top, respectively. The vertical load capacity of the glass sheets was supplemented by means of pipe stanchions which were kept in place except when photographs were to be made from the side.

A reference grid was established using hard-drawn aluminum wires that were spring-loaded to prevent sag from temperature change. The thrust of the spring anchorage was transmitted through the glass as an added safeguard against the opening of the butt joints.

FIG. 8.—EXPERIMENTAL DROP STRUCTURE WITH PITOT TUBE AND TRAVERSING EQUIPMENT IN USE

A final precaution in the design of the drop installation, taken to insure undistorted photography, was the arrangement of space so that the cameras might be set 10 to 15 ft away from the glass.

Point-gage depth measurements and pitot-tube velocity measurements were facilitated by the provision of a wheeled carriage supporting traversing equipment and running on pipe rails. One of these pipe rails was connected to the domestic water-supply lines in order to provide pressure for flushing the pitot-tube and manometer velocity-measuring system. The carriage was made heavy and stiff to insure dependability of the reference planes in measurement.

In making this provision, the designer unintentionally furnished the experimenters with a solid but movable working platform that was very convenient in making photographs from above the drop and channel system.

Most of the photographs were made with a 5-in. by 7-in. reflex camera and a studio-type 16 mm motion-picture camera. All still photographs were developed and examined in negative form before proceeding with successive stages of the experiment. The care taken to insure good quality in the photographic work was thoroughly justified by the fact that most of the final conclusions of the study were based on measurements of the flow outlines as recorded in the photographs.

Sketches and velocity-distribution records were prepared only in the extent that they were necessary to the interpretation of the photographic record of the experiments. The decision to conduct the experiments under such a policy was made because of the great contrast between the time required for the two types of data recording and the recognized brevity of the time allowed for active testing. Only in this way was it possible to complete so widespread an investigation in nine weeks of active testing.

Selected parts of the motion-picture record of the experiments have been combined into an educational film which has been used in the instruction of field technicians in the Pacific Southwest Region. This film is considered to be a valuable auxiliary to the written report of the experiment.

RESULTS OF THE SECOND SERIES OF EXPERIMENTS

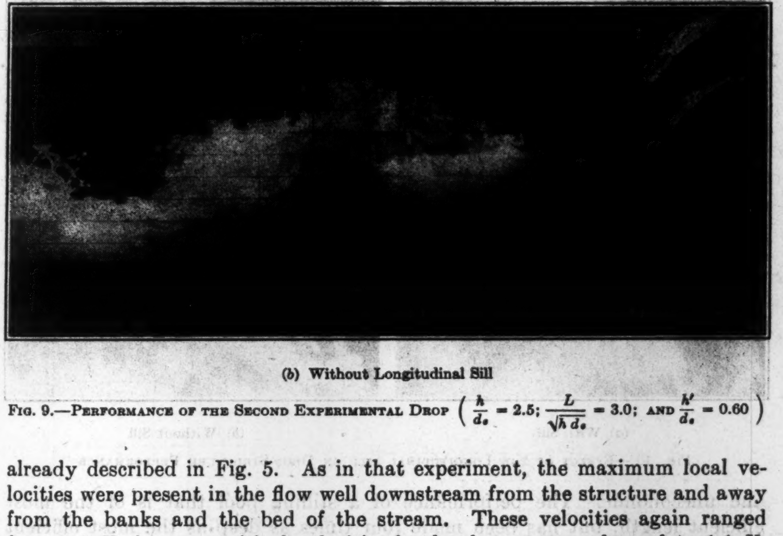
The first encouraging result of the second series of experiments was the duplication, through experiment, by different individuals with equipment of a different scale, of the results obtained in the first series of laboratory tests. Fig. 9(*a*) may be compared with Fig. 5 to demonstrate this point.

When the results of the first series were discussed, no photographs were presented that might show the purpose and result of the installation of the longitudinal sills on the apron. Fig. 9(*b*), taken from the second series, may be contrasted with Fig. 5 and Fig. 9(*a*) to show the performance of the structure without the longitudinal sills. Since the primary evidence of the improvement accomplished by the use of the longitudinal sills is the shifting of the high "plume" of rapidly moving water and spray from the bank to the center of the stream, these photographs do not show the difference as well as does Fig. 10, which demonstrates the performance with and without the longitudinal sill, respectively. Although the improvement in the flow conditions near the bank was sufficient justification for the use of longitudinal sills, it has been possible to detect an improvement in the over-all energy dissipation effectiveness of the stilling pool, as well. This improvement was noted in velocity measurements and in the decrease of the distance downstream to the beginning of the hydraulic jump for a given tailwater stage.

The tests of this second series covered a wide range of structures, from low drops with thick flows over their crests to high falls with thin sheets of water passing over them. In terms of the ratio of fall height to critical depth (calculated for the notch), $\frac{h}{d_c}$, the height range of the drops extended from 1.0 to 15.

Fig. 11 shows how stilling pools of appropriate length and depth, for falls of many different heights, were used to give performance equivalent to that

(a) With Longitudinal Sill



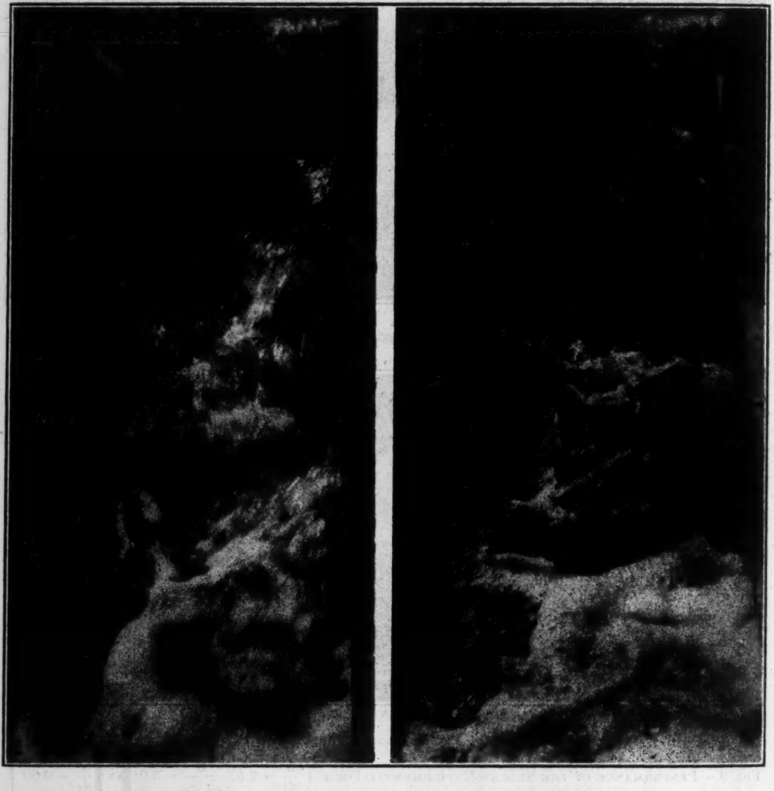
(b) Without Longitudinal Sill

FIG. 9.—PERFORMANCE OF THE SECOND EXPERIMENTAL DROP $\left(\frac{h}{d_c} = 2.5;\ \frac{L}{\sqrt{h\, d_c}} = 3.0;\text{ AND }\frac{h'}{d_c} = 0.60\right)$

already described in Fig. 5. As in that experiment, the maximum local velocities were present in the flow well downstream from the structure and away from the banks and the bed of the stream. These velocities again ranged from 1.6 V_c for supercritical velocities in the downstream channel to 1.1 V_c for submerged operation with the tailwater stage at 2 d_c (V_c = critical velocity for the downstream channel).

To obtain the most efficient structures, the second test program included pools that had other proportions than those shown in Fig. 11. Inadequate "stilling" and dangerous distribution of the flow over the end sill characterized structures that were too small, such as the shallow pool of Fig. 12(*a*) and the short pool of Fig. 13(*a*).

The long stilling pool of Fig. 13(*b*) gave (like the long pool of Fig. 7(*c*)) performance that was not measurably better than that of the pools already described as satisfactory. Therefore, the use of long pools is regarded as wasteful and uneconomic. The performance of a stilling pool that is of the most efficient length, but has been made four times as deep as the most efficient depth (see Fig. 12(*b*)), resembles the performance of the short structures of Figs 7(*a*) and 13(*a*). Although the energy-dissipation performance of the deep pool is fairly good, the direction of the flow over the end sill is such as to promote the formation of a deep scour hole immediately downstream from the



(*a*) With Sill (*b*) Without Sill

FIG. 10.—EFFECT OF THE LONGITUDINAL SILL ON DROP-STRUCTURE PERFORMANCE

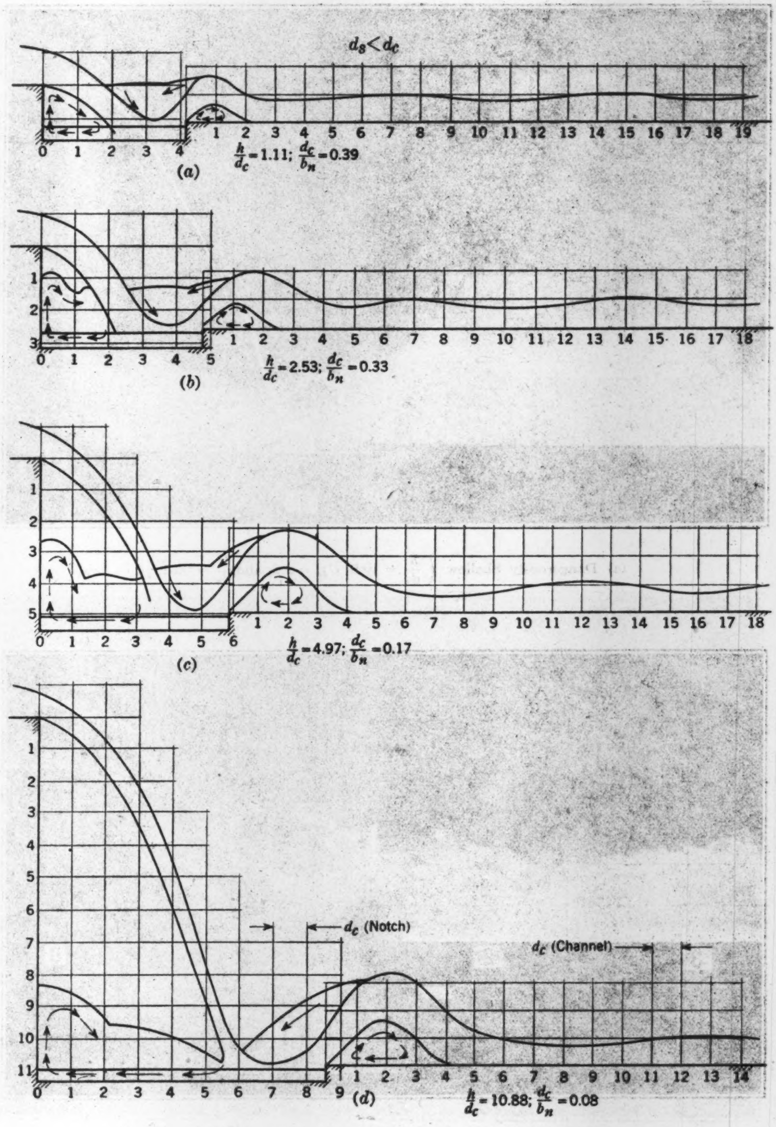
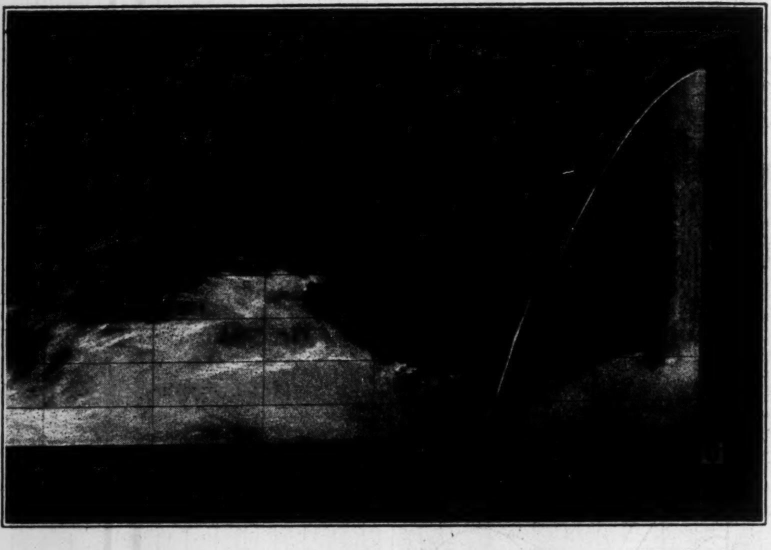
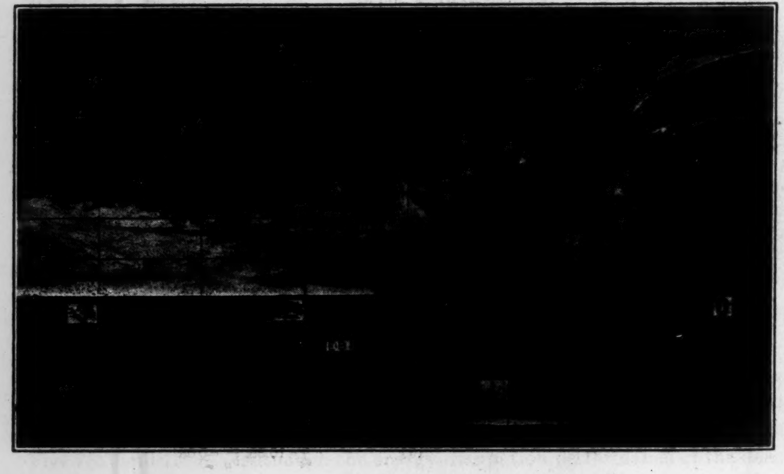

$d_s < d_c$
0 1 2 3 4
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19
$\frac{h}{d_c} = 1.11$; $\frac{d_c}{b_n} = 0.39$
(a)
1 2 3
0 1 2 3 4 5
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
$\frac{h}{d_c} = 2.53$; $\frac{d_c}{b_n} = 0.33$
(b)
1 2 3 4 5
0 1 2 3 4 5 6
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
$\frac{h}{d_c} = 4.97$; $\frac{d_c}{b_n} = 0.17$
(c)
1 2 3 4 5 6 7 8 9 10 11
d_c (Notch)
d_c (Channel)
0 1 2 3 4 5 6 7 8 9
1 2 3 4 5 6 7 8 9 10 11 12 13 14
(d)
$\frac{h}{d_c} = 10.88$; $\frac{d_c}{b_n} = 0.08$

FIG. 11.—FLOW PATTERNS FOR DROPS OF VARIOUS RELATIVE HEIGHTS

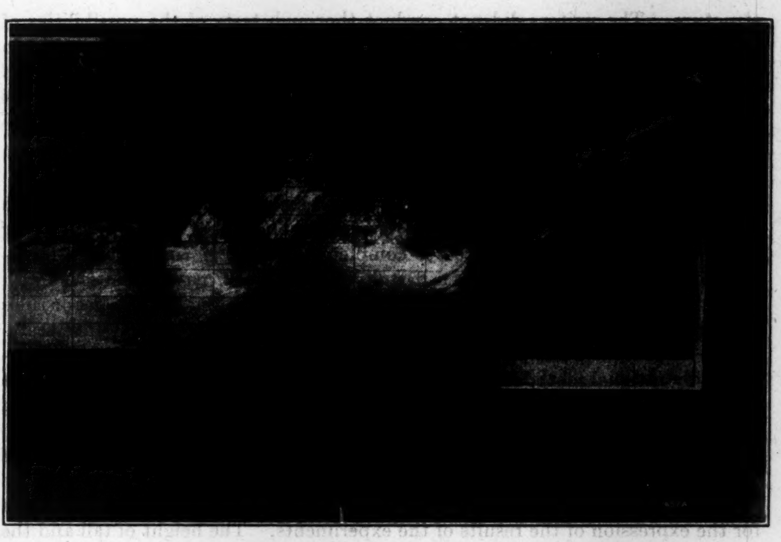


(a) Dangerously Shallow $\left(\frac{h}{d_c} = 10.9; C_L = 2.6; \text{and } \frac{h'}{d_c} = 0.31 \right)$



(b) Excessively Deep $\left(\frac{h}{d_c} = 2.5; C_L = 3.0; \text{and } \frac{h'}{d_c} = 1.95 \right)$

FIG. 12.—EFFECT OF DEPTH OF STILLING POOL



(*a*) Dangerously Short $\left(\frac{h}{d_c} = 1.0; \text{ and } \frac{h'}{d_c} = 0.30 \right)$

(*b*) Excessively Long $\left(\frac{h}{d_c} = 14.3; \text{ and } \frac{h'}{d_c} = 1.15 \right)$

Fig. 13.—Effect of Length of Stilling Pool ($C_L = 3.0$)

structure. The writers did not conduct thorough tests of the possibilities of deep stilling pools in which this behavior would be eliminated by increase in the pool length but they believe that the improvement in energy dissipation that might possibly be gained in this manner would not warrant the great increase in construction cost necessary to obtain it.

DEVELOPMENT OF DESIGN FORMULAS FROM EXPERIMENTAL DATA

Although the experiment programs described herein were successful in developing efficient drop structures of various proportions, the development of these designs was not considered to complete the study. In order that structures of any proportions required in gully-control work may be designed with equal facility and dependability, it is necessary that rational equations or rules be written that will unite the information from individual designs into statements of general principles. From these studies it has been possible to develop, by rational methods, rules governing the length of aprons and the height of end sills and describing the location and proportions of the flow-straightening longitudinal sills.

1. Apron Length.—Eq. 2, which was already in use for the determination of apron length at the time the first test program was started, has a sound enough rational basis to make it a good starting point in developing a design formula for the expression of the results of the experiments. The height of fall and the critical depth for the notch are retained as the controlling variables.

The results of the tests shown in Figs. 5 and 7(*a*) suggested the use of Eq. 2 with a coefficient of $C_L = 3.0$. The establishment of this value of the coefficient, together with the experimental finding that a similar equation with a coefficient of $C_L = 2.0$ could be used to describe the trajectory of the water falling from the crest, suggests that the apron length requirement consists of two parts—(*a*) the distance required for the falling water to reach the apron and (*b*) the length of apron required to establish the energy-dissipating flow pattern.

For the time between the two series of tests, $C_L = 3.0$ in Eq. 2 was made the standard for apron length determinations. That the single value of the coefficient used in this equation should be too great for one extreme type of drop structure and too small for another was not surprising. The unsatisfactory performance of designs arrived at through its use has already been shown in Fig. 13.

As information was produced in the second series of tests, attempts were made to analyze the changes that occurred in the performance of structures of increasing or decreasing height. The experiments on high falls with thin flows were conducted first.

Soon it appeared that the distance between the point of impingement of the nappe and the end sill should not necessarily be either equal to or proportional to $\sqrt{h\, d_c}$. It seemed more likely that this distance should be related to the thickness of the impinging sheet of water. If this sheet has an original thickness proportional to d_c, then, neglecting the effects of friction and of mixing of air and water, it may be shown that the acceleration of gravity will make the thickness after falling a distance, h, proportional to $d_c \sqrt{\frac{d_c}{h}}$. Assuming that this thickness controls the remaining length of the apron, an equation for the coefficient, C_L, of Eq. 2, may be written:

$$C_L = 2.0 + C_1 \frac{d_c \sqrt{\frac{d_c}{h}}}{\sqrt{h\, d_c}} = 2.0 + C_1 \frac{d_c}{h} \ldots\ldots\ldots\ldots\ldots (3)$$

This change in the coefficient proved too drastic (see Fig. 11, $\frac{h}{d_c} = 4.97$, $\frac{h}{d_c} = 10.88$) and, by cut-and-try methods, a new equation was determined which represented the data correctly:

$$C_L = 2.5 + 1.2 \frac{d_c}{h} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

From the success of Eq. 4 it must be concluded that the effect of the thinning of the falling jet on the additional length required after impingement is not so severe as it was at first assumed to be.

The extension of the tests into the low fall, thick flow range (see Fig. 11, $\frac{h}{d_c} = 1.11$), showed that the foregoing adjustment is not sufficient to account for all of the added length requirements of this type of fall. Here the physical picture is different; additional length is now required before impingement. It is not to be expected, in the first place, that the parabolic trajectory equation, with its origin at the crest of the drop, should apply satisfactorily to the description of the flow near the crest itself. The finite thickness of the flow alone is sufficient to make the nappe fall above and downstream from the location thus predicted.

A second feature of the flow changes the pattern in low falls: The equilibrium of forces and momentum changes in the vicinity of the point of impingement requires that the water under the fall should stand above the tailwater stage established by downstream conditions. This water under the fall contributes to the support of the nappe even before the reduction of the fall height creates support from submergence alone (see Figs. 9(*a*), 12(*b*), and 13(*a*)).

The development of an analytical expression to adjust the coefficient of the apron-length equation for these effects has been too complicated a task for this study. Instead, an empirical term has been developed to express the influence of those features of the flow that will require longer aprons for low-drop structures:

$$C_L = 2.5 + 1.1 \frac{d_c}{h} + 0.7 \left(\frac{d_c}{h}\right)^3 \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

If this coefficient is used, the complete equation for the stilling-pool length may be written:

$$L = \left[2.5 + 1.1 \frac{d_c}{h} + 0.7 \left(\frac{d_c}{h}\right)^3 \right] \sqrt{h\, d_c} \ldots\ldots\ldots\ldots (6a)$$

or

$$\frac{L}{\sqrt{h\,d_c}} = 2.5 + 1.1\,\frac{d_c}{h} + 0.7\left(\frac{d_c}{h}\right)^3 \ldots\ldots\ldots\ldots\ldots (6b)$$

For the convenience of designers Eq. 5 has been plotted in its dimensionless form in Fig. 14. In developing Eqs. 2 to 6 it has been assumed that the dam crest constitutes a control section. If shooting flow in the upstream channel is not forced to jump before reaching the crest, these pool-length rules are not applicable to the design of the structure in question.

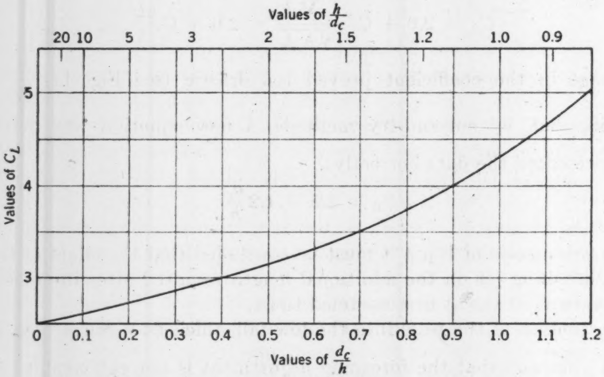


FIG. 14.—THE APRON LENGTH RULE (CURVE OF EQ. 6)

2. Height of Transverse End Sill.—The rules used for the determination of the end-sill height have been completely changed during the period of the laboratory work on drop structures. At the beginning of this period, the standard design of the Engineering Division of the Pacific Southwest Region called for a stilling pool whose depth was one eighth of the fall height, so that

$$h' = \frac{1}{8}\,h \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7a)$$

After the tests shown in Fig. 5 and Fig. 7(*b*) were made, this rule was discarded in favor of

$$h' = 0.10\,L = 0.3\,\sqrt{h\,d_c} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7b)$$

At the conclusion of the second series of experiments the third rule was stated:

$$h' = \frac{1}{2}\,d_c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7c)$$

in which d_c is still the critical depth at the weir notch. The weakness of the second rule is apparent in Fig. 13 which has been cited as belonging to the same stage of design development.

This form of the sill-height rule (Eq. 7*c*) was not anticipated before the experiments were completed and was accepted, with some reluctance at first, as the only reasonable statement of the results of the experiments. Rather than attempt a complete analysis, the writers will state their conception of the factors that make this relationship possible.

The height of the end sill of the stilling pool necessary to develop energy-dissipating action is apparently determined by the depth and energy content of the flow in the pool.

Other tests conducted in the laboratory have demonstrated[9] that the loss of kinetic energy incident to the impingement of the nappe on the apron is a large quantity and that this loss increases rapidly with increasing fall height so as to offset, to an appreciable extent, but not completely, the increased energy of the higher falls. From these findings it can be stated that the flow away from the impingement zone still will be shallower for high falls than for low drops. Such shallow flows, of course, will have greater momentum than the deeper ones. It was desired to turn these shallower, swifter flows upward and over the end sill in such a manner as to insure the development of the energy-dissipating flow and the ground roller so essential to the protection of the structure. For this purpose the end sill must be proportionally higher than for the deeper slower currents from the impingement of lower falls.

The results of the experiments would indicate that the increase in relative sill height with increasing velocity and decreasing flow depth in the pool is such that the same depth of stilling pool, or height of end sill, relative to the flow depth at the crest is required for all fall heights.

3. Location of Longitudinal Sills.—Longitudinal sills were introduced as flow-straightening devices in drop-structure stilling pools for the first time during the first series of experiments. The size and shape of the sills were established during the first stages of the experiments and have remained unaltered so far as recommended designs may be concerned. Sills, three quarters of the end-sill height, are the most satisfactory. The width of these sills is determined by the materials used and structural considerations in general. For ordinary reinforced concrete construction (sills less than 3 ft high) a 6-in. width is adequate.

When the second series of experiments was planned, it was believed that the operation of the experimental drop at many flow stages at a constant width would produce data from which the possibility of other arrangements of the longitudinal sills than the third-point spacing originally used might be investigated. The experiments failed to satisfy this prediction. The third-point spacing furnished the best performance from beginning to end of the series of experiments originally planned.

From this result of the tests and from observation of the effects of the lateral contraction of the flow at the crest of the drop, it was concluded that the spacing of the longitudinal sills, like the lateral contraction of the nappe, was determined by the horizontal configuration of the contraction alone and was independent of the fall height and the flow depth. Accordingly, an auxiliary series of experiments was conducted in which the amount of the contraction of the flow at the crest of the drop was varied, whereas the fall height and the flow depth were held constant.

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[9] "Energy Loss at the Base of a Free Overfall," by Walter L. Moore. See this volume of *Transactions*.

Fig. 15(*a*) shows the performance of the drop with the sill at the third point of the width of the hypothetical complete structure. The pronounced contraction of the flow and the presence of an offset for the ventilation of the nappe make this, the contraction used in all previous experiments, a very severe lateral effect. The distance between the edge of the nappe and the side-wall of the stilling pool may be considered to be a measure of this severity.

Figs. 15(*b*) through 15(*d*) show, continuously, the effect of the successive elimination of the causes of the contraction of the nappe. In each case the longitudinal sill has been placed in a position for maximum efficiency in flow straightening. In the case of parallel flow (Fig. 15(*d*)) this adjustment of the longitudinal sill has necessitated its complete removal since its ultimate location would be flush with the side-wall.

The importance of flow contraction at the drop-structure crest to the production of crosscurrents over the end sill was noted by Professor Etcheverry,[10] who recommended avoiding lateral contraction where possible. In gully-control work the shape of natural gully cross sections makes contraction difficult to avoid in any simple structure. The flow-straightening sills are the simplest and most economical devices that may be used to counteract contraction effects.

Although an investigation of nappe-ventilation provisions had not been anticipated in either series of drop-structure tests, the provision of ventilation by lateral offset or contraction, the two flow features governing the longitudinal-sill location, has become an inseparable part of the design problem.

It is possible to describe, with some confidence, the flow phenomena that govern the longitudinal-sill locations. As the contracted, or offset, flow passes from the crest of the drop to the stilling pool, the nappe fails to span the apron width uniformly. This lateral inequality becomes effective, at impingement, as a pair of lateral pressure gradients giving the flow over the end-sill components of velocity toward the banks of the gully. The function of the longitudinal sills is to prevent the development of such lateral components by separating the high-pressure and low-pressure regions of the impingement zone. It becomes clear, from this analysis, that the longitudinal sills must be placed near the third points when the pressures are changing over a large part of the apron width and nearer to the side when the region of pressure differences is small.

In the development of an equation for the spacing of the longitudinal sills as controlled by the notch contraction and the ventilation offset, it has proved more convenient to use an analogy with another flow contraction phenomenon than to attempt to follow step by step the process of producing the pressure gradients on the apron.

As the stream flow above the drop is contracted, it experiences a drawdown from the depth before contraction to a depth slightly less than the critical at the notch itself. The effect of curvature is to make this last depth depart from the calculated critical depth; but if this difference is neglected, the drawdown

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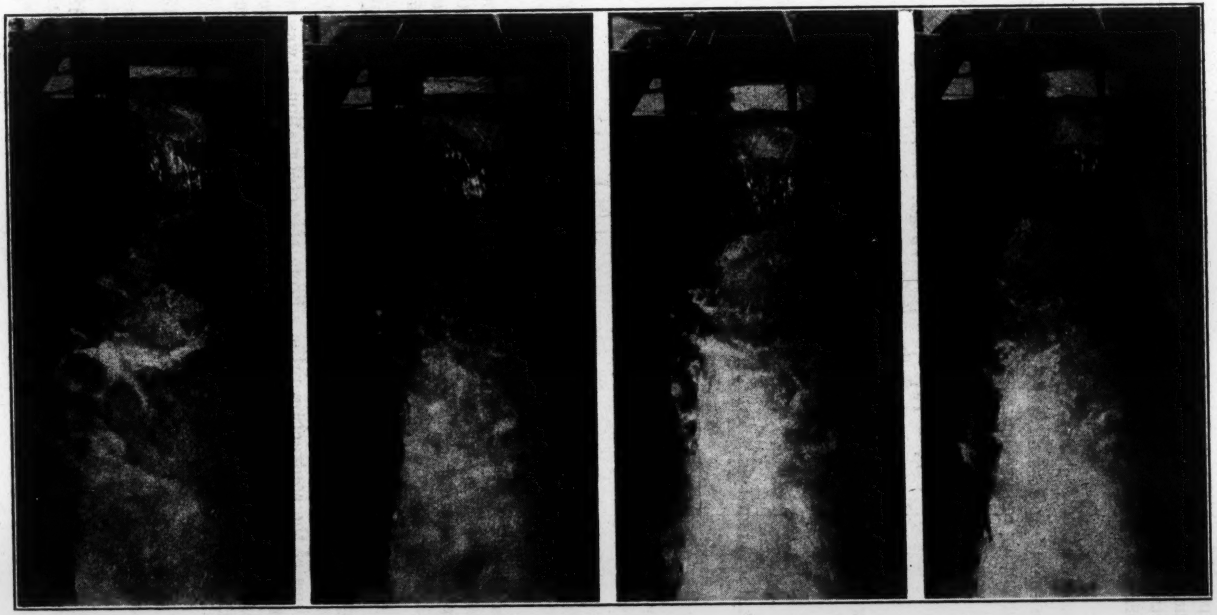
[10] "Irrigation Practice and Engineering," by B. A. Etcheverry, Vol. III, McGraw-Hill Book Co., Inc., New York, N. Y., 1916, pp. 238, 243.



(a) $\frac{b_w}{b_n} = 1.47$; $\frac{a}{b_n} = 0.031$; $\frac{x'}{b_n} = 0.32$ (b) $\frac{b_w}{b_n} = 1.38$; $\frac{a}{b_n} = 0$; $\frac{x}{b_n} = 0.21$ (c) $\frac{b_w}{b_n} = 1.11$; $\frac{a}{b_n} = 0$; $\frac{x}{b_n} = 0.12$ (d) $\frac{b_w}{b_n} = 1.00$; $\frac{a}{b_n} = 0$; $\frac{x}{b_n} = 0$

FIG. 15.—EXPERIMENTS ON THE CONTROL OF THE SPACING OF LONGITUDINAL SILLS BY LATERAL CONTRACTION

can be computed from the energy of a stream flowing at critical depth and the Bernoulli energy equation written for the section upstream from the contraction and the section at the notch.

The energy in the flow over the crest is

$$\frac{3}{2} d_c = d_s + \frac{V^2_s}{2\,g} \quad \text{(8)}$$

which is the energy upstream from the contraction.

From continuity requirements,

$$Q = V_n\, b_n\, d_n = V_c\, b_n\, d_c = V_s\, b_w\, d_s \quad \text{(9a)}$$

and

$$V_s = V_c \frac{b_n\, d_c}{b_w\, d_s} = \frac{b_n\, d_c}{b_w\, d_s} \sqrt{g\, d_c} \quad \text{(9b)}$$

Substituting in Eq. 8: $\frac{3}{2} d_c = d_s + \frac{1}{2} d_c \left(\frac{b_n}{b_w}\right)^2 \left(\frac{d_c}{d_s}\right)^2$; and

$$\frac{3}{2}\left(\frac{d_s}{d_c}\right)^2 = \left(\frac{d_s}{d_c}\right)^3 + \frac{1}{2}\left(\frac{b_n}{b_w}\right)^2 \quad \text{(10)}$$

Substituting the identity,

$$\frac{d_s - d_c}{d_c} + 1 = \frac{d_s}{d_c} \quad \text{(11)}$$

in Eq. 10 and transposing, the drawdown equation is obtained,

$$\left(\frac{d_s - d_c}{d_c}\right)^3 + \frac{3}{2}\left(\frac{d_s - d_c}{d_c}\right)^2 = \frac{1}{2}\,\frac{\left(\frac{b_w}{b_n}\right)^2 - 1}{\left(\frac{b_w}{b_n}\right)^2} \quad \text{(12)}$$

The tests devised for the study of sill location as a function of the notch contraction showed that no sills were required when there was no contraction and that increasing contraction moved the sills from their hypothetical position flush with the walls to positions near the third points of the apron width. Comparison of the experimental values of $\frac{x}{b_n}$ with the corresponding computed values of $\frac{d_s - d_c}{d_c}$ indicated that each had the same functional trend with changing values of the contraction ratio. From this observation it was concluded that the sill spacing, the departure of the nappe edge from the plane of the notch edge, and the drawdown were similar measures of the contraction of the flow at the crest. From the experiments shown in Fig. 15,

$$C_x = \frac{x}{b_n} = 0.60\,\frac{d_s - d_c}{d_c} \quad \text{(13a)}$$

and

$$\left(\frac{x}{b_n}\right)^3 + 0.90\left(\frac{x}{b_n}\right)^2 = 0.108\,\frac{\left(\frac{b_w}{b_n}\right)^2 - 1}{\left(\frac{b_w}{b_n}\right)^2} \quad \text{(13b)}$$

or,

$$C_x^3 + 0.90\,C_x^2 = 0.108\,\frac{\left(\dfrac{b_w}{b_n}\right)^2 - 1}{\left(\dfrac{b_w}{b_n}\right)^2} \quad \dots\dots\dots\dots\dots (14)$$

Since cubic equations of the type of Eq. 13*b* do not lend themselves to quick solution, a graph of Eq. 14 in its dimensionless form is used in design work.

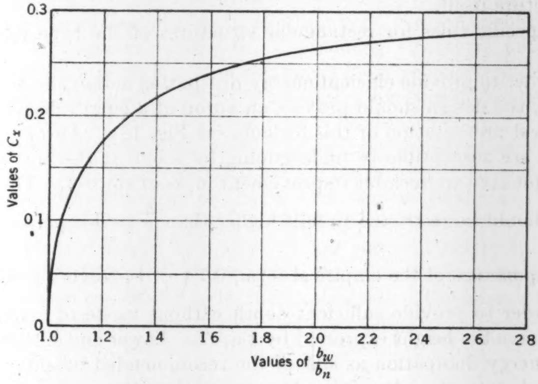


FIG. 16.—THE SILL-LOCATION RULE (EQ. 14)

This graph (Fig. 16) shows the relation between the sill spacing and the notch-contraction ratio.

The data on which Eq. 13*b* is based were determined from experimental drops with wing walls directed 30° upstream from the line of the crest. Although no experiments were made with walls of different orientation, sills located according to Eq. 13*b* should improve the flow conditions for almost any angle ordinarily met. The results of experiments with sill locations other than those recommended show that they still enabled the sills to improve flow conditions.

Eq. 13*b* gives the best sill location for drop structures that do not have an offset between the edge of the notch and the side-wall of the stilling pool. From what little information was obtained in tests it appears that the best results are produced if the distance between the sill and the edge of the notch, x, is increased by an amount equal to three times the notch offset, so that $x = C_x\,b_n$ and

$$x' = x + 3\,a \dots\dots\dots\dots\dots\dots\dots\dots (15)$$

The contraction of the flow not only makes the edge of the nappe depart from the plane of the notch edge, but also distorts the cross section of the nappe so that the flow is thickest near the center. The ventilation offset, for the purposes of these tests a small distance by comparison with the notch width, serves only to offset the nappe, the high pressure area on the apron, and the sills farther from the side-walls.

SUMMARY OF DESIGN RULES AND FORMULAS

Two general rules for all drop structures are:

1. The structure must drop the water within its confines and discharge it downstream in such a way as to cause a minimum of locally intensified erosion; and

2. The structure must discharge the stream in such a manner that flows less than, or equal to, the design discharge rate will not bring about the undermining of the structure itself.

Three specific rules for rectangular structures of the type treated in this paper are:

3. In order to provide efficient energy-dissipating action and to avoid waste of material, the design should provide an apron of a length derived by Eq. 6*a* (for graphical presentation of this formula see Fig. 14). Aprons shorter than this length are susceptible to undermining by scour at the end sill. Longer aprons do not give appreciable improvement in scour control. The application of Eq. 6*a* should be restricted to falls higher than $\frac{h}{d_c} = 1.0$ because of the increasing importance of the empirical term, $0.7\left(\frac{d_c}{h}\right)^3$, to very low falls.

4. In order to provide sufficient depth without waste of material the end sill should have the height expressed by Eq. 7*c*. Lower end sills do not give as complete energy dissipation as sills of the recommended height. Higher sills add to the danger of undermining by scour unless they are used with longer aprons than would be provided by Eq. 6*a*. The amount of improvement in scour control, which may possibly be obtained through simultaneous increase in depth and length of stilling pools, is believed to be so small as not to justify the increased cost of construction.

5. If the drop is laterally contracted, longitudinal sills should be installed on the apron at a distance x' from each side of the weir notch as given by Eq. 15, in which x is determined from the solution of Eq. 13*b* (for graphical presentation of formula, see Fig. 16). These recommended spacings are based on the use of wing walls placed at angles of 30° with the axis of the dam, but should be applicable to most wing-wall orientations commonly used. Sill locations other than those recommended are less effective in flow straightening and scour control, but give better performance than does the complete omission of the sills.

The longitudinal sills should be made three quarters of the height of the end sill of the stilling pool. The width of the sills is determined by the strength of the material used for their construction. Since Eq. 15 is empirical and is based on measurements at $\frac{a}{b_n} = 0.031$, its application should be restricted to the range $0 < \frac{a}{b_n} < 0.031$.

It is recognized that the foregoing design rules are based on hydraulic considerations only. Therefore, they must be regarded as minimum requirements susceptible to increase on the basis of other design considerations.

The foregoing rules and formulas have been in use by the Engineering Division of the Pacific Southwest Region, SCS, since July, 1940.

Gully-Control Installations of Free-Overfall Drop Structures

The gully-control structures most nearly resembling those used in the experiments are constructed of reinforced concrete and are of substantial size. Structures similar to the one shown in Fig. 1 have been built for falls from 5 to 30 ft in height, from 8 to 75 ft in width, and from 50 to 5,000 cu ft per sec in discharge rate. To all sizes of structures of this type, the design rules developed in this study are directly applicable. Obviously, structural and economic considerations will limit the height of structures for which the free-overfall type of spillway is feasible; but the hydraulic principles involved in the designs given are independent of scale for flows more than a few inches deep.

Smaller structures are often more economically constructed of masonry using local stone. The same rules that govern the outline of the water passages in the reinforced concrete structure have been adapted to these. The downstream face of a masonry drop normally has an appreciable batter. Since these drops are ordinarily designed to operate at low ratios of height to depth, the nappe will not strike the face of the wall and, therefore, the starting point of the pool length has been considered to be the crest of fall. The space taken up by the sloping section of the breast wall is not considered to be important in the action of the stilling pool, except that the nappe should fall free of the face for its entire height.

Under special conditions of exposure and operation it is desirable to construct rectangular drop structures of logs or of timber. Here again, the rules developed for the reinforced concrete structure will govern the hydraulic proportions of the structure.

The writers do not expect that the limitations in the applicability of Eq. 6*a* to low falls will be encountered often in gully-control drop design, because of the possibility of the use of other types of structures in place of the free-overfall and rectangular stilling-pool type they have discussed.

Application of Drop-Structure Rules to Other Types of Hydraulic Structures

The principles used in developing the rules for the design of rectangular drop structures for gully control are sufficiently general for application to many problems. There is no great difference between the methods that have been used in insuring energy dissipation and in protecting the structure from undermining and those commonly used in the design of spillways for major engineering structures. Although structural and economic considerations require departure from the rectangular construction discussed herein, the flow patterns for major spillways and stilling pools are still such that kinetic energy is converted to turbulence in the pool and that the flow over the end sill produces a reverse ground roller that protects the end of the structure.

At the other end of the scale of spillway structures used in hydraulic engineering are temporary "checks" constructed of various types of inexpensive materials, such as boards, loose rock, and brush. Only the promise of inexpensive construction has kept this type of dam a part of engineering practice. Perhaps the application of some of the ideas presented herein may help in the identification and elimination of some of the characteristic weaknesses of this type of construction. Already experience has pointed out several types of failure: Undermining due to scour downstream, undermining due to percolation, lateral by-passing, and structural disintegration brought about by hydrostatic or hydrodynamic forces.

The first of these causes of check-dam failures has been analyzed in this study of the rectangular drop structure and it would appear that the thoughtful application of the design rules presented in this paper should prevent failures from this cause alone. The other three causes of failure may be regarded as expressions of rules limiting the use of cheaply-erected checks of various materials.

CONCLUSION

The writers have presented hydraulic design rules and formulas now in use by the Pacific Southwest Region, SCS, U. S. Department of Agriculture, and have described the experimental and analytical development of these rules at the laboratory of the Cooperative Research Project of the Soil Conservation Service and the California Institute of Technology.

The analyses presented in the development of design rules for drop structures may be regarded as statements of progress in a long-term investigation of the mechanics of energy dissipation and of localized scour. The writers realize that many problems remain to be studied, and hope that this paper will serve to indicate some of them. Conspicuous among items considered for future study are: (*a*) The importance of lateral contraction and divergence of the flow in the structure to energy dissipation performance; (*b*) stability of flow over the end sill; (*c*) energy dissipation at the end sill; (*d*) analysis of eddy-production in the lateral separation zone at the banks below the end wall of the structure; and (*e*) development of methods for reducing the scouring power of the eddy system of Item (*d*).

The particular studies reported in this paper were made through the cooperation of field technicians and a laboratory research group. It is anticipated that this valuable method of design data development will be continued.

ACKNOWLEDGMENTS

The operations of the Soil Conservation Service in the Pacific Southwest Region are directed by Harry E. Reddick, Assoc. M. Am. Soc. C. E., Regional Conservator. Engineering activities in the Region are supervised by J. G. Bamesberger, Regional Engineer. The research work of the laboratory is under the direction of Robert T. Knapp, M. Am. Soc. C. E., for the California Institute of Technology, and Vito A. Vanoni, Assoc. M. Am. Soc. C. E., for the Soil Conservation Service.

Prior to August, 1938, the work of the laboratory in this particular field was directed by Mr. Christensen, to whom the writers are much in debt for the information developed in the first series of experiments. In the second series, the writers were assisted by Wilson B. Jones, Jun. Am. Soc. C. E., who made the experimental measurements and collected the data of that series.

DISCUSSION

John Hedberg,[11] Assoc. M. Am. Soc. C. E.—The authors have done a very commendable job in establishing rules for the design of drop structures. It is especially gratifying to find that the rules have been founded on the basic fundamentals of flow rather than upon purely visual inspection.

However, there are several factors involved in the study that the authors have not chosen to discuss but which deserve some attention. The first of these is the use of a half section of the structure instead of the whole. The center wall interferes with the lateral movement of eddies, and therefore a better alinement of flows probably results in the model tested than in the corresponding prototype. Is it justifiable to dismiss this factor?

There is also a question of whether a model study involving a large amount of entrained air yields comparable results. The authors have varied dimensions to find the best combination in their model, based upon the flow characteristics of a combined air and water mixture. Is it not possible that the mixture will be quite different in the prototype and consequently have somewhat different flow characteristics?

There remains the question of the influence of the surrounding air pressure on the flow through the structure and its effect on the model comparison. It is so much trouble to run tests at less than atmospheric pressure that it is not surprising that strict similarity of forces is usually ignored. However, the shape and size of the ground roller will depend on the surrounding air pressure. Since the protection of the structure depends on this roller, there is a real question of whether the violation of similarity here is not of some importance.

L. Standish Hall,[12] M. Am. Soc. C. E.—The research work reported by the authors to determine the proper design of drop structures is noteworthy both for the way in which the experiments were made and for the manner in which the results have been presented. The existing knowledge of the proper length of the basin below the drop has been extended so that a proper design is now possible under a wide range of conditions.

One of the most important features, in the writer's opinion, was the development of the longitudinal sill, which has proved very effective in eliminating bank scour below drops. It is remarkable that so simple a device can be so effective in solving a troublesome problem. Twenty drop structures built under the writer's direction have operated very satisfactorily and show no signs of bank erosion. Some of these structures, under flood conditions, are described elsewhere.[13]

The authors have suggested several names descriptive of these structures and have adopted the term "stilling basin." The terms "roller basin" or "tumble bay" would be more descriptive of the action of the water in passing through the basin. The action greatly resembles that occurring in a jump basin at the toe of a dam where the basin is too shallow.[14]

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[11] Associate Prof., Civ. Eng., Stanford Univ., Stanford University, Calif. Professor Hedberg died on August 3, 1942.

[12] Hydr. Engr., East Bay Municipal Utility Dist., Oakland, Calif.

[13] "Drop Structures for Erosion Control," by L. Standish Hall, *Civil Engineering*, May, 1942, p. 247.

[14] "Study of Stilling-Basin Design," by C. Maxwell Stanley, *Transactions*, Am. Soc. C. E., Vol. 99 (1934), p. 490.

Mention has been made of the difficulty of determining equilibrium slopes and stable grades between drop structures. Some data have been gathered on this phase of the problem that may be of interest. At the end of the 1941 runoff season, the silt grade was determined above several of the erosion-control dams on the San Pablo Creek (California) watershed, where an appreciable deposit had occurred. The ponds are only partly full at the present time (March, 1942), and the grade which the silt deposits finally will take after the ponds are eventually filled with detritus is of importance in determining the effective capacity of the structures. Sufficient observations have not been made to arrive at any definite conclusions, but it appears that the grade taken by the silt deposits is inversely proportional to the size of the drainage area. The maximum size of material carried by the creek also has an effect on the silt grade, coarser material taking a steeper slope. The size of rock fragments is governed in part by the type of rock formation in a specific drainage area. On the small drainage areas ranging up to 600 acres, the silt grade was found to vary between 1.3% and 2.3%, whereas on the larger areas it ranged from 0.25% to 1%. The results from the several areas are given in Table 1.

TABLE 1.—Silt Gradients above Drop Structures

Structure No.; SP:	Drainage area (acres)	Estimated runoff[a]	Silt grade (%)	G (in.)[b]
(a) West Side, San Pablo Reservoir				
26	321	253	2.1	12
26X1	321	253	1.8	3
11	177	161	2.3	3
11A	177	161	1.8	3
10	133	131	2.0	8
10A	133	131	2.0	8
(b) East Side, San Pablo Reservoir				
18	548	408	2.2	1
16	47	63	1.6	1
24	97	110	1.3	1
(c) Scow Canyon				
13	1,210	618	1.0	1.25[c]
(d) San Pablo Creek				
201D	8,700	3,000	0.4	6
501N	1,100	640	0.25	...[d]

[a] Estimated 25-yr storm runoff, in cubic feet per second. [b] Material graded from G inches down. [c] Material graded from fine silt to a 1.25-in. maximum. [d] Private dam above crossroads; material would pass a No. 10 screen.

After the pond above a drop structure has filled with debris so that sand and gravel pass over the crest during flood discharges, there appears to be a pronounced tendency to scour the central part of the stream bed between the longitudinal sills. On one or two structures, under the writer's observation, the placing of riprap or paving in this location is indicated as desirable within a few years.

This bed erosion has occurred on channels having a steep profile. In many cases the natural longitudinal profile of the stream ranges from 1 to 5%. Calculation indicates that the velocity in the channel below the structure is slightly above the critical. Under these circumstances, the "ground roller" would be greatly reduced in size from that observed in the laboratory model, in which the exit channel was on a relatively flat grade. This problem of bed erosion below the drop would arise on structures for gully control, and would not be of importance generally where the structure was used as a drop in an irrigation canal due to the controlled rates of flow, flatter slopes, and lower velocities in the latter.

J. E. Christiansen,[15] Assoc. M. Am. Soc. C. E.—Most hydraulic laboratory studies are confined to large important works, where possible savings in a single structure would pay the costs involved. The fact that appreciable savings may be effected by improving the design of small, relatively simple drop structures, used in large numbers, often has been overlooked. The paper by Messrs. Morris and Johnson is apparently the first record of a laboratory study to determine the proper proportions of such structures.

For several years the writer has been interested in the improvement of irrigation systems, especially farm systems and others composed of relatively small canals that ordinarily receive little engineering consideration. Especially significant in this field is the large number of small structures. Since they are individually inexpensive, they are seldom designed carefully. Often they are not designed at all—just built according to the ideas of the construction foreman. That such structures could be greatly improved is clearly evident from field inspections. One might make many improvements by applying sound engineering practice, but still more improvements might result from laboratory investigations of the type described by the authors.

Among the most common structures in irrigation systems in California are combination check gates and drops, which are used to take up the excess grade in the canals, and also to control the water-surface elevation for diversions. In design these resemble the drop structures described by the authors, but they are generally lower and therefore may have higher ratios of d_c/h. They differ also in that they have adjustable crests controlled by flashboards.

Typical check gates are those used by the Consolidated Irrigation District near Fresno, Calif. These structures are representative of those used in districts that have had the benefit of a good engineering department. Originally designed about 1925 by I. H. Teilman, M. Am. Soc. C. E., chief engineer of the District, they have been only slightly modified since then, and have proved entirely satisfactory.

Fig. 17 shows the general design of these check gates; Table 2 gives their principal dimensions. The letter symbols are the same as in Fig. 3. Representative views appear in Figs. 18 and 19. The former is a relatively small structure with single opening. Although it has been in use for some time, there is no

TABLE 2.—Principal Dimensions of Check Gates Used by Consolidated Irrigation District (See Fig. 17)

H	h[a]	L'	L	h'	A	B	C[a]	G	UW	LW
3	0	6	4.0	0.5	0	1.50	0	1.0	4	4
4	0.5	7	5.0	0.5	0	1.50	1.0	1.0	4	4
5	1.0	8	6.0	0.5	0.5	1.50	1.5	1.0	5	6
6	1.5	10	8.0	0.5	1.0	1.50	2.0	1.5	6	8
7	2.0	12	9.5	1.0	1.5	2.00	3.0	1.5	7	9
8	3.0	14	11.5	1.0	2.0	2.00	4.0	1.5	8	10
9	4.0	16	12.75	1.0	2.5	2.75	5.0	1.5	8	12
10	4.0	16	12.75	1.0	3.0	2.75	5.0	1.5	9	12
12	5.0	18	13.75	1.0	3.0	3.75	7.0	3.0	9	14
14	6.0	20	15.75	1.0	4.0	3.75	8.0	3.0	9	15
16	7.0	22	17.75	2.0	4.0	3.75	9.0	3.0	9	16

[a] Minimum values.

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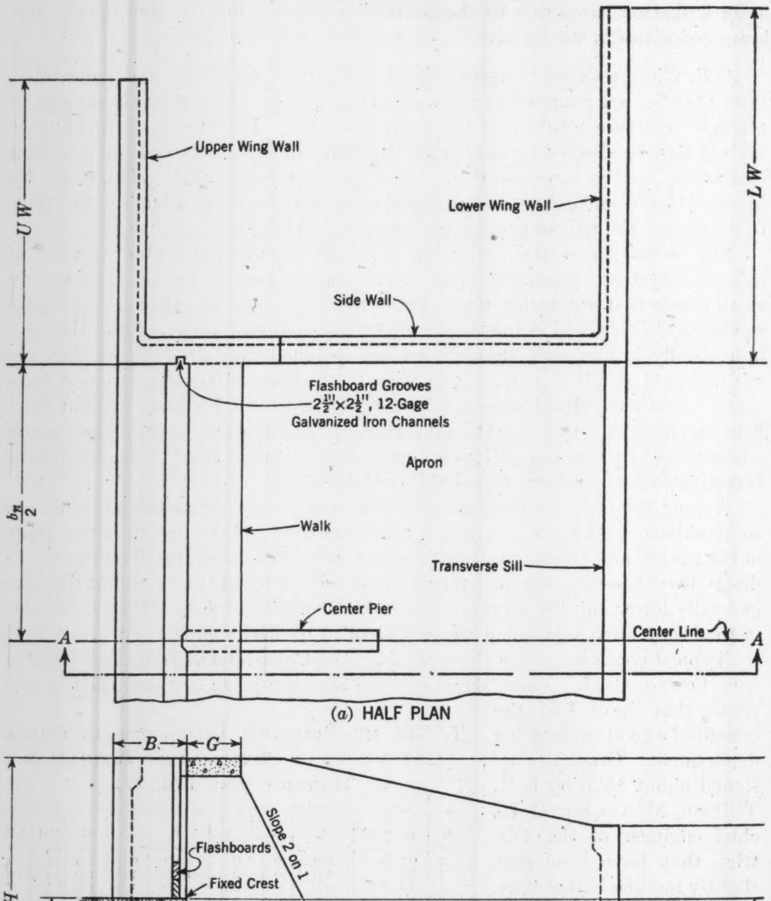
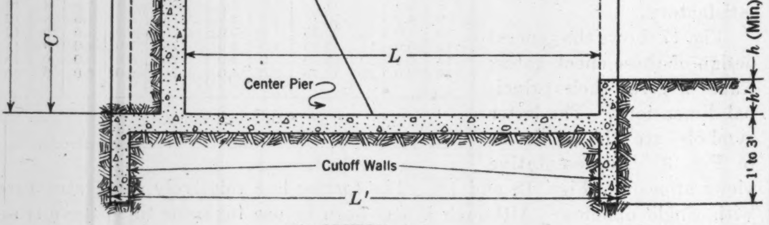
[15] Irrig. and Drainage Engr., U. S. Regional Salinity Laboratory, U. S. Dept. of Agriculture, Riverside, Calif. (formerly Asst. Irrig. Engr., Coll. of Agriculture, Univ. of California, Davis, Calif.).


Upper Wing Wall
U W
Lower Wing Wall
L W
Side Wall
Flashboard Grooves
$2\frac{1}{2}'' \times 2\frac{1}{2}''$, 12-Gage
Galvanized Iron Channels
Apron
$\frac{b_n}{2}$
Walk
Transverse Sill
Center Pier
A
Center Line
A

(*a*) HALF PLAN


B
G
Slope 2 on 1
H
Flashboards
Fixed Crest
C
L
h (Min.)
Center Pier
h'
Cutoff Walls
1' to 3'
L'

(*b*) SECTION *A-A*

FIG. 17.—STANDARD CHECK GATE USED IN CONSOLIDATED IRRIGATION DISTRICT

visible erosion on the downstream side. Fig. 19 shows one of the larger structures, the total height above the apron being 16 ft. There are eight openings, with a total width between the side-walls of 64 ft, 8 in. The photograph was taken shortly after the structure was completed and before water had passed through.

The main differences between these and the drop structures described by the authors are as follows:

(1) Upper wing walls at right angle to center line of structure;
(2) Crest a short distance downstream from upper cutoff and wing walls;
(3) Adjustable flashboard crest;
(4) Walkway over crest to handle flashboards; and
(5) No longitudinal sills.

Originally, the transverse sill was placed from 1 to 2 ft upstream from the lower cutoff wall, but more recently was moved to the end of the structure, thereby increasing the length of the apron. These check gates have been built in heights (top of walk to top of apron) from 3 ft to 20 ft, and in widths (inside of side-walls) from 3 ft, 8 in., to more than 60 ft. Standard practice is to make the widths 4 in. less than an even number of feet so that flashboards can be cut from even-dimensioned lumber without waste. For widths between 7 ft 8 in. and 15 ft 8 in. a center supporting pier is used, and the flashboards span both openings. For greater widths, two or more intermediate piers are used, and alternate ones are provided with flashboard grooves. Normally, wide struc-

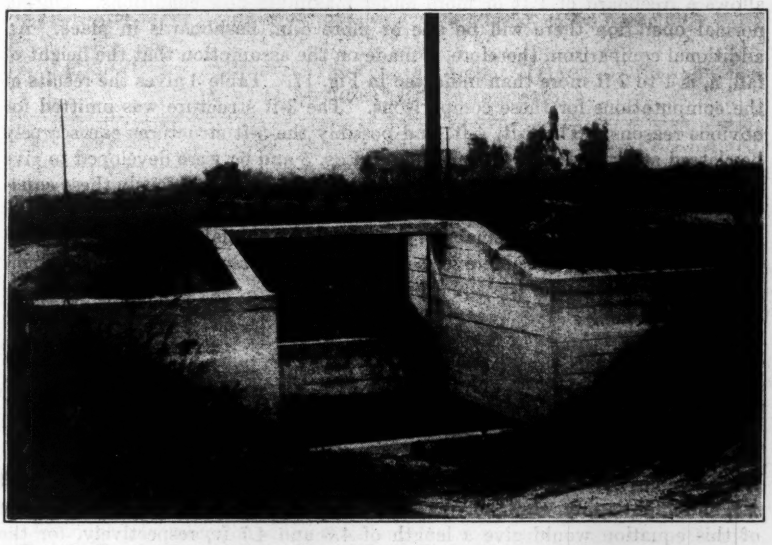


FIG. 18.—TYPICAL CHECK GATE AS USED IN SMALL CANAL—HEIGHT, 7 FT; WIDTH, 7 FT 8 IN.

tures are built with an odd number of intermediate piers, so that all flashboards span two openings.

To minimize the cost of these structures, the practice in the Consolidated District is to standardize the principal dimensions and use collapsible forms made up in panels that can be bolted together. In height, these standard sizes range from 3 to 16 ft. The length and most of the other dimensions are functions of the total height above the apron. Under special conditions, certain other dimensions are varied.

Longitudinal sills are not used and would possibly offer no advantage, since lateral contraction is eliminated largely by placing the crest a short distance downstream from the upper wing walls. There is no provision for ventilating the nappe. Where the drop is not great, the crest is often submerged, especially at high flows. Under these conditions wave action downstream is severe. The downstream wing walls are made longer than the upstream wing walls to give maximum protection against bank erosion downstream. Additional bank protection for a short distance is sometimes required.

To compare the apron length and the height of the transverse sill with the authors' recommendations as stated in Eqs. 6*a* and 7*c*, the critical depths, d_c, were computed from the design capacities furnished by Mr. Teilman. This allows a freeboard of 1 ft or more under maximum flow conditions. During normal operation there will be one or more 6-in. flashboards in place. An additional comparison, therefore, is made on the assumption that the height of fall, h, is 1 to 2 ft more than indicated in Fig. 17. Table 3 gives the results of the computations for these comparisons. The 3-ft structure was omitted for obvious reasons. The 3-ft, 4-ft, and possibly the 5-ft structures can scarcely be classed as drop structures. Although Eqs. 5 and 6*a* were developed to give additional length for conditions of low fall and deep flow, obviously these equations should be restricted to ratios of h/d_c that do not exceed 1.0, as mentioned by the authors in their summary of design rules. A relation of the same form as Eq. 4 might be applicable over a wider range of values of d_c/h. One might use, for example, one of the equations

$$C_L = 2.5 + \frac{2\, d_c}{h} \qquad \text{(16a)}$$

or

$$C_L = 2.2 + \frac{2\, d_c}{h} \qquad \text{(16b)}$$

Eq. 16*a* gives only slightly higher values of C_L for values of d_c/h between 0 and 1.1, and might apply reasonably well for values of d_c/h up to 2 or more. The use of this equation would give a length of 4.6 and 4.7 ft, respectively, for the assumed values of d_c and h for the 4-ft structure as compared with 7.3 and 4.3 ft (Table 3). Eq. 16*b* gives almost the same values of C_L as Eq. 5 for values of d_c/h between 0.4 and 0.9, and lower values of C_L outside of this range.

Although the authors recommend shorter aprons, experience indicates that the longer aprons are better than the shorter ones formerly used. Quite likely

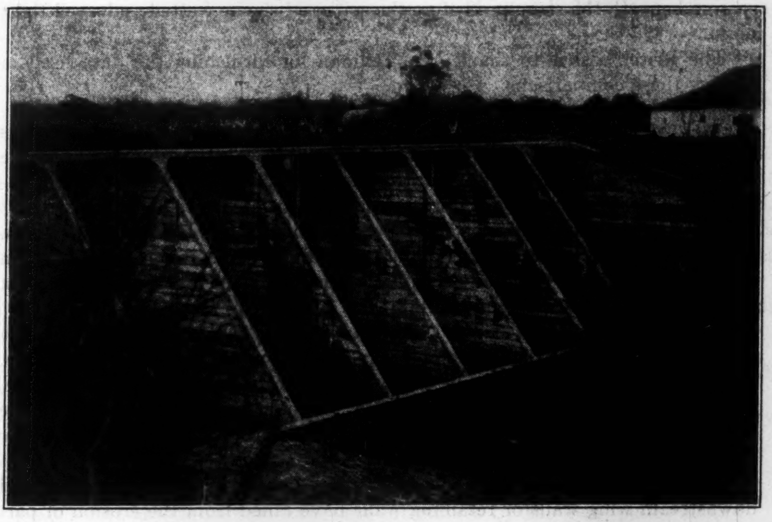


FIG. 19.—LARGE CHECK GATE SHORTLY AFTER COMPLETION—HEIGHT, 16 FT; WIDTH, 64 FT 8 IN.

TABLE 3.—COMPARISON OF APRON LENGTH AND HEIGHT OF TRANSVERSE SILL WITH AUTHORS' RECOMMENDATIONS

ACTUAL DIMENSIONS			FIRST COMPARISON[a]					SECOND COMPARISON[b]		
H	L	h'	d_c (max.)	h (min.)	C_L (Eq. 5)	L (Eq. 6a)	h' (Eq. 7c)	h	C_L (Eq. 5)	L (Eq. 6a)
4	5.0	0.5	1.0	0.5	10.30	7.3	0.50	1.5	3.45	4.3
5	6.0	0.5	1.2	1.0	5.03	5.3	0.60	2.0	3.31	5.1
6	8.0	0.5	1.4	1.5	4.08	5.9	0.70	2.5	3.25	6.1
7	9.5	1.0	1.6	2.0	3.74	6.7	0.80	3.0	3.18	7.0
8	11.5	1.0	1.8	3.0	3.31	7.7	0.90	4.0	3.06	8.2
9	12.75	1.0	2.0	4.0	3.14	8.9	1.00	5.0	2.98	9.4
10	12.75	1.0	2.0	4.0	3.14	8.9	1.00	5.5	2.93	9.7
12	13.75	1.0	2.1	5.0	3.01	9.7	1.05	6.5	2.88	10.6
14	15.75	1.0	2.1	6.0	2.92	10.4	1.05	8.0	2.80	11.5
16	17.75	2.0	2.3	7.0	2.89	11.6	1.15	9.0	2.79	12.7

[a] Based on values of d_c computed from design flows, and minimum values of h (no flashboards). [b] Based on same d_c and higher values of h (assuming flashboard in place).

the depth of water downstream is greater, and the velocities lower, than for the tests. Would this depth influence the desirable length of the apron?

Although any extension of the total length of the structure would increase its cost, the placement of the upstream cutoff wall a short distance upstream from the crest gives greater stability and additional insurance against undermining. This would seem to be a particular advantage in gully-control structures in the West, where during the long periods of drought the soil dries out and shrinks away from the walls. A sudden flow in the dry channel may result in water passing beneath the structure before the soil can become wetted and swell tight again.

The writer wishes to thank Mr. Teilman for furnishing the data used in making these comparisons.

Walter T. Wilson,[16] Assoc. M. Am. Soc. C. E.—During the years 1934 through 1938 the writer designed, supervised construction of, and made field inspections of, many gully-control structures. Literally hundreds of reinforced concrete and masonry erosion-control structures came to the writer's attention during this period. Had the research described in this paper been available at that time, great savings would have resulted not only from the economy of rational design, but also from the reduction of losses from under-design and subsequent failure.

Although the authors stressed the fact that they recognized that hydraulic limits are only part of the problem, it is relevant to mention a few of the others. The length of the floor must be great enough to provide adequate fill slope, top width of fill, and reasonable freeboard. The floor of the structure must be long enough to prevent overturning or "floating" of the structure from the hydrostatic head of the impounded water retained by it.

Lateral scour seems to have been given too little attention. Toe walls and downstream wing walls or retaining walls have failed from the erosion of part of the fill or of the original gully banks by the water discharging from the structure and forming a vortex. Attempts to control this lateral scour have included pervious pile wing dams radiating out from the sides of the structure.

The study of seepage past the structure may be included logically under the subject of hydraulics. The dimensions of the cutoff walls and other parts of the structure must be considered in the light of Darcy's law relating the static head to the path of flow or the length of seep or creep lines.

In the upper Mississippi region it was found that under many conditions the economical type of gully drop structure was either a drop inlet or a flume. Drop inlets, consisting of a box culvert with riser, have many advantages over the open-notch type of structure. One of the advantages is the provision for a roadway across the gully at the site. An early laboratory study of the hydraulics of gully-control structures was reported at the University of Wisconsin in 1934.[17]

The writer believes that the most effective energy dissipater is a fairly deep stilling pool. He has seen many structures with a horizontal floor that were either too high or too low relative to the gully floor. In one case the floor might be silted full, eliminating the effectiveness of an acrobatic outlet as an energy dissipater. In the other case the floor might be left hanging, with danger of undercutting.

A properly designed flume with the lower end carried far enough below the subgrade has many advantages. An adequate stilling pool forms naturally. Variations in the elevation and gradient of the gully bottom from season to season can be accommodated. The flume, consisting of a hypotenuse in contrast to the vertical and horizontal elements of a vertical drop structure, naturally requires less materials.

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[16] Asst. Hydrologic Engr., Weather Bureau, Washington, D. C.

[17] "Erosion Control Structures—Drop Inlets and Spillways," *Research Bulletin No. 122*, Univ. of Wisconsin, Madison, Wis., June, 1934.

N. A. CHRISTENSEN,[18] ASSOC. M. AM. SOC. C. E., AND DWIGHT GUNDER,[19] ESQ.—The exceedingly large number of drop structures needed in gully-control and irrigation work is reason enough for laboratory research of this kind. The actual value of the research becomes very evident when one considers the large number of failures of drops already built and also the large number which have been overbuilt in order to insure safety. The value of this research has already been demonstrated; but, if another wave of conservation and reclamation sweeps the United States after World War II, these researches will pay unusually large dividends.

The "half model" technique used in these experiments deserves further elaboration. The two principal advantages of this technique are:

(1) It permits accurate observation of conditions at the center line of the channel; and

(2) With a given model size, the quantity of water required is reduced by half.

The objections which might be raised are:

(1) Extraneous friction forces are introduced by the glass plate; and

(2) The turbulent action normally found at the center of the stream is eliminated by the rigid glass plate serving as a boundary.

The two advantages enumerated are obvious and need no further discussion. The following remarks will be confined to the disadvantages mentioned.

The friction forces introduced by the glass plate are a source of error, but in every model study of this kind friction forces are always a source of error because Froude's number is used as a criterion for similarity. With this number as a criterion, one is assured of the proportionality of the gravity and inertia forces in the prototype and model. The friction forces, on the other hand, are entirely neglected and considered insignificant as far as results are concerned. The smaller the model, the more serious is the error introduced by this assumption. It is quite likely that the increase in scale made possible by the "half model" technique more than compensates for the extra friction force introduced by the presence of the glass plate at the center line.

In order to appraise the importance of the second objection to the "half model" technique, attention is called to natural channels in which drop structures have been erected. From a study of these natural channels, it is obvious that the excessive scour which sometimes occurs just below the structure is caused by lateral currents of the stream itself and not by the haphazard turbulence which is present in most streams. If normal stream turbulence were the cause of the excessive scour, it would not be localized just downstream from the drop structure but would occur equally along the entire reach of the stream. These lateral currents and excessive scour are invariably associated with an unlevel water surface and the pressure gradients within the body of water which cause continuous currents having lateral velocity components. In most drop structures, a vertical plane of symmetry exists, which includes the center line of the structure. Not only is the structure itself symmetrical about this plane, but also the mean flow and the pressure field within the flow. In such structures a vertical plane may be inserted at the center line, and the mean flow will be essentially unchanged. It is true that the turbulent flow components will be reduced, but these components play only a secondary part in the excessive erosion that sometimes takes place just downstream from the drop structure.

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[18] Dean of Eng., and Chairman Eng. Div., Experiment Station, Colorado State College of Agriculture and Mechanic Arts, Fort Collins, Colo.

[19] Assoc. Prof. of Civ. Eng., Colorado State College of Agriculture and Mechanic Arts, Fort Collins, Colo.

The requirements necessary for the "half model" technique would seem to be:

(*a*) Geometric symmetry about the center line of the structure;
(*b*) Kinematic and dynamic symmetry of the mean flow; and
(*c*) Stability of the flow pattern normal to the plate at the center line.

Whenever these three conditions do not obtain, the "half model" technique should not be used.

In the paper, an inference was made that gully-control structures designed according to the principles set forth would require very little maintenance. For a series of drop structures to be permanently and completely successful, each structure must not only be soundly designed, but the reaches between the structures must also be completely stable. The question of what is a stable grade for the gully must first be answered before the height and spacing of the various drops can be determined. The importance of an accurate determination of a stable grade under the varying flow conditions of the gully should be emphasized as part of the over-all problem of gully control by means of structures. This problem is further complicated by the fact that the flow conditions of gullies may change rather rapidly due to various erosion-control practices.

The writers would like to request elaboration of certain parts of the paper. The statement made beginning nine lines from Eq. 7*c*—"The height of the end sill of the stilling pool necessary to develop energy-dissipating action is apparently determined by the depth and energy content of the flow in the pool"—seems to be somewhat inconsistent with Eq. 7*c*. This formula would call for the same sill height regardless of the height of drop. The height of drop, on the other hand, has much to do with the energy content of the flow in the pool. Eq. 7*c* seems to indicate that the height of the sill is determined by d_c alone, which in turn is a direct measure of quantity of flow. As far as Eq. 7*c* is concerned, the question of energy does not enter.

The last sentence of the third design rule (see heading "Summary of Design Rules and Formulas") seems to be somewhat inconsistent with the material contained in the article proper. The discussion involving Eqs. 3 to 6, as well as Fig. 14, seems to indicate that Eq. 6*a* would apply for values of $\frac{h}{d_c}$ smaller than 1. If the use of this formula is hazardous for values of $\frac{h}{d_c}$ smaller than 1, then a certain part of Fig. 14 should be deleted.

It should be noted also that, whereas the derivation of Eq. 6a starts with what appears to be a rational attack, the final equation is almost completely empirical, especially so in that it contains a purely empirical term. Further, the expression for the thickness of the falling sheet (given as proportional to $d_c\sqrt{\frac{d_c}{h}}$) is a rather rough approximation, which is satisfactory only if $\frac{d_c}{h}$ is relatively small. It can be seen at once that this expression, $d_c\sqrt{\frac{d_c}{h}}$, for the thickness approaches infinity as h becomes smaller. One is thus inclined to suspect that any equation using this thickness will be inaccurate in the region where $\frac{d_c}{h}$ is large. This is just what was found by the authors.

A slightly different attack on this problem will now be given. If the depth d_c is included in the calculation of the location of the point of impingement of the jet on the floor, and the other assumptions are as given in the paper, it is found that the horizontal distance x from the crest to the middle of the impinging sheet is $x = \sqrt{d_c\,(2\,h + d_c)}$. The thickness of the falling sheet under the same conditions is given by the continuity equation as

$$t = d_c\frac{\sqrt{d_c}}{\sqrt{2\,(h + d_c)}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17a)$$

and the horizontal width of this sheet is given by

$$w = t\sec\theta = \frac{d_c\,\sqrt{d_c}}{\sqrt{2\,(h + d_c)}} \times \frac{\sqrt{2\,g\,(h + d_c)}}{\sqrt{2\,g\left(h + \frac{d_c}{2}\right)}} = \frac{d_c}{\sqrt{2\,\frac{h}{d_c} + 1}}\ldots(17b)$$

in which θ is the angle between the vertical and the direction of the sheet filaments.

If one now assumes that the length of the apron is linearly dependent upon each of these equations (C and C_1 are empirical factors):

$$L = C\,\sqrt{d_c\,(2\,h + d_c)} + C_1\frac{d_c}{\sqrt{2\,\frac{h}{d_c} + 1}}$$

$$= \left(C + \frac{C_1}{1 + 2\,\frac{h}{d_c}}\right)\sqrt{d_c\,(2\,h + d_c)}\ldots\ldots\ldots\ldots\ldots(18)$$

Since the writers did not have the original data available for computations, a rough comparison of the apron lengths given by Eq. 18 and by Eq. 6a was made by assuming $C = C_1$. This gives

$$L = 2\,C\frac{\left(1 + \frac{h}{d_c}\right)\sqrt{d_c\,(2\,h + d_c)}}{1 + 2\,\frac{h}{d_c}} = 2\,C\frac{\left(1 + \frac{d_c}{h}\right)}{\sqrt{2 + \frac{d_c}{h}}}\sqrt{h\,d_c} = C'_L\,\sqrt{h\,d_c}\ldots(19)$$

The coefficient C'_L of $\sqrt{h\,d_c}$ in Eq. 19 corresponds to the C_L of the author's paper. Using $2\,C = 3.5$, the values of C'_L were computed and are compared with the values of C_L estimated in Fig. 14 (see Table 4). These results show a maximum variation of C'_L of about 5% from the values of C_L (excluding the doubtful value $\frac{d_c}{h} = 1.2$). Since this is accomplished without the inclusion of a purely empirical term such as the $0.7\left(\frac{d_c}{h}\right)^3$ in Eq. 6*a*, it might be considered as somewhat better than Eq. 6*a*.

TABLE 4.—VALUES OF C'_L COMPARED WITH VALUES OF C_L ESTIMATED IN FIG. 14

Coefficient	Ratio, $\frac{d_c}{h}$						
	0	0.2	0.4	0.6	0.8	1.0	1.2[a]
C'_L	2.5	2.8	3.15	3.45	3.8	4.1	4.35[a]
C_L	2.5	2.7	3.0	3.3	3.75	4.3	5.0[a]

[a] These values are in the range where the authors claim that C_L should not be used. The writers do not know what values should be used.

It should be noted that the foregoing results were obtained with just one empirical constant. Perhaps even better results could be obtained by adjusting both C and C_1.

The writers do not wish to claim that this is a rigorous rational derivation of the expression for apron length L, but it is felt that it may be helpful in further study of the problem of the design of gully-control structures, which the authors have so admirably attacked.

BORIS A. BAKHMETEFF,[20] M. AM. SOC. C. E., AND NICHOLAS V. FEODOROFF,[21] ESQ.—The paper presents an interesting endeavor to proportion structural design by basing the dimensioning on data derived from detailed observations of the physical aspects of the flow. It is evident from Fig. 6 that the dimensions of the stilling pool are principally conditioned by the horizontal distance from the wall at which the falling live vein strikes the apron, and implicitly by the angle at which such striking is effected. The latter seems in particular to determine the direction followed by the rebounding vein, and thus the length of the downstream channel required for the flow to assume more or less normal features.

It stands to reason that economy in construction, as well as more favorable outflow, would be secured if the falling vein were caused to strike the apron close to the wall and thus at a larger angle. This could be achieved by resorting to the so-called "adherent" type of nappes, which would follow the surface of the wall and which would strike the floor at the very toe of the structure. The general disadvantage of such nappes, of course, is their instability. In fact, when the brink of the fall has a sharp wedge, as in Figs. 3 and 6, the nappe, except for extremely low heads, will tend to detach itself from the wall, changing to the customary outline.

The situation can be remedied, however, by properly rounding the exit wedge. Indeed, observations show that, if the brink of the fall is shaped by inserting an arc of a circle, the adherent nappes become quite stable. Even a relatively small rounding seems to serve the purpose in securing a dependable vertical falling nappe, and that at comparatively high rates of flow. The situation is illustrated by Fig. 20, which depicts flow profiles obtained in the Fluid Mechanics Laboratory at Columbia University in New York, N. Y. The apparatus used in the experimenting has been well described elsewhere.[22] In Fig. 20(a) the wedge is rounded to a radius $r = 1$ in., whereas, in Fig. 20(b), $r = 3\frac{1}{2}$ in. In both figures the solid lines show the downstream surface profiles in free undisturbed outflow. The broken lines demonstrate the eventual effect

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[20] Prof., Civ. Eng., Columbia Univ., New York, N. Y.

[21] Research Associate, Civ. Eng. Dept., Columbia Univ., New York, N. Y.

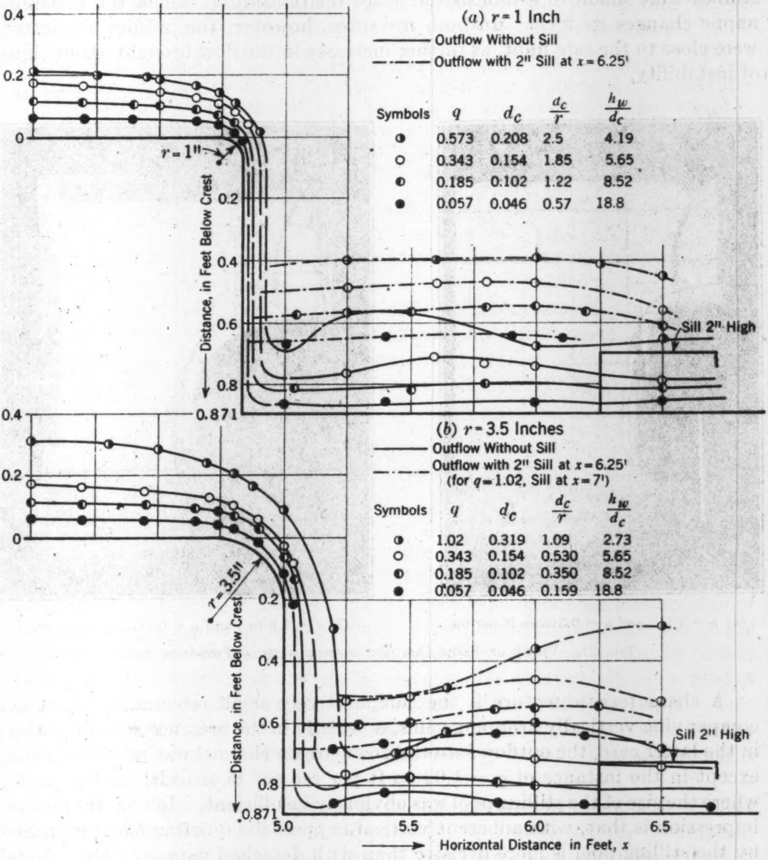

FIG. 20.—SURFACE PROFILES WITH AND WITHOUT SILL

of an interposed 2-in. sill forming a stilling basin. The unit width discharges are reflected in the parametric values of the critical depth d_c, and the last two columns present the dimensionless ratios $\frac{d_c}{r}$ and $\frac{h_w}{d_c}$, h_w being the height of the fall. Obviously, the foregoing ratios govern the flow forms. No exhaustive studies were made to establish the exact relationship at which the adherent nappe changes its form. In both instances, however, the profiles as plotted were close to the safe limit, as further increases in the flow brought about signs of instability.

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[22] "Energy Loss at the Base of a Free Overfall," by Walter L. Moore; see discussion by Boris A. Bakhmeteff and N. V. Feodoroff in this volume of *Transactions*.

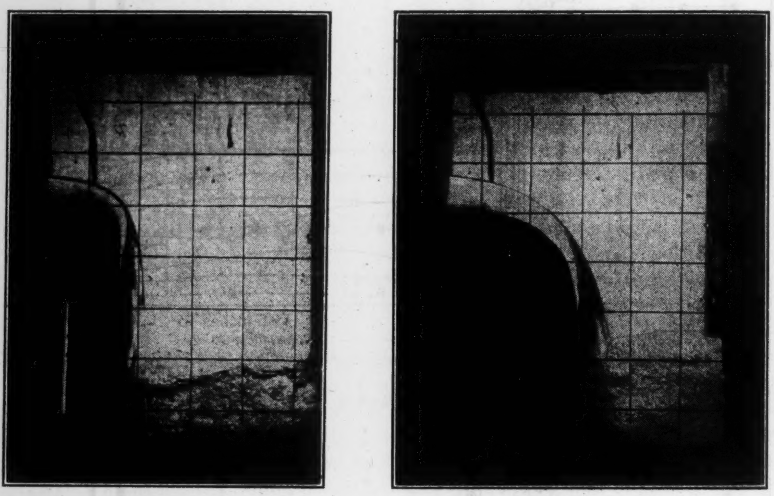
(a) $r = 1$ in.; and $q = 0.343$ cu ft per sec (b) $r = 3.5$ in.; and $q = 0.343$ cu ft per sec

FIG. 21.—VIEWS OF FLOW CHARACTERISTICS WITH A TWO-INCH SILL

A characteristic feature is the comparatively small rebounding effect accompanying vertically dropping veins, especially in the presence of a sill. Also, in the latter case, the outflow into the downstream channel was relatively calm, except in the instance of $q = 1.02$ cu ft per sec per ft of width in Fig. 20(*b*), where the size of the stilling pool was obviously insufficient. In fact, the general impression is that, with adherent vertical nappes, the quieting effect exercised by the stilling pool is more decisive than with detached nappes. The general physical aspects are partly illustrated by Fig. 21. The underpressures developed next to the rounded surface of the wedge were relatively small.

G. H. HICKOX,[23] M. AM. SOC. C. E.—A number of descriptions of stilling basins for large dams have been published, but the results have not been generally applicable. This paper should be of particular interest to the designer of small structures, the costs of which are not sufficient to support independent model studies, and the authors are to be commended on its presentation, and on the studies that preceded it.

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[23] Senior Hydr. Engr., TVA, Hydraulic Laboratory, Norris, Tenn.

It is believed that a word of caution is necessary, however, with respect to the general applicability of the results. The writer does not share with the authors the optimism implied in the following statements (see heading "Application of Drop-Structure Rules to Other Types of Hydraulic Structures"):

> "The principles used in developing the rules for the design of rectangular drop structures for gully control are sufficiently general for application to many problems. There is no great difference between the methods that have been used in insuring energy dissipation and in protecting the structure from undermining and those commonly used in the design of spillways for major engineering structures."

A casual reading of the foregoing statements might lead the reader to assume that the criteria developed in the paper were directly applicable to spillways for major engineering structures, although it is quite probable that the authors did not mean to imply this.

There are at least two important differences between the gully-control structures described in the paper and spillways for large dams. The first of these differences is contained in the statement of the authors (see heading "Development of Design Formulas from Experimental Data: 2. Height of Transverse End Sill")

> "* * * that the loss of kinetic energy incident to the impingement of the nappe on the apron is a large quantity and that this loss increases rapidly with increasing fall height so as to offset, to an appreciable extent, but not completely, the increased energy of the higher falls."

This may be true for the type of structure tested, in which the water falls directly on a horizontal apron, but it is not generally true for major structures. There, an effort is usually made to prevent this type of energy loss because of the tremendous impact forces that would exist immediately at the toe of the structure. Instead of striking on a horizontal floor, the flow is usually turned in a horizontal direction at the toe by means of a vertical curve or bucket having a radius, depending on the depth of flow, of from 35 ft to 100 ft. This gradual change of direction involves less loss of energy and means that a greater amount must be dissipated in the stilling basin. This difference is not on the side of safety.

The second important difference is in the assumption made with respect to tailwater elevations. The tests apparently were made on the assumption that flow below the structure would occur at relatively high velocities and small depths. Thus, the ground roller below the end sill was always formed under similar conditions. Dams on large rivers are usually relatively low, and the elevation of tailwater becomes an important factor. In many cases, tailwater for maximum discharge may be above the spillway crest. This is true at Dams Nos. D1, D2, and D3, Table 5, and at two other main dams on the Tennessee River, and doubtless in many other instances. When the tailwater is sufficiently high above the spillway crest, the type of flow changes and, instead of plunging beneath the tailwater to the apron on the river bed, continues near the surface for a time. The conditions for energy dissipation are thus completely changed and the criteria proposed are no longer applicable.

The criterion for the length of structure is based on the assumption that the discharge falls freely from the crest to the apron. This is not true for overflow spillways on gravity dams because the requirements of stability place the toe of the dam downstream from the normal trajectory of the overfalling sheet. As a result, the distance L from crest to end sill is necessarily increased. Table 5 is a comparison of values of L and h' as calculated from Eqs. 6a and 7c and as adopted on the basis of model tests for a number of dams built by the Tennessee Valley Authority. These dams were all built with horizontal aprons and plain end sills, and provide typical examples of both low-head and high-head dams to which a general relationship might be expected to apply. The lack of satisfactory agreement is evident.

TABLE 5.—COMPARISON OF ACTUAL DAM DIMENSIONS WITH EQS. 6a AND 7c

Key designation	h (ft)	d_c (ft)	L (FT)		h' (FT)	
			Actual	By Eq. 6a	Actual	By Eq. 7c
D1	38	30.7	130	129	5	15.4
D2	20	30.7	106	167	8	15.4
D3	48	24.5	150	108	5	12.2
D4	56	24.5	167	113	5	12.2
D5	237.5	17.3	420	165	17	8.6
D6	203.5	24.5	395	186	12	12.2

Another factor of importance in many spillway designs is the formation of waves below the spillway and their effect on bank erosion and navigation. The slope of the upstream face of the end sill has an important bearing on the production of waves as well as on the formation of the ground roller. On none of the dams listed in Table 5 is the upstream face of the end sill vertical. Until many more experiments have been made under a wide range of conditions, special investigations should be made for all major structures.

The writer is puzzled by criterion (1), which states (see heading "Criteria for Satisfactory Drop-Structure Performance"):

> "The kinetic energy of the effluent stream, as measured by the excess of local velocity over the velocity for absolute minimum specific energy, shall be a minimum."

The reason for the choice of the kinetic energy at the critical velocity as a reference base is not clear, since it is the total energy, and not the kinetic energy, that is a minimum at that point. It would also seem necessary to specify the point at which the energy (or velocity) was measured, since there is considerable dissipation of energy a short distance below the structure, as illustrated by Fig. 6. If the purpose of the structure is the prevention of erosion, is not a low bottom velocity the end to be desired?

It is noted that extensive use was made of photography for taking data. The possibilities of photography for this purpose are very great but, so far, comparatively little progress in its use as a laboratory tool has been reported. The authors are to be commended for their use of this versatile tool.

B. T. Morris,[24] Jun. Am. Soc. C. E., and D. C. Johnson,[25] Assoc. M. Am. Soc. C. E.—The interest that so many hydraulic engineers and laboratory specialists have shown in discussing a paper on the hydraulics of a common erosion-control structure has repaid the writers many times over for their efforts in bringing their work to publication. That the late Professor Hedberg should have found strength during his last illness to prepare the first discussion was a tribute that they shall not soon forget.

For the sake of clarity, the writers have organized their closure by subject rather than by specific reply to individual discussion.

Performance of Drop Structures and Spillways.—The importance of the trajectory of the overflow to the apron-length requirement was recognized and emphasized by Professor Bakhmeteff and Mr. Feodoroff. Most of the length determined by Eq. 6*a* is occupied by the freely falling nappe. Where such a nappe does not exist, apron-length rules based on free fall cannot be used.

Professor Bakhmeteff and Mr. Feodoroff have suggested the use of a rounded overfall crest to produce a clinging nappe and, consequently, a shorter apron; but clinging nappes indicate underpressure at the overfall crest. The writers would be quite apprehensive of the safety of any design in which underpressures, however small in the model, are important to the stability of the flow. That pressures below atmospheric must exist in the flow of Fig. 20 and Fig. 21 will be obvious to any one who seeks to locate the force producing the change in horizontal momentum—from a large quantity to zero. Perhaps very small structures might have sufficiently rounded crests to form stable clinging nappes, but structures with overflow depths of 3 ft to 5 ft would require underpressures of the order of − 4.5 ft to − 7.5 ft in average value over an area equal to the stream cross section. Moreover, these underpressures would not be evenly distributed. Variations in velocity and imperfections of crest profile may produce areas with negative pressure approaching or reaching the limit of barometric head less the vapor pressure of water and entrained air. Under such circumstances clinging flow would be unstable.

Dean Christensen and Professor Gunder have emphasized the uselessness of Eq. 6*a* when $\frac{h}{d_c}$ is less than 1.0. The writers agree that the graph of Eq. 5 in Fig. 14 should be a broken line beyond $\frac{d_c}{h} = 1.0$ or that the diagram should be restricted to $\frac{d_c}{h}$ less than 1.0. The overflow is continuously supported at $\frac{h}{d_c}$ less than 1.0 even when the tailwater depth is less than critical in the downstream channel. Because Eq. 6*a* is empirical in so far as it applies to supported nappes, it should not be used beyond the range of the experimental data on supported nappes.

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[24] Research Engr., C. F. Braun & Co., Alhambra, Calif.; formerly Associate Hydr. Engr., SCS, Pasadena, Calif.

[25] Sacramento, Calif.; formerly Asst. Regional Engr., SCS, Berkeley, Calif.

Mr. Hickox has demonstrated, forcefully, the fact that Eq. 6*a* cannot be applied to overfalls in which the nappe is supported continuously by an ogee crest or similar spillway section of the type he describes. However, the writers believe that the flow conditions on drop-structure aprons downstream from the free trajectory intersection are closely related to those on the aprons of large dams.

Mr. Hall has pointed out, directly and through reference to the valuable work of C. Maxwell Stanley, M. Am. Soc. C. E., the resemblance between the flow conditions on the drop-structure aprons and those in basins too shallow for hydraulic jump operation. In recent months, the senior writer and James T. Rostron, Jun. Am. Soc. C. E., have conducted experiments like those of Mr. Stanley, although with sluice entrance rather than chute entrance. By means of these experiments they have isolated a shock-wave system of flow much like the hydraulic jump but belonging peculiarly to channels with cross sills of height greater than 0.5 d_c. This phenomenon they have named the "forced shock-wave." Although their investigation is too extended to report herein, it may be stated that apron lengths and sill heights may be chosen so as to attach the forced shock-wave flow system to any high-velocity stream.

The writers believe that forced shock waves are produced at many structures. The close agreement (Table 3) between the values of h' given for Mr. Teilman's check gates by Mr. Christiansen and those given by Eq. 7*c* is evidence that these structures employ the forced shock-wave flow.

Flow Downstream from the Structure.—The manner in which the flow over the end sill produces a ground roller and a scour hole adjacent to the toe of the structure is discussed in the paper. Dean Christensen and Professor Gunder have added an excellent statement of the relative significance of large-scale eddy motion and the kind of turbulence that is characteristic of normal channel flow in producing scour at the toe of the structure. They have discussed also the importance of the estimated stable grade on which the layout of the entire system of structures in a gully is based before any individual drop can be designed. Mr. Hall has presented some data to offset the almost complete lack of factual information on which to base the estimation of the hypothetical stable grade.

Mr. Wilson states that "Lateral scour seems to have been given too little attention." Perhaps the writers did not make themselves clear. Even with properly proportioned stilling pools and longitudinal sills for the control of cross flow, lateral scour is the most important unsolved problem. The writers certainly have no ready-made answer. The senior writer has recorded elsewhere[26] his conception of the mechanics of lateral scour. Fig. 22, taken from that paper, shows scour at a model of a drop structure designed according to the rules already stated.

Mr. Hall reports scour of the central part of the stream bed between the longitudinal sills. This is contrary to the writers' experience and may be peculiar to the conditions he describes. Central scour should be far less dangerous than lateral scour unless the structure has been installed with too steep a downstream grade.

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[26] "Scour Control and Scour-Resistant Design for Hydraulic Structures," by Brooks T. Morris, *Transactions*, Am. Geophysical Union, Pt. I, 1942, p. 60.

Mr. Hall calls attention, as well, to the fact that the structure may discharge into a steep channel whose normal rate of flow is faster than critical velocity. Figs. 6(*a*) and 11 show such flow for various heights of drop.

From time to time during the life of a structure, its apron may be covered with silt and debris, particularly if the stable grade has been chosen conservatively. As the structure is self-cleaning, this condition is not dangerous.

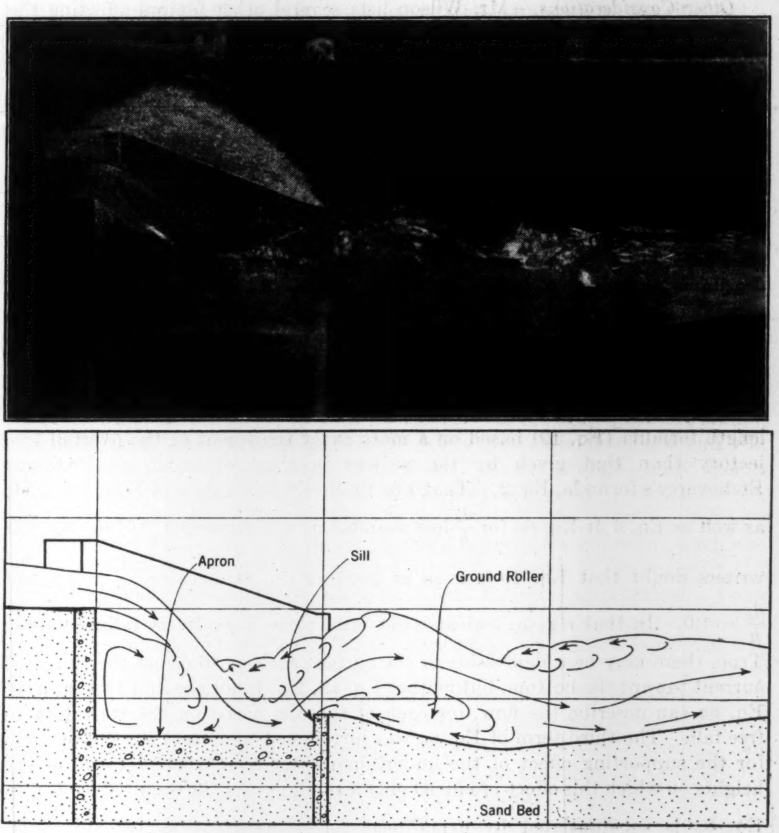

FIG. 22.—SCOUR HOLE AT A MODEL DROP STRUCTURE
(Reference Grid Spaces Are 0.2 Ft by 0.5 Ft)

On the other hand, if the steepness of the stable grade has been overestimated, the apron will remain above the channel and the structure will discharge with a second free fall. Although the structure may survive a short period of operation at this condition, the channel may suffer rapid enlargement. Perhaps Mr. Wilson had this situation in mind when he urged deep stilling pools. The writers are aware of a long-standing practice of deep stilling-pool construction in which $\frac{h'}{d_c}$ might range from 1 to 3 or even deeper. Although they are convinced of the necessity for deep settings and conservatively estimated stable grades, they have satisfied themselves by experiment and practice that deep pools (that is, high transverse sills) are not necessary.

Other Considerations.—Mr. Wilson lists several other factors affecting the apron length in addition to those presented by the writers. Not only must these factors always be taken into account, but also, for high dams $\left(\text{say } \frac{h}{d_c} \text{ greater than } 5\right)$ they may be expected to require greater length, L, than that given by Eq. 6*a*. Only when much of the stability of the structure is furnished by abutment resistance may Eq. 6*a* alone determine the apron length of a high drop.

Mr. Wilson calls attention to certain characteristics of flumes that have led him to favor that type of fall over the drop structure. The writers have used flumes in a few instances but only with recognition of two weaknesses of the flume as a drop installation. For one thing, the flume is essentially an energy conserving device. For another, it is susceptible to heavy damage from minor foundation shifts. The rectangular drop structure, particularly when built of reinforced concrete or framed timbers, is a caisson-like unit that can withstand undermining, tilting, and settling without structural disintegration.

Design Criteria.—Dean Christensen and Professor Gunder present an apron-length formula (Eq. 19) based on a more exact treatment of the overfall trajectory than that given by the writers in their extension of Professor Etcheverry's formula, Eq. 2. That Eq. 19 should fit the data of Figs. 6 and 11, as well as Eq. 4 or Eq. 6*a* for $\frac{d_c}{h}$ less than 0.8, is not surprising. However, the writers doubt that Eq. 19 is even as good as Eq. 6*a* between $\frac{d_c}{h} = 0.8$ and $\frac{d_c}{h} = 1.0$. In that region, and at lower falls, there is no freely falling nappe. True, there may be a depression of the surface such as to direct the principal current toward the bottom, but neither Eq. 19, Eq. 4, nor the first two terms of Eq. 6*a* can describe the flow, for each of these is based on the trajectory of free fall. The third term of Eq. 6*a* was introduced to correct the relationship for the supporting effect of the underlying water over a small range of fall heights on which this effect produced only a small part of the required length, L. Eq. 6*a* is substantiated by experiment for $\frac{d_c}{h}$ as great as 1.0 and $\frac{d_s}{d_c}$ as great as 2.0.

Mr. Christiansen and Mr. Hickox have reminded the writers that high tail-water stages can alter the flow over the stilling pool at low falls. An extreme case of altered flow would be the submergence of control. This might be anticipated, for example, at $\frac{h}{d_c} = 1.0$ and $\frac{d_s}{d_c} = 3.0$. The stilling pool would be only an indentation in the bottom for such a flow as that. So long as

$\frac{d_c}{h}$ remains equal to, or less than, 1.0 and $\frac{d_s}{d_c}$ equal to, or less than, 2.0, control exists at or near the drop structure crest and Eq. 6*a* is sufficient for the determination of *L*.

Dean Christensen and Professor Gunder have called attention to apparent discrepancies in the presentation of Eq. 7*c*. A little algebraic manipulation would show that Eq. 7*c* implies a unique relationship between the sill height, the depth on the apron, and the kinetic energy of flow such that

$$h' = \frac{d_c}{2} = \left[\frac{(V_1)^2}{2\,g}\right]^{1/3} \left[\frac{d_1}{2}\right]^{2/3} \qquad \text{(20)}$$

The new studies by the senior writer and Mr. Rostron indicate that $h' \geqq 0.5\, d_c$ is the criterion for stability of the forced shock wave below overfalls. Unless this wave is formed, the flow leaves the apron over the end sill without significant energy dissipation downstream from the free trajectory intersection.

Mr. Hickox discusses the effect of steepness of the upstream face of the end sill on the violence of motion downstream from it. In so doing, he reveals the most essential difference between the stilling pools for low dams with tranquil downstream channels and those for drop structures. Where the tailwater stage can be relied upon to stabilize a hydraulic jump over the spillway apron, the function of the end sill is only to invert the velocity profile and produce a ground roller. Neither the large amounts of energy dissipation nor the violent motion attendant to it are necessary. In the drop-structure stilling pool the shock wave acts at and immediately upstream from the end sill. Violent motion is characteristic; but, as Mr. Hickox implies, such violent motion is to be compared with that in the normal regime of the particular downstream channels involved.

Laboratory Technique.—Professor Hedberg raises three questions, each one of which is important to the interpretation of the results of laboratory tests. His first question deals with the reliability of the half-model technique. Dean Christensen, who originated the method in connection with these studies, has presented a statement of its value and limitations that the writers will not attempt to improve.

The second question has to do with air entrainment. Knowledge of the controlling factors of air entrainment is so meager that positive description of air entrainment relationships in model and prototype is not now possible. In general, the writers subscribe to the analysis[27] of air entrainment proposed by their former adviser and colleague, Professor Knapp. They have relied on relatively large-scale experiments to minimize the discrepancies of air entrainment behavior between model and prototype. Observation of the experimental drop structure indicated active air intake at all points where it might be expected in larger structures, except at the surface of the falling nappe itself.

The writers would expect difficulties from the effect of bulking on the density of the falling water relative to that in the pool to occur first. Such difficulties would become apparent in reduced capacity of a given apron length. However, such difficulties remain hypothetical. No trouble has yet been reported from the highest fall among drop structures built by the Pacific Southwest Region, SCS—a six-year-old structure with $h = 25$ ft and $\frac{h}{d_c} = 15$.

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[27] "Entrainment of Air in Flowing Water: A Symposium"; see discussion by Robert T. Knapp in this volume of *Transactions*.

The third question raised by Professor Hedberg has to do with the effect of the net absolute pressure on the stability of flow in the model and the prototype. Had their structure incorporated a curved surface or a plane surface with adverse pressure gradients in a zone of low total pressure, the writers might have worried over this element of similitude for the reasons set forth in their discussion of the clinging overfall. They believe that the regions of flow separation in the vicinity of the end sill are so located in reference to the corner of the sill as to be stable with respect to changes in the character of the fluid passing them. Unstable flow occurred only where the apron was critically short. No structure installed in the field has shown the anomalous performance necessary to confirm doubt of the writers' assumptions.

Mr. Hickox questions the value of criterion (1), which was used to compare the laboratory drop structures. The critical velocity was chosen as a reference base to give appropriate scale to local velocity measurements while still producing a dimensionless description of the magnitude of eccentricities in the velocity distribution. The writers did not make use of bottom velocity measurements because such quantities would apply only to specific localities in the artificial downstream channel. Instead they chose a criterion through which the maximum local velocity was taken to be a measure of the intensity of eddy motion and the availability of energy for further eddy production. The measurement of bottom velocity, or, preferably, velocity gradient, is useful in straight channels of uniform roughness where such measurements can be correlated with shear intensity and turbulence. However, as Dean Christensen and Professor Gunder have so neatly pointed out, localized scour is related more closely to large-scale and locally intensified turbulence or eddy motion than it is to the turbulence that is characteristic of channel flow.

Mr. Hickox' kind remarks in regard to the photographic procedure employed in these studies have been communicated to Hugh Stevens Bell of the SCS Laboratory staff at the California Institute of Technology. Mr. Bell, who is responsible for photographic control and procedures at the laboratory, advised and assisted the writers in many ways during the investigations.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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TRANSACTIONS

Paper No. 2199

DISTRIBUTION GRAPHS OF SUSPENDED-MATTER CONCENTRATION

By JOE W. JOHNSON,[1] ASSOC. M. AM. SOC. C. E.

WITH DISCUSSION BY MESSRS. H. A. EINSTEIN, G. C. DOBSON, THOMAS R. CAMP, FRANK S. BAILEY, AND JOE W. JOHNSON.

SYNOPSIS

Damages or benefits resulting from sedimentation at various points in a drainage basin may be due to different types of sediment, possibly coming from different sources. For instance, flood and drainage damage caused by stream-channel aggradation appears to be primarily a bed-load problem, involving the deposition of relatively coarse material. On the other hand, fine material is responsible for much of the deposition in irrigation and drainage canals and the increased cost of water purification in many public water-supply systems. Both fine and coarse material contribute to the deposits in reservoirs, but the finer grade of sediment normally is the chief source of damage, because it constitutes the greater portion of the load. Because different types of damage are caused in different proportions by the two classes of sediment, it is important in sediment-load investigations that, not only the total load of the stream be determined, but also the relative proportions of fine and coarse material. These are the subjects discussed in the paper.

GENERAL

With the exception of dissolved matter, the total solids load passing a particular cross section in a stream may be classified either as bed-material load or wash load, with a definite grain size representing the division between the two classes (3).[2] The bed-material load, or "bed load" for brevity, is that part of the total solids composed of the relatively coarse material (coarser than the division grain size), and it moves by rolling or sliding along the bed, or by long jumps between points of contact with the bed while in temporary suspension, at a rate related to the stream discharge. The wash load, except

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NOTE.—Published in October, 1942, *Proceedings*.

[1] Asst. Prof., Dept. of Mech. Eng., Univ. of California, Berkeley, Calif.; formerly, Hydr. Engr., SCS, Washington, D. C.

[2] Numerals in parentheses, thus: (3), refer to corresponding items in the Bibliography, in the Appendix.

for a negligible percentage, moves in suspension at a rate that bears no direct relation to discharge and is composed of the relatively fine material (finer than the dividing grain size). The wash load constitutes much the greater part of the material in a suspended-load sample taken at flood stage, and may be regarded as sediment in relatively permanent suspension.

The particular grain size that divides the sediment load into these two classes varies in different streams and, generally, throughout the length of a particular stream, although it has a definite value in a particular reach of a stream. For example, the dividing grain size in the Colorado River near Imperial Dam is 0.01 mm (fine silt) (10), whereas in the Enoree River near Greenville, S. C., the size is 0.351 mm (medium sand) (3). The dividing grain size can be determined from mechanical analysis curves of samples taken from the stream bed immediately after a flood, and is the finest grain size existing in appreciable quantities.

The bed material (coarse material) in a stream system ordinarily changes slowly in composition over a long period of years, if at all, and the quantity moved is mainly in response to discharge. The quantity of wash material (fine material), however, may vary greatly from year to year and from season to season, because of the changing conditions of supply that are independent of conditions of flow. The most important variables of these completely interrelated conditions of supply are vegetal cover, tillage methods, stage of soil erosion, and rainfall characteristics.

Observations made at the Soil Conservation Service (SCS) sediment-load laboratory near Greenville (2) have shown the relative magnitudes of the two types of sediment load in a representative Piedmont stream. Of the total sediment load passing the station in the Enoree River during its two years of operation, more than 90% was found to consist of wash material.

Although this ratio between the annual amounts of the two classes of sediment load is not expected to be the same for all streams, or even constant for the Enoree River, it probably represents fairly well the general order of magnitude for streams of this size in the humid sections of the United States where the amount of wash material subject to erosional attack is essentially unlimited.

Suspended Load

Vertical Distribution.—A general theory of the vertical distribution of suspended sediment in a turbulent stream of water has been established as a result of extensive investigations on the transportation of sediment, in both the United States and Europe. Based upon this theory, the total suspended-sediment load transported in a vertical section can be estimated from a determination of the concentration and the particle size distribution of the sediment at a single point in the vertical, and the hydraulic characteristics of the stream. In actual practice, however, the procedure is more simple because it is necessary to apply the theory only to sediment coarser than 0.075 mm (200-mesh), since material finer than this size appears to be uniformly distributed throughout the section in most streams, and a single accurate sample taken at any point in the stream gives the concentration applicable to the entire cross section.

Sampling and Analysis.—By accurate sampling and grade-size analysis of the suspended matter in a stream, it is possible to compute the total load of coarse material—material of sizes found in appreciable quantity in bed deposits—carried in temporary suspension; and, by making a sufficiently large number of observations over a wide range in stage, a relationship between coarse load and discharge can be established for a particular location. Estimates of the future suspended load of coarse material, therefore, can be made from this established relationship, and from a record of the discharge, without the necessity of further sampling. The load of fine material, on the other hand, can be estimated accurately merely from a dip sample. However, it bears no direct relationship to the discharge, and in the past continuous sampling has been necessary to estimate the total volume of fine material carried in a stream during a flood.

Observation Periods.—In small flashy streams almost all of the annual load of wash material is carried during a few days out of the year (6), and the content of wash material may vary many 100% within the period of an hour during the rising and falling stage of the stream. To the present time, no method of wide applicability, except frequent sampling, has been developed for estimating or computing, with even approximate reliability, the total suspended load carried by any small stream. It is true that on some large rivers an approximate relationship (1) has been developed between discharge and suspended load—presumably including both fine and coarse material—and, from this, estimates of total sediment transportation have been made. However, where such relationships have been developed, the period of observation, in general, has extended over only one or two years when conditions of supply apparently were approximately constant. Continued observations over a longer period of time usually have resulted in an increased scattering of data (9). The discussion to follow is concerned primarily with the transportation of the fine material in small streams in humid climates and gives a general description of some of the sources and factors controlling the supply of material.

Sources of Sediment.—The load of fine material in a stream appears to have its main source, not in the stream bed, but in other parts of the drainage basin. It is possible, of course, that a small amount of fine material in any particular flood may come from the stream bed, where it was deposited in pools during low stage, or where it became trapped between larger bed particles, and was released when the bed material started to move during a higher stage. In general, however, the major sources of sediment are cultivated lands, road cuts, roadside ditches, gullies, and caving stream banks. The amount of material that may enter a stream from these sources during a particular storm depends upon such complex, interrelated, and, in part, changing factors as topography, soils, shape and size of the drainage basin, land use, vegetal cover, and surface and underground storage, as well as upon rainfall intensity and duration. In any drainage system, however, certain of these factors may be considered constant, whereas others have a seasonal variation.

Results of Past Studies.—Numerous investigations of soil loss and runoff from relatively small agricultural plots give general information on the principles underlying the supply of erosional debris. These studies have shown, in general, that the sediment concentration in runoff water from a plot usually increases to a maximum within a very short time, and then decreases as the duration of application of water continues, approaching a constant minimum value asymptotically. The high initial sediment concentration appears to be the direct result of rain impact. However, rain packing and adhesion from wetting and the swelling of colloids, as the rainfall continues, as well as the variation of soil porosity with depth, appear to be the main factors tending to reduce concentration. The variation of the infiltration capacity with duration of rainfall also has been observed to follow a law similar to that of sediment concentration.

Erosion Principles.—Erosion by rain impact and overland flow applies more specifically to the land surface and the exposed side slopes of road cuts, ditches, and gullies, but once water is concentrated in relatively large, well-defined channels, the principles of stream erosion more nearly apply. Stream erosion probably is closely related to discharge, because, as a rule, the greater the discharge, the greater will be the velocity, and hence the capacity to dislodge and transport particles. In any event, however, and despite the complicated mechanics by which material is placed in suspension and transported to the outlet of the ordinary small plot, the concentration of fine material appears, in general, to vary with the surface runoff. The same reasoning should apply to a natural stream; although, obviously, the problem is more complicated, because rainfall distribution and areas of sediment production, which are in a sense a system of small plots, vary considerably in their characteristics throughout the drainage basin.

Hydrographs and Distribution Graphs.—It has been observed generally that, following a rainfall during which surface runoff occurs, the discharge of this surface runoff into stream channels simultaneously over a drainage basin results in pronounced rises in stream levels, followed by periods of decline. The concentrations of suspended matter in streams also have been observed to rise and decline in a similar manner (Fig. 1).

The plotted graph of stream flow, showing the rises and declines of runoff, has been used by many investigators as a basis for studying the phenomenon of surface runoff. L. K. Sherman, M. Am. Soc. C. E. (8), conceived the idea that, for a particular drainage area, the surface runoff from rainfall occurring within the same time interval will produce hydrographs, the ordinates of which will vary with the amount of the surface runoff. Mr. Sherman has given the term "unit graph" to the hydrograph of surface runoff that results from rain falling within a unit of time, as a day or an hour. If the unit hydrograph of surface runoff is modified to show the proportional relations of its ordinates in the percentage of total surface runoff, the resulting graph is termed a "distribution graph." The unit-hydrograph method is based upon the hypothesis that in a particular drainage basin surface runoff from rainfall occurring in a unit of time will produce hydrographs of approximately equal bases, and the ordinates will vary with the intensity of the net rainfall.

From the consideration, as suggested by plot studies, that surface runoff and suspended-matter concentration appear to be related to the same factors, an investigation was made as part of the SCS sediment-load studies to determine if any such relationship actually existed in a natural stream. Data from the East Fork of Deep River, near High Point, were used for this study, and

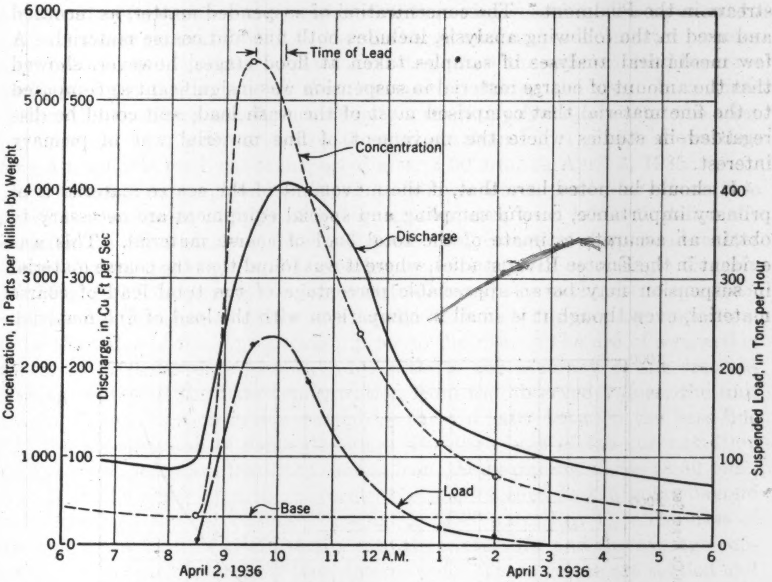

FIG. 1.—GRAPHS OF DISCHARGE, SUSPENDED-MATTER CONCENTRATION, AND SUSPENDED LOAD FOR A TYPICAL RISE ON EAST FORK OF DEEP RIVER, NORTH CAROLINA

upon examining the hydrographs and corresponding graphs of suspended-matter concentration for all isolated stream rises it was found that there was a "distribution graph" of concentration analogous to the distribution graph used in analyzing stream flow as discussed previously herein, and that it could be used to advantage in studying the suspended load of streams.

DEVELOPMENT OF DISTRIBUTION GRAPH

The data used in this study were obtained on the East Fork of Deep River during an extensive suspended-matter sampling program conducted by the United States Geological Survey in cooperation with the SCS and the North Carolina Agricultural Experiment Station. Collection of water samples for the determination of sediment concentration was made once a day during periods of low water, and at more frequent intervals during periods of storm runoff. Fig. 1 illustrates a typical example of a stream rise, and shows the usual frequency of sampling considered necessary to obtain sufficient information for the determination of variations in content of suspended matter. Other basic data secured included the stream-discharge and precipitation records from three standard and three recording gages located in and about the 13.9-sq-mile drainage basin.

The basic data from the gaging station are sufficiently complete to give valuable general information on the movement of suspended matter in a typical stream in the Piedmont. The concentration of suspended matter, as recorded and used in the following analysis, includes both fine and coarse material. A few mechanical analyses of samples taken at flood stages, however, showed that the amount of coarse material in suspension was insignificant as compared to the fine material that comprised most of the wash load, and could be disregarded in studies where the movement of fine material was of primary interest.

It should be noted here that, if the movement of the coarse material is of primary importance, careful sampling and special equipment are necessary to obtain an accurate estimate of the total load of coarse material. This was evident in the Enoree River studies, where it was found that the coarse material in suspension may be an appreciable percentage of the total load of coarse material, even though it is small in comparison with the load of fine material.

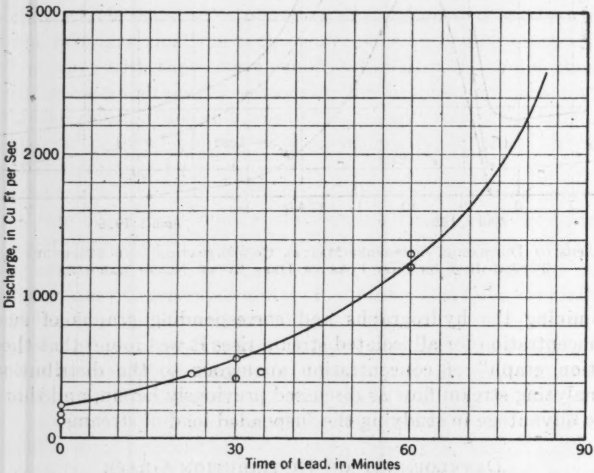

FIG. 2.—RELATION BETWEEN MAXIMUM DISCHARGE AND THE TIME THAT MAXIMUM SUSPENDED-MATTER CONCENTRATION PRECEDES THE MAXIMUM DISCHARGE

Fig. 1 illustrates the variation of concentration during a typical isolated rise, and shows the general conditions under which suspended-matter content increases very rapidly from the beginning of the rise until about the peak stage. The maximum content either occurs somewhat in advance of, or coincides with, the maximum discharge of the stream, depending upon the stream discharge (Fig. 2). The intervening period of time, between the occurrence of the peak concentration and the occurrence at the peak discharge, is termed the "time-of-lead."

After the peak concentration has been reached, the content of suspended matter decreases with the falling stage of the stream, but the decrease usually is at a less rapid rate than the increase during the rising stage. In general, the recession curve of concentration has a long "tail," as evidenced by the muddy appearance of a stream for a considerable time after discharge has returned to approximately normal flow. As the discharge and suspended-matter concentration continue to decrease, a point is reached where the sediment load becomes negligible in comparison with that transported during the rise. In a small stream this condition usually occurs only a relatively few hours after the crest has passed. This is illustrated by the load graph in Fig. 1, which shows that only a negligible load was transported after 3:00 a.m. on April 3, 1936.

A distribution graph is calculated as follows: For a particular stream rise in which a sufficient number of water samples have been taken to define the variation in concentration throughout the rise, curves of suspended-matter concentration, discharge, and suspended load are plotted on a single diagram as in Fig. 1. To segregate the concentration due wholly to the rise in question, a horizontal base line is drawn at an ordinate, which appears, by inspection, to be the value of the concentration prior to the rise. (The use of a recession curve, as in stream-flow studies, adds little to the accuracy of the results.) Having separated the base concentration from the observed values, the unit graph of concentration is represented by the ordinates between the base line and the actual graph of concentration. The time base of the concentration graph is considered arbitrarily to extend from the beginning of rise (8:30 p.m., Fig. 1) to the time when the concentration has receded to the point beyond which the load can be considered negligible (3:00 a.m., Fig. 1). The base of the unit graph is divided arbitrarily into time intervals, and the average concentration for each interval is then determined. These values are totaled and the percentage of the total occurring during each of the periods is computed. Time intervals of 30 min and 1 hr were found suitable for representing accurately the shape of the graphs for rises occurring in the East Fork of Deep River.

Distribution graphs were prepared for all the isolated rises that occurred in the East Fork of Deep River during the four years of record from April, 1934, to June, 1938. After reducing these various graphs to a constant time interval, it was found that the shape of the graphs appeared to be a function of the period of stream rise, and somewhat of the intensity of the rainfall. The period of rise in this case is defined as the time intervening between the beginning of surface runoff and the occurrence of the peak discharge. Fig. 3 shows the distribution graphs for ten storms in which the period of rise was 2 hr or less.

These various graphs were prepared from rises that occurred during all seasons of the year, and, consequently, represent various conditions of land use, vegetal-cover, soil-moisture, and rainfall characteristics. Rises with the highest runoff and soil loss generally occurred during April and September. In September the losses appeared to be the result of torrential rains, whereas in April a high initial soil-moisture content was the most important factor. Two rises occurred during July as a result of high-intensity rains, but the runoff and subsequent erosion were not as severe as for April and September, probably because of the dry and porous condition of the soil, but perhaps also because the total rainfall was less and the evaporation and transpiration probably were greater.

The average distribution graph of the superimposed graphs in Fig. 3 is tabulated and plotted in Fig. 4, together with the average distribution graphs

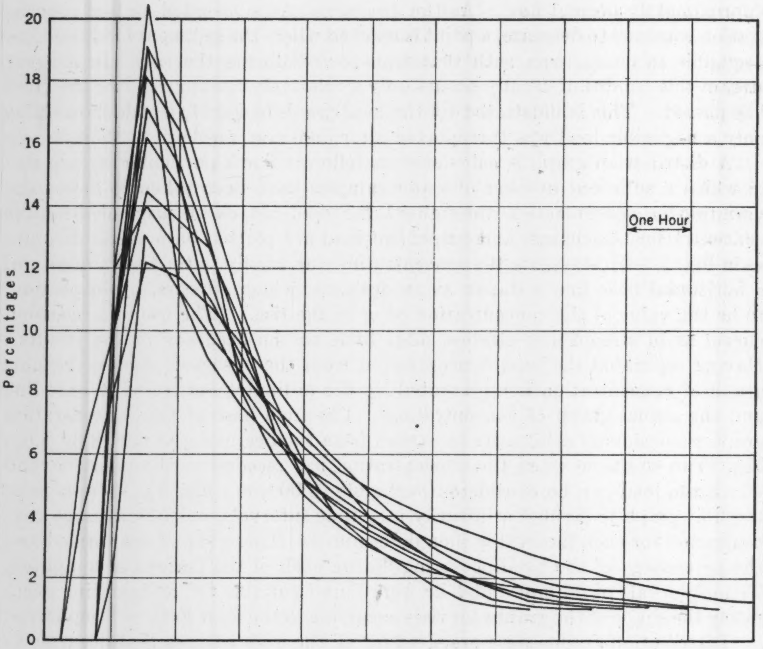

FIG. 3.—SUPERIMPOSED DISTRIBUTION GRAPHS OF SUSPENDED-MATTER CONCENTRATION FOR TEN RISES, ON THE EAST FORK OF DEEP RIVER, IN WHICH THE PERIOD OF STREAM RISE WAS 2 HR OR LESS

(Time Interval of One-Half Hour Used in Preparation of Graphs)

for other periods of rise, the percentages in all cases being expressed for a time interval of 1 hr. The graph for a period of rise of 2 hr or less is characteristic of high-intensity rains of short duration, when rain impact probably is responsible for placing most of the sediment into suspension. The graphs for the longer periods of rise are characteristic of long duration, low-intensity rains, where rain impact probably is of secondary importance in placing sediment in suspension.

Rises resulting from uniformly distributed rainfall should be used, where possible, in preparing a set of distribution graphs. Although rainfall records are not necessary in the application of the graphs, such information is of importance in judging the relative reliability of the computations as affected by rainfall distribution.

Period of Stream Rise, in Hours:					
<2	3	4	5	6	7
Half-hour intervals		One-hour intervals			
1.0	4.7	1.5	2.0	0.5	0.4
9.2	8.7	8.0	4.5	2.4	3.1
(16.6)[a]	10.0	15.0	8.9	5.0	4.8
13.9	(11.5)[a]	(16.8)[a]	11.8	7.3	6.3
11.5	10.0	14.5	(12.1)[a]	9.2	7.4
9.0	8.4	11.8	11.4	(10.0)[a]	(8.0)[a]
7.1	7.4	9.2	9.3	9.5	7.7
5.7	6.4	7.5	7.4	8.4	7.2
4.8	5.3	5.6	6.0	7.2	6.7
4.0	4.6	4.0	5.0	6.2	6.1
3.4	3.7	2.8	4.1	5.4	5.6
2.7	3.1	1.8	3.4	4.7	5.1
2.2	2.7	1.0	2.8	4.0	4.4
1.9	2.4	0.5	2.3	3.4	3.9
1.6	2.1	....	2.0	2.9	3.5
1.4	1.9	....	1.7	2.6	3.2
1.2	1.7	....	1.4	2.2	2.8
1.1	1.5	....	1.2	1.8	2.5
0.9	1.4	....	1.0	1.6	2.3
0.8	1.3	....	0.9	1.4	2.1
....	1.2	....	0.8	1.3	1.8
....	....	....	....	1.1	1.5
....	....	....	....	1.0	1.4
....	....	....	....	0.9	1.2
....	....	....	....	....	1.0

[a] Peak percentages are shown in parentheses.

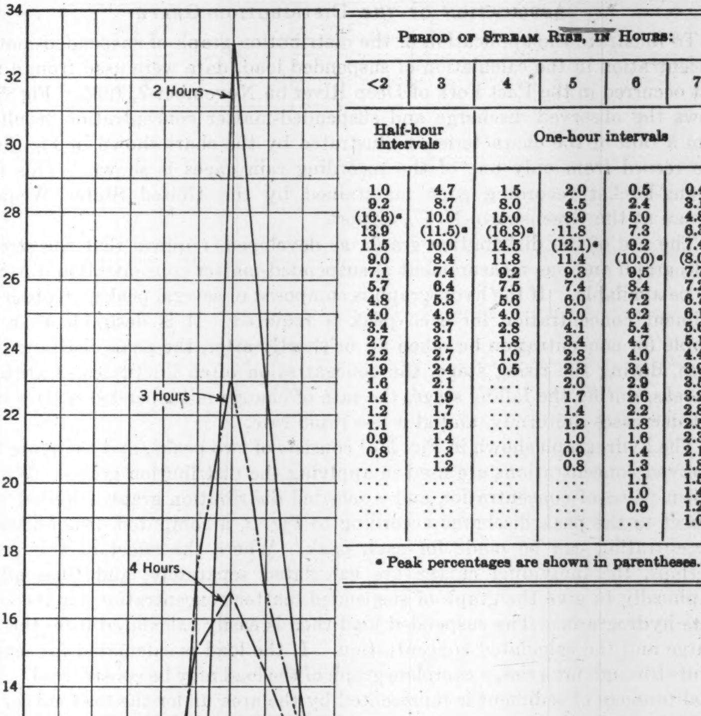

Fig. 4.—Average Distribution Graphs of Suspended-Matter Concentration, in East Fork of Deep River, for Various Periods of Stream Rise

APPLICATION OF THE DISTRIBUTION GRAPH

To illustrate the application of the distribution graph of suspended-matter concentration in the calculation of suspended load, data were used from a rise that occurred in the East Fork of Deep River on November 7, 1935. Fig. 5(*b*) shows the observed discharge and suspended-matter concentration resulting from a rain of the characteristics illustrated by the chart shown in Fig. 5(*a*). The record from only one of the recording rain gages is shown. This is a tipping-bucket recording gage maintained by the United States Weather Bureau at the Greensboro, N. C., airport.

The use of the distribution graph, as developed, requires that the stream hydrograph and one measurement of suspended-matter concentration during a rise be available. If the hydrograph is composed of several peaks, an observed sediment concentration for each peak is required. It is desirable that the sample for concentration be taken at, or shortly after, the peak discharge, because, during the rising stage, the concentration often fluctuates erratically; whereas, during the falling stage, the rate of change in suspended-matter content decreases uniformly, and at a less rapid rate.

The hydrograph shown in Fig. 5(*b*) consists of two peaks, and therefore two observed concentrations are used in applying the distribution graph. With a known value of concentration and a selected distribution graph oriented with respect to the peak discharge according to Fig. 2, a computed distribution of concentration can be made for each peak. Where the effect of two peaks overlaps, the individual curves are calculated separately, and then added graphically to give the graph of suspended-matter concentration for the complete hydrograph. The suspended load then is easily calculated from the discharge and the calculated concentration. If the load is calculated for several points throughout a rise, a complete graph of the load may be constructed. The total tonnage of sediment is represented by the area under the load curve. If desired, this load then may be converted to volume by selection of a volume-weight ratio.

In the first rise, shown in Fig. 5(*b*), the period of rise is approximately 4 hr. With a peak discharge of 70 cu ft per sec, the time-of-lead is zero according to Fig. 2, and the peak concentration therefore is assumed to coincide with the peak discharge. The 4-hr distribution graph, shown in Col. 3 of the tabulation in Fig. 4, is used and plotted in Fig. 5(*c*), with the peak coinciding with the time of peak discharge. It so happens that the observed concentration of 860 ppm, as selected for use in the calculations, was taken at the instant of peak discharge; consequently, it is the concentration at which the curves of calculated and observed concentration will arbitrarily be made to coincide. By multiplying this peak concentration of 860 ppm by the ratio of the ordinate percentages for each time interval of the distribution graph to the peak percentage, a calculated curve (curve a, Fig. 5(*c*)) of concentration is obtained for the first rise. It may be noted that the recession part of this curve extends over a period of time sufficiently long to contribute to the total amount of sediment in suspension during the second rise; consequently, to distribute the concentration in the second rise, a segregation of the concentration resulting from the first rise will be required.

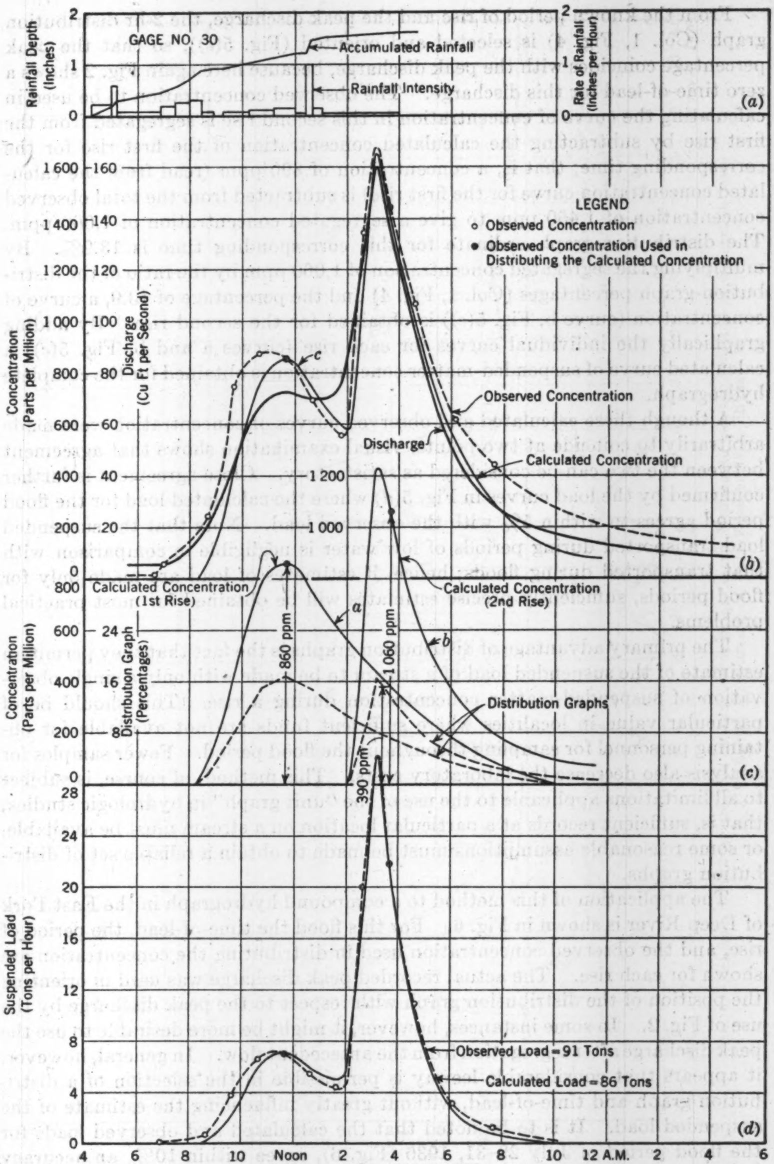

FIG. 5.—ILLUSTRATION OF METHOD OF CALCULATING THE SUSPENDED LOAD OF A STREAM BY USE OF DISTRIBUTION GRAPHS OF SUSPENDED-MATTER CONCENTRATION

From the known period of rise and the peak discharge, the 2-hr distribution graph (Col. 1, Fig. 4) is selected and oriented (Fig. 5(*c*)), so that the peak percentage coincides with the peak discharge, because here again Fig. 2 shows a zero time-of-lead for this discharge. The observed concentration to be used in calculating the curve of concentration in this second rise is segregated from the first rise by subtracting the calculated concentration of the first rise for the corresponding time; that is, a concentration of 390 ppm (read from the calculated concentration curve for the first rise) is subtracted from the total observed concentration of 1,450 ppm to give a segregated concentration of 1,060 ppm. The distribution-graph ordinate for this corresponding time is 13.9%. By multiplying the segregated concentration of 1,060 ppm by the ratio of the distribution-graph percentages (Col. 1, Fig. 4) and the percentage of 13.9, a curve of concentration (curve b, Fig. 5(*c*)) is obtained for the second rise. By adding graphically the individual curves for each rise (curves a and b, Fig. 5(*c*)) a calculated curve of suspended-matter concentration is obtained for the complete hydrograph.

Although these calculated and observed curves of concentration were made arbitrarily to coincide at two points, visual examination shows that agreement between the two can be considered as satisfactory. Close agreement is further confirmed by the load curves in Fig. 5(*d*) where the calculated load for the flood period agrees to within 5% with the observed load. Note that the suspended load transported during periods of low water is negligible in comparison with that transported during floods; hence, if estimates of load are made only for flood periods, sufficiently precise estimates will be obtained for most practical problems.

The primary advantage of distribution graphs is the fact that they permit an estimate of the suspended load of a stream to be made with only a single observation of suspended-matter concentration during a rise. This should be of particular value in localities where sufficient funds are not available for obtaining personnel for sampling throughout the flood period. Fewer samples for analysis also decrease the laboratory costs. This method, of course, is subject to all limitations applicable to the use of the "unit graph" in hydrologic studies; that is, sufficient records at a particular location on a stream must be available, or some reasonable assumptions must be made to obtain a reliable set of distribution graphs.

The application of this method to a compound hydrograph in the East Fork of Deep River is shown in Fig. 6. For this flood the time-of-lead, the period of rise, and the observed concentration used in distributing the concentration are shown for each rise. The actual recorded peak discharge was used in orienting the position of the distribution graph with respect to the peak discharge by the use of Fig. 2. In some instances, however, it might be more desirable to use the peak discharge after segregation from the antecedent flow. In general, however, it appears that considerable leeway is permissible in the selection of a distribution graph and time-of-lead, without greatly influencing the estimate of the suspended load. It is to be noted that the calculated and observed loads for the flood period of July 29–31, 1936 (Fig. 6), agree within 10%, an accuracy which, for most practical purposes, may be considered as satisfactory.

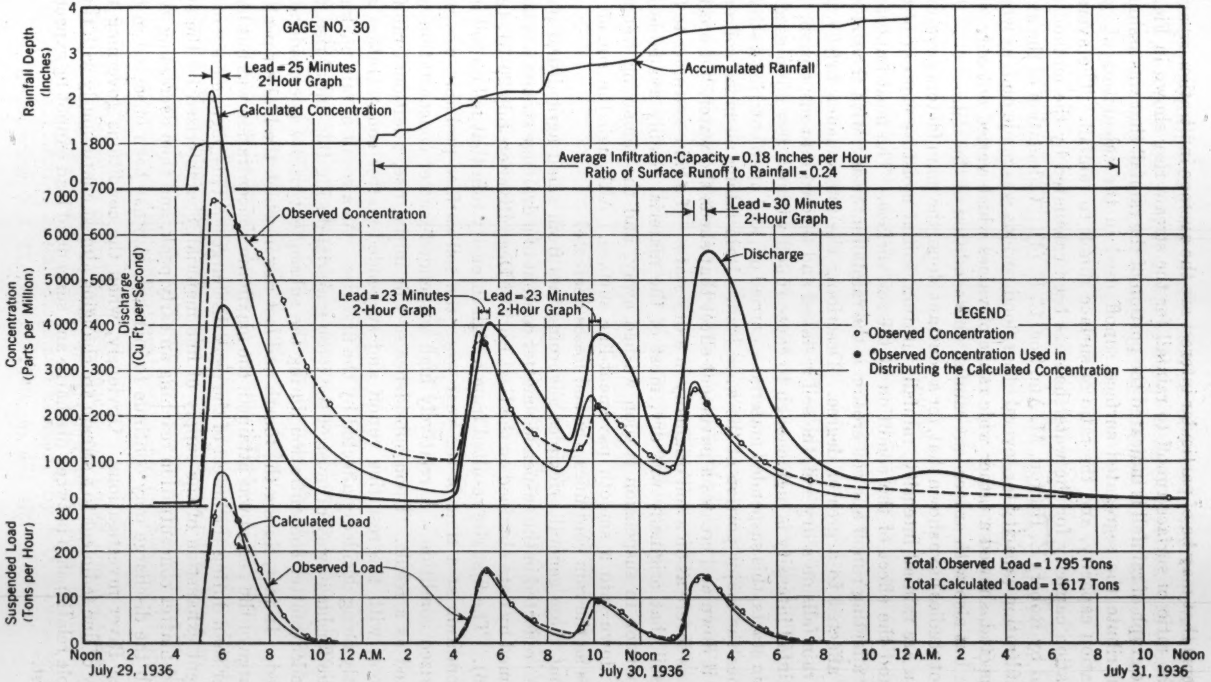

FIG. 6.—EXAMPLE OF THE APPLICATION OF THE DISTRIBUTION-GRAPH METHOD TO THE CALCULATION OF SUSPENDED LOAD IN A COMPOUND HYDROGRAPH ON EAST FORK OF DEEP RIVER

Items that may be of particular interest are the average infiltration capacity and the ratio of surface runoff to rainfall for the stream rise, shown in Fig. 6. On the graph of rainfall is indicated the portion of the rainfall that was assumed to contribute the segregated surface runoff used in the calculation of both infiltration capacity, and the ratio of surface runoff to rainfall. The average infiltration capacity for the watershed has been calculated by the method described by Robert E. Horton, M. Am. Soc. C. E. (4). Calculation of the average infiltration capacity for several other flood periods was made and, as would be expected, showed a rather wide range of values which varied not only with season but also with antecedent rainfall and condition of the soil.

Plot studies have shown that, for a constant slope, the main factors governing erosion are rainfall intensity, rainfall duration, initial moisture content of the soil, and the effect of the condition of the soil surface. The most important factor affecting runoff and soil erosion is the rainfall intensity, with the soil loss being affected to a greater degree. Describing the phenomenon briefly (7), when rain falls on a dry soil, the soil is slaked and thrown into suspension. If the rainfall intensity is high enough to cause runoff within a few minutes after the rain starts, this suspended material is carried off, and upon entering a stream becomes the major portion of the wash load. Rain falling on moist soil only packs it down and creates a pavement effect that sheds the water, but erosion is not as severe as on a soil originally in a dry state. If a rain continues to fall on a soil that originally was dry, most of the material readily available for transportion in suspension is soon washed away, and the remaining wet soil packs down into a smooth pavement-like surface. Although the runoff increases as the rain continues, the soil losses decrease.

That these general principles governing loss from small agricultural plots are also reflected in the suspended-matter content of a drainage stream has been confirmed by data obtained on the East Fork of Deep River on July 29–31, 1936 (Fig. 6). The first rise resulted from a high-intensity rain that fell on soil in a dry condition at a time of the year when the soil surface probably was in a pulverized condition. A relatively high suspended-matter concentration was attained as a result. Upon subsidence of this initial rise, additional rainfall occurred, with the resulting runoff and suspended-matter concentration apparently being decidedly affected by the first rise. As a result of rain packing, and the flushing away of most of the loose material during this initial rise, the suspended-matter concentration during the subsequent rises was considerably reduced. In fact, during the fourth and last rise in which the highest peak of discharge of the period was attained, the maximum concentration was but little greater than during the lowest of the peak discharges. This phenomenon of an apparent exhaustion of the supply of fine material, characterized by the suspended-matter concentration reaching an early peak, and then decreasing even though the discharge may continue to increase, also has been observed in the Enoree River investigations. Obviously, when the conditions governing the supply of fine sediment to a stream are changing almost continuously with time, a simple relationship between discharge and suspended load cannot be expected to exist.

The eventual solution of the problem of erosion will require considerable information on the rainfall characteristics during each storm. In reporting the results of 1940 studies on raindrops and erosion, J. O. Laws (5) made the following statement:

> "* * * intensity and duration records by themselves do not constitute an adequate description of rain. To adequately describe rain so far as erosion and infiltration are concerned, it is necessary in addition to know the size and velocity of its drop. The relative significance of the drop characteristics depends upon a multitude of factors, such as cover, slope, and soil type, and has yet to be evaluated."

CONCLUSIONS

1. By using a distribution graph of suspended-matter concentration in connection with the observed hydrograph and a single observed suspended-matter concentration, the total suspended load of fine material transported during a particular stream rise can be calculated with reasonable accuracy. Previous records are necessary, of course, in developing a set of distribution graphs for a particular location.

2. Although the shape of the distribution graph depends on the period of rise and, to some extent, on the intensity of rainfall, unless considerable basic data are available to define the effect of intensity adequately, an average distribution graph for various periods of the rise should be used.

ACKNOWLEDGMENTS

The basic data for the East Fork of the Deep River drainage basin were obtained by the Quality of Water Division of the United States Geological Survey in cooperation with the Soil Conservation Service and the North Carolina Agricultural Experiment Station.

APPENDIX

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DISCUSSION

H. A. Einstein,[3] **Assoc. M. Am. Soc. C. E.**—A method is outlined in this paper for obtaining a dependable record of suspended load with only a fraction of the number of samples required without its use. This procedure reduces the rather high cost for any individual station considerably, and it will prove to be of greatest significance for any future suspended-load sampling program. The paper is theoretically important because it shows, more clearly than ever has been done before, the necessity of dividing the sediment load of a stream into bed-material load and wash load. The limiting size between the two is to be determined for any station before load measurements are started, because it divides the total load into two fully independent parts that follow strictly separate laws of transportation and, therefore, are to be determined according to entirely different rules.

The concept "time-of-lead" is introduced by Professor Johnson for the time interval between maximum concentration and maximum discharge, and it is explained by the rapidly decreasing availability of fine sediment in the field during a rain. In watersheds as small as those studied by the author, this original "time-of-lead" as created in the field can be found in any section of the watershed, since mere addition of sediment loads and discharges will not materially affect this time interval. In large streams, however, another influence that tends to change this time interval becomes effective. The sediment in suspension travels with the average speed of the water, or somewhat slower (especially the coarse particles, which are concentrated near the bed where the velocities are less than average). Changes of discharge travel with wave velocity, which is considerably higher than the flow velocity in most streams. This fact sometimes enables the maximum of discharge to overtake the maximum concentration. This phenomenon occurs most frequently in parts of streams below long stretches without tributaries. Probably the best example has been published by Alfred Dale Lewis under the title "Silt Observations of the River Tigris," (11) in which he describes a solitary flood of the River Tigris at Amara, Iraq, in January, 1918. The flood peak occurred on January 12, with a silt concentration equal to that before the flood. Only after the peak had passed did the concentration start to increase; and it reached its maximum on January 15, three days after the peak of the flood. Such extreme cases are rare, but the effect of the difference between the velocities of the sediment and of the flood peak is common especially in larger rivers.

There is another reason why the logic of applying the unit-graph method to large rivers is problematic. This method basically assumes a normal distribution over the entire watershed of the precipitation that causes the rise. The larger the watershed, the smaller will be the probability that the distribution of the precipitation during any one storm occurs according to the normal pattern. Even in a watershed as small as that of Enoree River (65 sq miles), mentioned by the author, storms will be found that deviate widely from the normal. This fact can be observed very readily in this river, because its normal flood has two distinct peaks, a smaller one from rain in the lower part of the watershed and a somewhat higher second peak from the rain in the upper part of the basin. These two sections are distinctly separated by a stretch of the river without major tributaries. Some few storms occurred in this watershed with precipitation in either the lower or the upper part of the basin only. The resulting hydrographs showed only one peak, and, therefore, could not be handled by the unit-graph method. Cases of this kind are rare in small physiographically uniform watersheds but become more and more frequent the larger and the more complex the watershed under consideration.

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[3] Hydr. Engr., SCS, U. S. Dept. of Agriculture, Greenville, S. C.

In the practical application of the unit-graph method as proposed by the author, it is important to bear in mind, therefore, that, although its application to streams with small and uniform watersheds has been proved to give good results with a minimum of cost, its applicability to streams with large and complex watersheds seems questionable and for this reason needs further study.

G. C. DOBSON,[4] M. AM. SOC. C. E.—A reasonably accurate estimate of the total suspended load of fine material transported by a natural stream involves a discouragingly expensive program of observations and sampling over a long period of time. Any suggestions for reducing the length or cost of these sampling programs without material loss of accuracy are very welcome. Professor Johnson's paper is a valuable contribution in this respect. Although his method depends on the usual field program until sufficient data are available for the development of a set of distribution graphs, the requirement thereafter of only one observed suspended-matter concentration for each rise or flood would reduce the cost materially and would eliminate some troublesome field problems. The assumptions involved seem in general to be as reasonable as those supporting the use of the "unit graph" which supplied the idea underlying Professor Johnson's method.

In the literature on transportation of sediment there are many references to the fact that no definite relation can be found between discharge and the load of suspended sediment. Unfortunately, little information is given as to why this relationship cannot be found. The author answers this question definitely and satisfactorily by citing the different factors affecting the availability and entrainment of the fine and coarse material moving as suspended load. It follows from this that a stream rarely, if ever, transports suspended matter to its capacity. In fact, it has not been demonstrated that a stream has a definite transporting capacity.

In the discussion of the two component parts of suspended load and the factors governing their availability, entrainment, and transportation, some implications, not developed by the author, seem too important to pass without discussion. If the wash load, which can constitute as much as 90% of the total suspended load, is carried in relatively permanent suspension and is composed of very fine material (finer than any found in appreciable quantities in the stream bed), practically all of this wash load must move continuously downstream during a flood, in suspension, at a velocity practically equal to that of the stream until it is deposited on parts of the flood plain where the velocity is greatly reduced, in a reservoir, in a lake, or in some other artificial or natural silt trap.

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[4] Senior Soil Conservationist, SCS, Greenville, S. C.

From the author's statements it seems logical to conclude that a very large part of the sediment entering a reservoir during a flood is composed of particles that had traveled from the points where they entered the stream channel to the reservoir in one continuous journey. The remainder of the sediment—much the smaller part—is made up of bed material in suspension and bed load, each governed by its own law of transportation, the particles traveling intermittently and requiring a long time to make the journey. This time factor should be of considerable importance to those interested in conserving bottom agricultural land, the capacities of reservoirs, or other downstream values because it means that any measures that would prevent the fine sediment, wash load, from entering the river system would be very efficient and immediate in their effect. There exists a rather prevalent belief that, because of the enormous amount of sediment now in river channels, such preventive measures would become effective only after a long period of time.

Thomas R. Camp,[5] **M. Am. Soc. C. E.**—The labor of measuring the silt load carried by a stream may be greatly reduced without undue sacrifice in precision, by the method presented in this paper. In regions subject to soil erosion, measurements of silt transport are necessary in order to gage the effectiveness of works constructed to reduce soil erosion. If the program is to be effective, a large number of stations must be established on streams and tributaries, and continuous records should be kept of the rate of flow of "farm land" toward the ocean. As a conservation measure, this program is just as important as stream gaging. The author is to be commended, therefore, for furnishing a means by which more stations may be operated for the same cost.

It has been found convenient, in Professor Johnson's experience, to separate "suspended load" from "bed load" and virtually to ignore the latter. Since it is the fine silt from farm land which is of interest in the conservation program, that is what should be measured. A word of caution is in order, however, concerning the temptation to draw conclusions regarding the mechanics of sediment transport from observations made in the field. The material that appears to be permanently in suspension is there because of its size and not because of its origin; and the material that appears to be transported only by bed-load movement remains at the bottom because of its size and not because of its origin.

The mechanics of the transport process is the same for all sizes of materials, except for the changes in drag coefficient of the suspended particles with size. The material that moves only along the bottom does so because the transport capacity of the eddies in the stream is inadequate to move it upward. The finer material which appears to be permanently in suspension is not permanently in suspension; it is continuously settling out (as it must if it is heavier than water) and is continuously being carried upward by turbulence.

The entire process of transportation, both on the bed and in suspension, depends for its continuance upon scour from the bed. There is no such thing as transportation in suspension, as such, for particles whose density differs from that of water. William E. Dobbins (12), Jun. Am. Soc. C. E., has demonstrated, both by theory and experiment, that material which appears to be permanently in suspension will settle out readily, under the same conditions of turbulence in the stream, if scour from the bottom is prevented. The equilibrium condition in which the concentration of suspended matter at any depth remains substantially constant with time can be maintained only if the rate of settling out across any horizontal plane—including the bottom—is exactly balanced by the rate of upward transport of sediment.

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[5] Associate Prof., San. Eng., Mass. Inst. Tech., Cambridge, Mass.

The mechanics of turbulent transport of sediment in suspension has been studied widely and is understood fairly well. A. Shields (13) has presented a satisfactory rational development for the critical tractive force at which bed-load movement starts. No satisfactory attack, however, has yet been made on the problem of the relations between the rate of scour from the bed and the hydraulic characteristics of the stream. Many measurements have been made, in the laboratory, of the transport capacity of streams, but, since nearly all the studies have been confined to bed-load movement, little information is available on the total transport capacity of a stream. For a better understanding of the hydraulic conditions governing the rate of scour, a combined theoretical and laboratory study is needed.

Frank S. Bailey,[6] **Assoc. M. Am. Soc. C. E.**—The use of a distribution graph of suspended-matter concentration is described in this paper together with the observed hydrograph and a single observed suspended-matter concentration for calculating the total suspended load of fine material transported during a particular storm. These phenomena deserve the attention of all persons interested in the collection of such data.

Having noticed the resemblance of distribution graphs of different storms for a particular stream, Professor Johnson shows quite convincingly that sufficiently accurate results may be obtained at much less labor and expense than formerly have been considered necessary.

In these days, especially when so many men are being taken into the armed services, a method of obtaining needed information, which eliminates much unnecessary effort, should be utilized whenever possible. Of course, the various changing conditions on watersheds which the author mentions always should be considered in applying his method.

Extraordinary storms may occur in which the single observation may be found inadequate, but for average cases the writer believes Professor Johnson has developed a valuable idea, which should be tested carefully in various localities.

Joe W. Johnson,[7] **Assoc. M. Am. Soc. C. E.**—As previously stated, this paper "is concerned primarily with the transportation of the fine material in small streams in humid climates." The restriction of the use of distribution graphs to humid regions is due merely to the fact that a study of their application to streams in other localities has not been made. Because the unit-graph method assumes a normal distribution of rainfall over the drainage basin, the practical application of the method is limited to small streams. As stated by Mr. Einstein, streams like the Enoree River which show different hydrographs (depending on whether the rainfall distribution is normal or concentrated either at the upper or lower end of the drainage basin) cannot be treated to advantage by the unit-graph method.

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[6] With Stone & Webster Eng. Corp., Boston, Mass.

[7] Asst. Prof., Dept. of Mech. Eng., Univ. of California, Berkeley, Calif.

An analysis of the suspended-matter records obtained in the Enoree River during floods, resulting from rains of uniform distribution, permitted the construction of a set of distribution graphs. As in the East Fork of Deep River, the shape of these graphs varied regularly with the period of stream rise. A usable relationship between the "time-of-lead" and the rate of rise, similar to Fig. 2, was also obtained, the "time-of-lead" in this instance being the period of time between the occurrence of the maximum suspended-matter concentration and the higher second peak (as mentioned by Mr. Einstein) of the hydrograph. No doubt, a set of curves could be obtained for the condition in which rainfall occurs only at the upper end of the drainage basin, and another set could be obtained for rainfall occurring only at the lower end. However, unless a single set of curves can be obtained which covers the larger percentage of cases, other approaches to the problem of estimating the wash load of a stream should be exploited.

The use of turbidity determinations is another method of reducing the labor of determining the wash load of streams. The turbidity of a water sample, as measured by the usual type of commercial turbidimeter commonly used in water works practice, is known to be related to the concentration of suspended sediment. This relationship is usually expressed as the ratio between the suspended-matter concentration in parts per million and the turbidity in parts per million of silica and is referred to as the coefficient of fineness. The value of the coefficient of fineness, however, is not constant for a particular stream. In a detailed investigation of the use of turbidity readings in suspended-load studies on the Enoree River (14), R. G. Grassy concluded that, if turbidity readings are supplemented by a few gravimetric determinations of suspended-matter concentration, accurate estimates of suspended load can be made. The method consists of first taking duplicate water samples at intervals throughout a stream rise. Turbidity determinations then are made on one sample from each pair taken, the second sample being stored temporarily. The values of turbidity are next plotted against time, and, where breaks occur in the trend of the turbidity curve, gravimetric determinations of suspended-matter concentration are made on the second sample. Values of the coefficient of fineness are calculated, and a curve of the coefficient for the duration of the stream rise is plotted and used to correct all intermediate turbidity measurements. The corrected turbidity measurements are then used to calculate the suspended load in the same manner as suspended-matter concentrations. The method requires continuous sampling throughout a flood, but the laboratory cost of analysis is much less than if all samples were analyzed by the gravimetric method. The method is also adaptable to either large or small streams, the only variable being the frequency with which samples are taken.

Mr. Dobson has extended, very clearly, the conception of the movement of the wash load in a stream by describing the continuous journey of this fine material from the point where it enters the stream to the point of deposition. The energy required to transport the wash material is probably a very small percentage of the total energy of the stream, so that once the material is washed into a stream it is transported by even the most feeble velocities. Should water-carrying wash load be diverted from a stream into an irrigation canal a great percentage of the material would be transported completely through the system to the fields. For example, S. F. Fortier and H. F. Blaney, M. Am. Soc. C. E., in the measurements of the silt load of the Colorado River in suspension in the irrigation canals of Imperial Valley found that (15):

> "In all cases this material (fine material that passes the 300-mesh sieve) was found to be equally distributed throughout the vertical section for all velocities under which tests were made, including mean velocities of less than two-thirds foot per second in small ditches. In other words, any velocity that is practical for an irrigation canal will carry in suspension the greater part of the silt transported by the waters of Colorado River, and most of it, therefore, passes on to the irrigated land. The amount of suspended silt deposited in the canals is a very small proportion of the total amount in suspension, notwithstanding the fact that large quantities of bed silt are removed each year."

Professor Camp is quite correct in stating that all material is always tending to settle out and that continuance of transportation depends upon scour from the bed. For practical purposes, though, the fine material is not derived from the bed by scour; in fact, in a canal lined with concrete practically the entire wash load carried into the canal will be transported throughout the entire length of the canal. Only the coarser fractions of the suspended load are related to the composition of the bed material and the hydraulic characteristics of the stream.

It was not the writer's intention to ignore the bed load in his discussion because he considered it unimportant but rather because such a discussion would be beyond the scope of the paper. For practical considerations, the total sediment load of a stream consists of three parts: The wash load, the coarse material in suspension, and the coarse material rolling or sliding along the bed. The methods used in estimating these loads differ greatly. The wash load may be estimated by the use of distribution graphs (as described in the paper) or by the use of turbidity data. A practical method of estimating the load of coarse suspended material has been developed by E. W. Lane, M. Am. Soc. C. E., and A. A. Kalinske, Assoc. M. Am. Soc. C. E. (16). The material that rolls or slides along the bed may be computed by the ψ-ϕ relation developed by H. A. Einstein, Assoc. M. Am. Soc. C. E., from flume studies (17) and later extended in the range of application by measurements in natural streams (18)(19).

The method and accuracy of estimating the suspended load of a stream depend on the purpose for which the data are obtained. A reliable estimate of the average annual suspended load transported past a given point is possible only if adequate records covering a considerable period of time are available. When the construction of a new project is contemplated, however, such records are seldom available, and it is rarely practicable to delay the project several years while the necessary measurements are being made. The result is that estimates are based on short-term or fragmentary records at the site, or on records of another locality where conditions are assumed to be comparable. The estimated silting rate of a reservoir, for example, may be based, in the absence of long-term data, on an average annual suspended load obtained from two or three years of record. The reliability of this value of the average load depends, primarily, on whether the factors controlling the supply of sediment to the stream were normal during the period of sampling. An investigator interested in obtaining an estimate of the average annual suspended-load contribution of a stream should examine the data in detail and evaluate them in the light of the more important variables which control the supply of sediment to the stream (20). Whereas an estimate of the average annual contribution of suspended load of a stream requires considerable investigation, an estimate of the average annual contribution of bed load can be made accurately and conveniently from the application of a suitable bed-load formula and a flow-duration curve (19)(21). Because the rate of bed-load transportation is related to discharge, a curve showing this relationship can be constructed by using a formula, such as the Einstein ψ-ϕ relationship. Used in conjunction with the flow-duration curve, this curve permits the average annual bed-load contribution of a stream to be computed. Should a flow-duration curve for a particular stream not be available, it is possible in some instances to use the flow-duration curve for a neighboring stream (21).

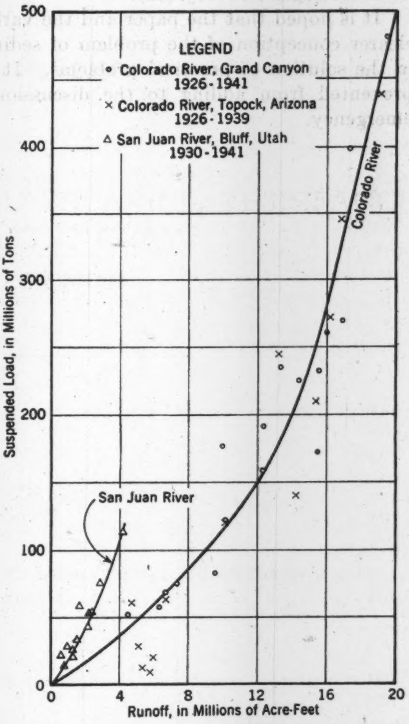

FIG. 7.—RELATION BETWEEN ANNUAL SUSPENDED LOAD AND RUNOFF FOR THE COLORADO RIVER AND SAN JUAN RIVER

Although the instantaneous value of the suspended load in a stream bears no direct relationship to the discharge, it may be possible with relatively long-term records to obtain a usable relationship between the annual suspended load and the annual runoff. For example, Fig. 7 shows a plot of runoff and silt content on the Colorado River as measured at various stations by the U. S. Geological Survey (22).

It is hoped that the paper and the various discussions will contribute to a clearer conception of the problem of sediment transportation and will assist in the solution of practical problems. It is regretted that others have been prevented from adding to the discussion because of the present national emergency.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852

TRANSACTIONS

Paper No. 2200

EARTH PRESSURE AND SHEARING RESISTANCE OF PLASTIC CLAY

A SYMPOSIUM

PAGE
FOREWORD 966

Liner-Plate Tunnels on the Chicago (Ill.) Subway.
BY KARL TERZAGHI, M. AM. SOC. C. E. 970

Earth-Pressure Measurements in Open Cuts, Chicago (Ill.) Subway.
BY RALPH B. PECK, JUN. AM. SOC. C. E. 1008

Earth Pressure on Tunnels.
BY W. S. HOUSEL, M. AM. SOC. C. E. 1037

WITH DISCUSSION BY MESSRS. RALPH H. BURKE, L. G. LENHARDT, GEORGE E. SHAFER, M. E. CHAMBERLAIN, A. A. EREMIN, A. E. CUMMINGS, D. M. BURMISTER, GREGORY P. TSCHEBOTARIOFF, D. P. KRYNINE, L. F. COOLING, KARL TERZAGHI, RALPH B. PECK, AND W. S. HOUSEL.

NOTE.—Published in June, 1942, *Proceedings*.

FOREWORD

At the meeting of the Soil Mechanics and Foundations Division of the Society in Chicago, Ill., in October, 1941, three papers presented as different parts of the program contained interesting material on the subject of shear in plastic clays. Subsequently, it was decided to assemble these papers into a composite group designed for widespread discussion by the entire membership. The present Symposium is the result.

Valuable experience on the Chicago subway has been recorded by Karl Terzaghi, M. Am. Soc. C. E., and Ralph B. Peck, Jun. Am. Soc. C. E. The first paper concerns tunnel sections and the second open-cut sections.

Work described in these two papers was done by the Department of Subways and Superhighways, City of Chicago, under Edward J. Kelly, M. Am. Soc. C. E., Mayor, and Philip Harrington, M. Am. Soc. C. E., commissioner of the Department. Ralph H. Burke, M. Am. Soc. C. E., was chief subway engineer during the major part of the construction program, until Charles E. De Leuw, M. Am. Soc. C. E., became acting chief subway engineer. Joshua d'Esposito, M. Am. Soc. C. E., was project engineer for the U. S. Public Works Administration. Soil mechanics activities were administered by R. S. Knapp, who was in administrative charge of the Survey Section.

Professor Terzaghi was consulting engineer in matters pertaining to soil mechanics and Mr. Peck was in direct charge of soil investigations and the field observations. The parts of these papers dealing with the theory of earth pressure, with the interpretation of field data, and with the application of the findings to the design of the bracing in open cuts (for example, parts II to VI of the Peck paper) are based on memoranda prepared by Professor Terzaghi.

Contractors engaged in the work contributed materially to the success of the studies. On the Chicago subway these were the Healy Subway Construction Corporation, M. J. Boyle and Company, John C. Tully Company, Kenny Construction Company, M. Pontarelli and Sons, John March and Company, Robert R. Anderson Company, and Paschen Contractors, Inc. In addition, equipment for some of the tests on the Chicago subway was made available by the firm of Spencer, White and Prentis, Inc., through the kindness of Lazarus White, M. Am. Soc. C. E.

The third paper in the Symposium, by W. S. Housel, M. Am. Soc. C. E., contains detailed directions for determining the distribution of earth pressures on tunnels, which is part of the general research program in soil mechanics at the University of Michigan in Ann Arbor. This program consists of a more or less continuous series of investigations coordinated with construction projects and sponsored by the outside engineering organizations directly interested. The results reported by Professor Housel are derived from a project undertaken by the Department of Water Supply, City of Detroit, Mich. Laurence G.

Lenhardt, M. Am. Soc. C. E., general manager and chief engineer of that Department, was one of the originators of a soil mechanics research program in the Detroit area and has since been one of its most consistent supporters.

During the past several years the Armco Drainage Products Association has become actively associated in this work through their interest in earth pressure in underground excavations. A study of the accumulated earth-pressure measurements has been made and some related investigation undertaken and completed under the sponsorship of the Association. George Shafer, Assoc. M. Am. Soc. C. E., chief engineer, has been directly concerned in that development.

Notation.—The following letter symbols, adopted for standard use in the Symposium and subsequent discussion, conform essentially to American Standard Letter Symbols for Mechanics, Structural Engineering, and Testing Materials,[1] prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1932. In particular, they follow, as a guide, the Society's *Manual of Engineering Practice No. 22*, entitled "Soil Mechanics Nomenclature." Some restrictive definitions as modified by subscripts are given in the Symposium where they first appear.

A = area;

B = one half the width of a tunnel;

C = overload ratio; $S_1/S_c = C_1$; $S/S_A = C_2$;

c = cohesion of clay: c_s = cohesion of a substitute clay layer;

D = diameter of a tunnel section;

d = depth: d_1, d_2, etc., are the vertical distances between the ground surface and struts 1, 2, etc.;

F = resultant force normal to a curved sliding surface;

H = height and depth, as depth to the center of tunnel:

  H_b = depth to bottom of tunnel;

  H_c = depth of a cohesionless layer;

  H_t = depth to top of tunnel;

  H_z = depth below the surface to any point;

  H_1 = height of the tunnel arch springing above the invert; also H_1, H_2, H_3, etc. = depth from ground surface to struts 1, 2, 3, etc.;

  H_2 = distance from lowest strut to bottom of excavation;

K = pressure ratio, or coefficient of earth pressure at rest:

  K_A = pressure ratio for active earth pressure;

  K_s = hydrostatic pressure ratio for cohesionless material;

  K_0 = ratio between the horizontal and the vertical unit pressure at a given point in a sedimentary deposit;

L = length; length of a curved sliding surface: L_p = horizontal distance between vertical profiles of struts;

l = lever arm of a force with respect to its center of rotation:

  l_c = lever arm of force S at the ground surface;

  l_r = lever arm of the force P_A;

  l_w = lever arm of weight W;

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[1] ASA-Z10a—1932.

N = center of pressure = ratio of the height of the center of gravity of the measured strut loads above the bottom of the cut, to the total depth of the cut;

n = coefficient of progressive failure;

P = a force:

P_A = total lateral pressure from the surface to a depth H_s;

P_b = a reaction;

P_F = the magnitude of a resultant earth pressure analogous to a fluid of equal weight;

P_l = lateral pressure of a fluid having a density equal to γ_a;

p = unit pressure intensity; also unit static pressure intensity:

p_a = air pressure in excess of atmospheric pressure;

p_h = horizontal pressure;

p_N = maximum normal pressure;

p_v = top vertical pressure;

p'_2 = lateral pressure;

p_3 = bottom vertical pressure under temporary equilibrium;

Q = total measured lateral pressure in one profile of bracing; also, total vertical load supported by sheeting: Q_1, Q_2, etc. = individual measured strut loads;

q = unconfined compressive unit strength or bearing capacity:

q_a = average strength for depth H;

q_c = compressive strength of clay;

q'_c = strength of relatively undisturbed clay samples as determined by laboratory tests;

q_D = bearing capacity of footings;

q_m = minimum normal strength;

q_r = strength of clay within given block;

q_s = strength of a fictitious clay layer assumed to replace a sand layer;

r = radius:

r_i = inside radius of a thick-walled cylinder;

r_o = outer radius of a thick-walled cylinder;

S = effective shearing resistance of clay along one of the potential surfaces of sliding:

S_A = average shear;

S_c = cumulative average shear from the ground surface to the top of the tunnel;

S_s = shearing resistance of a sand layer;

S_v = average vertical shear;

t = time;

W = total weight; one half the weight of clay (solids and water combined) within a given shaded area, per foot length of tunnel; also weight of a wedge of soil mass within a failure zone:

W_s = total weight of earth in a corner fillet;

W_t = total weight of a tunnel;

w = water content:

w_L = liquid limit;
w_p = plastic limit;

γ = density; also unit weight of soil:

γ_a = average density at a depth H;
γ_c = average density of clay layer;
γ_1 = density of sand deposit;

Δ = clearance allowed for jack, in making stress measurements;
ϵ = strain;
θ = angle of inclination from the horizontal;
σ = stress intensity:

σ_A = active earth pressure;
σ_h = horizontal stress on a vertical section;
σ_v = vertical stress on a horizontal section;

ϕ = angle of internal shearing resistance of clay: ϕ_1 = angle of shearing resistance in clay.

—ED.

LINER-PLATE TUNNELS ON THE CHICAGO (ILL.) SUBWAY

BY KARL TERZAGHI,[2] M. AM. SOC. C. E.

SYNOPSIS

The initial subway system designed for Chicago, Ill., includes two routes, one called the State Street route and the other known as the Dearborn Street route. The locations of these two routes with reference to the lake shore and the Loop District are shown in Fig. 1. Each route consists of station sections which are connected with each other by a double tube. Each tube has a clear width of 15 ft and a clear height of 15 ft. The bottom of the tubes is approximately 50 ft below the street surface. Between about 15 ft and 60 ft below the street surface, the soil consists of blue clay, the consistency of which ranges between stiff and very soft. The system comprises a total of 7.7 miles of double tubes and station sections. Of this total, 5.7 miles have been constructed by the light liner-plate method and 2.0 miles by the use of shields. In both types

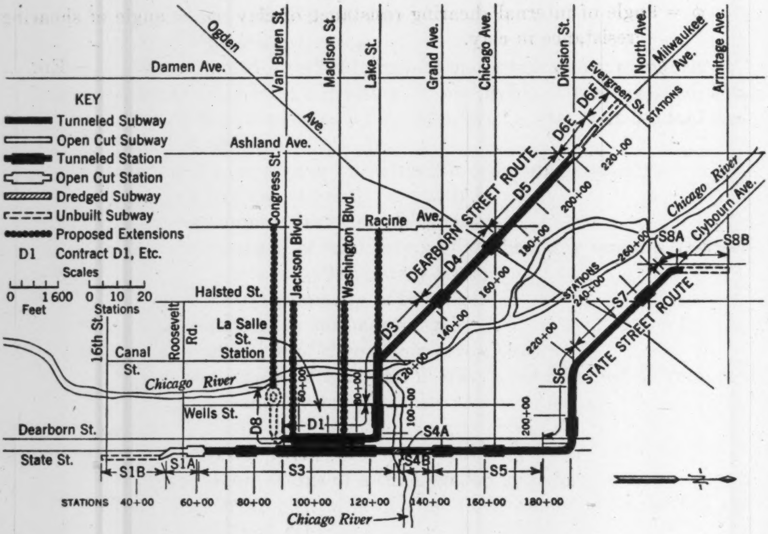
KEY
Tunneled Subway
Open Cut Subway
Tunneled Station
Open Cut Station
Dredged Subway
Unbuilt Subway
Proposed Extensions
D1 Contract D1, Etc.
Scales
0 1600
Feet
0 10 20
Stations
Ogden Ave.
Van Buren St.
Madison St.
Lake St.
Grand Ave.
Chicago Ave.
Division St.
North Ave.
Milwaukee Ave.
Armitage Ave.
Damen Ave.
Ashland Ave.
Evergreen St.
STATIONS
D6F
D6E
DEARBORN STREET ROUTE
D5
D4
D3
Congress St.
Jackson Blvd.
Washington Blvd.
Racine Ave.
Halsted St.
La Salle St. Station
Roosevelt Rd.
16th St.
Canal St.
Chicago River
Wells St.
Dearborn St.
State St.
D8
D1
S1B
S1A
S3
S4A
S4B
S5
S6
S7
S8A
S8B
STATE STREET ROUTE
Clybourn Ave.
STATIONS
40+00
60+00
80+00
100+00
120+00
140+00
160+00
180+00
200+00
220+00
240+00
260+00
Chicago River

FIG. 1.—INITIAL SUBWAY SYSTEM; CHICAGO, ILL.

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[2] Winchester, Mass.

of tunnels an air pressure of about 12 lb per sq in. in excess of atmospheric pressure was maintained.

The streets, under which the subway is located, carry street-car and elevated lines and are used by heavy trucks. Buildings from one to eighteen stories border the streets from one end of the subway routes to the other. The foundations of the buildings vary from spread footings 4 ft to 6 ft below the ground surface to caissons founded on bedrock. To serve these buildings utilities of every description are located between the street surface and the roof of the tubes.

Before construction was started, the following questions had to be answered:

(*a*) What soil investigations should be made in order to obtain a general idea of the soil conditions along the proposed routes?

(*b*) Which sections can be constructed by means of the light liner-plate method and where will it be necessary to use the more expensive shield method?

(*c*) What are the intensity and the distribution of the clay pressure on the permanent tubes?

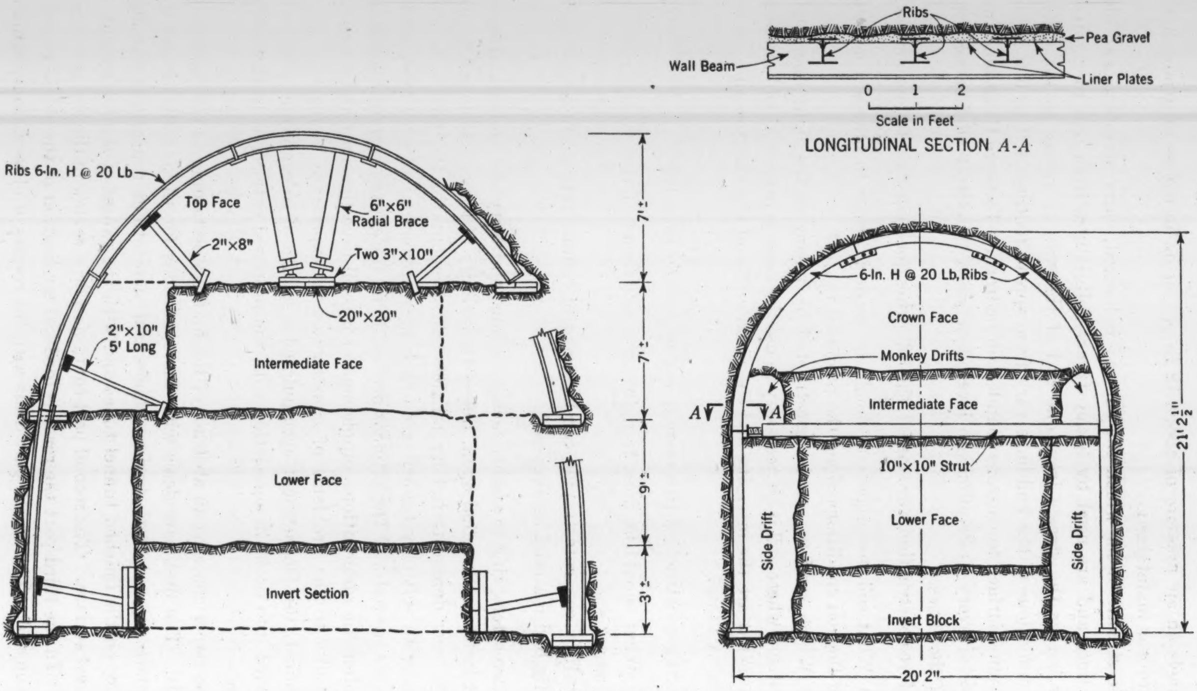
During construction two more questions arose—namely,

(*d*) What should be done to reduce the settlement of the street surface to a minimum? and

(*e*) How can catastrophic consequences of mining into a pocket of unexpectedly bad material be avoided?

When attempting to answer such questions, no theory and no method based on soil tests can be relied upon, unless the validity of the conclusions has previously been demonstrated by reliable measurements in the field. In connection with soft clay tunneling no such measurements had ever been made. Therefore the Subway Department decided to fill the gaps in current knowledge by systematic observations supplemented by soil investigations and to apply the results to the problems of construction as experience increased. As a by-product, the Department accumulated a wealth of information which can be turned into profit in connection with the proposed extensions to the initial system.

The paper contains an abstract of the findings pertaining to the liner-plate tunnels. The first part deals with the conditions for the equilibrium of the liner-plate support, with various theoretical conceptions concerning the clay pressure on the finished tunnel tubes and with the causes of the settlement of the street surface. The second part contains a brief review of the soil investigations. In the third part the reasons for local application of the shield method are discussed. The fourth part deals with the results of observations, made in


Ribs
Pea Gravel
Wall Beam
Liner Plates
0
1
2
Scale in Feet
LONGITUDINAL SECTION *A-A*
Ribs 6-In. H @ 20 Lb
Top Face
6"×6"
Radial Brace
2"×8"
Two 3"×10"
20"×20"
2"×10"
5' Long
Intermediate Face
Lower Face
Invert Section
7'±
7'±
9'±
3'±
6-In. H @ 20 Lb, Ribs
Crown Face
Monkey Drifts
A
A
Intermediate Face
10"×10" Strut
Side Drift
Lower Face
Side Drift
Invert Block
21'2½"
20'2"

(*a*)

(*b*). CONTRACT S7

FIG. 2.—CHARACTERISTIC TUNNELING OPERATIONS

an experimental section, concerning the clay pressure on the finished tubes, and the fifth concerns the settlement of the street surface. Another paper concerned with the shield tunnels is appearing elsewhere.[2a]

I.—Clay Pressure on Temporary and on Permanent Tunnel Support

Liner-Plate Method of Tunneling in the Chicago Subway.—Fig. 2 illustrates the two principal liner-plate methods used on the Chicago subway. The ribs consist of 6-in., 20-lb I-beams spaced 2 ft to 2.5 ft on centers. The space between the ribs is bridged by the liner-plates. The individual sections of the liner-plates were either very long and were bolted to the web of the ribs, or they were narrow (lagging plates) and freely supported on the outside flanges of the ribs. Since it is not possible to make the excavation exactly fit the liner-plates, they were inevitably separated from the clay by an empty space varying from zero to several inches. Three different methods were used for filling this space: Pumping a sand-cement grout through holes in the liner-plates; blowing pea gravel through holes in the liner-plates by means of compressed air; or driving wooden wedges into the open space between the clay and the lagging plates.

According to the method illustrated by Fig. 2(*a*) each rib was installed in sections starting at the crown. In every stage of construction the two lower ends of the arch ribs rested on footings except during the brief periods when new sections are added. Mining started at the crown (top face) and ended with the excavation of the invert section.

When the method illustrated by Fig. 2(*b*) was used, the attack was made by first excavating two unlined monkey drifts. The bottom of each monkey drift served as a support for a wall beam with a length of 6 ft to 16 ft. The length of the monkey drift ranged between one and one and one-half times the length of the wall beam, which in turn transferred the load on to the footings of the vertical sections of the ribs.

In both types of tunnels, the size of the footings ranged from 0.8 and 2 sq ft per lin ft of the tunnel. The small footings consisted of square blocks and the large ones were continuous. To strengthen the lateral support of the clay, horizontal struts were inserted between the ribs below midheight of the tunnel or between the wall beams as shown in Fig. 2(*b*). Customarily, two or three shifts excavated the working face for a distance of 25 to 40 ft. The invert was then cleared, reinforcing steel laid, and the concrete invert poured. In the next step the concrete arch was concreted, after which sand-cement grout was injected through pipes in the wall to fill the voids between the concrete and the liner-plates and the remaining open spaces between the liner-plates and the clay.

Conditions for the Stability of the Liner-Plate System.—Figs. 3(*a*) and 3(*b*) show simplified sections through the temporary support of the clay in a Chicago liner-plate tunnel. In both figures the shaded area represents the block of clay that moves downward if the support fails. If, by chance, the lower part of the tunnel is located within an exceptionally stiff layer of clay between soft layers, the potential surfaces of sliding consist of vertical planes through the edges of the bottom of the tunnel and the failure occurs approximately as shown in Fig. 3(*a*). Let

---

[2a] "Shield Tunnels of the Chicago Subway," by Karl Terzaghi, *Journal*, Boston Soc. of Civ. Engrs., Vol. 29, July, 1942.

W = one half of the weight of the clay (solid and water combined) located within the shaded area, per foot of length of the tunnel;
A = one half of the area of the footings, per foot of length;
B = one half the width of the tunnel;
q_D = bearing capacity of the footings per square foot;
S = the effective shearing resistance of the clay along one of the potential surfaces of sliding, per foot of length of the tunnel; and
p_a = the air pressure in pounds per square foot, in excess of the atmospheric pressure.

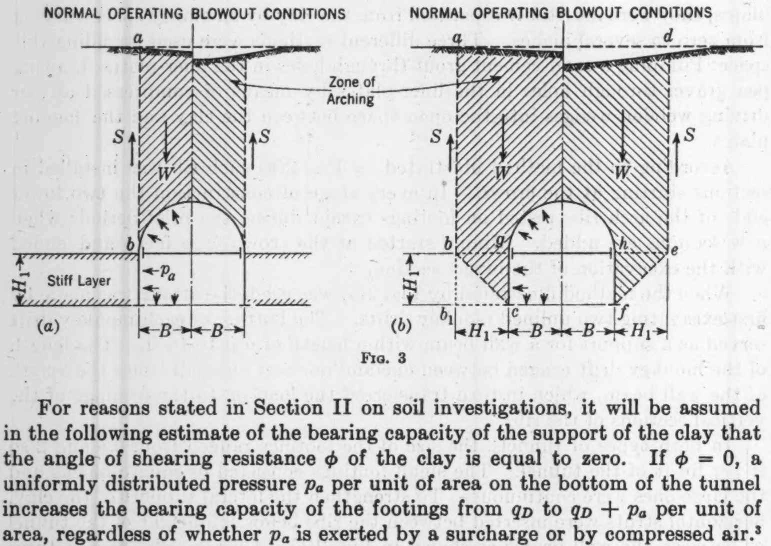

NORMAL OPERATING BLOWOUT CONDITIONS
NORMAL OPERATING BLOWOUT CONDITIONS
a
a
d
Zone of Arching
S
W
S
S
W
W
S
b
b
g
h
e
H1
Stiff Layer
p_a
H1
b1
c
f
(a)
B
B
(b)
H1
B
B
H1

FIG. 3

For reasons stated in Section II on soil investigations, it will be assumed in the following estimate of the bearing capacity of the support of the clay that the angle of shearing resistance ϕ of the clay is equal to zero. If $\phi = 0$, a uniformly distributed pressure p_a per unit of area on the bottom of the tunnel increases the bearing capacity of the footings from q_D to $q_D + p_a$ per unit of area, regardless of whether p_a is exerted by a surcharge or by compressed air.[3] The forces that act on the clay adjoining the tunnel under normal working conditions are shown on the left-hand side of Fig. 3(*a*). The temporary support of the roof fails unless

$$W - S = p_a B + A\,(q_D + p_a) \qquad \text{(1a)}$$

After a complete blowout $p_a = 0$, whereupon the stability of the support requires that

$$W - S = A\,q_D \qquad \text{(1b)}$$

If the clay on both sides of the tunnel is not exceptionally stiff, the width of the block of clay that sinks down in case of a failure has approximately a width of $2(B + H_1)$, as indicated in Fig. 3(*b*). For the purpose of a rough evaluation of the stability conditions, the potential surfaces of sliding can be replaced by two sets of plane surfaces represented by the lines a b c and d e f, respectively. The prism of clay located below the horizontal section b g is acted upon by a vertical load $W - S - p_a B$. Part of this load is carried by the prismatic block b b_1 c g and the remainder by the footings of the ribs. The average unconfined compressive strength q_r of the clay located within the block is known from tests on cylindrical samples with a height of 3.5 in. and a diameter of $1\frac{7}{8}$ in. The height of the block is equal to its width, H_1. No lateral displacement can occur along the surface b g and the tendency to bulge toward the tunnel is resisted by the stiffness of the adjoining ribs and by the adhesion between the base of the footings and the clay. If necessary, the resistance against lateral yield of the lower ends of the ribs can be increased by horizontal braces. The flexural strength of the ribs combined with the resistance of the footings against lateral displacement increases the unit load required for producing a failure of the block from q_r to at least $2\,q_r$. A horizontal unit pressure p_a on the vertical side of the block further increases this value from $2\,q_r$ to $2\,q_r + p_a$. The bearing capacity of the footings is $A\,(q_D + p_a)$. As a condition for the adequate support of the roof of the tunnel:

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[3] "Erdbaumechanik auf bodenphysikalischer Grundlage," by Karl Terzaghi. Leipzig, Deuticke. 1925.

$$W - S - B\,p_a = (2\,q_r + p_a)\,H_1 + A\,(q_D + p_a) \ldots\ldots\ldots (2a)$$

A complete blowout reduces p_a to zero, whereupon

$$W - S = 2\,q_r\,H_1 + A\,q_D \ldots\ldots\ldots\ldots\ldots\ldots (2b)$$

In either case the failure would be associated with a bending and buckling of the vertical legs of the ribs, preceded by a sinking of the footings. Other types of failure are conceivable, but they are not likely to occur because almost everywhere on the Chicago subway the stiffness of the clay increases quite rapidly with increasing depth below the bottom of the tunnels.

Eqs. 1 and 2 represent the results of a crude estimate. However, more elaborate computations would be entirely out of proportion with the approximations involved in the estimate of the shearing resistance of irregularly stratified masses of clay that have been explored by means of drill holes spaced at 300 ft.

Clay Pressure on the Finished Tubes.—In the vicinity of the working face the clay is in a state of slow, viscous flow toward the tunnel. After a few days this movement becomes imperceptible. However, if the mining operations were discontinued, the movement would undoubtedly proceed at a very low rate, and the support would adjust itself to the movement of the clay. This is a process of slow deformation produced by a constant state of stress in the clay. The construction of the concrete tube introduces a practically rigid obstacle within the clay. If the moving clay meets a rigid obstacle, the pressure of the clay on the obstacle gradually increases, and the major part of the load, which was previously carried by arching, is transferred to the obstacle. This conclusion is in accordance with the observations on the Chicago tunnels, and it precludes the application of the reasoning illustrated by Fig. 3 and expressed by Eqs. 1 and 2 to the computation of the clay pressure on the reinforced concrete tubes. Still less applicable is Rankine's earth-pressure theory because it is based on the assumptions that the clay is everywhere in a state of plastic equilibrium and that the mining operations do not change the state of stress in the clay. All these facts are known to most engineers who have had experience in tunnel construction; yet no other theories are available. Hence, in the past, tunnel tubes have been designed on the basis of empirical rules whose deficiencies were compensated by a wide margin of safety. Experience acquired in the course of the construction of the Chicago sewer system led to the tentative conclusions that the roof of the tunnels carries the full weight of the overburden and that the ratio between the horizontal and the vertical normal stress at a given point of the clay may have any value between 1/3 and 2/3. The sewer tunnels which have been designed on the basis of these assumptions have stood up very well and, since most of them are rather small, the thickness of the walls does not appear to be excessive. However, when applying the same rule to the design of large tubes such as those of the Chicago subway, one finds that the tunnels require very thick walls provided with heavy reinforcement and one faces the question of whether the computed dimensions may not be excessive.

It would be idle to attempt to answer this question on purely theoretical grounds, because the final state of stress in the tubes depends to a large extent on the state of stress in the clay after mining, which in turn depends not only on the method of mining but also on the workmanship. However, the mechanical properties of the soft clay, as disclosed by observations in the tunnel, suggest the following possibility, illustrated by Fig. 4. This figure shows a simplified section through a tunnel tube designed on the assumption that the ratio between the horizontal and the vertical pressure in the clay is everywhere equal to an arbitrarily selected constant—for instance 7/8. If the real value of this ratio is smaller (for instance, 1/2) a slight increase of the width of the tunnel is sufficient to increase the pressure ratio in the clay adjoining the tunnel from 1/2 to 7/8. A small change of the horizontal diameter does not injure the tunnel tube unless the walls are very thick. Hence, it is likely that a thin concrete shell with a nominal reinforcement should suffice permanently for withstanding the clay pressure. On account of the economic importance of this possibility, the Subway Department authorized the construction of two experimental sections for investigating the deformations associated with the stress adjustment in the clay adjoining a flexible tube. One of the sections is located in a liner-plate tunnel, on contract S-6 (Fig. 1) and the other one in a shield tunnel, on contract S-3. The results of the observations on the liner-plate section are described, subsequently, in Section IV.

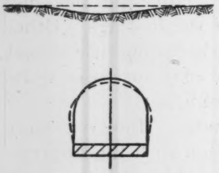
FIG. 4

Settlement of the Street Surface Above Liner-Plate Roof Supports.—During the process of mining, the arch ribs and the compressed air carry only the difference between the weight of the clay located above the excavation and the shearing forces which oppose the downward movement of the clay (see Fig. 3). This was demonstrated, for instance, by strain-gage measurements in the legs

8/12 · 7/8

of the arch ribs on contract D-3. The weight transmitted by the arch ribs on to the footings represented the equivalent of an overburden of 5 ft; the compressed air carried the equivalent of 15 ft; and the total depth of the overburden was 55 ft. As a matter of fact, without effective cooperation of arching above the tunnel, the liner-plate method of construction would not be conceivable. For mechanical reasons arching is inevitably associated with a troughlike depression of the surface of the ground located above the tunnel. Since the volume of the clay remains practically unchanged, the ground surface cannot settle unless the clay advances simultaneously toward the excavation. The movement can be resolved into three components, a downward movement (crown settlement), a horizontal yield toward the working face (axial squeeze), and an inward movement of the sides of the tunnel (lateral squeeze). On account of the mechanical properties of undisturbed clays a very small yield suffices to develop all the strength the clay has. Any further yield of the clay either has no influence on the pressure on the support, or it increases the pressure as it does in open cuts. Hence a certain part of the settlement of the ground surface is inevitable and useful. Everything in excess of this part is harmful and should be avoided.

If the support of the roof of the tunnel is adequate, the rate of the yield of the clay toward the excavation should decrease rapidly like the rate of settlement of an adequately dimensioned footing on clay. On the other hand, if the load on the liner-plate support is excessive, the roof deflects downward at a constant or an accelerated rate until failure occurs. Therefore the rate of subsidence of the roof can be utilized as a means of determining whether or not the support is loaded to the point of failure. The methods of observation and interpretation are discussed in Section V.

II.—SOIL INVESTIGATIONS

Geological Conditions.—Between the ground surface at an average elevation of about 12 ft above the level of Lake Michigan and at a depth of roughly 100 ft below the surface, the subsoil of Chicago consists of an irregularly stratified glacial deposit whose surface is covered with a layer of silt, silty sand, and artificial fill with a thickness of from 10 to 15 ft. The bedrock consists of Niagaran Limestone. Within the glacial deposit the following principal strata can be distinguished:

Stratum	Description
a	A top crust of hard, yellowish or bluish clay with a thickness of 3 to 10 ft.
b	An irregularly stratified bed of soft blue clay whose stiffness increases in a general way with increasing depth below the top crust. The thickness of this bed ranges between 60 ft and 80 ft. The bed contains a variable percentage of small rock fragments (sand to pea-gravel size, rarely greater) and isolated, more or less distorted pockets and lenses of silt and sand.
c	A layer of hard, gray, sandy material, commonly called hardpan, which may or may not be separated from the overlying stratum b by a layer of very hard, bluish clay.

The bottom layer c seems to represent the remnants of a ground moraine. Its base is usually separated from the surface of the bedrock by a layer that consists of sand intermingled with silt, gravel, and boulders. In some places the base layer contains natural gas and appears to be very permeable. The soft clay b located above the hardpan is assumed to have originated by direct deposition out of the melting ice at a time when the lake level was higher than at present. The stiff top crust (a) was formed by desiccation and local oxidation of the soft blue clay during a temporary low stage of the lake which occurred some time after the ice had retreated permanently from the Chicago area. During the same stage flowing water eroded little valleys, here and there, into the soft blue clay. These valleys have subsequently been filled with sand and silty sand. Near the intersection of South State and 16th streets, the bottom of one of these gullies was found to be about 34 ft below the present lake level. It was separated from the underlying soft blue clay by a hard crust similar to the stratum a which is commonly encountered immediately above or below the present lake level.

Practically all the tunnels are located within the soft blue clay (stratum b). The hardpan layer c was not encountered except at the river crossings.

Type and Depth of Test Borings.—Since the stiffness of the bottom has a decisive influence on the bearing capacity of the temporary support of the roof, all the test borings were carried to a depth of about 10 ft below the bottom elevation, unless the bottom was located below the surface of the hardpan. The drilling equipment consisted of that commonly used for wash borings, with the additions required for securing continuous 2-in. seamless tube samples of the clay in 2.5-ft sections. The diameter of the casing was 2.5 in. The first operation consisted in breaking the pavement. The hole was made to a depth of about 6 ft with a post hole digger; and below this depth and down to the surface of the uppermost clay stratum (a), a washing bit was used. The samplers were forced into the clay by static load. As a supplement to the soil tests the static load required to force the sampler into the clay has been recorded. If the maximum load of about 540 lb did not suffice to produce penetration, the sampler was driven into the clay by a 125-lb weight used as a hammer (2.5 ft drop and about 40 blows per min). However, this procedure was only used in some of the holes for penetrating the hardest part of the top crust.

The bottom of all the tunnels except those at the river crossings is at a considerable elevation above the base of the soft clay stratum b. Therefore only one of the test borings was driven to bedrock (which was encountered at a depth of about 100 ft) and one of them was discontinued within the hardpan stratum, at a depth of about 80 ft. Both these borings are close to the La Salle Street station. All the others terminated within the lower and stiffer part of the soft blue clay.

Spacing of Drill Holes and Variation of Soil Properties Between the Drill Holes.—It was the writer's intention to make one drill hole every 200 ft. However, on account of the time limit set for the completion of the survey, this distance was increased to about 300 ft. The average depth of the drill holes was approximately 60 ft; the total length of the drill holes on the initial system is 8,376 ft, and 3,548 ft have been added along the proposed extension lines. Fig. 5(*a*) shows an ideal section through the principal strata with two drill holes 300 ft apart. Considering the geological origin of the deposit and the resulting erratic structure, it appears hopeless to draw from the test-boring records any conclusions concerning the soil conditions within the space between the holes. However, when dealing with irregularly stratified water-laid deposits, the writer has been repeatedly impressed by the fact that the average soil properties do not change, in a horizontal direction, nearly as erratically as the individual records would lead one to expect. The observations on the Chicago subway have revealed the same fact in connection with the ice-laid deposit of this region. This is illustrated by Figs. 5(*b*) and 6.

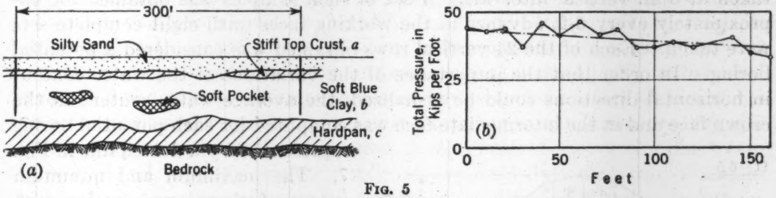

FIG. 5

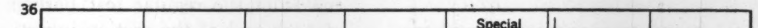

FIG. 6.—AVERAGE WATER CONTENT, TUNNEL HEADING SAMPLES

Fig. 5(*b*) represents the variation of the lateral pressure on the timbering in an open cut with a length of 160 ft and a depth of about 36 ft, on contract S-1A (south end of South State Street, Fig. 1). The ordinates represent the measured lateral pressure in kips per linear foot of the cut. They are indicative of the average shearing resistance of the soil on both sides of the cut. The individual values do not deviate by more than ±16% from those represented by the straight, broken line.

Fig. 6 shows the variations of the natural water content of the clay over a length of 50 ft of the crown face and the intermediate face of the tunnel on contract S–5 (Fig. 1) between Station 144+59 and Station 155+03. The dotted vertical lines in Fig. 7 represent the lines along which the samples were taken at 6-in. vertical intervals. A set of such samples was obtained for approximately every 5-ft advance in the working faces until eight complete sets were taken. Each of the 24 vertical rows obtained was considered a potential boring. In order that the importance of the variations of the water content in horizontal directions could be visualized, the average water content in the crown face and in the intermediate face was computed for each potential boring represented by a dotted line in Fig. 7. The maximum and minimum values of these averages for each face at a given position of the heading are plotted in Fig. 6. The ordinates of the straight line a b represent the values of the water content which were to be expected on the basis of interpolation between the values found for material from the regular test borings at each end of the section investigated. Fig. 6 shows that the greatest difference between the real values and those represented by the lines a b does not exceed ±7%, corresponding to a variation of the water content of about ±2%.

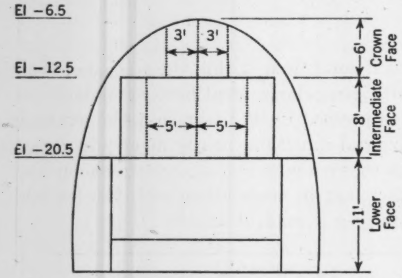

FIG. 7.—LOCATION OF TUNNEL HEADING SAMPLES (Spacing, 6 In. Vertically)

These data and the results of all the other variation surveys seem to justify the conclusion that every group of five or six text borings, driven along a section with a length of 1,200 or 1,500 ft, can be considered indicative for the general character of the entire section.

Soil Testing.—The testing program was governed by the practical purpose of the investigations. Observations were made: (*a*) To learn in advance whether the soil conditions on the different contracts were "good," medium, or "bad"; (*b*) to determine on which contracts it would be necessary to use a shield; and (*c*) to correlate the results of noteworthy observations and experiences with the properties of the clay adjoining the seat of the observations—otherwise the results would have been practically worthless.

The difficulties associated with the mining operations depend almost exclusively on the shearing resistance of the clay located above, on both sides, and immediately below, the bottom of the tunnel. During the ten years since 1932 the shearing resistance of clay has been thoroughly studied by many investigators. Important contributions were made in 1936 by L. Rendulič[4] and in 1939 by M. J. Hvorslev.[4,5] All students of the subject have demonstrated, for both undisturbed and remolded samples, that the shearing resistance of the clay increases rapidly with the pressure under which the clay has been consolidated in the laboratory. It is very probable—although it has not yet been conclusively substantiated by field observations—that the disturbed clay in a rolled or puddled earth dam exhibits similar characteristics. However, during his entire professional career, the writer has not yet seen any evidence that aforementioned conclusions based on the test results also apply to soft clay deposits in a natural state. Furthermore, he has found repeatedly (in striking contrast to what one should expect from the test results) that slopes on soft, undisturbed clays fail if the average shearing stress on the potential surface of sliding becomes roughly equal to one half of the average nonconfined compressive strength of fairly undisturbed samples, regardless of the depth of the overburden.[6] On the other hand, no one has ever published any field data that would demonstrate adequately the existence of a pressure-conditioned shearing resistance in soft clay deposits in other localities. Theoretical arguments do not count in connection with this practical issue. Hence, before construction was started, the writer considered the nonconfined compressive strength of fairly undisturbed samples more indicative for the shearing resistance of the clay in the field than any other quantity that can be determined by laboratory tests.

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[4] "Relation Between Void Ratio and Effective Principal Stresses for a Remolded, Silty Clay," by L. Rendulič, *Proceedings*, International Conference on Soil Mechanics, Cambridge, Mass., 1936.

[5] "Torsion Shear Tests and Their Place in the Determination of the Shearing Resistance of Soils," by M. J. Hvorslev, *Proceedings*, A. S. T. M., Vol. 39, 1939, p. 999.

Observations during construction merely strengthened the writer's original confidence in the compressive strength test. The digest of the pressure measurements in open cuts demonstrated for the Chicago clays the existence of a simple statistical relation between the lateral pressure and the average nonconfined compressive strength which appears to be analogous to the relation between shearing resistance and compressive strength for soft clay slides, which has been mentioned previously herein. This is the subject of the second paper in the Symposium, by R. B. Peck, Jun. Am. Soc. C. E. No evidence has been found that the angle of shearing resistance of the clay is appreciably in excess of zero degrees at any point. The discussion of the physical causes of this phenomenon is beyond the scope of this paper.

In order to make the routine soil tests, the Subway Department established a soils laboratory under the personal direction of Mr. Peck. It occupied an area of 30 ft by 20 ft and was equipped with a very satisfactory humid room. The personnel consisted of eight young assistants, most of whom worked part of the time in the laboratory and part of the time in the field (on cut measurements, observations in the experimental sections, etc.).

By loading a great number of nonconfined seamless tube samples ($3\frac{1}{2}$ in. long, $1\frac{7}{8}$ in. in diameter) to the point of failure within a time ranging between 2 and 20 min, it was found that, within this range, the time factor is immaterial. Therefore it was decided to run the tests within the shortest time compatible with satisfactory technique. This time was 5 min.

Other series of tests demonstrated that the average nonconfined compressive strength of seamless tube samples is equal to 75% of the compressive strength of samples that have been carefully cut with a knife out of the same clay in the tunnel. On the other hand, the pressure measurements in the open cuts led to the conclusion that progressive failure reduces the average compressive strength of the clay by about 25%. Since the angle of shearing resistance is practically equal to zero, the shearing resistance of the clay is approximately equal to one half of its compressive strength. This supports the conclusion that the shearing resistance against progressive shear failure in the field is approximately equal to one half of the average compressive strength of the seamless tube samples. Complete remolding reduced the nonconfined compressive strength of undisturbed tunnel samples to about one fourth of this value.

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[6] "Stability of Slopes of Natural Clay," by K. Terzaghi, *Proceedings*, International Conference on Soil Mechanics, Cambridge, Mass., Vol. I, pp. 161–165.

Although the time for determining the nonconfined compressive strength of one seamless tube sample does not exceed 5 min, the construction program did not permit making a compression test on every 6-in. section of every sample. However, by plotting a great number of nonconfined compressive-strength values against natural water content, it was found that a fairly consistent statistical relation exists between these two quantities for the soft Chicago clay. Therefore the following procedure was adopted: Every 2.5-ft seamless tube sample was cut into six sections. The natural water content was determined for each of the six sections, but the nonconfined compression test was made on only one of them.

The data thus obtained were used for plotting curves of equal compressive strength on vertical sections through the center line of the tunnel. Fig. 8(*a*) shows such a plot for the South State Street part of the State Street route and Fig. 8(*b*) for the North State Street and the Division Street part. The plots reveal at a glance the difference between the soil conditions south and north of the Chicago River. On South State Street, in the Loop District, the bottom of the tunnel is almost entirely located above the 0.8 kg per sq cm line, indicating relatively soft bottom and that more than 85% of the length of the roof is located within very soft material with a compressive strength of less than 0.4 kg per sq cm. On the other hand, along the northern part of the route, shown in Fig. 8(*a*), the bottom of the tunnel is almost consistently firm and the major part of the roof is located within medium stiff clay.

As a supplement to the aforementioned routine tests, Atterberg limit and triaxial compression tests were made. Since these tests had no direct bearing on the practical purpose of the investigations, they were made only on a limited number of samples. They furnished the following data, which may serve for comparing the Chicago clay with soft clay deposits in other localities. The plastic limit w_p of the uppermost and softest part of the soft clay deposit (designated as stratum b) was about 19% and that of the lower, stiffer part 18%. The corresponding values of the liquid limit, w_L, were 40% (exceptionally up to 50%) for the uppermost and about 30% for the lower part. For the hardpan on contract D-3 the values $w_L = 18\%$ and $w_p = 14\%$, indicating a nonplastic material, and for the hard blue clay at the upper boundary of the hardpan the values $w_L = 21\%$ to 25% and $w_p = 16\%$ have been obtained.

Triaxial compression tests at constant water content after complete consolidation under different hydrostatic pressures up to 4 atmospheres have been made on two sets of samples. One set was for the stiffer clays in the lower part of the stratum b and the other one for the softest material which is occasionally encountered in the uppermost part of this stratum. The tests on the first series were made by P. C. Rutledge at Purdue University in Lafayette, Ind., and those on the second series by D. W. Taylor, Assoc. M. Am. Soc. C. E., at Massachusetts Institute of Technology in Cambridge, Mass. The tests yielded the values 11° (softest clay) and 18° (stiffer varieties), respectively, for the angle of shearing resistance. These values are only of theoretical interest, because they have no relation to the shearing resistance of the clays in connection with cut and tunnel problems.

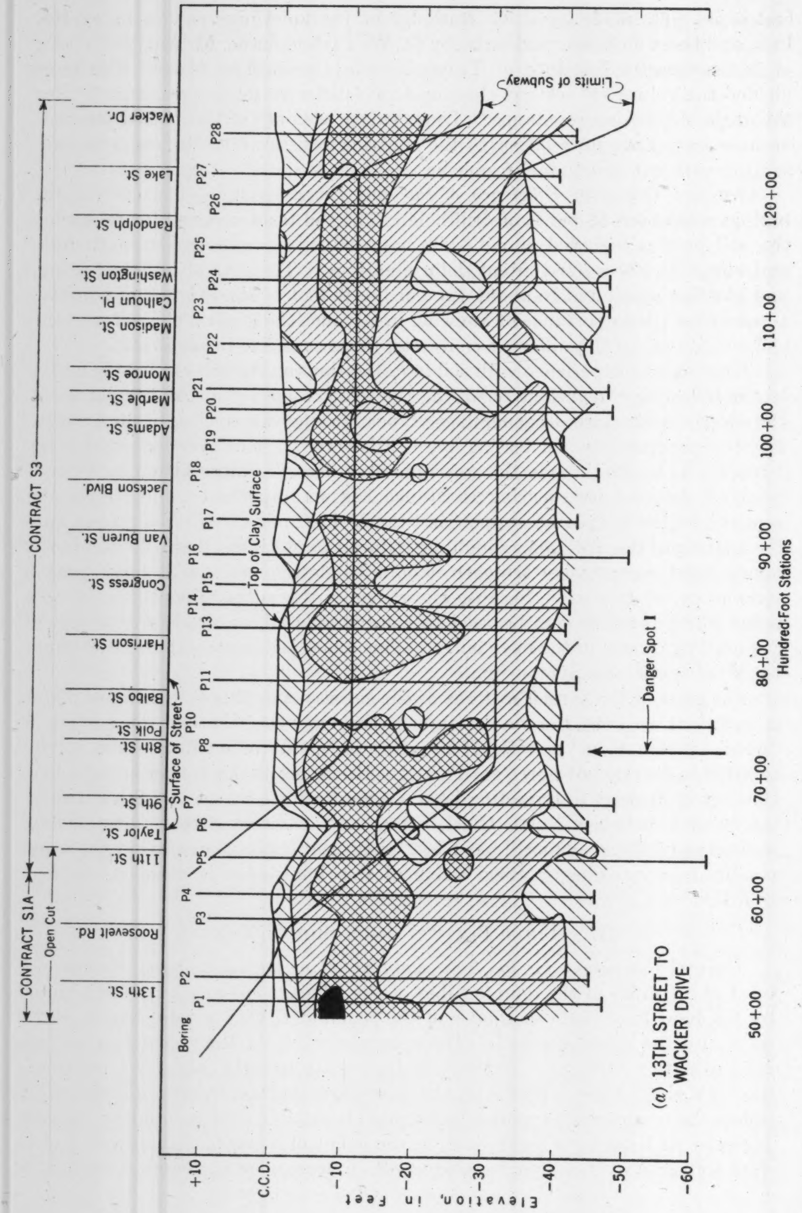
Cost and Degree of Efficiency of the Soil Investigations.—The cost of the borings was about $2 per ft of drill hole. The soil tests required for preparing the soil profiles included about 6,000 nonconfined compressive-strength tests and about 25,000 water-content determinations. The cost of boring, testing, and plotting amounted to about 42¢ per linear foot of single track of subway. Considering the importance of knowing in advance the general character of the soil conditions for the different contracts, these costs are insignificant.

Two years of experience with soil investigations on the subway led the writer to the following conclusions regarding the relative merits and shortcomings of the adopted procedure: In general, the type of boring, the spacing of the borings, and the program for soil testing were satisfactory. However, the stiff crust (stratum a) located above the soft blue clay deserves more attention than it received, because its shearing resistance has an important influence on the pressure on the temporary support in tunnels and open cuts. Along open cuts the spacing of the drill holes should be reduced to not more than 100 ft, and the nonconfined compressive strength should be determined on at least 2 or 3 samples per 2½-ft section, because knowledge of this subject has advanced to a point where the data can be used to considerable advantage in the design of the bracing system in the open cuts. (This phase is demonstrated in the paper by Mr. Peck in this Symposium.)

For most of the tunnel sections of the subway a drill-hole spacing of 300 ft is sufficient because the results inform the engineer whether the section is "good," medium, or "poor." If a section is "good" or medium, he is not interested in finding out how "good" it is. However, at the danger spots, where the clay is weakest (intersection between South State Street and 8th Street or intersection between North Clark Street and Division Street) intermediate borings should be made until one has a fairly clear conception of what the worst conditions are. The practical value of this knowledge is discussed in Section III.

III.—Liner-Plate Versus Shield Tunnel

General Conditions for the Safe Application of the Liner-Plate Method.—A total of 5.7 miles of the double tubes of the Chicago subway were constructed by the light liner-plate method and the remainder, with a total length of 2.0 miles, by the shield method. The contract price for the shield tunnels was approximately $580, and that for the liner-plate tunnels only $270, per linear foot of a single tube. However, the liner-plate method cannot be used safely unless the temporary support of the tunnel is capable of withstanding the clay pressure, at least for a short time, in the event of a complete blowout, and in very soft ground it cannot be used at all on account of inadequate support of


(a) 13TH STREET TO WACKER DRIVE

CONTRACT S1A
Open Cut
CONTRACT S3
13th St.
Roosevelt Rd.
11th St.
Taylor St.
9th St.
8th St.
Polk St.
Balbo St.
Harrison St.
Congress St.
Van Buren St.
Jackson Blvd.
Adams St.
Marble St.
Monroe St.
Madison St.
Calhoun Pl.
Washington St.
Randolph St.
Lake St.
Wacker Dr.
Surface of Street
Top of Clay Surface
Limits of Subway
Boring
P1
P2
P3
P4
P5
P6
P7
P8
P10
P11
P13
P14
P15
P16
P17
P18
P19
P20
P21
P22
P23
P24
P25
P26
P27
P28
Danger Spot I
+10
C.C.D.
−10
−20
−30
−40
−50
−60
Elevation, in Feet
50+00
60+00
70+00
80+00
90+00
100+00
110+00
120+00
Hundred-Foot Stations

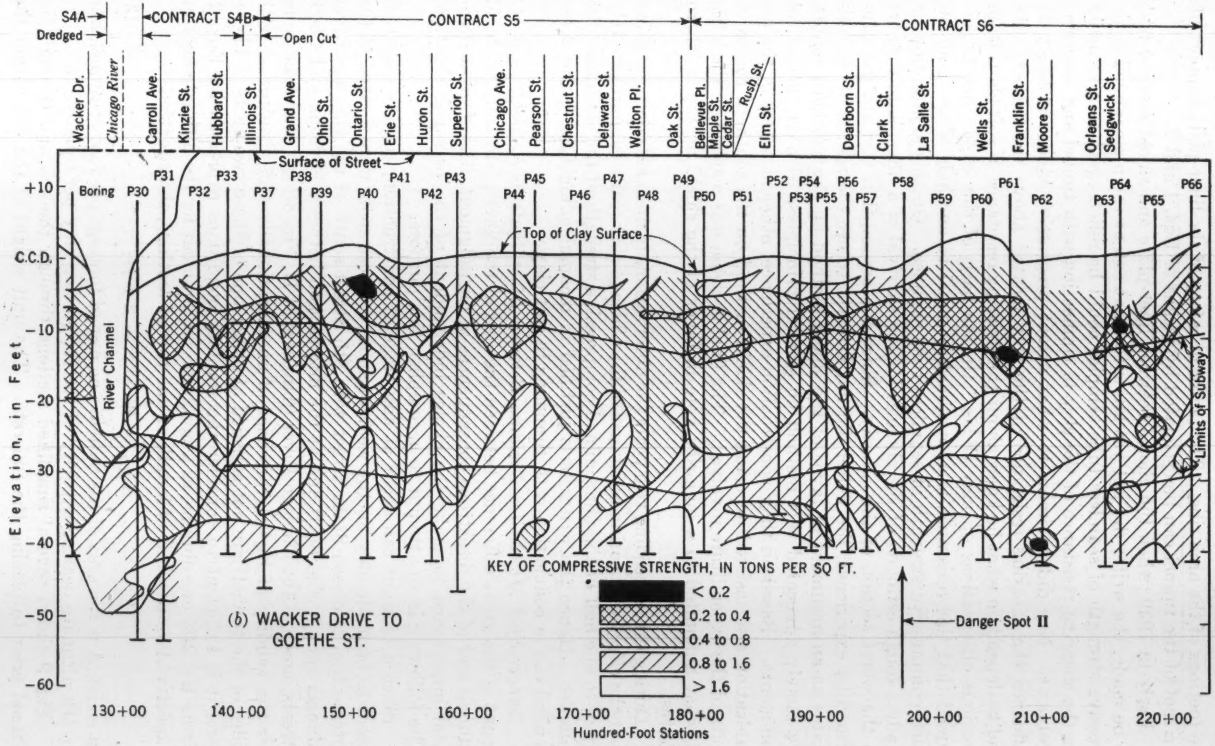


FIG. 8.—PROFILES, UNCONFINED COMPRESSIVE STRENGTH, ON THE STATE STREET ROUTE

the footings of the arch ribs. The factor of safety of the liner-plate support of the roof of the tunnel can be estimated by means of Eqs. 1 and 2.

Early in 1939, when the choice between liner-plate and shield method had to be made, the writer did not yet know to what extent the nonconfined compressive strength of seamless tube samples could be relied upon as a measure for the shearing resistance of the clay in connection with tunnel and open-cut problems. Nor did he know to what extent progressive failure is likely to influence the shearing resistance of the clay located above the tunnel. He had to depend on what he knew about the shearing resistance of soft clays from previous observations on clay slides. On the basis of this experience he assumed that this resistance is approximately equal to one half of the nonconfined compressive strength of seamless tube samples. On this assumption he made a rough estimate of the chances for the survival of a liner-plate support in the worst section of the subway. Since the assumption was not yet confirmed by experience, the writer did not attempt to use the results of his estimates as an argument in favor of the shield method on the southern routes. He merely advocated this method on general principles, without theoretical comments. Based on their construction experience, other members of the organization also insisted on using the shield method on the southern routes in spite of the expenditure involved. The final decision was reached on the basis of what amounted to a majority vote in favor of the shield on the southern routes.

During the succeeding two years the observations in open cuts demonstrated that the effective shearing resistance of the Chicago clay really is approximately equal to one half of the nonconfined compressive strength of the seamless tube samples. Hence the writer feels justified in recommending Eqs. 1 and 2 for general use in connection with tunneling in soft clay.

Danger-Spot Method for Deciding Between Liner-Plate and Shield.—On account of the great difference between the cost of the liner-plate and the shield tunnels and of the disastrous consequences of misjudgment, it is of great practical importance to make the choice independent of the personal equation. The following is an attempt to accomplish this purpose.

Since a blowout can occur at any stage of construction, the investigation needs to be made only for the weakest spots. These spots were located at the intersection of South State Street with 8th Street south of the Chicago River (danger spot I, Fig. 8(*a*)) and at the intersection between Division and Clark streets, north of the Chicago River (danger spot II, Fig. 8(*b*)). The conditions for the validity of Eqs. 1 (presence of an exceptionally stiff stratum on both sides of the tunnel) are not satisfied at either one of these two spots. Therefore, it is sufficient to consider the stability conditions expressed by Eqs. 2. From Eq. 2(*b*) one obtains, as a minimum requirement for the ultimate bearing capacity $A\, q_D$ per unit of length of one row of footings, the equation

$$A\, q_D = W - S - 2\, q_r\, H_1 \qquad (3)$$

in which A is the area of the base of one row of footings per foot of tunnel, and q_D the ultimate bearing capacity of the supporting clay per square foot.

Fig. 9 shows several simplified sections through the soil and through the tunnel prior to placing the concrete. The unit weight of the soil within the

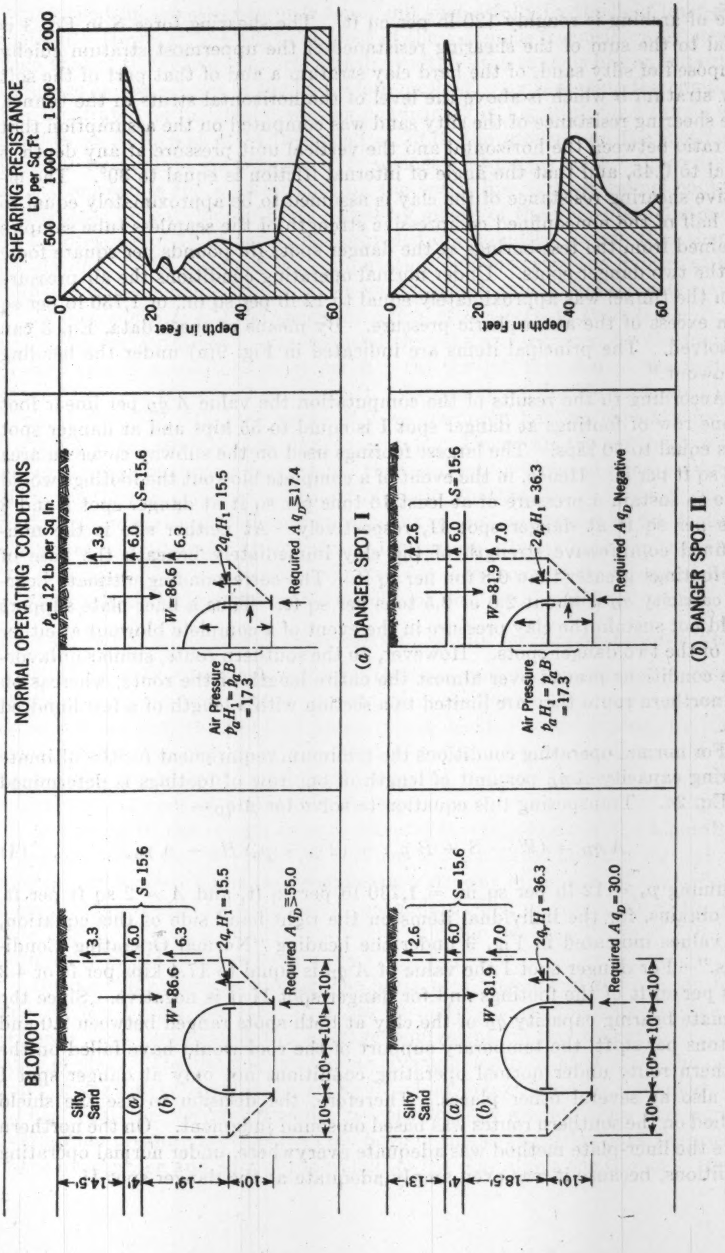

BLOWOUT
NORMAL OPERATING CONDITIONS
SHEARING RESISTANCE Lb per Sq Ft
0
500
1000
1500
2000
Depth in Feet
0
20
40
60
Silty Sand
(a)
(b)
14.5'
4'
19'
10'
10'
S=15.6
3.3
6.0
6.3
W=86.6
$2q_rH_1=15.5$
Required $Aq_D \geqq 55.0$
p_a=12 Lb per Sq In.
Air Pressure $p_aH_1=p_aB$ =17.3
Required Aq_D=17.4
(a) DANGER SPOT I
13'
4'
18.5'
10'
2.6
7.0
W=81.9
$2q_rH_1=36.3$
Required Aq_D=30.0
Required Aq_D Negative
(b) DANGER SPOT II

FIG. 9.—SIMPLIFIED SECTIONS PRIOR TO PLACING CONCRETE
(All loads and forces given in kips per foot)

zone of arching is roughly 120 lb per cu ft. The shearing force S in Eq. 3 is equal to the sum of the shearing resistance of the uppermost stratum chiefly composed of silty sand, of the hard clay stratum a and of that part of the soft clay stratum b which is above the level of the horizontal struts in the tunnel. The shearing resistance of the silty sand was computed on the assumption that the ratio between the horizontal and the vertical unit pressure at any depth is equal to 0.45, and that the angle of internal friction is equal to 30°. The effective shearing resistance of the clay is assumed to be approximately equal to one half of the nonconfined compressive strength of the seamless tube samples obtained from the test borings at the danger spots (in pounds per square foot) for the two danger spots. Under normal operating conditions the air pressure p_a in the tunnel was approximately equal to 12 lb per sq in., or 1,730 lb per sq ft in excess of the atmospheric pressure. By means of these data, Eq. 3 can be solved. The principal items are indicated in Fig. 9(*a*) under the heading "Blowout."

According to the results of the computation the value $A\ q_D$ per linear foot of one row of footings at danger spot I is equal to 55 kips and at danger spot II is equal to 30 kips. The largest footings used on the subway cover an area of 2 sq ft per ft. Hence, in the event of a complete blowout the footings would have to sustain a pressure of at least 16 tons per sq ft at danger spot I and 8 tons per sq ft at danger spot II, respectively. At neither site is the nonconfined compressive strength of the clay immediately beneath the base of the footings greater than 0.8 ton per sq ft. The corresponding ultimate bearing capacity q_D is about 2.0 or 2.5 tons per sq ft. Thus a liner-plate support could not sustain the clay pressure in the event of a complete blowout at either one of the two danger spots. However, on the southern route, similar unfavorable conditions prevail over almost the entire length of the route, whereas on the northern route they are limited to a section with a length of a few hundred feet.

For normal operating conditions the minimum requirement for the ultimate bearing capacity $A\ q_D$ per unit of length of one row of footings is determined by Eq. 2*a*. Transposing this equation to solve for $A\ q_D$—

$$A\ q_D = (W - S - B\ p_a) - (2\ q_r + p_a)\ H_1 - A\ p_a \quad \text{.........} (4)$$

Assuming $p_a = 12$ lb per sq in. $= 1{,}730$ lb per sq ft, and $A = 2$ sq ft per ft, one obtains, for the individual items on the right-hand side of this equation, the values indicated in Fig. 9 under the heading "Normal Operating Conditions." For danger spot I the value of $A\ q_D$ is equal to 17.4 kips per ft or 4.3 tons per sq ft of the footings and for danger spot II it is negative. Since the ultimate bearing capacity q_D of the clay at both spots ranged between 2.0 and 2.5 tons per sq ft, the temporary support of the roof would have failed on the southern route under normal operating conditions not only at danger spot I but also at several other places. Therefore, the decision to use the shield method on the southern routes was based on sound judgment. On the northern route the liner-plate method was adequate everywhere, under normal operating conditions, because it was even amply adequate at the danger spot II.

All the Chicago sewer tunnels were built successfully by the light liner-plate method. However, the width of most of these tunnels is considerably smaller than that of the subway tunnels and the width of the tunnels has an important influence on the success of the liner-plate method. By repeating the computation illustrated for "normal operating conditions" in Fig. 9(*a*), for a tunnel with a width and a height of 14 ft instead of 20 ft, the pressure on the footings is found to be equal to 1.8 tons per sq ft against 4.3 tons per sq ft. Since danger spot I represents the weakest spot on the entire line, a 14-ft tunnel could be constructed beneath South State Street by means of the liner-plate method at an air pressure of 12 lb per sq in., provided one could be certain that there is no blowout. The result of the computation demonstrates the danger of generalizing the experiences made when constructing tunnels through soft clay.

Influence of Size of Footings on the Safety of Liner-Plate Procedure.—In the preceding computations it was assumed that the area covered by the footings of each row is equal to 2 sq ft per ft. On the job the area of the footings ranged between 0.8 and 2 sq ft per ft. According to the blowout conditions illustrated in Fig. 9(*b*), the value of $A\ q_D$ (required bearing capacity for the footings) is negative for the danger spot II. Hence on the northern routes under normal operating conditions, the smallest footings are large enough. However, in the event of a serious blowout, the temporary support on the roof of the tunnel may fail only on account of inadequate size of the footings. Therefore it is advisable to specify, in the future, a lower limit for the size of the footings. This size can be computed for each contract by means of Eq. 3 on the basis of the least favorable conditions disclosed by the soil profile.

IV.—CLAY PRESSURE AFTER CONCRETING AND THE DESIGN OF TUNNEL TUBES

Purpose of Investigations.—The writer's conceptions regarding the pressure of the clay on the tunnel tubes are stated in Section I. To verify or modify these conceptions, the Subway Department decided to determine, by direct measurement, the change in the shape of a fairly flexible experimental tube designed on the basis of an arbitrarily selected value for the ratio between the horizontal and vertical pressure in the clay.

According to the results of experimental investigations in the laboratory, it appears likely that the ratio K_0 between the horizontal and the vertical unit pressure at a given point in a sedimentary deposit of sand is of the order of magnitude of 0.45. In a normally consolidated bed of clay such as the soft blue clay in Chicago, this ratio is likely to range between 2/3 and 7/8. In both types of soil it is less than unity.

During the process of mining, the horizontal pressure in the clay on both sides of the tunnel decreases from its normal value to the value of the air pressure that represented the average equivalent of the weight of a 15-ft layer of clay. The overburden had an average depth of 35 ft. On the other hand, immediately beyond the zone of arching represented by a shaded area in Fig. 3(*b*) the vertical pressure is considerably greater than the normal overburden pressure because the clay beyond this zone carries both the weight of the overburden and part of the weight of the clay above the tunnel. Hence the process of mining tends to reduce the average value of the ratio from K_0 to a smaller value K_1.

When the temporary support of the clay is replaced by a relatively rigid tunnel tube, the average value K gradually changes and approaches a final value K_2. There is no reason for assuming that K_2 should be greater than K_0 which ranges between 2/3 and 7/8. It is most likely that K_2 is approximately 2/3.

To establish an important difference between the stress conditions which exist in the vicinity of a rigid tube and those imposed upon the clay by a flexible tube, it was decided to give the flexible experimental tube a circular cross section. If it is subjected to an equal all-around pressure, the shape of the tube remains unchanged. On the other hand, if the horizontal pressure in the clay is less than the vertical pressure, the tube flattens until a similar state of stress is established. A tube with very thick walls is likely to fracture during the process of adjustment unless it is heavily reinforced, whereas a tube with thin walls undergoes only elastic deformations. To design a tube in such a manner that it does not break necessitates knowing the magnitude of the deformations involved. Therefore the core of the investigations was

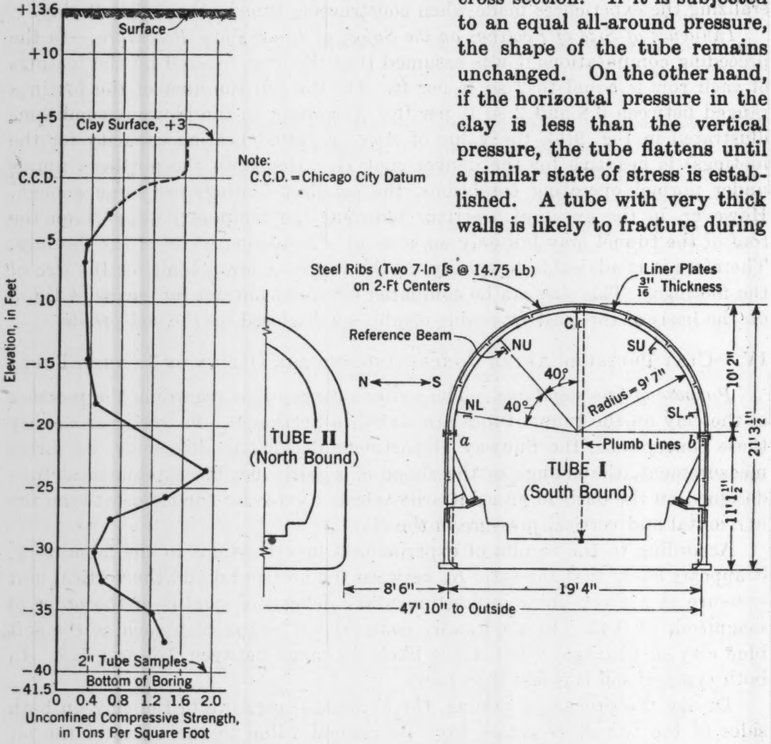

+13.6
Surface
+10
+5
Clay Surface, +3
C.C.D.
-5
-10
-15
-20
-25
-30
-35
-40
-41.5
2" Tube Samples
Bottom of Boring
Elevation, in Feet
0 0.4 0.8 1.2 1.6 2.0
Unconfined Compressive Strength, in Tons Per Square Foot
Note: C.C.D. = Chicago City Datum
Steel Ribs (Two 7-In [s @ 14.75 Lb) on 2-Ft Centers
Liner Plates 3/16" Thickness
Reference Beam
C
NU
SU
N ⟷ S
40°
40°
Radius = 9' 7"
NL
SL
a
b
Plumb Lines
TUBE II (North Bound)
TUBE I (South Bound)
10' 2"
21' 3½"
11' 1½"
8' 6"
19' 4"
47' 10" to Outside

FIG. 10.—DISTORTION MEASUREMENTS, EXPERIMENTAL SECTION S-6

the measurement of the deformations. As a by-product, information on the intensity and the distribution of the contact pressure on the base of the invert was obtained.

Experimental Section on Contract S-*6*.—The experimental section, shown in Fig. 10, is located on contract S-6 between Station 201+47 and Station 201+68 in the southbound tube, which was the first constructed. It has a length of 21 ft. Above the wall beams the tube is composed of steel ribs on 2-ft centers curved to an inside radius of 9 ft 7 in., with steel liner-plates bolted to the webs of the ribs to span the space between the ribs. The manner of erecting the ribs and of the liner-plates was the same as that for contracts where ribs and large liner-plates were used for temporary lining. The ribs were composed of two 7-in. channels riveted together back to back in order to provide sufficient area to carry the direct compression at a moderate unit stress. The plates were fabricated from 3/16-in. sheet steel.

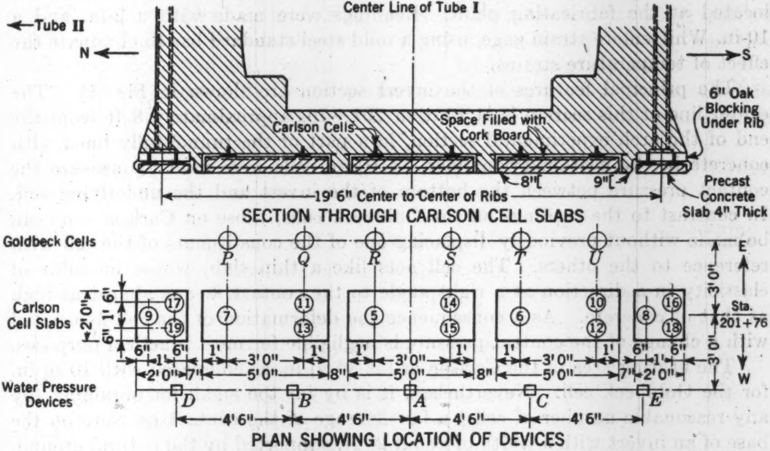

Center Line of Tube I
←Tube II
N←
→S
6" Oak Blocking Under Rib
Carlson Cells
Space Filled with Cork Board
8"[
9"[
Precast Concrete Slab 4" Thick
19' 6" Center to Center of Ribs
SECTION THROUGH CARLSON CELL SLABS
Goldbeck Cells
P
Q
R
S
T
U
E
W
Carlson Cell Slabs
9
17
19
7
11
13
5
14
15
6
10
12
8
16
18
6"
1'
2' 0"
3' 0"
5' 0"
7"
8"
Sta. 201+76
Water Pressure Devices
D
B
A
C
E
4' 6"
PLAN SHOWING LOCATION OF DEVICES

Fig. 11.—Location of Pressure Measuring Devices, Experimental Section, Contract S-6

Below the wall beam, normal construction procedure involves the erection of vertical steel ribs supported on wood bearing blocks. To avoid complications in the erection procedure, this method was adhered to and the concrete invert was poured as usual. The brackets were subsequently fastened to the vertical ribs below the wall beams. The hinge was supported on the concrete invert. After erection of the brackets, the vertical steel ribs were burned through above the invert. The space between the liner-plates and the clay was filled with wood blocking grouted with sand-cement grout.

To measure the distortion of the ribs, five steel reference beams were located as shown in Fig. 10. They are designated by NL, NU, C, SU, and SL. The ends of the reference beams were supported by steel sleeves embedded in the concrete of the adjacent permanently-lined sections. The action of the experimental section was made independent of the adjacent concrete sections by means of sliding joints. The distance between the ribs and the reference beams was measured between two gage points by means of a pair of calipers and a steel scale reading directly to 0.02 in.

In addition to the distortion measurements, provision was made for strain readings on the ribs themselves. On the circular section above the wall beam, eight 2-in. and four 10-in. gage lines were located on the center line of each rib to determine the compressive stress in the ribs. Many others were located on the ribs below the wall beam. In all, 840 strain-gage lines were established. The holes were drilled and initial readings were made while the steel was still located at the fabricating plant. Readings were made with a 2-in. and a 10-in. Whittemore strain gage, using a mild steel standard bar to eliminate the effect of temperature strains.

The principal features of the invert section are shown in Fig. 11. The center line of this section is at Station 201+76, approximately 8 ft from the end of the arch experimental section, in a part of the tunnel fully lined with concrete in the customary manner. Carlson cells were used to measure the contact pressure between the bottom of the invert and the underlying soil. In contrast to the readings on the Goldbeck cells, those on Carlson cells can be made without previously displacing one of the constituents of the cell with reference to the others. The cell acts like a thin slab, whose modulus of elasticity in a direction at a right angle to the contact face is almost as high as that of concrete. As a consequence the deformation of the cell associated with a change of the contact pressure is negligible for most practical purposes.

The bearing area of the Carlson cell is 30 sq in. as compared with 10 sq in. for the Goldbeck cell. Nevertheless, it is by far too small for obtaining (by any reasonable number of cells) a fair average of the contact pressure on the base of an invert with a width of about 20 ft, supported by the natural ground. In order to overcome this difficulty, the writer proposed constructing larger measuring units that consist of heavily reinforced, rectangular concrete slabs. The dimensions of these units are given in Fig. 11. Each unit rests on the natural ground and the intensity of the contact pressure on each unit is measured by means of three Carlson cells located between the top surface of the concrete slab and the concrete invert. Two of the units, placed beneath the steel ribs, are 2 ft by 2 ft in plan and were pre-cast so that the entire assembly of the slab and the cells could be set beneath the rib instead of the wooden bearing block customarily used. Great care was taken to prevent the concrete from entering the assembly. The remaining slabs were 2 ft by 5 ft in plan. They were cast in place on the invert soil in channel forms 8 in. deep. Surrounding each slab was a second channel frame 9 in. deep with a clearance between the two frames of 1/2 in. This space was filled with porous cork board to prevent the entrance of concrete. On top of the measuring slabs, steel plates were bedded at the location of the Carlson cells, and the cells were seated on the plates with plaster of Paris. The part of the slab not covered by Carlson cells was built up to a height of the outer channel frame with cork board and covered with a metal sheet to prevent the invert concrete from coming into contact with the slabs except over the cells. Cork board was chosen because it was sufficiently rigid to compress only 0.1 in. under the weight of the fresh invert concrete, but sufficiently compressible to carry no appreciable part of the load after the concrete had set.

Because of the wide previous use of Goldbeck cells, it was considered desirable to make an installation of these devices in the invert section and to compare the readings with those obtained by means of the Carlson cells and the strain gage readings in the ribs. Accordingly, one Goldbeck cell was placed opposite each Carlson cell location in the central part of the invert as indicated in Fig. 11 (cells P to U). Each Goldbeck cell was attached to a length of flexible high-pressure tubing in the laboratory and all connections made, so that no piping or field connections were necessary. Each cell was placed face down on the clay surface and carefully bedded. It was then anchored in position and eventually covered by the invert concrete.

Finally a group of five water-pressure measuring devices was installed for the purpose of measuring the hydrostatic pressure in the water contained in the clay below the invert.

Results of Observations.—During the period of observation the following events are of special importance:

Date (1940)	Description	Cumulative time t, in days
February 27	Erection of arch section started..........	....
February 29	Invert section completed, initial readings on Carlson cells......................	....
March 20	Initial readings on reference beams for deformation measurement.............	0
March 23	Hinges in arch section freed	3
March 27	Working face of the second tube passed by the Experimental Section located in the first tube...........................	8
March 30	Temporary drop of air pressure from 11.5 to 8 lb per sq in. while the second tube was not yet concreted	10
July 19	Air pressure removed.....................	121

Fig. 12 shows the deformation of the tube at different stages before and after the removal of the compressed air. During the process of mining, the arching above the roof of the tunnel increases the vertical pressure in the clay in the vicinity of the boundaries of the zone of arching over its normal value. While the second tube was mined part of the weight of the clay located above this tube was transferred to the north half of the experimental section. The increase of the vertical clay pressure on this part caused by mining the second tube produced the unsymmetrical deformation of the experimental section, shown in Fig. 12.

Fig. 13 represents the relation between the time and the horizontal displacement of the points NL and SL (Fig. 10), respectively. Both points moved outward, which indicates that the initial ratio between the horizontal and the vertical pressure in the clay was smaller than unity. A considerable part of this movement occurred at time $t = 8$ days, while the second tube was being mined. Since the second tube was located on the north side of the experimental section the point NL moved much more than the point SL. After a period of about 500 days both points became practically stationary. The final displacement of NL was 0.375 in., and that of SL was 0.210 in.

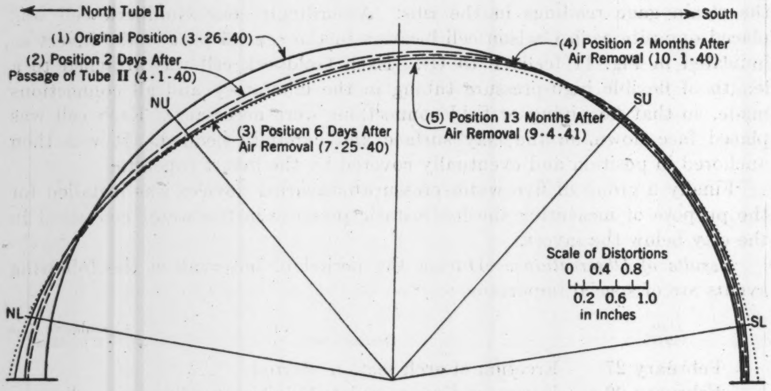

North Tube II
South
(1) Original Position (3·26·40)
(2) Position 2 Days After Passage of Tube II (4·1·40)
(3) Position 6 Days After Air Removal (7·25·40)
(4) Position 2 Months After Air Removal (10·1·40)
(5) Position 13 Months After Air Removal (9·4·41)
NU
SU
NL
SL
Scale of Distortions
0 0.4 0.8
0.2 0.6 1.0
in Inches

FIG. 12.—DISTORTION MEASUREMENTS, EXPERIMENTAL SECTION, SOUTHBOUND TUBE I, CONTRACT S-6

Hence the horizontal diameter of the tube did not increase by more than 0.58 in., which is an insignificant amount and much smaller than the writer anticipated. Almost one half of the increase occurred during the first two weeks after the hinges were freed.

The results of the pressure-cell readings are shown in Fig. 14. The two horizontal lines represent ideal cell readings computed on the assumption that there is no arching above the tunnel and that the contact pressure is uniformly distributed over the base of the invert. They correspond to the states before

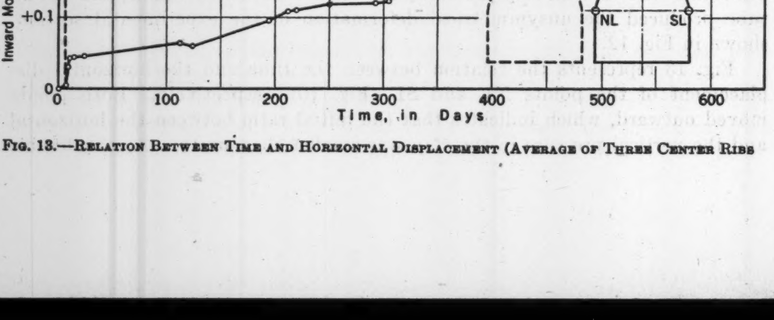


FIG. 13.—RELATION BETWEEN TIME AND HORIZONTAL DISPLACEMENT (AVERAGE OF THREE CENTER RIBS

and after the removal of the air pressure respectively. At the end of the first month after the section was mined, the arching still carried a considerable part of the weight of the overburden. This is shown by the position of the line marked 3–26–40 with reference to the upper horizontal line. However, as time went on the arching effect decreased and the contact pressure approached the difference between the weight of the overburden and the air pressure. Removal of the air induced a temporary revival of the arching effect. The final value is again fairly close to the weight of the overburden. Along the outer edges of the invert the unit contact pressure is somewhat greater than the average contact pressure on the central part of the invert. This is chiefly a consequence of establishing the footings for the arch ribs before the invert is poured. While the second tube was mined, the unit pressure along the northern edge of the tube became permanently greater than that along the southern edge. The reason is identical with that for the unsymmetrical deformation of the rib. However, all these deviations from the uniform pressure distribution are not important enough to deserve practical consideration.

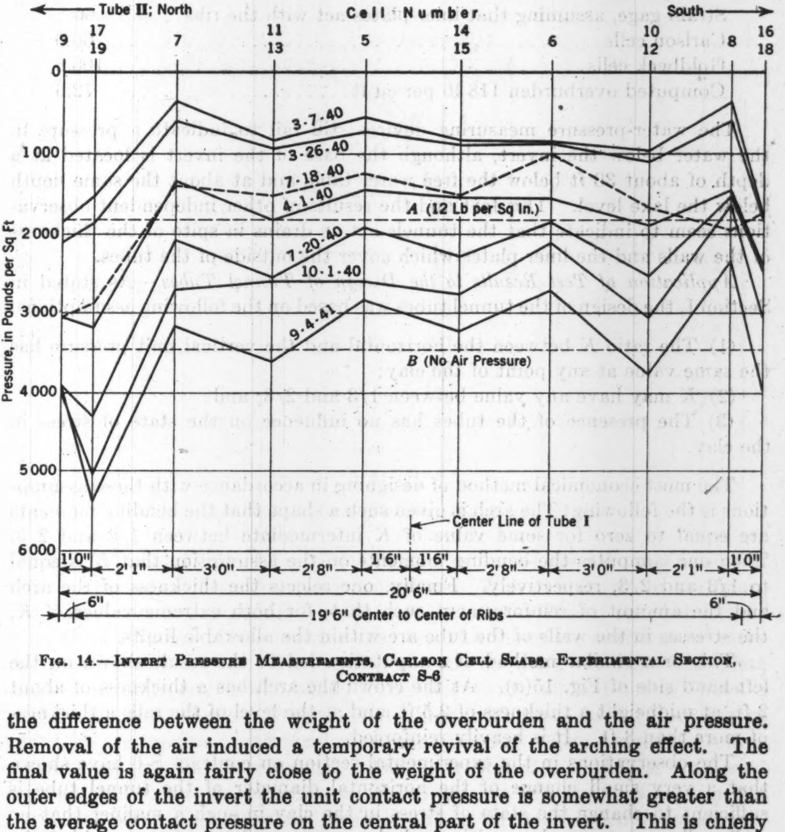


FIG. 14.—INVERT PRESSURE MEASUREMENTS, CARLSON CELL SLABS, EXPERIMENTAL SECTION, CONTRACT S-6

All the Goldbeck cells remained in operation and gave fairly consistent results. However, the individual readings are so scattered that they give no indication of the true distribution of the contact pressure as disclosed by the Carlson cells.

The following data represent the summary of the results of measurements of the contact pressure made in September, 1940:

Method	Stress, in kips per ft
Strain gage, assuming that liner-plates act with the ribs	66
Carlson cells	66.0
Goldbeck cells	106.5
Computed overburden 118 lb per cu ft	72.6

The water-pressure measuring devices still fail to indicate a pressure in the water below the invert, although the base of the invert is located at a depth of about 30 ft below the free water table and at about the same depth below the lake level. This fact and the results of other independent observations seem to indicate that the tunnels act as drains in spite of the thickness of the walls and the liner-plates which cover the outside of the tubes.

Application of Test Results to the Design of Tunnel Tubes.—As stated in Section I, the design of the tunnel tubes was based on the following assumptions:

(1) The ratio K between the horizontal and the vertical unit pressure has the same value at any point of the clay;

(2) K may have any value between 1/3 and 2/3; and

(3) The presence of the tubes has no influence on the state of stress in the clay.

The most economical method of designing in accordance with these assumptions is the following: The arch is given such a shape that the bending moments are equal to zero for some value of K intermediate between 1/3 and 2/3. Then one computes the bending moments on the assumption that K is equal to 1/3 and 2/3, respectively. Finally, one selects the thickness of the arch and the amount of reinforcement such that, for both extreme values of K, the stresses in the walls of the tube are within the allowable limits.

This or a similar method of computation led to the result shown on the left-hand side of Fig. 15(*a*). At the crown the arch has a thickness of about 2 ft, at midheight a thickness of 2.5 ft, and at the level of the rails a thickness of more than 3 ft. It is heavily reinforced.

The observations in the experimental section on contract S-6 have shown that a very small change of the horizontal diameter of the tunnel tube is sufficient to change the state of stress in the clay in such a manner that no stiff walls are required to withstand the pressure of the clay, provided the following conditions are satisfied: First, the shape of the arch should be such that the bending moments in the arch are equal to zero for any arbitrary value of K intermediate between unity and 2/3 or less. Second, the arch should be thin enough to follow the deformation of the clay without rupture. However, even a local failure does not endanger the tunnel. Railway tunnels with thick walls have survived a settlement at the crown of more than 1 ft and an irregular distortion of their shape. In the case under consideration the deformations do not exceed a fraction of an inch. These empirical facts make it possible to reduce the quantity of concrete and of reinforcement for the arch to a fraction of what is required according to the customary methods of design and, in addition, to reduce the quantity of excavation quite substantially.

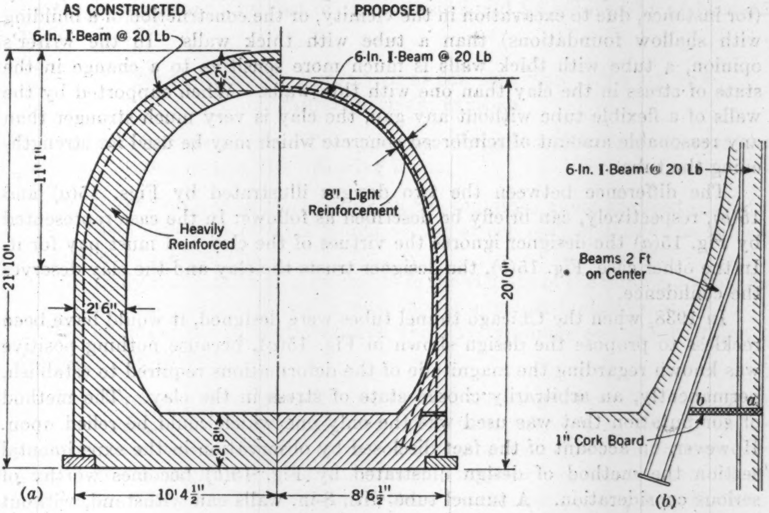


FIG. 15.—COMPARISON OF PROPOSED AND ACTUAL DESIGNS

The right-hand side of Fig. 15(*a*) illustrates one of several possibilities for taking advantage of the foregoing conditions. The arch shown in the figure has a thickness not exceeding 8 or 10 in. No harm is done if the shell is only 6 in. thick in some places because the steel ribs alone are sufficient to carry the pressure of the clay. The mining is done by using the light liner-plate method without any modification. The ribs (6-in., 20-lb I-beams) have a slightly elliptical shape. They do not differ from those used before except that they are provided with brackets as shown in Fig. 15(*b*). After the tunnel is mined the invert C_1 is concreted. The lower ends of the brackets are buried in concrete. After the concrete of the invert has set, the vertical legs are burned at point a whereby the pressure due to the overburden is transferred on to the brackets. Then the surface of the concrete between a and b is covered with cork board, to provide some flexibility for the connection between the arch and the invert. Finally the arch is poured.

It has been mentioned that almost one half of the deformation of the experimental section occurred within the first two weeks after the hinges were freed. The subsequent increase of the horizontal diameter of the tunnel did not exceed 0.3 in. Therefore the concreting of the arch should be postponed until the second tube has been mined or, preferably, until the air is turned off.

Once the ends of the brackets are buried in the concrete, even a complete blowout has no more harmful effect, because the invert is designed to carry the weight of the entire overburden, without the assistance of compressed air.

One might object that a tunnel tube with thin walls may be less likely to suffer the consequences of a subsequent change in the state of stress in the clay (for instance, due to excavation in the vicinity, or the construction of a building with shallow foundations) than a tube with thick walls. In the writer's opinion, a tube with thick walls is much more sensitive to a change in the state of stress in the clay than one with thin walls. When supported by the walls of a flexible tube without any gap, the clay is very much stronger than any reasonable amount of reinforced concrete which may be used for strengthening the tube.

The difference between the two designs illustrated by Figs. 15(*a*) and 15(*b*), respectively, can briefly be described as follows: In the case represented by Fig. 15(*a*) the designer ignores the virtues of the clay and must pay for it. In the other case, Fig. 15(*b*), the designer trusts the clay and the clay deserves the confidence.

In 1938, when the Chicago tunnel tubes were designed, it would have been reckless to propose the design shown in Fig. 15(*a*), because nothing positive was known regarding the magnitude of the deformations required to establish, permanently, an arbitrarily chosen state of stress in the clay. The method of computation that was used was the only one which could be relied upon. However, on account of the fact disclosed by observation in the experimental section the method of design illustrated by Fig. 15(*b*) becomes worthy of serious consideration. A tunnel tube with 8-in. walls can withstand, without serious harm, deformations ten times greater than those observed. Furthermore, the deformations of a tube with a slightly elliptical cross section can be expected to be smaller than those of a tube with a circular section.

Since the clay is not uniform the shape which the section of a flexible tube tends to assume changes from place to place in an unpredictable manner. However, the character of the clay changes very gradually both in a horizontal and in a vertical direction and the concrete is capable of very considerable plastic deformation without losing its strength. Therefore, the writer believes that the variable character of the lateral support of the tunnel tube is a factor of minor importance as long as the magnitude of the change of the horizontal diameter remains within the limits disclosed by the observations in the experimental section.

It goes without saying that Fig. 15(*b*) merely serves to illustrate the existing theoretical possibilities. In practice the imperfections of the methods of construction must be taken into consideration. They place a lower limit on the thickness of the walls of the tunnel.

A second experimental section was established in a shield tunnel, between Station 87+55 and Station 88+00 on contract S-3 (Fig. 1) (see Footnote 2*a*). The results of the observations in this section were at least as encouraging as those already described. The cost of both experimental sections and of the observations to date amounted to about $35,000—or 45¢ per linear foot of single-track subway. A considerable part of the money was spent on concreting these sections after construction was finished, which was not included in the writer's program. As a matter of fact, it would have been a great asset to engineering science if permission had been granted to keep the experimental sections permanently accessible. By taking advantage of the facts disclosed by the observations, the cost of the tunnels could be reduced by at least $30 per ft.

V.—SETTLEMENT OF STREET SURFACE AND THE RÔLE OF THE SQUEEZE OBSERVATIONS

Settlement Observations.—The program for the settlement observations included, among others, the following items:

(1) Establishment of level points on the buildings along the routes;

(2) Establishment of level points on the street surface above the subway on profiles spaced 20 ft;

(3) Daily readings on all level points in the vicinity of the mining operations;

(4) Measurement of lateral movements of adjacent buildings; and

(5) Construction of plots showing the level readings and a tabulation of the daily movements.

The bench marks were established beyond the range of possible movements due to mining. Three readings were taken with reference to the city datum and the mean was used as the elevation of the bench. Level marks were then placed on all columns, walls, machinery, or other important parts of the buildings. The initial elevations of these marks were checked in the same manner as the benches.

In the vicinity of the seat of mining operations, readings were taken daily or as often as needed. At times readings were made every two hours for a 24-hr period. Level parties were on call at all times to take care of emergencies. Experience showed that the settlement does not extend beyond a distance of about 75 ft from the lot line on each side of the street. Hence few readings were made beyond this distance.

The results of the settlement observations were condensed into settlement profiles containing the maximum settlement of the street surface due to mining for each of the two tubes individually, the maximum total settlement 3 to 20 days after removal of air pressure, the air pressure at the time of mining, the distance between the end of the finished tube and the working face, and finally general information concerning the soil conditions. The files of the Subway Department also contain complete settlement profiles on 20-ft intervals. The cost of the settlement observations amounted to $2.75 per linear foot of single-track tunnel.

General Characteristics of the Street Settlement.—Figs. 16(*a*) and 16(*b*) represent characteristic settlement profiles of the street surface above a twin tube, and Figs. 16(*c*) and 16(*d*) are similar profiles taken above double tubes. In each figure the ordinates of the lower curve represent the settlement due to the construction of the first tube (I = settlement until the arch was concreted and I′ = settlement until the working face of the second tube approached the profile). The ordinates of the upper curves represent the supplementary settlement due to the construction of the second tube. (Curve II is the settlement between the time when the working face approached the profile and the time when the arch was concreted; and II′ is the settlement due to second tube until the air was turned off.) The broken-line curves represent reversed images of the curves I. They facilitate a comparison of the settlement due to mining the first tube with that produced by mining the second tube. If the movements produced by these two mining operations were identical, the broken-line curve should be identical with curve II.

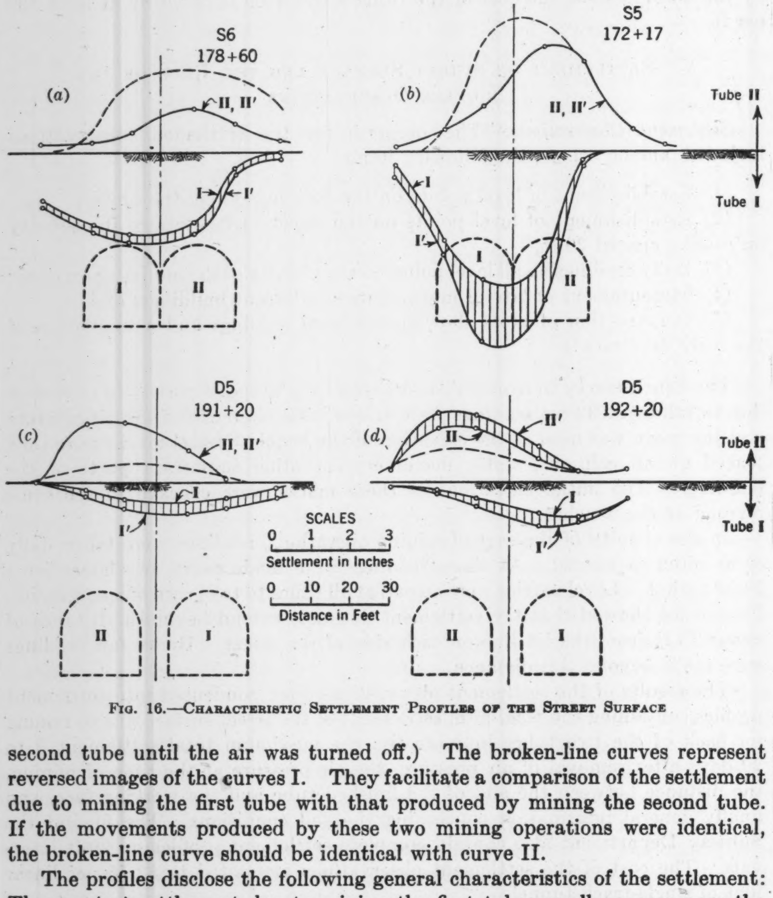


FIG. 16.—CHARACTERISTIC SETTLEMENT PROFILES OF THE STREET SURFACE

The profiles disclose the following general characteristics of the settlement: The greatest settlement due to mining the first tube usually occurs near the middle of the street and not above the center line of the tunnel. This seems to indicate that the change of stress in the sandy top layer increases the susceptibility of this layer to being compacted by the street traffic. This interpretation is corroborated by the observation that the settlement of the street surface is commonly greater than that of the clay surface underlying the sand. On the other hand, the greatest settlement due to mining the second tunnel is usually above the center line of this tunnel because at the time the second tunnel was mined the sand was already compacted.

The greatest settlement due to mining the second tube on a twin tube section is almost invariably much smaller than that due to mining the first one. Figs. 16(*a*) and 16(*b*) (and subsequently, Fig. 18) serve as examples. If the subway consists of a double tube the settlement over the second tube is everywhere very much greater than that caused by mining the first tube, as shown in Figs. 16(*c*) and 16(*d*). These rather surprising phenomena seem to be associated with the archings above the tunnel. If the second tunnel of a twin tube is mined, one side of the zone of arching is located above the first tube which represents a practically rigid support. On the other hand, while the second tunnel of a double tube is mined, one of two supports of the mass of clay which arches over the tunnel consists of a block of clay that has been disturbed while mining the first tube. Every disturbance inevitably reduces the rigidity of the clay. Hence twin tubes appear to be preferable to double tubes.

Squeeze Measurements.—The settlement of the street surface and that of the adjoining buildings with shallow footings represent an undesirable but inevitable accompaniment of tunneling in general. Even the expensive shield method cannot eliminate this feature. To be able to discriminate between avoidable and unavoidable settlement, one must examine the individual factors that contribute toward the settlement. This was accomplished by measuring the different components of the movement of the clay toward the tunnel. For future reference the measurements were supplemented by recording all the data that may help to identify the conditions under which the observations were made.

The following is a list of the items that constituted a complete set of squeeze measurements:

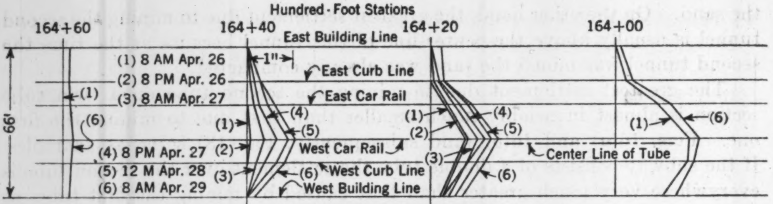
(*a*) Complete Record of Excavation and Construction Progress.—This record consisted of an hourly tabulation of the location of each of the portions of working face and other pertinent data.

(*b*) Measurement of Lateral Inward Squeeze of Clay Walls.—As soon as the clay on the sides of the tunnel was exposed by excavation, a steel rod was driven into the clay on each side of the tunnel and the distance between the rods was measured at suitable time intervals. Simultaneously the lateral movement of the legs of the arch ribs was observed and recorded.

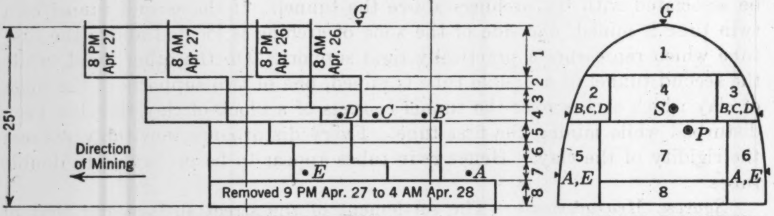
(*c*) Downward Movement of the Tunnel Crown.—A steel rod was driven in a vertical direction into the clay in the roof and the elevation of the lower end of the rod was determined at suitable time intervals with a level.

(*d*) Settlement of the Lining.—Level readings were made on the wall beams and on the footing blocks in order to determine whether or not vertical movement of the temporary lining occurred.

(*e*) Inward Movement of the Working Face.—Steel spearheads were set in advance of the face before mining operations in a given heading were undertaken. These spearheads were driven into the working face to a distance up

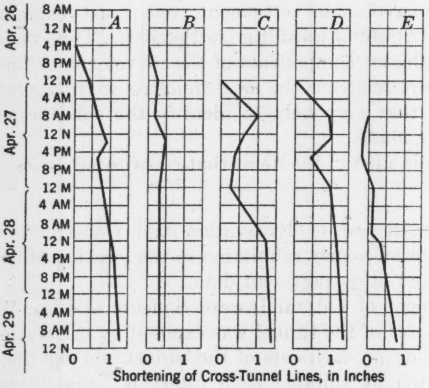


(a) SETTLEMENT PROFILES


G
8 PM Apr. 27
8 AM Apr. 27
8 PM Apr. 26
8 AM Apr. 26
25'
Direction of Mining
D
C
B
E
A
Removed 9 PM Apr. 27 to 4 AM Apr. 28
1
2
3
4
5
6
7
8
G
1
2
B,C,D
4
S
3
B,C,D
P
5
A,E
7
6
A,E
8

(b) EXCAVATION PROGRESS; LOCATION OF REFERENCE POINTS

(c) KEY TO EXCAVATION PROGRESS AND LOCATION OF REFERENCE POINTS



Point (Fig. 17(c))	Location	Time of setting	Settlement (in.)
G	Crown reference points	11:00 a.m.; April 26[b]	2⅛
S	Spearheads; No. 1 set 10 ft beyond face	3:00 p.m.; April 25	1⅞[a]
	No. 2 set, during mining, 9 ft beyond face	1:00 p.m.; April 29	1[a]
P	At working face after mining	No movement during 24 hr while face was standing	

[a] Toward excavation during mining. [b] Read at noon, April 29.

FIG. 17.—RECORDS OF SQUEEZE TEST, SOUTHBOUND TUBE, APRIL 26–29, 1939

to 12 ft ahead of excavation by means of a 1-in. pipe. A smaller steel rod which could be inserted inside the pipe was then used to push the spearhead away from the end of the pipe and leave it surrounded on all sides by clay. After the spearhead was ejected from the pipe, but while the rod was still in contact with its rear end, the distance between the spearhead and a reference point in the completed part of the tunnel was measured. A second measurement was made when the spearhead was uncovered by excavation. The difference between the two measurements is equal to the inward movement of the spearhead during mining. In one instance the spearhead was connected with a steel wire that protruded beyond the original position of the working face. Thus it was possible to ascertain the position of the spearhead at different stages of mining.

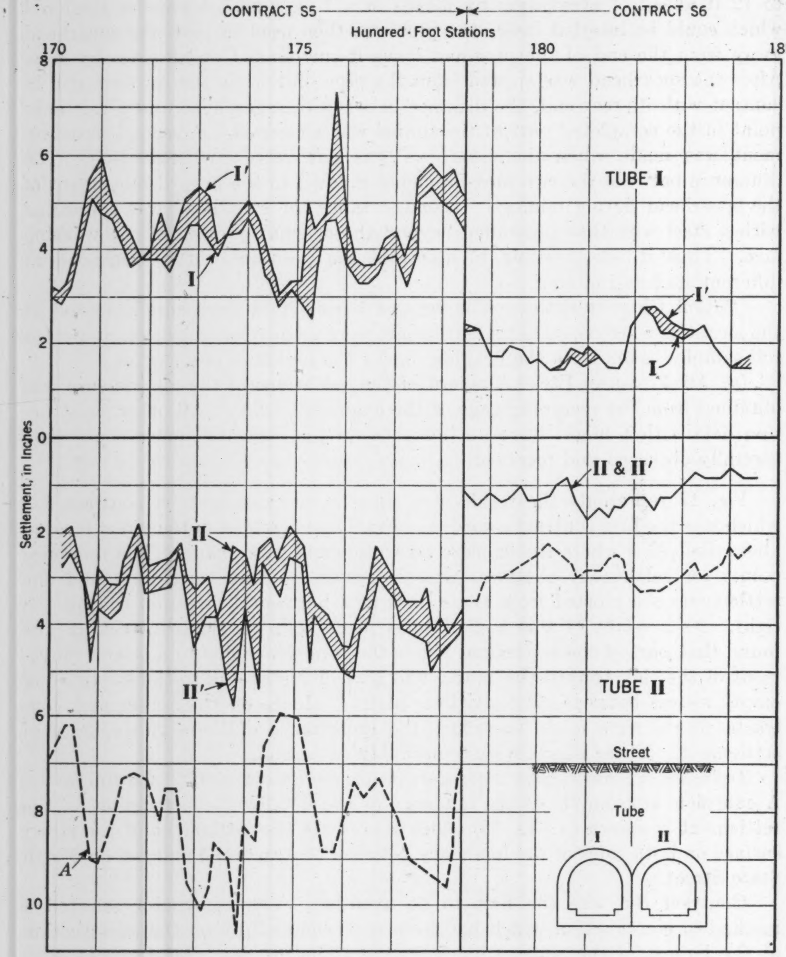
(*f*) Soil Characteristics.—The regular boring data were supplemented at the localities where squeeze measurements were made by soil tests on numerous soil samples taken from the heading during the period of observation.

(*g*) Air Pressure, Etc.—A record of the variations of the air pressure was obtained from the recording gage at the engineer's office. All other construction details that might have an influence on the measured movements were carefully observed and recorded.

Fig. 17 represents the result of a squeeze measurement on contract S-5 which was the first contract awarded. Fig. 17(*a*) is a plan of the street located above the place where the squeeze measurements were made. The reference points for settlement observations are located on transverse lines and the settlements are plotted from these lines in a horizontal direction toward the right. As a result of this and of subsequent tests on this contract, it was found that part of the settlement above the tunnel was due to a lateral movement of the clay toward the tunnel and part of it was due to the existence of empty spaces between clay and liner-plates. Hence on the subsequent contracts efforts were made to reduce the importance of these two sources of settlement. These efforts were remarkably successful.

Influence of Construction Procedure on the Settlement of the Street Surface.—A conspicuous example of the influence of the details of construction on the settlement is shown in Fig. 18, which represents the settlement of the street surface on both sides of the boundary between contracts S-5 and S-6 on North State Street.

Contract S-5 was the first to be awarded. The contractor selected a method of construction which had been used successfully on the construction of the largest of the sewer tunnels in the Chicago area. Mining was done without the aid of a wall beam, by means of the procedure illustrated by Fig. 2(*a*). However, each operation of mining the side pockets and erecting rib segments below already completed portions removed most of the support afforded by the rib to the clay overlying the tunnel. As a result it was frequently found that the crown segment settled as often as a lower portion was added to the rib. By the time the footing blocks at the base of the ribs were placed, the ribs had undergone a considerable downward movement. As a consequence, on numerous occasions, some crown ribs were at a considerably



Curve	Identity; Maximum:
I	Settlement prior to pouring concrete in Tube I, southbound
I′	Settlement prior to the approach of Tube II, northbound
II	Increase in settlement between the approach of Tube II and the beginning of concreting
II′	Increase in settlement between the passing of Tube II and the removal of air pressure
A	Total settlement 12 to 25 days after the removal of air pressure

FIG. 18.—SETTLEMENT AS INFLUENCED BY DETAILS OF CONSTRUCTION

lower elevation than others. The size of the footings was equal to 1 sq ft per ft of the length of the row. Sand-cement grout was injected through holes in the liner-plates from an air line. One gained the impression that the grout did not penetrate all the empty spaces. However, the influence of these details of construction on the settlement of the street surface was not known because in former days no systematic records of the settlement of the street surface were kept. Therefore, it was decided to investigate this influence and to modify the method of construction on the later contracts in accordance with the results of the observations.

Once the relations between the underground operations and the settlement of the street surface were clearly recognized it was a relatively simple matter to reduce the settlement by appropriately modifying the procedures on the subsequent contracts, including the adjoining contract S-6. One step consisted of inserting a wall beam between the curved and the straight part of the arch ribs, approximately at midheight of the tunnel. The wall beams prevented the settlement of the crown. By bracing the wall beams against each other with horizontal struts it was also possible to reduce the lateral movement of the clay toward the tunnel. Another important step consisted in changing the method of grouting. On contract S-6, instead of attempting to fill the space between clay and liner-plates with sand-cement grout, wood wedges were driven into this space until the voids were almost completely eliminated. In the third step a mining templet was used, cut from a piece of 1-in. by 8-in. lumber which served to check the accuracy of the upper working face without actually lifting the crown rib up to the face. Mining was thus simplified to a considerable extent and made more accurate.

The effect of these improvements on the settlement of the street surface is demonstrated by Fig. 18. The soil conditions were practically the same on both sides of the boundary between contracts S-5 and S-6. Above the north end of contract S-5 the settlement of the street surface amounted to 6 in. and more. By taking advantage of the experiences gained during the construction of S-5 it was possible to reduce the settlement to 2½ or 3½ in. above the adjoining section of S-6.

Although the settlement above contract S-6 was very much smaller than it was above S-5, there was ample evidence, that the pressure of the clay on the temporary support of the clay on S-6 was also smaller. These and similar observations lead to the following conclusions: To obtain the best results both in the tunnel and on the street surface, the clay should be given as little opportunity for deformation as possible. Even if everything has been done to prevent deformation, the deformations are still important enough to mobilize all the strength inherent in the clay. In this respect the undisturbed soft clay behaves like a solid with a high modulus of elasticity. Very small deformations suffice to stress it to the point of incipient failure. Any further deformation reduces the strength of the clay. On several occasions the writer got the impression that the soft clay actually breaks before it becomes plastic, because both grout and air were found to escape from the tunnel through a thick bed of clay toward the surface without evidence for the presence of a pre-existing communication between the tunnel and the surface through a vein of foreign material.

By reducing the deformation of the clay to the smallest amount compatible with the general method of construction one also reduces (on Chicago tunnels) the danger of breaking the stiff top crust of the soft clay. According to the diagrams of blowout conditions in Fig. 9, the shearing resistance of this crust (stratum a) represents almost 40% of the total shearing force associated with arching above the roof. This crust cannot stand as much deformation as other strata. As soon as it breaks on account of excessive settlement of the crown of the tunnel, the shearing force S decreases instantaneously. In the writer's opinion the remarkably low values of the settlement above contract S-6 were due partly to the stiff top crust remaining intact. This opinion is corroborated by the observation that the crown of the tunnel on S-5 settled not continuously but by jerks which occurred as often as the working face advanced through a distance of about 20 or 25 ft.

Squeeze Measurements as Danger Signals.—The squeeze measurements described herein proved to be very enlightening for both the engineers and the contractors. Therefore it became customary for the contractors to request squeeze measurements whenever the clay in the tunnel seemed to behave abnormally or the settlement of the street surface increased for no apparent reason. Thus the squeeze measurements served unintentionally as a safeguard against accidents.

It has been stated before that the strength of the clay above the tunnels is due largely to the hard upper crust (see Fig. 9). It is further known that the present stage of Lake Michigan was preceded by a low-water stage during which the bed of clay was exposed to surface erosion. The test borings were spaced 300 ft and they can easily miss such a detail as a buried gully. They are also likely to miss isolated pockets of semi-liquid material surrounded by soft clay (see Fig. 5(*a*)). Hence at any stage of construction the working face might have approached an exceptionally soft spot or a buried gully, the existence of which was not known. The first indication of the pending encounter between the tunnel and the bad spot consists in an abnormal increase in the settlement of the street surface, which always induced a call for a squeeze measurement. The characteristics of the normal flow of the clay toward the tunnel were known to the engineers by experience. Hence any abnormal phenomenon such as a flow at an accelerated rate would have induced them to request speedy construction of a bulkhead to be followed by an exploration of the danger zone by means of supplementary test borings. Once the character of a danger zone is known one can modify the method of mining accordingly.

CONCLUSIONS

(1) In general, the original program for the test borings and the soil tests was satisfactory. In connection with future investigations of a similar nature in the Chicago district, the following modifications appear desirable. Along open cuts the spacing between drill holes should be reduced from the normal value of about 300 ft to not more than 100 ft. In the danger zones disclosed

by the soil profile, supplementary borings should be made and the stiff top crust of the soft blue clay deserves more attention than it received.

(2) In several places on South State Street the liner-plate method of construction would have led to failure even in case an air pressure of 12 lb per sq in. had been permanently maintained. A blowout would have had catastrophic consequences almost everywhere on this route. Early in 1939, when the contracts were let, it was not yet possible to predict these conditions on the basis of the results of soil tests. Therefore the decision to use the shield method on this route was reached by necessity on the basis of what amounted to a majority vote; opinions were divided. The uncertainties which then existed have been eliminated by developing a method, called the "danger-spot method," which permits a decision on the basis of the results of soil tests as to whether or not the liner-plate method can safely be used on a given contract.

(3) Observations on an experimental section in a liner-plate tunnel have demonstrated the possibility of safely reducing the thickness of the arch and the amount of reinforcement to a fraction of that which is required according to the customary methods of design. The proposed method of design takes advantage of the mechanical properties of soft, undisturbed clay.

(4) The soft Chicago clay in an undisturbed state behaves like a solid material with a high modulus of elasticity. On a tunnel with a width of 20 ft, deformations of a few inches seem to suffice to stress the clay to the point of incipient failure. Hence, by reducing the movement of the clay toward the tunnel to the smallest amount compatible with the method of construction, everything can be gained and nothing is lost. The methods for accomplishing this purpose on the job have been discussed in detail.

(5) Valuable information has been secured by measuring, from time to time, the movement of the clay toward the tunnel and the movement of the members of the temporary support. The test results placed the efforts toward reducing the settlement of the street surface on a rational basis, and the practice of making the test whenever the clay appeared to behave abnormally reduced the risk of accidents due to mining into bad spots whose existence had not been known.

EARTH-PRESSURE MEASUREMENTS IN OPEN CUTS, CHICAGO (ILL.) SUBWAY

By Ralph B. Peck,[7] Jun. Am. Soc. C. E.

Synopsis

Systematic field observations on open-cut portions of the Chicago subway provided a constant check on the loads carried by the bracing as well as the soil movements associated with the excavations. Results of the measurements are given in this paper, together with a comparison of the measured earth pressures with generally accepted theories. It was found that the magnitude of the total lateral pressure was in satisfactory agreement with either the plane or general wedge theories for purely cohesive soils, having no effective internal friction, but that the distribution of the pressure was non-hydrostatic. Measured movements of the sheeting were found to be in accordance with those theoretically necessary for non-hydrostatic distribution. Simple rules are given which are believed to be applicable to the design of bracing for similar cuts in plastic clay deposits.

Part I.—Field Measurements

Although the Chicago subway was constructed largely by tunneling methods, a number of special structures were built in open cut. The excavations, frequently of considerable magnitude, were in congested districts near the downtown area in a clay deposit of variable stiffness. It was considered of vital importance to obtain definite information concerning the safety of the bracing of the temporary structures, and to determine the nature and magnitude of the movements of the soil mass and ground surface associated with the excavation. Furthermore, it was considered desirable to obtain quantitative data which could be used as the basis for economical design of the bracing for future excavations. As a result, a systematic program of measurements was undertaken to determine the necessary information.

The open cuts varied in type of bracing, in dimensions, in soil properties, and in method of construction. All these factors were included in the information obtained. The proposed excavations were always outlined by steel piling before any material was removed. Both continuous steel sheeting and H-piles spaced at suitable distances for the insertion of timber lagging were used. All of the excavations were braced with horizontal wales and struts, some of steel and others of wood. The excavations varied in width from 15 ft to 55 ft, in depth from 35 ft to 46 ft, and in length from 100 ft to 1,300 ft. Although the

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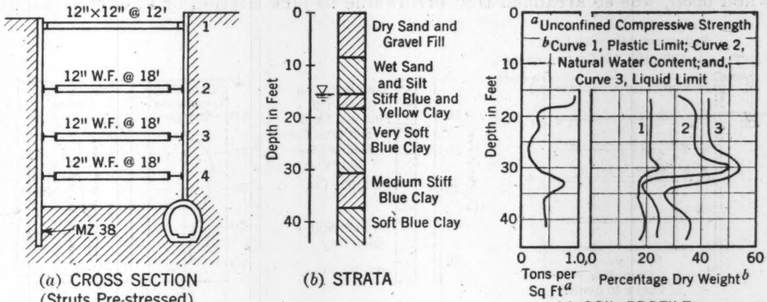
[7] Research Asst. Prof., Soil Mechanics, Univ. of Illinois, Urbana, Ill.

construction procedures were all standard practice, each cut was made by a different contractor, with inevitable variations in detail.

The measurements included systematic observations of the settlement of

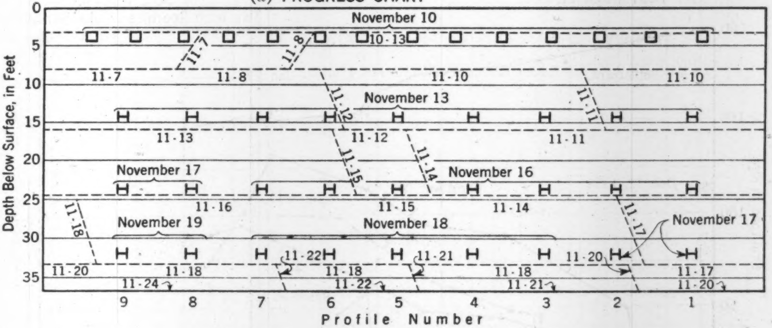

12"×12" @ 12'
12" W.F. @ 18'
12" W.F. @ 18'
12" W.F. @ 18'
1
2
3
4
MZ 38
0
10
20
30
40
Depth in Feet
Dry Sand and Gravel Fill
Wet Sand and Silt
Stiff Blue and Yellow Clay
Very Soft Blue Clay
Medium Stiff Blue Clay
Soft Blue Clay
a Unconfined Compressive Strength
b Curve 1, Plastic Limit; Curve 2, Natural Water Content; and, Curve 3, Liquid Limit
Depth in Feet
1
2
3
0
1.0
0
20
40
60
Tons per Sq Ft a
Percentage Dry Weight b

(a) CROSS SECTION (Struts Pre-stressed)

(b) STRATA

(c) SOIL PROFILE Two-Inch Seamless Tube Samples

(d) PROGRESS CHART



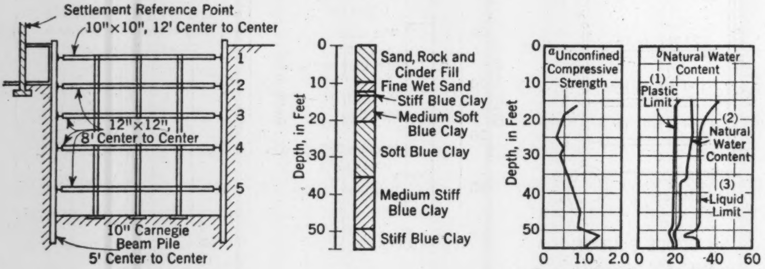
(e) SUMMARY OF STRUT LOADS, IN KIPS (ESTIMATED ERROR ±15%)

Depth (ft)	Date (Nov. 1939)	PROFILE (SEE FIG. (d))								
		9	8	7	6	5	4	3	2	1
4.0	20–27	27	....	33	44	30	....	44	....	....
14.5	20	322	257	270	254	292	220	262	242	226
	27	....	252	249	188	238	....	....	....	....
23.5	20	147	185	161	....	230	164	188	....	140
	27	185	222	199	....	269	218	210	....	157
31.5	20	58	49	40	77	98	81	145	112	150
	27	182	....	252	295	355	248	323	266	262

FIG. 19.—STRUT LOADS IN OPEN CUT, CONTRACT S-3

the adjacent ground surface, of the lateral movements of the sheeting during the excavation process, of the rise of the soil at the bottom of the cuts, and of the settlements of the sheeting itself. In each cut, at least one vertical profile of bracing was selected for determination of the loads carried by the struts.

In all, more than fifty profiles were subjected to measurement, some repeatedly. For steel struts the loads were usually determined by strain gage, and for wooden struts a hydraulic jacking arrangement was used. Hydraulic jacking, when used, was so arranged that errors due to jack friction or to over-jacking


Settlement Reference Point
10"×10", 12' Center to Center
1
2
3
4
5
12"×12",
8' Center to Center
10" Carnegie Beam Pile
5' Center to Center
Depth, in Feet
0
10
20
30
40
50
Sand, Rock and Cinder Fill
Fine Wet Sand
Stiff Blue Clay
Medium Soft Blue Clay
Soft Blue Clay
Medium Stiff Blue Clay
Stiff Blue Clay
a Unconfined Compressive Strength
b Natural Water Content
(1) Plastic Limit
(2) Natural Water Content
(3) Liquid Limit
0
1.0
2.0
Tons per Sq Ft a
0
20
40
60
Percentage Dry Weight b

(*a*) CROSS SECTION AT TEST PROFILE

(*b*) STRATA

(*c*) SOIL PROFILE
Two-Inch Seamless Tube Samples

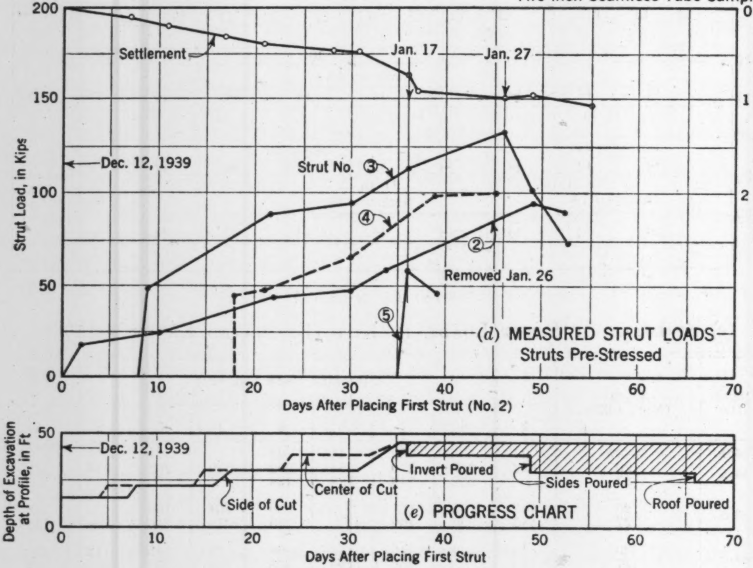


FIG. 20.—STRUT LOADS IN OPEN CUT, CONTRACT S-4B

were eliminated. A detailed discussion of the techniques of measurement and the errors involved was presented elsewhere by the writer.[8] The soil properties adjacent to each cut were determined from the results of laboratory tests on

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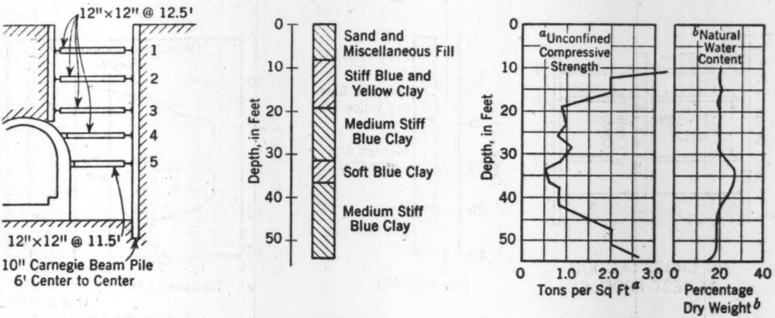
[8] "The Measurement of Earth Pressures on the Chicago Subway," by Ralph B. Peck, *Bulletin*, A. S. T. M., August, 1941, p. 25.

2-in. diameter samples obtained in seamless tubes from borings taken at 300-ft intervals along the subway right of way.[9] For the comparison of results of the open-cut measurements, the most important physical property of the clay was the unconfined compressive strength, determined on samples 2 in. in diameter and 3.5 in. long at 2.5-ft intervals in the vertical direction. The samples obtained from the borings were disturbed to a certain extent because of the small diameter of the tubes, and an extensive series of tests was conducted to determine the loss in strength during the sampling process. As a result, it was determined that the average unconfined compressive strength of large relatively undisturbed blocks of clay was approximately 1.35 times the value ob-


12"×12" @ 12.5'
1
2
3
4
5
12"×12" @ 11.5'
10" Carnegie Beam Pile
6' Center to Center
0
10
20
30
40
50
Depth, in Feet
Sand and Miscellaneous Fill
Stiff Blue and Yellow Clay
Medium Stiff Blue Clay
Soft Blue Clay
Medium Stiff Blue Clay
a Unconfined Compressive Strength
b Natural Water Content
Depth, in Feet
0
1.0
2.0
3.0
Tons per Sq Ft a
0
20
40
Percentage Dry Weight b

(*a*) CROSS SECTION AT TEST PROFILE

(*b*) STRATA

(*c*) SOIL PROFILE
Two-Inch Seamless Tube Samples

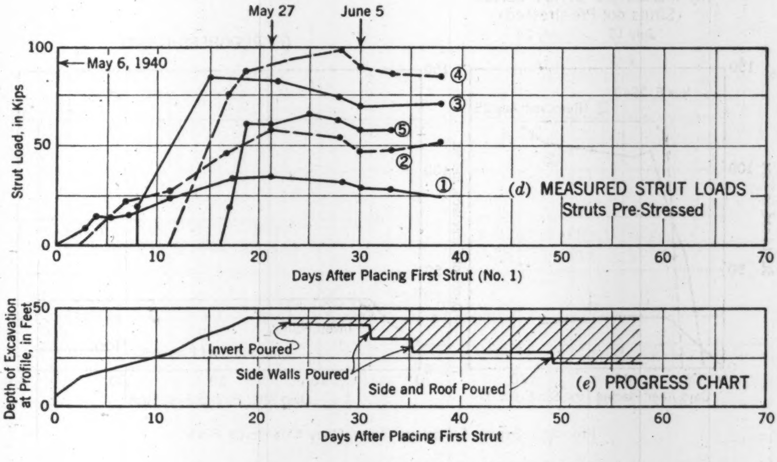


FIG. 21.—STRUT LOADS IN OPEN CUT, CONTRACT S-9C

tained from the seamless tube samples on the same soil. It was also considered important to determine the order of magnitude of the error in the average unconfined compressive strength for a single boring, introduced because samples were tested only at 2.5-ft intervals. On certain borings, two samples were tested in each 2.5-ft interval and the average value of compressive strength determined from all of the tests. This value was compared with the average for alternate samples, representing the usual 2.5-ft spacing. It was found that the difference was about 6%, and that the average compressive strength of the entire boring was probably determined within 10% by the use of the 2.5-ft spacing.

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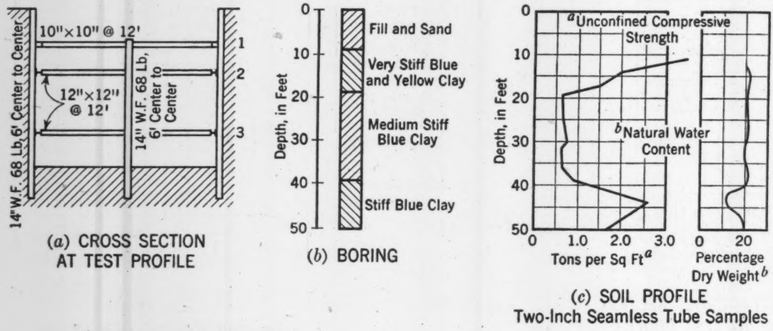
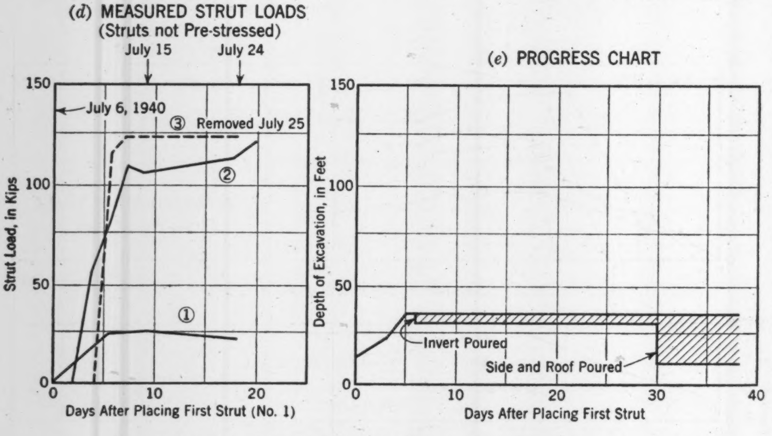
[9] For details of the soil exploration program, see "Soil Tests Check Chicago Subway Work," by Ralph B. Peck, *Engineering News-Record*, December 7, 1939, p. 87.



(*a*) CROSS SECTION AT TEST PROFILE

(*b*) BORING

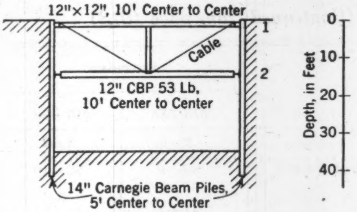
(*c*) SOIL PROFILE
Two-Inch Seamless Tube Samples


(*d*) MEASURED STRUT LOADS
(Struts not Pre-stressed)

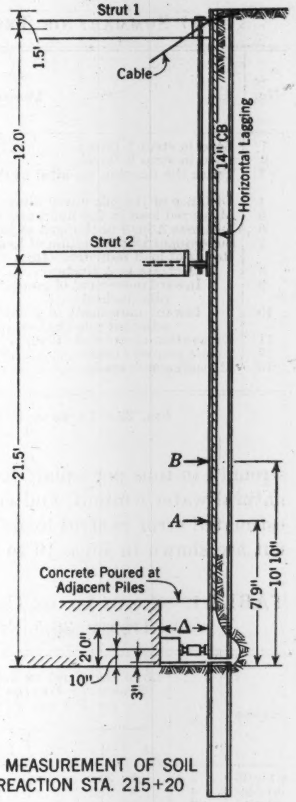
(*e*) PROGRESS CHART

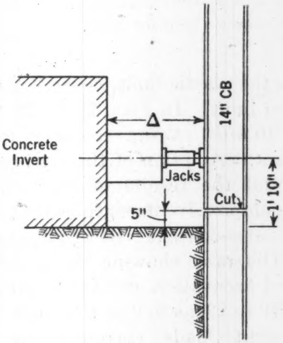
FIG. 22.—STRUT LOADS IN OPEN CUT, CONTRACT S-8A

The pertinent information and test results for each open cut are shown in Figs. 19 to 26. In Fig. 19 the progress of the excavation is indicated by the dates along the broken lines. The remaining dates indicate when the struts were placed. In Fig. 19(*c*) the left-hand curve gives the unconfined compressive


12"×12", 10' Center to Center
1
Cable
2
12" CBP 53 Lb,
10' Center to Center
14" Carnegie Beam Piles,
5' Center to Center
Depth, in Feet
0
10
20
30
40

(*a*) TYPICAL CROSS SECTION AT TEST PROFILE


Strut 1
1.5'
Cable
12.0'
14" CB
Horizontal Lagging
Strut 2
21.5'
B
A
10' 10"
Concrete Poured at
Adjacent Piles
7'9"
Δ
2'0"
10"
3"


Δ
14" CB
Concrete
Invert
Jacks
Cut
5"
1' 10"

(*b*) METHOD FOR OBTAINING SOIL REACTION, STA. 213+85

(*c*) MEASUREMENT OF SOIL REACTION STA. 215+20

(*d*) SOIL PROFILE

Two-Inch Seamless Tube Samples

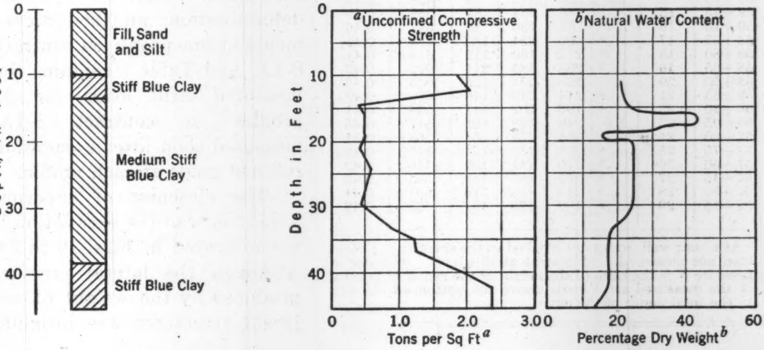


Fig. 23.—continued on page 1014.

(*e*) SUMMARY OF TEST RESULTS (*Continued from page 1013*)

No.	Description	STATIONS:	
		213+85	215+20
1	Load in strut 1 (kips)	9±5	1±2
2	Load in strut 2 (kips)	167±10	137±5
3	Twice the reaction (in kips) at the embedded part of the pile	83±4	82±10
4	Distance of the pile cutoff above the invert (inches)	5	3
5	Measured load in the hydraulic jacks (kips)	41±2	41±5
6	Distance Δ held unchanged at both stations	✓	✓
7	Subsequent test reduction of load (kips)	8	0
	Result of load reduction (Item 7):		
8	Decrease in Δ (inches)	0.2	0.21
9	Inward movement of point A, Fig. (*c*), as measured on the cut pile (inches)	....	0.4
10	Inward movement of point B, Fig. (*c*), as measured on the adjacent pile (inches)	....	0.0013
11	Excavation completed (1940)	Oct. 8	Oct. 24
12	Invert poured (1940)	....	Nov. 12
13	Measurements made	Oct. 14–18	Nov. 13

FIG. 23.—LATERAL EARTH PRESSURE MEASUREMENTS, CONTRACT D-6E

strength in tons per square feet; and curve 1 is the plastic limit, curve 2 is the natural water content, and curve 3 is the liquid limit. In Figs. 20 to 22, the estimated error in strut loads is approximately 10 kips. Cross sections of each cut are shown in Figs. 19 to 25, together with a description of the soil profile and the results of unconfined compressive strength and natural water-content determinations. Diagrams showing the progress of excavation are shown in Figs. 19 to 22, as well as the measured strut loads corresponding to various excavation stages for contracts S-3, S-4B, S-9C, S-8A, and D-6E (for location, see Fig. 1). Figs. 25 and 26 show the excavation progress, strut-load determinations and soil movements at one profile on contract S-1A, and Table 1 contains the measured strut loads for 21 profiles in contract S-1A, measured soon after excavation reached grade at each profile.

The closeness of superimposed loads to the excavation is not indicated in Figs. 19 to 24. Although the lateral pressure produced by the weight of adjacent structures was insignifi-

TABLE 1.—STRUT LOADS, CONTRACT S-1A; DEPTH 36.5 FT

Station	Profile	LOAD (IN KIPS) IN THE STRUT, AT DEPTHS (IN FT) OF:				Total load, Q	Center of pressure,[d] N
		4	13	23	31		
61+05	3	...[b]	342	...[c]	...[b]	....	....
60+90	4	...[b]	335	...[c]	...[b]	....	....
60+45	7	...[b]	466	...[c]	...[b]	....	....
60+30	8	8.5	...[b]	...[c]	...[b]	....	....
60+15	9	...[b]	...[b]	...[c]	237	....	....
59+85	11	...[b]	338	82	232	652	0.43
59+70	12	3.8	223	316	89	628	0.43
59+25	15	...[b]	286	284	82	652	0.45
59+10	16	...[b]	258	196	88	542	0.46
58+95	17	...[b]	300	254	130	684	0.45
58+80	18	...[b]	271	199	142	612	0.44
58+65	19	...[b]	294	185	214	693	0.42
58+50	20	...[b]	228	254	123	605	0.43
58+35	21	...[b]	244	249	146	639	0.42
58+20	22	...[b]	196	202	147	545	0.41
58+05	23	1.0	249	193	105	547	0.45
57+90	24	...[b]	261	224	83	568	0.51
57+75	25	...[b]	278	146	124	548	0.46
57+60	26[a]	...[b]	170	147	109	426	0.42
57+45	27	...[b]	259	187	148	594	0.43
57+30	28	...[b]	219	166	116	501	0.44
57+15	29	...[b]	189	173	131	493	0.42

[a] At a stair well; wales not blocked to the sheeting. [b] Strut load not measured. [c] No strut at this level. [d] Center of pressure N is the ratio of the height of the center of gravity of the measured strut loads above the bottom of the cut, to the total depth of the cut.

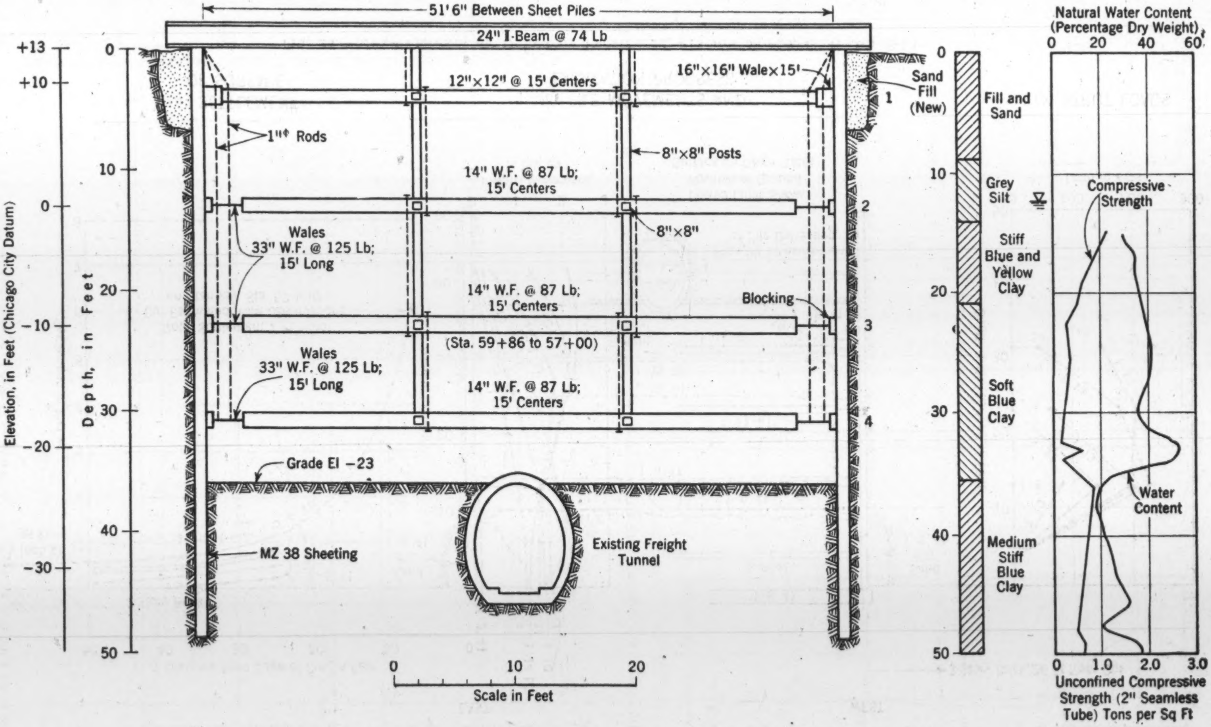
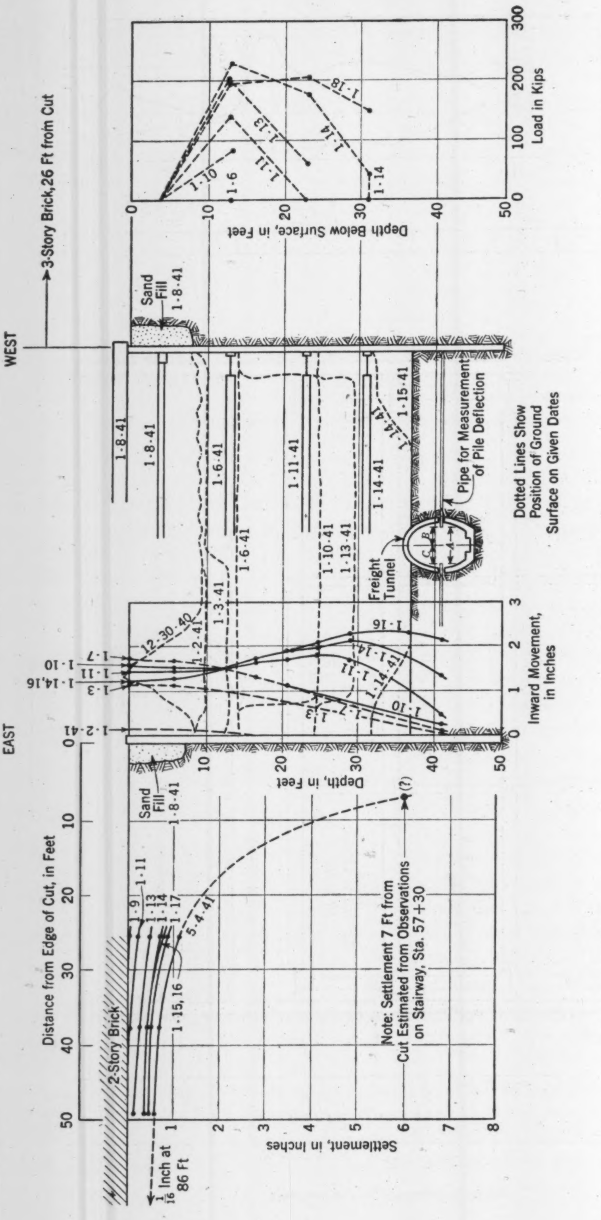

51' 6" Between Sheet Piles
24" I-Beam @ 74 Lb
+13
+10
0
−10
−20
−30
Elevation, in Feet (Chicago City Datum)
Depth, in Feet
0
10
20
30
40
50
12"×12" @ 15' Centers
16"×16" Wale×15'
1" Rods
8"×8" Posts
14" W.F. @ 87 Lb; 15' Centers
8"×8"
Wales 33" W.F. @ 125 Lb; 15' Long
14" W.F. @ 87 Lb; 15' Centers
Blocking
(Sta. 59+86 to 57+00)
Wales 33" W.F. @ 125 Lb; 15' Long
14" W.F. @ 87 Lb; 15' Centers
Grade El −23
MZ 38 Sheeting
Existing Freight Tunnel
0
10
20
Scale in Feet
Sand Fill (New)
1
2
3
4
Fill and Sand
Grey Silt
Stiff Blue and Yellow Clay
Soft Blue Clay
Medium Stiff Blue Clay
Natural Water Content (Percentage Dry Weight)
0
20
40
60
Compressive Strength
Water Content
0
1.0
2.0
3.0
Unconfined Compressive Strength (2" Seamless Tube) Tons per Sq Ft

FIG. 24.—OPEN-CUT BRACING AND SOIL PROFILE, CONTRACT S-1A, STATIONS 61 + 36 TO 57 + 00 (STRUTS PRE-STRESSED)



(a) SETTLEMENT OF SURFACE

(b) PILE MOVEMENTS AND EXCAVATION PROGRESS

(c) STRUT LOADS

FIG. 25.—MEASUREMENTS ON OPEN-CUT PROFILE 22, STATION 58+20, CONTRACT S-1A

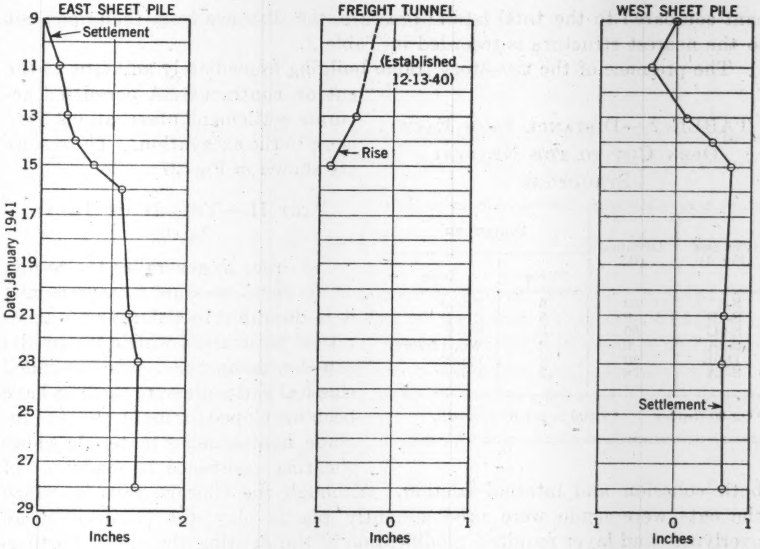


FIG. 26.—SETTLEMENT OF SHEET PILE WALLS, AND RISE OF FREIGHT TUNNEL (See Fig. 25(*b*))

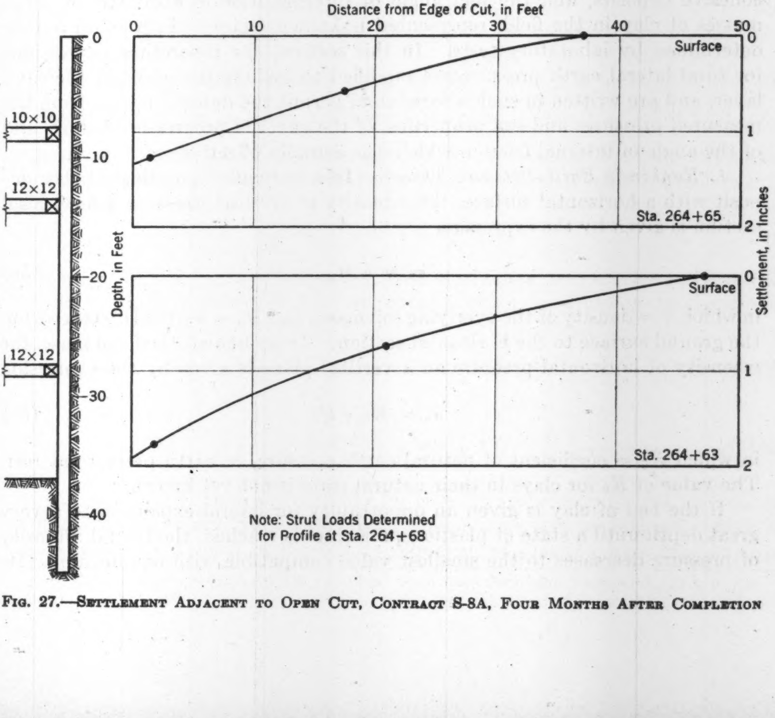


FIG. 27.—SETTLEMENT ADJACENT TO OPEN CUT, CONTRACT S-8A, FOUR MONTHS AFTER COMPLETION

cant compared to the total lateral pressure, the distance from each open cut to the nearest structure is recorded in Table 2.

The presence of the two-story frame building immediately adjacent to the cut on contract S-8A permitted accurate settlement observations very close to the excavation. The results are shown in Fig. 27.

TABLE 2.—DISTANCE FROM EACH OPEN CUT TO THE NEAREST STRUCTURE

Cut (see Fig. 1)	Distance (ft)	STRUCTURE	
		Stories	Type
S-3	50	5	Brick[a]
S-4B	9	3	Brick[b]
S-9C	50	3	Brick
S-8A	0	2	Wood frame
D-6E	40	2	Brick
S-1A	26	2	Brick

[a] On 60-ft piles. [b] 7,000 lb per foot of wall.

PART II.—THEORY OF TOTAL LOADS

In order to generalize the results of the earth-pressure measurements, it is desirable to determine a theoretical basis upon which the results can be compared. The so-called classical earth-pressure theories have been developed for many years to include homogeneous materials whose shearing resistance is made up of both cohesion and internal friction. Although the Chicago soils in which the cuts were made were predominantly plastic clay, the presence of an overlying sand layer required modification of the existing theories. Furthermore, the influence of progressive failure in reducing the average strength of cohesive deposits, and the real angle of internal friction exhibited by large masses of clay in the field, represented unknown factors which could not be determined by laboratory tests. In this section, the theoretical expressions for total lateral earth pressure are modified to include the effect of the sand layer, and are written in such a form as to permit the determination, from the measured pressures and soil properties, of the effect of progressive failure, and of the angle of internal friction which was actually effective.

1. Rankine's Earth-Pressure Theory.—In a normally consolidated clay deposit with a horizontal surface, the intensity of vertical stress on a horizontal section is given by the expression

$$\sigma_v = \gamma H_z \qquad (5a)$$

in which γ = density of the overlying soil mass; and H_z = vertical distance from the ground surface to the horizontal section. In an undisturbed soil mass, the intensity of horizontal pressure on a vertical plane is given by the expression

$$\sigma_h = K_0 \gamma H_z \qquad (5b)$$

in which K_0 = coefficient of natural earth pressure, or earth pressure at rest. The value of K_0 for clays in their natural state is not yet known.

If the bed of clay is given an opportunity for lateral expansion to a very great depth until a state of plastic equilibrium is reached, the lateral intensity of pressure decreases to the smallest value compatible with equilibrium. Its value, known as the active earth pressure, is

$$\sigma_A = \gamma H_s \left[\tan^2\left(45° - \frac{\phi}{2}\right) - \frac{2c}{\gamma H_s} \tan\left(45° - \frac{\phi}{2}\right)\right] \ldots\ldots (6)$$

in which ϕ = effective angle of internal friction of the clay; and c = cohesion of the clay. The total lateral pressure from the surface to depth $H_s = H$ is equal to

$$P_A = \int_0^H \sigma_A \, dH_s = \tfrac{1}{2} \gamma H^2 \tan^2\left(45° - \frac{\phi}{2}\right) - 2 c H \tan\left(45° - \frac{\phi}{2}\right) \ldots (7)$$

Eqs. 6 and 7 were probably first derived by Résal (1910) as an extension of Rankine's theory. A derivation in English was given by the late William Cain, M. Am. Soc. C. E., in 1916.[10] Eq. 7 may be expressed in another useful form: The unconfined compressive strength of a clay sample, q, may be expressed in terms of the cohesion and effective angle of internal friction by the equation

$$q = \frac{2c}{\tan\left(45° - \frac{\phi}{2}\right)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Substituting Eq. 8 in Eq. 7, the expression

$$P_A = \tfrac{1}{2} \gamma H^2 \left(1 - \frac{2q}{\gamma H}\right) \tan^2\left(45° - \frac{\phi}{2}\right) \ldots\ldots\ldots\ldots (9)$$

is obtained.

The resultant earth pressure given by Eqs. 7 and 9 is horizontal, and its point of application is below the lower third-point. For $H = \frac{2q}{\gamma}$, it is at infinite depth below the bottom of the cut. The magnitude of the resultant earth pressure is often compared with that of a fluid of equal unit weight, which exerts a total lateral pressure

$$P_F = \tfrac{1}{2} \gamma H^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

The ratio of P_A to P_F is known as the hydrostatic pressure ratio for active earth pressure, and is denoted by K_A. Its value is given by either of the equations

$$K_A = \tan^2\left(45° - \frac{\phi}{2}\right) - \frac{4c}{\gamma H} \tan\left(45° - \frac{\phi}{2}\right) \ldots\ldots\ldots (11a)$$

or

$$K_A = \left(1 - \frac{2q}{\gamma H}\right) \tan^2\left(45° - \frac{\phi}{2}\right) \ldots\ldots\ldots\ldots\ldots (11b)$$

In actual experience, opportunity for lateral expansion sufficient to develop a state of plastic equilibrium throughout the soil mass is never realized. Such a movement would result in the full development of shearing resistance on two intersecting sets of sliding planes, each of which makes an angle of $45° - \frac{\phi}{2}$ with the vertical as indicated in Fig. 28. Actually the expansion is restricted to a wedge-shaped zone extending behind the wall supporting the earth mass, and having a depth limited approximately to the depth H of the wall. If sufficient movement of the wall is permitted to develop the shearing resistance along the plane which intersects the wall at depth H, the validity of Eqs. 11 is not impaired, provided that two other conditions are also fulfilled. A bond must exist between the lateral support and the clay, capable of resisting tension to a depth, according to Eq. 6, of

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[10] "Earth Pressure, Retaining Walls and Bins," by William Cain, New York, N. Y., 1916, p. 186.

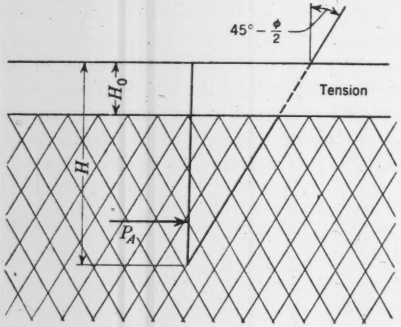

FIG. 28

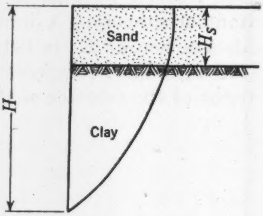

FIG. 29

$$H_0 = \frac{q}{\gamma} \dots\dots\dots\dots\dots\dots\dots\dots (12)$$

and the deformation conditions must be such that the point of application of the resultant pressure is identical with the theoretical location. In an actual open cut, these latter two conditions are never satisfied, because bracing is always inserted near the ground surface, and lateral expansion is greatly restricted. The zone of tension in the upper part of the deposit no longer can exist, the center of pressure rises, and Eq. 6 is no longer applicable. The sliding surface becomes curved, and the resultant earth pressure is not horizontal. Ignorance of these factors can be expected to introduce errors into the computed values of the total pressure. The theory is adequate to serve, nevertheless, as the starting point in the analysis of the experiment results.

Eqs. 11 refer strictly to a homogeneous clay mass possessing everywhere an unconfined compressive strength q and an angle of internal friction ϕ. Natural clay deposits are not homogeneous, and the unconfined compressive strength is variable. To use equations containing the quantity q, it is then necessary to substitute in place of it the arithmetic average of the compressive strength of the clay, denoted by q_a. In the case of the Chicago clays, another complication is introduced by the presence of a deposit of cohesionless material in the upper portion of the soil mass. For computation, it is convenient to transform the cohesionless layer into an equivalent layer of clay possessing the same shearing resistance along the potential sliding surface as the existing sand, but having an angle of internal friction equal to that of the actual clay deposit.

The shearing resistance of the sand along the portion of the sliding wedge located above the clay is equal to

$$S_s = \sigma \tan \phi_1 \ldots\ldots (13)$$

in which: σ = the average normal stress on the sand portion of the sliding surface; and ϕ_1 = angle of internal friction of the sand. The average normal stress (neglecting the inclination of the shearing surface which actually is almost vertical, as indicated in Fig. 29) is equal approximately to

$$\sigma = K_s \times \tfrac{1}{2} \gamma_1 H_s \ldots\ldots (14a)$$

and the average shearing resistance

$$S_s = K_s \times \tfrac{1}{2} \gamma_1 H_s \tan \phi_1 \ldots\ldots (14b)$$

In Eqs. 14: K_s = hydrostatic pressure ratio for the cohesionless material; γ_1 = density of the cohesionless material; and H_s = depth of cohesionless layer. The shearing resistance of the substitute clay layer must also be equal to S_s and is given by the expression

$$S_s = c_s + \sigma \tan \phi \ldots\ldots (15)$$

in which: c_s = cohesion of the substitute clay layer; and ϕ = angle of internal friction of the clay deposit. Substituting Eq. 14*b* for S_s in Eq. 15 and solving for the cohesion, it is found that

$$c_s = \tfrac{1}{2} K_s \gamma_1 H_s (\tan \phi_1 - \tan \phi) \ldots\ldots (16a)$$

The unconfined compressive strength of the substitute clay layer would then be, from Eq. 8,

$$q_s = \frac{2\,c}{\tan\left(45° - \dfrac{\phi}{2}\right)} = \frac{1}{\tan\left(45° - \dfrac{\phi}{2}\right)} \times K_s \gamma_1 H_s (\tan \phi_1 - \tan \phi) \,.\,.\, (16b)$$

The average unconfined compressive strength of the entire deposit from the surface to depth H would become

$$q_a = \frac{1}{H} \left[\frac{K_s \gamma_1 H_s^2}{\tan\left(45° - \dfrac{\phi}{2}\right)} (\tan \phi_1 - \tan \phi) + (H - H_s)\, q_c \right] \ldots (17)$$

Eqs. 16*a* and 17 contain the quantity K_s, which can only be estimated, by a procedure to be described subsequently. They afford only estimates of the influence of the sand layer, rather than exact evaluations.

One further soil property must be considered in the adaptation of the theory to an actual soil mass. The value of the average unconfined compressive strength used in the computations is based upon tests of individual samples as representative of the clay deposit as possible. For each sample, the maximum strength is recorded as the unconfined compressive strength. For most natural clays, the maximum strength is attained at relatively small strain, and the compressive strength decreases rapidly to a constant value which persists throughout the additional strain, as shown in Fig. 30. Since some of the layers in a natural deposit are stiffer than others, at a given deformation the clay at certain points along the sliding surface may have attained its maximum resistance and experienced the decrease in strength, whereas at other points the maximum may not have been reached. As a result, the effective shearing resistance along the surface is always less than the average computed from the maximum strengths of individual samples. When sufficient deformation has taken place to reach the constant portion of the stress-strain curve for each component of the soil, it is apparent that the average resistance is only a fraction of that determined from the tests. The ratio between the shearing resistance actually developed along the sliding surface, and that determined by averaging the separate shearing resistances of the individual samples is denoted by n. Eq. 17 can be modified to take account of this phenomenon (known as progressive failure) by replacing q_c by $n\, q'_c$, in which q'_c is the average unconfined compressive strength of practically undisturbed clay samples as determined in the laboratory. No reduction is experienced in the sand layer because the stress-strain curve for loose sands, as exist over the Chicago clay deposits, does not exhibit a peak before attaining a constant shearing resistance as deformation proceeds. The final form of Eq. 17 is therefore

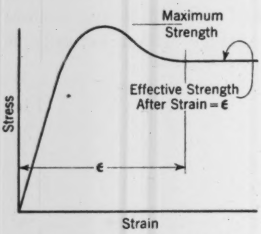

FIG. 30.—STRESS-STRAIN CURVE

$$q_a = \frac{1}{H}\left[\frac{K_s\, \gamma_1\, H_s^2}{\tan\left(45° - \frac{\phi}{2}\right)}(\tan\phi_1 - \tan\phi) + (H - H_s)\, n\, q'_c\right] \quad ..(18a)$$

and the theoretical relation Eq. 11*b* becomes

$$K_A = \left(1 - \frac{2\, q_a}{\gamma_a\, H}\right)\tan^2\left(45° - \frac{\phi}{2}\right) \quad \ldots\ldots\ldots\ldots(18b)$$

in which the average density, γ_a, is determined by the equation

$$\gamma_a = \frac{1}{H}\left[H_s\, \gamma_1 + (H - H_s)\, \gamma_c\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots(19)$$

γ_c being the average density of the clay.

It is important to note that Eq. 18*b* represents a linear relationship between K_A and $\frac{2\, q_a}{\gamma_a\, H}$, as shown in Fig. 31. Furthermore, the intercept of the straight line on the K_A-axis must be equal to $\tan^2\left(45° - \frac{\phi}{2}\right)$, and the intercept on the

$\frac{2\,q_a}{\gamma_a\,H}$ -axis must be equal to unity. **These facts are of the greatest importance, because they enable the values of ϕ and n for the clay deposit to be determined from the experimental values of earth pressure.** The procedure is described in Part III.

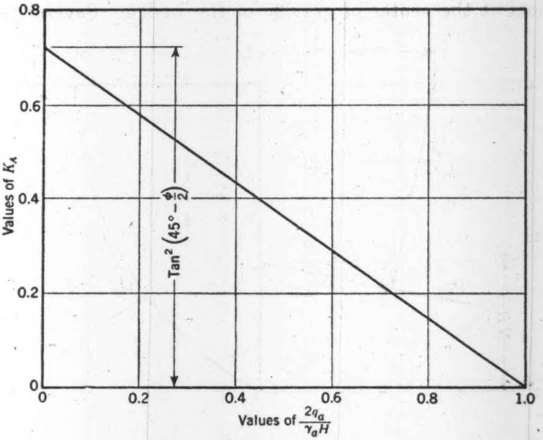

FIG. 31

2. General Wedge Theory for Earth Pressures.—When an open cut is braced with struts, the deformation at the top of the cut is greatly restricted and the conditions for tensile stresses in the upper part of the soil mass are no longer satisfied. Pressure exists everywhere within the wedge, and the center of pressure is correspondingly raised. The sliding surface is no longer plane, but takes the shape of a curve which intersects the surface at a right angle. The resultant earth pressure is not horizontal, in general, but is inclined to the horizontal at an angle θ.

The total pressure against the sheeting under these circumstances may be computed by consideration of the equilibrium of the failure wedge, under the assumption that the shape of the curved boundary is that of a logarithmic spiral. This curve was chosen for its close approach to observed conditions and its simplicity, because the resultant of the normal and shearing forces on the curved surface passes through the center of the spiral and may be eliminated from computations if this point is taken as the center of moments. For the special case when $\phi = 0$, the logarithmic spiral becomes a circle with its center on a horizontal line at the elevation of the ground surface. The forces considered are shown in Fig. 32. They include W, the weight of the wedge; P_A, the resultant earth pressure; S, the total shearing resistance along the curved sliding surface; and F, the resultant normal force on the curved surface. Denoting the lever arms of these forces with respect to the center of rotation

O by the symbol l, the value of P_A is found to be

$$P_A = \frac{W\,l_w - S\,l_c}{l_r} \qquad (20)$$

The total shearing resistance S is equal to the product of one half the average unconfined compressive strength and the length L of the curved surface. The weight W acts at the center of gravity of the wedge. Several positions are assumed for the center of rotation along the horizontal line, and the maximum value of P_A is taken as the active earth pressure. The value of the hydrostatic pressure ratio is defined as the ratio of the horizontal component of P_A to the lateral pressure exerted by the fluid of equal density. Thus

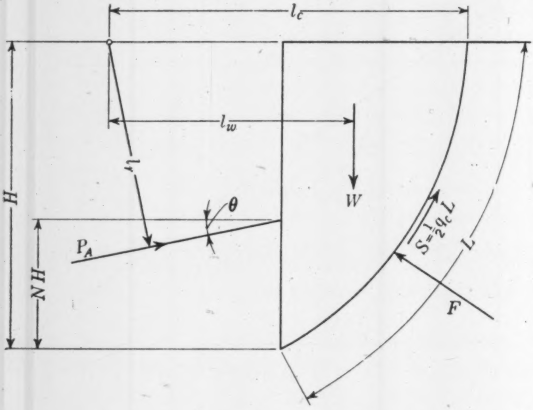

FIG. 32

$$K_A = \frac{P_A \cos\theta}{\frac{1}{2}\,\gamma_a\,H^2} \qquad (21)$$

Values of K_A have been computed for different values of the inclination of the resultant earth pressure, for different positions of the center of pressure, and for different values of cohesion. The results are shown in graphical form in Fig. 33. Each of these charts pertains to a particular value of the inclination of the resultant pressure. Values of K_A may be determined for any location of the center of pressure, and for any value of cohesion, expressed in terms of the quantity $\frac{H\gamma}{2q}$. It should be recalled that Fig. 33 is constructed on the assumption that $\phi = 0°$. The values of K_A determined by Rankine's theory are also shown in Fig. 33. They appear as horizontal lines, inasmuch as the values are independent of the location of the center of pressure. For either of the values of θ it is observed that a particular value of the position of the center of pressure gives results by the general wedge theory which are practically identical to those given by Rankine's theory. For values of $\theta = 0°$, and $20°$, the respective centers of pressure for this condition are approximately at 0.37 *H* and 0.45 *H*. Similar charts have also been prepared for other values of ϕ, in which case the cohesion of the clay is determined by Eq. 8. However, the agreement with the field data is more satisfactory in the case of Fig. 33, and the other charts are not shown.

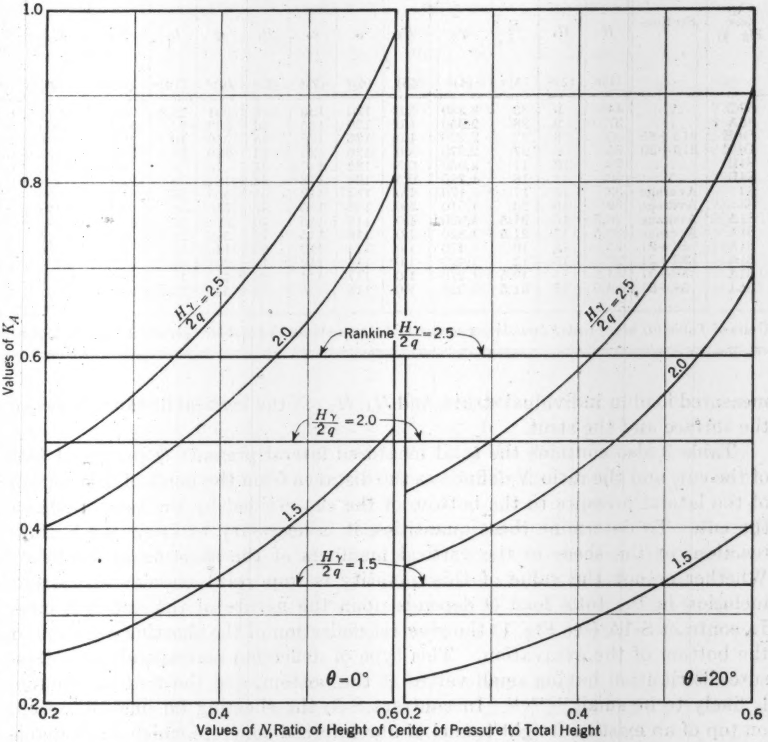

FIG. 33.—RESULTS OF EARTH PRESSURE COMPUTATIONS; GENERAL WEDGE THEORY ($\phi = 0°$)

PART III.—COMPARISON OF MEASURED AND THEORETICAL TOTAL LOADS

The essential data regarding the open cuts are shown in Figs. 19 to 27 and Table 1. The results are summarized in tabular form in Table 3. In Table 3(*a*), the symbol *H* indicates the total depth of the cut; H_s the thickness of the overlying sand layer, q'_c the average unconfined compressive strength of practically undisturbed clay samples between the clay surface and the bottom of the cut; K_s the hydrostatic pressure ratio for the sand; P_l the lateral pressure of a fluid having a density equal to γ_a; and L_p the horizontal distance between vertical profiles of struts. In Table 3(*b*), the symbols Q_1, Q_2 . . . indicate the

TABLE 3.—COMPARISON OF MEASURED AND

Cut (see Fig. 1)	Station	(a) TOTAL LATERAL PRESSURE												
		H (1)[a]	H_s (2)[a]	$H - H_s$ (3)[a]	q'_e (4)[b]	K_s (5)	γ_s (6)[c]	γ_e (7)[c]	P_l (8)[d]	Q (9)[d]	L_p (10)[a]	P_A (11)[e]	K_A (12)	$\frac{2 q_e}{\gamma_e H}$ (13)
S9C	....	44	9	35	3,280	1.0	130	126	124	331	12.5	26.5	0.214	0.748
S8A	....	37	9	28	2,950	1.0	130	125	86	258	12	21.5	0.250	0.783
D6E	213+85	35	8	27	2,270	1.0	126	123	75	259	10	25.9	0.346	0.664
D6E	215+20	35	8	27	2,270	1.0	126	123	75	220	10	22.0	0.293	0.664
S4B	....	23	12	11	2,050	0.5	122	116	31	73	8	9.13	0.295	0.701
S4B	....	30	12	18	1,730	0.5	124	118	53	181	8	22.7	0.428	0.526
S3	Average	33	16	17	1,170	1.0	114	112	61	558	18	31.0	0.508	0.511
S3	Average	39	16	23	1,310	1.0	118	115	87	750	18	41.7	0.478	0.443
S1A	Average	36.5	15	21.5	1,320	1.0	118	115	76	626	15	41.7	0.547	0.464
S1A	Average	36.5	15	21.5	1,320	1.0	118	115	76	586	15	39.1	0.515	0.464
S1A	58+21	25	15	10	1,470	1.0	114	112	35	140	15	9.33	0.267	0.722
S1A	58+21	27	15	12	1,420	1.0	114	112	41	264	15	17.6	0.429	0.661
S1A	58+21	34.5	15	19.5	1,270	1.0	117	114	68	448	15	29.9	0.433	0.484
S1A	58+21	36.5	15	21.5	1,320	1.0	118	115	76	550	15	36.6	0.482	0.464

Units of Table (a) are: [a] feet; [b] pounds per square foot; [c] pounds per cubic foot; [d] kips = 1,000 lb; [e] kips per foot.

measured load in individual struts, and H_1, H_2 . . . the vertical distance between the surface and the strut.

Table 3 also contains the total measured lateral pressure Q per lineal foot of the cut, and the ratio N defined as the distance from the point of application of the lateral pressure to the bottom of the cut divided by the total depth of the cut. To determine these quantities, it is necessary to know the bottom reaction, or the shear in the vertical members of the sheeting at depth H. Whether or not the value of this quantity is important enough to warrant inclusion in the total load Q depends upon the nature of the different cuts. In contract S-1A (see Fig. 1) the greatest deflection of the sheeting occurred at the bottom of the excavation. This type of deflection corresponds to a pressure distribution having small values at the bottom, and the bottom reaction is likely to be small.[11, 12, 13] In contract S-3, the sheeting on one side rested on top of an existing freight tunnel of unreinforced concrete which was capable of supporting an inconsequential lateral load after exposure. Similarly in contract S-4B the results of the strut loads indicate plainly that the intensity of lateral pressure decreased from the midheight to the bottom of the cut and that the major part of this pressure has been taken by the struts. On the contrary, the vertical H-piles on contract D-6E penetrated a stiff layer below the bottom of the cut, and the large clearance between the lower strut and the bottom of the cut induced a deflection curve with a maximum value at some distance above the bottom of the cut. As a result, the bottom reaction was considered likely to be of considerable importance, and it was measured by the

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[11] "Distribution of the Lateral Pressure of Sand on the Timbering of Cuts," by Karl Terzaghi, *Proceedings*, International Conference on Soil Mechanics, Vol. I, Cambridge, Mass., 1936.

[12] "A Fundamental Fallacy in Earth Pressure Computations," by Karl Terzaghi, *Journal*, Boston Soc. of Civ. Engrs., April, 1936, p. 78.

[13] "General Wedge Theory of Earth Pressure," by Karl Terzaghi, *Transactions*, Am. Soc. C. E., Vol. 106, 1941, pp. 68, 94.

THEORETICAL TOTAL LOADS IN OPEN CUTS

(b) INDIVIDUAL STRUT LOADS

H	Strut 1		Strut 2		Strut 3		Strut 4		Strut 5		N	Q
	H_1	Q_1	H_2	Q_2	H_3	Q_3	H_4	Q_4	H_5	Q_5		
(14)	(15)	(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)	(26)
44	5.4	35	12.6	58	19.5	82	25.8	91	33.0	65	0.52	331
37	7.8	25	12.2	109	28.1	124	....	....	....	....	0.48	258
35	5.5	9	13.6	167	33.2	(83)[f]	....	....	....	....	0.44	259
35	1.5	1	13.5	137	34	(82)[f]	....	....	....	....	0.40	220
23	0	0	11	25	18.6	48	....	....	....	....	0.30	73
30	0	0	11	43	18.6	88	27.4	50	....	....	0.36	181
33	4	36	14.5	260	23.5	173	31.5	89	....	....	0.42	558
39	4	36	14.5	232	23.5	209	31.5	273	....	....	0.42	750
36.5	4	8	13	381	....	....	31	237	....	....	0.46	626
36.5	4	2	13	251	23	203	31	130	....	....	0.44	586
25	4	0	13	140	....	....	....	....	....	....	0.48	140
27	4	0	13	205	23	59	....	....	....	....	0.45	264
34.5	4	0	13	230	23	178	31	40	....	....	0.46	448
36.5	4	0	13	196	23	204	31	150	....	....	0.40	550

[f] Values in parentheses obtained by burning H-pile free at bottom of cut.

method indicated in Fig. 23. For cuts S-8A and S-9C, it was concluded that the bottom reaction was not important, although it is possible that in S-9C a small reaction might have been required to produce the state of stress measured in the bracing. As a result, the cut on contract D-6E was the only one in which the bottom reaction was taken into consideration in the computations.

To compute the real value of K_A, the average density of the soil throughout the height H must be determined by Eq. 19. For this computation, the density of the sand above the clay surface was estimated at 110 lb per cu ft. The density of the clay was computed from the average natural water content and a specific gravity of the solid matter assumed equal to 2.74. The values of γ_c and γ_a are given in Table 3(*a*).

The theoretical equation (Eq. 18*a*) for the average equivalent unconfined compressive strength of the deposit contains two values, K_s and $\tan \phi_1$, which can only be estimated. Since the sand above the clay is usually loose and somewhat silty, it is likely that ϕ_1 varies between the limits of 30° and 35°. It is most likely closer to 30°, but both values have been used in subsequent calculations. The value of K_s depends upon the lateral strain that develops in the sand during excavation. It must be selected for each cut individually, because of its dependence upon the method of excavation and bracing. As an example of the method of evaluation, cut S-9C is used. In the vicinity of the cut, the depth of sand was 9 ft. For $K_s = 1$, the earth pressure exerted by the sand is $\frac{1}{2} \times 110 \times 9^2 = 4{,}450$ lb per ft = 4.45 kips per ft. The top row of struts was located at a depth of 5.4 ft, and the next at 12.6 ft below the surface. Thus the top row received almost the full lateral pressure of the sand. The greatest load in this strut was 35 kips, or 2.80 kips per ft, corresponding to a hydrostatic pressure ratio $K_s = 0.63$. However, while excavation proceeds, the vertical H-beams pivot about the second row of struts, and by virtue of the stiffness of the H-beams the strut load is relieved. As a result, the pressure actually exerted by the sand is greater than that indicated by the strut load, and a value of $K_s = 1.0$ was estimated. Conditions were similar for all other cuts except contract S-4B, where the top strut was inoperative and the pressure of the sand was entirely carried by bending in the upper part of the H-beams. Therefore, much greater opportunity for lateral expansion of the sand existed in S-4B than in the other cuts, and a value of $K_s = 0.5$ was estimated correspondingly. An error of 25% in the estimated values of K_s would result in an error in the computed shearing resistance of less than 2% on contracts S-9C, S-8A, and S-4B, less than 10% on D-6E, and approximately 15% on S-1A and S-3.

In Table 3, the results of measurements on contract S-4B after the cut exceeded a depth of 30 ft are not included because measurements were not made on all profiles simultaneously, and certain strut loads would have to be determined by interpolation. This operation is not considered reliable enough in comparison with the other data. On cuts subsequently measured, care was taken to obtain readings on all struts on the same day when possible.

Table 3(*a*), therefore, contains the experimental values of K_A, and all of the quantities necessary to compute the value of q_a in Eq. 18(*a*), except the unknown properties of the clay, ϕ and n. The most probable values of ϕ and n are those which produce the most satisfactory agreement with the theoretical straight-line relationship shown in Fig. 31, provided a straight-line relationship is found to exist. In Figs. 34(*a*), 34(*b*), and 34(*c*), the experimental values of K_A are plotted against values of $\frac{2\,q_a}{\gamma_a\,H}$, in which q_a has been computed by Eq. 18*a* on the assumption that $\phi = 17°$. Values of n equal to 0.50, 0.75, and 1.0 were also assumed. A straight-line relationship is seen to exist, indicating that the reasoning leading to Eq. 18*b* is satisfactory. However, in none of these cases do the plotted test results indicate a satisfactory agreement with the theoretical relationship for $\phi = 17°$, shown by the heavy line. It appears evident that the value of ϕ must be appreciably less than 17°. In Figs. 34(*d*), 34(*e*), and 34(*f*), a similar series of plots has been made, in which q_a was determined on the assumption that $\phi = 0°$. In this case, it is observed that the experimental data are in excellent agreement with the theory when $n = 0.75$. It was found by similar computations that no other combination of ϕ and n besides $\phi = 0°$ and $n = 0.75$ gave a reasonable fit to the theoretical relationship. In the charts of Fig. 34, the value of ϕ_1 was taken as 30°. Repetition of the computations with a value of $\phi_1 = 35°$ led to almost identical results. The solution of Eq. 18*b*, therefore, results in the conclusions that, for all practical purposes, in the Chicago subway cuts, the value of ϕ for the clay is equal to 0°, and the factor $n = 0.75$.

The correctness of the values for ϕ and n is influenced by the fact that the upper end of the line is largely determined by points representing contracts S-3 and S-1A, in which an error in the value of K_s has an important effect. Nevertheless, if the position of the line is determined from only those profiles in which a 25% error in K_s involves a negligible error in the result, the value of ϕ cannot possibly exceed 3°, and the value of n is not significantly altered. It

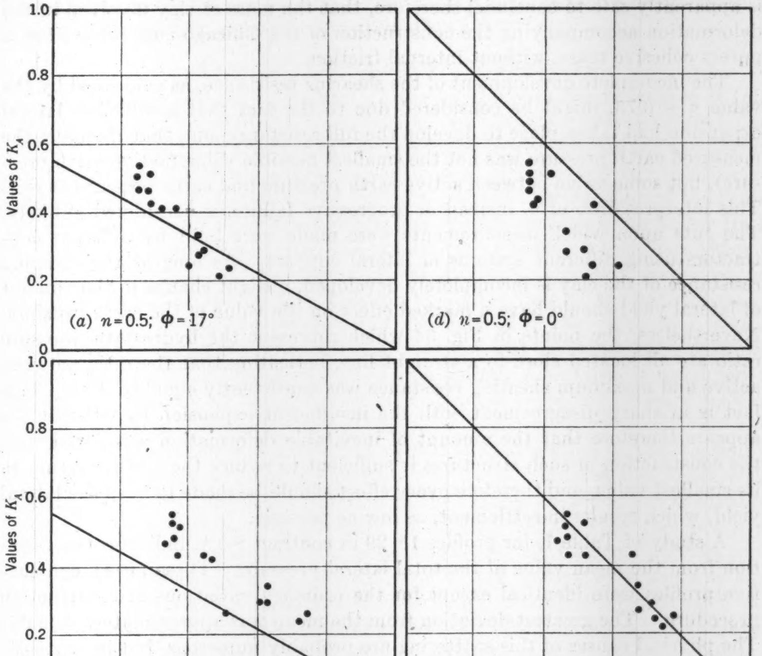
Values of K_A
1.0
0.8
0.6
0.4
0.2
0
(a) $n=0.5$; $\phi=17°$
(d) $n=0.5$; $\phi=0°$
(b) $n=0.75$; $\phi=17°$
(e) $n=0.75$; $\phi=0°$
(c) $n=1.0$; $\phi=17°$
(f) $n=1.0$; $\phi=0°$
0.2
0.4
0.6
0.8
1.0
Values of $\frac{2q_a}{\gamma_a H}$

FIG. 34

is apparently safe to conclude, therefore, that the mass of clay involved in the deformation accompanying the construction of the Chicago cuts behaved as a purely cohesive mass, without internal friction.

The incomplete development of the shearing resistance, as evidenced by the value $n = 0.75$, might be considered due to the fact that insufficient lateral expansion had taken place to develop the full resistance, and that therefore the measured earth pressure was not the smallest possible value (active earth pressure), but some value between active earth pressure and earth pressure at rest. This interpretation of n, instead of progressive failure, is considered unlikely. The cuts upon which measurements were made were built by different contractors using different systems of lateral support. As long as the shearing resistance of the clay is incompletely developed, a slight change in the amount of lateral yield should have a marked effect on the value of the earth pressure. Nevertheless, the points in Fig. 34 which represent the hydrostatic pressure ratio are all located close to a straight line, indicating that the ratio between active and maximum shearing resistance was consistently equal to 0.75. This fact is in sharp disagreement with the insufficient expansion hypothesis. It appears therefore that the amount of inevitable deformation associated with the construction of such structures is sufficient to reduce the earth pressure to its smallest value, and therefore every effort should be made to keep the lateral yield, which results in settlement, as low as possible.

A study of Table 1, for profiles 15–29 in contract S-1A, indicates the variation from the mean value of the total lateral pressure. These fifteen consecutive profiles were identical except for the ordinary variations in construction procedure. The greatest deviation from the mean was approximately ± 20%. The physical causes of this scattering are probably numerous, but in any case, provision in design should be made to take it into account.

It is of interest and importance to compare the measured results of the total lateral pressure with the values computed by the general wedge theory, which takes into account the curvature of the sliding surface, the location of the center of pressure, and the inclination of the resultant earth pressure. Inasmuch as no direct measurements were made of the vertical load carried by the sheeting, the value of θ in the experiments in unknown, and only a general comparison is possible. It is observed that the average center of pressure for all the measurements was at $0.43\,H$. This value, according to the curves of Fig. 33, gives results by the general wedge theory when $\theta = 20°$ which are on the average nearly equal to those given by Rankine's theory. It should be emphasized, however, that the center of pressure in Rankine's theory can never be as high as the lower third point, and for small values of $\frac{H\gamma}{2q}$ may even be below the bottom of the cut. The fact that the total lateral pressure is approximately equal to the Rankine value, even when the distribution of the pressure is entirely different, is not altogether surprising after a study of Fig. 33, from which it can be ascertained that the effect of the curvature of the sliding surface is to increase the total lateral pressure, whereas the effect of the vertical component of the pressure is to reduce it. Actually, the observed total pressures were found to fit the general wedge theory for $\theta = 20°$ slightly better than the plane wedge theory, represented by Eq. 18*b*.

PART IV.—DISTRIBUTION OF PRESSURE

It has been mentioned in Part II that if the shearing resistance is fully developed along the sliding surface forming one of the boundaries of the failure wedge, the total earth pressure can be computed by considering the equilibrium of the wedge. This procedure, however, does not produce any information concerning the distribution of the lateral pressure. The position of the center of pressure depends upon the deformation conditions within the wedge itself. Professor Terzaghi[14] has stated the conditions necessary for various types of pressure distributions. He has shown that if the supporting wall does not yield in a manner approaching rotation about the base of the cut, through a sufficiently large angle to reduce all of the material in the wedge to a state of plastic equilibrium, then the center of pressure will be higher than that predicted by the theory of plasticity. In particular, if yielding is restricted at the top of the cut by the presence of struts, the center of pressure may be expected to rise.

Theoretical considerations do not permit the calculation of the location of the center of pressure, or of the loads carried by individual struts, because the deformation conditions are a function of the manner in which the excavation and bracing are performed. Inspection of Fig. 25, showing the measured deformations accompanying the excavation of one Chicago subway cut, demonstrates that the type of yielding is of the nature to produce a high center of pressure, but quantitative information about the distribution can only be obtained from measurements in the field. An important fact, however, is disclosed by Fig. 25. Almost all of the deformation at a given elevation occurs before excavation reaches that depth. This yielding obviously cannot be prevented entirely by the use of struts. This observation leads to the conclusion that minor variations in construction procedure cannot greatly influence the nature of the deformation, and thus may not have a pronounced effect on the location of the center of pressure. It also leads to the conclusion that most of the deformation occurs before the placement of wales and struts, and that it is unlikely that important arching in the horizontal direction will develop.

Inspection of Table 3(*b*) shows that the measured centers of pressure for the various open cuts were quite consistent, and were in general slightly below midheight of the cuts. In the cut S-4B the highest strut that was acted upon by earth pressure was 11 ft below the surface, and the next one at a depth of 19 ft. Hence the center of pressure while the cut was at shallow depths was necessarily low.

Table 1 gives the individual strut loads measured on contract S-1A. The deviation of the center of pressure from its average value is less than 15%. However, the variation in individual strut loads is very much greater. That this is due to minor details of construction was demonstrated in many instances such as the following:

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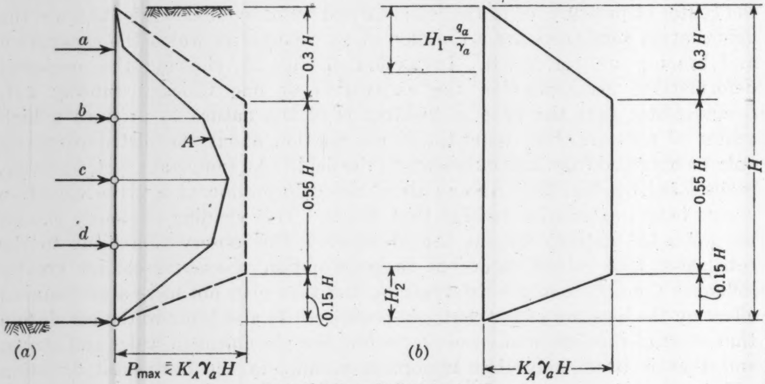
[14] "A Fundamental Fallacy in Earth Pressure Computations," by K. Terzaghi, *Journal*, Boston Soc. of Civ. Engrs., April, 1936, p. 78.

On contract S-1A, excavation reached a depth of 24 ft in the bay between profiles 11 and 12 (see Table 1), and a strut at the 23-ft level was installed in each profile on the same day. In profile 11, excavation on the next day proceeded to a depth of 33 ft and a strut was placed immediately at the 31-ft level. In profile 12, excavation did not proceed from the 24 to the 33-ft level until the third day, at which time the entire bay to profile 13 was mined to 33 ft. The lower strut in profile 12 was not placed until 8 hr after the 33-ft depth was reached. As a result, the lower strut in profile 11 attained a final load of 232 kips, whereas the corresponding strut in profile 12 never exceeded 89 kips. Nevertheless, the center of pressure was at 0.43 *H* in both profiles.

It is apparent, therefore, that it is useless to attempt to compute the real distribution of lateral pressure over the sheeting. Of far greater practical importance is the statistical investigation of the variation in strut loads actually measured, in order to determine the maximum loads that may be expected under ordinary construction procedures. A useful method of investigation is by means of an equivalent beam, as demonstrated by Professor Terzaghi[15] in analyzing the results of measurements on the Berlin subway. The method is shown in Fig. 35. The vertical members of the sheeting are assumed to be hinged at each strut except the uppermost one, and a hinge is assumed to exist at the bottom of the cut. The abscissas of the pressure diagram "A" represent the intensity of horizontal pressure required to produce the measured strut loads. A study of such diagrams for all of the measured profiles disclosed that the maximum abscissa never exceeded the value $K_A \gamma_a H$. Every measured set of strut loads resulted in a different pressure diagram "A," all of which were found to lie within the boundaries of the trapezoid indicated by the dotted lines. Thus, if strut loads are computed on the basis



Equivalent Beam for Strut Loads: (*a*) from Tests; and (*b*) for Design

FIG. 35

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[15] "General Wedge Theory of Earth Pressure," by Karl Terzaghi, *Transactions*, Am. Soc. C. E., Vol. 106 (1941), p. 93.

of this trapezoid, they will most probably be on the safe side. The area of the trapezoid is approximately 50% in excess of the computed total lateral pressure, which is equal to

$$P_A = K_A \times \tfrac{1}{2} \gamma_a H^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

per unit of length of the cut. The excess takes care of the statistical deviation of individual strut loads from the average.

PART V.—SETTLEMENT CONSIDERATIONS

The settlement of the ground surface is a direct result of the lateral squeeze of the clay toward the cut. Although construction conditions made it impossible to determine accurately the complete record of soil movements at the sides and bottom of the cut at the same profiles where complete settlement cross sections were available, a study of the partly incomplete results at numerous profiles established the approximate equivalence of the settlement volume and the volume of lateral squeeze. The most complete information at any of the measured profiles of movements of the sides and bottom of a cut is assembled in Figs. 25 and 26. In this particular cut, the presence of an existing freight tunnel, located as shown in Fig. 25, permitted measurements to be made of the lateral movement of the sheeting at the level of the tunnel while excavation had not yet reached the bottom of the cut. It is evident that the lateral squeeze extended for a considerable distance below the bottom of excavation.

The movements encountered at this profile are particularly interesting because they provide information from which can be deduced the maximum excavation that should be permitted before placement of the uppermost strut. On January 3, 1941, the earth pressure on the sheet piles was practically zero, because the trench, dug to permit driving the sheet piles, was not yet backfilled. In spite of this fact, the tops of the piling moved toward the cut more than 1 in. because of the unbalanced weight of the earth outside of the excavation. Most of this movement occurred while the depth of excavation increased from 10 to 13 ft, corresponding to an increase of the unbalanced part of the load from 1,100 to 1,440 lb per sq ft. The average unconfined compressive strength of the clay between a depth of 13 ft and the bottom of the cut was 1,340 lb per sq ft. Hence, the important deformation of the sheeting began when the unbalanced load became approximately equal to the unconfined compressive strength of the underlying clay. The disturbance of the clay was deep-seated, as evidenced by the appearance of horizontal cracks in the side walls of the freight tunnel when excavation reached a depth of approximately 8 ft. It seems possible that the abrupt and extensive yielding represents the spontaneous change in the clay from its relatively rigid undisturbed state to a plastic state. Installation of the vertical strut before reaching a depth H_1 equal to

$$H_1 = \frac{q_c}{\gamma_a} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$$

would probably have resulted in the prevention of this sudden change and effected a reduction in the corresponding settlement. The struts actually placed at a depth of 4 ft after excavation had reached 13 ft never received an appreciable load. On the other hand, in contract S-3, which was adjacent to contract S-1A with practically identical soil conditions, the top struts were placed when excavation reached a depth of only 8 ft. These struts experienced a substantial load, and no such sudden increase in lateral deflection of the sheeting, or of settlement, occurred. As a matter of fact, in every other open cut observed, the top strut was placed before a depth equal to H_1 was reached, and in each of these cases, the spontaneous settlement was absent. On contract D-6E, the soil was excavated 13 ft before struts were placed, exactly as in S-1A, but the value of H_1 for this case was 18 ft. The maximum movement experienced by the piling for the complete excavation of this cut was 0.5 in.

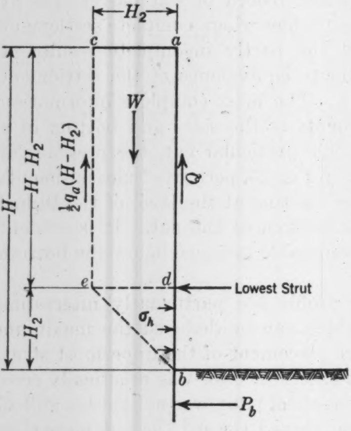

FIG. 36

The maximum permissible spacing between the bottom strut and the bottom of the excavation is less simple to formulate. No satisfactory analytical formulation has as yet been devised, which gives results in close accordance with the observations. The factors involved may be recognized by study of the simplified diagram in Fig. 36.

The total vertical force acting on a horizontal section (e d) at the level of the lowest strut is equal to the weight W of the prism a d e c minus the shearing resistance along c e, which is equal approximately to $\frac{1}{2} q_a (H - H_2)$, and minus the vertical load Q carried by the sheeting. The width of the prism is taken as H_2, equal to the distance between the lowest strut and the bottom of the cut, because the failure surface e b is likely to be inclined at 45° when $\phi = 0$. The intensity of vertical pressure at midheight of the space between b and d is, therefore,

$$\sigma_v = (H - H_2)\,\gamma_a - \frac{1}{2} q_a \frac{(H - H_2)}{H_2} - \frac{Q}{H_2} + \frac{1}{2}\gamma_c H_2 \ldots\ldots (24)$$

in which: γ_a = average density of material above lowest strut; γ_c = average density of material between b and d; and q_a = average effective unconfined compressive strength of material above lowest strut. The lowest value of the normal stress σ_h on the vertical face b d which is compatible with the average vertical stress σ_v is

$$\sigma_h = \sigma_v - q_c = (H - H_2)\,\gamma_a - \frac{1}{2} q_a \frac{(H - H_2)}{H_2} - \frac{Q}{H_2} + \frac{1}{2}\gamma_c H_2 - q_c \ldots (25)$$

in which q_c = unconfined compressive strength of material between b and d.

For the particular case when the entire vertical load is carried by shearing stresses on the wall and on the plane c e, the value of σ_h becomes equal to zero (the most desirable condition) when

$$H_2 = \frac{2\,q_c}{\gamma_c} \quad \text{(26)}$$

or, in other words, when the distance below the bottom strut equals the critical height of the material. (The critical height is the theoretical height at which a vertical bank of cohesive material will stand without support.)[16] If the vertical load is greater than zero, or if it is desired to increase H_2 beyond the critical height of the material, it is necessary that the vertical sheeting be stiff enough to carry the horizontal load σ_h, and that the clay beneath the bottom of the cut be adequate to carry the reaction P_b with sufficient factor of safety. Calculation for the Chicago open cuts indicates that the method of Eq. 23 is likely to be ultraconservative, but it is seen that until measurements are available for evaluation of the shearing forces on the sheeting, a theoretical solution, even of more refined nature, would not be of great practical value.

Repeated calculations of the distances H_2 as excavation and successive struts advanced would determine the maximum vertical spacing between struts in order to prevent excessive lateral squeeze and settlement. This distance was empirically checked for the open cuts on the Chicago subway by systematic observations of inward movement of the sheeting. Excessive movements were readily determined and corrected by relocating struts. An instructive example was again furnished by contract S-1A. Between a depth of 24 and 34 ft, the critical height of the clay ranged between 10 and 15 ft. In the first part of the cut, the vertical distance between the second and third row of struts, at depths of 13 and 31 ft, was equal to 18 ft, considerably greater than the critical height. The corresponding lateral movement and settlement were very important. As a consequence, an intermediate row of struts was established which reduced the spacing at the softest part of the clay to 8 ft. Thereafter the lateral squeeze dropped from 3½ to 2¼ in. at this elevation, and the settlement on adjacent buildings was reduced approximately 35%. The total pressure on all the struts of one vertical row remained practically identical.

On one early open cut, 46 ft deep, upon which measurements were not made, a lateral yield of 10 in. developed, and evidence of considerable distress in some of the bracing appeared. From this observation, together with those already described, it may be concluded that for the Chicago clays, a lateral yield of slightly less than 0.25% of the height of the cut is all that is necessary to reduce the earth pressure to the active value, and that the magnitude of the pressure remains substantially unchanged for a yield up to 1%. Beyond this value, the lateral pressure again increases. Furthermore, it is apparent that nature takes care that a yield of at least 0.25% will occur, regardless of the type of bracing.

PART VI.—RULES FOR DESIGN

For open cuts constructed in a normally consolidated clay soil in accordance with standard construction procedure, the following rules for design seem justified on the basis of the Chicago subway experiments:

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[16] See "Poussée des Terres," Pt. II, by J. Résal, Paris, 1910, p. 77.

1. From test borings, secure samples for unconfined compression tests and other soil data needed to establish the value q in the equation (compare Eq. 18a)

$$q_a = \frac{1}{H}\left[\gamma_1 H^2_s K_s \tan \phi_1 + (H - H_s)\, n\, q_c\right] \ldots\ldots\ldots (27a)$$

and for clays with no overlying sand layers,

$$q = n\, q_c \ldots\ldots\ldots (27b)$$

For Chicago, $n = 0.75$. For other locations, the value of n must be determined on the basis of a few representative field tests.

2. Compute the total lateral pressure by use of the formula

$$K_A = 1.2\left(1 - \frac{2\, q_a}{\gamma_a\, H}\right) \ldots\ldots\ldots (28)$$

The factor 1.2 in Eq. 28 merely takes account of the scattering from the mean value. It should not be taken to indicate that K_A can be greater then 1.0 for a perfect fluid.

3. Construct the pressure diagram Fig. 35, with maximum abscissa, $K_A\, \gamma_a\, H$, and compute the maximum probable strut loads. The struts may be designed for these loads with a conservative factor of safety, inasmuch as provision for all normal variations has been made in the equations. Wales should be designed with no reduction for horizontal arching.

4. Determine allowable depth of excavation prior to placing the first row of struts, from Eq. 23—$H_1 = \frac{q_c}{\gamma_a}$.

Conclusive information concerning the maximum permissible vertical spacing between remaining struts has not been obtained. The proper depth of penetration for the piling depends to a large extent on the stiffness of the clay, and the change in stiffness below the bottom of the cut. Therefore, no general rules can be established for determining this depth.

It should be cautioned that although the results of the Chicago tests are likely to be applicable to other deposits of normally consolidated clays, they almost surely will not apply to clays which have been overconsolidated by heavy pressures in their previous geological history, nor to clays possessing cracks and fissures. They are not likely to apply to shafts or to other excavations in which the deformation conditions of the sheeting are likely to be restricted or significantly different from those in the excavations described. Although the general applicability of Eqs. 11 for total pressure in the Chicago clays has been strongly indicated, the final test in any other locality resides in field measurements.

EARTH PRESSURE ON TUNNELS

By W. S. Housel,[17] M. Am. Soc. C. E.

Synopsis

An investigation of earth pressure on tunnels was undertaken in Detroit, Mich., in 1930, when a number of pressure cells were installed in tunnels, shafts, and retaining walls. Goldbeck pressure cells were used, and practically all installations were in structures built in or on the plastic clay deposits typical of this and several other cities bordering on the Great Lakes.

Earth pressures were observed for periods varying from a few months to several years. Some installations became inaccessible, as the structures were placed in service, and some others were destroyed. In other cases it has been possible to extend the measurements over longer periods, and for the Central Avenue Tunnel, in Detroit, which is discussed in this paper, the records have been continued to the present time, covering a period of approximately ten years.

The compilation and analysis of this particular set of observations, supplemented by shearing resistance tests at the location of the tunnel, were completed in September, 1941. During the years that the data were accumulating, and particularly during the two years 1940 and 1941, a number of analyses have been made in an attempt to develop a logical and accurate method of representing the pressure variations that were observed. In this way, gradually, the "circuit method" of computing earth pressure on tunnels has been evolved. This method consists of establishing the static equilibrium of adjacent elements of mass at the top, side, and bottom of the tunnel in terms of the shearing resistance of the soil. The circuit is then closed by checking from the top to the bottom of the tunnel, treating the tunnel itself as a free body in equilibrium under the combination of stresses conceived to be acting in the surrounding mass.

When such a circuit is closed it means simply that the equations of static equilibrium have been completely satisfied by the selected system of stresses. To be of practical value such a method must be capable of reproducing the pressure variations that have been observed and must do so in terms of measurable soil properties. It is the purpose of this paper to present the experimental background of such a method, develop it, and demonstrate its applicability under the observed conditions.

The extent to which the method may be applied to the computation of

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[17] Lt. Comdr., CEC, U. S. N. R., Office of Superintending Civ. Engr., Norfolk, Va.

earth pressure under more general conditions must be demonstrated by other applications. In so far as the fundamental basis is nothing more than the application of the laws of static equilibrium, general applicability scarcely can be denied. The concept used to represent soil resistance, however, does involve a viewpoint that has been the subject of considerable deliberation and debate. In this connection it should be stated that the paper is confined to a discussion of plastic clay. According to the generally accepted definition of a plastic solid, the intent, therefore, is to discuss materials in which shearing resistance is a function of true cohesion and independent of internal friction. In other words, it is assumed that the soil involved in this investigation is a purely cohesive material. Discussion of the method and its general applicability doubtless will be focused largely on the question of whether or not soil resistance may be so represented, and, if so, how wide a range of soil texture may be included in that classification.

Experimental Results

A pressure-cell installation at the Central Avenue Tunnel has produced a consistent series of earth-pressure measurements extending over a long enough period of time to give a complete picture of the changing condition of equilibrium in such a disturbed soil mass. Soil conditions were quite uniform, there were no apparent disturbances other than the tunnel construction, and the use of the tunnel was such as to permit continued observation for an unlimited time. Other installations were complicated by nonuniform soil conditions, other major excavations in the immediate vicinity, or restrictions on the period of observation.

Fig. 37 shows the general construction features of the tunnel, which is of monolithic concrete with an inside diameter of 9.5 ft and an outside diameter slightly less than 13 ft. The tunnel was built as a by-pass for a large water main beneath a main-line railroad. The soil conditions as indicated by four original borings were quite uniform and consisted of approximately 12 ft of sand and fill at the surface, underlain by a plastic blue lacustrine clay for the entire depth to hardpan. A fifth boring was made in 1940 to obtain samples for shearing resistance tests, the results of which will be used in the present study. The tunnel was located at considerable depth, there being approximately 60 ft of cover. Three rings of pressure cells were installed at the location shown in Fig. 37, all within a distance of some 200 ft.

The arrangement of the cells in the three sections is shown in Fig. 38. The various borings indicated that the soil conditions were substantially uniform, so that it is possible to combine the results to give a desirable duplication of measured pressures. The composite diagram of pressure-cell locations (Fig. 38(*d*)) indicates that the total of twenty-two cells provides three duplicate readings at the top and bottom of the tunnel, five at the spring line, and, with one exception, two readings at each of six intermediate points.

Shearing Resistance Tests.—In Fig. 39 are shown the results of the shearing resistance tests plotted as a shear profile. After penetration through the sand overburden, samples were taken every 5 ft down to hardpan at a depth considerably below the tunnel. The shearing resistance as measured by these tests was quite uniform although there was some variation due, presumably, to the stratification characteristics of clay deposits in this area. The shear varied from approximately 400 lb per sq ft at the top of the clay to an average of 150 lb per sq ft for the remainder of the depth to the top of the tunnel From the top of the tunnel to hardpan the clay was somewhat stiffer, the average being taken as 200 lb per sq ft. The shearing resistance of the sand overburden has been estimated as equivalent to the weight transfer for a 30° sand arch or wedge.

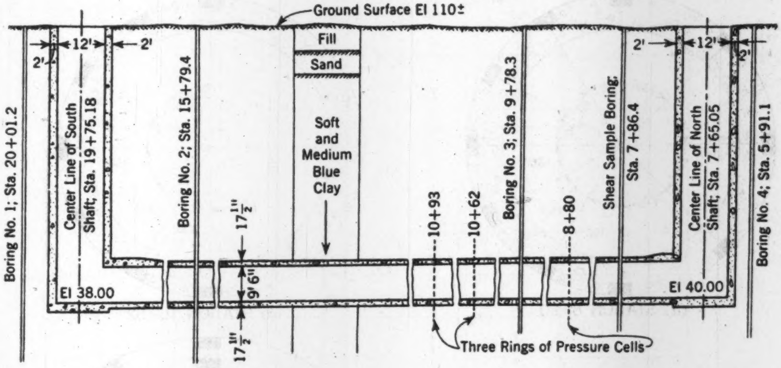
Ground Surface El 110±
Fill
Sand
Soft and Medium Blue Clay
2'
12'
2'
2'
12'
2'
Boring No. 1; Sta. 20+01.2
Center Line of South Shaft; Sta. 19+75.18
Boring No. 2; Sta. 15+79.4
17 1/2"
9'6"
17 1/2"
El 38.00
10+93
10+62
Boring No. 3; Sta. 9+78.3
8+80
Shear Sample Boring; Sta. 7+86.4
Center Line of North Shaft; Sta. 7+65.05
Boring No. 4; Sta. 5+91.1
El 40.00
Three Rings of Pressure Cells

FIG. 37.—GENERAL SECTION OF CENTRAL AVENUE TUNNEL, DETROIT, MICH.

The shearing resistance of the clay was measured by the ring-shear test which has been described in detail in other publications.[18,19] This type of test uses static loading, and the shearing resistance is measured without the application of any pressure normal to the shear plane. The test thus conforms strictly to the accepted definition of a purely cohesive material without internal friction. In this connection the volumetric analyses of the samples are shown in Fig. 39. It may be noted that the voids varied from 40% to 55% by volume, and these voids were filled with water as the air voids were practically negligible.

The heavy dashed line in Fig. 39 represents an average shear profile designated as the design value shearing resistance from which are taken shear values used in the subsequent analysis of earth pressure. These values conform to the yield value as ordinarily defined and determined by the adopted test procedure. There are two significant figures taken from this shear profile which will be used subsequently. These are (1) the shearing resistance at the level of the tunnel, which is taken as 200 lb per sq ft, and (2) the average vertical

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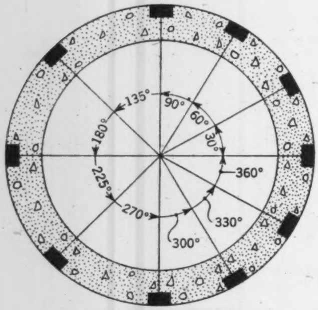
[18] "The Shearing Resistance of Soil—Its Measurement and Practical Significance," by W. S. Housel, *Proceedings*, A.S.T.M., Vol. 39, 1939, p. 1084.

[19] "Laboratory Manual of Soil Testing Procedures," published by Edwards Brothers, Ann Arbor, Mich., 1939.

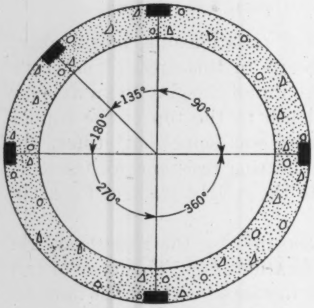
shear from the surface of the ground to the top of the tunnel, which is approximately 300 lb per sq ft.

Earth-Pressure Measurements.—The results of the earth-pressure measurements are shown in Fig. 40, the pressure-cell readings being plotted in polar

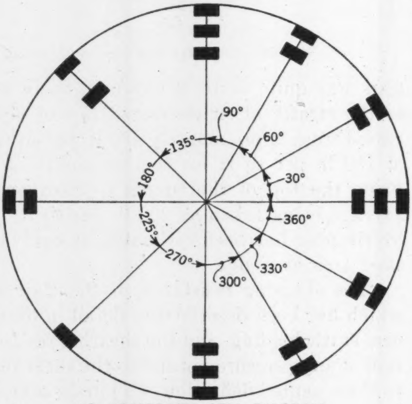


(*a*) STATION 8+80

(*b*) STATION 10+62



(*c*) STATION 10+93



(*d*) COMPOSITE CELL DIAGRAM

FIG. 38.—LOCATION OF PRESSURE CELLS

coordinates. The pressure diagrams or pressure rings are plotted to a scale of 1 in. = 4,000 lb per sq ft so that the two circles shown represent 4,000 and 8,000 lb per sq ft, the center being the origin. The individual pressure readings are represented by indicating the range of the readings at each point on the circumference, with the average values also designated. The numbers shown in small circles indicate the number of readings at each point that are included in the average.

At the first two dates (December 4 and December 10, 1930) an air pressure of 27 lb per sq in. was being carried in the tunnel. The pressure readings and particularly the increase between those two dates would indicate that air pressure was not sufficient to balance the active earth pressure completely and that gradual yielding was taking place. It should be noted that the pressure readings were made inside the tunnel with cells and measuring devices referred to the existing air pressure as a datum. The pressures recorded by the cells previous to December 12, 1930, are earth pressures in excess of the pressure maintained in the tunnel.

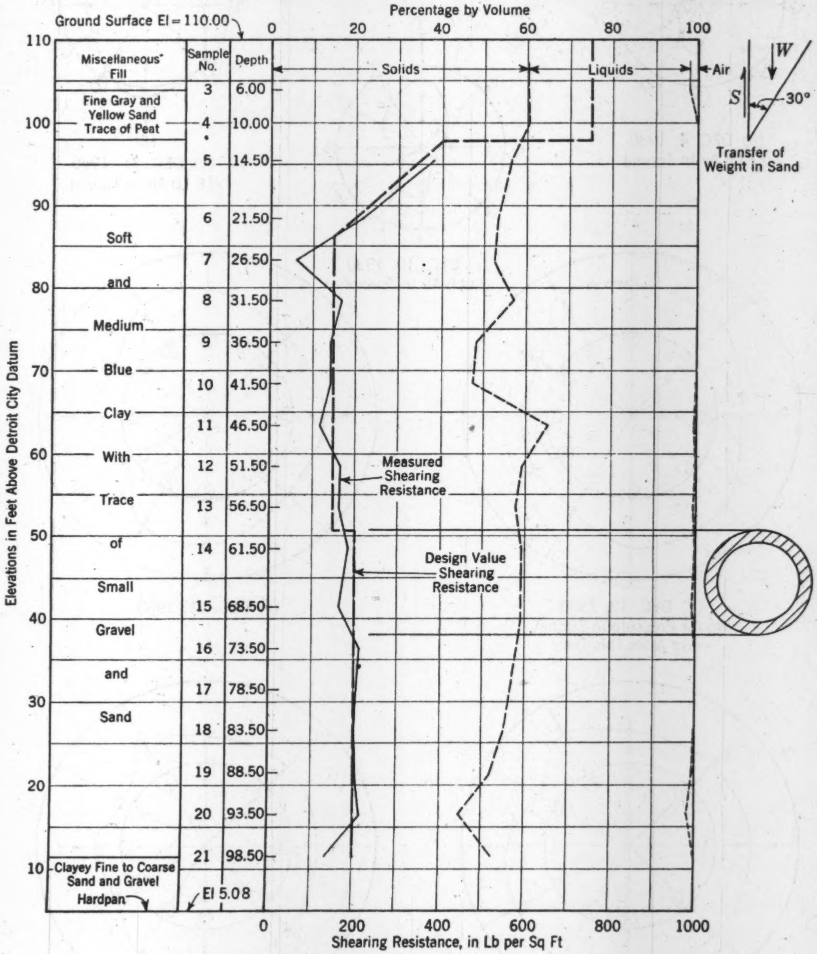
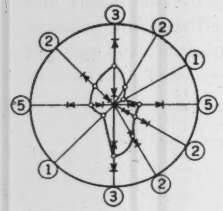
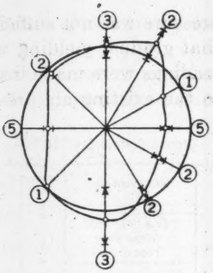


FIG. 39.—SHEARING RESISTANCE TESTS

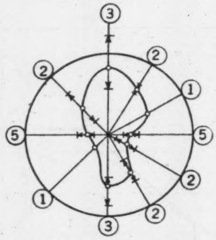
Starting on December 10, 1930, and for the succeeding two days, the com-


3
2
2
1
5
5
2
1
2
3

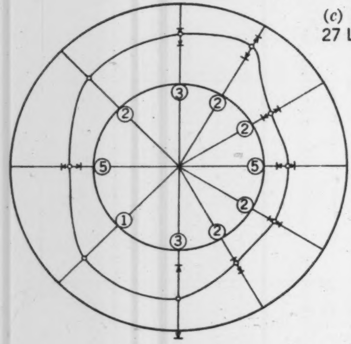
(*a*) DEC. 4, 1930
27 Lb Air in Tunnel


3
2
2
1
5
5
2
1
2
3

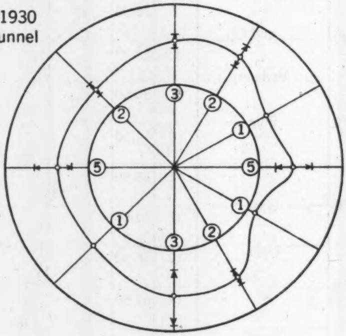
(*b*) DEC. 11, 1930
16 Lb Air in Tunnel


3
2
2
1
5
5
2
1
2
3

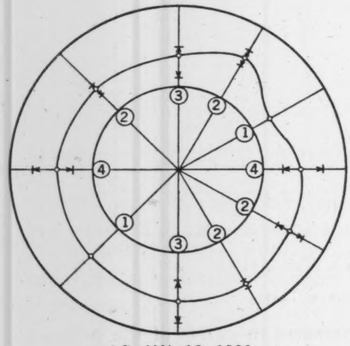
(*c*) DEC. 10, 1930
27 Lb Air in Tunnel


3
2
2
2
5
5
2
1
2
3

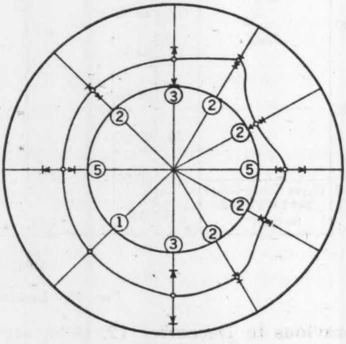
(*d*) DEC. 12, 1930
No Air Pressure in Tunnel
on or After This Date


3
2
2
1
5
5
1
1
2
3

(*e*) DEC. 31, 1930

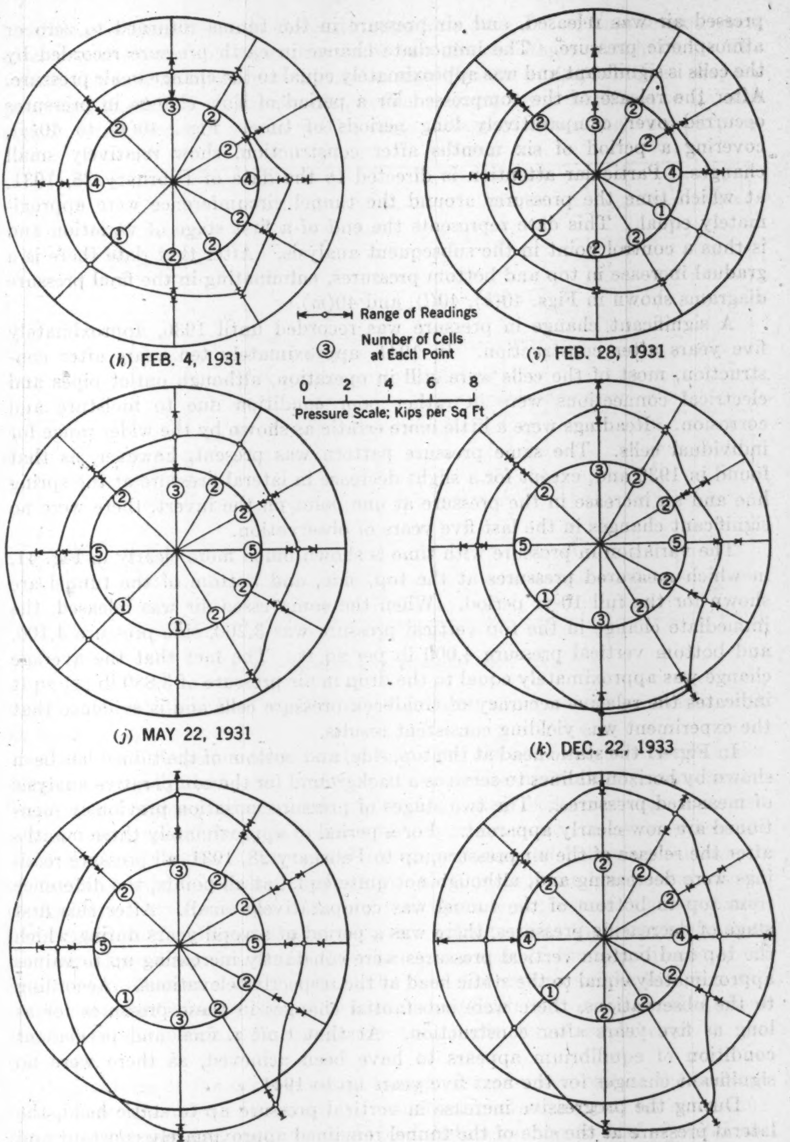

3
2
2
1
4
4
2
1
2
3

(*f*) JAN. 13, 1931


3
2
2
2
5
5
2
1
2
3

(*g*) JAN. 16, 1931

FIG. 40.—EARTH-PRESSURE



(*h*) FEB. 4, 1931

(*i*) FEB. 28, 1931

(*j*) MAY 22, 1931

(*k*) DEC. 22, 1933

(*l*) FEB. 7, 1936

(*m*) JAN. 3, 1941

pressed air was released, and air pressure in the tunnel returned to zero or atmospheric pressure. The immediate change in earth pressure recorded by the cells is significant and was approximately equal to the change in air pressure. After the release of the compressed air a period of slow change in pressures occurred over comparatively long periods of time. Figs. 40(*f*) to 40(*j*), covering a period of six months after construction, show relatively small changes. Particular attention is directed to the date of February 28, 1931, at which time the pressures around the tunnel circumference were approximately equal. This date represents the end of a first stage of variation and is thus a control point in the subsequent analysis. After that date there is a gradual increase in top and bottom pressures, culminating in the final pressure diagrams shown in Figs. 40(*k*), 40(*l*), and 40(*m*).

A significant change in pressure was recorded until 1936, approximately five years after construction. In 1941, approximately ten years after construction, most of the cells were still in operation, although outlet pipes and electrical connections were in rather poor condition due to moisture and corrosion. Readings were a little more erratic as shown by the wider range for individual cells. The same pressure pattern was present, however, as that found in 1936 and, except for a slight decrease in lateral pressure at the spring line and an increase in the pressure at one point on the invert, there were no significant changes in the last five years of observation.

The variation in pressure with time is shown much more clearly in Fig. 41, in which measured pressures at the top, side, and bottom of the tunnel are shown for the full 10-yr period. When the compressed air was released, the immediate change in the top vertical pressure was 3,200, side pressure 4,100, and bottom vertical pressure 4,000 lb per sq ft. The fact that the average change was approximately equal to the drop in air pressure of 3,880 lb per sq ft indicates the relative accuracy of Goldbeck pressure cells and is evidence that the experiment was yielding consistent results.

In Fig. 41 the static head at the top, side, and bottom of the tunnel has been shown by horizontal lines to serve as a background for the comparative analysis of measured pressures. The two stages of pressure variation previously mentioned are now clearly apparent. For a period of approximately three months after the release of the air pressure, up to February 28, 1931, all pressure readings were decreasing and, although not quite equal at all points, the difference from top to bottom of the tunnel was comparatively small. After this first stage of decreasing pressures, there was a period of several years during which the top and bottom vertical pressures were constantly increasing up to values approximately equal to the static head at the respective elevations. According to the observations, there were substantial changes in these pressures for as long as five years after construction. At that time a final and permanent condition of equilibrium appears to have been achieved, as there were no significant changes for the next five years up to 1941.

During the progressive increase in vertical pressure up to static head, the lateral pressure at the side of the tunnel remained approximately constant and very much less than the static head at that elevation. Thus, in Fig. 41 is shown a consistent variation in earth pressure that is indicative of changing conditions of equilibrium in the earth mass for a period of at least several years after construction. There appear to be two distinct stages of behavior. The first involves a temporary condition of equilibrium during which it may be presumed that maximum resistances in the mass are being mobilized to transfer weight away from the tunnel. The second stage of increasing vertical pressures, culminating in final equilibrium, represents some transfer of weight back to the tunnel.

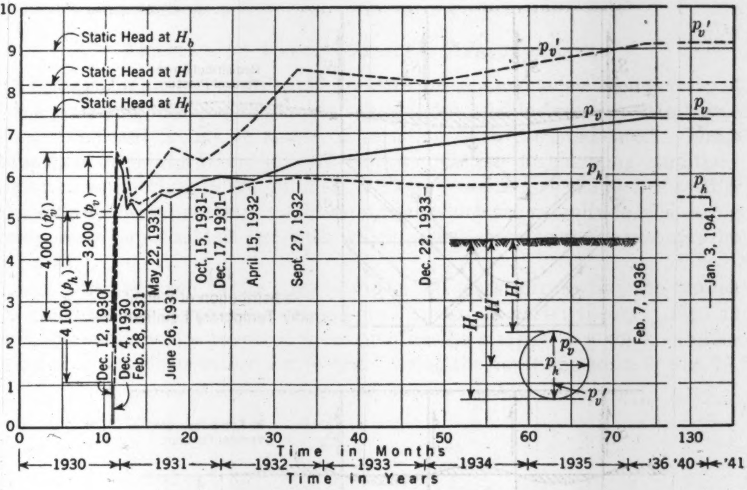
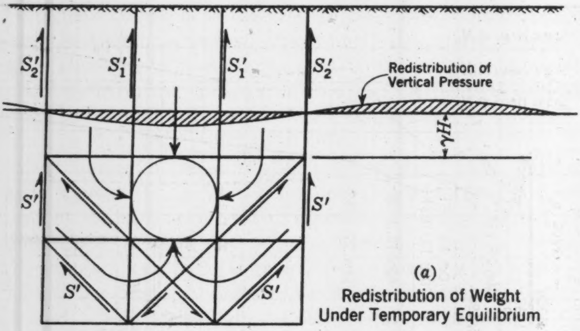


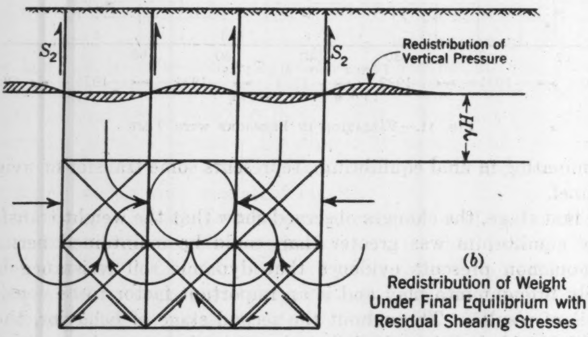
FIG. 41.—VARIATION IN PRESSURE WITH TIME

In the first stage, the changes observed show that the weight transfer under temporary equilibrium was greater than could be maintained permanently. This phenomenon presents evidence that dynamic soil resistance has been temporarily brought into play and is an important factor to be considered in an analysis of results. Throughout the second stage of behavior, the side of the tunnel is shielded against the full static earth pressure and remains the point of minimum active pressure. The fact that this lateral pressure is substantially less than static head after final equilibrium indicates that there are permanent or residual shearing stresses in the mass as a result of the disturbance of its internal equilibrium. The fact that the tunnel structure weighs less than the displaced earth by approximately 5,300 lb per lin ft also indicates that there must be some permanent changes in the conditions of equilibrium in the soil mass.

An analysis of the earth-pressure measurements must explain the manner in which weight is transferred away from the tunnel during and for some time after construction and then is shifted back before coming to final equilibrium. Fig. 42 is an attempt to picture this weight transfer schematically in terms of certain conditions that must be involved. In Fig. 42(a) conditions are indicated that may be considered as controlling during the earlier stage of temporary equilibrium. It is conceived that displacement is toward the tunnel from all directions, with soil resistance being mobilized on all sides in opposition to this motion. Thus, weight is being transferred from the entire region above and on either side of the tunnel. Under these circumstances active pressure tends toward minimum values at all points on the tunnel circumference as the mobilized shearing resistance approaches maximum. A considerable part of this resistance is dynamic and in accordance with the properties of plastic solids in which the shearing stress in excess of the yield value is proportional to the rate of movement. This trend, as previously shown, reaches a limit some three months after construction, at which time controlling values may be established.



(*a*)
Redistribution of Weight
Under Temporary Equilibrium



(*b*)
Redistribution of Weight
Under Final Equilibrium with
Residual Shearing Stresses

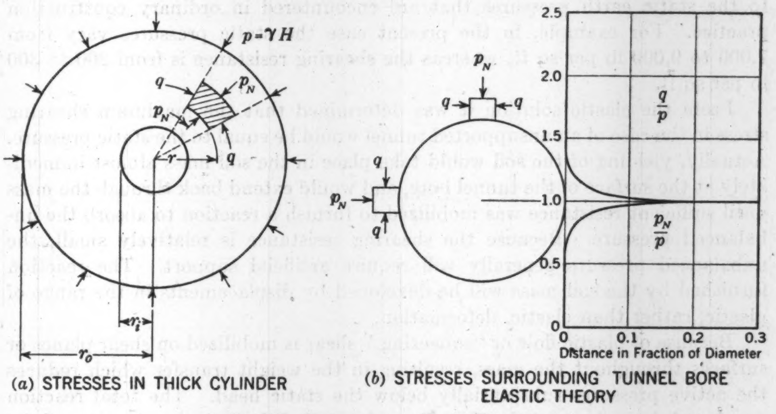
FIG. 42.—WEIGHT TRANSFER IN TUNNEL EXCAVATION

Fig. 42(*b*) shows conditions conceived to exist as a final condition of equilibrium. In the first place, the shearing resistance values have decreased to the yield value which is the maximum that may be maintained under static equilibrium. Thus, due to subsidence on either side of the tunnel, some weight has been transferred back to the tunnel, building up vertical pressure at the top and bottom. The lateral pressure at the side is still minimum and, as a matter of fact, has been almost constant at all times. In setting up a system of stresses for the final condition of equilibrium, it is assumed that any tendency for motion would be toward the sides of the tunnel from points of high pressure at the top and bottom, as well as from static pressure outside of the elements adjacent to the tunnel.

ANALYSIS OF EARTH-PRESSURE MEASUREMENTS

The basis for analyzing the earth-pressure measurements has been portrayed by the preceding discussion of experimental data. Aside from the data themselves, significant trends or stages of variation have been indicated. These trends have been interpreted in an attempt to describe the changing conditions of internal equilibrium in the soil mass. There remains now the task of setting up specific equations of static equilibrium from which the magnitude of shearing stresses may be estimated and with which actual shearing resistance may be compared.

As a first approach to the stresses in the soil mass surrounding the tunnel, it is enlightening to review a related problem from elastic theory. Fig. 43 shows the relationships involved in computing the stresses in a thick cylinder subjected to a uniform external pressure. Using the notation shown in Fig. 43, S. Timoshenko[20] gives the maximum and minimum normal stresses, p_N and q_m, respectively, as:



(*a*) STRESSES IN THICK CYLINDER (*b*) STRESSES SURROUNDING TUNNEL BORE ELASTIC THEORY

FIG. 43.—STRESSES IN A THICK CYLINDER

$$p_N = \frac{p\, r_o^2}{r_o^2 - r_i^2}\left(\frac{r_i^2}{r^2} - 1\right) \ldots\ldots\ldots\ldots\ldots\ldots (29a)$$

and

$$q_m = \frac{p\, r_o^2}{r_o^2 - r_i^2}\left(1 + \frac{r_i^2}{r^2}\right) \ldots\ldots\ldots\ldots\ldots\ldots (29b)$$

---

[20] "Strength of Materials," by S. Timoshenko, Pt. II, D. Van Nostrand Co., 1930.

When r_o is equal to infinity and r is equal to r_i, as would be the case at the surface of a tunnel bore in an indefinite mass, these equations reduce to $p_N = 0$ and $q_m = 2\,p$.

The maximum shearing stress is equal to one half the difference in the principal stresses and is thus equal to static pressure, p. Applying these equations to the tunnel problem also reveals (as shown in Fig. 43(*b*)) that the principal stresses become practically equal and the shearing stress becomes zero, within a distance from the surface of the tunnel bore of less than one fourth of the diameter of the tunnel.

Several important deductions may be made from this elastic solution. In the first place, the pressure acting normal to the tunnel bore is a principal pressure, from which it follows that under normal conditions there would be no tangential or shearing forces on this surface. In the second place, at the surface of the tunnel bore there is the usual principal stress difference, equal to twice the shearing resistance of the material. When movement of the soil mass is toward the tunnel, the normal pressure would be the minimum principal stress. In the third place, the stress differential at the surface of the tunnel is dissipated within a relatively short distance from the tunnel.

The stress relationships from elastic theory obviously hold only as long as the shearing resistance of the material involved is not exceeded. In plastic clay, shearing resistance is frequently of a low order of magnitude as compared to the static earth pressures that are encountered in ordinary construction practice. For example, in the present case the static pressures vary from 7,000 to 9,000 lb per sq ft, whereas the shearing resistance is from 200 to 300 lb per sq ft.

From the elastic solution it was determined that the maximum shearing stress in the case of an unsupported tunnel would be equal to the static pressure. Actually, yielding of the soil would take place in the soil mass almost immediately at the surface of the tunnel bore, and would extend back through the mass until sufficient resistance was mobilized to furnish a reaction to absorb the unbalanced pressure. Because the shearing resistance is relatively small, the unbalanced pressure generally will require artificial support. The reaction furnished by the soil mass will be developed by displacements in the range of plastic, rather than elastic, deformation.

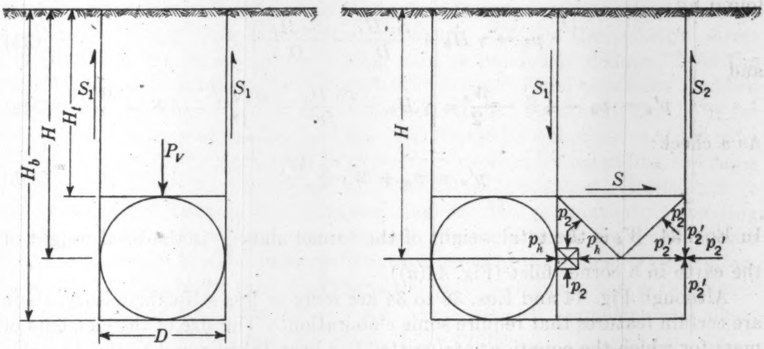
Because of plastic flow or "squeezing," shear is mobilized on shear planes or surfaces throughout the mass, resulting in the weight transfer which reduces the active pressures substantially below the static head. The total reaction developed depends upon the shearing resistance of the soil, the diameter of the tunnel, the depth beneath the surface, and the weight of the soil. With these various factors determined, the analysis of earth pressure can be reduced to the selection of a stress system consistent with static equilibrium, which is capable of reproducing the measured pressures.

Such a system of stresses is shown in Fig. 44 and constitutes what has been previously called the "circuit method" of computing earth pressure on the tunnel lining. For the temporary condition of equilibrium the stress system shown involves the maximum transfer of weight away from the tunnel.

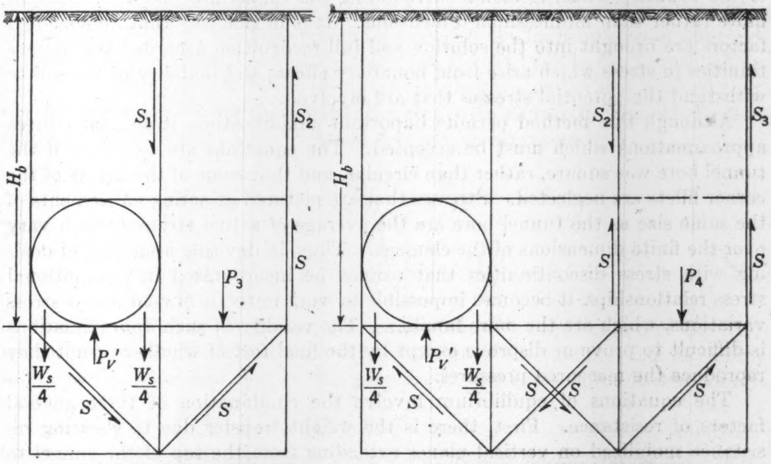
For the top vertical pressure (Fig. 44(*a*)):

$$p_v = \gamma H_t - \frac{2 S_1 H_t}{D} \quad \ldots\ldots\ldots\ldots\ldots\ldots (30)$$

in which $S_1 = C_1 S_v$; S_v is the cumulative average shear from the ground surface to the top of the tunnel; and γ is the soil weight per cubic foot.


H_b
H
H_t
S_1
S_1
P_V
D
H
S_1
S_2
S
p_2
p'_2
p_h
p_h
p_2
p'_2
p_2
p'_2
p_2
p'_2

(*a*) TOP VERTICAL PRESSURE (*b*) LATERAL PRESSURE


S_1
S_2
H_b
S
P_3
$\frac{W_s}{4}$
P_V'
$\frac{W_s}{4}$
S
S
S_2
S_3
H_b
S
S
P_4
$\frac{W_s}{4}$
P_V
$\frac{W_s}{4}$
S
S
S
S

(*c*) BOTTOM VERTICAL PRESSURE UNDER TEMPORARY EQUILIBRIUM (*d*) BOTTOM VERTICAL PRESSURE UNDER FINAL EQUILIBRIUM.

FIG. 44.—COMPUTATION OF EARTH PRESSURE

For lateral pressure (Fig. 44(*b*)):

$$p'_2 = \gamma H + \frac{S_1 H_t}{D} - \frac{S_2 H_t}{D} \quad \ldots\ldots\ldots\ldots\ldots\ldots (31)$$

in which $S_2 = C_2 S_v$; $S = C_2 S_A$; and S_A = average shear in the region of the tunnel. If $p_2 = p'_2 - 2S$ and $p_h = p_2 - 2S$, Eq. 31 becomes

$$p_h = \gamma H + \frac{S_1 H_t}{D} - \frac{S_2 H_t}{D} - 4S \qquad (32)$$

The bottom vertical pressure under temporary equilibrium (Fig. 44(*c*)) is found by

$$p_3 = \gamma H_b + \frac{S_1 H_t}{D} - \frac{S_2 H_t}{D} - S \qquad (33)$$

and

$$p'_v = p_3 - 4S - \frac{W_s}{2} = \gamma H_b + \frac{S_1 H_t}{D} - \frac{S_2 H_t}{D} - 5S - \frac{W_s}{2} \qquad (34a)$$

As a check:

$$p'_v = p_v + W_t + \frac{W_s}{2} \qquad (34b)$$

In Eqs. 34, W_t is the total weight of the tunnel and $\frac{W_s}{4}$ is the total weight of the earth in a corner fillet (Fig. 44(*a*)).

Although Fig. 44 and Eqs. 30 to 34 are more or less self-explanatory, there are certain features that require some elaboration. The size of the elements of mass for which the equations are written has been taken equal to the diameter of the tunnel. Thus, elements surrounding the tunnel are taken as cubes of finite rather than infinitesimal dimensions. It is in this way that dimensional factors are brought into the solution and full recognition accorded the discontinuities in stress which arise from boundary effects and inability of the soil to withstand the potential stresses that are involved.

Although this method permits important simplification, it also introduces approximations which must be accepted. The equations are set up as if the tunnel bore was square, rather than circular, and thus some of the effects of the corner fillets are neglected. Stresses that are pictured as acting on elements of the same size as the tunnel bore are the average of actual stresses which vary over the finite dimensions of the element. Thus, in devising a method of dealing with stress discontinuities that cannot be incorporated in conventional stress relationships, it becomes impossible to work in terms of continuous stress variations, which are the more familiar. The validity of such approximations is difficult to prove or disprove except by the final test of whether or not they reproduce the measured pressures.

The equations of equilibrium involve the combination of three general factors of resistance. First, there is the weight transfer due to shearing resistance mobilized on vertical planes extending from the top of the tunnel to the surface of the ground. This transmission of force around the tunnel bore is sometimes referred to simply as arching action. The second influence to be considered likewise may be referred to as arching of the lateral pressure around the tunnel. It is shown, however, as shear on a horizontal plane at the top of the tunnel.

Both of these factors are functions of plastic displacements and the resulting mobilization of shearing resistance. They both operate to diminish the static earth pressure in the region of the tunnel, but they have been evaluated in two separate steps. The static pressure at the depth H is shown at the back of the vertical column that is affected by lateral movement toward the tunnel. It is computed by evaluating the magnitude of the weight transfer by shearing forces on vertical planes of slipping. The horizontal arching action is then evaluated as a pressure differential made available by mobilization of horizontal shearing resistance. Finally, there is the third factor of resistance which makes available a further decrease in lateral pressure. This is the principal stress differential at the surface of the tunnel bore as previously deduced from the elastic solution based on stresses in a thick cylinder. Final equations are then written for the summation of all resistance factors. As stated, Eq. 30 is for the vertical pressure at the top, and Eq. 32 is for the lateral pressure at the side.

Eq. 34*a* for the bottom vertical pressure is derived by extending the same method of analysis to an element of mass below the tunnel. Resistance to upheaval at the bottom of the tunnel includes the summation of shearing resistance mobilized on the sides of the vertical column beside the tunnel, which tends to move down when upheaval at the bottom of the tunnel is impending. In addition, there is the pressure differential on two elements below the tunnel and the weight of the corner fillets. Eq. 34*b* is the check equation and closes the pressure circuit from top to bottom of the tunnel.

For final equilibrium, the equations for the top vertical and the lateral pressure on the side are the same as for temporary equilibrium (Eqs. 30 to 32). As previously stated, the final pressures represent the limiting case of a second stage of behavior in which weight is being transferred back to the tunnel. The top and bottom are now points of maximum pressure, and it is conceived that the tendency for motion is toward the side. As shown in Fig. 44(*d*), this tendency controls the bottom vertical pressure as expressed by

$$p_4 = \gamma H_b + \frac{S_2 H_t}{D} - \frac{S_3 H_t}{D} - 2S \qquad \text{(35)}$$

Then

$$p'_v = p_4 - \frac{W_s}{2} = \gamma H_b + \frac{S_2 H_t}{D} - \frac{S_3 H_t}{D} - 2S - \frac{W_s}{2} \qquad \text{(36)}$$

in which $S_3 = C_2 S_v$. As before, a check is afforded by application of Eq. 34*b*, which then closes this circuit from top to bottom of the tunnel. Eq. 36 differs substantially from Eq. 34*a* for the pressure at the same point in temporary equilibrium.

Relation of Shear Stress to Yield Value.—In all of the equilibrium equations the shearing stresses are set up in general terms, and specific values are yet to be established. The correlation of specific values of shearing stress and the shearing resistance of the soil as measured by test is the primary objective of the investigation. This correlation may be made most conveniently in terms of the ratio of shearing stresses computed from the equilibrium equations to the yield value of the soil as measured. This ratio has been designated by C_1 and C_2 and has been called the overload ratio, inasmuch as shearing stresses mobilized under field conditions are frequently in excess of the yield value.

The two controlling values of shearing resistance are 200 lb per sq ft for the average shear, S_A, at the depth of the tunnel, and 300 lb per sq ft for the average vertical shear, S_v, for the depth from the surface of the ground to the top of the tunnel. By substituting the measured values of earth pressure and shearing resistance in the equilibrium equations, the values of C_1 and C_2 may be computed for the entire range of observation. It should be noted that C_1 refers to the shearing stress at the sides of a vertical column above the tunnel, whereas C_2 refers to shearing stresses that are mobilized in opposition to both lateral movement and upheaval into the tunnel excavation. The relative magnitude of these two shear stresses depends upon the dimensions involved. In the present case, where the diameter of the tunnel is fairly large with respect to the depth beneath the surface, lateral movement or upheaval probably will be the controlling influence.

Typical computations are given in the Appendix to illustrate the solution of the equilibrium equations and to serve as an example of the accuracy with which these equations reproduce measured pressures. The computations for cases 1 and 2 are for the limiting cases of temporary and final equilibrium, respectively, and thus represent controlling values of shearing resistance. The range of variation in shear stress is shown in Fig. 45 by the change in C_1 and C_2 with time.

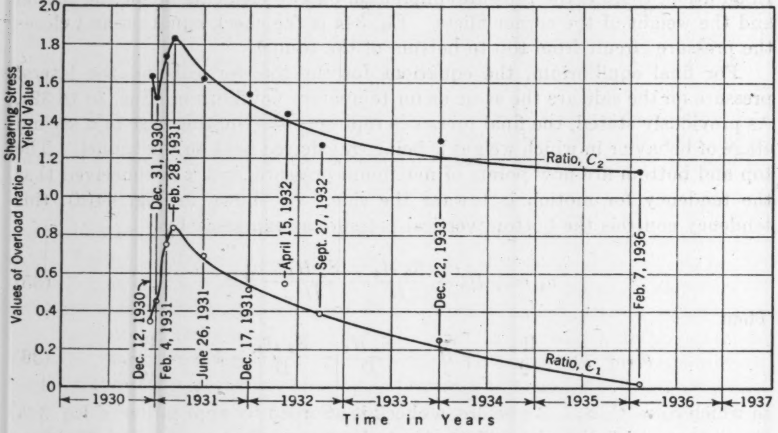
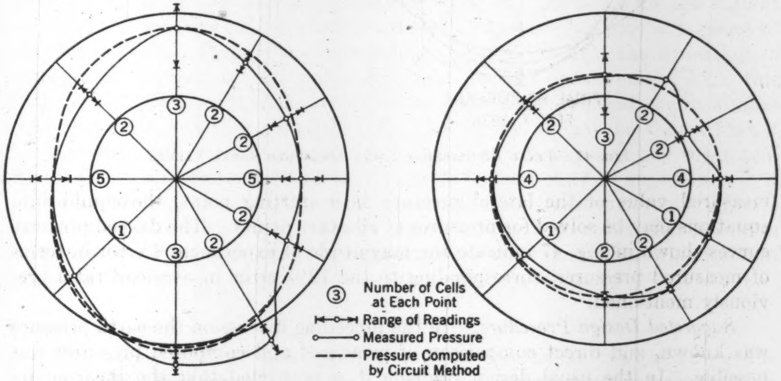


FIG. 45.—VARIATION IN SHEAR STRESS WITH TIME

The maximum and minimum values are the controlling factors for practical application. The maximum shear stress is represented by the overload ratio, C_2, equal to 1.84, and occurs at the end of the first stage of pressure variation more than three months after the air pressure is released. From this value of the overload ratio it is concluded that, due to movement of the soil mass toward the tunnel, a shearing stress approaching twice the yield value has been mobilized. A very substantial part of this must be dynamic resistance and not permanently available. This is borne out by the fact that the value of C_2 decreases almost to unity for final equilibrium, the actual value being 1.12. The final value of the overload ratio is a significant figure as it represents the over-all accuracy with which the proposed method reproduces the observed earth pressure in terms of the independent measurement of shearing resistance. It means, specifically, that for a perfect check the shearing resistance should be 12% greater than the yield value that was actually found by laboratory test. It is impossible to state whether this margin of error is in the shear test or the method of computing the pressure, but it is felt that it is well within practical limits. The values of C_1 are not controlling and they simply indicate the weight transfer away from the top of the tunnel at various stages. The fact that C_1 approaches zero in this particular case indicates that the pressure at the top finally became almost equal to the static head.

Further proof of the validity of the equations used in the circuit method of computing pressure depends on the accuracy with which they reproduce all of the observed pressures. Comparisons of computed and measured pressures for both temporary and final equilibrium are shown in Fig. 46. The computed pressures are shown by dashed curves and have been found by substituting the shear-stress values deduced from the previous analysis. Normal pressure, p_N, at intermediate points on the circumference has been found by combining components of the lateral and vertical pressure in accordance with the following equation, in which θ is the angle with the horizontal:


③
②
②
②
⑤
⑤
②
①
③
②
③ Number of Cells at Each Point
Range of Readings
Measured Pressure
Pressure Computed by Circuit Method
②
②
③
②
④
④
①
①
②
②

FINAL EQUILIBRIUM
Feb. 7, 1936

TEMPORARY EQUILIBRIUM
Feb. 28, 1931

FIG. 46.—COMPARISON OF MEASURED AND COMPUTED PRESSURES

$$p_N = p_h \cos^2 \theta + p_v \sin^2 \theta \tag{37}$$

Fig. 46 indicates only that the equations used are capable of reproducing the pressure variations around the entire circumference. The closure of the pressure circuit is reasonably close and indicates that the equations are consistent as a group.

A more important experimental verification is afforded by the comparison shown in Fig. 47. In this case the measured yield value of the soil has been used to obtain the computed values of the earth pressure. The solution is given as case 3 in the Appendix. It is assumed that in final equilibrium the controlling shear values are equal to the measured yield values. Taking a measured value of the lateral pressure as a starting point, the equilibrium equations may be solved for pressures at all other points. The dashed pressure curves shown in Fig. 47 indicate the magnitude of experimental error in terms of measured pressures, corresponding to the 12% error in overload ratio previously mentioned.

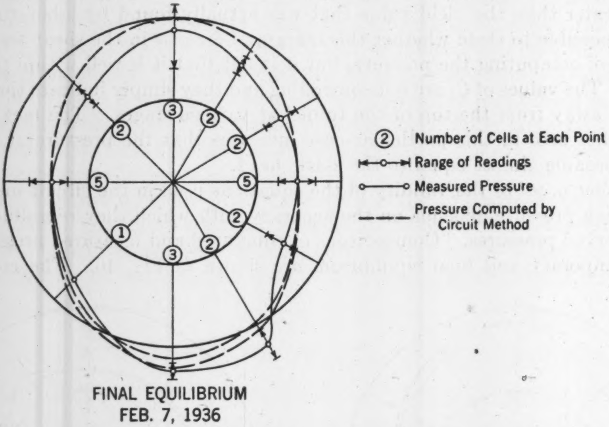


FIG. 47.—FINAL EQUILIBRIUM USING MEASURED SHEAR VALUES

Suggested Design Procedure.—In the preceding discussion the earth pressure was known, and direct comparison of measured and computed pressures was possible. In the usual design problem it is presumed that the shearing resistance of the soil, its weight, and the general dimensions of the tunnel will be known and that the earth pressure must be computed. An examination of the equations reveals too many unknowns for a direct solution, and some simplifying condition must be introduced.

One of the most common problems in tunnel construction is that of estimating the air pressure that will be required under a given set of physical conditions. In this case the internal pressure supplied will be equal at all points in the tunnel, and this imposed condition makes the equilibrium equations of the circuit method determinate.

In specifying the shearing resistance to be used in computing the air pressure required, the time element is perhaps the most important factor. It obviously would be unnecessary to use a high enough air pressure to hold shear stresses down to the yield value. In the present observations, an overload ratio of almost 2 represented available resistance over a period of three months. By allowing the same displacement in a shorter period of time, an even higher overload ratio is possible.

In previous investigations the writer has had a number of opportunities to evaluate overload ratios for progressive settlement of large buildings and has found them to be of the order of 2, or slightly more, without dangerously high rates of settlement. Similar study of embankments and open excavations has indicated a general range of from 2 to 3 for these ratios. Based on these observations, the writer for some time has been using a limiting value of 2.5 for the design of bracing in open excavation or for embankment slopes where some settlement was permissible. Analyses of such designs that have gone through the construction stage appear to indicate that overload ratios approaching or in excess of 3 involve rates of displacement that are excessive and may be approaching the danger zone as far as slides are concerned. It is difficult to supply general design criteria for such a problem, as each individual case should receive special study. Among other factors, the volume of the mass and general dimensions will affect materially the rate of movement produced.

The Central Avenue Tunnel itself supplies the best example of a permissible overload ratio for the specific conditions involved. As noted in the earliest results reported, the tunnel was built under compressed air with a pressure of 27 lb per sq in. It also was found that this pressure held the squeezing to permissible limits but did not completely prevent movement. With this balanced air pressure it is shown in case 4 of the Appendix that the maximum overload ratio was approximately 2.8. Conversely, it may be shown that if the overload ratio for design had been assumed as 2.5, as previously suggested, the required air pressure would have been 30 instead of 27 lb per sq in.

This is a fair example of present practice in tunnel excavation and appears to be quite representative. It seems probable that there are similar examples available from the construction experience of other engineers which would yield valuable design data if subjected to a similar analysis.

CONCLUSION

The following conclusions should serve to summarize the preceding discussion of earth pressure on tunnels and to emphasize the practical value of shearing resistance tests in evaluating these pressures:

(1) A consistent record of measured earth pressure over a period of ten years indicated that final equilibrium was not achieved until five years after construction.

(2) There were two separate stages of pressure variation. For the first three months after construction pressures were decreasing, indicating a continued mobilization of available soil resistance. Thereafter, vertical pressure increased for several years until static equilibrium was achieved and soil resistance values returned to the yield value as measured by independent shearing resistance tests.

(3) The minimum pressures recorded occurred at the end of the first stage of variation, at which time they were practically the same at all points around the circumference of the tunnel.

(4) At the time of final equilibrium the vertical pressures were about equal to the static head, or even slightly greater, whereas the lateral pressure, which varied but little throughout the observations, was substantially less than the static head.

(5) A method of computing the earth pressure was devised which reproduced the pressures at final equilibrium within practical limits by the use of shear values measured by independent laboratory tests.

(6) Analysis of the temporary condition of equilibrium shortly after construction revealed that dynamic resistance was brought into play with shear stresses approaching twice the yield value.

(7) Analysis of equilibrium involving the use of compressed air indicated that the required shear stresses would approach three times the yield value.

(8) Assumptions were suggested for temporary conditions during construction. These involved an overload ratio of not more than 2.5 for design when some movement and settlement may be permitted.

APPENDIX

TYPICAL COMPUTATIONS TO ILLUSTRATE THE SOLUTION OF EQUILIBRIUM EQUATIONS

Case 1. Temporary Equilibrium.—

p_v = 5,065 lb per sq ft S_v = 300 lb per sq ft H_t = 59.2 ft
p_h = 5,345 lb per sq ft S_A = 200 lb per sq ft H = 65.6 ft
p'_v = 5,650 lb per sq ft W_t = 860 lb per sq ft H_b = 72.0 ft
γ = 125 lb per cu ft W_s = 340 lb per sq ft D = 12.84 ft

$$p_v = \gamma\, H_t - \frac{2\, S_1\, H_t}{D}\ (S_1 = C_1\, S_v)$$

$$5{,}065 = 125 \times 59.2 - \frac{2 \times C_1 \times 300 \times 59.2}{12.84} = 7{,}400 - 2{,}760\, C_1$$

$$C_1 = \frac{7{,}400 - 5{,}065}{2{,}760} = 0.84$$

$$p_h = \gamma\, H + \frac{S_1\, H_t}{D} - \frac{S_2\, H_t}{D} - 4\, S\ (S_2 = C_2\, S_v \text{ and } S = C_2\, S_A)$$

$$5{,}345 = 125 \times 65.6 + \frac{0.84 \times 300 \times 59.2}{12.84} - \frac{C_2 \times 300 \times 59.2}{12.84}$$

$$- 4 \times C_2 \times 200 = 8{,}200 + 1{,}160 - 1{,}380\, C_2 - 800\, C_2$$

$$C_2 = \frac{8{,}200 + 1{,}160 - 5{,}345}{2{,}180} = 1.84$$

$$p'_v = \gamma\, H_b + \frac{S_1\, H_t}{D} - \frac{S_2\, H_t}{D} - 5\, S - \frac{W_s}{2}$$

$$p'_v = 125 \times 72.0 + \frac{0.84 \times 300 \times 59.2}{12.84} - \frac{1.84 \times 300 \times 59.2}{12.84}$$

$$- 5 \times 1.84 \times 200 - \frac{340}{2} = 9{,}000 + 1{,}160 - 2{,}540 - 1{,}840$$

$$- 170 = 5{,}610 \text{ lb per sq ft} < 5{,}650 \text{ lb per sq ft.}$$

Check:

$$p'_v = p_v + W_t + \frac{W_s}{2} = 5{,}065 + 860 + \frac{340}{2} = 5{,}065 + 860 + 170$$

$$= 6{,}095 \text{ lb per sq ft} > 5{,}650 \text{ lb per sq ft.}$$

Case 2. Final Equilibrium.—

$$\begin{array}{lll} p_v = 7{,}320 \text{ lb per sq ft} & S_v = 300 \text{ lb per sq ft} & H_t = 59.2 \text{ ft} \\ p_h = 5{,}805 \text{ lb per sq ft} & S_A = 200 \text{ lb per sq ft} & H = 65.6 \text{ ft} \\ p'_v = 9{,}120 \text{ lb per sq ft} & W_t = 860 \text{ lb per sq ft} & H_b = 72.0 \text{ ft} \\ \gamma = 125 \text{ lb per cu ft} & W_s = 340 \text{ lb per sq ft} & D = 12.84 \text{ ft} \end{array}$$

$$p_v = \gamma H_t - \frac{S_1 H_t}{D} \; (S_1 = C_1 S_v)$$

$$7{,}320 = 125 \times 59.2 - \frac{2 \times C_1 \times 300 \times 59.2}{12.84} = 7{,}400 - 2{,}760\, C_1$$

$$C_1 = \frac{7{,}400 - 7{,}320}{2{,}600} = 0.03$$

$$p_h = \gamma H + \frac{S_1 H_t}{D} - \frac{S_2 H_t}{D} - 4\, S \; (S_2 = C_2 S_v \text{ and } S = C_2 S_A)$$

$$5{,}805 = 125 \times 65.6 + \frac{0.03 \times 300 \times 59.2}{12.84} - \frac{C_2 \times 300 \times 59.2}{12.84}$$

$$- 4 \times C_2 \times 200 = 8{,}200 + 42 - 1{,}380\, C_2 - 800\, C_2$$

$$C_2 = \frac{8{,}200 + 42 - 5{,}805}{2{,}180} = 1.12$$

$$p'_v = \gamma H_b + \frac{S_2 H_t}{D} - \frac{S_3 H_t}{D} - 2\, S - \frac{W_s}{2}$$

$$p'_v = 125 \times 72.0 + \frac{1.12 \times 300 \times 59.2}{12.84} - \frac{1.12 \times 300 \times 59.2}{12.84}$$

$$- 1.12 \times 200 \times 2 - \frac{340}{2} = 9{,}000 + 1{,}545 - 1{,}545 - 450 - 170$$

$$= 8{,}380 \text{ lb per sq ft} < 9{,}120 \text{ lb per sq ft.}$$

$$p'_v = p_v + W_t + \frac{W_s}{2} = 7{,}320 + 860 + \frac{340}{2} = 7{,}320 + 860 + 170$$

$$= 8{,}350 \text{ lb per sq ft} < 9{,}120 \text{ lb per sq ft.}$$

Case 3. Final Equilibrium Using Measured Shear Values.—

$$\begin{array}{ll} p_h = 5{,}805 \text{ lb per sq ft} & C_2 = 1 \\ S_v = 300 \text{ lb per sq ft} & S_2 = S_3 = S_v \\ S_A = 200 \text{ lb per sq ft} & S = S_A \end{array}$$

$$p_h = \gamma H + \frac{S_1 H_t}{D} - \frac{S_2 H_t}{D} - 4 S$$

$$5{,}805 = 125 \times 65.6 + \frac{S_1 \times 59.2}{12.84} - \frac{300 \times 59.2}{12.84} - 4 \times 200$$

$$S_1 = \frac{5{,}805 + 1{,}380 + 800 - 8{,}200}{4.61} = -\ 47 \text{ lb per sq ft.}$$

$$p_v = \gamma H_t - \frac{2 S_1 H_t}{D} = 125 \times 59.2 - \frac{2 \times (-\ 47) \times 59.2}{12.84}$$

$$= 7{,}400 + 435 = 7{,}835 \text{ lb per sq ft} > 7{,}320$$

$$p'_v = \gamma H_b + \frac{S_2 H_t}{D} - \frac{S_3 H_t}{D} - 2 S - \frac{W_s}{2}$$

$$p'_v = 125 \times 72 - 400 - 170 = 9{,}000 - 370 = 8{,}430 \text{ lb per sq ft} < 9{,}120.$$

Check:

$$p'_v = p_v + W_t + \frac{W_s}{2} = 7{,}835 + 860 + 170 = 8{,}865 \text{ lb per sq ft} < 9{,}120.$$

Case 4. Temporary Equilibrium Under Balanced Air Pressure.—With a balanced air pressure of 27 lb per sq in. in the tunnel, find various overload ratios.

$$27 \text{ lb per sq in.} = 3{,}888 \text{ lb per sq ft} = p_v = p_h = p'_v$$

$$S_v = 300 \text{ lb per sq ft.} \qquad S_A = 200 \text{ lb per sq ft.}$$

Temporary Equilibrium.—

$$p_v = \gamma H_t - \frac{2 S_1 H_t}{D}$$

$$3{,}888 = 125 \times 59.2 - \frac{2 \times C_1 \times 300 \times 59.2}{12.84} = 7{,}400 - 2{,}760\ C_1$$

$$C_1 = \frac{7{,}400 - 3{,}888}{2{,}760} = 1.27$$

$$p_h = \gamma H + \frac{S_1 H_t}{D} - \frac{S_2 H_t}{D} - 4 S$$

$$3{,}888 = 125 \times 65.6 + \frac{1.27 \times 300 \times 59.2}{12.84} - \frac{C_2 \times 300 \times 59.2}{12.84}$$

$$-\ 4 \times 200 \times C_2 = 8{,}200 + 1{,}750 - 1{,}380\ C_2 - 800\ C_2$$

$$C_2 = \frac{9{,}950 - 3{,}888}{2{,}180} = 2.78$$

$$p'_v = \gamma H_b + \frac{S_1 H_t}{D} - \frac{S_2 H_t}{D} - 5 S - \frac{W_s}{2}$$

$$3{,}888 = 125 \times 72.0 + \frac{1.27 \times 300 \times 59.2}{12.84} - \frac{C_2 \times 300 \times 59.2}{12.84}$$

$$-\ (5 \times C_2 \times 200) - 170 = 9{,}000 + 1{,}750 - 1{,}380\ C_2 - 1{,}000\ C_2 - 170$$

$$C_2 = \frac{6{,}692}{2{,}380} = 2.81.$$

DISCUSSION

Ralph H. Burke,[21] **M. Am. Soc. C. E.**—The study of soil mechanics gives promise of effecting the next major advance in the design and construction of underground structures. The past three or four decades have witnessed great improvements in the manufacture of cement and steel, and to a lesser extent a standardization in the choice of aggregates, so that uniformity in the ordinary construction materials may now be assured. The technique of concrete proportioning has become fully recognized so that, with proper engineering organization, the failure of structures due to careless or ignorant workmanship is eliminated. The analysis of stresses, even though many underground structures are indeterminate, has been the subject of extensive research and has become rationalized to such an extent that full dependence may be placed on the structures to withstand their assumed loads. Design assumptions based upon anticipated earth pressures, however, still are irrational in the extreme and contain about as many unknowns as must have confronted the builders of the Egyptian pyramids.

The results of soil studies on the Chicago subway, as outlined in Mr. Peck's paper, are a valuable contribution to the engineering profession in a field that is still relatively unexplored. They indicate that, at a moderate expenditure, soil samples carefully taken and properly analyzed in the laboratory can be safely relied upon to disclose the physical characteristics of the soil. The field measurements have demonstrated that upon these test results reasonably sure assumptions as to the value of the lateral pressures against open cuts can be predicated. The results also indicate that, in tunneling through a wide variety of Chicago clays, with water contents ranging from 15% to 60%, the structures tend to deflect to the shape that will produce horizontal pressures approaching the full vertical loads. In circular tubes that are sufficiently flexible, for example, it follows that the bending moments so commonly assumed to exist are not to be found in the finished structure. The results further indicate that, in an open cut, by the proper spacing of wales and bracing and by pre-stressing the struts to prevent deflections under load, the adjacent ground movements or settlement can be minimized.

These results go further than theory. They are of practical value. They mean dollars saved in the elimination of wasteful design. They mean dollars saved in the prevention of damage from faulty construction.

The design assumptions adopted for the Chicago subway were in accordance with widely accepted practise. Although it was felt that they were ultra-conservative, they were fully justified by precedent. In fact, in the absence of exact data at the time the work was started, any departure from the current practise would have been unwarranted.

The method most commonly used is to assume that the soil is a perfect fluid having an arbitrary density. This density is usually the result of majority opinion and has no rational background. To the credit of the engineering profession, it is usually assumed so great that underground structures do not fail when so designed. As a further factor of safety, instead of adopting only one value for density, stresses are analyzed on the basis of horizontal pressures, equal, respectively, to $\frac{1}{3}$ and to $\frac{2}{3}$ of vertical pressures, and the structures in each instance are made adequate to meet the most severe of these conditions. It is apparent that these assumptions disregard cohesion and other soil properties and are independent of the depth of the work. The experimental results on the Chicago subway indicate that both the soil properties and the depth are factors that affect pressures. In general, the fluid theory may be said to be irrational and in most instances will lead to uneconomical design.

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[21] Chf. Engr., Chicago Park Dist., Chicago, Ill.

The chief inherent fault of the fluid hypothesis is that the total lateral pressure is thus assumed to increase as the square of the depth, and to have a value modified somewhat by whatever angle of internal friction is assumed. In fact, however, the test results show that total pressures at moderate depths are less than those computed on the fluid hypothesis, although at extreme depths the reverse may be true. Further, the Chicago clays are shown to act in connection with excavation work as if they have an angle of internal friction of 0°. It follows that any assumptions that neglect the significant soil properties are irrational and, although generally the accepted assumptions are on the safe side, occasions may arise wherein such assumptions would lead to unsafe designs. Mr. Peck's suggestions that the unconfined compression strength of the soil and the depth of the cut be used as factors in determining total lateral pressure, and that pressure diagrams be developed from this data, seem to offer a logical and relatively simple approach to the fixing of this very essential element of design.

Many of the other results of soil tests on the Chicago subway have proved of great value in establishing safe construction procedures. The rates of movement of the clay in tunnel headings were observed by spearheads driven into the clay in advance of the headings, and from these measurements practical cycles of operations were established. Similarly, lateral movements were determined from so-called "squeeze" tests. By close observation of the extent of the movements and the time intervals involved, practical limits for spacing of lateral braces were fixed and measures taken for pre-stressing the tunnel linings to avoid undue or unsafe ground settlement.

In open-cut excavations on the Chicago subway, extended observations were made as outlined in detail in Mr. Peck's paper. This supplied the necessary data for the determination of the safe vertical spacing of the wales and bracing, the depth of penetration of the sheeting, and the magnitude of prestressing of the struts required to prevent untoward movements of the adjacent ground and to keep settlements within practical limits.

Although much of the experimental data obtained from soil tests on the Chicago subway were collected during the progress of the work, and were thus too late to affect the design, this branch of the engineering service was of great benefit on construction procedures. It is further felt that the results will be of inestimable value for future designs both to the city departments in new public works and also to the engineering profession as a whole. These tests not only furnish more exact knowledge of the physical properties of Chicago clays than has heretofore been available, but they also point the way to a more rational basis of design and to safer construction procedures.

L. G. LENHARDT,[22] **M. AM. SOC. C. E.**—The Central Avenue Tunnel, described by Professor Housel, was designed to convey a 60-in. steel water main beneath the classification yards and main-line tracks of the Michigan Central Railroad Company and two grade separations. There were twenty-two tracks at grade to be passed under and seven tracks carried on the separations, which supported the "hump" tracks of the Michigan Central classification yards. The volume of traffic on the tracks, the character of the soil, and probable future grade separation led to the decision of carrying this important transmission main beneath these obstacles in the tunnel.

The tunnel as constructed was some 1,200 ft long between center lines of shafts. The inside diameter was 9 ft 6 in. with an average wall thickness of about 20 in. Reinforcing steel was used which consisted of 1-in. square bars on the inside top and bottom quadrants of the lining and ¾-in. round bars on the outside at the spring-line quadrants. Spacing averaged about 12-in. centers. Longitudinal steel consisted of ¾-in. round bars at 18-in. centers.

The soil encountered at tunnel depths was a typical soft Detroit blue clay which is known as "squeezing" clay. Therefore the specifications required the work to be done with compressed air. With the pressures used there was generally little subsidence of the ground above the tunnel, and slight subsidence of grade-separation abutments was recorded. No particular construction difficulties were encountered other than the usual difficulties attendant on compressed air tunneling in soft blue clays where unremitting vigilance is the price of successful construction.

Mining operations were conducted during the day with a usual progress of 12 ft to 15 ft per shift. Crown planking was not used except beneath the railroad tracks where some eight to ten planks were carried. The concrete lining was placed, on the night shift, to within about 3 ft of the tunnel face so that at no time, except over week ends, would more than 3 ft of tunnel be exposed for a period of 24 hr. Over week ends the tunnel face was well breasted.

During the progress of the tunneling, grout was forced over the concrete bulkheads every other day through grout holes spaced at a maximum of 10-ft centers in the crown with additional grout piping at the upper quarter points where they were believed necessary. A few grout holes were also placed at the spring line and at the invert as checks; but no grout could be placed except through the crown grout holes and occasionally at the upper quarter points. When the tunnel lining operations were completed grout holes were cleaned out and the tunnel grouted to refusal.

It is the writer's opinion that the grouting tended toward uniformity of soil pressures during the early months. Grouting pressures did not find their ultimate release until some time after the tunnel was constructed. This was noticeable as holes drilled through the tunnel lining after construction showed water and air under pressure. This could only have originated from the grouting as there was no water encountered in the tunneling operations. Upon first making the tap outside of the lining the water was plainly under pressure but it was quickly relieved and these holes soon dried up.

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[22] Gen. Mgr. and Chf. Engr., Dept. of Water Supply, Detroit, Mich.

The characteristic elliptical pressure observed in the later measurements was also observed in pressure cells installed in other tunnels during construction of the Springwells project.[23] Unfortunately access to these cells is no longer available due to the use of the tunnels.

The pressure cell measurements again emphasize the necessity for speed in soft ground tunneling operations. Further, the Central Avenue Tunnel is a relatively rigid structure and, as such, is essentially different from thin-walled structures, which, due to large deflections, utilize the passive resistance of the soil to maintain uniform pressures around their circumference.

Professor Housel has presented an ingenious analysis of an elastic structure in a plastic clay soil which cannot help but challenge attention. It should be stated that his analysis is based entirely on the concept of soft clay as a truly cohesive material. The fact that he closes his analyses through the various conditions of loading merits careful consideration.

George E. Shafer,[24] Assoc. M. Am. Soc. C. E.—From the standpoint of a practicing engineer as contrasted with the soil-research engineer, it is refreshing to read a paper on soil mechanics that is not filled with theoretical mathematical formulas and that can be followed intelligently by the engineer who has the practical problem of determining the probable load that will be supported by an undergound structure such as a tunnel. Professor Housel has applied theory to a 10-yr experiment and has arrived at a workable solution to an age-old problem in which the structure to be built will be in plastic soil. This is a valuable contribution to the profession, as any one will appreciate who has had the problem of determining pressures on conduits installed by the tunnel method.

True, Anson Marston, Past-President and Hon. M. Am. Soc. C. E., and his associates at the Iowa Engineering Experiment Station have conducted research over a long period of years on pressures on conduits, and the result of that monumental work[25] has made the design of certain types of conduits a science instead of guesswork; but the Iowa research was concerned with conduits placed in open trenches or on top of the ground and then covered with earth, whereas Professor Housel's work deals with structures on which the load comes not from earth placed on or against the structure by man but by readjustment of the plastic earth itself when part of its support has been removed.

Engineers who encounter the problem of tunnel pressures for the first time are surprised to learn that tunnel structures are designed more by rule of thumb than by actual design data. The reason, of course, is that until quite recently very little was known as to the magnitude of the pressure to be supported, and still less as to the distribution of that pressure around the periphery of the tunnel.

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[23] "The Springwells Filtration Plant," by Eugene A. Hardin, *Transactions*, Am. Soc. C. E., Vol. 101 (1936), Fig. 1, p. 346.

[24] Chf. Engr., Armco Culvert Mfrs. Assn., Middletown, Ohio.

[25] *Bulletin No. 96*, Eng. Experiment Station, Iowa State College, Ames, Iowa.

There are two distinct parts to the problem of pressures on tunnels (as revealed in Professor Housel's paper): One is the initial pressure, which governs the design of the temporary lining of the tunnel; and the second is the maximum final pressure that will govern the design of the permanent lining.

Some years ago when the organization with which the writer is associated entered the metal tunnel liner field, the writer was amazed to discover that there was no rational basis on which to design liner-plates and no way but costly experience for the contractor to determine how much strength his plates would require. The practice was (and still is) to place the "burden of proof" on the contractor, who was expected to use his experience on other jobs to determine what liner-plates he should buy for this particular contract. He may buy heavier plates than necessary, in which case no one will ever know how much money might have been saved if science had replaced guesswork. If the contractor is optimistic, the chances are that the plates will be too light and he is then faced with all kinds of trouble such as the danger of physical injury to workmen, putting in compressed-air apparatus, getting structural reinforcement members, or trying to get the plate manufacturer to take back a few carloads of plates and supply heavier ones overnight.

There are, of course, certain limitations to the application of the method proposed by Professor Housel. In the first place, it applies only to soils in which the shearing resistance is a function of true cohesion and independent of internal friction. In the language of the practicing engineer, this means what is generally classified as clay, and except for tunnels in bedrock probably includes a large percentage of the cases encountered in practice. Then, as stated by Professor Housel, the pressure equations do not permit a direct solution without the introduction of simplifying conditions. If air is to be used, this is one of the simplifying conditions through the introduction of equal pressures at the top, side, and bottom of the tunnel. In making use of measured shear values, it is necessary to select what the author terms an "overload ratio." Fortunately, enough experience is already available to permit the engineer to establish this ratio with reasonable success.

The investigation has brought to light one phenomenon which may puzzle many, and that is the possibility of a decreasing load on the structure for the first three months of its life. However, that is not as important to the engineer or contractor responsible for the temporary lining as is the fact that the initial load on the temporary lining does not increase for this period, which should give sufficient time for placing the permanent tunnel structure.

The results on time required to reach final equilibrium check quite closely with the results of Dean Marston's experiments, previously mentioned,[25] although naturally the final loads could not be expected to be similar.

Given the shearing values of the soil (which, of course, will have to be determined from soil samples obtained at the site for that purpose) and sufficient other information to permit assigning a reasonably accurate "overload ratio," the engineer now has a new tool for use in designing both a temporary lining for a tunnel and the permanent structure as well.

Other applications of this same principle doubtless will be discovered in connection with other types of structures subjected to loads produced by earth masses in which the condition of equilibrium has been disturbed.

Frequently, one hears the criticism that the studies of soil mechanics research men are too theoretical and abstruse. Here is one excellent example of a practical application of soil mechanics, an application that can remove a lot of guesswork and save contractors and owners sizable sums of money.

M. E. **Chamberlain**,[26] Assoc. M. Am. Soc. C. E.—During the construction of the Chicago subway, the writer as one of the consultants followed with interest and profit the soil investigations outlined in this Symposium. To state that construction methods during the building of these works were influenced, to a great degree, by the results of these investigations would be an understatement. The farsightedness of Messrs. Harrington and Burke in instituting these investigations contributed greatly to the safety of construction and economy. Careful examination of the results will make for greater safety and a considerable saving in future construction. Design of future construction, in this class of ground, may be approached with a much clearer understanding of the problems involved.

In the main, as noted, the material found north of the Chicago River is a stiff blue clay at the depths involved in the construction, and it is to the capacity of this material to arch and resist shear that the successful use of the so-called "liner-plate" or "light-plate" method may be attributed. The upper stratum of hard yellow clay must have added little except weight to the overburden since, for years, it has been broken along the full length of the work during the laying of public utilities and other structures.

It is a matter of common knowledge that the stiff blue clays found in this region have a very strong structure, as evidenced by the many thousands of feet of small circular tunnels that have been built without temporary support. Some of these tunnels have had an outside diameter of as many as 12 ft. In these works the permanent lining is extended almost to the face of the excavation at all times, thus reducing to a minimum the amount of ground open at any one time. The time that any ground is without support is about the same as that required to place a temporary lining. The settlement on these projects is usually confined to an area of about twice the width of the structure, and the amount of settlement increased in the softer clays.

To combat this settlement a method of construction known as the liner-plate method was developed, in which light steel plates were placed in contact with, and jacked hard against, the ground as soon as it was exposed. The plates were supported first on the bench and then on a needle beam placed at or near the center of the tunnel. After the predetermined length of tunnel had been excavated and the temporary lining placed, the jacks were removed as the concreting advanced. Grouting of the permanent lining was completed daily to the end of the finished concrete lining. The outstanding feature of this method was that no part of the overburden was without support at any time prior to the completion of the permanent lining.

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[26] Lt. Col., Corps of Engrs., U. S. Army, Sault Ste. Marie, Mich.

A large number of circular liner-plate tunnels have been constructed in clay and other materials in the twenty years 1922–1942 without any considerable amount of settlement, including the trunk sewer (16.5 ft in outside diameter) in the City of Detroit. In four of the Detroit tunnels there was only the slight uniform settlement of the pavement caused by the initial movement of the clay that had been found to be unavoidable in Chicago. It is interesting to note that, in Chicago also, the settlement was confined to an area of less than twice the width of the structure.

The use of the liner-plates became a very popular method of construction until the desire for long traveler forms for concrete, greater daily progress, and the application of plates to large-diameter tunnels caused the introduction of internal supporting ribs. The introduction of ribs, and the elimination of the complete 24-hr cycle of operation from the start of excavation to the completion of the permanent lining, changed the liner-plate method back to the old heading and bench method.

For a number of years the use of steel ribs with steel-plate lagging has been standard in Chicago. Many sewer tunnel sections of the Sanitary District have been built with complete success. Since the ground conditions in the northerly sections of the subway work varied but little from those which prevailed in a greater part of the Sanitary District, no sufficient reason could be advanced for not using a similar type of structure. The fact that the cross section of the subway tunnels was greater than the cross section of those previously built did present many problems, as the works were to be built in a thickly populated part of the city, and settlement had to be reduced to a minimum.

It was recognized that the use of any method of construction involving open heading and bench excavation depends in great measure on the capacity of the ground to arch and to support itself for at least a period of time necessary to the placement of the temporary lining. As no records of actual measurements of the magnitude or direction of the loads to be handled by the temporary lining were available, it was necessary to depend on the results observed in work previously constructed.

Work was started on contract S-5 (Fig. 1) in what was considered good clay. Steel ribs with liner-plates bolted to them, but set away from the clay to provide space for packing and grouting, were used. It became apparent in the early stages of this work that the ground was not being held during the several operations essential to the completion of the full rib. This fact contributed to excessive settlement and caused the development of weight that on several occasions resulted in severe distress of the ribs.

The aforementioned situation offered a fertile field for investigation, and actual measurements of the loads being developed were made. Careful study of these data made it possible to improve methods immediately, and on contract S-6, in clay of the same stiffness, it was possible to reduce settlement materially. The principal changes that were made on contract S-6 were the development of a plate that could be blocked or wedged to contact with the clay when placed, and the use of wall plates at the spring line. Of all the improvements, the most important was the use of the wall plates, as it was possible by means of jacks to place the full arch in bearing and hold it until the posts could be placed and wedged to the foot blocks. These changes became standard for the remaining work.

Many of the difficulties that now seem inherent in work of this character can be eliminated, or greatly reduced, by careful study of data such as those contained in this Symposium. It is to be hoped that similar investigations may be made in other classes of ground.

South of the Chicago River, after a careful examination of available data having to do with the subsurface conditions, and after evaluation of the possible difficulties due to heavy loading of the large structures along the line of the work, it was decided that the clay was too soft and plastic to permit the use of the method of construction that had been started on the northerly contracts. Therefore, the shield method was adopted, and it proved very satisfactory.

The clay in both the sections driven with shields was stiff enough to permit line and grade to be maintained with the jacks and a judicious use of taper rings. This fact made it possible to control the grade of the street by the amount and speed of the flow of the plastic clay through the pockets of the shield. The surface of the street was under continuous observation of level parties and, when the rise or fall of the street exceeded predetermined amounts, the heading was notified and corrections were made in the openings of the shield. At a few points it was found that the clay was so soft that the surface would not respond to pressure, indicating that the use of any but the shield method would have allowed complete collapse.

It is interesting to note that, for almost the full length of the shield-driven tunnel, the volume of pea gravel used to fill the void caused by the tail of the shield exceeded the theoretical void. This would indicate that the action of the shield caused rupture of the clay even in these soft areas.

There is one conclusion drawn by Professor Terzaghi from the investigations on this work with which the writer is not entirely in accord. In the matter of future design of tunnel structures of the magnitude of the ones shown in Fig. 15, he shows a thin-walled structure, the design of which he states is based, in part, on the strength of the clay through which the tunnel is driven. If it were possible to state that the ground through which any considerable length of tunnel is to be driven will be uniform, and that in the future there will be no disturbance of the ground adjacent to the structure, the writer would be willing to explore the possibility of such a change with more enthusiasm. Consideration must also be given to the fact that concrete and other lining material may not measure up to laboratory standards. The personal equation which makes for perfection in workmanship must also be kept in mind at all times.

A. A. Eremin,[27] **Assoc. M. Am. Soc. C. E.**—In the computation of the soil pressures on the tunnel lining, Professor Terzaghi correctly has stated (see heading "I.—Clay Pressure on Temporary and on Permanent Tunnel Support: Clay Pressure on the Finished Tubes") that "It would be idle to attempt to answer this question on purely theoretical grounds." However, any one making an experimental study of the various factors affecting the distribution of the tunnel pressures cannot attain a reliable result without a rational basic theory. Furthermore, in order that the method of computation may be applied to the various cases of tunnels and tunnel loadings, the assumptions must have the least possible number of limiting conditions.

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[27] Associate Bridge Engr., Bridge Dept., Div. of Highways, State Dept. of Public Works, Sacramento, Calif.

In Fig. 3(*b*) Professor Terzaghi assumes that the width of the soil prism that produces a load on the tunnel lining is equal to the sum of the tunnel width and twice the height of the tunnel walls. Generally, the height of the tunnel walls varies. In highway tunnels, the entire tunnel section often is formed by the arched section. Evidently, in tunnels with low walls, Professor Terzaghi's method of constructing the soil prism will not represent the characteristic form for the pressures on a tunnel lining. In Eq. 2 Professor Terzaghi assumes that the vertical load carried by the soil at the back of tunnel walls is equal to $2\, q_r\, H_1$. It is to be noted that this is true only in the case of failure or when the tunnel footings are yielding. Obviously, the footing pressures shown in Fig. 9 at the initial step of failure will be much greater than that shown by Professor Terzaghi.

In his paper Professor Housel has developed the "circuit method" for computing the pressures in the tunnel lining, based on various approximations. He states (see heading "Analysis of Earth-Pressure Measurements") that "The validity of such approximations is difficult to prove or disprove except by the final test of whether or not they reproduce the measured pressures." True, it is difficult to prove the relation between Eq. 32 and Eq. 34*a*. In Eq. 32 the shear factor expressing the horizontal pressure on the tunnel lining as transformed from the vertical pressures in the earth prism is $4\, S$. In Eq. 34*a*, on the other hand, the shear factor expressing the upward vertical pressure on the invert arch in the tunnel lining as transformed also from the vertical pressures in the earth prism (or with respect to a change in direction of stresses by an angle of 180°) is only $5\, S$.

Eq. 34*b* is developed for use in checking the pressures on the invert arch of the tunnel lining. Since friction on the sides of the circular tunnel lining is negligible, Eq. 34*b* could be considered as an approximate equation for the computation of the pressures on the inverted arch, whereas Eq. 34*a* is more suitable for checking the assumed shear factors.

A. E. CUMMINGS,[28] M. AM. SOC. C. E.—The papers included in this Symposium contain some of the most worth-while information that has ever been published about the behavior of soft plastic clay. In addition, the correlations between laboratory tests and field measurements offer an excellent opportunity for comparing two fundamentally different methods of laboratory test procedure.

On the Chicago subway work, the shearing resistance of the clay was determined by means of unconfined compression tests. Soil samples for this test usually are obtained in thin-walled seamless steel tubes. In the unconfined compression test, a small cylinder of clay is placed in the testing machine and tested to failure just as a concrete cylinder is tested for its ultimate compressive strength. In some cases, the Chicago test specimens failed on a shear plane passing diagonally across the specimen. In most cases, the specimens did not fail completely but simply bulged like a barrel. Whenever bulging of this kind occurred, loading was continued until the height of the specimen was reduced by 20% of its original height. The shearing resistance was defined as one half the load at which the specimen failed completely or as one half the load that produced 20% strain in those specimens that bulged without failing.

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[28] Research Engr., Raymond Concrete Pile Co., New York, N. Y.

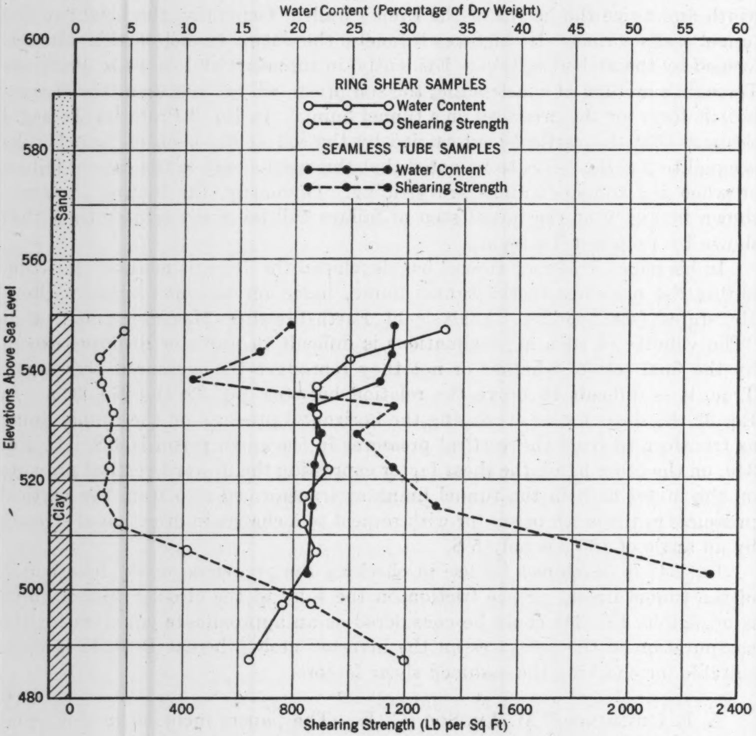

FIG. 48

On the Detroit tunnel work, the shearing resistance of the soil was determined by means of the ring-shear test. Soil samples for this type of test are obtained in the field in sectional brass tubes which are designed to fit the laboratory testing apparatus. The tube is about 7 in. long and is cut crosswise into three sections. The two end sections are each about 3 in. long and the center section is about 1 in. long. The tube containing a soil sample is placed in a horizontal position in the testing machine and the two end sections are clamped in place. Load is applied to the center section so that the soil sample is tested like a rivet in double shear. This type of test has shown that deformation under a given load practically stops in a few minutes when the load is very small. As the load is increased, deformation finally becomes progressive and the soil continues to yield without any increase in load. The test results are plotted in such a manner that the curves indicate the shearing resistance at which progressive yielding of the soil begins.

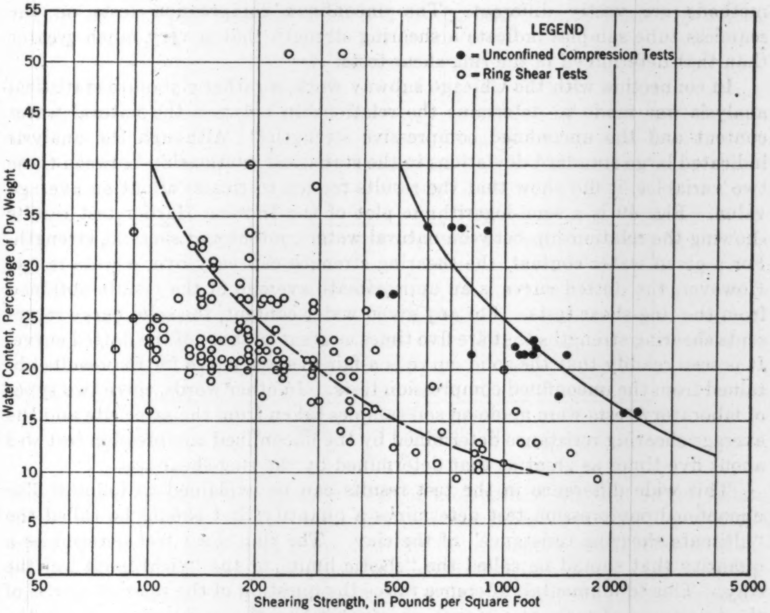

FIG. 49

It is seen readily that these two types of tests are fundamentally different so that it would be a matter of considerable interest to determine what relationship might exist between the shearing resistances determined by the two methods. It has been the writer's privilege to work with all three of the Symposium authors for many years, but he is familiar with only one case in which both types of test were made on soil samples secured from the same site. This project was that near Indiana Harbor, Ind., on the south coast of Lake Michigan. On this job a total of 120 samples were secured for ring-shear tests and 17 samples for unconfined compression tests. The ring-shear samples were taken in the standard type of apparatus used by Professor Housel and the tests were performed in his laboratory. The samples for the unconfined compression tests were taken in seamless tubes by the same methods that were used on the Chicago subway and these samples were tested by Mr. Peck in his laboratory.

The soil conditions over the entire site are quite uniform. A bed of sand about 40 ft thick overlies a bed of plastic clay 60 to 70 ft thick. The upper part of the clay bed is quite soft but the clay becomes increasingly hard as the depth increases. Fig. 48 is a composite diagram showing the results of tests on samples taken from two borings which were about 75 ft apart. As far as natural water content is concerned, the two sets of samples differed very little. On the other hand, the shearing strengths determined by the two different test methods are vastly different. The unconfined compression tests on the seamless tube samples indicate a shearing strength that is very much greater than that determined in the ring-shear tests.

In connection with the Chicago subway work, a rather elaborate statistical analysis was made to determine the relationship between the natural water content and the unconfined compressive strength.[29] Although the analysis indicated large standard deviations in the statistical relationship between these two variables, it did show that the results tended to cluster about an average value. Fig. 49 is a semi-logarithmic plot of the Indiana Harbor test results showing the relationship between natural water content and shearing strength. For a given water content, the shearing strength can vary over a wide range. However, the dotted curve is an approximate average of the results obtained from the ring-shear tests. For any given water content, the solid curve represents shearing strengths that are five times as great as those of the dotted curve. It is seen readily that the solid curve is a fair average curve for the results obtained from the unconfined compression tests. In other words, these two types of laboratory tests were made on soil samples taken from the same site and the average shearing resistance determined by the unconfined compression test was about five times as great as that determined by the ring-shear test.

This wide difference in the test results can be explained as follows: The unconfined compression test determines a quantity that should be called the "ultimate shearing resistance" of the clay. The ring-shear test determines a quantity that should be called the "elastic limit" or the "yield point" of the clay. This fundamental difference raises the question of the relative merits of the two types of tests as a means of investigating the problems to which the test results are to be applied. In tunnel work and in large open cuts such as are described in the Symposium, there is always some flow in the clay bed. In other words, the stresses in the clay approach the ultimate strength of the clay so that movements occur which are often of considerable magnitude. If the ultimate strength of the clay has been determined in the laboratory by unconfined compression tests, the designer will know at what stress he can expect the clay to fail completely. If the laboratory determines the elastic limit or the yield point of the clay by means of the ring-shear test, the designer will be forced to use some kind of correction factor such as the "overload ratio" described by Professor Housel.

Consideration of the significance of the overload ratio leads to the conclusion that it must depend on several factors which can vary over a wide range. In the first place, the numerical value of the overload ratio must depend on the ratio between the stress at which the clay fails and the stress at which it starts to flow. In the second place, the numerical value of the overload ratio must depend on the effects of progressive failure in the soil mass in the field.

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[29] *Proceedings*, Purdue Conference on Soil Mechanics and Its Applications, 1940, p. 149.

As to the ratio between failure stress and yield stress, this can be practically unity but it might be as great as 4 or 5 if not greater. For example, Fig. 50 shows the results of several unconfined compression tests on undisturbed samples of the very fine grained clay that underlies Mexico City, Mexico. This soil had a natural water content of about 400%; a plastic limit of about 125%; and a liquid limit of about 500%. As can be seen from Fig. 50, the stress-strain curves are practically straight lines all the way out to the failure load. In spite of its plastic properties, this clay behaves almost like a brittle material. There is no yield point until the failure load is reached, so that the ratio between failure stress and yield stress is practically unity. On the other hand, Fig. 49 indicates that for the Indiana Harbor clay, the failure stress determined by the unconfined compression test is about five times the yield stress determined by the ring-shear test. It is entirely reasonable to expect that the ratio between failure stress and yield stress might be even greater than 5 in some cases. Accordingly, a designer, basing earthwork computations on the yield value determined in the ring-shear test, would have a difficult problem in choosing a proper value for the overload ratio. The tendency, almost certainly, would be to stay on the conservative side.

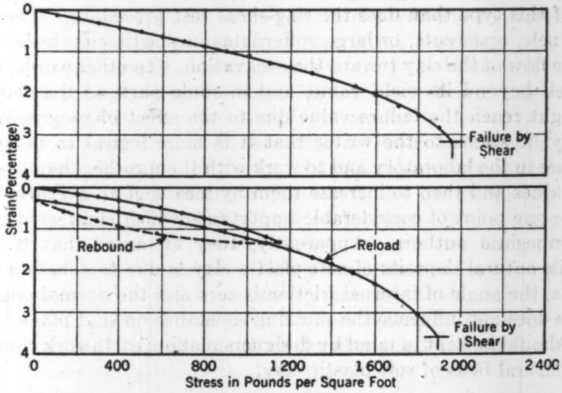

FIG. 50

The effect of progressive failure in the soil mass in the field can also vary over a considerable range. The term "progressive failure" refers to the fact that the soil does not necessarily fail everywhere at the same time. In some cases, progressive failure occurs because the stresses are much higher in some parts of the clay bed than they are in others. For example, if a square bearing block is used to make a load test on plastic clay, the soil will fail at the corners and edges of the plate before it fails under the center of the plate. In other cases, progressive failure occurs because the shearing strength of the clay varies in different parts of the clay bed. In such a case, the weaker layers might fail first, leaving the stronger layers to carry more than their share of the load. It is very probable that progressive failure is often caused by a combination of these two factors. In any case, the effect of progressive failure would have to be taken into account in the choice of a numerical value for the overload ratio. Here again, the tendency on the part of the designer would be to stay as far as possible on the conservative side because he is dealing with failure stresses whereas the laboratory tests have determined the yield stresses.

From considerations of this kind, the writer has come to the conclusion that the unconfined compression test procedure represents a more direct approach to problems of this type than does the ring-shear test procedure. The construction of tunnels, open cuts, or large cofferdams in plastic clay beds always involves some flow of the clay toward the excavation. In other words, the clay is stressed well beyond its yield value, and in some parts of the clay bed the stresses might reach the failure value due to the effect of progressive failure. Accordingly, it seems to the writer that it is more logical to determine the failure values in the laboratory and to work with them rather than to determine the yield values and then to increase them by means of an overload ratio.

There is one point of considerable importance which is referred to by each of the Symposium authors. Apparently, they all agree that the shearing resistance in natural deposits of soft plastic clay is due to cohesion only. In other words, the angle of internal friction is zero and the normal pressure on a given plane does not influence the shearing resistance on that plane. This is a point that should be kept in mind by designers making earthwork computations involving natural beds of soft plastic clay.

D. M. BURMISTER,[30] ASSOC. M. AM. SOC. C. E.—This Symposium is an outstanding contribution to the knowledge of the forces and physical factors involved in tunnel and open-cut construction in plastic clays. The authors are to be congratulated for the presentation of this valuable information to the profession. The importance of their investigation is threefold: (1) A rational scientific approach is offered for the investigation and the practical solution of such problems; (2) usable criteria are suggested for safe and economical design and construction; and (3) control criteria and procedures to be used during construction are developed to meet and anticipate changes in subsurface conditions for the worse and to minimize settlement due to tunneling and open-cut operations.

The writer was impressed greatly by the comprehensive program of soil investigations conducted on a time schedule that permitted the use of the information for construction purposes. This was obviously not a research problem, but a practical problem of soil mechanics on a mass-production basis. The construction experience itself is evidence of the thoroughness and the efficiency with which this program was planned. It is interesting to note that reliance was placed primarily on the results of two simple soil tests—moisture content and unconfined compression—which had a direct bearing on the problem in hand. The method of presenting and interpreting the data is especially noteworthy because of its value in clearly visualizing subsurface conditions and in indicating the nature of the difficulties involved.

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[30] Asst. Prof., Civ. Eng., Columbia Univ., New York, N. Y.

Professor Terzaghi has made a number of statements of experience in connection with the construction of the Chicago subway which are of great importance. First, he states that when dealing with irregularly stratified water-laid deposits, and also the ice-laid deposits of the Chicago region, the average soil properties do not change, in a horizontal direction, nearly as erratically as the individual records would lead one to expect. Thus, considerable reliance can be placed on the data regarding subsurface conditions obtained from quite widely spaced borings, as commonly used. The greatest uncertainty lies in the possibility of missing some abrupt change for the worse in the thickness of the deposit or in stratification. Analyses of foundation conditions for a number of important projects, covering quite large areas, by the writer have borne out similar conclusions, including the consolidation properties of clays and critical density-pressure relations for deposits of fine sands.

In a second conclusion of great consequence, Professor Terzaghi states that, on account of the mechanical properties of undisturbed clays, a very small yield suffices to develop all the strength the clay has. In discussing the lateral pressures in the Chicago clays, Mr. Peck states that a lateral yield of slightly less than 0.25% of the height of the cut is all that is necessary to reduce the earth pressure to the active value. This is in contrast to the opinion commonly held that opportunity for lateral expansion sufficient to develop the major part of the shearing strength of soft clays is never realized. Obviously, the yield required will depend on the strength of the clay itself and on the stress-strain properties. The zone in which the lateral yield must take place may be considerably more restricted or limited in extent than formerly believed. In the closing discussions, it would be of great value to include relations, if consistent ones were found, between the strain at the maximum unconfined compressive strength and the measured distortion in the tunnel section, the "squeeze" into the tunnel, and the lateral yield in the open-cut section.

A third conclusion of great importance is that the medium soft clay acts as if the angle of friction were equal to zero and that the shearing strength did not increase with the depth of overburden. Hence, the unconfined compressive strength was considered to be more indicative of the shearing strength of the clay. Professor Terzaghi states that he has not seen any evidence of the existence of a direct relation between shearing strength and depth of overburden in soft clay deposits. This same phenomenon exists in the softer Detroit clay to an even more marked extent, as discussed by Professor Housel and as shown in Fig. 39. The writer has discussed a similar experience[31] in connection with the investigations of the subsurface conditions of the very soft deposit of organic silty clay at the Flushing Meadow Park site for the New York (N. Y.) World's Fair. The moisture contents, shearing strengths, and preconsolidation pressures were constant throughout almost the full depth of the deposit, some 40 to 60 ft in thickness below the surface root mat, whereas in a natural state of complete consolidation one would expect that the water content would decrease and the preconsolidation pressure, and hence the shearing strength, would increase almost directly with the depth in the deposit. Such a radical departure may be the normal natural state of consolidation under very slow geologic loading; or it may be explained on the basis of the presence of artesian water pressure in the sand strata beneath the deposit. Such a condition, with an infinitesimal flow upward, would induce permanent hydrostatic stresses in the pore water, which would limit consolidation so that a certain proportion of the overburden pressure at every depth would be carried permanently by pore water pressure. However, the shearing strength, in any case, was found to be substantially directly proportional to the preconsolidation pressure evaluated from consolidation tests, whether it is less than, equal to, or greater than the existing overburden pressures, which vary with the geologic load history of the deposit. This preconsolidation pressure represents the effective stress acting on the grain structure of the clay. In a closed system (which is always the case in a large mass of relatively impermeable clay), any increase in stresses from any cause above the preconsolidation pressure is carried entirely by excess hydrostatic stresses in the pore water. This is a consequence of the characteristic phenomena in clays—a tendency toward a decrease in volume under shearing stress—which always accompanies shearing deformations. Hence, at constant volume shearing, the clay behaves as though the angle of friction equaled zero, and only the shearing strength due to the pre-existing effective stresses can be mobilized. Because of this phenomenon, the unconfined compression test on reasonably undisturbed samples yields the most representative and reliable measure of the natural shearing strength of the clay, as stated by Professor Terzaghi. The writer prefers to think of the maximum shearing stress for such a condition as being determined by one half the difference of the principal stresses—$S_{max} = \frac{1}{2}(p_1 - p_3)$. Since the applied lateral pressure p_3 is zero in the unconfined compression test, the maximum shearing strength is equal to one half of the maximum applied vertical stress, which is the compressive strength of the clay. In obtaining representative values from unconfined compression tests, it has been the writer's experience that the shearing strength obtained by testing a specimen having the full diameter of the undisturbed sample may be only 75% of that obtained from a prism carefully cut from the central part of the sample with a maximum width, about one half of the diameter of the undisturbed sample. Also, it has been his experience that direct quick shear tests yield values which may be 90% or less than those obtained from unconfined compression tests on specimens cut from an immediately adjacent part of the sample. This is believed to be a consequence of the progressive nature of the failure conditions inherent in the direct shear test method. Also, this may account for the fact noted by Professor Housel that, for a perfect check between measured and estimated pressures on the tunnel, the shearing resistance should be 12% greater than the yield value that was found experimentally by ring-shear laboratory tests.

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[31] "Laboratory Investigations of Soils at Flushing Meadow Park," by Donald M. Burmister, *Transactions*, Am. Soc. C. E., Vol. 107 (1942), p. 187.

The outstanding achievement of this Symposium is the development of rational methods of analysis, design, and control of construction procedures for tunnel linings and sheeting and bracing in open cuts in plastic clay, which are in such close agreement with actual measurements. The most important practical aspects for future construction are the measurements of: (*a*) The distortions and pressures in the tunnel lining in the Chicago clays and the settlement of street surfaces; (*b*) the magnitude and distribution of pressures in open cuts and the yield in the bracing system permissible to limit adjacent surface settlement; and (*c*) the distribution of pressures in the tunnel lining in the Detroit clay, the variation with time, and the time required for equilibrium condition to be attained.

In order to make these measurements of greater practical value for future construction, the properties of the Chicago and Detroit clays should be identified more positively. It is suggested that diagrams showing the range of soil properties and conditions should be included, if fairly consistent relations exist, for example: (1) Maximum unconfined compressive strength or shearing strength versus moisture content, relative density—$D\,d = \frac{W_L - W_N}{W_L - W_P}\,100\%$, and corresponding strain at maximum strength; (2) maximum shearing strength and corresponding strain versus the measured distortion in the tunnel lining, the squeeze into the tunnel, the pressure on the tunnel lining, and the ratio between the vertical and horizontal pressures observed; and (3) maximum shearing strength and corresponding strains versus the measured pressures and yield in the bracing system in open cuts.

GREGORY P. TSCHEBOTARIOFF,[32] M. AM. SOC. C. E.—The studies described in this Symposium mark an important milestone in the development of applied soil mechanics. Well-organized, full-scale observations were made during and after construction in combination with extensive laboratory and field tests. This research made possible a scientific comparison of the observed facts with those which should be expected if existing theoretical conceptions were valid. Only continued studies of this kind on future structures in many locations promise to give foundation and excavation engineers a real knowledge of the actual, instead of the assumed, factors of safety provided by their designs.

In this discussion, the writer mainly wishes to indicate some generalizations made by the authors of the papers of this Symposium which do not appear to be justified entirely in the light of experience elsewhere. In so doing, the writer does not wish to detract, in any way, from the importance of the work performed. On the contrary, his conclusion will be that more studies of this kind are essential.

The papers of this Symposium are discussed in the order of their presentation. The studies on the Chicago subway described by Professor Terzaghi and Mr. Peck undoubtedly have contributed greatly to the rationalization of current construction procedures and future designs in that region.

In addition, conclusions of general interest were reached. One such conclusion is: "No evidence has been found that the angle of shearing resistance of the clay is appreciably in excess of zero degrees at any point" (see heading, "II. Soil Investigations: Soil Testing"). In this connection, Professor Terzaghi stated that

> "* * * he has found repeatedly * * * that slopes on soft, undisturbed clays fail if the average shearing stress on the potential surface of sliding

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[32] Asst. Prof., Civ. Eng., Princeton Univ., Princeton, N. J.

> becomes roughly equal to one half of the average nonconfined compressive strength of fairly undisturbed samples regardless of the depth of the overburden.[6] On the other hand, no one has ever published any field data that would demonstrate adequately the existence of a pressure-conditioned shearing resistance in soft clay deposits in other localities."

The foregoing generalizations are not justified entirely. Because of such statements, some engineers believe that the shearing strength of all soft clay deposits never is pressure conditioned and that the component of possible shearing resistance caused by the intensity of normal pressure is to be neglected entirely in all cases. Such a concept appears to be quite unwarranted and should be avoided.

Data published contemporaneously with the Symposium provide material for contrary conclusions. For example, L. F. Cooling and H. Q. Golder, of the British Building Research Station, have investigated the failure of an earth dam during construction[33] at Chingford, Essex, England. The dam was underlain by a natural layer of undisturbed soft clay. In their paper, it is stated:

> "Theoretically, if a load is applied to a clay sample, the shear strength will increase as the sample consolidates under the applied load. * * * Evidence confirming these theories was obtained. The mean shear strength of the clay which had been under 26 feet of bank was 5.2 lbs. per square inch, whilst under 11 feet of bank it was only 3.7 lbs. per square inch. The mean moisture content under 26 feet of bank was 80.0 per cent; under 11 feet of bank, it was 87.8 per cent. * * * At the time of the slip, most of the bank material had been in position only about 37 days, and the mean shear strength of the clay was 2.0 lbs. per square inch. The later samples were taken after 8 months, and the mean shear strength under 26 feet of bank was 5.2 lbs. per square inch. That is an increase of 3.2 lbs. per square inch under an applied pressure of 1.34 ton per square inch."

Kenneth E. Fields and William L. Wells,[34] Associate Members, Am. Soc. C. E., have described a similar case. A levee was built over a soft clay layer in a manner calculated to produce failure. Pore pressures were measured continuously in the clay. Laboratory tests on the clay were followed by an analysis of the induced failure after it occurred. The results indicate definitely that considerable frictional resistance must have been present if pore pressures were taken into account and that the only question is—what was the exact relative value of this frictional resistance.

At first glance, the results of the failure studies at Chingford and at Pendleton would appear to be in complete contradiction to the conclusions drawn from the Chicago observations by Professor Terzaghi. A closer examination, however, will show that they may place only certain limitations on some of the generalized statements made by Professor Terzaghi (see heading, "II. Soil Investigations: Soil Testing") and that these limitations do not apply necessarily to soil and loading conditions similar to the ones encountered during the work on the Chicago subways.

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[33] "The Analysis of the Failure of an Earth Dam during Construction," by Leonard Frank Cooling and Hugh Quentin Golder, *Journal*, Institution of Civ. Engrs., London, November, 1942, pp. 51–53.

[34] "Pendleton Levee Failure," by Kenneth E. Fields and William L. Wells, *Proceedings*, Am. Soc. C. E., December, 1942, p. 1763.

Some of the curves giving the change of water content of the Chicago clay with depth (for instance, Figs. 21(*c*) and 22(*c*)) show that there is scarcely any decrease of water content with depth. Similar conditions were observed in other localities, and this phenomenon has been explained tentatively by Professor Terzaghi himself.[35] It was attributed to the peculiar structure of certain very slowly deposited natural soft clays. This structure was created by the gradual building up of the bond strength between the clay particles accompanied by a gradual transfer of stress from the adsorbed water films to the solids. Apparently, no consolidation took place after a certain point, since the weight of the gradually deposited overburden increased at so slow a rate that the resultant increase of bond strength could keep pace with it.

Assuming that the Chicago clay deposits have been formed in a manner corresponding to the foregoing hypothesis, the results of the Chicago subway observations do not appear surprising. The excavation in open cuts probably was not accompanied by a disturbance of the adjoining clay sufficient both to produce any further appreciable consolidation during the period of measurements and to increase its shearing strength. The vertical pressures in the clay mass remained unchanged at all times; and the lateral pressure decreased only slightly as a result of the yielding of the piles.

The Chingford and the Pendleton conditions were entirely different. Considerable additional vertical surcharge was applied to the clay at a relatively rapid rate. Consolidation naturally resulted and was accompanied by an increase of shearing strength. It follows, therefore, that the shearing strength of soft clays can be pressure conditioned if an additional considerable load is applied rapidly. Therefore, the foregoing generalized statement (see heading, "II. Soil Investigations: Soil Testing") should be limited accordingly.

A second limitation of this generalized statement should be made, since it does not appear to apply to all types of soft clays. For instance, varved clays apparently can have appreciable frictional resistance, although varved clays belong to the category of soft clays. The writer had the opportunity to investigate a slide in a riverbank through a bed of soft varved clay. Making allowance for all the uncertain factors in that investigation, it, nevertheless, could be stated very definitely that cohesive forces alone, as determined by the unconfined compressive strength tests, could not possibly have been responsible both for the stability of the bank during previous years and for the stability of all other similarly located sections which had not failed. The shearing strength necessary to provide such stability was from 50% to 100% higher than the shearing strength due to cohesion. The failure immediately followed the driving of a few piles in the vicinity. A tentative explanation is that the thin layers of silt-like rock flour in varved clays provided frictional resistance, but were at the same time largely responsible for its sudden decrease as a result of vibrations.

A third point that the writer would like to emphasize is related to the assumption that Chicago clay had no frictional resistance at all. This appears possible, and even likely, but cannot be considered proved definitely. Some diagrams (for instance, Fig. 20(*c*)) indicate an increase of the water content with depth, showing that the structure of the clay was not always of the type outlined in the previous discussion. Furthermore, the zero value for the angle of frictional resistance of the clay was arrived at after a certain shearing strength was assigned to the overlying sand. The assumptions concerning the shearing resistance of this sand layer do not appear to be entirely convincing. Since some movement of the upper part of the sheet piling was observed prior to the installation of the first row of struts, such a movement easily might have reduced strongly the horizontal pressure component in the sand layer, so that the shearing strength assigned to it may have been too high. This, in turn, would indicate that the actual shearing strength of the clay was higher than the one assumed, and that some frictional resistance was present. So far as future designs in Chicago are concerned, this point is of little practical importance. The studies there have established the actual values of earth pressures which can be used for further designs in that region. However, the point emphasized by the writer should be considered in relation to designs in other localities where no such sand layers are present above the clay. Further studies of this type are extremely important in order to clarify the question concerning the amount of frictional resistance which different types of clay may have under varying conditions.

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[35] "Undisturbed Clay Samples and Undisturbed Clays," by Karl Terzaghi, *Journal*, Boston Soc. of Civ. Engrs., July, 1941, pp. 215, 219–222, and 229.

The results of Professor Housel's observations on the Detroit sewer tunnels are of considerable interest, especially since the studies have been conducted over a period of years. The results appear to indicate a change of pressure with time, interpreted as signifying plastic deformations which gradually increased the pressures on the tunnel lining. However, the diagrams of the paper show considerable scattering of the readings on the Goldbeck cells. It would be interesting to learn, from the authors' closing discussion, the manner in which the readings were made. Professor Terzaghi[36] has expressed some doubt as to the efficiency of the Goldbeck cells when measuring pressures in clay, because the soil pressure registered is a function of the rate of application of the air pressure on the cell membrane. Furthermore, it appears conceivable, although not very likely, that the readings increased with time as a result of a progressive stiffening and increased resistance of the clay to displacement by the cell membrane, whereas the static pressure of the clay actually remained unchanged. Professor Housel made his observations with the most efficient equipment available at the time. The results are of considerable practical importance and should be given due consideration. However, it is to be hoped that similar measurements will be repeated with the more modern and precise electric resistivity pressure cells now available.

In conclusion, the writer wishes to compliment the authors of the three papers on their outstanding contribution to engineering knowledge in the field of applied soil mechanics.

D. P. Krynine,[37] M. Am. Soc. C. E.—Each of the four parts of this Symposium, namely (*A*) Foreword (nomenclature and notations), (*B*) "Liner-Plate Tunnels in the Chicago (Ill.) Subway," by Karl Terzaghi, (*C*) "Earth

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[36] "Measurements of Pore Water Pressure in Silt and Clay," by Karl Terzaghi, *Civil Engineering*, January, 1943, p. 34.

[37] Research Associate in Soil Mechanics, Dept. of Civ. Eng., Yale Univ., New Haven, Conn.

Pressure Measurements in Open Cuts, Chicago (Ill.) Subway," by Ralph B. Peck, and (*D*) "Earth Pressure on Tunnels," by W. S. Housel, respectively, will be discussed separately.

(*A*) *Nomenclature and Notations.*—According to the "Foreword," the symbol ϕ stands for the angle of internal shearing resistance of clay, ϕ_1 being the angle of shearing resistance in clay. The difference between these two terms is not clear. In Manual 22,[38] the term "angle of shearing resistance" does not appear at all. It follows that this term was introduced sometime in 1941 or 1942, and it would be helpful to have it explained. Professor Terzaghi uses the term in question, whereas, in interpreting the symbol ϕ, Professor Peck uses the expression "effective angle of internal friction of the clay" in presenting Eq. 6. Moreover, when dealing with the symbol in Eq. 13, Professor Peck defines it as the "angle of internal friction of the sand."

The writer has no intention of defending the term "angle of internal friction" as applied to plastic clays since he feels that, in this case, the term in question is incorrect; but he also believes that there is no justification for the new term "angle of shearing resistance" since the conception of the term "angle," which is rather tangible in the case of cohesionless and cohesive sands and similar materials, has nothing to do with the shearing resistance of plastic clays.

(*B*) *Paper by Professor Terzaghi.*—In the paper by Professor Terzaghi, formulas that control stability of the lining are given in Eqs. 1*b* and 2*b*, derived for the case of a blowout. In general, equations of the type of Eq. 1*b*, in which the left side represents the total bearing capacity of the structure and there is only one member on the right side, are unquestionably correct. This is not the case with equations of the type of Eq. 2*b*, in which the right side represents the sum of bearing capacities of two or more supporting agencies. In a general case, the bearing capacity of the entire structure may be less than the sum of the bearing capacities of the supporting agencies represented at the right side of the equation. To make this statement clear, assume a rigid beam on two supports (Fig. 51), the ultimate bearing capacity of the supports being 10 and 100 units, respectively. As the beam is gradually loaded, the entire structure possibly will fail when the load on the beam is more than 20 units—that is, far less than the value of 10 + 100 = 110 units. Both supports will fail simultaneously, however, if the increasing load is distributed between the supports in the ratio 10 : 100.

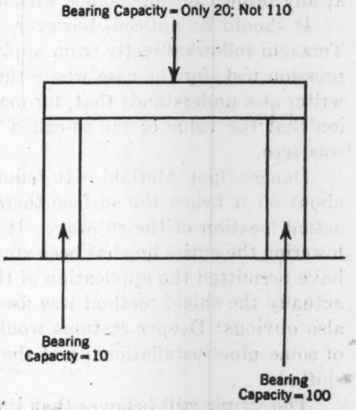

Fig. 51.—The Ultimate Bearing Capacity of a Structure May or May Not Be the Sum of the Ultimate Bearing Capacities of Its Supports

Before using equations of the type of Eq. 2*b*, their validity must be proved in each particular case. The lining of the Chicago subway was successful because the following system was satisfied:

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[38] "Soil Mechanics Nomenclature," *Manual of Engineering Practice No. 22*, Am. Soc. C. E., 1941.

$$W - S = R_1 + R_2 \dots\dots\dots\dots\dots\dots\dots (38)$$

in which $R_1 \gtreqless 2\, q_r\, H_1$ and $R_2 \gtreqless A\, q_D$. Loads R_1 and R_2 are carried by the clay at the back of the tunnel and by the footings, respectively.

Clay Pressure on the Finished Tubes.—The difference between thick-walled and thin-walled concrete tubes is discussed by Professor Terzaghi under the heading, "Clay Pressure on Temporary and on Permanent Tunnel Support: Clay Pressure on the Finished Tubes"; and he makes the statement "that a thin concrete shell with a nominal reinforcement should suffice permanently for withstanding the clay pressure." The advantage offered by thin-walled tubes has been known for many years in the theory of pipe culverts. A deflected flexible pipe approaches the pressure line corresponding to the gradually increasing lateral pressure[39] just as shown in Fig. 4.

Soil Testing.—The statement made by Professor Terzaghi under the heading, "Soil Investigations: Soil Testing," really deserves the attention of the profession, especially of those engineers who are working on foundations and earth structures. Slopes on soft, undisturbed clays fail if the average shearing stress on the surface of sliding becomes roughly equal to one half of the average nonconfined compressive strength of fairly undisturbed samples, regardless of the depth of the overburden. Commendably, Professor Terzaghi has applied this statement to tunnels and cuts. The procedure used in Chicago for determining the shearing resistance is simple and efficient, and can be duplicated at any project of importance without great effort and expense.

It should be noticed, however, that the statement advanced by Professor Terzaghi follows directly from applying Mohr's circle to the nonconfined compression test, for the case where the "angle of internal friction," $\phi = 0$. The writer also understands that, for many years, Professor Housel was of the opinion that the value of the so-called "angle of internal friction" in plastic clays was zero.

Danger-Spot Method.—It follows from Figs. 8 and 9 that at a depth of about 55 ft below the surface there is considerably stronger clay than at the actual location of the subway. It is not clear whether or not the problem of lowering the entire line has been studied in some detail. Such a lowering would have permitted the application of the liner-plate method in many places where actually the shield method was used. The disadvantages of this lowering are also obvious: Deeper stations would increase maintenance costs and the cost of some pipe installations; and the entrances to the tunnel perhaps would be shifted.

The writer still believes that the proper way to use Eq. 3 is to replace the values of $2\, q_r\, H_1$, and $A\, q_D$ by R_1 and R_2, respectively, from Eq. 38.

Experimental Section on Contract S-6.—This is interesting work, indeed, and Professor Terzaghi is to be commended for having done it so carefully. It is regrettable, however, that, since the abundant research material presented does not include pressures on the arch and data referring to the bending or buckling of vertical legs, a complete picture of the "deflected structure" cannot be established. Two circumstances, however, can be detected from Figs. 12 and 14: (1) All five positions of the "deflected structure" practically intersect at two points, so that the arch in Fig. 12 may be considered as having two hinges; and (2) the footings of the "deflected structure" were not horizontal but slightly inclined in the north-south direction; hence the bottoms of the legs were not strictly vertical. In fact, in the final position 9-4-41 (Fig. 14), the average reading of cells 17-19 was about 34% larger than that of cell 9; and the average reading of cells 16-18 was about 72% larger than the reading of cell 8. The northern footing of the experimental section was about one third

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[39] "Design of Pipe Culverts," by D. P. Krynine, *Proceedings*, Highway Research Board, 20th Annual Meeting, 1940.

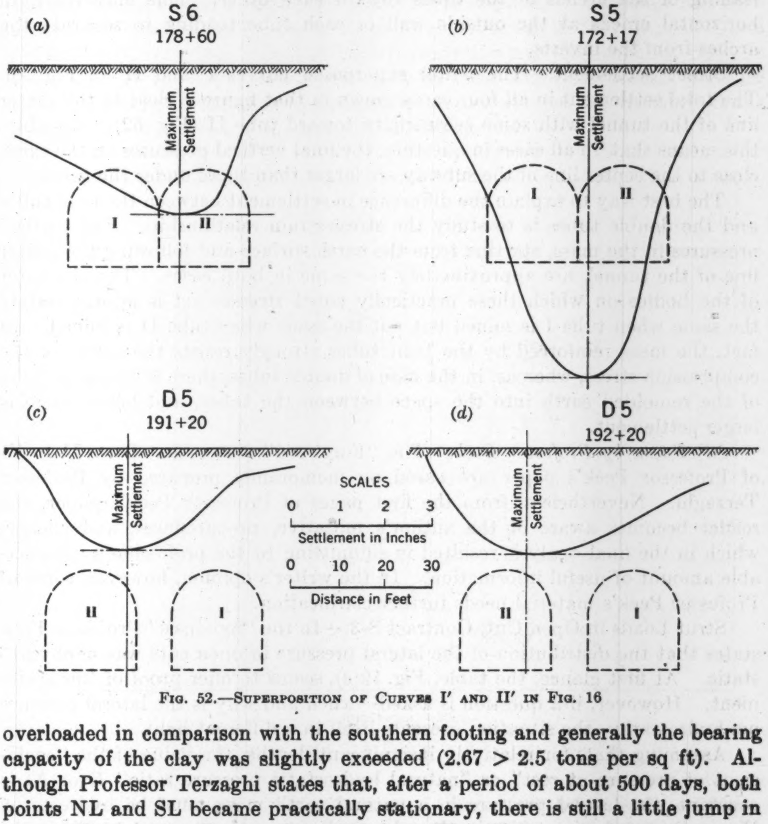
(a)
S 6
178+60
Maximum Settlement
I
II
(b)
S 5
172+17
Maximum Settlement
I
II
(c)
D 5
191+20
Maximum Settlement
II
I
(d)
D 5
192+20
Maximum Settlement
II
I
SCALES
0 1 2 3
Settlement in Inches
0 10 20 30
Distance in Feet

FIG. 52.—SUPERPOSITION OF CURVES I′ AND II′ IN FIG. 16

overloaded in comparison with the southern footing and generally the bearing capacity of the clay was slightly exceeded (2.67 > 2.5 tons per sq ft). Although Professor Terzaghi states that, after a period of about 500 days, both points NL and SL became practically stationary, there is still a little jump in curve NL corresponding to about 535 days (Fig. 13).

The writer sincerely hopes, however, that, by placing a scale on his qualitative considerations referring to the experimental section, they will furnish negligible results.

Application of Test Results to the Design of Tunnel Tubes.—The writer agrees with Professor Terzaghi that the design shown in the right side of Fig. 15 is more economical and not less safe than that shown in the left side of Fig. 15. Precisely the same idea is followed in highway engineering when flexible pipes of corrugated metal are placed under embankments instead of heavy concrete pipes.

Professor Terzaghi's reasoning is quite correct for a single tube. In the case of thin double tubes, however, there may be a possibility of a rather considerable downward movement between the tubes (as explained subsequently under the heading, "Street Settlement") accompanied by the mutual leaning of the arches of the tubes toward each other. This may result in horizontal cracks at the outside wall of each tube tending to separate the arches from the inverts.

Street Settlement.—The writer superposed curves I′ and II′ in Fig. 16. The total settlement in all four cases shown in that figure is close to the center line of the tunnel with some eccentricity toward tube II (Fig. 52). Possibly, this means that, in all cases in question, the final vertical pressures on the earth close to the center line of the subway are larger than those under the tubes.

The best way to explain the difference in settlement between the twin tubes and the double tubes is to study the stress-strain relationship. The vertical pressures in the mass, starting from the earth surface and following the center line of the tunnel, are approximately the same in both cases. The character of the bodies on which these practically equal stresses act is approximately the same when tube I is mined but not the same when tube II is mined. In fact, the mass reinforced by the twin tubes strongly resists the action of the compression stress, whereas, in the case of double tubes, there is simple packing of the remolded earth into the space between the tubes, and hence there is larger settlement.

(*C*) *Paper by Professor Peck.*—The "Foreword" states that Parts II to VI of Professor Peck's paper are based on memoranda prepared by Professor Terzaghi. Nevertheless, from the first pages of Professor Peck's paper, the reader becomes aware of the author's initiative, preparedness, and energy, which in the final analysis resulted in submitting to the profession a considerable amount of useful information. In the writer's opinion, however, some of Professor Peck's material needs further clarification.

Strut Loads in Open Cut, Contract S-3.—In the "Synopsis" Professor Peck states that the distribution of the lateral pressure in open cuts was nonhydrostatic. At first glance, the table, Fig. 19(*e*), seems to offer proof of this statement. However, if a question is asked—when and why is the lateral pressure nonhydrostatic—the situation presents itself in a different light.

Assuming that, for relatively insignificant depths, the value of the "coefficient of pressure at rest" or "natural hydrostatic pressure ratio" K or K_0 is constant, the lateral pressure in a natural earth mass must be hydrostatic. Hence it was hydrostatic at the place where continuous sheet piling and H-piles were to be driven. During and after this driving, however, friction and shearing stresses developed in the mass in the same way as when a nail is driven into a wooden plank. When the excavation was being made, new shearing stresses developed in the mass because the earth mass tended to move toward the sheet piling which was being bent, and conservative nature opposed this disturbance by providing new shearing stresses. It is possible (but this remains to be proved) that all these systems of shearing stresses were self-balanced. In this case the lateral pressure on the sheet piling equaled the natural hydrostatic pressure; otherwise these two values would have been somewhat different. Because of the action of the shearing stresses, the pressure on the sheet piling was not hydrostatic, but redistributed (see data of November 20 in Fig. 19(*e*) and in Fig. 53).

Shearing stresses provided by nature to oppose movements within the mass may decrease or even completely vanish as time passes. This fact has been proved, for instance, by Professor Housel's observations in Detroit. The data in Fig. 19(*e*), referring to November 27, prove it also, quite conclusively. It seems that at the lower levels (El. 23.5 and El. 31.5), the lateral pressure tended to return to the natural hydrostatic shape, whereas the recovery at the intermediate layers (El. 14.5) was not so pronounced.

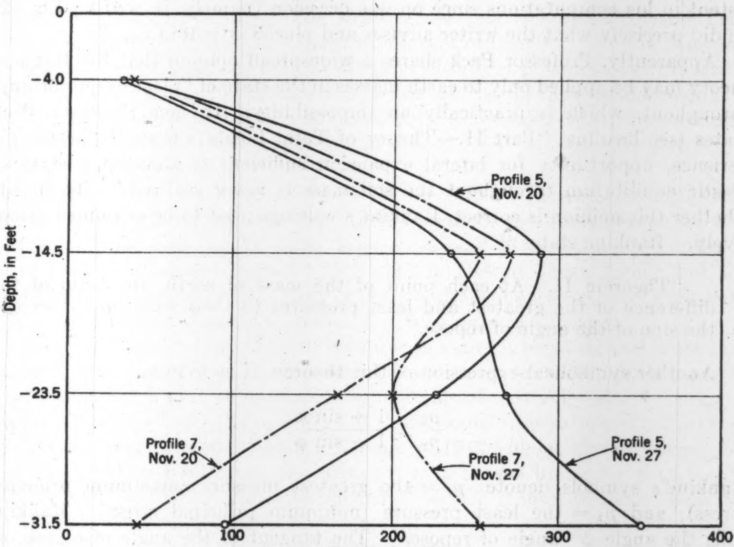

FIG. 53.—STRUT LOADS IN OPEN CUT, CONTRACT S-3 (DATA FROM FIG. 19(*e*))

Several questions arise in studying Fig. 19(*e*): How much time is needed for complete recovery of a cut from shearing stresses, if it recovers at all? If an excavation must stay braced for a long time, is it to be designed against both redistributed and hydrostatic pressure? In what cases, if any, does this phenomenon of recovery take place behind permanent flexible walls?

Rankine Earth Pressure Theory.—According to Professor Peck, Eqs. 6 and 7 were probably first derived by Résal (1910) as an extension of the Rankine theory. The writer has had the privilege of attending lectures by Résal, and he simply cannot admit that Résal did not know that Eq. 7 was published by Captain Français in 1820,[40] at which time Rankine was about one year old.

In writing Eq. 6 Professor Peck knew that, for his clay, $\phi = 0$. He should have made this statement and should have introduced this value into Eq. 6. This procedure would have simplified Eqs. 7, 8, 9, 11*a*, 11*b*, 15, 16*a*, 16*b*, 17, 18*a*, and 18*b*, and would have eliminated superfluous elementary mathematics. Thus the value of the paper would have been considerably increased. As an example, consider Eq. 18*b* if modified as the writer advises:

$$K_A + \frac{2\, q_a}{\gamma_a\, H} = 1 \qquad (39)$$

Obviously this is a straight-line relationship between K_A and $\frac{2\, q_a}{\gamma_a\, H}$, and Fig. 31 becomes quite unnecessary. Incidentally, Professor Peck is not entirely consistent in his computations since on one occasion (namely, in writing Eq. 12) he did precisely what the writer advises and placed $\phi = 0$ in Eq. 6.

Apparently, Professor Peck shares a widespread opinion that the Rankine theory may be applied only to earth masses in the state of "plastic equilibrium" throughout, which is practically an impossibility. In fact Professor Peck states (see heading, "Part II.—Theory of Total Loads") that "In actual experience, opportunity for lateral expansion sufficient to develop a state of plastic equilibrium throughout the soil mass is never realized." To decide whether this opinion is correct, Rankine's writings need to be examined attentively. Rankine states:[41]

> "Theorem II. At each point of the mass of earth, the ratio of the difference of the greatest and least pressures to their sum cannot exceed the sine of the angle of repose."

Another symbolical expression of this theorem is as follows:

$$\frac{p_1}{p_2} \overset{=}{<} \frac{1 + \sin\phi}{1 - \sin\phi} \qquad (40)$$

Rankine's symbols denote: p_1 = the greatest pressure (maximum principal stress); and p_2 = the least pressure (minimum principal stress). Rankine calls the angle ϕ "angle of repose." The tangent of the angle of repose, ϕ, according to Rankine,[42] equals the coefficient of friction. In other words, Rankine's "angle of repose" is what is now called "angle of friction."

The double sign in the Rankine formula (Eq. 40) clearly means that, according to Rankine, an earth mass may (sign =) or may not (sign <) be in a state of "plastic equilibrium."

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[40] *Mémorial de l'officier du génie*, No. 4, 1820, p. 165, 4th line from the top.

[41] "A Manual of Applied Mechanics," by W. J. M. Rankine, 15th Ed., London, 1898, pp. 213 and 214.

[42] *Ibid.*, p. 212.

In applying this theorem to a semi-infinite earth mass, Rankine made an incorrect statement; namely, that the lateral pressure, p_2, in such a mass is a minimum, which really means that the mass is in the state of plastic equilibrium. This statement is to be rejected; but theorem II is not affected by this rejection. It is still valid and may be applied to any mass, in the state of plastic equilibrium or otherwise.

General Wedge Theory for Earth Pressures.—The wedge that tends to separate from the backfill behind a wall or behind a brace is bounded by a curved surface. This was stated by Coulomb as early as 1773. Considerable work in determining the shape of this surface was done in Germany, starting, to the writer's knowledge, with Kötter (1888). Prandtl (1920) used the logarithmic spiral and so did Professor Terzaghi (1942). As is well known, Swedish engineers simply assumed that the sliding surface in the two-dimensional representation is an arc of a circle. After all, this is what Professor Peck does, but, instead of proceeding directly, he first uses the logarithmic spiral, places $\phi = 0$ in its equation, and thus indirectly arrives at an arc of a circle. The writer does not see why this should be done.

Distribution of Pressure.—The writer agrees heartily with Professor Peck that (see heading, "Part IV.—Distribution of Pressure") "* * * it is useless to attempt to compute the real distribution of lateral pressure over the sheeting." In this case, as in the case of other structures, all that should be done is to design a structure that would be (*a*) safe and (*b*) economical. The statistical material contained in the paper by Professor Peck will be very helpful in reaching this objective.

(*D*) *Paper by Professor Housel.*—To the writer's knowledge, European literature contains no research data for the pressure on circular tunnels. The first work published (1937) along these lines was on the Lincoln Tunnel in New York.[43] Professor Housel's field work in Detroit began earlier, but the results were published later. More or less simultaneous research work concerning tunneling using the shield method has been done in Chicago.[2a]

The paper by Professor Housel contains so many interesting and important features that, to discuss it, it would be necessary to write another paper. The writer wishes only to emphasize some items of particular significance.

Duration of Observations.—Fig. 40 shows some observations that covered the period from December, 1930, to January, 1941—about eleven years. No one has previously observed the pressure on a tunnel lining for that long a period; and this circumstance permitted Professor Housel to draw some important conclusions.

Variation of Pressure with Time.—Fig. 41 is an interesting diagram showing the recovery of an earth mass disturbed by the tunneling. It seems that the mass tends to return to its original state of stress which is analogous to the healing capacity of a living organism. However, just as scars and spots remain on the skin after the healing of a wound, so there may be permanent or residual shearing stresses in the mass as the result of the disturbance of its internal equilibrium.

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[43] "Lincoln Tunnel: The Field Measurements and Study of Stresses in the Tunnel Lining," by G. M. Rapp and A. H. Baker, Port of New York Authority, July, 1937 (lithoprinted).

Hypothesis of Shearing Stresses in the Mass.—It follows from the conditions of equilibrium of a small cube within an earth mass that the weight of this cube is balanced by both the vertical pressure and the vertical shearing stress. Thus the vertical pressure at a point of a mass is not the same as the weight of the mass above that point. If the shearing stress is increased because of some disturbance in the mass such as tunneling, the vertical pressure will decrease accordingly. This explains the phenomenon sometimes termed "arching" or the sidewise transfer of weight of the mass over the tunnel. Evidently, the graph of "redistribution of vertical pressure" as shown in Fig. 42 is but a very crude approximation. . A vast field is open to the research workers in the province of these shearing stresses: Their physical nature should be investigated and the laws controlling their increase and decrease formulated. Nowhere, however, has the problem of the action of shearing stresses over a tunnel been approached in such detail as in Professor Housel's paper, and he is to be highly commended for this achievement.

Conclusion.—The cities of Chicago and Detroit are to be complimented for organizing soil investigations and research measurements during the tunnel building in those cities. This is precisely what large cities and government organizations should do when undertaking considerable building projects. Both cities are also to be congratulated for the able choice of their research workers who produced a mass of useful information contained in this Symposium.

L. F. COOLING,[44] ESQ.—There has been no opportunity for the writer and his colleagues at the Building Research Station to make detailed measurements of earth pressures such as those described in the Symposium, but they had occasion to investigate a number of practical problems relating to soft clay soils that involved a study of the shearing resistance of plastic clays. The results had a bearing on the conclusions outlined by Professor Terzaghi in the section entitled "II. Soil Investigations: Soil Testing."

For stability problems in soft clay soils the Building Research Station used a simple method of testing in which the compression strength of samples was measured in the field by means of a portable apparatus. In the analysis of the problems the shear strength was taken as equal to half the compression strength with $\phi = 0$. The apparatus and technique have been described elsewhere.[45, 46] In the light of that prior experience, the applicability of this method to practical problems involving soft clays has been considered satisfactory, but, since it has not been backed by such substantial evidence as that afforded by the careful observations in Professor Peck's paper, the findings in the Symposium are of special interest to the writer.

Typical data on various problems may be briefly indicated as follows:

(a) Table 4 shows data obtained in two problems relating to limiting slopes in natural clays. The first was a drainage cut in a soft alluvial clay in Somerset, England, which failed during construction. The second was a dredged slope in an estuary in Northern Ireland where the soil was a soft silty clay. These results suggest that a shear strength equal to half the compression strength with $\phi = 0$ gives the best agreement with fact.

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[44] Building Research Station, Watford, England.

[45] "Portable Apparatus for Compression Tests on Clay Soil," by L. F. Cooling and H. Q. Golder, *Engineering*, Vol. 149, January, 1940, p. 57.

[46] "Soil Mechanics and Site Exploration," by L. F. Cooling, *Journal*, Institution of Civ. Engrs., Vol. 18 (1941–42), March, 1942, p. 37.

(b) When a structural load is applied quickly, in comparison with the time required for consolidation of a clay stratum, the clay again behaves as if it had a constant shear strength with $\phi = 0$.

An opportunity for checking this point was afforded by the investigation of a failure which occurred during the construction of a building in Scotland.[47] The clay beneath the most heavily loaded footing failed in shear. This clay had a natural water content of 60%, liquid limit 70%, plastic limit 27%, and an average compression strength of about 700 lb per sq ft. The progress of construction was such that the actual failing load could be estimated closely, and this was found to agree well with the theoretical value obtained by analyses based on shear strength equals 0.5 compression strength and $\phi = 0$.

TABLE 4.—LIMITING SLOPES IN NATURAL CLAYS

No.	Slope	Depth (ft)	AVERAGE VALUES (%) FOR:			SHEAR STRENGTH (LB PER SQ FT)				FACTOR OF SAFETY	
								EQUILIBRIUM SHEAR			
			Liquid limit	Plastic limit	Water content	Required[a]	Measured[b]	C	ϕ	On one-half compression	Equilibrium shear
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
1	1 on 2	16	71	27	55	300	300 to 320	94	24°	1.1	1.5
2	1 on 3.5	36	67	28	50	250	240 to 400[c]	180	25°	1.2[d]	2.3

[a] Average shear strength required from analysis, $\phi = 0$. [b] Half-compression strength. [c] Average 300 lb per sq ft. [d] Approximate.

Another example relates to the failure of an earth dam during construction.[48] In this case the foundation layer beneath the bank was a highly colloidal clay about 3 ft thick. It had a natural water content of 90%, liquid limit 145%, plastic limit 36%, and a compression strength of about 580 lb per sq ft. The failure occurred when the bank (slope 1 on 2.5) had reached a height of 26 ft. At the time of the slip most of the bank material had been in position for only about thirty-seven days so that there had been little time for consolidation to take place in the foundation clay layer. An analysis of the bank, based on shear strength equal to 0.5 compression strength and $\phi = 0$, gave a factor of safety close to unity.

(c) If the structural load acts for a sufficient time to allow consolidation to take place then the strength of a soft clay, of course, does increase.

Information on this point was obtained in the investigation of the dam cited,[48] when the foundation clay layer was examined after stripping the bank for rebuilding some eight months after the failure. Whereas the original compression strength was about 580 lb per sq ft, the mean compression strength at points which have been under 26 ft of bank was about 1,500 lb per sq ft with a water content of 80%; and, at points which had been under 11 ft of bank, the corresponding values were 1,060 lb per sq ft with a water content of about 88%. Comparison between the observed increase in strength and the theoretical increase due to consolidation showed reasonable agreement in view of the variation in the conditions obtaining at the site.

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[47] "The Investigation of the Bearing Capacity of a Soft Clay Soil," by A. W. Skempton, *Journal*, Institution of Civ. Engrs., Vol. 18, June, 1942, p. 307.

[48] "The Analysis of the Failure of an Earth Dam during Construction," by L. F. Cooling and H. Q. Golder, *Journal*, Institution of Civ. Engrs., Vol. 19, November, 1942, p. 38.

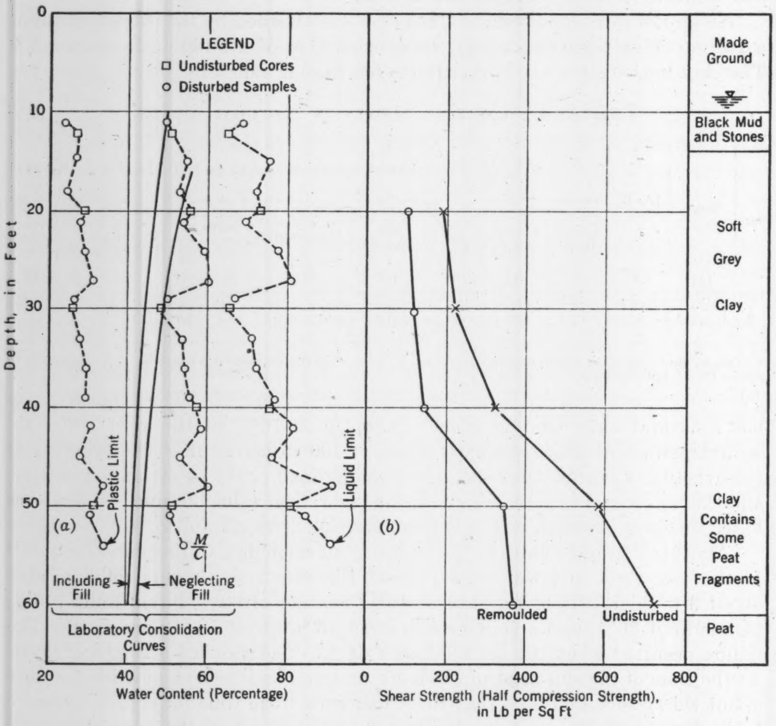

Fig. 54

(d) As a matter of general interest it may be worth while to refer to a striking property of deep natural deposits of soft clay as pointed out by Professor Terzaghi[49]—namely, that the moisture content does not decrease with depth of overburden to the extent that would be expected from the results of laboratory tests. The Building Research Station has explored a number of such deposits, and typical data from one example are given in Fig. 54 and Table 5. These show the results of tests on samples from various depths in a boring in a soft alluvial clay deposit near Portsmouth, England. Although the conditions at the site were complicated a little by the fact that the clay had been covered, for about ten years, by about 10 ft of fill, the results are considered to be of interest. It will be noted from Fig. 54(*a*) that the slight variations in water content down the profile are reflected in changes in the liquid limit. The liquidity index remains practically constant at 0.58 ± 0.04. The full lines on Fig. 54(*a*) show limits for the theoretical variation in water content with depth calculated from consolidation test results, one assuming that the weight of the fill is fully effective and the other neglecting the weight of the fill. The probable curve will lie between these two, but the actual water content is significantly higher than the theoretical values. Fig. 54(*b*) shows the variation with depth of the shear strength (that is, with half-compression strength) determined by nonconfined compression tests. Reference to Table 5 shows that these values are very much lower than those calculated on the basis of the results of equilibrium shear tests on samples, even when the weight of the fill is neglected. The effect of remolding on the compression strength also is indicated in Table 5.

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[49] "Undisturbed Clay Samples and Undisturbed Clays," by Karl Terzaghi, *Journal*, Boston Soc. of Civ. Engrs., Vol. 28, No. 3, July, 1941, p. 211.

(e) Finally, the writer would like to emphasize the cautionary note in the last paragraph of Professor Peck's paper regarding the applicability of his results to overconsolidated and stiff-fissured clays. The Building Research Station investigated a number of failures of retaining walls backed by London clay (a clay of the stiff-fissured type), and these showed the special nature of the problems involved. For instance, in two examples relating to walls about 40 ft deep, theoretical analyses indicated that the average compression strength required for stability (factor of safety = 1) was about 1,700 lb per sq ft. The results of tests on a large number of samples taken from these sites showed in both cases that, in the body of the clay, the average compression strength down the soil profile was about 3,800 lb per sq ft, which is more than twice the required value. However, examination on the site revealed the presence in the clay of softened-fissure planes, of very local extent, in which the compression strength was as low as 500 lb per sq ft, and it was evident that the stability of the backing and the pressure on the wall were controlled largely by these planes of weakness.

TABLE 5.—TESTS ON DEEP NATURAL DEPOSITS OF SOFT ALLUVIAL CLAY, NEAR PORTSMOUTH, ENGLAND

Sample	Depth (ft)	OVERBURDEN PRESSURE (TONS PER SQ FT		One-half compression strength (lb per sq ft)	Water content	Liquid limit	Plastic limit	One-half remolded compression strength (lb per sq ft)	Water content	Drop on remolding (%)	EQUILIBRIUM SHEAR			
		With fill	Without fill								Lb per sq ft	φ	SHEAR AT OVERBURDEN (LB PER SQ FT)	
													With fill	Without fill
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
2	20	0.62	0.22	190	55.1	73	29	105	55	44	...	..	....	....
3	30	0.82	0.42	225	48.5	66	26	120	48.5	48	85	25°	920	500
4	40	1.00	0.60	325	57.2	76	30	150	55.3	54	230	21°	1,080	730
5	50	1.20	0.80	595	51.2	80.9	36	350	47.9	41	115	27°	1,450	1,000
6	60	1.40	1.00	735	62.3	103.1	..	375	62.3	49	546	16°	1,455	1,195

KARL TERZAGHI,[50] M. AM. SOC. C. E.—The testimony presented by Mr. Burke in behalf of the practical benefits derived from the soil investigations and field observations on the Chicago subway deserves the attention of all engineers who will have to direct similar construction operations in the future. In his discussion Mr. Burke expresses the opinion that the fluid theory for the design of tunnel tubes is irrational and, in most instances, will lead to uneconomical design. The writer agrees thoroughly with this statement, and he hopes that the factual data contained in his paper will lead to a thorough revision of the conventional methods of design.

Mr. Shafer states that Professor Housel's paper revealed two distinct phases in the development of the pressure on the tunnel—the initial phase and the final phase. In the paper on the Chicago subway, the transition from the initial to the final phase is illustrated by Fig. 14. During construction a further record of this transition was obtained by periodic strain-gage measurements on the ribs of the lining in the experimental section.

During the first phase, which lasts as long as the clay is supported by the temporary lining of the tunnel, the shearing resistance of the clay is fully active. This condition makes it possible to estimate the pressure on the temporary lining if the average shearing resistance of the clay along vertical sections is known. Previous experience indicates that a shear failure in an undisturbed, soft clay is resisted only by the cohesion of the clay. The observations during construction substantiated this assumption.

In the past, the shearing resistance due to cohesion was determined by two different methods—the unconfined compression test and the ring shear test. Mr. Shafer mentions only the second method. The results furnished by these two methods are very different for the same clay; yet no rational basis for choosing between the methods was available until Mr. Cummings published the results of pertinent experimental investigations in one of the discussions of this Symposium.

He has shown that the unconfined compression test informs the experimenter as to the failure stress, whereas the ring shear test discloses the yield stress. The earth pressure on temporary supports in cuts and tunnels and the stability of slopes on clay are determined not by the yield but by the failure stress, and the stress at failure, in the field, is roughly equal to one half of the unconfined compressive strength of slightly disturbed samples such as seamless tube samples. This has been demonstrated by field observations by different investigators for very different clays. The following instances may be mentioned: Open cuts in the Chicago clay which seems to be part of a ground moraine (see Professor Peck's paper), a slide in a glacial clay of lacustrine origin in Southern Germany (observations by the writer), shear failure in a flood plain clay north of London, England (Mr. Cooling's discussion), and shear failure in a soft marine clay in Scotland, due to overloading a footing, as reported by Mr. Cooling.

In order to derive the failure stress from the results of an unconfined compression test, it suffices to divide the test result by two, regardless of the nature or the geological origin of the clay. On the other hand, to derive the failure stress from the results of a ring shear test, it is necessary to multiply the result by an empirical value called the overload ratio, which may range between one and five. This ratio is very different for different clays and can be determined only by an additional experimental investigation, unless it is known already from tests with the same clay. Furthermore, in contrast to the failure stress the yield stress depends on the deformation characteristics of the clay, and experience has shown that the elastic properties even of undisturbed samples in the laboratory may be entirely different from those of the same clay in the field. Finally, it has been found by experiment that the failure stress is practically independent of the rate of load application, whereas it is known that the relations between stress and strain for clays depend to a large extent on the time factor. In the writer's opinion these conditions leave no doubt which one of the two methods is preferable from a practical point of view.

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[50] Winchester, Mass.

The agreement between the time required to reach final equilibrium in Dean Marston's experiments[25] and in those on the Detroit tunnels seems to be purely accidental, because this time depends on three independent variables—the nature of the soil, the size of the tunnel, and the method of tunneling.

Colonel Chamberlain assumes that the crust of hard, yellow clay has been broken over the full length of the tunnel tubes during the installation of public utilities and other structures. This opinion does not seem to be justified because the public utility facilities were installed either in open trenches whose bottom was located above the surface of the hard clay or in tunnels that were excavated below the crust. The punctures due to the construction of shafts are few and far apart. Therefore, the writer believes that the presence of the stiff crust reduced the settlement quite considerably wherever it survived the tunneling operations without injury. Furthermore, if the crust had not been essentially intact, the heave of the street surface, due to shoving the shield on the contracts south of the Chicago River, would have been far in excess of the actual amount.

One of the outstanding features of the Chicago project was the successful application of the liner-plate method. Since Colonel Chamberlain was intimately associated with the development of this method from the very start, his review of the history of the method is of great interest.

At the end of his discussion Colonel Chamberlain expresses his reluctance to depart radically from the present method of designing tunnel tubes. Fortunately there is a wide margin between a radical departure from these methods and no change at all. At the time when the plans were prepared, the stress conditions in clay surrounding the tunnel tubes were unknown. Under such conditions, sound engineering requires design on the basis of arbitrary, but sufficiently conservative, assumptions. However, as experience increases, sound engineering further requires a modification of design assumptions in accordance with findings. The design of the Chicago subway was based on the assumption that the ratio between the horizontal and the vertical pressure in the clay may range between the extreme limits of 1 : 3 and 2 : 3. Observations on the finished tunnel tubes and in the experimental section disqualified the assumption that the pressure ratio could be as low as 1 : 3 at any point.

These observations also disclosed the important fact that an insignificant deformation of the tunnel tubes suffices to reduce the bending moments in the walls of the tubes to a small fraction of the values they would have in the walls of a perfectly rigid tube. On account of these findings, the next steps should be (1) to make the tubes more flexible by reducing the thickness of the walls, (2) to design the tubes on the assumption that the pressure ratio may range between unity and 2 : 3, and (3) to increase very substantially the allowable stresses in the steel and in the concrete because the observations leave no doubt that the real bending moments in the walls of the tubes will be much smaller than those computed.

In this connection, a recent development in the design of the tunnel tubes for the London subway may be of interest. The City of London rests on a bed of stiff, overconsolidated clay whose thickness increases from zero at the West India Docks on the east to 300 ft beneath Hampstead Heath on the west. Most of the tunnel tubes are in clay. The plastic limit of the clay ranges between 22% and 24%, the natural water content between 24% and 28%, and the liquid limit between 60% and 80%. Geological evidence suggests a preconsolidation load of 20 tons to 30 tons due to the weight of sedimentary strata which were subsequently removed by erosion.[51] On account of the intense preconsolidation, the London clay is very much stiffer than the Chicago clay. However, the strength of the clay is impaired by hair cracks and silt partings, and the initial stresses in the clay caused by overconsolidation give rise to some trouble in the tunnel due to swelling.

All the tunnels in the London clay have been constructed by means of the shield method. The advance of the shield is followed immediately by the erection of the permanent cast-iron lining. The design of the lining has not changed during the 70 years since, in 1873, Greathead constructed the Tower subway in London. Since a cast-iron lining is very rigid, a slight, unbalanced pressure suffices to break the segments. On various occasions, such ruptures actually have occurred.

In 1937, after most of the contracts for the construction of a new section of the subway were let, the increasing shortage of raw material for the cast-iron lining induced one of the firms to propose the installation of a permanent tunnel lining which was to be constructed entirely from pre-cast reinforced concrete segments. The walls of the finished tunnel tube consist of a 2-in. reinforced concrete skin, stiffened by circumferential and longitudinal ribs with a height of about 3 in. Hence, along the ribs the total thickness of the permanent lining is about 5 in. and between the ribs it is 2 in. Each ring is 20 in. wide and consists of six segments.[52] The design was accepted by the London Passenger Transport Board and 2.75 miles of tunnel have been equipped with the new type of lining, which constitutes a really radical departure from tradition.

Since the design of all existing tunnels is based on more or less arbitrary assumptions, the thickness of the walls of the tunnels is determined not by the magnitude and distribution of the acting forces but by the nature of the assumptions. The following data illustrate this statement:

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[51] "A Laboratory Study of London Clay," by L. F. Cooling and A. W. Skempton, *Journal*, Institution of Civ. Engrs. (London), January, 1942, pp. 251–275.

[52] "Tunnel Linings, with Special Reference to a New Form of Reinforced Concrete Lining," by G. L. Groves, *Paper No. 5304*, *Journal*, Institution of Civ. Engrs. (London), Vol. 20, No. 5, March, 1943, pp. 29–64.

Tunnel	Thickness, in inches
Walls of the Lincoln Tunnel; inner diameter 28 ft; soft Hudson River silt	18
Walls of the Chicago subway shield tunnels; inner diameter 20 ft; soft clay	24
Ribs of the new section London subway; inner diameter 12 ft; stiff, fissured clay	3
Shell of the new section, London subway	2

Since the walls of the Chicago tubes are much thicker and much more heavily reinforced than those of tubes in both softer and stiffer clays, considerable concessions can be made to economy without trespassing on the boundaries of the realm covered by experience.

In this connection it should be mentioned that the gradual deformation of the tunnel tube is associated with a consolidation of the clay along those sides of the tube which tend to move out. When a clay consolidates, both the failure stress and the yield value increase very considerably. Hence, neither the failure stress nor the yield value for the clay in its original state have any bearing on the conditions for the stability of the finished tube.

Mr. Eremin discusses the assumptions on which the writer's Eqs. 2 are based. He states that these equations are valid only if the footings of the ribs are yielding. They yielded markedly on each contract except on those parts of contract D-3 where they rested on hardpan. Mr. Eremin also claims that the soil prism shown in Fig. 3(*b*) does not represent the characteristic form, if the tunnel has a semicircular cross section. In such a tunnel, the height of the vertical walls is zero. Hence the footings do not need to carry anything beyond the difference between W and S in Fig. 3(*a*). In other words, if H_1 in Fig. 3(*b*) is equal to zero, Fig. 3(*b*) becomes identical with Fig. 3(*a*) and Eqs. 2 become identical with Eqs. 1.

Mr. Cummings' discussion will be a source of great relief to all those engineers who were mystified by the simultaneous existence of two experimental methods which, for the same property of the same clay, yielded two entirely different values. Considering the vital bearing of the cohesion value on earth pressure and the stability of slopes, it is hoped that his data will be supplemented before long by similar data on clays of different geological origin.

Professor Burmister presents a review of the facts revealed by the observations recorded in the Symposium and adds a series of questions concerning the physical properties of the Chicago clays. A study of the statistical relation between the natural water content w of the Chicago clays and the unconfined compressive strength q_u was made by Professor Peck. This relation can be, and has been, used to estimate the q_u-value on the basis of the results of water-content determinations. However, after a competent experimenter has made about fifty compression tests on different samples from the same stratum, he is able to estimate fairly closely the q_u-value for other samples from the same stratum. The average deviation of the estimated from the real values is considerably smaller than the average scattering of the points about the statistical $(w - q_u)$-curve. Therefore the writer feels that direct estimates deserve preference over estimates based on statistical relations.

Professor Peck has made a similar study for a glacial clay in Cleveland, Ohio. The Cleveland clay is overconsolidated, whereas the Chicago clay is normally consolidated. Otherwise, the clays were very similar. Analysis of the test results has shown that the average q_u-value for the Cleveland clay is roughly equal to 3.6 times the q_u-value for a Chicago clay with the same natural water content.

For undisturbed samples of the soft Chicago clays, the strain corresponding to the maximum stress was roughly equal to 8%. From observations on the job, it seemed fairly evident that the strength of the same clays in a natural state began to decrease as soon as the strain exceeded about 1%. Hence it seemed futile to attempt any correlation between the results of stress-strain observations in the laboratory and those in the field.

Professor Tschebotarioff quotes the writer's statement (see heading "II. Soil Investigations: Soil Testing") "* * * no one has ever published any field data that would demonstrate adequately the existence of a pressure-conditioned shearing resistance in soft clay deposits * * *." He remarks that the statement can cause misunderstandings. Lifted from its context, the statement really can be misleading. Therefore the writer wishes to emphasize that it applies only to the initial shearing resistance of undisturbed, soft clays whose natural water content is practically independent of depth. As soon as the clay starts to consolidate (for instance, under the influence of an artificial load), it is no longer undisturbed, and its shearing resistance increases with decreasing water content approximately in the same manner as it does in the laboratory, under similar conditions of loading and drainage.

As examples of pressure-conditioned shearing resistances, Professor Tschebotarioff mentions the earth dam of Chingford, north of London, England, the Pendleton Levee in Mississippi, and some observations of his own.

To the writer the word Chingford had a familiar ring, because he acted as a consultant to the contractors during the reconstruction of this dam. The two other examples are not quite to the point because they refer to shear failures in clay strata which contain horizontal sand or silt partings. If a clay stratum contains such partings, a great part of the surface of sliding is within such a parting and the average resistance against failure depends to a large extent on the intensity and distribution of the excess hydrostatic pressure over the partings. In this connection the reader is referred to the writer's current discussion of the Pendleton Levee failure.[53]

When investigating a slide in a riverbank, Professor Tschebotarioff evaluated the shearing resistance of the varved clay by means of unconfined compression tests. In the writer's opinion, this method is rather unsuitable for investigating varved clays.

In his discussion of the interpretation of the open-cut data, Professor Tschebotarioff states that the assumptions concerning the shearing resistance of the sand layer are not entirely convincing. The paper on the open cuts was written by Professor Peck, but the assumptions were made by the writer. In 1929 the writer made a large-scale experimental study of the influence of the yield of a lateral support on the earth pressure of sand,[54] and since that time he has had various opportunities to make observations in open cuts in sand. On the strength of these varied experiences, he feels that the error associated with his estimate is not important enough to be of any practical consequence.

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[53] *Proceedings*, Am. Soc. C. E., June, 1943, p. 949.

Professor Krynine's statement that the combined bearing capacity of four supports can be very much smaller than the sum of the bearing capacity of each of them is perfectly correct and much to the point. As a matter of fact, it is easy to imagine a considerable variety of conditions which would invalidate Eqs. 1 and 2. In Chicago, observations in the tunnels indicated that the shearing resistance and the bearing capacity of the clay remained practically constant over a relatively wide range of deformation. Therefore Eq. 2*b* was justified, but Professor Krynine's request that the validity of the equation should be investigated in each particular case also is justified.

Professor Krynine's allusion to flexible culverts is very appropriate. The soil surrounding culvert pipes is much more compressible than the soil adjoining the walls of an expertly constructed tunnel. Therefore, experiences with flexible culverts indicating the beneficial effects of flexibility are directly applicable to tunnels.

The last paragraph of Professor Krynine's comments should be supplemented by the statement that the purely empirical rule $\phi = 0$ loses its validity as soon as the clay begins to consolidate.

The layout of the tunnel tubes was completed long before the writer learned that the City of Chicago intended to build a subway. The justification for Eq. 3 is identical with that for Eq. 2*b*.

The investigations included a great number of strain-gage observations on the arch ribs. However, there was no incentive for presenting them in detail, because they agreed, within the range of observational error, with the results of the computation of the stresses based on the pressure-cell readings and the measured deflections. They also showed that the forces were far too small to cause buckling. For these reasons, all the information of practical import can be derived from the data shown in Figs. 12 and 14.

The observations left no doubt that the deformations of a flexible tunnel lining caused by the construction of a second tube are far more important than those that would occur during and after the construction of a single tube in the same clay. However, a few weeks after the second tube was installed, the deformations of the first tube, caused by the construction of the second one, ceased to increase. Therefore, the detrimental effects produced by constructing the second tube can be avoided by postponing the installation of the permanent lining, as proposed in the paper.

The relation between settlement and pressure distribution is not quite so simple as Professor Krynine suggests, because the intricate time effects make it practically impossible to establish a direct relation between these two types of quantities. Professor Krynine's explanation of the causes of the difference between the settlement of the street surface above twin tunnels and above double tunnels is essentially identical with the writer's opinion. However, the term "remolded" is a trifle too strong. The disturbance of the block of clay between two tunnels certainly did not exceed the disturbance of the best "undisturbed" samples that can be obtained.

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[54] "Large Retaining-Wall Tests," by K. Terzaghi, *Engineering News-Record*, Vol. 112, 1934, pp. 136–140, 259–262, 316–318, 403–406, and 503–508.

The abnormal position of the dotted curve marked 7–18–40 in Fig. 14 with reference to the others gave rise to some queries which call for an answer. On March 26 the air pressure in the tunnel dropped temporarily to 8 lb per sq in. When the measurements represented by the curve 4–1–40 were made, the air pressure was not yet completely restored. Between April 1 and July 18 the pressure on the base of the invert decreased while the tunnel was filled with compressed air at the normal operating pressure of about 11.5 lb per sq in. and assumed the values given by the ordinates of the dotted curve. The first readings after the removal of the air pressure are represented by the curve marked 7–20–40.

The data in Table 4 confirm the writer's experience that the shearing resistance of a soft, undisturbed clay in the field, prior to consolidation, is roughly equal to one half of the unconfined compressive strength of slightly disturbed samples.

To Mr. Cooling's data in Table 5 could be added the description of a very soft post glacial marine clay by B. K. Hough, Jr.,[55] Assoc. M. Am. Soc. C. E. Between a depth of 15 ft and 50 ft below the surface the natural water content of this clay was practically independent of depth and well above the liquid limit. However, the fact that the water content of the clay is independent of depth does not necessarily imply that the shearing strength is also independent of depth. In one locality, it was found that, due to the submerged weight of the overburden, the shearing strength of a soft clay increased with depth by amounts equal to about 0.07 times the effective unit pressure. In other clays no increase has been noticed and the reason is still unknown. In this connection the following observation may be mentioned. The soft, glacial clays of Chicago differ from those in Cleveland merely by their load history. The Chicago clays are normally consolidated whereas those of Cleveland carry a temporary surcharge of at least 3 tons per sq ft. The identity of the liquidity indexes of the two clays demonstrates that the overburden had practically no influence on the natural water content; yet the unconfined compressive strength of a Cleveland clay is roughly equal to 3.6 times that of a Chicago clay at equal natural water content. Most of the Cleveland samples fail by shear along inclined surfaces whereas most of the Chicago samples fail by bulging.

At the end of his discussion Mr. Cooling gives an example of the effect of fissuration on the average shearing strength of stiff clays. Considering the importance of this subject the writer wishes to add a few examples from his own experience. From four slides in fissured clays which he analyzed the following data were obtained:

The shearing resistance of intact samples ranged between 5 and 22 tons per sq ft; yet the average shearing resistance of the clay along the surfaces of sliding in the field ranged between the narrow limits of 0.22 and 0.32 tons per sq ft. An analysis of the Sevenoaks slide on the Southern Railway south of London, England, showed that the average shearing resistance along the surface of sliding was between 0.24 and 0.30 tons per sq ft, which is that of a soft clay; yet the cut was located in a material which, in an undisturbed state, resembled a soft shale. At the walls of test pits it could be seen that the sliding mass consisted of stiff chunks embedded in softer material. Within a sample of less than 1 cu ft the unconfined compressive strength of the clay ranged between 0.24 and 1.7 tons per sq ft, corresponding to a range of the failure stress between 0.12 and 0.85 tons per sq ft. The position of the surface of sliding indicated that the average shearing resistance of the material increased rapidly with increasing depth beneath the slope.

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[55] *Proceedings*, Am. Soc. C. E., December, 1942, pp. 1833–1836.

In conclusion, the writer wishes to express his gratitude to the discussers who gave him an opportunity for a stimulating exchange of opinions and especially to Mr. Cummings who added a set of data of lasting value to the contents of the Symposium.

RALPH B. PECK,[56] JUN. AM. SOC. C. E.—The several discussions contributed materially to the value of this Symposium. Mr. Burke has stated that the use of the equivalent fluid method for the calculation of the lateral pressure exerted by a cohesive soil has little to recommend it. The writer agrees with this opinion. It is apparent from Fig. 35(*a*) that the concept of a fluid leads to an erroneous conception of the distribution of the pressure. Furthermore, Eq. 18 indicates that, for a given soil, the density of the "equivalent" fluid for computation of total pressure varies with the depth of the cut. This is even more evident from Fig. 55, which is a graphical representation of Eq. 18. In this chart, γ_w is the density, in pounds per cubic foot, of the equivalent fluid which exerts the same total lateral pressure as the real soil. For example, if the overlying sand layer is 0.2 H in thickness and the value of q'_c is 2,000 lb per sq ft, the equivalent fluid for a cut 25 ft deep weighs 25 lb per cu ft, whereas that for a cut 50 ft deep weighs 82 lb per cu ft. Thus, the use of some arbitrary value, such as 40 lb per cu ft, leads to lack of economy for shallow cuts and to lack of safety for deep cuts.

Mr. Burke also states that the soil mechanics program of the Chicago subway existed for very practical reasons—to increase safety, reduce settlements, and to save money in design and construction. It was not set up for the purpose of determining fundamental information about the nature of clays. Any advances in the latter direction were considered desirable, but incidental. The writer believes that no single test or observation was made which did not have an immediate practical purpose.

Mr. Cummings has made an outstanding contribution to the Symposium with his discussion of shear values obtained from the compressive strength tests as used in Chicago and the ring shear tests as used in Detroit. From time to time, the results of field measurements, which have been correlated with the results of shear tests made by one of these methods, have been published. Mr. Cummings' discussion makes it possible, for the first time, to evaluate the field measurements on a common basis. Mr. Cummings prefers the use of the compression test in connection with tunnel and open-cut problems for a reason which has seemed logical to the writer. Since the soil near such structures is in the plastic, or failure, state, the laboratory test results should be based on a failure criterion.

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[56] Research Asst. Prof. of Soil Mechanics, Univ. of Illinois, Urbana, Ill.

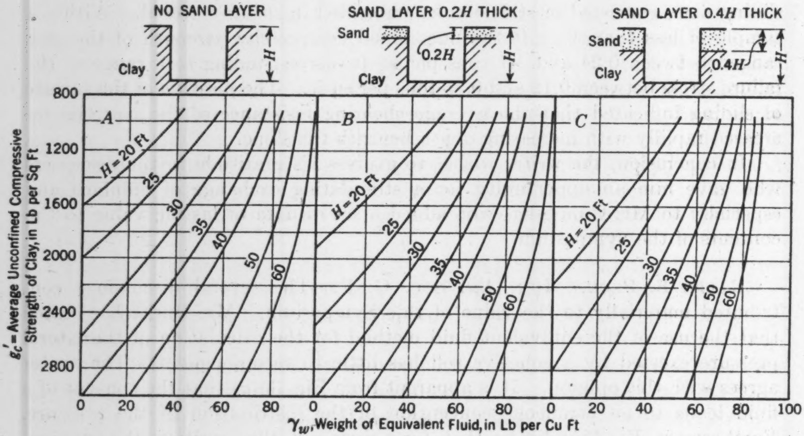

FIG. 55.—DESIGN CURVES FOR EARTH PRESSURE AGAINST OPEN CUTS, CHICAGO SUBWAY EXPERIMENTS

Professor Burmister suggests that the properties of the Chicago and Detroit clays should be given in more detail. For Chicago clays, the statistical relation between water content and unconfined compressive strength is shown in Fig. 56. The other relations which he suggests have been published elsewhere.[29] Professor Burmister also requests information concerning the maximum shearing strength and corresponding strains versus the measured pressures and yield in the bracing systems in open cuts. The relation between shearing strength and measured pressures is given by Fig. 34(*e*). No specific information can be given regarding the yield in the bracing systems in relation to the strength of the clay, other than that following Eq. 26, because the amount of yield depends largely on the depth and spacing of the struts, and the time and excavation stage at which they were placed.

Professor Tschebotarioff is chiefly concerned with the evidence for considering the angle of internal friction of soft clays to be negligible. In particular, he states that, for the Chicago clays, the existence of no frictional resistance is an assumption rather than a proved fact. His statement is based on the belief that the shearing resistance of the sand which overlies the clay was assigned too high a value and that the real shearing resistance of the clay was consequently underestimated. He cites the fact that movement of the sheet piling toward the excavation was observed prior to the installation of the first row of struts and that such movement might reduce strongly the horizontal component of pressure in the sand layer and decrease the shearing strength. In other words, he considers it likely that the value of K_s in Eq. 14—the hydrostatic pressure ratio for the cohesionless material—was smaller than the value (usually 1.0) assigned to it by the writer.

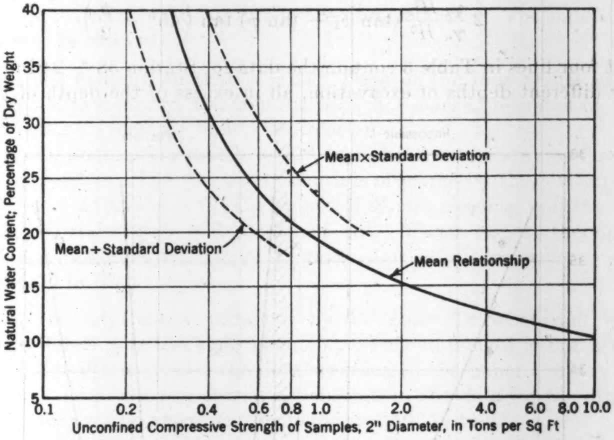

FIG. 56

As a matter of fact, the value of $K_s = 1$ was no arbitrarily assigned quantity. It was arrived at after long and careful study of all of the factors believed to be involved, and it is justified by a number of separate considerations.

Professor Tschebotarioff is correct in his general statement that if a lower value of K_s is used, it will be necessary to assign a frictional resistance to the clay layers to account for the measured strut loads. However, there are certain specific conditions which K_s must satisfy if any such adjustment is attempted. These are as follows:

1. If the angle of internal friction of the sand is 30° (the lowest reasonable value), the value of K_s cannot be less than the active value $K_s = \tan^2\left(45° - \frac{30°}{2}\right) = 0.333$. Obviously, it cannot be negative.

2. The value of K_s may decrease from its original value toward the active value in the first stages of excavation. However, after the struts in the upper part of a cut have been placed, and excavation continues, the part of the sheeting in the upper part of the cut moves back against the sand (see Fig. 25). Therefore, as a cut becomes deeper, after the sand layer has been passed, the value of K_s should increase somewhat. It should certainly not continue to decrease as the depth of the cut increases.

It can be shown, on the basis of the test results, that it is impossible to satisfy these conditions unless the angle of internal friction of the clay is very small.

The relation between K_s and ϕ is given by Eq. 18. For convenience, the equation may be solved for K_s:

$$K_s = \frac{\tan^2\left(45^\circ - \frac{\phi}{2}\right)\left(1 - \frac{2}{\gamma_a}\frac{H - H_s}{H^2} n\, q'_c\right) - K_A}{2\,\frac{\gamma_1}{\gamma_a}\frac{H^2_s}{H^2}(\tan\phi_1 - \tan\phi)\tan\left(45^\circ - \frac{\phi}{2}\right)} \quad \ldots\ldots\ldots(41)$$

The last four lines in Table 3 contain the data for Station 58 + 21 in cut S1A, for four different depths of excavation, all in excess of the depth of the sand

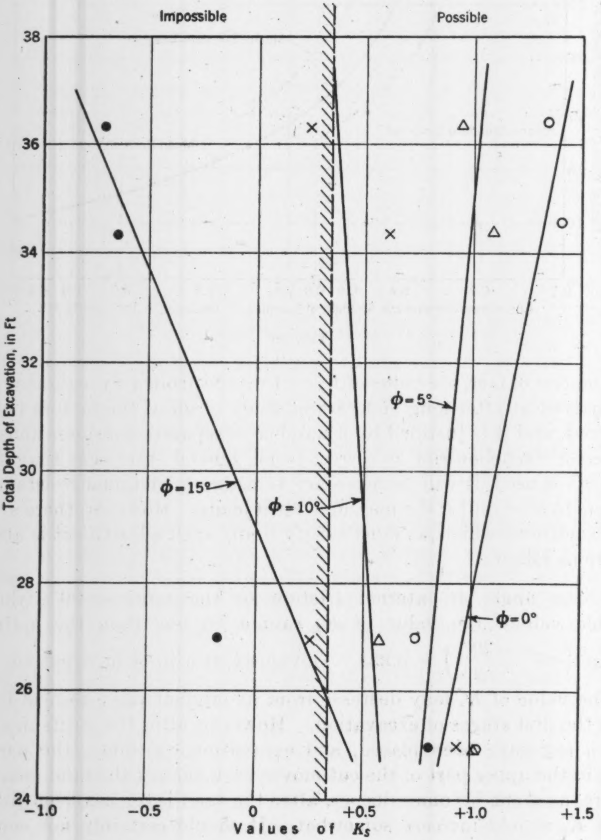

FIG. 57

layer. Assume that for the clay $\phi = 15^\circ$, and proceed, as suggested by Professor Tschebotarioff, to ascertain the corresponding value of K_s to satisfy the measured loads. For the stage $H = 36.5$ ft, $\gamma_a = 118$, $\gamma_1 = 110$, $H_s = 15.0$,

$n = 0.75$, $q'_c = 1{,}320$, $K_A = 0.482$, and $\phi_1 = 30^\circ$—Eq. 41*a* yields $K_s = -0.73$. Hence, in order to assign to the clay a value $\phi = 15^\circ$, one must assume K_s to be negative. This is an absurdity. Proceeding with the other depths of cut and repeating the calculations for other assumed values of ϕ, the values of K_s plotted in Fig. 57 are obtained. There is considerable scattering of the points, as would be expected, but the trends are unmistakable. The following conclusions may be drawn:

1. For the assumption that $\phi = 15^\circ$, three of the four values of K_s are impossible. Furthermore, the trend is toward a marked decrease of K_s with depth, whereas an increase should be expected.
2. For the assumption that $\phi = 10^\circ$, two of the values are less than 0.333, and are impossible. The trend with depth is obscured by the scattering.
3. For $\phi = 5^\circ$ and $\phi = 0^\circ$, the values of K_s are possible, and the trend is toward a slight increase with depth of cut. These values, therefore, are reasonable. Furthermore, a good average value of K_s for either $\phi = 0^\circ$ or $\phi = 5^\circ$ would be about unity.

Hence, the adjustment proposed by Professor Tschebotarioff leads to absurdities unless ϕ is less than about 5°. This statement is not arbitrary. It is inevitable if one studies the data recorded in the paper. Many similar studies on other cuts led to similar conclusions. The data for the other cuts at different excavation stages can be found in Figs. 19 to 22.

Professor Tschebotarioff has suggested that, if no sand existed over the clay, there would be no question regarding the shearing resistance of the sand. The ratio of depth of sand to depth of cut for the cuts represented in Table 3 varies from 20% to 60%. If the influence of the sand layer is evaluated erroneously, one would expect to find evidence of that fact within this large range; yet the deviation from the line of best fit of the points in Fig. 34(*e*) bears no relation to the relative depth of sand.

It is generally accepted that, if the lateral support of a sand mass can move laterally about 0.1% of the height of the mass, the value of K_s will approach the active value. It seems reasonable that, once the sand is in the active state, a backward movement of 2.5 times that amount (see Fig. 25) should increase K_s to a point well above the value for earth pressure at rest (about 0.5 or 0.6). Therefore, a value of 0.5 or 0.6 is the minimum which the writer could consider probable, and the sum total of his experience with the Chicago subway cuts over a three-year period leads him to consider unity the most probable value. The vibration from street traffic, and the presence of cranes and trucks running alongside the cuts during the construction period, add to the general impression.

In the first attempt to analyze the data, it was assumed that $\phi = 17^\circ$. The writer had no prejudice toward $\phi = 0^\circ$; he was forced to this value by the data. Professor Tschebotarioff's belief that the value of ϕ was made zero by a suitable choice of K_s is not tenable if the measurements are permitted to speak for themselves. If all of the data are analyzed in a critical manner, it will be found that the only set of values which is reasonably consistent with the data for all of the open cuts, at all of the excavation stages, is $\phi = 0^\circ$, $K_s = 1$, and $n = 0.75$.

By way of contrast, Professor Krynine believes that the writer should have chosen $\phi = 0$ in the first place, because the mathematical expressions in the paper would have been simplified. To the writer, however, the value of ϕ associated with the behavior of large masses of clay in the field was not a quantity to be determined by assumption. Professor Krynine may have considered it most likely that $\phi = 0$; Professor Tschebotarioff evidently considered it probable that ϕ is greater than zero. This diversity of opinion was widespread, and it could not be resolved on the basis of theoretical considerations. Therefore, the writer felt compelled to derive the theoretical equations for the general case in which ϕ is not necessarily equal to zero and to make the decision on the basis of the field data. To lose this opportunity to determine ϕ on the basis of field measurements would be too great a price to pay for mathematical simplicity.

In recent years, several writers have attempted to clarify the differences between various earth-pressure theories by historical excursions into the writings of early students of the problem. Professor Krynine's remarks concerning Rankine's theory are a case in point. The writer believes that this approach is likely to be fruitless for two reasons: The discussion is purely academic, and the information confuses the present concepts.

In the writer's opinion, the earth-pressure theories which have had sufficient merit to survive belong to one of two categories. The theories in the first category assume that the entire soil mass (or at least that the part of the soil mass within the sliding wedge) is in a state of plastic equilibrium. Since, as Professor Krynine states, Rankine investigated the mathematical conditions for plastic equilibrium, it is customary to associate Rankine's name with any theory in this category. The theories in the second category require no assumptions regarding the state of stress within the wedge—the results depend entirely on the external conditions for the equilibrium of a sliding mass. These theories are customarily called the wedge theories. They include Coulomb's theory, because Coulomb treated the earth-pressure problem in this manner at an early date.

It is quite probable that neither Rankine nor Coulomb would recognize the theories commonly associated with their names at the present time, and there is little point in insisting on historical purity so that they could. Engineers must designate the useful theories by some commonly accepted names, and it seems entirely appropriate to associate the names of two pioneer contributors with the present theories which embody their chief contributions.

The writer neglected to report a significant detail in the construction of the open cut on contract S-3. Therefore, he is indebted to Professor Krynine for noting the large increase in the loads carried by the lower struts in this cut within a period of only one week, when the depth of excavation increased only 2 ft. Fig. 19(*a*) shows a cross section through the cut. A small freight tunnel is located below the bottom of the sheet piles on the right-hand side of the cut. On November 20, 1939, when the strut loads were first measured, the excavation had not uncovered the freight tunnel and the tunnel lining was intact. The additional 2 ft of excavation, preceding the measurements of November 27, exposed the top of the tunnel. Since the tunnel lining was of unreinforced concrete, it possessed practically no flexural strength, and under the unbalanced loading it cracked. Once the lining had broken, the space occupied by the freight tunnel had the same effect as an additional excavation. Therefore the presence of the broken freight tunnel effectively increased the depth of the cut. Rather arbitrarily, the writer concluded that the effective depth of the cut extended to a point at about midheight of the freight tunnel, and for this reason the value $H = 39$ ft is recorded in Table 3, instead of the actual depth of excavation, 36 ft. For a depth of 39 ft the center of pressure is at 0.42 H, which is not different from the location of the center of pressure on November 20. Therefore, the increase in strut loads is believed to be the result of an increase in the depth of the cut, rather than a decrease of the shearing resistance of the soil with time.

The routine measurement of strut loads on several open cuts extended over a period of more than two weeks after excavation reached grade. Within this period there was no significant change in the strut loads. In a few locations, excavations stood open for more than six months. There are no quantitative data regarding the change in lateral pressure with time, but the writer's general impression, based on close observation of the cuts, is that within this period the load transferred to the sheeting did not change appreciably, even when the cuts were located in the softest clays. It should be emphasized, however, that the writer's paper does not refer in any way to the loads acting on the permanent structures constructed in the open cuts.

Mr. Cooling's remarks, together with the examples of the behavior of soft clay, constitute a very worth-while addition to the Symposium.

In conclusion, the writer wishes to thank the discussers for their stimulating comments. The experience on the Chicago subway demonstrated convincingly that the determination of strut loads and movements in a systematic manner resulted in better and more economical construction. Therefore, the writer hopes that such determinations will become standard practice. In addition, the results cannot help but add greatly to the store of knowledge on this subject.

W. S. HOUSEL,[57] M. AM. SOC. C. E.—The various discussions have been a substantial contribution to the subject of the Symposium. As a whole these discussions support and amplify the conclusions drawn by the authors, although there are several important points which will be considered in more detail on which there is less unanimity of opinion. As one of the authors, the writer wishes to express his appreciation for the interest shown and the contributions made by the discussers.

Inasmuch as the three papers deal with some phases of the subject which are closely interrelated, it seems appropriate and necessary for the authors of the three papers to comment upon related conclusions in their closing discussions. There is one conclusion drawn by Professor Terzaghi which the writer believes should be taken with some reservation, and that is the statement (see heading, "I. Clay Pressure on Temporary and on Permanent Tunnel Support: Clay Pressure on the Finished Tubes") that "* * * it is likely that a thin concrete shell with a nominal reinforcement should suffice permanently for withstanding the clay pressure." This tentative deduction is reinforced by a more positive statement in the final conclusions to which it is added that this "proposed method of design takes advantage of the mechanical properties of soft, undisturbed clay."

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[57] Lt. Comdr., CEC, U.S.N.R., Office of Superintending Civ. Engr., Norfolk, Va.

The advantages of flexible pipe culverts in mobilizing so-called passive resistance in soil masses is fairly well recognized and is commented upon by several discussers. However, permanent stability would seem to be limited to the ability of the soil to withstand the residual stresses imposed on the mass without continued deformation. It would then appear that the ability of plastic clay to furnish such permanent resistance would be limited definitely to its yield value and any additional residual stress would imply continued deformation at a rate prescribed by the time rate of shear relationship of the particular material involved. In this connection it would seem more accurate and in conformity with the accepted definition of plastic solids to comment on the plastic properties of the soft, undisturbed clay rather than its mechanical properties. Mechanical properties of soil, as the writer understands them, refer to granular behavior, including internal friction, which is not a property of plastic clay of the nature under discussion to more than a negligible degree, a point on which all three investigators apparently now agree.

Experimental evidence of the temporary character of stresses in the soil mass in excess of the yield value is contained in the 10-yr record of earth-pressure observations presented by the writer. Decreasing pressures indicative of increasing plastic resistance were consistently recorded for the first several months. Subsequently, for about five years, pressures at the top and bottom of the tunnel increased to values approximately equal to the full static head, and the decreased lateral pressure which was maintained at the side could be accommodated by residual stresses approximating the yield value of the soil. The implication seems clear that, if the tunnel structure continued to yield, lower earth pressure and higher stresses in the soil mass probably could have been maintained for some indefinite period of time. The extent to which the design of a relatively permanent structure could accept such continued deformation is extremely problematical and is certainly an unexplored field.

The writer was interested particularly in the analysis of earth-pressure measurements on Chicago subway cuts because it provided an excellent opportunity to compare the results with similar earth-pressure observations on Detroit tunnels. The results of these two entirely independent investigations should be very encouraging to those who are interested in the solution of such problems especially because of the direct verification of several of the important conclusions that were drawn in each case. This is all the more striking because the theoretical basis from which the two authors approach the problem at least in the beginning represents viewpoints which have not been in accord for some time past.

In dealing with plastic clays the writer has insisted for some years that many (or perhaps most) of these materials were, for all practical purposes, purely cohesive, meaning that shearing resistance was essentially independent of internal friction. In referring to the writer's present analysis of earth-pressure measurements the statement (see "Synopsis") is made that "Discussion of the method and its general applicability doubtless will be focused largely on the question of whether or not soil resistance may be so represented, and, if so, how wide a range of soil texture may be included in that classification."

In expectation that this particular subject would again be the main subject of discussion the writer scarcely anticipated that the results accumulated in the Chicago work would agree so completely with the investigations made in Detroit. This is all the more convincing because Mr. Peck's theoretical approach is based on the conventional treatment of shearing resistance as a combination of internal friction and cohesion, and it is the results of the actual measurements which demonstrate the negligible magnitude of the internal friction as a factor of resistance.

Both investigations follow essentially the same logic or procedure in developing their subject, and this procedure consists of the following steps:

1. A systematic and comprehensive set of factual data has been accumulated.
2. A theoretical basis of analysis of these data is framed in terms of conceptions of soil resistance to which the authors subscribe. In so far as validity of the procedure is concerned, the fact that these conceptions are not the same is only incidental and does not affect the final result.
3. The theoretical equations are solved by substituting the measured values of the primary variables and computing the corresponding values of the factors of soil resistance. These factors are then correlated in each case by ratios between the stresses found to be acting and the measured resistance of the soil.

Professor Peck uses a factor n as a ratio of unconfined compressive strength to computed compressive stress. This ratio is less than unity in his analysis, indicating that the available shearing resistance of the soil, as measured in the particular test involved, is not completely mobilized. The writer uses an overload ratio C which is a ratio of measured yield value to stresses computed from equilibrium equations. The overload ratio has values substantially greater than unity, indicating that, under the conditions investigated in the Detroit tunnel, the soil was stressed well beyond the point of progressive failure during and for some time after construction.

Herein lies the only substantial disparity in the results of the two investigations, and some examination of this point is extremely significant from the standpoint of practical application. In the writer's opinion the disagreement does not indicate any fundamental error in either method of analysis, as both are almost identical processes and are equally valid. It does indicate, however, that the basis of comparison or the methods of measuring shearing resistance of the soil produce completely different results.

That such a wide disparity existed in the results of the transverse shear test conducted under static load and the unconfined compression test probably has been generally recognized for some time. In one instance within the writer's knowledge where a direct comparison was available, the shearing resistance, as determined by the unconfined compression test, was between three and four times the value obtained by the transverse shear test. Although one test or the other could be selected as the standard of comparison and the source of discrepancy thus removed, it would appear to be of some practical importance to select the test which most accurately represents the actual behavior of the soil mass.

Although disturbance of the soil sample is undoubtedly a factor, it is not sufficient to account for more than a small part of the total discrepancy. It is the writer's opinion that in excavations of the magnitude of those under discussion in both Chicago and Detroit, the soil mass is stressed well beyond the yield value. Progressive displacement necessarily must be present and, as shown by earth-pressure measurements in Detroit, final equilibrium may not be achieved for several years after construction. The time covered by the Chicago measurements is not great enough to establish this point, but the magnitude of the settlement so soon after excavation would indicate continued movement for a considerably longer time. The probability of progressive movement involving shearing resistance in excess of the yield is in better agreement with general construction experience in squeezing soils where speed of operation is a most essential factor in successful prosecution of the work.

Other substantiating data for the viewpoint expressed are available from the analysis of progressive settlement of large structures. Three such analyses made by the writer within the past few years have shown overload ratios as high as two for permanent structures with measured rates of settlement of no more than 1 in. in five years. Analyses of excavations in which progressive movement has been substantially higher indicated overload ratios as high as three. The overload ratio of approximately 2.8 for the Central Avenue Tunnel given as the last example in the Appendix to the writer's paper is typical of several such analyses of tunnel operations which are representative of construction practice and which afford a direct comparison between the air pressure actually used and yield value shearing resistance measured by independent transverse shear tests. Further experimental verification of the direct correlation between yield value shearing resistance and full-scale structures has been reported,[58,59,60] as well as similar correlation with full-scale loading tests.[61,62]

This is a subject discussed in some detail by Mr. Cummings, who has also collaborated with the writer on several pertinent investigations which have not been published for general information. Even though the writer has the highest regard for Mr. Cummings' grasp of the subject and practical judgment in such matters, he cannot avoid taking an opposing view on two conclusions expressed in Mr. Cummings' discussion—

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[58] "Measurement of Cohesive Soil Properties Applied to Engineering Design," by W. S. Housel, *Civil Engineering*, August, 1942, p. 442.

[59] "The Shearing Resistance of Soil—Its Measurement and Practical Significance," by W. S. Housel, *Proceedings*, A.S.T.M., Vol. 39, 1939, p. 1084.

[60] "Soil Mechanics in Engineering," by W. S. Housel, *Proceedings*, Eng. Conference on Soils for Engrs., Michigan State College, East Lansing, March, 1941.

[61] "Report from the Soil Mechanics Laboratory at the University of Michigan on Testing Apparatus, Technique of Testing and Investigations in Progress," by W. S. Housel, *Proceedings*, International Conference on Soil Mechanics and Foundation Eng., Vol. II, pp. 24–28; Vol. III, pp. 99–101.

[62] "Design of Flexible Surfaces," by W. S. Housel, *Proceedings*, 23d Annual Highway Conference, Univ. of Michigan, Ann Arbor, February, 1937.

1. That the unconfined compression test procedure represents a more direct approach to problems of this type than does the ring-shear test procedure; and
2. That it is more logical to determine failure values in the laboratory and to work with them rather than to determine yield values and then to increase them by means of an overload ratio.

In this discussion the procedure for selecting the critical shearing resistance value was described by the following definition:

> "The shearing resistance was defined as one half the load at which the specimen failed completely or as one half the load that produced 20% strain in those specimens that bulged without failing."

The writer would characterize this procedure as conveniently indefinite and wholly lacking in the precision required to tie down such an elusive factor as the yield value of plastic clay. Nothing is said about rate of loading or observation of the load versus rate of shear relation, which time effects are the very essence of plastic behavior. When, as in the present case, the real problem hinges on progressive displacement in the soil mass, it is extremely difficult to see how one can avoid the control and observation of all of the variable factors which are essential parts of the problem. In the writer's opinion, the failure to do so is the primary reason for the inadequacy of many load tests that have been run and the greatest handicap to successful utilization of shear tests being attempted at the present time. The result of this failure seems fairly obvious, and the example cited in which shear values obtained by one method are four or five times the other is quite typical. The comparative values, however, may be expected to vary for various materials and for every change in rate of loading during tests. The example of the peculiar properties of Mexico City clay makes this apparent, but still presents no evidence that carefully controlled time-deformation measurements would not have been perfectly applicable.

If the preceding line of reasoning is valid, the next logical step is a correlation of laboratory shear values with construction practice, which by force of necessity has searched out, though blindly, the upper limits to which the soil mass may be stressed without dire results. Practical experience in tunneling in clay as well as in other types of excavation has emphasized that time is of the essence and many things can be done quickly which lead to disaster when construction is delayed. Likewise, the magnitude of the forces involved and the subsequent settlement or subsidence extending over considerable periods of time make it apparent that almost any excavation of magnitude must cause stresses considerably in excess of the yield value. The evidence in all three papers presented in the Symposium confirms this conclusion and further support is given by the correlations referred to previously. Under these conditions the writer can see no reason for shying away from an overload ratio. After all, that is simply being realistic and practical. Having admitted that there is a yield value that can be measured by some test procedure, it would seem to be a definite "bench mark" that may serve as the first step in eliminating uncontrolled guessing. The overload ratio then becomes nothing more than a correlation factor translating laboratory tests into full-scale operation and the means of controlling the hazards involved within practicable limits.

Some further advantage in the ring-shear tests arises from the fact that the smaller samples may be obtained with little change in commercial sampling practice. In general practice the greater difficulty and expense in obtaining the larger undisturbed samples in conjunction with construction operations frequently limit the number of samples and provide a less representative measure of the character of the mass as a whole.

There are several other points raised by other discussers which require comment. Mr. Eremin mentions an inconsistency in Eq. 32 for lateral pressure on the tunnel and Eq. 34*a* for pressure on the invert. The criticism is well taken, but the writer was simply at a loss to improve the development of the equations. The problem of formulating relations to express the pressure variation measured in terms of measured shear values without violating basic principles of mechanics was peculiarly perplexing and was the subject of many months of study. In order to express arching action (which is a function of the diameter of the tunnel) in terms of stress, it was advantageous to deal with elements of mass of the same dimension. To do so required dealing in terms of average stresses on the boundaries of these large cubes although it is recognized that these boundary stresses vary from point to point. Such approximations have been successful in developing bearing capacity relations in interpreting load tests and appear to be equally useful in the present case. Two methods presented themselves in formulating an expression for lateral pressure. The cube at the side could be taken with the horizontal and vertical faces as principal planes establishing the principal stress differential of 2 S between vertical and lateral pressure. The arching action around the tunnel bore in a horizontal direction could then be included by superimpsoing horizontal shear forces on the top and bottom faces to represent diminishing pressure from the back vertical face to the tunnel bore. This led to the fundamental objection that the planes of maximum shear would rotate and the value of maximum shear would be greater than the shear value of the soil.

Consequently the effort was made to find another method of combining the incongruous factors involved. The approximation used was considered to be a less objectionable compromise. The pressure in the region back of the cube is shown as subject to a decrease resulting from a maximum horizontal shear caused by arching action, establishing a principal stress differential of 2 S with principal planes at 45° to the horizontal. A second principal stress differential at the face of the tunnel bore is taken from the elastic solution for stresses in thick cylinders giving a total differential of 4 S.

The combination of stresses which hold for the elastic range to which are added plastic resistance developed by subsequent yielding is beyond the reach of so-called rigorous mathematics in cases such as that under discussion, and the writer has done his best to bridge the gap. That the result is perhaps a crude approximation is admitted, but it is believed to be useful until someone masters the complexities involved more successfully.

In developing the expression for the invert pressure, vertical arching action did not appear to be involved beyond that already taken into account by weight redistribution, and the writer retained the method which he prefers and which involved the use of the large element of mass.

Professor Tschebotarioff raised two questions to which the writer may supply an answer, the first concerning the pressure-cell readings. Earth-pressure readings with the Goldbeck cells were made by bleeding air slowly into the cells until the air pressure was just sufficient for the movable weighing face of the cell to break electrical contact. Care was exercised to cut off the air supply immediately without exceeding this pressure. The movement of the weighing face is very small, something on the order of 0.001 in., and on previous experiments it was found that the procedure outlined made it possible to check earth-pressure readings immediately. In other cases it was found that, when excessive air pressure was allowed to enter the cell, such readings could not be checked for some period of time. These observations were taken to indicate that deformation of the soil in contact with the cell could be controlled to no more than negligible proportions. Further than this the magnitude of the readings and the consistency of results are evidence that they have not been thrown out of focus by inaccuracy in the pressure cell used. Such scattering of results as was experienced, particularly after some period of years, could be traced rather definitely to faulty connections and condensation of moisture in the cells, some of which eventually became useless.

Professor Tschebotarioff also inquired as to the general characteristics of the Chicago and Detroit clay. Professor Terzaghi gave some information in this regard in his paper. Soil index test data for a typical Detroit clay were obtained during cooperative shear tests made several years ago in which a number of investigators cooperated, and typical results have been reported.[58]

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STATISTICAL ANALYSIS IN HYDROLOGY

By L. R. Beard,[1] Jun. Am. Soc. C. E.

With Discussion by Messrs. L. Standish Hall, H. Alden Foster, Paul V. Hodges, Joe W. Johnson, R. H. Blythe, Jr., R. W. Davenport, Edward J. Bednarski, Ralph W. Powell, E. J. Gumbel, and L. R. Beard.

Synopsis

The basic principles of the duration curve are reviewed in this paper in such a manner as to outline the various ways in which it is properly used in hydrologic analyses. Some of the details and differentiations discussed are of small significance compared to inaccuracies existing in the greater part of the hydrologic data available today. However, the proper use of the duration curve is essential to a fundamental understanding of statistical analysis and, since proper use is no more difficult than improper use, the differentiations are deemed relevant from an engineering standpoint.

Conception of the Duration Curve

In its application to flood-control work, the duration curve may be looked upon as a refinement of the practice of basing the design on the largest flood of record. The difficulty of this old practice lies not only in the uncertainty that the maximum occurrence is representative for the period of record, but also in the absence of criteria for determining the variation of magnitude with period of occurrence. Both of these deficiencies are remedied by the proper use of the duration curve.

The theory of the duration curve is relatively simple. The occurrences of the future (those that are "in store" for the future) are considered to constitute a series, arranged in the order of descending magnitude. If there are an infinite number of occurrences in store, then this series may be represented by a continuous curve such as that shown in Fig. 1. Furthermore, if it is considered that there is no cycle or trend in magnitude with respect to time, it may be assumed that occurrences which happen within a given period have each been selected at random from the store of occurrences. Lastly, since the number of occurrences in store is considered to be infinite, the selection of a finite number

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Note.—Published in September, 1942, *Proceedings*.

[1] Associate Engr., U. S. Engrs., War Dept., Los Angeles, Calif.

of occurrences will have no effect on the distribution of magnitudes within the remaining series of occurrences.

Since they have been chosen at random from the store of occurrences, the occurrences of record (those to be used in the analysis) may be used to estimate, by the laws of probability, the curve represented in Fig. 1. With this curve defined, forecasts concerning future occurrences may be made.

CONSTRUCTION OF THE DURATION CURVE

The curve shown in Fig. 1 may represent all of the occurrences ever to happen or it may represent all of the yearly maximum occurrences ever to happen, or any one of many other series. This curve is called a probability or distribution curve and is similar in nature to the normal probability curve of Gauss and Laplace. Equal areas under the curve represent equal numbers of occurrences. The duration curve is the integral curve of the probability curve. Thus, if magnitude of occurrence is plotted against cumulative area under the probability curve, the duration curve is obtained. Fig. 2 shows the duration curve that corresponds to the probability curve shown in Fig. 1. The maximum value of the duration curve ordinate is equal to the total area

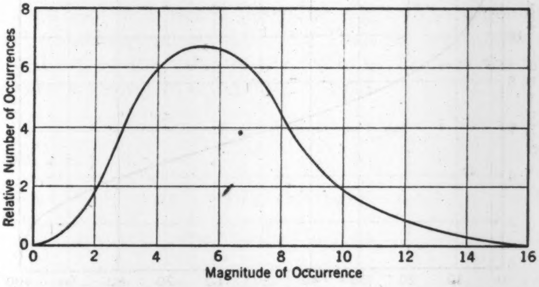

FIG. 1.—DISTRIBUTION CURVE

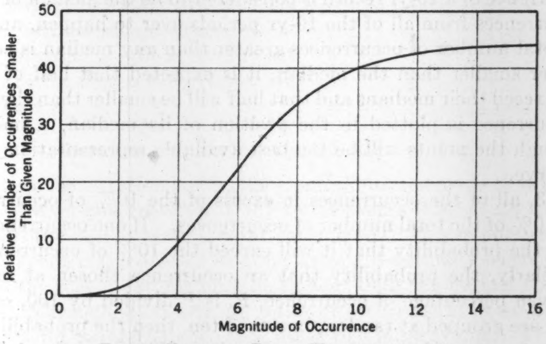

FIG. 2.—DURATION CURVE (UNCONVENTIONAL)

under the probability curve. If this value is called 100%, then the ordinate scale of Fig. 2 will range from zero to 100%, and this scale is called the percentage of occurrence scale. Convention interchanges the ordinate and abscissa scales and reverses the order of the percentage of occurrence scale such that the maximum ordinate corresponds to 0% of occurrence. The duration curve of Fig. 2 is shown conventionally in Fig. 3.

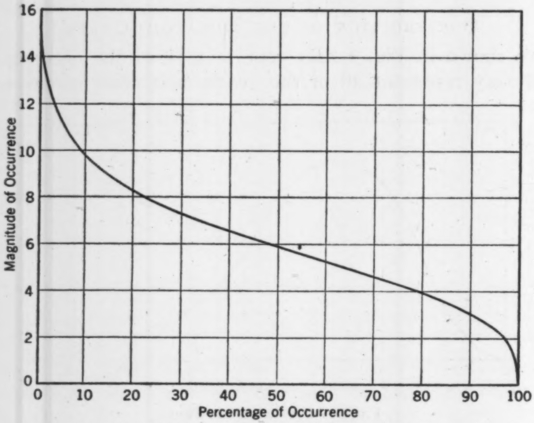

FIG. 3.—DURATION CURVE (CONVENTIONAL)

In constructing the duration curve from the occurrences of record, it is assumed that each occurrence is representative for the period—that is, the largest occurrence in a 10-yr period is considered to be the median of the largest occurrences of all of the 10-yr periods ever to happen. Likewise, the second largest occurrence of a 10-yr record is considered to be the median of the second largest occurrences from all of the 10-yr periods ever to happen, and so forth. Since the total number of occurrences greater than any median is equal to the total number smaller than the median, it is expected that half of the occurrences will exceed their medians and that half will be smaller than their medians. If each occurrence is plotted in the position of its median, a smooth curve drawn through the points will be the best available representation of the true duration curve.

In Fig. 3, all of the occurrences in excess of the 10% of occurrence value constitute 10% of the total number of occurrences. If one occurrence is chosen at random, the probability that it will exceed the 10% of occurrence value is 10%. Similarly, the probability that an occurrence chosen at random will exceed a given percentage of occurrence, P, is P divided by 100, or p. If all occurrences are grouped at random in sets of ten, then the probability that the maximum occurrence of a set chosen at random will exceed the median of the maximum occurrences of all of the sets is 50% (since there are as many above a median as there are below). Therefore, it may be stated that the probability that none of the ten occurrences chosen at random will exceed the median of the maximum occurrences of all of the sets is 50%.

Let P equal the per cent of occurrence of the median. Then q, or $(1 - p)$, will be the probability that an occurrence chosen at random will not exceed the median. The probability that ten consecutive occurrences chosen at random will be smaller than the median is therefore q^{10} and is known to equal 50%, or 0.50. From this, the value of q is 0.933, and p equals 0.067. The percentage of occurrence corresponding to the median of the maximum occurrences from all of the sets is therefore 6.7%.

In a similar manner, it is possible to determine the percentage of occurrence that defines the median of all occurrences consisting of the second maximum occurrence from each set. In this case, the probability that at least nine of ten occurrences chosen at random will fall below a given value is 50%. The probability that none will exceed the value is q^{10}. The probability that exactly one will exceed the value is $10\,p\,q^9$. Then the probability that at least two of ten will exceed the value is one (certainty) minus the two foregoing probabilities and is known to equal 0.50. Thus,

$$1 - (q^{10} + 10\,p\,q^9) = 0.50 \ldots\ldots\ldots\ldots\ldots\ldots (1a)$$

Since p equals $1 - q$,

$$1 - (q^{10} + 10\,q^9 - 10\,q^{10}) = 0.50 \ldots\ldots\ldots\ldots\ldots (1b)$$

and, therefore,

$$10\,q^9 - 9\,q^{10} = 0.50 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

A solution by successive approximations yields a value of 0.836 for q and 0.164 for p. The percentage of occurrence corresponding to the median of the second largest occurrences from the sets is therefore 16.4. Values for the third largest, and others, may be computed similarly.

The "plotting position" of an occurrence is its estimated percentage of occurrence. Plotting positions for any set of ten arbitrarily chosen occurrences are given in Table 1.

TABLE 1.—PLOTTING POSITIONS FOR SERIES OF TEN OCCURRENCES

No. of occurrence	Plotting position (%)
1	6.7
2	16.4
3	25.8
4	35.5
5	45.2
6	54.8
7	64.5
8	74.2
9	83.6
10	93.3

INTERPRETATION OF THE DURATION CURVE

In plotting the duration curve, the assumption was made that the recorded occurrences were representative of any set of events that would occur in a period equal to the period of the record considered. This is not true, however. The largest event may be twice as great as would normally occur in the given period, and the second largest may be smaller than the normal second largest occurrence. Throughout the entire series of occurrences, however, these discrepancies should counterbalance each other to a large extent. The longer the record, the more accurate is the balancing out. Since the series plotted is not accurately representative, it will not plot as a smooth curve. If the points that appear high were lowered somewhat and the points that appear low were raised in such a manner that they all lie on a smooth curve, the points then would be a much better representation of a normal series of occurrences for a period equal to the period of record. In plotting the curve, the largest of ten occurrences was assumed to equal the median of the maximum occurrences of all ten sets. If the occurrence as plotted did not lie on the smooth curve, the assumption was in error, and the point on the smooth curve having the same percentage of occurrence was the new estimate of the median.

Suppose that the "store" of occurrences for the future is arbitrarily divided into sets of ten occurrences each. If one of these sets is chosen at random, there is a 50% chance that the maximum occurrence will exceed the median of the maximum occurrences of all sets. This is obvious since there are as many occurrences to choose from in excess of the median as there are smaller than the median. Thus, the median of the maximum occurrences of all of the sets is the smallest value that probably will not be exceeded in a single set chosen at random. This value for a ten-occurrence set is read on the smooth curve at 6.7%.

Since the smooth curve obtained is the best possible representation of the true duration curve, and since the curve is independent of the number of occurrences used in its construction, a normal series comprising any number of occurrences can be obtained from the curve. Thus, a normal series of one hundred occurrences can be obtained by merely picking off values at the respective points at which each of one hundred occurrences would be plotted (their plotting positions).

It often has been necessary to determine the frequency with which a given value is exceeded. If by the duration curve it is determined that the value in question has a 10% occurrence, then, on the average, 10% of the occurrences will exceed that value. Thus, if fifty occurrences happen in ten years, five can be expected to exceed the value.

THE NORMAL DURATION CURVE

The necessity of extrapolating duration curves of short-period records to make long-period determinations has arisen in almost every hydrologic investigation in which duration curves were used. Unless the curve can be described mathematically, the extrapolation is necessarily based upon a few occurrences at the end of the curve, and, therefore, is liable to large error. If the form of the curve can be determined beforehand, however, every point can play its part in determining the extrapolation.

Many attempts have been made to describe the duration curve mathematically. Of the many equations suggested, only one has a rigid mathematical derivation. This is the Gauss-Laplace normal distribution curve. The fundamental assumption upon which this law has been formulated is that the arithmetic mean of the variate is equal to the median of the variate. The justification of this assumption for many cases has been proved unquestionably by innumerable observed distribution curves. However, the attempts that have been made to fit rainfall or runoff data to the curve have resulted in definite discrepancies. The result is that many individuals have taken it upon themselves to find a more suitable distribution curve to use in rainfall and runoff analysis. Unfortunately, the ordinary treatment of the problem has not been to attempt to discover the discrepancies in the Gaussian law and thereby correct the fundamental assumptions, but rather to adapt the law by imposing limits or using mathematical variations of the equation. Most of these do not adhere to the simple assumptions that form the basis of the law of Gauss. They do not have a rigid derivation, therefore, and will fit observed data only by accident.

Outstanding discussions of the problem here involved appeared in the *Transactions* of the Society in 1927[2] and 1936.[3] Particularly enlightening is the discussion by Arne Fisher.[4] Referring to Thiele, Mr. Fisher states:

> "He mentions that, if the distribution of the variate, x, as actually observed, is found to be skew, or to deviate from the normal probability curve, x may be considered as a 'fictitious' observation of the 'pseudo-normal' variate, $t = h(x)$, which latter function is supposed to follow the Gaussian error law; that is, if $t = h(x)$ is distributed according to a known normal error law of exponential form, it is possible to find the frequency distribution of the inverse function, $x = g(t)$."

Again Mr. Fisher states:

> "Thiele's investigations led him to the obvious fact that if the geometric mean of the observations yields the most probable value, the arithmetic mean of the logarithms of the observations must yield the most probable value of the logarithms of the observations, so that if the logarithms of the observations are assumed to be normally distributed, a law of error can be found for the observations themselves by a logarithmic transformation of the variate. Such a law must necessarily lead to the geometric mean of the observations as the most probable value."

A logarithmic transformation of the variate often has been used with exceptional success. However, as is certain to happen when short-period records are analyzed, many variations from the ideal curve were observed. As it was not apparent that there was theoretical justification for using the logarithmic transformation, it was considered that other curves could be used which would have as good a basis. However, a substitute law seldom has been formulated that did not receive more adverse criticism than praise. Evidence supporting or refuting the applicability of the normal distribution curve to rainfall and runoff analysis is weak. To indicate the applicability or inapplicability of the normal distribution curve to rainfall and runoff occurrences would necessitate obtaining a good many 1,000-yr records.

When a series of rainfall occurrences consisting of every occurrence of record is plotted, the existence of the annual cycle may, and probably does, impart a degree of "skewness" to the series, since it cannot be stated that each occurrence is independent of every other. If the series selected consists of the maximum occurrence of each year, however, the series is not affected by the annual cycle. Such a series, having the advantage of a normal distribution, often can be used instead of the cumbersome series consisting of every occurrence recorded. If it is necessary to construct a duration curve of all occurrences, the duration curve of yearly maximum occurrences may be used to extrapolate the curve if extrapolation appears difficult, since one curve is a function of the other.

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[2] "Straight Line Plotting of Skew Frequency Data," by R. D. Goodrich, *Transactions*, Am. Soc. C. E., Vol. 91 (1927), p. 1.

[3] "An Asymmetric Probability Function," by J. J. Slade, Jr., *ibid.*, Vol. 101 (1936), p. 35.

[4] *Ibid.*, p. 72.

The writer has constructed duration curves for six rainfall records from southern California stations varying in length from forty-nine to sixty-two years. Four duration curves of 24-hr rainfall for these stations are shown in Fig. 4. The data were plotted on logarithmic-probability paper (on which a duration curve will be a straight line if the logarithm of the variate follows the normal probability curve). Some good results have been obtained, and some sizable deviations exist. This, however, constitutes neither a proof of the applicability of the law nor a condemnation of the law. The records are far too short to serve as a basis for testing the law itself. They do, however, illustrate the typical duration curve and show what type of deviations from the normal may be expected.

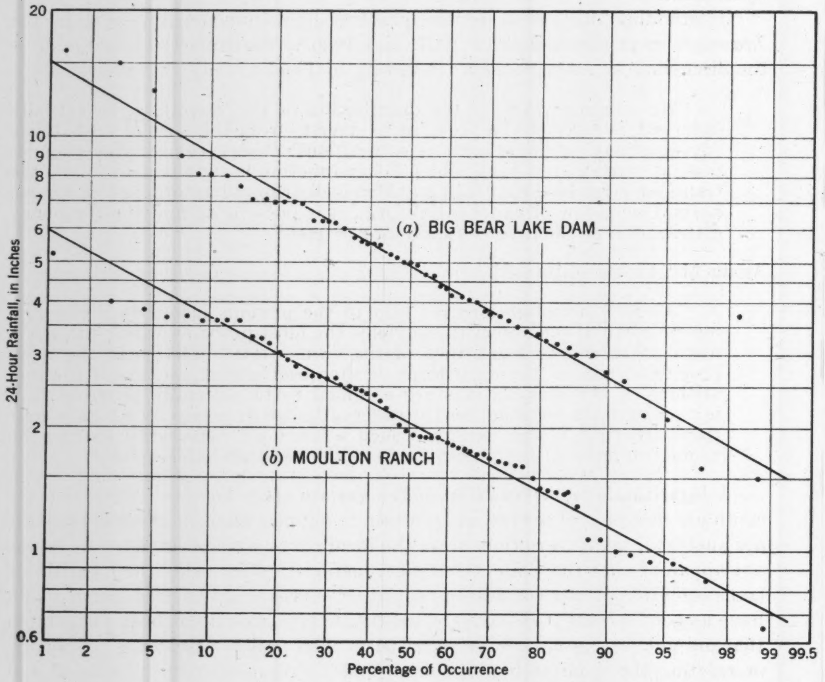
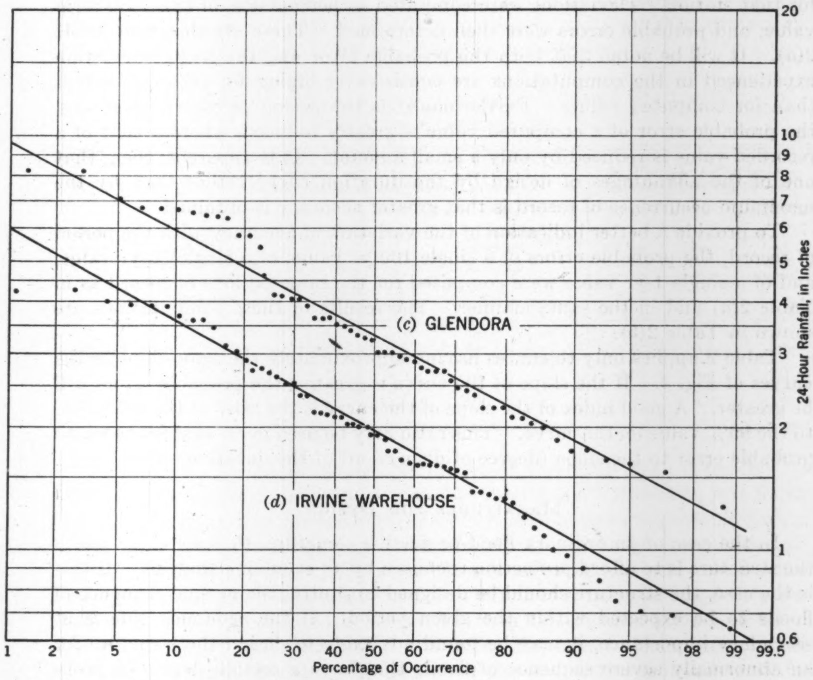


FIG. 4.—DURATION CURVES OF 24-HR RAINFALL;

ACCURACY OF THE DURATION CURVE

The writer has attempted an analysis of the accuracy obtained in using a duration curve. After choosing six long-period rainfall records of stations in the vicinity of Los Angeles, Calif., duration curves of yearly maximum, daily rainfall totals were prepared for the entire period of each station. In addition to these, the periods were broken down into 10-yr, 15-yr, 20-yr, 25-yr, 30-yr, and 40-yr periods, and duration curves for each period for each station were constructed. The average period of record for the stations was about sixty years. Each station, therefore, had approximately six 10-yr records, four 15-yr records, three 20-yr records, two 30-yr records, and one 40-yr record. Duration curves were drawn as straight lines on logarithmic-probability paper.



VICINITY OF LOS ANGELES, CALIF.

The first step in the analysis was to determine for a 10-yr period whether the 10-yr normal maximum occurrence computed by the duration curves had greater correlation than did the actually recorded 10-yr maximum occurrences. The values obtained from the duration curves for a given station were averaged, and the percentage deviation of each value from the mean was computed. The thirty-three deviations thus obtained from the six stations were used to compute the probable error of a single observation. By a similar procedure, the probable error of a single recorded 10-yr maximum occurrence was computed. The probable errors thus obtained were 12.6% for a duration-curve value and 17.5% for a recorded value.

The probable errors for values of the longer periods could not be computed satisfactorily in the same manner since there were not enough occurrences for each station to establish a good mean value. However, probable errors for these periods were computed by assuming that the correct value for a given station is that indicated by the duration curve for the entire period of record for that station. Deviations were computed as percentages of this long-term value, and probable errors were then determined. These are shown in Table 2(*a*). It will be noted that both the probable error and the largest deviation experienced in the computations are consistently higher for recorded values than for computed values. Furthermore, as the period of record lengthens, the probable error of a computed value is greatly reduced, whereas that of a recorded value is reduced by only a small amount. It is apparent, then, that one of the advantages of design by the duration curve rather than by the maximum occurrence of record is that greater accuracy is obtained.

To provide a better indication of the variation of accuracy with the period of record, the probable errors of a single 100-yr value, of a single 50-yr value, and of a single 1-yr value were computed for the same periods of record as in Table 2(*a*) and in the same manner. The results of these computations are shown in Table 2(*b*).

Table 2 applies only to curves having approximately the same slope as the curves of Fig. 4. If the slope of the curve is greater, the probable errors will be greater. A good index of the slope of the curve is the ratio of the 1% value to the 99% value on the curve. This ratio may be used in an analysis to relate probable error to the slope (degree of dispersion) of the duration curve.

Magnitudes for Design

In the case of an ordinary flood-protection structure, the period for which the structure is to afford protection is chosen by an economic analysis. If this is the case, the structure should be designed to control the normal sequence of floods to be expected within the given period. If the economic side is of secondary importance, it may be found advisable to design the structure for an abnormally severe sequence of floods that have a certain degree of probability of occurring during the life of the structure. The extreme case is that in which the structure is designed for the maximum possible flood which can take place within the lifetime of the structure.

A loosely used term in this regard is "maximum probable" occurrence. This term is not definable unless one states the degree of probability intended. "Probable," in the ordinary sense of the word, means greater than a 50–50 chance of happening or greater than 50% probable. However, some engineers think of "maximum probable" as much greater than this. One attempts to conceive of something possible, but having no probability of happening. This is impossible to do since zero probability is synonymous with impossibility. It would seem advisable in view of this misunderstanding to eliminate the phrase, "maximum probable," from the hydrology vocabulary.

In deriving the values shown in Table 1, the writer assumed that the value sought was that which has a 50% chance of being exceeded within a given period. However, if the value sought, for instance, is that which has a 25% chance of being exceeded in one hundred years, then, instead of using possibly a value having a 50% chance of being exceeded in, say, two hundred years (as is often erroneously done), a value can be derived easily that will determine a flood having a 25% chance of being exceeded in one hundred years. This is: $P = 100(1 - 0.75^{0.01}) = 0.287$. Thus, it is not only inadvisable, but also unnecessary, in dealing with a structure to be designed for 100-yr protection, to consider any frequency other than the 100-yr frequency.

TABLE 2.—COMPUTED AND OBSERVED NORMAL MAXIMUM PROBABLE OCCURRENCES

Years of record	(a) PROBABLE ERROR AND MAXIMUM DEVIATION; COMPUTED AND RECORDED						(b) PROBABLE ERROR OF A SINGLE COMPUTATION (%)		
	Normal occurrence (years)	No. of curves	Probable Error (%)		Largest Deviation (%)		100-yr occurrence	50-yr occurrence	1-yr occurrence
			Duration curve	Recorded	Duration curve	Recorded			
10	10	33	13.0	17.9	33	67	19.1	17.1	12.5
15	15	21	13.0	17.7	35	51	17.5	15.9	7.8
20	20	15	11.2	15.6	37	42	14.4	13.0	4.5
25	25	12	5.7	14.0	16	33	7.0	6.2	3.2
30	30	11	6.3	13.0	18	32	7.4	7.2	3.4
40	40	6	4.2	14.3	10	30	5.0	4.3	2.6

To protect a dam against possible failure, it would be ideal, if feasible, to design a spillway for the maximum possible flood flows. However, it is found that this quantity, if estimated conservatively, is so huge as to prohibit design. In his paper entitled "Maximum Probable Floods on Pennsylvania Streams," C. F. Ruff,[5] M. Am. Soc. C. E., states: "The term 'maximum probable flood' is used to describe not the largest flood possible, but a flood so large that the chance of its being exceeded is no greater than the hazards normal to all of man's activities." This definition has been criticized as being too vague. However, the basic idea seems to be a good one and, if the probability so defined can be put down as a percentage probability such as 10% (one chance in ten), then the flood under consideration can be determined mathematically, although a large probable error may exist at such a remote point on the duration curve. This large probable error may cause some to hesitate to use the quantity, but there is no type of analysis available by which the error can be reduced. Therefore, it may be stated that a so-called "maximum probable" flood can be determined by the use of the duration curve if the degree of probability of occurrence within a given period of time is stipulated.

DISCUSSION

In his paper entitled "Duration Curves," H. Alden Foster,[6] M. Am. Soc. C. E., states: "Having only a limited number of items from which to plot a continuous curve, the assumption is made that this continuous curve would pass through the center of each block of the actual record." It is apparent that this assumption is slightly in error, since the values thus obtained for Table 1 would be 5, 15, 25, etc., instead of 6.7, 16.4, 25.8, etc. As the number of items increases, however, the error in this assumption rapidly becomes negligible, and curves plotted by the approximate method will be very close to those plotted in the correct plotting positions. Since computation of plotting positions for various numbers of occurrences requires considerable time, it has been impossible to include a table of plotting positions. For the same reason, it may be found to be impracticable for an individual to compute plotting positions for certain studies. In this case, a good approximation may be made by computing the plotting position of the largest occurrence and estimating the plotting positions of the remainder of the occurrences, using for a guide the approximate plotting positions referred to by Mr. Foster and expressed by the equation:

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[5] *Transactions*, Am. Soc. C. E., Vol. 106 (1941), p. 1453.

[6] *Ibid.*, Vol. 99 (1934), p. 1220.

$$P = \frac{2\,m - 1}{2\,n} \qquad (3)$$

It will be noted that the true plotting positions are always closer to 50% than are the approximate ones, and that the difference between the true and approximate-values becomes smaller as 50% is approached.

In using the duration curve for flood-control design, the design flood is usually defined as that which will be exceeded once (on the average) within the lifetime of the structure. Since common sense dictates that a flood which probably will be exceeded is ordinarily a poor design flood, it appears advisable to adopt the design flood definition as the minimum flood which will probably not be exceeded within the life of the structure. Table 3 gives percentage of occurrence values (one of which was determined in the foregoing analysis) of design floods for various periods for which structures may be designed.

TABLE 3.—PERCENTAGE OF OCCURRENCE VALUES

Life of structure (years)	Percentage of occurrence (design flood)
1,000	0.07
500	0.14
300	0.23
200	0.35
100	0.69
50	1.38
25	2.73
10	6.70
1	50.00

It has been common practice, in order to facilitate an analysis, to plot only the upper portion of a duration curve for which a great number of occurrences are available. From an engineering viewpoint, this procedure is generally justified because it is approximately correct and eliminates a great deal of work that in most cases is unnecessary. However, the construction of a curve from a partial series has two disadvantages—plotting positions are approximate, and a partial series cannot have a normal distribution. These disadvantages may in some cases (in which more accurate work is justified by design considerations and available data) outweigh the advantage of expediency.

Probably one of the most important weaknesses of the duration curve results from its use as a mechanical process, the accuracy of which is often overestimated and sometimes underestimated. One therefore hesitates to use the curve for a record of short duration. However, in addition to the recorded occurrences, it is possible to use all types of rational comparisons and analogies to assist in constructing the curve itself. If this can be done properly, a very satisfactory design can be obtained. Any method used in hydrologic design must be flexible and adaptable to varying quality of data. Hydrology has not advanced far enough but that judgment must still play a major part in any analysis.

CONCLUSIONS

Conclusions drawn from the preceding analyses and discussions are as follows:

(*a*) In order to represent best a distribution curve using a finite number of occurrences, the occurrences should not be plotted with equal spacing;

(*b*) Following from conclusion (*a*), the plotting of a series of occurrences above a base value as a complete series is not mathematically rigorous;

(*c*) The use of the Gaussian normal distribution curve has in no way been proved to be inapplicable to a properly chosen rainfall or runoff series;

(*d*) The duration curve can be used to determine the occurrence that probably will not be exceeded within a given period as well as that occurrence which will be exceeded (on the average) once within a given period; and

(*e*) The "maximum probable" flood, adequately defined, can be determined by use of the duration curve.

DISCUSSION

L. Standish Hall,[7] M. Am. Soc. C. E.—In presenting applications of statistical analysis to hydrology, this paper should prove valuable in many respects. The accuracy of the projection of a short-term record by statistical methods to determine the probability of the occurrence of an event over a longer period of time has long been open to question. In fact, even on the basis of statistics, the probable error of such a projection is large.[8]

The theory of the frequency distribution curve, on which the duration curve is based, is predicated on certain assumptions indicated by the author. These conditions are not rigidly applicable to hydrological data, since it is tacitly assumed:

(1) That there is no difference in any essential respect between the conditions and methods from which the observations are drawn; and, if the observations have been made at different epochs, that there has not been any essential change during that period;

(2) That the conditions that regulate the appearance of any event must be the same for every series, and also for every observation in the series; and

(3) That the individual events must be completely independent of one another.

Of these conditions, the second and third are marked limitations. To satisfy the first condition in stream-flow measurements, no changes should have occurred from the transfer of the measurements from one location to another, or in the accuracy with which gage heights are observed or stream flow measured or estimated. It is well known that this condition has not been fulfilled, since there has been a progressive improvement over the period of available records with regard to all of these items. Recording gages have replaced staff gage readings; improvements have been made in the technique of measuring stream flow and of estimating flood discharges by surface measurements, slope observations, or water-surface drop at contracted openings.[9] In general, no attempt has been made to review previous flood discharge observations in the light of more recent knowledge. The same situation also applies to rainfall observations and most long records are a composite of observations at several locations in a given locality made under varying conditions of accuracy, due to exposure to wind and height of gage above ground. Also, assuming that the measurements of stream flow were all of equal accuracy, there have been changes on many streams due to the works of man causing changes in channel storage by reservoirs, levees, and channel improvement, changes of permeability of the surface by roads, buildings, cultivation of land, soil erosion, and deforestation by lumbering or by fire.

The satisfaction of the second condition requires that no cyclic changes affect the hydrologic record. Although the climatic divisions of the world appear, in general, to be constant, there are known minor fluctuations having a duration ranging from a few years to many centuries. Of the shorter cycles the best known are the 11-yr sunspot cycle and the 35-yr Brückner cycle. Of the longer meteorological periodicities, the glacial and inter-glacial cycles are the most pronounced; but even within historic times there is definite evidence of wet and dry cycles. In the western United States evidence of climatic changes have been derived from the width of the annual growth of the Big Trees of California, checked and adjusted by the variations in the levels in salt lakes of the desert area and by archeological remains. As a result of the correlation of these data, the cycles of precipitation have been traced back to 1,000 B.C., indicating the occurrence of alternate wet and dry periods, each having durations of several hundreds of years.[10] Deviations from the mean extending over cycles of such length could not be ascribed to fortuitous chance.

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[7] Hydr. Engr., East Bay Municipal Utility Dist., Oakland, Calif.

[8] *Transactions*, Am. Soc. C. E., Vol. 91 (1927), p. 58.

[9] "The Measurement and Computation of Flood Discharge," by Carl G. Paulsen, *Transactions*, Am. Geophysical Union, 1939, Pt. II, p. 177.

The third condition that individual events be completely independent of each other cannot be realized in the hydrologic cycle. Since climate is a continuous process, each event is influenced to a greater or less degree by the weather of the preceding days, weeks, months, and in some instances even years. In particular, flood discharges are circumscribed in this respect. A given precipitation may result in either a moderate runoff or a flood of unprecedented magnitude, depending on the temperature and precipitation of the preceding weeks and months, as these may affect the soil infiltration rate, or the accumulation of snow on the ground of a sufficiently shallow depth to permit of melting during a warm rain. This complex set of circumstances surrounding the occurrence of major floods results in a marked skewness of the data, which in turn affect unfavorably the accuracy of prediction from short records of future flood frequency. In some flood records of 30 to 50 years length, there has been included a maximum so greatly in excess of the frequency array that its occurrence could not possibly be predicted by mathematical analysis. More accurate flood-prediction frequencies would be possible if more study were made of precipitation intensities in conjunction with a correlation with the other factors affecting runoff. In general, rainfall records do not exhibit the degree of skewness evidenced by flood-flow records, and hence can be projected with relatively greater accuracy. For example, in a tropical or semi-tropical climate lacking the influence of snow accumulation, the runoff from identical storms can vary considerably, depending on whether the storm occurs on soil either dry, of average moisture content, or saturated. The last condition will naturally produce the maximum runoff, and assuming that this frequency is one fourth, a precipitation occurring once in twenty-five years would produce a maximum flood only once in one hundred years. Such full use of available data, although involving much statistical analysis, would undoubtedly yield results of greater dependability. In regions having accumulations of snow on the ground, another variable is added involving such factors as saturation of the soil before the snow fell, water content of the snow, depth of accumulation, temperature, and precipitation. A favorable coincidence of these factors produces a flood of unprecedented magnitude.

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[10] "Climatic History," *Encyclopædia Britannica*, 14th Ed., Vol. 5, p. 827.

The author's determination of the location of the plotting position of each of a series of occurrences is based on the theoretical expression of the binomial expansion as the form of the frequency distribution.[11] The lack of a complete mathematical treatment makes this portion of the paper obscure, and it would be difficult for one to compute the plotting position of any number of occurrences without the author's formula. This is unfortunate, as the author has not included a complete table of plotting positions.

The basis of the formula is the proposition that the frequencies of 0, 1, 2, $\cdots n$ successes of n events are given by the successive terms of the binomial expansion $(q + p)^n$; namely:

$$q^n + n\,q^{n-1}\,p + \frac{n\,(n-1)}{1 \times 2}\,q^{n-2}\,p^2 + \frac{n\,(n-1)\,(n-2)}{1 \times 2 \times 3}\,q^{n-3}\,p^3 \ldots \quad (4)$$

in which p represents the frequency of successes and q the frequency of failures. The plotting position is determined by equating the summation of successive terms of the binomial series to $\frac{1}{2}$.

There is some question in the writer's mind as to the propriety of this assumption. Considering, for example, a 10-yr period, the first term does not indicate the probability of equal chances of failure or success to secure the median of the events occurring one tenth of the time, but rather the equal chances of failure weighted against the success of obtaining 1, 2, 3, or 4 successes in the series.

This is illustrated in the following tabulation, which shows the calculated distribution for $n = 20$, $p = 0.1$, and $q = 0.9$, or the expansion of $(0.9 + 0.1)^{20}$:

No. of successes	Probability of p per 100 series	No. of successes	Probability of p per 100 series
0	12.16	5	3.18
1	27.02	6	0.89
2	28.51	7	0.20
3	19.01	8	0.04
4	8.98	9	0.01

Two successes would be normal and would have the greatest probability, but cases of one success are almost as frequent and even nine successes may occur about once in 10,000 trials. The probability of obtaining 2 or more successes in 100 series of 20 events each is 60.82 as against 39.18 chances of no successes or only one success.

In all cases in which p does not equal q, an increase in n naturally raises the number of successes most probable of occurrence, but at the same time it increases the dispersion and decreases the skewness. This is revealed if the distribution for $n = 20$ is compared with that for $n = 100$ or the expansion of the series $(0.9 + 0.1)^{100}$; namely:

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[11] "An Introduction to the Theory of Statistics," by G. Udny Yule, Charles Griffin & Co., Ltd., 1917, pp. 291-295.

No. of successes	Probability of p per 100 series	No. of successes	Probability of p per 100 series
0	...	12	9.88
1	0.03	13	7.43
2	0.16	14	5.13
3	0.59	15	3.27
4	1.59	16	1.93
5	3.39	17	1.06
6	5.96	18	0.54
7	8.89	19	0.26
8	11.48	20	0.12
9	13.04	21	0.05
10	13.19	22	0.02
11	11.98	23	0.01

In this case, ten successes would be normal and would have the greatest probability, but the frequency of from 8 to 11 successes is nearly equal. The skewness of the series is much less pronounced than in the previous example. The probability of obtaining 10 or more successes in 100 series of 100 events each is 54.87 as against 45.13 chances of obtaining from zero to nine successes. It will be noted that practically no chance of failure exists in a 100-yr record. To have a probability of 1 in 100 of no success for $p = 0.1$, a 44-yr record is necessary. This indicates the length of record necessary for a similar probability for $p = 0.01$.

The chances of no successes in the series $(q + p)^n$ where $n = 1/p$ increases as p becomes smaller as follows:

p	q	n	Probability of no successes per 100 trials
0.10	0.90	10	34.87
0.05	0.95	20	35.85
0.02	0.98	50	36.42
0.01	0.99	100	36.60

In other words, the chances of failure to obtain during the once-in-a-hundred event in a series of 100 trials are greater than the chances of failure to obtain during the once-in-ten event in a series of 10 trials. This is offset by the probability of one or more successes in each series of trials. Hence, in any given series, it is justified to assume that each occurrence is representative of the period and can be considered the median for $1/n$th of the period.

In selecting the data to form the duration curve for maximum daily precipitation or maximum flood discharges, the maximum for each of the n years can be chosen or n maximum events can be selected regardless of the year in which they occurred. Assuming that the "store" of occurrences or "universe" were known, then, with records of 10 to 50 years' length according to the binomial expansion, the chance of failure would occur in about 35% of the years and the successes would be grouped in the remaining 65% of the years. The selection of the n maximum events would be the proper manner of arranging the data. In records of this type it is usual to find wet years with two, or even three, occurrences that exceed the maximums of the dry years. In fact, on many western streams it is not unusual to find wet years having 50% or more of the daily discharges in excess of the maximum daily flow of the driest years. Under these circumstances the selection of the maximum occurrence of each calendar year obviously does not afford the best use of the available data.

The author indicates that the Gauss-Laplace normal distribution curve is the only one having a rigid mathematical derivation. Many rigid mathematical treatments are possible based on assumptions differing from those forming the basis of the normal curve of error. It is possible that many distributions covering all types may follow this law, but, in general, hydrologic data differ too greatly from it to make possible any forced agreement.

The author states (see heading "The Normal Duration Curve") that the "logarithmic transformation of the variate often has been used with exceptional success." This assumes that the geometric mean is the true average of the series, and that the logarithms follow the normal law of error. The geometric mean of a series is always less than the arithmetic mean, and most hydrologic data have a distribution with the mode less than the mean. This does not indicate that the geometric mean is the most probable value. In many series having a large number of zeros the geometric mean obviously does not apply, as for example in the distribution of daily precipitation. A very cursory examination of such data will indicate zero as the most probable value. Such data cannot possibly follow the normal law of error.

The method used in plotting the duration curves in Fig. 4 is not stated, and if the author's method has been used the location of the plotting positions would be facilitated by appending a table. The selection of the data (which is 24-hr rainfall, in inches) is perhaps unfortunate, due to the fact that in many wet years there might be two or more occurrences of 24-hr precipitation in excess of the maximum occurrences in dry years. The method used in preparing the statistical array is not stated. A discontinuous type of occurrence, as the ones illustrated, is best treated by other methods.[12] For flood discharges, for example, a combination of probability methods to determine rainfall, combined with "rational" methods for flood runoff or a unit hydrograph is most advisable. However, this criticism does not affect the validity of the author's method of analysis.

The discussion of the accuracy of the duration curve is very valuable, but the author does not explain his method of determining the probable error and the largest deviation to sufficient extent to indicate the applicability of his results to other statistics. Much of the author's paper is argumentative rather than analytical, but this is not without value, since it presents the need for a complete analysis of statistical methods as applied to hydrology in order to evaluate the worth of various probability methods brought forth over the past 25 or 30 years and to determine the proper application of the various types of probability studies to engineering statistics. The investigation of this field of hydrology would merit further study.

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[12] "Methods of Estimating Floods" from "Floods in the United States," *Water Supply Paper No. 771*, U. S. Geological Survey, 1936, pp. 28–67.

H. Alden Foster,[13] **M. Am. Soc. C. E.**—A discussion of various questions connected with the use of duration curves in the analysis of hydrological records is contained in this paper. A number of these questions have been treated in some detail by the writer in previous publications.[14, 15, 16] At the risk of some repetition, the following comments are submitted:

Mr. Beard devotes considerable space to a demonstration of his conclusion that the generally accepted method for plotting items of the duration curve by Eq. 3 is in error, although he concedes that, in general, the presumed error is negligible. The writer has been unable to follow the demonstration of Mr. Beard's theorem.

In the first place, he states (see heading "Construction of the Duration Curve") that "the largest occurrence in a 10-yr period is considered to be the median of the largest occurrences of all of the 10-yr periods ever to happen." In the writer's opinion, this statement is not correct. As the duration curve is generally considered, the largest occurrence in a 10-yr period should be taken as the median of the 10 largest occurrences in a 100-yr record, or the median of the 100 largest occurrences in a 1,000-yr record.

To make this clear, assume 10 duration series, each covering 10 years, taken from a 100-yr record:

$$\begin{array}{l} a_1\, b_1\, c_1 \cdots\cdots h_1\, i_1\, j_1 \\ a_2\, b_2\, c_2 \cdots\cdots h_2\, i_2\, j_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{10}\, b_{10}\, c_{10} \cdots\cdots h_{10}\, i_{10}\, j_{10} \end{array}$$

Also assume that these 10 series have been arranged so that the a terms are in descending order of magnitude from a_1 to a_{10}. By Mr. Beard's method, if only one 10-yr series is available, the a term in that series is assumed as equal to the median of the 10 values, a_1 to a_{10}.

Now it is quite possible that one or more of the ten b terms may be greater than the smallest a term, a_{10}. Consequently, if a duration series is prepared using the entire 100-yr record, the first 10 terms in this combined series may include some of the b terms, in which case it will not include all of the a terms. Hence, the median of the "largest occurrences of all of the 10-yr periods" (a_1 to a_{10}) will be smaller than the median of the 10 largest occurrences in the 100-yr period.

It is true (see heading "Construction of the Duration Curve") that "the probability that the maximum occurrence of a set chosen at random will exceed the median of the maximum occurrences of all the sets is 50%." The probability that the said maximum occurrence will be less than the median is also 50%. However, the probability that "none of the ten occurrences chosen at random will exceed the median of the maximum occurrences of all the sets" is not 50% as stated by the author. The 50% probability previously noted applies to the 10 maximum occurrences (the aforementioned a terms); so that the probability that any one of the ten terms in one series (such as a_1 to j_1, for example) will exceed the median of the ten *a* terms is $0.50 \times 0.10 = 0.05$, or 5%. Hence the probability that none of the ten occurrences will exceed the median should be 95% instead of 50%.

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[13] With Parsons, Brinckerhoff, Hogan & Macdonald, New York, N. Y.

[14] "Theoretical Frequency Curves and Their Application to Engineering Problems," by H. Alden Foster, *Transactions*, Am. Soc. C. E., Vol. LXXXVII (1924), p. 142.

[15] "Duration Curves," by H. Alden Foster, *ibid.*, Vol. 99 (1934), p. 1213.

[16] "Floods in the United States," *Water Supply Paper No. 771*, U. S. Geological Survey, 1936 (Chapter on Methods for Estimating Floods, by H. Alden Foster).

It seems obvious to the writer, therefore, that Mr. Beard's method of computing the percentage of occurrence corresponding to the median of the maximum occurrences from all of the sets cannot be justified. The same conclusion applies to his method for computing the other plotting positions. In the writer's opinion, Eq. 3 gives the theoretically correct plotting position. Incidentally, this method has been quite generally accepted for the plotting of duration curves in hydrological studies during the 25 years since the first World War.

The writer is in full agreement with most of Mr. Beard's discussion of duration curves, except as to the general use of the normal, or Gaussian, probability curve. It has been quite well established, both in actuarial studies and in various statistical studies in other sciences, such as biology, that unsymmetrical distributions of many types do occur. There seems to be no reason why similar distributions should not occur in hydrology and meteorology. In fact, it is the writer's observation that the unsymmetrical, or skew, distribution is the rule rather than the exception, in hydrological data. The mathematical form of the skew probability or duration curve has been discussed extensively elsewhere,[14, 15, 16] and as noted by Mr. Beard, and would seem to be outside the scope of the present discussion.

Mr. Beard's study of the accuracy of the duration curve is of interest. The writer fully concurs in his statement (see heading "Accuracy of the Duration Curve") that "one of the advantages of design by the duration curve rather than by the maximum occurrence of record is that greater accuracy is obtained." The discussion of the use of such terms as "maximum probable occurrence" is also well taken.

In conclusion, Mr. Beard should be commended for his description of the duration curve and its application to problems in hydrology. Although the writer does not agree with Mr. Beard's method of locating plotting points for the curve, this discrepancy is of only minor significance in the practical application of the curve.

PAUL V. HODGES,[17] M. AM. SOC. C. E.—That the straight-line logarithmic curves fit the rainfall records very well, and that the logarithms of the observations may be assumed to be normally distributed, are demonstrated in this paper. In Figs. 4(*b*) and 4(*d*), it appears, however, that curves developed by some other method, as for instance either the Slade,[3] Foster,[6] or Hazen[18] method, would give even closer agreement with the rainfall records.

Rainfall and runoff statistics for a period of record represent the samples for basing a prediction of what may be expected. The coefficient of variation and coefficient of skew of these samples represent their characteristics, although the coefficient of skew for available periods of record may be in error and therefore not truly representative.

To obtain a straight-line logarithmic curve it is often necessary to change both the coefficient of variation and coefficient of skew, and the question arises as to whether such a change would not change the characteristics of the sample so that it would fail to be truly representative.

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[17] U. S. Engrs. Office, Omaha, Nebr.

[18] "Flood Flows," by Allen Hazen, John Wiley & Sons, Inc., 1930, p. 188.

In the development of curves by the Slade, Foster, and Hazen methods, it is often necessary to adjust the coefficient of skew to obtain a curve that best fits the data, but the computed coefficient of variation is a true representative of the sample as far as known, and should not be changed.

In Fig. 5 the yearly runoff of the Gila River at San Carlos Reservoir is shown. The coefficient of variation as computed for these data was 0.876. In drawing the curve for the Foster method this value of the coefficient of variation was used, but the coefficient of skew of 2.3 was found to give the best fit.

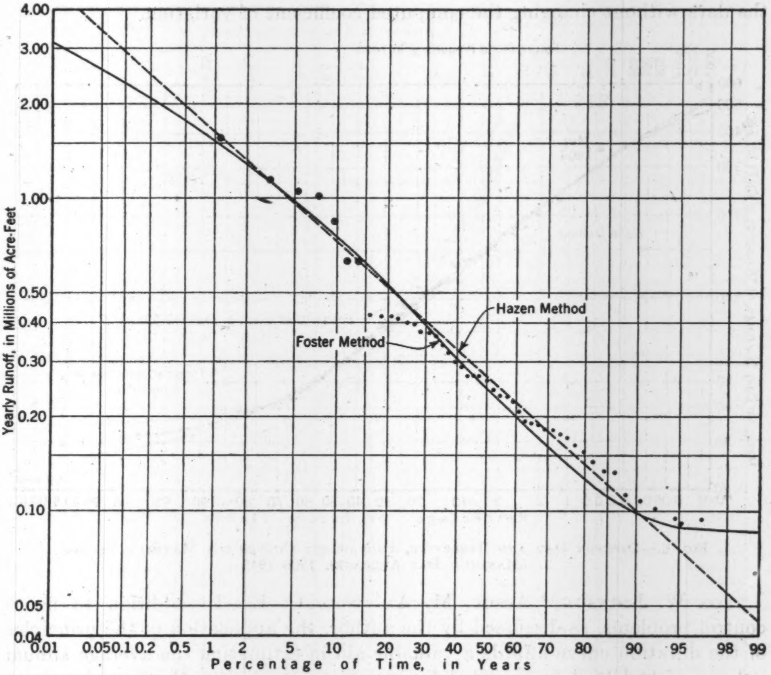

FIG. 5.—PROBABILITY CURVES FOR YEARLY RUNOFF IN ACRE-FEET, GILA RIVER AT SAN CARLOS RESERVOIR, ARIZONA, 1895–1941

To obtain a straight line with the foregoing value of the skew coefficient, a coefficient of variation of 0.82 must be used, according to the Hazen factors, thus changing the characteristics of the sample. Both curves appear to fit the data except for the lower values.

A probability study of floods of the Red River near Denison, Tex., is shown in Fig. 6. Flows that averaged 30,000 cu ft per sec daily or more were selected from the tabulation of daily discharge and the mean flood was 72,400 cu ft per sec (maximum daily average), with a coefficient of variation of 0.664.

During the period, 1906–1942, there have been 117 such occurrences. These floods occurred during different seasons of the year, the largest being in April, May, and June. They are considered to be a homogeneous series of flood quantities.

The curves in Fig. 6 were developed by the Slade and Foster methods and fit the data very much closer than any straight line.

In view of the foregoing examples it appears that rainfall, runoff, and flood occurrences may or may not indicate straight-line logarithmic curves, and that the various methods of developing curves are all valuable tools for use with different statistical characteristics. It also appears that a straight-line logarithmic curve would be justified only in cases where it can be made to fit the data without changing the computed coefficient of variation.

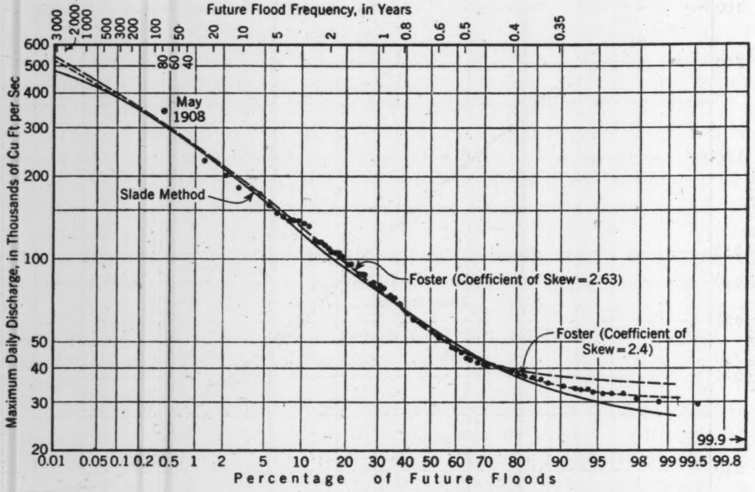

FIG. 6.—DENISON DAM AND RESERVOIR, PROBABILITY CURVES FOR MAXIMUM FLOODS (MAXIMUM DAY AVERAGE), 1906–1942

JOE W. JOHNSON,[19] ASSOC. M. AM. SOC. C. E.—In addition to flood-control problems, as discussed by the author, the application of the principles of the duration curve affords a valuable aid in estimating the average annual volume of bed load transported by streams. It is often the case, however, that estimates of bed-load transportation are required for streams for which discharge records are not available. A flow-duration curve, consequently, cannot be constructed for the stream in question, but it may be possible, in some instances, that the duration curve from an adjacent stream for which discharge records are available can be used. For example, Thorndike Saville, M. Am. Soc. C. E., and John D. Watson, Assoc. M. Am. Soc. C. E.,[20] found that, by using a ratio of the discharge to the mean discharge as the ordinate,

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[19] Asst. Prof., Dept. of Mech. Eng., Univ. of California, Berkeley, Calif.

[20] "An Investigation of the Flow-Duration Characteristics of North Carolina Streams," by Thorndike Saville and John D. Watson, *Transactions*, Am. Geophysical Union, 1933, pp. 406–425.

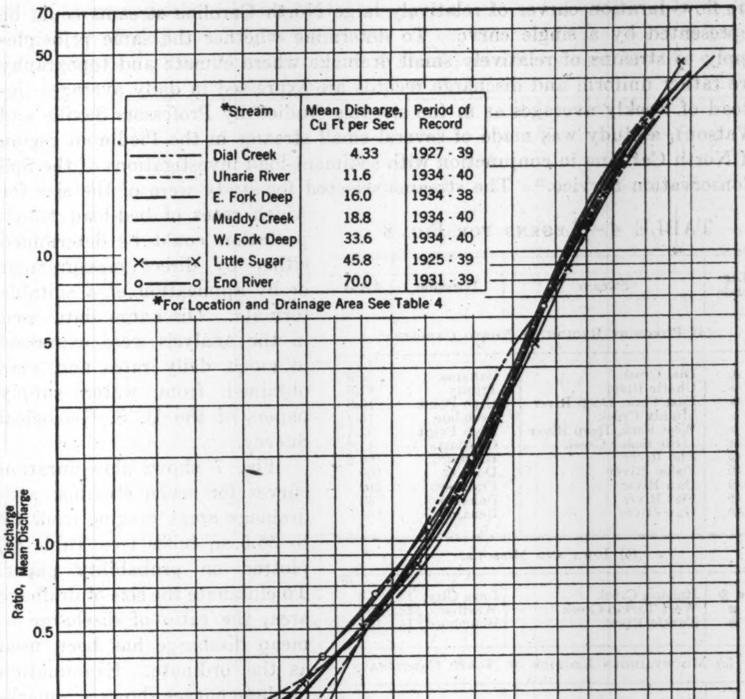

FIG. 7.—FLOW DURATION CURVES FOR SEVERAL SMALL STREAMS IN THE PIEDMONT REGION OF NORTH CAROLINA

the flow-duration curves of relatively large North Carolina streams could be represented by a single curve. To determine whether the same principles apply to streams of relatively small drainage where climate and topography are rather uniform and discharge records are expressed in daily averages (instead of weekly averages as in the streams studied by Professors Saville and Watson), a study was made of several small streams in the Piedmont region of North Carolina in conjunction with sediment-load investigations of the Soil Conservation Service.[21] The streams selected for study were of the size for which rates of bed-load transportation could be determined either by direct measurement or by application of a suitable formula. Discharge data used in the analysis were expressed in mean daily rates and were obtained from water supply papers of the U. S. Geological Survey.

TABLE 4.—LEGEND FOR FIG. 8

Legend[a]	Stream	Location	Area[b]
	(*a*) PIEDMONT REGION OF NORTH CAROLINA		
①	Dial Creek	Bahama	4.9
●	Uharie River	Trinity	11.3
◆	East Fork, Deep River	High Point	13.9
-◇-	Muddy Creek	Archdale	16.2
✦	West Fork, Deep River	High Point	32.1
②	Little Sugar Creek	Charlotte	41.4
③	Eno River	Hillsboro	66.5
④	Fisher River	Dobson	109
⑤	Dan River	Francisco	119
⑥	Flat River	Bahama	150
⑦	Haw River	Benaja	168
	(*b*) IOWA AND MISSOURI		
■ ⑧	Ralston Creek	Iowa City, Ia.	3
-■-	West Tarkio Creek	Westboro, Mo.	200
■	Tarkio River	Blanchard, Ia.	105
	(*c*) MOUNTAINOUS REGIONS OF NORTH CAROLINA[c]		
○	Coweeta Creek 9	Asheville	2.79
ϕ	Coweeta Creek 8	Asheville	2.93
-○- 1⃞	Beetree Creek	Swannanoa	5.46
✦ 2⃞	South Fork, Mills River	Pink Beds	9.99
3⃞	Linville River	Branch	65
△	French Broad River	Rosman	67.9
4⃞	Tye River	Roseland, Va.[c]	68
	(*d*) STATE OF WASHINGTON		
-△-	Missouri Flats Creek	Pullman	27.5
▲	Four Mile Creek	Pullman	71.9
□	South Fork Palouse River	Pullman	81.1
	(*e*) WISCONSIN		
-⊟-	Little La Crosse River	Leon	77.1
⌽	Coon Creek	Coon Valley	77.2

[a] Numerals in circles or squares denote a period of ten years or more. [b] Drainage area, in square miles. [c] All in North Carolina except Roseland (Va.).

Fig. 7 shows flow-duration curves for seven streams, with drainage areas ranging from 4.9 to 66.5 sq miles (see Table 4), plotted on probability paper. To eliminate the size of drainage area, the ratio of discharge to mean discharge has been used as the ordinate. Examination of these curves shows a remarkably close agreement for the upper 40% of the time—that is, for the higher rates of discharge, the various curves are almost identical. For that part of the curves representing the lower rates of flow, however, the curves depart radically from each other. These conditions are to be expected because, for low rates of flow, the drainage basin characteristics, such as soil types, vegetal cover, tillage methods, and stage of soil erosion, influence the rate of runoff greatly. However, during periods when high rates of precipitation cause high runoff rates, the drainage basin characteristics affect the runoff very little.

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[21] "Studying Sediment Loads in Natural Streams," by Gilbert C. Dobson and Joe W. Johnson, *Civil Engineering*, February, 1940, pp. 93–96.

Of interest in this respect is the comparison of the curves for the Uharie River and Muddy Creek with the curves for the East and West forks of Deep River (Fig. 7). The drainage basins of the Uharie River and Muddy Creek are in the slate belt of North Carolina where the depth of soil above bedrock is usually only from 0 to 3 ft. Consequently, there is little space for groundwater storage, and runoff is rapid and flashy in character. The low-water flows are relatively small; hence, the duration curves for these two streams plot relatively lower than the other streams in Fig. 7. On the other hand, the

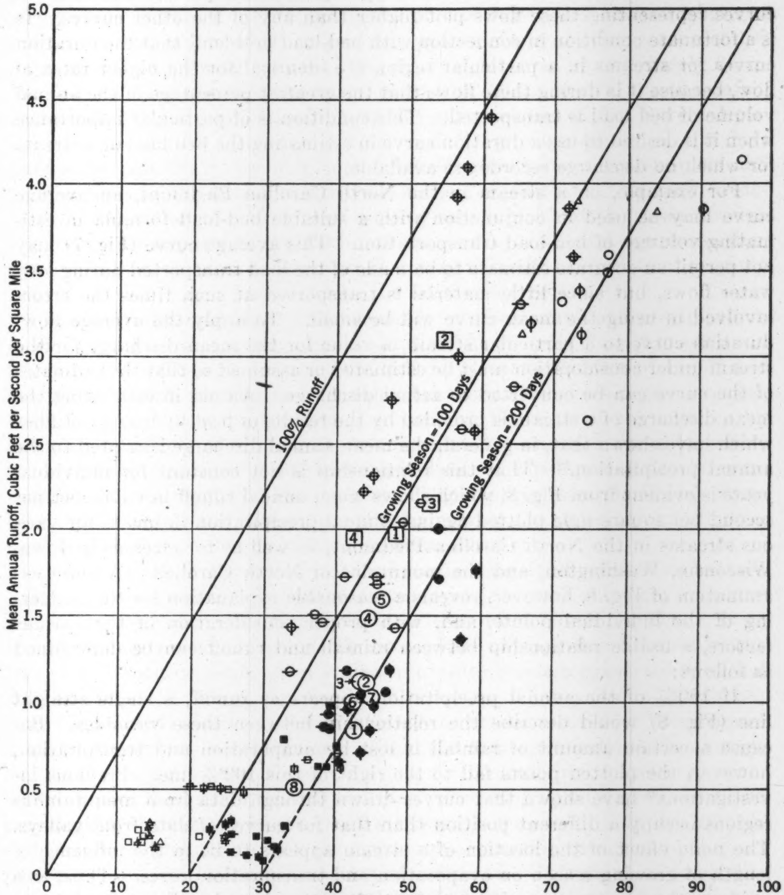

FIG. 8.—RELATION BETWEEN MEAN ANNUAL RUNOFF AND ANNUAL PRECIPITATION FOR STREAMS IN VARIOUS LOCALITIES IN THE UNITED STATES

drainage basins of the East and West forks of Deep River are in the granitic schist region of the Piedmont where weathering is relatively deep and a considerable depth of soil is available for ground-water storage. Thus, the low-water flows in these streams are relatively high and the part of the duration curves representing these flows plot higher than any of the other curves. It is a fortunate condition in connection with bed-load problems that the duration curves for streams in a particular region are identical for the higher rates of flow, because it is during these flows that the greatest percentage of the annual volume of bed load is transported. This condition is of particular importance when it is desired to use a duration curve in estimating the bed load of a stream for which no discharge records are available.

For example, on a stream in the North Carolina Piedmont, an average curve may be used in conjunction with a suitable bed-load formula in estimating volumes of bed-load transportation. This average curve (Fig. 7) may not permit an accurate estimate to be made of the load transported during low-water flows, but since little material is transported at such times the errors involved in using the mean curve will be small. To apply the average flow-duration curve to a particular stream, a value for the mean discharge for the stream under consideration must be estimated or assumed so that the ordinates of the curve can be converted to actual discharge. An aid in estimating the mean discharge of a stream is provided by the results of past hydrologic studies which have shown that, in general, the mean annual discharge is related to the annual precipitation.[22] That this relationship is not constant for individual years is evident from Fig. 8, which shows mean annual runoff in cubic feet per second per square mile plotted against annual precipitation in inches for various streams in the North Carolina Piedmont, as well as for streams in Iowa, Wisconsin, Washington, and the mountains of North Carolina. A close examination of Fig. 8, however, reveals a reasonable explanation for the scattering of the individual points, and, with proper consideration of the various factors, a usable relationship between rainfall and runoff can be determined as follows:

If 100% of the annual precipitation appears as runoff, a single straight line (Fig. 8) would describe the relationship between these variables. Because a certain amount of rainfall is lost by evaporation and transpiration, however, the plotted points fall to the right of this 100% line. Previous investigations[23] have shown that curves drawn through data from mountainous regions occupy a different position than that for curves of data from valleys. The main effect of the location of a stream appears to be in the influence of length of growing season on evaporation and transpiration losses. Thus, as a guide to the construction of curves from the data plotted in Fig. 8, a plot was made of the annual water loss (represented by the horizontal distance from the 100% curve to the plotted point) against the average length of growing season for the county, or counties, in which the drainage basin of the stream is located. (Data on the length of the growing season were obtained from the 1941 *Yearbook of Agriculture* in which a climatic summary for practically every county in the United States is given.[24]) A straight line was then drawn through the plotted points. If it is now assumed that a certain base rainfall is required to supply evaporation and transpiration losses, and that practically all precipitation greater than this amount appears as runoff, curves may be constructed as in Fig. 8. Thus, from the plot of water loss against length of growing season, a loss of 20 in. seemed apparent for a growing season of 100 days, and for a growing season of 200 days the loss was approximately 30 in. By drawing lines through the base amounts of 20 in. and 30 in. of rainfall, and parallel to the 100% runoff curve, the curves shown in Fig. 8 were determined. It is to be noted that the curve for a growing season of 100 days passes through the group of points obtained from mountainous regions and from regions in the northern latitudes of the United States. It may be observed further that the curve for a growing season of 200 days passes through the data from streams in the Piedmont and Iowa. As a first approximation, these curves have been constructed as straight lines; however, further study may show that curved lines describe the relationship better.

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[22] "The Elements of Hydrology," by A. F. Meyer, John Wiley & Sons, Inc., p. 446.

[23] "Results of Stream Measurement," by F. H. Newell, Fourteenth Annual Rept., U. S. Geological Survey, Pt. 2, 1892–1893, pp. 89–155.

It is acknowledged that the relationship between runoff and precipitation, as shown in Fig. 8, is merely a generalization that reveals class likenesses and obscures the differences between the individual characteristics of runoff and the character and distribution of rainfall (as well as the effect of temperature, vegetal cover, topography, soil, and subsoil) on the disposal of rainfall. Although the method of arriving at an estimated flow-duration curve for a particular stream upon which all hydrologic data are missing may not be compatible with the results of the more recent hydrologic research, it appears to be the only simple and direct approach to provide the necessary information used in estimating volumes of bed-load transportation.

The use of Figs. 7 and 8 in bed-load studies may be illustrated as follows: Assume that an estimate is desired of the average annual volume of bed load transported in a particular stream in the Piedmont of North Carolina. If no discharge data are available on the stream, the average flow-duration curve from Fig. 7 must be used. The ordinates of this curve are converted to actual discharges by the use of Fig. 8. From the map of average annual rainfall, shown in the 1941 *Yearbook of Agriculture*,[24] the annual rainfall for the drainage basin of the stream is determined. From the same publication, the average length of growing season is also determined. By entering the abscissa scale of Fig. 8 with the annual rainfall and then proceeding upward until the curve representing the average length of growing season is reached (only the 100-day and 200-day curves are shown), the mean annual runoff in cubic feet per second per square mile is read from the scale of ordinates. This value of mean discharge, multiplied by the area of the drainage basin in square miles, gives the estimated mean discharge. Multiplying each value along the ordinate scale of the mean duration curve (Fig. 7) by the value of the mean discharge, a flow-duration curve with ordinates expressed in cubic feet per second now is available for the stream under consideration.

A table that shows the frequency of occurrence of various discharges (distribution curve) can be prepared from the flow-duration curve—that is, the total range in discharge is divided into a convenient number of parts and the percentage of time that the mean discharge in each part prevails is then determined and tabulated. The time that each discharge prevails per year is computed. The volume of bed load that is transported per unit time for each discharge may be determined either by a suitable bed-load formula (such as the ϕ-ψ function of H. A. Einstein,[25] Assoc. M. Am. Soc. C. E.) or by a relationship between load and discharge established by direct measurement[26] on the stream under investigation. The rate of transportation per unit of time for a particular discharge, multiplied by the total time that the discharge prevails per year, gives the annual volume of bed load. The sum of the amounts for the various discharges gives the average annual volume of sediment transported as bed load.

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[24] "Climate and Weather Data for the United States," *Yearbook of Agriculture*, 1941, U. S. Government Printing Office, Washington, D. C., pp. 685–1228.

In connection with the computations for the duration curves shown in Fig. 7, it is of interest to note that, for Muddy Creek and the East and West forks of Deep River, curves were computed directly from the stage recorder charts as well as from the records of mean daily discharge. The procedure of determining, directly from the recorder charts, the percentage of time that a particular discharge is equaled or exceeded gives the correct curve, because instantaneous discharges are used instead of daily averages. A comparison of the curves as calculated by the two methods showed that, for a particular stream, the curve based on instantaneous discharges was practically coincident with the curve based on mean daily discharges for lower rates of flow; was slightly below the mean daily curve for flows in the vicinity of the average discharge; and was above the mean daily curve for the highest discharges. The relative location of the two curves is to be expected because, during periods of low flow, the instantaneous and the mean daily discharges are practically the same, and the duration curves naturally coincide. For the high rates of flow, the highest values of discharge are obtained when instantaneous discharges are considered; hence, the duration curve based on instantaneous discharges will lie above the curve based on mean daily flows. A plot of the two curves on probability paper shows that, for all practical purposes, the curves can be considered to coincide. Because of the ease with which curves may be calculated from tabulations of mean daily discharges, the curves shown in Fig. 7 were computed on that basis.

The duration curves shown in Fig. 7 are based on records which are probably far too short to permit the development of a curve that is typical of the North Carolina Piedmont. In fact, for only a few rivers in the United States is it possible to obtain records for a period sufficiently long to include a reasonably wide range of conditions. The author's analysis of duration curves, especially as to their accuracy and the type of deviations that may be expected, is of particular value in connection with the applications of the duration curves to bed-load studies as illustrated in this discussion. In such an application, where the shape of the curve in the range of the higher discharges is of primary importance, the material presented by the author is of great value.

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[25] "Formulas for the Transportation of Bed Load," by H. A. Einstein, *Transactions*, Am. Soc. C. E., Vol. 107 (1942), pp. 561–577.

[26] "The Flow of Water on a Movable Bed," by H. A. Einstein, presented before the 2d Hydraulic Conference, Univ. of Iowa, Iowa City, Iowa, June, 1942.

R. H. Blythe, Jr.,[27] Esq.—Failure of practical application to meet conditions that are the fundamental bases of the several mathematical theories of flood-frequency curves makes mathematical refinement in the fitting of such curves a meaningless labor. Furthermore, flood-frequency curves find their most justifiable use in the solution of economic problems, such as the estimation of average annual damage, rather than in the forecasting of the maximum flood for design purposes. For these economic problems, the lower and better-defined parts of the curves are usually at least as important as the extremities, which show the frequency of the larger floods. Extrapolation of the upper end of such a curve is certainly an unjustifiable procedure in view of the discrepancy between natural conditions and the assumptions underlying the theories.

Since these things are so, a curve fitted freehand to plotted points would seem to be the logical method of expressing the flood-frequency information contained in the usual record of flood flows; but at this point the question of the plotting position of the points arises. Currently, there seem to be two major opinions on this subject, and a number of minor ones, including the hypothesis presented in the paper.

To put the problem simply and concretely, consider a series of twenty occurrences. It makes no difference whether they are maximum annual floods, the twenty largest floods of record, or some other similar series. These occurrences are assumed to represent a small sample of an indefinitely large population. The mean of this sample can be considered an estimate of the mean of the population of events specified in the same way as the sample. Also, assume that the maximum observed event in the sample affords some information about the largest 5% of the population. In this sample a value equal to this maximum has occurred with a frequency of 5%. Therefore, the supporters of the "California method"[28] (a very simple and ordinary concept to be called a "method") contend that the maximum flood should be plotted as having a frequency of 5%; but Mr. Foster and his followers[16] point out that there is no reasonable basis for assuming that this one particular maximum event in a sample of twenty represents the exact 5% frequency point. They contend, rather, that it represents the entire class of events occurring with a frequency of 5% and less. They state that, since it represents the interval 5% to 0, a reasonable point for plotting is the midpoint of this frequency interval, or 2½%. This seems more logical than the California technique. Statistically, the assumption is that the one maximum occurrence observable in a series of twenty is the median value of the twenty values that would compose the upper 5% of a sample of 400, or the median value of the upper 5% of the infinite population it is taken to represent.

There seems to be still more logic in assuming that the maximum observed occurrence in one sample of twenty is the mode of the distribution of extreme values in a number of samples of twenty observations. This follows from the fact that the mode, by definition, occurs most frequently, and therefore any single value chosen from a non-normal distribution is more likely to be the mode than the average or the median. In any normal distribution the three coincide, but the distribution of the extreme values of samples drawn from a normal population is non-normal.

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[27] U. S. Dept. of Agriculture, Forest Service, Washington, D. C.

[28] "Floods in the United States," *Water Supply Paper No. 771*, U. S. Geological Survey, 1936.

Studies of the distribution of the extreme values of samples of n observations taken from an infinite normal population have been made by L. H. C. Tippett,[29] who gives tables of the distribution of the extreme values of samples of several sizes from $n = 3$ to $n = 1{,}000$. The values of the variate are given in terms of the standard deviation of the original population; also given are tables showing the mean, standard deviation, and β_1 and β_2 (coefficients of skewness and kurtosis) of the distributions for $n = 2$ to $n = 1{,}000$. From these tables it was possible to compute the value of the mode by the formula:

$$\text{Mode} = \bar{x} - \sigma \frac{\sqrt{\beta_1}\,(\beta_2 + 3)}{2\,(5\,\beta_2 - 6\,\beta_1 - 9)} \quad \ldots\ldots\ldots\ldots\ldots (5)$$

in which $\bar{x}$ = the mean, and σ = the standard deviation. Values of the mode thus computed are shown in Table 5. It should be noted that they are expressed in terms of multiples of the standard deviation.

If these are the most likely values for the extreme observation of a single sample, then the probability corresponding to this modal value is the most likely estimate of the probability (or frequency) of the extreme observation of the single sample. The probabilities corresponding to the modal values for samples of different sizes are given in Col. 3, Table 5. They were found by looking up the modal values in tables of the normal distribution.

By this hypothesis, the tabulated probabilities are the plotting positions of the extreme value of a sample. Also listed in Table 5 (Col. 4) are the plotting positions according to Mr. Foster's hypothesis. It will be noted that the differences are not large. It would seem logical for practical purposes to adopt his method as giving a reasonable and simple approximation of the best estimate. Because of the failure of actual flood occurrences to form a normal, distribution, and because of other discrepancies between theory and practice the labor involved in calculating the mode and its probability is unjustifiable. Of course, Table 5 could be used, and values for other sizes of sample could be interpolated.

TABLE 5.—MODE, PROBABILITY, AND PLOTTING POSITION OF EXTREME VALUE IN SAMPLES OF DIFFERENT SIZES

Size of sample (n) (1)	Mode of extreme values (2)	Probability (3)	Plotting position (4)
2	0.50851	0.306	0.250
5	1.06553	0.143	0.100
10	1.42421	0.077	0.050
20	1.74305	0.041	0.025
60	2.18877	0.014	0.0083
100	2.37639	0.0087	0.005
200	2.60510	0.0046	0.0025
500	2.90778	0.0018	0.0010
1,000	3.11482	0.00092	0.0005

Unfortunately, the problem of where to plot the next highest and succeeding floods remains to be solved. This is principally a matter of theoretical interest and is beyond the mathematical abilities and patience of the writer.

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[29] "On the Extreme Individuals and the Range of Samples Taken from a Normal Population," by L. H. C. Tippett, *Biometrika*, Vol. XVII, Pts. III and IV, p. 364.

R. W. Davenport,[30] **M. Am. Soc. C. E.**—An interesting illustration of the significance of plotting flood data, by the method presented by Mr. Beard, is afforded by a study made by the writer. The frequency relation, or curve, of maximum annual floods is developed graphically on probability paper for a certain river measurement station on the basis of available records and an estimated extrapolation of the relation to cover very rare floods. The derived relation is assumed to represent flood occurrences of the future or those that are "in store." For purposes of the study the writer developed the highest 1% of these flood occurrences into terms of 100 rates of discharge selected on the scale of frequency so that each term represented a range of equal frequency of occurrence.

The practical significance of the rates of discharge so selected is that, if a 10,000-yr period is conceived as so ideal that the 10,000 maximum annual floods would plot smoothly on the foregoing curve, the selected rates would represent the 100 highest annual floods in the 10,000 years, or the 100-yr floods, as they are called herein.

Although the average frequency of the selected floods is once in 100 years, it is apparent that if the 10,000-yr period were considered as one hundred distinct 100-yr periods, the distribution of the 100-yr floods among such periods would not be uniform. In fact, some 100-yr periods would have none of the 100-yr floods, some would have one, and some would have two or more. Assuming that rare floods occur fortuitously (it is enlightening to explore the assumption although it may vary more or less from the fact), statistical theory affords a means, through the application of the normal probability (binomial) distribution, to compute the theoretical distribution of the 100-yr floods among the one hundred 100-yr periods. Such distribution would be approximately as follows:

100-yr periods	100-yr floods per period	100-yr floods
37	0	0
36	1	36
19	2	38
6	3	18
2	4	8
100		100

The one hundred 100-yr flood values were assigned numbers consecutively from 1 to 100 and, continuing further on the assumption of fortuitous occurrence, random selections were made of numerous (26) sets of 100 numbers within that range. In each set the first thirty-six numbers were taken to represent the maximum floods in the respective thirty-six 100-yr periods in which one 100-yr flood occurred (second term, third column of the foregoing tabulation). The next thirty-eight were taken to represent the nineteen groups of two 100-yr floods which occurred at the rate of two floods per century. The higher of each group of two was selected and the lower discarded.

Following a similar procedure for the groups of three and the groups of four, the list was reduced to sixty-three terms which afford a basis for evaluating the sixty-three greatest of the maximum floods for 100-yr periods in the assumed 10,000 years. This left thirty-seven 100-yr periods of the one hundred in the 10,000-yr period in which the maximum floods were smaller in magnitude than any of the 100-yr floods.

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[30] Hydr. Engr., Div. of Water Utilization, Dept. of the Interior, U. S. Geological Survey, Washington, D. C.

The twenty-six sets of sixty-three flood values obtained as indicated were each arranged in order of magnitude and those of the same numerical order were averaged to produce a master set representing, in a sense quite limited, of course, experience of a 260,000-yr period.

According to the reasoning on which the plotting procedure outlined by Mr. Beard is based, the probability is 0.50 that the median of the 100 greatest floods for 100-yr periods will not be exceeded in any single 100-yr period. The probability that it will not be exceeded in a given year is $(0.50)^{1/100}$ or 0.9931. Thus, in plotting the maximum annual floods for a given 100-yr period, the greatest value would be plotted at 0.9931 rather than at 0.9950 as is contemplated in some other plotting procedures.

The flood data studied showed the following results (in cubic feet per second):

Average median of 100 greatest floods for twenty-six 100-yr periods	656,000
Flood of 0.9931 probability for all maximum annual floods "in store"	655,000
Flood of 0.9950 probability for all maximum annual floods "in store"	680,000

The close agreement of the first two of these results tends to demonstrate the soundness of Mr. Beard's procedure. Similar close agreement between flood values at corresponding plotting points was found throughout the range of frequencies considered. The results illustrate that essentially the magnitude and frequency relation of the array of greatest floods for 100-yr periods as arrived at by random selection can be derived directly from the relation based upon the array of maximum annual floods described in the first paragraph of this discussion.

One would be fortunate to have a record of floods on a stream in the United States for as many as 100 years. The maximum of the record would be a single sample of the many greatest floods for 100-yr periods which in a very long record may, so far as present knowledge goes, cover a rather wide range of magnitude. In the foregoing example the median of these values is 656,000 cu ft per sec. The distinctive feature in the plotting method presented by the author is that the assumed single sample would be plotted in the same position as the long-time median, 656,000 cu ft per sec, rather than at the midpoint of the highest 1% of the maximum annual floods, for which the value in the example is 680,000 cu ft per sec. Thus, in this instance the average discrepancy involved would be about 3.8%.

The distribution of the greatest floods for 100-yr periods would be skew. The plotting position for a single sample may conceivably be influenced by various practical considerations inherent in the particular problem to which the data are applied. For example, the position that would give the smallest average deviation from the ultimate curve, signs considered, would be that corresponding to the mean of the greatest floods for the 100-yr periods. It is evident that this position would be at a higher probability value than that for the median, but it is not subject to ready determination. Moreover, such plotting position, if known, would have the disadvantage that more than one half of a large group of 100-yr maxima would plot below the ultimate curve. The plotting position corresponding to the modal value of the skew distribution would seemingly be similarly indeterminate and would lead to more than half of the 100-yr maxima being plotted above the ultimate curve. Using the median as in Mr. Beard's method, the plotting position is readily determinable, and one half of the maxima would plot below and one half would plot above the ultimate curve.

As suggested in the "Synopsis," the differentiation between plotting a single 100-yr maximum at a percentage of occurrence 0.9950 or 0.9931 is of "small significance compared to inaccuracies existing in the greater part of the hydrologic data available today." It is of even smaller significance when one considers that, in so far as fortuity rules, the sample of the greatest flood experienced in a 100-yr period may vary by a large percentage from the long-time median value of such greatest floods.

The writer believes that the principal value of the method is in its fundamental soundness and its clarifying influence on reasoning and concepts, which influence may spread much beyond the consideration of the matter that may be immediately affected. The author is to be commended for his interesting and valuable contribution.

Edward J. Bednarski,[31] Assoc. M. Am. Soc. C. E.—The topic of the paper deals mostly with the application of statistical analysis to hydrology in order to obtain from the records of the past an advance judgment on the expectations in the future.

At any stage in the development of science there are the events that are predictable and the events that are not. This should not be lost from the sight of hydrologists working as statisticians. The American experience table on mortality gives information on life expectancy for men of every age, but there are no tables giving information as to when any particular individual will die. This problem still belongs to the realm of prophecy and is unpredictable by the theory of probability. The chances that a given person will win the Irish sweepstakes are very small; no one can predict who will win; but one thing is certain—the winner must have a ticket. This fact is predictable. Accordingly as yet no one can predict on the basis of past records the order of magnitude of the flood that might occur in a given locality this or next year.

In connection with this, the concept of "probable maximum flood," as defined by Mr. Ruff[5] and quoted by the author is very important. The flood should be "so large that the chance of its being exceeded is no greater than the hazards normal to all of man's activities." The writer would like very much to know the magnitude of this normal chance.

The literature on the subject shows that the persons confronted with the problem of advanced judgment in matters of hydrology do not consider the situation in this light but try to have some scientific justification for all their decisions. The theory of probability appears to satisfy the need.

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[31] Structural Engr., Los Angeles, Calif.

There is a tendency in general to accord preference to the solution of the problem if the solution is the result of complicated, lengthy, and tedious computations—carefully, numerically checked. The mathematical theory of probability with regard to statistics offers a fertile field with its three cardinal equations of distribution, supplemented by a host of others supposed to be easier in application and to reflect better the actual trend of the natural phenomena recorded in the past. Not so long ago a general frequency function was offered in the form of an integral expressing a duration curve applicable to cases with one or two boundaries or no boundaries. Incidentally, Professor Slade[3] calls the representation of probability by a duration curve "usual convenience," thus corroborating the author's recommendations for its use.

It was with a sense of deep relief, therefore, that the writer found the simple description of a new way to compute the "plotting positions" (the term used by Mr. Foster[32]). To make the computations more automatic the writer converted the reasoning into a mathematical expression as follows:

$$\sum_{i=1}^{n} (C_{i-1})^m \, (q_n)^{m-(i-1)} \, (p_n)^{(i-1)} = \tfrac{1}{2} \ldots\ldots\ldots\ldots\ldots (6)$$

This expression happens to be a variable component of

$$\sum_{i=1}^{m} (C_i)^m \, q^{m-i} \, p^i = (q + p)^m = 1 \ldots\ldots\ldots\ldots\ldots (7)$$

Essentially, Eq. 6 is the same as Eq. 4; but the comparison of Eqs. 6 and 7, in the form presented, shows strikingly the basic difference between the binomial law of distribution and the method of plotting positions, as developed by Mr. Beard. The solution of Eq. 6 for $n = 1$ and for values of m consecutively equal to 1, 10, 25, 50, 100, 200, 300, 500, and 1,000, results in the percentages shown in Table 3 of the paper. Here is a noteworthy circumstance. The approximate solution of these values can be obtained from the equation

$$p = 100 \, \frac{n - 0.33}{m} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

which is almost identical to that presented in 1927[33] by William P. Creager and Joel D. Justin, Members, Am. Soc. C. E., namely,

$$p = \frac{100 \, (n - 0.50)}{m} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Eq. 9 in turn is identical to the expression given by Mr. Foster[32] in 1934 as

$$p = \frac{100 \, (2 \, m - 1)}{2 \, n} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

in which the notations m and n have the respective meaning of n and m in the foregoing formulas.

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[32] *Transactions*, Am. Soc. C. E., Vol. 99 (1934), p. 1217.

Eq. 9 determines,[33] "according to the law of probabilities, the probable percentage of future floods that will equal or exceed a given flood." In the Foster paper, Eq. 10 gives a plotting position[32] "on the %-of-time axis," which is the same. In Table 3 of the paper, the same formula, with a simple substitution of the constant $\frac{1}{2}$ by $\frac{1}{3}$, solves the problem of determining the flood with regard to the expected life of the structure. This is a startling coincidence. Table 6 shows the values of p computed by Eqs. 6, 8 and 9.

TABLE 6.—COMPARISON OF PERCENTAGES (p) COMPUTED BY EQS. 6, 8, AND 9

Eq.	VALUES OF m:							
	10	25	50	100	200	300	500	1,000
6	6.7	2.74	1.39	0.68	0.35	0.24	0.14	0.07
8	6.7	2.68	1.35	0.67	0.35	0.23	0.135	0.067
9	5.00	2.00	1.00	0.50	0.25	0.17	0.10	0.05

Another remarkable property of Eq. 6 as it is represented by Table 1 is that, if plotted on the ordinary cross-section paper, it produces a straight line; but if plotted on the logarithmic probability paper it resembles very much the curve of peaks presented by Professor Slade.[3]

From the reasoning which results in the computation of the data in Table 1, one may conclude that it represents either a duration curve or a curve of medians for maximums of various degrees of magnitudes. It can be represented with a fair degree of accuracy by an equation of a straight line

$$p = a\,n + b \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

in which a and b are functions of m. For $m = 10$, $a = 9.62$ and $b = 2.92$.

The values computed according to Eq. 11 are given in Table 7, together with the values of Table 1.

TABLE 7.—COMPARISON OF p COMPUTED BY EQ. 11 WITH TABLE 1

Description	VALUES OF n:									
	1	2	3	4	5	6	7	8	9	10
Table 1........	6.7	16.4	25.8	35.5	45.2	54.8	64.5	74.2	83.6	93.3
Eq. 11........	6.7	16.32	25.94	35.56	45.18	54.80	64.42	74.04	83.66	93.28

Similarly the "plotting positions" for values of m other than 10 could be computed. In his reference to the formula mentioned by Mr. Foster, Mr. Beard seemingly assumes that the data of Table 1 determine the duration curve. The derivative of Eq. 11 representing it should produce a corresponding distribution curve. The derivative is a constant. This means that the distribution is the same for the floods of all magnitudes—quite a surprising result.

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[33] "Hydro-Electric Handbook," by W. P. Creager and Joel D. Justin, John Wiley and Sons, Inc., New York, N. Y., 1927, Eq. 70, p. 171.

The writer agrees with the statement of the author (see heading "The Normal Duration Curve") that "a duration curve will be a straight line if the logarithm of variate follows the normal probability curve" if plotted on the logarithmic probability paper. This is the property of the paper; but the writer does not understand why the author advocates the computation of the "plotting positions" by a method which in no way conforms with the normal probability curve, a method that must be abandoned in favor of Eq. 10, when computations become too laborious.

If the curve represents the variations of the position of medians for the floods of various degrees of magnitude, some way should be found to determine a maximum probable deviation in order to get the value of the probable maximum flood. Unfortunately, the writer was unable to discover in the paper any indication of it.

Mr. Beard is to be complimented on his attempt to avoid arcana of mathematics in trying to find a simple way to form an advance judgment regarding the size of the design flood in connection with the expected age of the structure designed to control it.

RALPH W. POWELL,[34] M. AM. SOC. C. E.—It seems very desirable that references be included in the discussion to other recent publications on this subject, such as those of Prof. E. J. Gumbel[35] and Bradford F. Kimball,[36] and to a type of plotting paper[37] for flood frequency studies based on the former.

The distribution curve for stream flow is very much skewed. Although it is true that the distribution curve of the logarithms of the flow is much nearer normal, and, therefore, that the lines in Fig. 4 are approximately straight, the writer feels that it is better to use a ruling based on a distribution which more nearly fits the facts. As to the "plotting positions" it would seem that those recommended by the author are not enough different from those used by Mr. Foster,[38] the late Allen Hazen,[39] M. Am. Soc. C. E., *et al.*, to be worth the trouble of computing them, especially since the theoretical basis seems open to question. As a matter of fact, plotting at the lower edge of the band instead of the middle seems to the writer to have much to recommend it. As he understands it, this gives a "frequency" corresponding to Fuller's method and the "California method"—what Mr. Horton calls the "recurrence interval" rather than the "exceedance interval."

E. J. GUMBEL,[40] ESQ.—Two important questions have been raised by Mr. Beard:

(*a*) What are the adequate statistical theories available for distributions observed in hydrology; and, especially, which theoretical distribution must be chosen for the largest observed values; and

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[34] Associate Prof. of Mechanics, Ohio State Univ., Columbus, Ohio.

[35] "The Return Period of Flood Flows," by E. J. Gumbel, *Annals of Mathematical Statistics*, Vol. XII (1941), pp. 163–190; "Probability-Interpretation of the Observed Return-Periods of Floods," by E. J. Gumbel, *Transactions*, Am. Geophysical Union, 1941, pp. 836–850; and "Statistical Control-Curves for Flood Discharges," *ibid.*, 1942, pp. 489–509.

[36] "Limited Type of Primary Probability Distribution Applied to Annual Maximum Flood Flows," by Bradford F. Kimball, *Annals of Mathematical Statistics*, Vol. XIII (1942), pp. 318–325.

[37] "A Simple Method of Estimating Flood Frequency," by Ralph W. Powell, *Civil Engineering*, February, 1943, p. 105.

[38] *Transactions*, Am. Soc. C. E., Vol. 99 (1934), p. 1217.

[39] "Flood Flows," by Allen Hazen, John Wiley & Sons, Inc., 1930, p. 13.

[40] Visiting Prof., The Graduate Faculty, The New School for Social Research, New York, N. Y.

(*b*) How should the observed frequencies be corrected to account for the small number of observations.

The writer wishes to show first that the solutions proposed by Mr. Beard are too general and, therefore, inadequate for specific problems. In the course of this proof he proposes a general method for obtaining the corrections. The actual corrections to be used depend upon the theoretical distributions chosen for representing the observations.

Theoretical Distributions.—There is no general solution to the quest for theoretical distributions to be applied to hydrological observations. In principle, such data are statistical observations. The difference from other statistical observations is a function of specific conditions such that the usual statistical methods cannot always be applied.

Certain hydrological distributions are subject to the normal probability law. In some cases, this will hold only for transformed variables. Mr. Beard has given an example of a logarithmically transformed normal distribution. It is a question of definition whether this still can be called a "normal distribution." Other distributions may be exponential, or of the Pearson type. Mr. Beard's contention that only the normal law is a true probability law is not valid since it denies the entire progress of statistics during the past three decades.

Mr. Beard believes that the distribution of the flood discharges is normal. In reality, numerous observations on many streams have shown that the distribution of the flood discharges is skew. The distribution rises quickly to a maximum—the most probable annual flood discharge, the mode—and reduces slowly. The mean annual flood discharge is found to be larger than the mode. The cumulative frequency converges more slowly toward unity than is the case with the normal curve.

To obtain the theoretical distribution of the flood discharges, one must realize how this distribution is observed. The largest of the 365 observations of the daily discharges herein termed "flood" is selected and repeated for n years. In general, the largest among N observations is selected, repeated n times, and an observed distribution of the largest value is thus obtained. The corresponding theoretical distribution is obtained as follows: Let $W(x)$ be the probability of a value equal to, or less than, x; let $w(x) = W'(x)$ be the density of probability (herein termed "distribution"). Then the probability that N observations will be equal to, or smaller than, x is $[W(x)]^N$. This is the probability that x will be the largest among N observations. Therefore, for an initial distribution $w(x)$ the distribution $w_N(x)$ of the largest value is

$$w_N(x) = N[W(x)]^{N-1}\, w(x) \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

The largest value is a statistical variable with a distribution of its own, which, in general, is different from the initial distribution, from which the largest values were taken. If the initial distribution is known, it is possible to calculate the distribution of the largest value, as a function of N. This has been done by L. H. C. Tippett[41] for the normal distribution and for small values of

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[41] "On the Extreme Individuals and the Range of Samples Taken from a Normal Population," by L. H. C. Tippett, *Biometrika*, December, 1925, p. 364.

N. To obtain the limiting distribution of the largest value that is valid for large numbers of observations N, the variable x in the initial distribution $w(x)$ is assumed to be unlimited. This holds for the normal law, and for most of the aforementioned distributions.

At first, this idea of unlimited variables sounds paradoxical. For the practical purpose of reproducing the observations, the distinction between limited and unlimited variables does not play an important rôle as the theoretical distributions converge rapidly toward zero; but the distinction is important in dealing with the largest value. If the initial variable is unlimited, the largest value also is unlimited. From the very moment that the idea of unlimited variables is accepted and the assumption is made that the distribution of the daily discharges belongs to this type, there is no such thing as a maximum possible flood.

Different authors[42] have shown, by different methods, that the probability $[W(x)]^N$ converges for initial distributions of an unlimited variable subject to few restrictions toward

$$W^*(x) = \text{Exp}\,[-\,\text{Exp}\,(-\,y)] \quad \dots\dots (13)$$

in which the reduced dimensionless variable y is

$$y = \alpha\,(x - u) \quad \dots\dots (14)$$

The two parameters u and $\frac{1}{\alpha}$ depend upon the initial distribution $w(x)$ and upon the number N. Both have the dimension x. They will be interpreted subsequently as a mean value and as a measure of dispersion, respectively.

For the normal curve, the probability of a value less than three times the standard deviation is 0.99865, whereas $W^*(x)$ reaches this value only for five times the standard deviation. The distribution $w^*(x)$ of the largest value, obtained from Eq. 13 by differentiation, is

$$w^*(x) = \alpha\,\text{Exp}\,[-\,y - \text{Exp}\,(-\,y)] \quad \dots\dots (15)$$

This is a skew distribution of an unlimited variable. All moments converge. The distribution has a maximum for $y = 0$, that is, $x = u$. This is the mode. The mean value of the reduced variable is $\bar{y} = 0.57722$.

If the two parameters are estimated from the observed distribution of the largest value, Eq. 15 may be used even when the initial distribution $w(x)$ is unknown. The parameter u is the most probable largest value for N observations. If Eq. 15 is applied to the flood flow, u is the most probable annual flood. It may be estimated through the relation

$$u = \bar{u} - 0.58721\,\theta \quad \dots\dots (16a)$$

in which $\bar{u}$ stands for the mean annual flood. The value θ, the "mean deviation," is the mean of the absolute deviations calculated from the mean annual flood.

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[42] See Bibliography in "The Return Periods of Flood Flows," by E. J. Gumbel, *The Annals of Mathematical Statistics*, Vol. XII, No. 2, June, 1941, p. 163.

The other parameter $\frac{1}{\alpha}$ has been shown[43] to be

$$\frac{1}{\alpha} = 1.01731\,\theta \qquad (16b)$$

Another interpretation of $\frac{1}{\alpha}$ is obtained by introducing the return period. For observations recorded in time, the theoretical return period $T(x)$ of a value equal to, or larger than, x is defined by

$$T(x) = \frac{1}{1 - W(x)} \qquad (17)$$

In treating flood discharges, the return period is counted in years, and $W(x)$ is replaced by $W^*(x)$. Combining Eqs. 13 and 14 with Eq. 17, the return period of a flood, equal to, or larger than, x, is found to converge for large values of x toward

$$\log_e T(x) = \alpha\,(x - u) \qquad (18)$$

Eq. 18 holds with sufficient accuracy for observations of twenty years and more. Under this condition, the flood discharge x occurring in Eq. 18 is the most probable flood discharge corresponding to the return period $T(x)$. The logarithm of the return period is a linear function of the most probable flood discharge. The parameter is the slope of this straight line. Furthermore, from Eq. 18, if the return period is multiplied by e, the most probable flood increases by $\frac{1}{\alpha}$.

The theory of largest values accommodates well the observations taken from the Columbia, Tennessee, Mississippi, Cumberland, Connecticut, Rhine, and Rhône rivers.[42,43,44] Thus answering the first question raised by the paper, this theory should be chosen for the observed distribution of the flood discharges.

Corrections of the Observed Frequencies.—The second question is: Which corrected frequency and return period shall be attributed to the mth value for an arbitrary number of observations n, and especially to the mth flood discharge?

Let $x_1, x_2, \cdots x_m \cdots x_{n-1}, x_n$ be observations arranged in increasing order of magnitude. The index m is the serial number. To the mth observation x_m, the observed frequency $\frac{m}{n}$ is attributed; but it is equally legitimate to arrange the observations in a decreasing order of magnitude. The mth observation increasing is the $(n - m + 1)$th observation decreasing. The observed frequency decreasing is $\frac{n - m + 1}{n}$, which corresponds to an observed frequency $\frac{m - 1}{n}$ increasing.

Just as there are two ways of counting the observations, there are two ways of plotting the cumulative frequency curve (the duration curve), which represents the absolute or relative number of observations smaller than, or larger than, a certain value. In the first case, the curve starts with 0 and ends with 100%—the total number of observations. (Fig. 2 ends with 44%, whereas Mr. Beard states correctly that it should end with 100%.) In the second case, the frequency curve starts with 100%, and ends with 0.

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[43] "Statistical Control—Curves for Flood Discharges," by E. J. Gumbel, *Transactions*, Am. Geophysical Union, 1942, p. 489.

[44] "Probability-Interpretation of the Observed Return-Periods of Floods," by E. J. Gumbel, *ibid.*, 1941, p. 836.

In the same sense as for the frequencies, there are two ways of plotting the observed return periods, namely, the so-called exceedance intervals

$$T'(x_m) = \frac{n}{n - m} \quad \text{(19)}$$

defined for $1 \leqq m \leqq n - 1$, and the recurrence intervals

$$T''(x_m) = \frac{n}{n - m + 1} \quad \text{(20)}$$

which may be plotted on probability paper for $2 \leqq m \leqq n$.

For small serial numbers, the differences between the two ways of plotting are small as the return periods then are approximately

$$T = 1 + W \quad \text{(21)}$$

in which W is nearly zero and T nearly unity; but, for large serial numbers, especially for the extreme values, the differences between the two ways of plotting are large. For the penultimate value the two observed return periods are n and $\frac{n}{2}$, respectively.

To solve this dilemma Mr. Beard assumes that the observed mth value is the median of all possible mth values. This is certainly legitimate; but he constructs an analogy of any observed distribution with a binominal distribution. In reality, the probability of an observed mth value taken from an arbitrary distribution cannot be identified with the probability that an event happens $m - 1$ times in an alternative. He advocates the same correction to the mth value, decreasing and increasing, independent of whether the distribution is symmetrical or not. Mr. Beard calculates the corrections for ten observations. It would be difficult to use his method for an arbitrary number of observations, as this requires the solution of $\frac{n}{2}$ equations of a complicated type. His proposal has no logical basis; and, moreover, is impracticable. Mr. Beard seems to have recognized the second drawback. Therefore, he proposes, as an alternative, the Foster-Hazen[45] correction which consists of a compromise between m and $m - 1$, and which attributes a frequency $m - \frac{1}{2}$ to the mth observation. The corrected intervals

$$T(x_m) = \frac{n}{n - m + \frac{1}{2}} \quad \text{(22a)}$$

may be used for $1 \leqq m \leqq n - 1$. The return period of the $(n - m + 1)$th observation (which is the mth observation, decreasing) is then

$$T(x_{n-m+1}) = \frac{2\,n}{2\,m - 1} \quad \text{(22b)}$$

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[45] "Flood Flows," by Allen Hazen, John Wiley & Sons, Inc., New York, N. Y., 1930.

It seems doubtful whether Eq. 22*b* can be used for the last value. In this case, one attributes the return period

$$T(x_n) = 2\,n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (22c)$$

to the largest among *n* observations. In reality, only *n* observations are available, and no statistical device can increase the number of observations beyond *n*. Besides, it does not seem safe to attribute a return period 2 *n* to an event which already has happened within *n* years.

According to Eq. 22*b* the return period of the *m*th value (decreasing) for *n* observations is equal to the return period of the $(m + 1)$th value for $n\,\frac{2\,m + 1}{2\,m - 1}$ observations. This relation is by no means satisfactory; common sense leads to another, simpler idea. The return period of the last value for *n* observations should be equal to the return period of the penultimate value for 2 *n* observations.

Messrs. Hazen and Beard infer that a general correction for any theoretical distribution is valid. The writer's solution is based on the following considerations:

(*a*) The correction must decrease with the number of observations *n*. This relation should be stated in such a way that the correction can easily be calculated as a function of *m* and *n*.

(*b*) For symmetrical distributions the corrections for two observations that are symmetrical about the mean must have symmetrical properties. As this will not hold for an asymmetrical distribution, the corrections must depend upon the theoretical distribution.

The *m*th value among *n* observations taken from an initial distribution $w(x)$ is a statistical variable which has a distribution $v_n(m, x)$, depending upon *m* and *n* and $w(x)$. The *m*th observation is assumed as a mean value $\mathbf{x}_m$ of the distribution $v_n(m, x)$. Therefore, one attributes to the *m*th observation the probability $W(\mathbf{x}_m)$ of this mean considered as one of the values of the initial variable.

For *n* observations, the *m*th value increasing is such that there are $m - 1$ values before the *m*th value and $n - m$ values after the *m*th value. These probabilities are, respectively, $[W(x)]^{m-1}$ and $[1 - W(x)]^{n-m}$. The *m*th value has the density of probability $w(x)$. Finally, the number of combinations must be taken into account. Therefore, the distribution $v_n(m, x)$ is

$$v_n(m, x) = \binom{n}{m} m\,[W(x)]^{m-1}\,[1 - W(x)]^{n-m}\,w(x) \ldots\ldots\ldots \quad (23)$$

The most probable *m*th value $\mathbf{x}_m$ obtained by logarithmic differentiation is the solution of

$$m - 1 - (n - 1)\,W(\mathbf{x}_m) = -\frac{w'(\mathbf{x}_m)}{w^2(\mathbf{x}_m)}\,W(\mathbf{x}_m)[1 - W(\mathbf{x}_m)] \ldots\ldots \quad (24)$$

The probability $W(\mathbf{x}_m)$ of the most probable *m*th value is a function of *m*, *n*, and the initial distribution $w(x)$. It is reasonable to assume that the observed *m*th value is the most probable one. Then one must attribute to the observed

mth value x_m the corrected frequency, obtained from Eq. 24:

$$\mathbf{W} = W(\mathbf{x}_m) \tag{25}$$

which is the probability of the most probable mth value. Eq. 24 may be written in the form

$$m - n\,\mathbf{W} = \Delta \tag{26}$$

in which the right side

$$\Delta = 1 - \mathbf{W} - \frac{w'(\mathbf{x}_m)}{w^2(\mathbf{x}_m)}\,\mathbf{W}\,(1 - \mathbf{W}) \tag{27}$$

herein termed the "correction," depends upon the initial distribution. The corrected frequency $\mathbf{W}$ and the corrected return period $\mathbf{T} = T(\mathbf{x}_m)$ of the mth observation are

$$\mathbf{W} = \frac{m - \Delta}{n}\text{; and } \mathbf{T} = \frac{n}{n - m + \Delta} \tag{28}$$

Eqs. 28 shows that the first postulate is fulfilled.

The numerical values of the correction Δ may be calculated from Eq. 27 for selected values of the corrected frequency $\mathbf{W}$. The serial number m as a function of $\mathbf{W}$ obtained through Eqs. 28 will not be an integer. The corrected frequencies $\mathbf{W}$ corresponding to integer numbers m are obtained by linear interpolations. If n is large, this tedious work may be avoided. The correction Δ as a function of $\frac{m}{n}$ is slightly different from Δ considered as a function of the frequency $\mathbf{W}$. Neglecting this difference, the calculated values of Δ are attributed to $\frac{m}{n}$ instead of $\mathbf{W}$; and, for given values of m, the corrected frequencies $\mathbf{W}$ are obtained from Eqs. 28.

By this procedure, Eq. 24 gives the corrected frequency for any number of observations, any serial number, and any initial distribution. This settles the author's second question.

To obtain the correction for symmetrical distributions, a reduced variable z is introduced in Eqs. 28—

$$z = \frac{x - \bar{x}}{\sigma} \tag{29}$$

in which $\bar{x}$ stands for the arithmetic mean and σ for the standard deviation. The probabilities $W(-z)$ and $W(z)$ are then related by

$$W(-z) + W(z) = 1 \tag{30a}$$

The same symmetry holds for the correction $\Delta(-z)$ and $\Delta(z)$. From Eq. 27:

$$\Delta(-z) + \Delta(z) = 1 \tag{30b}$$

For symmetrical distributions, the corrections show the same symmetry as the probabilities. This was the second postulate.

For an exponential distribution

$$w(x) = \alpha \operatorname{Exp}(\alpha\, x) \tag{31}$$

the corrected frequency of the mth observation is

$$\mathbf{W} = \frac{m - 1}{n} \tag{32}$$

and the corrected return periods are the recurrence intervals, Eq. 20. For the uniform distribution, in which $w'(x) = 0$:

$$\mathbf{W} = \frac{m-1}{n-1} \qquad \text{.........................(33)}$$

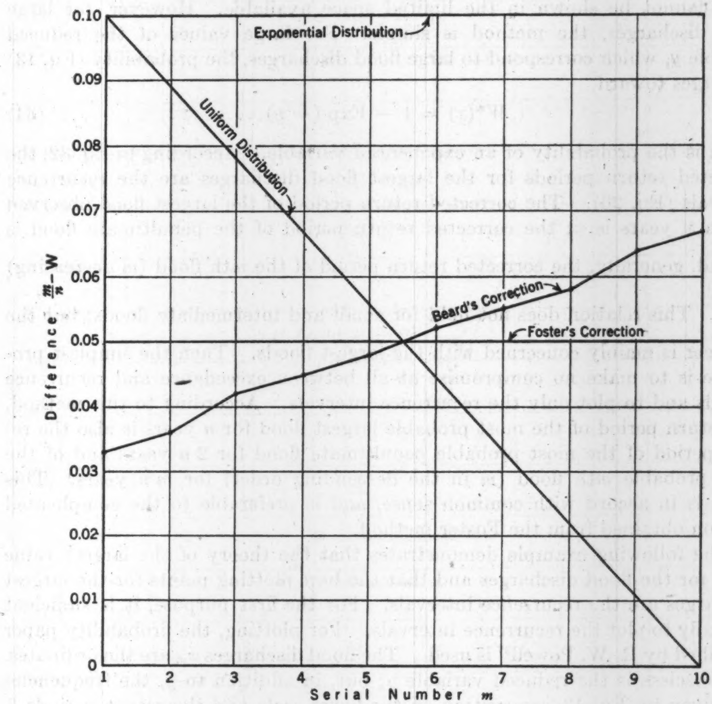

FIG. 9.—DIFFERENCES OF THE OBSERVED AND CORRECTED FREQUENCIES FOR TEN OBSERVATIONS

The differences $\frac{m}{n} - \mathbf{W}$ of the observed frequency $\frac{m}{n}$ and the corrected frequency W are plotted in Fig. 9 for different distributions and for ten observations. According to Hazen, the difference is equal to $\frac{1}{20}$, independent of m. The irregularities in Mr. Beard's curve are probably due to the fact that he did not calculate a sufficient number of decimals. The two straight lines for the uniform and exponential distributions are obtained from Eqs. 32 and 33. Fig. 9 shows that, for small numbers of observations, the correction depends strongly upon the distribution. The construction of a general correction does not make sense for a small number of observations.

Application to the Flood Flow.—The remaining step is to determine the best plotting points for the flood discharges, and to decide which compromise (if any) should be made between the recurrence and the exceedance intervals. For this purpose, the theory of the largest value, Eq. 13, is used. The calculation of the corrected frequencies and return period for an arbitrary mth flood cannot be shown in the limited space available. However, for large flood discharges, the method is simple. For large values of the reduced variable y, which correspond to large flood discharges, the probability (Eq. 13) converges toward

$$W^*(x) = 1 - \text{Exp}\,(-\,y) \qquad (34)$$

which is the probability of an exponential variable. According to Eq. 32, the corrected return periods for the largest flood discharges are the recurrence intervals (Eq. 20). The corrected return period of the largest flood observed within n years is n; the corrected return period of the penultimate flood is $\frac{n}{2}$; and, generally, the corrected return period of the mth flood (m decreasing) is $\frac{n}{m}$. This relation does not hold for small and intermediate floods; but the engineer is mainly concerned with the largest floods. Then the simplest procedure is to make no compromise at all between exceedance and recurrence periods and to plot only the recurrence intervals. According to this method, the return period of the most probable largest flood for n years is also the return period of the most probable penultimate flood for $2\,n$ years and of the most probable mth flood (m in the descending order) for $m\,n$ years. This result is in accord with common sense, and is preferable to the complicated relation obtained from the Foster method.

The following example demonstrates that the theory of the largest value holds for the flood discharges and that the best plotting points for the largest discharges are the recurrence intervals. For the first purpose, it is sufficient generally to plot the recurrence intervals. For plotting, the probability paper described by R. W. Powell[37] is used. The flood discharges x_m are the ordinates. The abscissa is the reduced variable y; but, in addition to y, the frequencies W^*, given by Eq. 13, are written on the lower scale and the return periods T on the upper scale. The discharges corresponding to $y = 0$ and $y = 0.577$ are the most probable and the mean annual floods, respectively; but these theoretical values need not occur among the observed discharges. The possible positions of these discharges are shown by broken lines in Fig. 10.

If the theory of the largest values holds, Eq. 14 should be a straight line for the observed flood discharges x expressed as a function of y. The two constants occurring in Eq. 14 are the flood discharge u corresponding to $y = 0$, and the slope $\left(\frac{1}{\alpha}\right)$ of the straight line. The two constants may be calculated by Eqs. 16 or obtained from the graph itself.

These methods may be applied to the flood discharges of the Mississippi River observed in Vicksburg in the fifty years, 1890–1939: The observations x_m, arranged in increasing order of magnitude, are given in a previous publica-

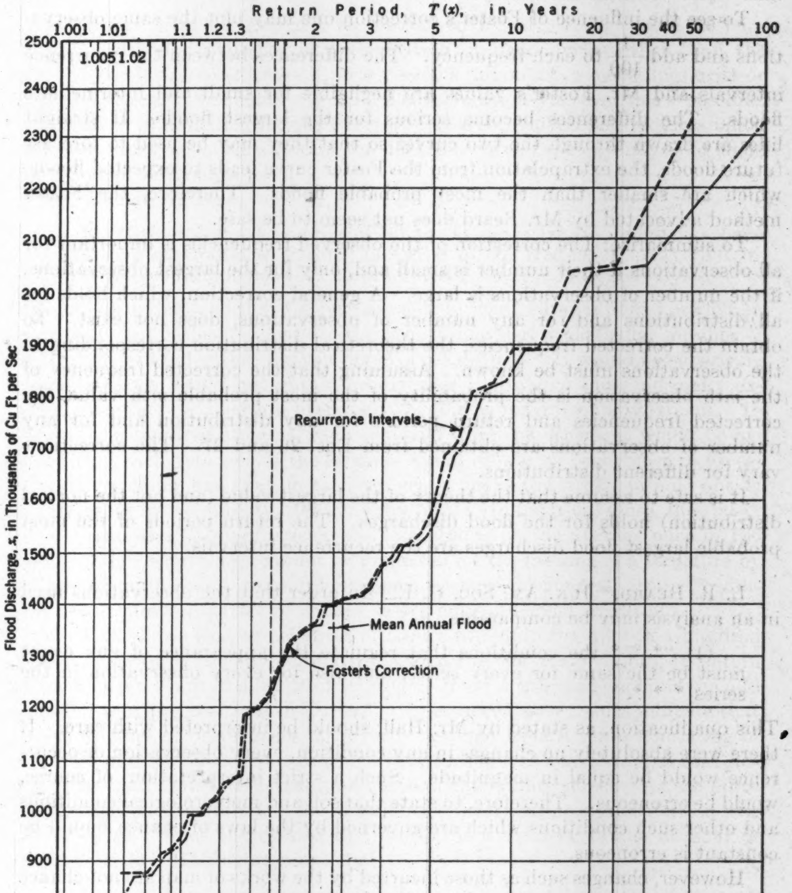

FIG. 10.—FLOOD DISCHARGES, MISSISSIPPI RIVER AT VICKSBURG, MISS., 1890-1939

tion.[42] The observations plotted in Fig. 10 are scattered around a straight line. This confirms the theory.

To see the influence of Foster's correction one may plot the same observations and add $\frac{1}{100}$ to each frequency. The differences between the recurrence intervals and Mr. Foster's values are negligible for small and intermediate floods. The differences become serious for the largest floods. If straight lines are drawn through the two curves so that they may be used to forecast future floods, the extrapolation from the Foster curve leads to expected floods, which are smaller than the most probable floods. Therefore, the Foster method advocated by Mr. Beard does not seem to be safe.

To summarize: The correction of the observed frequencies is important for all observations if their number is small and, only for the largest observations, if the number of observations is large. A general correction, which holds for all distributions and for any number of observations, does not exist. To obtain the corrected frequencies, the theoretical distribution corresponding to the observations must be known. Assuming that the corrected frequency of the *m*th observation is the probability of the most probable *m*th value, the corrected frequencies and return periods for any distribution and for any number of observations are obtained from Eqs. 26 and 27. The corrections vary for different distributions.

It is safe to assume that the theory of the largest value (and not the normal distribution) holds for the flood discharges. The return periods of the most probable largest flood discharges are the recurrence intervals.

L. R. BEARD,[46] JUN. AM. SOC. C. E.—In order that the observations used in an analysis may be comparable,

> (1) "* * * the conditions that regulate the appearance of any event must be the same for every series, and also for every observation in the series * * *."

This qualification, as stated by Mr. Hall, should be interpreted with care. If there were absolutely no changes in any condition, every observation or occurrence would be equal in magnitude. Such a strict interpretation, of course, would be erroneous. Therefore, to state that soil and meteorological conditions and other such conditions which are governed by the laws of chance should be constant is erroneous.

However, changes such as those incurred by the works of man are not chance occurrences. In general they are conditions which will prevail to a greater or lesser degree during the coming years and which did not prevail with equal influence during the period of recorded occurrences. The influence of such conditions should be estimated carefully. It appears best, in general, to adjust the observations of record to conform with expected conditions in the future and then to use the adjusted values in the construction of a duration curve.

In this connection, it should be noted that the variation of the accuracy with which observations are made is not similar to the variation of the conditions under which the observations are made. A variation in accuracy should not bear on the validity of the analysis, but simply on the accuracy of the analysis.

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[46] Associate Engr., U. S. Engrs., War Dept., Los Angeles, Calif.

The effect of cyclic changes, as mentioned by Mr. Hall, should be considered. Although evidence of precipitation or flood cycles other than the annual cycle is not conclusive, if, in a person's judgment, a cycle does exist or is assumed to exist, it should be given full consideration wherever this is possible. The problem of adjusting for the effect of cycles is involved and requires much study but must be solved whether the duration-curve type of analysis or some other type of analysis is used.

Mr. Hall states that

> (2) "In some flood records of 30 to 50 years length, there has been included a maximum so greatly in excess of the frequency array that its occurrence could not possibly be predicted by mathematical analysis."

It should be understood that the purpose of a duration curve is not to predict an occurrence, but rather to compute the probability that a particular number of occurrences will fall within a given range of magnitude. It would be surprising if some streams would not obtain occurrences well above those normally expected since there is a certain probability of this happening on any stream. In a group of a hundred streams, each removed from the others, it would be expected that each year one of the streams would obtain an occurrence in excess of its 100-yr value; and that, in each ten years, one of the streams would obtain an occurrence in excess of its 1,000-yr value, etc. (In the usual sense, the 100-yr value is read at 1% of occurrence and is exceeded on the average once in 100 years.) Such occurrences cannot be predicted for the lifetime of a structure by analysis of rainfall and ground conditions, because, in general, ground conditions are as much a matter of chance as are meteorological conditions.

Mr. Hall shows that the chances of failure to obtain the once-in-a-hundred event in a series of 100 trials are greater than the chances of failure to obtain the once-in-ten event in a series of 10 trials. This is true and explains the odd values for plotting positions in Table 1. In essence, these values indicate that a finite number of occurrences, arbitrarily selected from a series consisting of an infinite number of occurrences which vary throughout a large range of magnitude, does not follow the distribution curve of the infinite series. As the finite number approaches infinity, the distribution curve approaches that of the infinite series. Since the normal distribution is a function of the number of occurrences, Mr. Hall's statement that:

> (3) "* * * in any given series, it is justified to assume that each occurrence is representative of the period and can be considered the median for 1/*n*th of the period"

is erroneous.

Mr. Hall stated that

> (4) "* * * the selection of the maximum occurrence of each calendar year obviously does not afford the best use of the available data"

because in some records

> "* * * it is usual to find wet years with two, or even three, occurrences that exceed the maximums of the dry years."

The virtue of the selection of the maximum occurrence of each year is that the effect of the annual cycle is eliminated thereby. The fact that the resulting curve is not skew may justify the discarding of many large occurrences, especially when the curve must be extrapolated.

Mr. Hall states:

> (5) "Many rigid mathematical treatments are possible based on assumptions differing from those forming the basis of the normal curve of error."

This is true, of course, but, to the writer's knowledge, none have been offered which appear to fit observed distributions. The distribution derived by Gauss applies surprisingly well when the data are properly selected. A series (mentioned by Mr. Hall) which contains a great number of occurrences of zero magnitude is not a properly selected series because, if the magnitude is zero, the occurrence does not exist. The duration curve should be constructed of occurrences of finite magnitude, and proper account should be taken of the proportion of time during which there are no occurrences.

The use of

> (6) "* * * probability methods to determine rainfall, combined with 'rational' methods for flood runoff or a unit hydrograph * * *"

has presented insurmountable difficulties in many instances. The various factors such as the selection of rainfall patterns, snow and snowfall data, ground conditions, infiltration rates, channel storage, reservoir storage, operations, etc., are sources of errors which, when combined properly, very often form large total errors. It is very difficult to select the worst possible runoff conditions, and it is many times more difficult to select a combination of runoff conditions that will obtain a 100-yr flood from a 100-yr storm. Perhaps the future will bring more accurate methods, but at present the duration curve of runoff, combined with methods of comparing runoff characteristics, appears to the writer to be relatively accurate and entirely satisfactory where sufficient runoff data are available.

Mr. Foster suggests that, instead of considering the largest occurrence in a 10-yr period to be the median of the largest occurrences of all of the 10-yr periods ever to happen,

> (7) "* * * the largest occurrence in a 10-yr period should be taken as the median of the 10 largest occurrences of a 100-yr record, or the median of the 100 largest occurrences of a 1,000-yr record."

However, the largest occurrence of a 10-yr period will with equal probability be larger or smaller than the median of the ten maximum occurrences of ten 10-yr records and therefore probably (according to Mr. Foster's demonstration) will be smaller than the median of the ten largest occurrences of a 100-yr record. It is erroneous to assume, therefore, that it will equal that median.

Mr. Foster states that it is true

> (8) "* * * that 'the probability that the maximum occurrence of a set chosen at random will exceed the median of the maximum occurrences of all the sets is 50%.' The probability that the said maximum occurrence will be less than the median is also 50%. However, the probability that 'none of the ten occurrences chosen at random will exceed the median of the maximum occurrences of all the sets' is not 50% as stated by the author."

If the probability that the maximum of ten occurrences chosen at random will be less than a given value is 50%, the probability that none of the ten occurrences will exceed the given value must also be 50%. If the maximum occurrence is less, then all of the occurrences must be less. The second and third sentences in quotation (8) appear to be contradictory. Mr. Foster substantiates quotation (8) on the assumption that either a particular occurrence of ten, or none of ten occurrences of a series, must exceed the median of the maximum occurrences of many series. This is not true, since any other one of the ten occurrences also may exceed the median. Most of the possibilities were neglected, and hence Mr. Foster's answer of 95% instead of 50%.

The fact that many unsymmetrical distributions do occur in nature cannot be denied. When there are factors other than chance involved, the series will generally obtain some degree of skewness. However, it is the writer's belief that the skew distribution has been the rule rather than the exception in hydrologic data, because proper methods of selecting and plotting data have not been adhered to and because care has not been taken to avoid the types of series which are affected by factors other than chance.

Mr. Hodges has presented examples of curves whereby various methods of curve fitting could be compared. The series which could be expected to be normal (Fig. 5) is fitted reasonably well by both the Foster and Hazen (normal distribution) methods. The series (Fig. 6) consisting of values above a base amount could not be expected to be normal, and therefore is fitted best by one of the methods illustrated in Fig. 6. Perhaps if the complete series were plotted, the curve would be more nearly a straight line.

It cannot be denied that the various series presented by the writer in Fig. 4 could be fitted better by some curve more adaptable than the normal distribution curve. Of course, it is possible to fit a curve through all of the points. The question is: "Which curve is more truly representative of the actual distribution sought?" Is it better to make the assumption that the curve should have a normal distribution or simply to assume that the curve is smooth and has no abrupt change in curvature?

Since writing this paper, the writer has plotted about 120 duration curves from data published in *Water Supply Paper No. 771*, each curve representing a period of stream-flow record between 24 and 96 years long. It was interesting to note that most of the curves showed a tendency to curve downward at either end. However, a large minority of them showed the opposite tendency. A few of the fits were bad, a few excellent, and the remainder reasonable. Would it be better to assume that each stream has its own characteristic distribution which is not necessarily similar to the remainder or to assume that the distribution should be normal and that deviations from normal occur as a matter of chance and should be corrected?

One school of thought proposes that the probability of throwing "heads" with a penny should not be assumed to be 0.5, but should be determined by a great number of throws, because the penny is not perfectly symmetrical. This truth cannot be discounted, but it is pertinent to ask: "How many throws are required to establish the empirical probability more accurately than the assumed probability?" If, in ten throws, six were heads, would the probability of 0.6 for throwing a head be a better estimate than 0.5? Similarly, if a 50-yr record for a stream exists, would it be better to assume a normal distribution, as has been approximated by almost all long-period stream records, or to rely entirely on the one record and allow the points to determine the curve? Certainly the normal distribution is not exactly representative, as nothing is perfect, but it appears to be well within reason.

Mr. Blythe would discard curve-fitting refinements because of the

> (9) "* * * discrepancy between natural conditions and the assumptions underlying the theories."

However, the writer has not witnessed evidence that such discrepancies are large enough to make attempts at curve fitting futile. To discard the logical tool without evidence of its futility appears to the writer to be less justifiable than to use it even though that use may involve a small amount of "meaningless labor." In addition to the asserted advantage of better definition, a straight-line relation has the additional advantage of being definable by a point and a slope. This condition permits easier comparison and correlation among curves.

Mr. Blythe discusses the use of a mode in preference to a median and states that

> (10) "* * * the mode, by definition, occurs most frequently, and therefore any single value chosen from a non-normal distribution is more likely to be the mode than the average or the median."

Although the mode occurs more frequently, its chance of occurrence in the types of series which are being discussed is infinitesimal (if not zero), and the advantage of using the mode rather than the mean or median, by Mr. Blythe's reasoning, is practically zero. When the possible number of magnitudes is infinite, one must deal with a range of magnitudes rather than with a single magnitude. For this reason, the median can be put to a great number of uses mathematically, whereas the mean and mode are functions of the distribution, which is unknown until the points are plotted.

The writer was interested in the study described by Mr. Davenport, whose resourcefulness in developing a sound procedure for testing a method, which is apparently highly debatable on a theoretical basis, is to be admired. To note that the result is in substantial agreement with the theory discussed in this paper is reassuring.

Mr. Bednarski is interested in the derivation of a maximum probable flood, and would like to see a determination of a maximum probable deviation for a duration-curve series.

The first definition of "probable" given by Webster's Unabridged Dictionary is: "Having more evidence for than against; supported by evidence strong enough to establish presumption, but not proof, of its truth." These two clauses differ obviously, and the three other definitions which follow indicate that many different interpretations can be made of the phrase, "maximum probable." If the simple definition of "having a probability of being exceeded greater than 50%" is adhered to, then the maximum probable flood would be defined by stipulating a period of design. This is not what is generally considered as maximum probable.

When maximum probable is meant to be "estimated maximum possible," it should be designated as "maximum possible" since all computed quantities are estimated. When "maximum probable" means "considerably larger than that which has a 50% probability of being exceeded but smaller than maximum possible," then the meaning is not defined mathematically. To define such a quantity mathematically, the percentage probability may be stipulated, as suggested in this paper. Such a percentage may vary, depending on the type of design contemplated, and, therefore, could not be stipulated here.

Professor Gumbel has selected two points on which the paper, as presented, differed most with common practice—the computation of plotting positions and the selection of a normal distribution law. In discussing the latter, he has presented a method which, he states, should be chosen for the observed distribution of flood discharges. As near as the writer can determine, this method is justified in the sentence which contains Eqs. 13 and 14. The theory appears to be purely empirical, but the results seem satisfactory from an engineering standpoint. However, this does not in itself exclude the possibility that the Gaussian distribution is applicable.

Since it is not the burden of the writer to justify the theoretical soundness of the Gaussian law and its applicability in innumerable cases (these facts have been generally accepted), the curves presented in Fig. 11 should be sufficient evidence to show that the law is also applicable to hydrology. The data of Fig. 11(*b*) were obtained from the longest record of flood flows published in *Water Supply Paper No. 771*, and the data of Fig. 11(*a*) are those presented by Professor Gumbel.

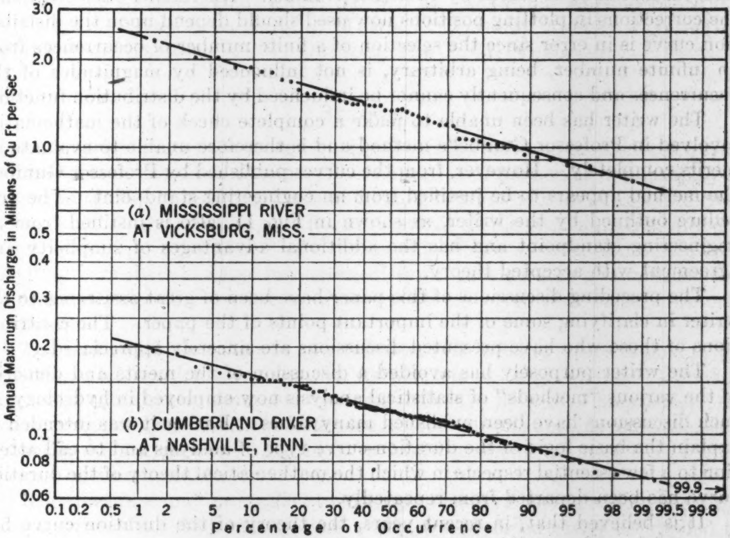

FIG. 11.—DURATION CURVES OF MAXIMUM ANNUAL FLOW

In proposing a return to a modified form of the "Hazen method," the writer did not intend to deny the entire progress of statistics during the past three decades but rather wished to demonstrate that Gauss and Laplace might have had more to offer in this instance than do modern schools. Such general and unwarranted statements as "In reality, numerous observations on many streams have shown that the distribution of the flood discharges is skew" have unfortunately destroyed the general faith in this application of the classic theory.

In connection with the computation of plotting positions, Professor Gumbel's statement that the system advocated by the writer is impracticable has no bearing on the validity of the system. The plotting positions given in Table 1 are derived by the accepted laws of probability from the only plausible basic assumption. Therefore, the system is not an invention of the writer but is rather a solution of a mathematical equation. The fact that the solution becomes laborious has no more bearing on its validity than does the fact that π is not an integer prove that it is incorrect. In recognizing the impracticability of computing plotting positions accurately, the writer has suggested an approximate method which is entirely satisfactory for all purposes. This approximate method is not a compromise between the recurrence-interval and exceedence-interval methods, as stated by Professor Gumbel. His further statement that the corrections in plotting positions now used should depend upon the distribution curve is in error since the selection of a finite number of occurrences from an infinite number, being arbitrary, is not influenced by magnitudes of the occurrences and consequently cannot be influenced by the distribution function.

The writer has been unable to make a complete check of the mathematics involved in Professor Gumbel's method and is therefore unable to evaluate its merits completely. However, from the curves published by Professor Gumbel, the method appears to be justified from an engineering standpoint. The procedure outlined by the writer, as shown in Fig. 11, also is justified from an engineering standpoint and has the additional advantages of simplicity and agreement with accepted theory.

The preceding discussions of this paper have been of great assistance to the writer in clarifying some of the important points of the paper. The contributions of those who have presented discussions are sincerely appreciated.

The writer purposely has avoided a discussion of the merits and demerits of the various "methods" of statistical analysis now employed in hydrology, as such discussions have been published many times. Rather it was intended to explain the basic logic of the duration-curve type of analysis and to call attention to a few essential respects in which the mathematical theory of the duration curve has been departed from repeatedly.

It is believed that, in recent years, the theory of the duration curve has been developed so sufficiently that its unqualified, but proper application, can be justified from an engineering standpoint. Many alternative methods used in hydrologic design, such as the application of enveloping curves and transposed storms, do not have mathematical significance. Such arbitrary designs are not justified from an engineering standpoint if the desired magnitude of a flood can be stipulated in such terms that the flood can be derived mathematically and can be given significance thereby.

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NUMERICAL PROCEDURE FOR COMPUTING DEFLECTIONS, MOMENTS, AND BUCKLING LOADS

By N. M. Newmark,[1] Assoc. M. Am. Soc. C. E.

With Discussion by Messrs. Bruce Johnston, M. S. Ketchum, Jr., John B. Wilbur, Ralph W. Stewart, Stefan J. Fraenkel, Alfred S. Niles, Camillo Weiss, A. A. Eremin, Myron L. Gossard, Robert A. Williamson, I. Oesterblom, C. W. Dunham, and N. M. Newmark.

Synopsis

A numerical procedure for computing the deflections and moments in beams and columns is described herein. The method is of particular applicability in determining critical buckling loads and configurations of bars of variable cross section loaded in various ways. For such problems the procedure becomes one of successive approximations. By means of a simple modification of the data entailing very little increase in numerical work, considerably greater accuracy is obtainable by this procedure than by others of similar nature hitherto available.

The numerical procedure is approximate, but leads to exact moments (or deflections) when the loading diagram (or diagram of "angle changes") is made up of segments that are bounded by straight lines or by arcs of parabolas. By taking more arbitrary divisions in the length of a bar one obtains more accurate results in the general case. For most practical problems no more than five or six segments are necessary.

The procedure may be applied to other problems which depend on the same general principles. In mathematical terms, the procedure may be applied to the process of numerical integration of certain types of differential equations, in some cases directly, and in other cases by a sequence of successive approximations.

The essential features of the procedure are not new. The writer's first acquaintance with the concepts involved in this paper came some years ago from lectures in graduate courses at the University of Illinois, Urbana, Ill.,

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Note.—Published in May, 1942, *Proceedings*.

[1] Research Asst. Prof., Civ. Eng., Univ. of Illinois, Urbana, Ill.

by Hardy Cross and H. M. Westergaard, Members, Am. Soc. C. E. Specific procedures have been discussed previously in engineering literature; for example, the application of similar graphical and analytical procedures to buckling of bars has been made by L. Vianello,[2] F. Engesser,[3] and others[4]; and the application of a graphical procedure to vibration of bars and shafts has been indicated by A. Stodola.[5,6] Methods of obtaining increased accuracy with a numerical procedure have been suggested by A. S. Niles,[7] Assoc. M. Am. Soc. C. E., and R. V. Southwell,[8,9] among others. The procedure suggested by Professor Southwell is in some respects similar to that described herein. However, the generalization of the procedure, the manner of application to specific problems, the treatment of functions with cusps or discontinuities, the simplified procedure for continuous functions with continuous derivatives, and the method of computing shears, slopes, or first derivatives, are essentially new, are more useful or more accurate than previous methods, and to the writer's knowledge have not been described previously.

PART I.—COMPUTATION OF MOMENTS IN BEAMS

Introductory.—The calculation of the values of a function of a single variable, when the magnitude of the second derivative of the function is known, is a fundamental part of a group of physical problems, examples of which are the determination of the deflection of a string, or of a beam, and the computation of moments in a beam due to given loads. Analogies may be drawn between these and similar problems, since generally they may all be solved by the same procedures.

The method of computation described herein is a numerical procedure permitting as accurate a determination as is desired of the values of a function for specific values of the variable. The method is described in terms of calculation of moments in a beam for a given system of loads, but the application to other problems is also indicated, and particular application is made to the problem of buckling of bars.

Treatment of Concentrated Loads.—A fundamental part of the procedure depends on the rapid and systematic calculation of shears and moments in a beam subjected to a series of concentrated loads. Essentially, the process is to compute the shears from one end of the beam to the other by adding or subtracting the successive loads, then to compute the moments by adding or subtracting the successive shears, multiplied by the length of beam over which the shear acts. The latter step is simpler if all the lengths between points of application of the concentrated loads are the same. However, the general case is not difficult, and the modification of the procedure described herein, to handle the general case, is obvious and will not be discussed.

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[2] "Graphische Untersuchung der Knickfestigkeit gerader Stäbe," by L. Vianello, *Zeitschrift des Vereins Deutscher Ingenieure*, Vol. 42, 1898, pp. 1436–1443.

[3] "Über die Knickfestigkeit von Stäben veränderlichen Trägheitsmomentes," by F. Engesser, *Zeitschrift des österreichischen Ingenieur- und architektenvereins*, Vol. 61, 1909, pp. 544–548.

[4] A discussion of some of these methods is given in "Theory of Elastic Stability," by S. Timoshenko, New York, N. Y., 1936, pp. 84–88 and 131–133.

[5] "Steam Turbines," by A. Stodola and L. C. Loewenstein, 2d Revised Ed., New York, N. Y., 1906, pp. 185–186.

[6] See also, for example, "Mechanical Vibrations," by J. P. Den Hartog, New York, N. Y., 1934, pp. 174–178.

[7] "Airplane Structures," by A. S. Niles and J. S. Newell, 2d Ed., New York, N. Y., 1938, Vol. I, pp. 133–136, and Vol. II, pp. 126–134.

[8] "Relaxation Methods Applied to Engineering Problems, I. The Deflexion of Beams Under Transverse Loading," by K. N. E. Bradfield and R. V. Southwell, *Proceedings*, Royal Soc. of London, Series A, Vol. 161, 1937, pp. 155–181, especially pp. 163–165.

[9] "Relaxation Methods Applied to a Spar of Varying Section, Deflected by Transverse Loading Combined with End Thrust or Tension," by R. J. Atkinson, K. N. E. Bradfield, and R. V. Southwell, *Reports and Memoranda No. 1822*, Aeronautical Research Committee, London, 1937.

To avoid confusion, a definite sign convention will be adopted in the work that follows. Moments will be considered positive when producing compression in the upper fibers of the beam. Shears will be considered positive when the resultant force to the left of a section is upward. Loads will be considered positive when the load acts upward. The latter convention is chosen so as to permit successive calculation of shears or moments always by adding, respectively, loads or shears, from left to right, and by subtracting the proper quantities from right to left.

When the shear or moment at any point is known the calculation can always be started from that point, but when only the moments at two points are known, the calculation of shears cannot be performed directly. However, a linear moment diagram, which corresponds to a constant shear, and therefore to no load, can always be added to the moments computed from some arbitrary shear chosen to start the calculations. Therefore, one may obtain the desired conditions relatively simply by merely adding a straight-line moment diagram as a correction, where it is needed.

The procedure is simplified by omitting the multiplication of the shears by the distance between loads until the end of the computations. That is, one can consider the loads as numerical quantities all multiplied by a common factor. The shears will be obtained from the loads, and will contain the same common factor. Then the moments will be computed as numerical quantities all multiplied by a common factor, which is the factor for the loads multiplied by the distance between loads.

The calculations are illustrated by the group of problems shown in Fig. 1. The units in which the loads are measured and the length of the panels are omitted purposely: These may have any values. The beam is divided into six equal segments, and the loads are shown in Fig. 1(*a*). The loading is the same for the different problems, but the manner of support and the method of performing the computations vary in the problems. In Fig. 1(*b*) the beam is cantilevered from the right end. Therefore at the left end the shear is zero and the moment is zero. In Figs. 1(*c*), 1(*d*), and 1(*e*) the beam is simply supported. In Fig. 1(*c*) are given linear correction moments which may be added to the moments in Fig. 1(*b*) to satisfy conditions of simple support; that is, zero moment at the two ends of the beam. The same result is obtained in Fig. 1(*d*), starting with the loads but choosing the shears so as to obtain the correct moments directly. In Fig. 1(*e*) the procedure is carried through in what might be a more usual calculation. One starts with a shear of five at the left end, merely as a guess. Then a proper correction to the moments is written in. The details of the calculations are self-explanatory.

Treatment of Distributed Loads.—When distributed loads are applied to the beam, one can choose equivalent concentrated loads that produce the same shears and moments at certain specified sections of the beam, and thereby handle the problem with the aforementioned single procedure. In so far as statics is concerned, the beam with the distributed load applied directly is equivalent to a system of simply supported stringers resting on the beam, and transmitting the distributed load to the beam as a series of concentrations which are the stringer reactions. The statical equivalence is illustrated in

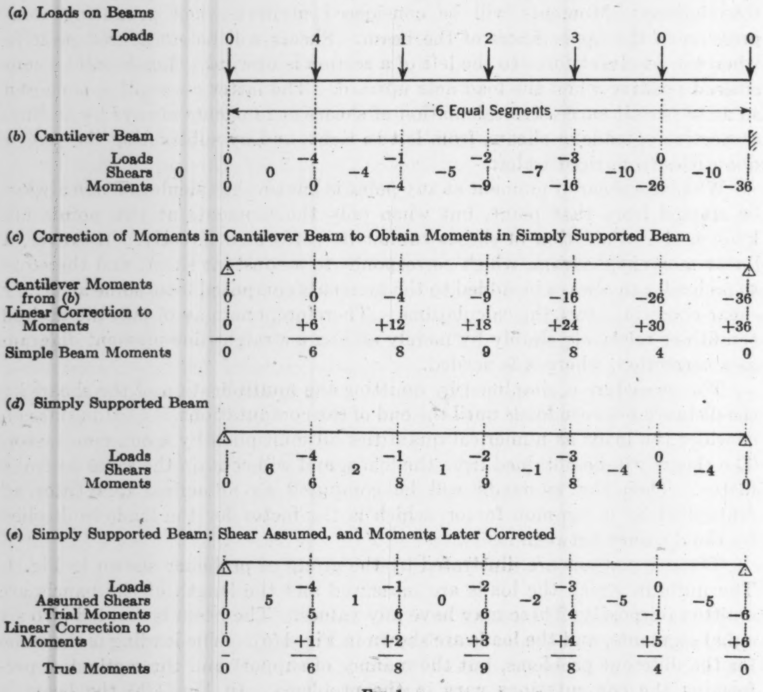

(a) Loads on Beams
Loads 0 4 1 2 3 0 0
6 Equal Segments
(b) Cantilever Beam
Loads 0 −4 −1 −2 −3 0
Shears 0 0 −4 −5 −7 −10 −10
Moments 0 0 −4 −9 −16 −26 −36
(c) Correction of Moments in Cantilever Beam to Obtain Moments in Simply Supported Beam
Cantilever Moments from (b) 0 0 −4 −9 −16 −26 −36
Linear Correction to Moments 0 +6 +12 +18 +24 +30 +36
Simple Beam Moments 0 6 8 9 8 4 0
(d) Simply Supported Beam
Loads −4 −1 −2 −3 0
Shears 6 2 1 −1 −4 −4
Moments 0 6 8 9 8 4 0
(e) Simply Supported Beam; Shear Assumed, and Moments Later Corrected
Loads −4 −1 −2 −3 0
Assumed Shears 5 1 0 −2 −5 −5
Trial Moments 0 5 6 6 4 −1 −6
Linear Correction to Moments 0 +1 +2 +3 +4 +5 +6
True Moments 0 6 8 9 8 4 0

FIG. 1

Fig. 2. It may be observed that the moment or shear at any section of the original beam is equal to the moment or shear at any section through the beam and stringer of the beam-stringer substitute.

One obtains correct moments and, with some care in separating the two subreactions that make up the substitute concentrated load at a point, one obtains correct shears in the original beam at the points of support of the fictitious stringers, by considering a substitute structure loaded only by concentrated loads which are the reactions on the fictitious stringers. One also obtains correct reactions at the ends of the beam.

For a load diagram which consists of straight-line segments, the equivalent concentrated loads are readily determined directly. Formulas for the equivalent loads[10] are stated in Fig. 3. To illustrate the use of the procedure for such a load diagram, several simple problems are shown in Fig. 4. In Fig. 4(*a*), a uniform load on a simply supported beam is considered. Solutions are given in Fig. 4(*b*) for a triangular load diagram on a cantilever beam, and in Fig. 4(*c*) for a triangular load diagram on a simple beam. In Figs. 4(*a*) and 4(*b*) the shears are computed at intermediate points; consequently the equivalent concentrations are shown in two parts. In Fig. 4(*c*) only the shears at the supports (that is, the reactions) are computed, and therefore only the total equivalent loads are shown. Note that the moments in Fig. 4(*c*) could have been obtained from Fig. 4(*b*) by adding a linear moment diagram.

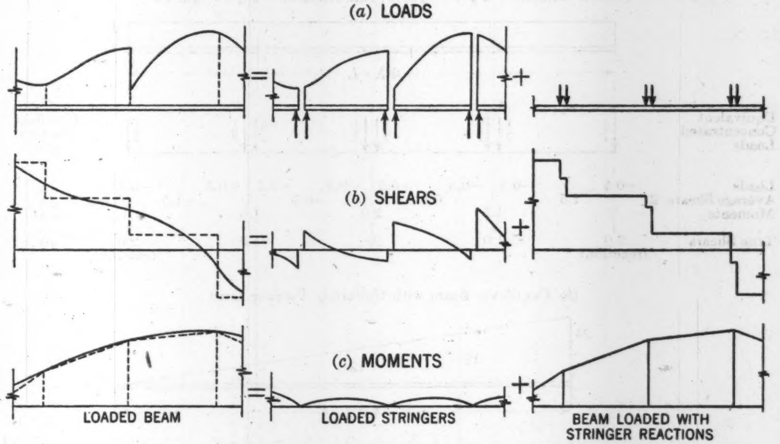
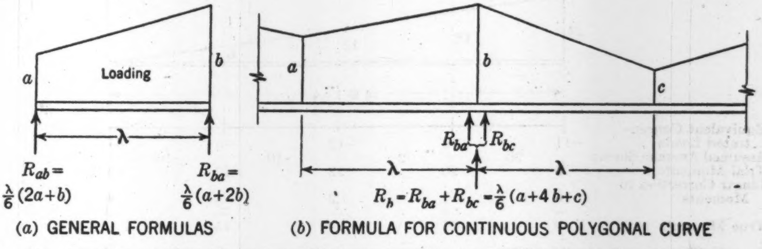

(a) LOADS
(b) SHEARS
(c) MOMENTS
LOADED BEAM
LOADED STRINGERS
BEAM LOADED WITH STRINGER REACTIONS

FIG. 2.—STATIC EQUIVALENCE OF BEAM WITH BEAM AND STRINGERS


a
Loading
b
λ
$R_{ab} = \frac{\lambda}{6}(2a+b)$
$R_{ba} = \frac{\lambda}{6}(a+2b)$
a
b
c
R_{ba}
R_{bc}
λ
λ
$R_b = R_{ba} + R_{bc} = \frac{\lambda}{6}(a+4b+c)$

(*a*) GENERAL FORMULAS (*b*) FORMULA FOR CONTINUOUS POLYGONAL CURVE

FIG. 3.—FORMULAS FOR EQUIVALENT CONCENTRATED LOADS, FOR POLYGONAL LOADING CURVE

One can obtain formulas for more complicated types of load distribution. For practical purposes a distribution that varies according to the ordinates to

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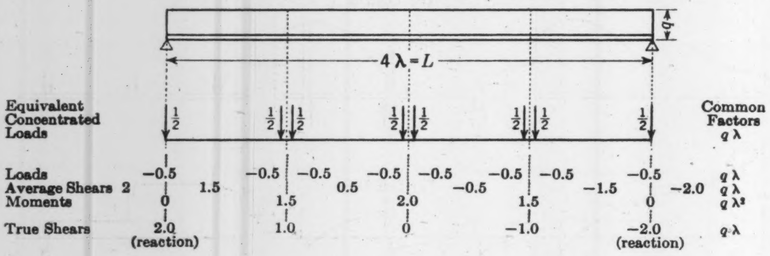
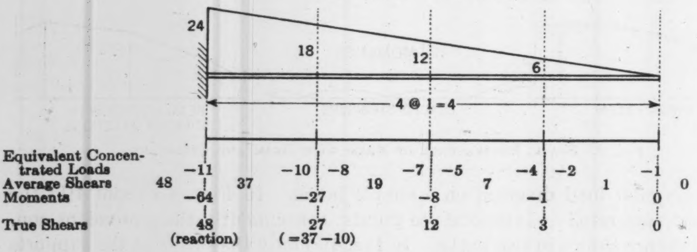
[10] The same formula for an equivalent concentration for a polygonal loading curve has been given in "Die graphische Statik der Baukonstruktionen," by H. Müller-Breslau, Vol. 2, Pt. II, 2d Ed., Leipzig, 1925, p. 44.

an arc of a second-degree parabola is sufficiently general, since it is possible to approximate almost any curve by segments of second-degree parabolas. Formulas for the equivalent concentrations for such a load are given in Fig. 5, in terms of three ordinates to the load distribution curve. The formula in Fig. 5(*b*) for a smooth loading curve was presented by A. Nádai in 1925[11]; and, what amounts to an equivalent formula for the smooth loading curve, with

(*a*) Simply Supported Beam with Uniform Load
(Center moment $= 2\,q\,\lambda^2 = q\,L^2/8$; End reaction $= 2\,q\,\lambda = q\,L/2$)


q
4 λ = L
Equivalent Concentrated Loads
1/2 1/2 1/2 1/2 1/2 1/2 1/2 1/2
Common Factors qλ
Loads −0.5 −0.5 −0.5 −0.5 −0.5 −0.5 −0.5 −0.5 qλ
Average Shears 2 1.5 0.5 −0.5 −1.5 −2.0 qλ
Moments 0 1.5 2.0 1.5 0 qλ²
True Shears 2.0 (reaction) 1.0 0 −1.0 −2.0 (reaction) qλ

(*b*) Cantilever Beam with Uniformly Varying Load


24
18
12
6
4 @ 1 = 4
Equivalent Concentrated Loads −11 −10 −8 −7 −5 −4 −2 −1
Average Shears 48 37 19 7 1 0
Moments −64 −27 −8 −1 0
True Shears 48 (reaction) 27 (=37−10) 12 3 0

(*c*) Simple Beam with Uniformly Varying Load

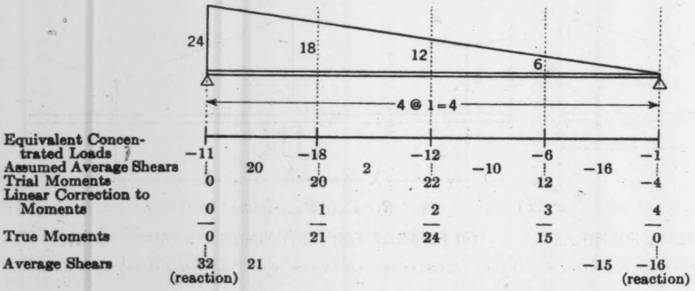

24
18
12
6
4 @ 1 = 4
Equivalent Concentrated Loads −11 −18 −12 −6 −1
Assumed Average Shears 20 2 −10 −16
Trial Moments 0 20 22 12 −4
Linear Correction to Moments 0 1 2 3 4
True Moments 0 21 24 15 0
Average Shears 32 (reaction) 21 −15 −16 (reaction)

FIG. 4.—APPLICATION OF EQUIVALENT LOADS

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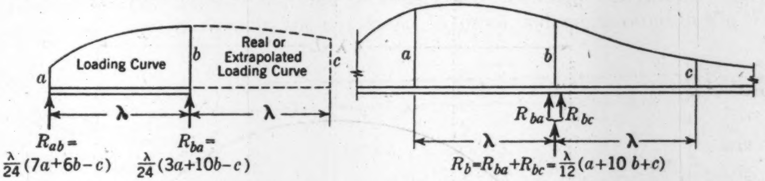
[11] "Die elastischen Platten," by A. Nádai, Berlin, 1925, p. 209, Eq. 13.

additional terms, was derived by Professor Southwell in 1937.[8] A derivation of the formulas in Fig. 5 appears in the Appendix of this paper. It is noted that one of the three ordinates in Fig. 5(*a*) need not be an actual ordinate to the loading curve, but can be an extrapolated value. The formulas in Fig. 5 reduce to those in Fig. 3 when $a + c = 2b$, or when the parabola becomes a straight line. In general, the formulas in Fig. 5 may be used for any distributed loading to give a reasonably good approximation, and they are used in the remainder of this work.


Loading Curve
Real or Extrapolated Loading Curve
a
b
c
λ
λ
$R_{ab} = \frac{\lambda}{24}(7a+6b-c)$
$R_{ba} = \frac{\lambda}{24}(3a+10b-c)$
a
b
c
R_{ba}
R_{bc}
λ
λ
$R_b = R_{ba} + R_{bc} = \frac{\lambda}{12}(a+10b+c)$

(*a*) GENERAL FORMULAS (*b*) FORMULA FOR SMOOTH LOADING CURVE

Fig. 5.—Formulas for Equivalent Concentrated Loads

Problems in which use is made of the formulas in Fig. 5 are given in Part II. A simpler manner of using the results in Fig. 5 for curves that have no discontinuities, nor abrupt changes in slope, is also illustrated in the material that follows.

Part II.—Calculation of Deflection of Beams

General Relations and Definitions.—A direct analogy can be drawn relating loads, shears, and moments in a beam to "angle changes," slopes, and deflections of a beam,[12] where the "angle changes" are the quantities giving the change of slope per unit length—that is, values of moment M divided by modulus of elasticity E and by moment of inertia I for an elastic beam with small deflections. The following sign convention is adopted in order that the analogy may hold without change of signs:

The "angle change" is defined as $-\frac{M}{EI}$; a positive "angle change" is considered as an upward load and therefore as a positive load. Then positive slope corresponds to an increase in deflection from left to right, and corresponds to a positive shear. Finally, positive deflection is taken as downward, and corresponds to a positive moment. A "concentrated angle change" corresponds to an abrupt change in slope at a point, and may be considered in the calculations without difficulty.

As a simple example of the use of the procedure, consider the deflection of a simply-supported beam of constant cross section subjected to uniform load, as in Fig. 6. The moment diagram is a parabola. Therefore the procedure will yield exact results with as many or as few segments as are desired. The calculations are shown for four segments in the length of the beam. The correct center deflection would have been obtained even if only two segments had been considered. Note that the constant factors in moment, angle changes, slopes, and deflections are written as multipliers at the right of the calculations. The equivalent concentrated angle changes at the ends of the beam need not be computed if only the deflections are desired.

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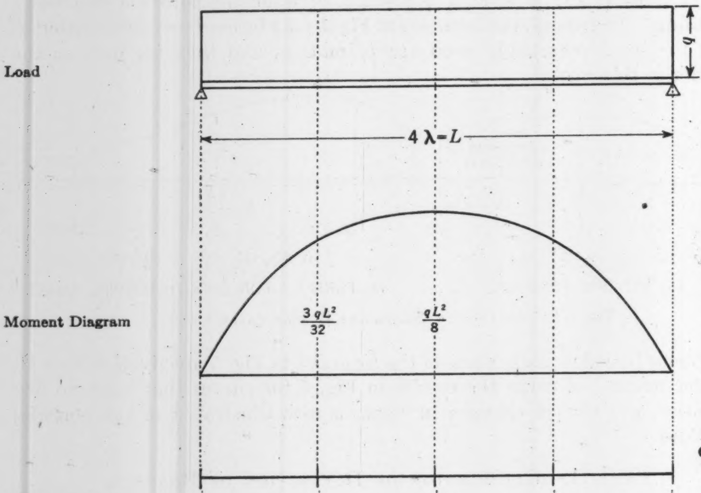
[12] See, for example, "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, New York, N. Y., 1932, pp. 28–30.



										Common Factors
Ordinates to Moment Diagrams	0		3		4		3		0	$\frac{q\,L^2}{32}$
Ordinates to Angle-Change Diagrams	0		−3		−4		−3		0	$\frac{q\,L^2}{32\,E\,I}$
Equivalent Concentrated Angle Changes	−7		−34		−46		−34		−7	$\frac{q\,L^2}{32\,E\,I}\,\frac{\lambda}{12}$
Average Slope		57		23		−23		−57		$\frac{q\,L^2}{32\,E\,I}\,\frac{\lambda}{12}$
Deflection	0		57		80		57		0	$\frac{q\,L^2}{32\,E\,I}\,\frac{\lambda^2}{12}$

Center Deflection $= \frac{80\,q\,L^2}{32\,E\,I}\,\frac{\lambda^2}{12} = \frac{5\,q\,L^4}{384\,E\,I}$

Quarter Point Deflection $= \frac{57}{80}$ **Center Deflection**

Slope at End of Beam $= (57 + 7)\,\frac{q\,L^2}{32\,E\,I}\,\frac{\lambda}{12} = \frac{q\,L^3}{24\,E\,I}$

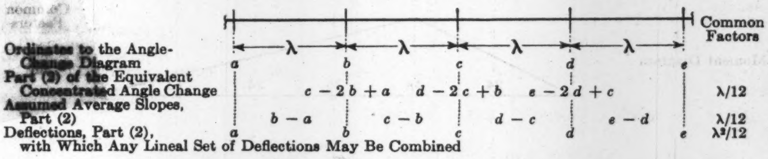
FIG. 6.—DEFLECTIONS FOR SIMPLY SUPPORTED BEAM WITH UNIFORM LOAD

Simplified Procedure for Smooth Angle-Change Curves.—It can be shown that, for the determination of deflections (or moments) alone, a simpler procedure may be used which avoids the calculation of the equivalent concentrated angle changes (or loads) from a distribution of angle changes (or loads) that has no discontinuities nor abrupt changes in slope in the region considered.

From the formula in Fig. 5(*b*) applying to a smooth curve, one has the relation:

$$R_b = \lambda\, b + \frac{\lambda}{12}\,(a - 2\,b + c) \;\dots\dots\dots\dots\dots\dots (1)$$

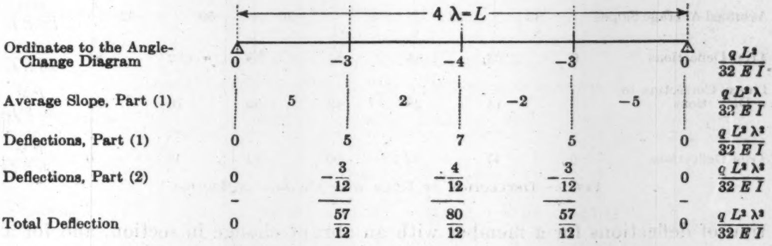
Then one can consider the equivalent concentration at any point such as b as made up of two parts: (1) The ordinate to the curve at the point multiplied by λ; and (2) a correction which is $\frac{\lambda}{12}(a - 2b + c)$. The correction loads at all the points, however, produce a deflection that is proportional to the original angle-change curve; actually, deflections at the various points due to the correction are $\frac{\lambda^2}{12}$ times the value of the distributed angle change at the point, plus any linear diagram required to satisfy the boundary conditions. This is obvious from the form of the equation. A proof is demonstrated in Fig. 7.



Common Factors
λ
λ
λ
λ
Ordinates to the Angle-Change Diagram
a
b
c
d
e
Part (2) of the Equivalent Concentrated Angle Change
$c - 2b + a$
$d - 2c + b$
$e - 2d + c$
λ/12
Assumed Average Slopes, Part (2)
$b - a$
$c - b$
$d - c$
$e - d$
λ/12
Deflections, Part (2), with Which Any Lineal Set of Deflections May Be Combined
a
b
c
d
e
λ²/12

FIG. 7.—DERIVATION OF PART (2) OF THE DEFLECTIONS FOR A SMOOTH CURVE OF ANGLE CHANGES

The problem of Fig. 6 is solved in Fig. 8 by use of the modified procedure. It is noted that the equivalent concentrated angle changes are not written; consequently the slopes must be computed from the original distributed angle changes multiplied by λ, as indicated in the factors to the right of the calculations. One should be careful that part (2) of the deflection is written with its proper sign. It is always $\frac{\lambda^2}{12}$ times the ordinates to the curve of distributed angle changes, and has the same sign as the distributed angle change.



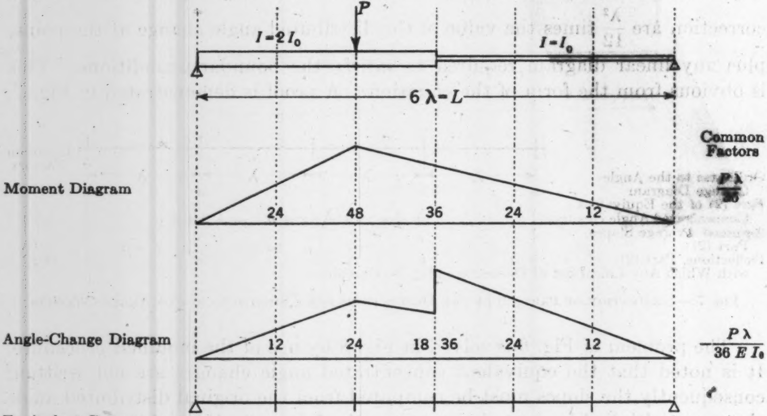
4 λ=L
Ordinates to the Angle-Change Diagram
0
−3
−4
−3
0
$\frac{qL^2}{32EI}$
Average Slope, Part (1)
5
2
−2
−5
$\frac{qL^2\lambda}{32EI}$
Deflections, Part (1)
0
5
7
5
0
$\frac{qL^2\lambda^2}{32EI}$
Deflections, Part (2)
0
$-\frac{3}{12}$
$-\frac{4}{12}$
$-\frac{3}{12}$
0
$\frac{qL^2\lambda^2}{32EI}$
Total Deflection
0
$\frac{57}{12}$
$\frac{80}{12}$
$\frac{57}{12}$
0
$\frac{qL^2\lambda^2}{32EI}$

FIG. 8.—ALTERNATIVE PROCEDURE FOR PROBLEM OF FIG. 6

When the original angle-change diagram is linear (that is, either constant or uniformly varying) it is unnecessary to consider part (2) of the deflections since one may add a linear set of deflections to make the net effect of $\frac{\lambda^2}{12}$ times the original angle changes and the added linear deflection zero. Then one may add whatever other linear deflections are required to satisfy the conditions of the problem.

Where there is a break in the curve or a discontinuity in slope, it is impossible to use this simplified procedure without modification. To avoid confusion the general procedure is recommended for such problems.

Further examples of the use of the general procedure and of the modified procedure are given in Figs. 9 and 10, which illustrate respectively the calculation of deflections for a member with an abrupt change in section, and for a member of varying cross section. The deflections in Fig. 9 will be exact since the $\frac{M}{EI}$-curve is composed of straight-line segments. However, the deflections in Fig. 10 are not exact since the curve of angle changes is not composed of straight-line or parabolic segments. More nearly correct results are obtained by taking more divisions in the length of the beam. The number of divisions actually taken (six) will yield results that are very accurate as is shown by comparison with a solution having twice the number of divisions, in Fig. 10(b), and with an "exact" solution in Fig. 10(c).

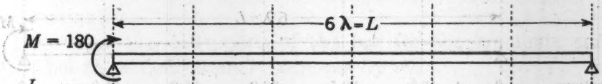


Equivalent Concentrated Angle Changes		−12	−21	−26	−24	−12		$\frac{P\lambda^2}{36EI_0}$
Assumed Average Slopes	33	21	0	−26	−50	−62		$\frac{P\lambda^2}{36EI_0}$
Trial Deflections	0	33	54	54	28	−22	−84	$\frac{P\lambda^3}{36EI_0}$
Linear Correction to Deflections	0	14	28	42	56	70	84	$\frac{P\lambda^3}{36EI_0}$
True Deflections	0	47	82	96	84	48	0	$\frac{P\lambda^3}{36EI_0}$

FIG. 9.—DEFLECTION OF BEAM WITH CHANGE IN SECTION

Analyses of Statically Indeterminate Beams.—By superposing the effects of different end moments one can solve the problem of a statically indeterminate beam also. For example, in Fig. 11(*a*) is shown the same beam as in Fig. 10, with a moment applied to the opposite end. The end slopes due to the moments applied in Figs. 10 and 11 are easily computed and are indicated on the figures. The calculation of end slope from the equivalent concentrated angle change at the end leads to a much greater accuracy than is possible by other means—for example, by methods involving differences of various orders of the final deflections only, as suggested by Professor Southwell.[13]

(a) Deflection of Beam of Variable Section Subjected to Moment at Left End



$\frac{I}{I_0}$	1		2		3		4		5		6		7	Common Factors
Ordinates to Moment Diagram	180		150		120		90		60		30		0	
Ordinates to Angle-Change Diagram	−180		−75		−40		−22.5		−12		−5		0	$1/EI_0$
Assumed Average Slope, Part (1)		100		25		−15		−37.5		−49.5		−54.5		λ/EI_0
Trial Deflections, Part (1)	31.5		131.5		156.5		141.5		104.0		54.5		0	λ^2/EI_0
Deflections, Part (2)	−15.0		−6.3		−3.3		−1.9		−1.0		−0.4		0	λ^2/EI_0
Linear Correction to Deflections	−16.5		−13.8		−11.0		−8.3		−5.5		−2.8		0	λ^2/EI_0
Final Deflection	0		111.4		142.2		131.3		97.5		51.3		0	λ^2/EI_0
End Slopes:														
From Average Slope	111.4												−51.3	λ/EI_0
From Equivalent Concentrated Angle Change at End	69.6												−0.8	λ/EI_0
Total End Slope	181.0												−52.1	λ/EI_0

(b) Results of Solution with 12 Divisions of Length $\frac{\lambda}{2}$; (values given only for alternate points corresponding to those above)

	1	2	3	4	5	6	7	Common Factors
Final Deflections	0	110.7	141.4	130.8	97.1	51.1	0	λ^2/EI_0
End Slopes	177.4						−51.9	λ/EI_0

(c) Exact Solution by Integration

	1	2	3	4	5	6	7	Common Factors
Deflections	0	110.63	141.37	130.76	97.08	51.12	0	λ^2/EI_0
End Slopes	176.75						−51.89	λ/EI_0

FIG. 10.—DEFLECTION OF BEAM OF VARIABLE SECTION BY MODIFIED PROCEDURE

The end slopes in Figs. 10 and 11 differ slightly from the exact values obtained by integration. A much better agreement with the exact values is obtained if a greater number of segments in the length of the beam are used. There is a rapid change in the values of the angle-change curve at the left end of the beam in Figs. 10 and 11, and consequently a greater error in the slopes for this end than for the right end, by the approximate procedure. It should also be pointed out that the slope at the right end in Fig. 10(*a*) should be equal to the slope at the left end in Fig. 11(*a*), by Maxwell's theorem of reciprocal deflections. The difference is due to the fact that the procedure involves some slight inaccuracies, which amount to analyzing slightly different structures in the two cases.

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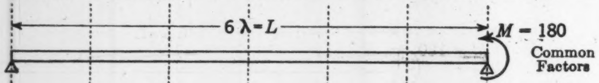
[13] "Relaxation Methods Applied to Engineering Problems, I. The Deflexion of Beams Under Transverse Loading," by K. N. E. Bradfield and R. V. Southwell, *Proceedings*, Royal Soc. of London, Series A, Vol. 161, 1937, pp. 166–167.

From the moments and slopes in Figs. 10 and 11, one can find, for example, the stiffness and the carry-over factor for the left end of the beam by adjusting the moments at the ends to give the proper conditions as shown in Fig. 11(*c*).

(*a*) Deflection of Beam of Fig. 10 for Moment $M = 180$ at Right End, with Beam Divided into 6 Segments



								Common Factors
Final Deflections	0	48.3	82.5	96.9	88.9	56.9	0	λ^2/EI_0
End Slopes	51.2						−69.7	λ/EI_0

(*b*) Exact Solution by Integration

Deflections	0	48.48	82.66	97.03	88.99	56.98	0	λ^2/EI_0
End Slopes	51.89						−69.73	λ/EI_0

(*c*) Combination of Fig. 10(*a*) and Fig. 11(*a*) to Obtain Stiffness and Carry-Over Factor for Left End of Beam

Left end		Right end	
181.0	End slopes, $M = 180$ at left end	−52.1	λ/EI_0
−38.2	End slopes, $M = -0.747 \times 180$ at right end	+52.1	λ/EI_0
142.8	Total slope, $M = 180$ at left end, $M = -0.747 \times 180$ at right end	0	λ/EI_0

Carry-over factor = − 0.747

$$\text{Stiffness} = \frac{180}{142.8}\frac{EI_0}{\lambda} = 1.261\frac{EI_0}{\lambda}$$

(*d*) Exact Values of Carry-Over Factor and Stiffness for Left End of Beam, by Integration

Carry-over factor = − 0.7442

$$\text{Stiffness} = 1.3031\frac{EI_0}{\lambda}$$

FIG. 11.—CALCULATION OF STIFFNESS AND CARRY-OVER FACTOR

For comparison, "exact" values of stiffness and carry-over factor are shown in Fig. 11(*d*), obtained by integration. The agreement is close although only six segments were used in the approximate procedure.

PART III.—DEFLECTION OF BEAMS WITH AXIAL LOADS; BUCKLING OF COLUMNS

General Procedure.—With an accurate procedure available for computing deflections of a beam when the moments are known, it is possible to set up a relatively simple procedure for handling deflections of bars subjected to axial loads as well as lateral loads, by successive approximations. In so far as the final deflections are concerned, the effect of lateral loads on the bar is the same as the effect of initial deflection of the bar from a straight line.

The following method of analysis is recommended for the general case:

(1) Divide the bar into a number of segments. Compute the deflections of the bar due to the lateral loads only, and add these deflections to the initial

deviations from a straight line. Let the total deflection with no axial loads be denoted by the symbol w_i.

(2) Guess at an assumed additional deflection, w_a, which is to represent the effect of the axial forces on the bar. Let the sum of w_a and w_i be denoted by w_0; that is,

$$w_0 = w_i + w_a \qquad \text{(2)}$$

(3) Compute the moments due to the axial loads on the bar, corresponding to the deflections w_0.

(4) Determine the deflections of the bar for the moments computed in step (3). Let these deflections be denoted by w'_a.

(5) Compare w'_a and w_a. If they are equal, w_a is the correct additional deflection of the bar, and w_0 is the correct total deflection of the bar. If they are not equal, repeat steps (2) to (5) until a desired measure of agreement is reached. One may take the values of w'_a in step (4) as a new set of assumed values of w_a, or one may modify these values in order to hasten the process and obtain agreement more rapidly between the assumed deflections and the resulting deflections.

It is necessary to point out that the procedure will work to advantage only when w'_a is a better approximation to the true additional deflections than w_a; in other words, the procedure works best when the sequence of successive approximations converges. It may not work at all when the sequence diverges or oscillates. One can formulate conditions that will insure convergence; but for practical purposes it will be evident that one either approaches a definite result or does not; and if the calculations approach a definite answer, it is the correct answer. Various "tricks" are possible in solving problems in which convergence is slow, or in which there is actually divergence of the results. However, such problems are not common. The writer does not wish to confuse this presentation with too elaborate a set of procedures for exceptional cases. It is sufficient to point out that if by any means w'_a and w_a can be made equal at all division points, one has the correct deflections. By trial, or by a systematic procedure, or by use of simultaneous equations, the two sets of values can always be made equal (even when the routine procedure of using the results in (4) as a new step (2) diverges), since one may take any arbitrary set of values of w_a.

Examples of the general procedure are given subsequently herein. Usually it is possible to obtain a good set of values of w_a if one has available a solution of the problem of pure buckling of the particular bar considered. For this reason a discussion of pure buckling will be given first.

Treatment of the Problem of Pure Buckling Without Lateral Loads.—Consider a bar subjected only to axial loads, without lateral load or initial deflection. Then the quantities w_i in step (1) of the general procedure are zero. The axial loads are to be determined so that an assumed set of deflections w_a corresponds to the same set of deflections w'_a, which means that the deflected bar is in a position of neutral equilibrium, and is on the point of reaching a position of stable (or possibly unstable) equilibrium which is different from the original undeflected position.

The special procedure becomes:

(*a*) Guess at a set of deflections w_a.

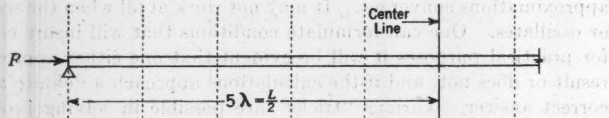
(*b*) Compute moments corresponding to the deflections w_a and an assumed set of values of the axial loads. It is convenient to consider generalized loads. That is, one considers the symbol P to represent all of the axial loads, or the system of such loads, acting on the bar. By assigning values to P, one assigns values to each of the individual loads which P represents.

(*c*) From the moments in step (*b*), compute deflections w'_a.

(*d*) Compare the deflections w'_a and w_a. If they are proportional—that is, if they can be made identically equal for a particular value of P, or for a particular set of values of the axial loads—there is a critical buckling load and the configuration of the bar corresponds to that load.

Again, certain questions may be raised regarding the convergence of a sequence of computations,[14] but these are beyond the scope of this paper; moreover, for most practical problems the difficulties do not arise.

Consider, for example, the calculation of the critical buckling load for a simply supported bar of constant cross section loaded at the ends with axial compressive forces. The calculations for an assumed parabolic deflection curve, symmetrical about the center line, are shown in Fig. 12. Only half the bar is considered since the structure and the deflections are symmetrical. It is seen that the ratio of w_a to w'_a is not constant; the different values of this ratio are recorded, and give the value of P required to produce equality of deflections at the particular points. A repetition of the calculation with new



														Common Factors
Assumed Deflections, w_a	0		36		64		84		96		100[a]		96	
Distributed Angle Changes	0		−36		−64		−84		−96		−100		−96	P/EI
Average Slope, Part (1)		330		294		230		146		50		−50		$P\lambda/EI$
Deflections, Part (1)	0		330		624		854		1,000		1,050		1,000	$P\lambda^2/EI$
Deflections, Part (2)	0		−3.0		−5.3		−7.0		−8.0		−8.3		−8.0	$P\lambda^2/EI$
Resultant Deflections, w'_a	0		327.0		618.7		847.0		992.0		1,041.7		992.0	$P\lambda^2/EI$
Ratio, $\frac{w_a}{w'_a}$	(0.1200)[b]		0.1101		0.1034		0.0992		0.0968		0.0960		0.0968	$EI/P\lambda^2$

(*a*) Average ratio, neglecting ratio for ends $= 0.1017 \frac{EI}{P\lambda^2}$, $\therefore\ P_{cr} = 10.17 \frac{EI}{L^2}$

(*b*) Ratio of sums of deflections, $\frac{\Sigma w_a}{\Sigma w'_a} = 0.0998 \frac{EI}{P\lambda^2}$, $\therefore\ P_{cr} = 9.98 \frac{EI}{L^2}$

(*c*) Best ratio by least squares, $\frac{\Sigma w_a w'_a}{\Sigma w'_a w'_a} = 0.0987 \frac{EI}{P\lambda^2}$, $\therefore\ P_{cr} = 9.87 \frac{EI}{L^2}$

[a] Beam and deflections symmetrical about center line. [b] Ratios of end slopes.

FIG. 12.—CRITICAL BUCKLING LOAD FOR BAR OF CONSTANT SECTION, STARTING WITH ASSUMED PARABOLIC DEFLECTION CURVE

values of w_a, equal to, or proportional to, the values of w'_a shown in Fig. 12, would give more nearly uniform ratios. The best value of the critical load may be taken as the average of the ratios, or as some weighted average; in Fig. 12, three different values of the critical load, computed in different ways, are reported. From similar calculations, it is the writer's conclusion that in most cases a reasonably good approximation to the critical load is the ratio of the sums of the ordinates to the curves of w_a and w'_a.

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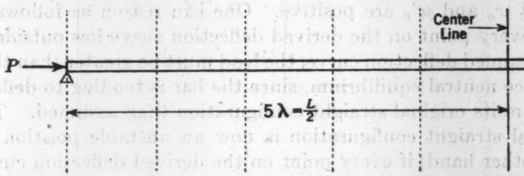
[14] See, for example, "Zur Konvergenz des Engesser-Vianello-Verfahrens," by A. Schleusner, Berlin, 1938.

A more uniform set of ratios with a correspondingly better approximation to the critical load is given with a curve that more nearly approaches the true buckling configuration. In Fig. 13 a set of values is assumed for w_a approximately proportional to the values of w'_a determined in Fig. 12. The result is practically exact. In both Figs. 12 and 13, the true value of the critical load is $\pi^2\ E\,I/L^2$; or, $9.870\ E\,I/L^2$.



												Common Factors
Assumed Deflections, w_a	0		31		59		81		95		100	
Distributed Angle Changes	0		−31		−59		−81		−95		−100	$P/E\,I$
Average Slope, Part (1)		316		285		226		145		50		$P\,\lambda/E\,I$
Deflection, Part (1)	0		316		601		827		972		1,022	$P\,\lambda^2/E\,I$
Deflection, Part (2)	0		−2.6		−4.9		−6.8		−7.9		−8.3	$P\,\lambda^2/E\,I$
Resultant Deflection, w'_a	0		313.4		596.1		820.2		964.1		1,013.7	$P\,\lambda^2/E\,I$
Ratio, $\frac{w_a}{w'_a}$	. .		0.0989		0.0990		0.0987		0.0985		0.0986	$E\,I/P\,\lambda^2$

$$\text{Average ratio} = 0.0988\,\frac{E\,I}{P\,\lambda^2}$$

$$\text{Ratio of sums} = 0.0987\,\frac{E\,I}{P\,\lambda^2}$$

$$\therefore\ P_{cr} = 0.0987\,\frac{E\,I}{\lambda^2} = 9.87\,\frac{E\,I}{L^2}$$

FIG. 13.—CRITICAL BUCKLING LOAD FOR BAR OF CONSTANT SECTION

Obviously it is possible to find different patterns of deflections corresponding to different values of critical loads for the same bar. In general, only the lowest critical load is of significance as far as pure buckling is concerned, since the higher loads must correspond to essentially unstable positions of equilibrium; but, if an initial deflection curve is assumed that contains no components of the configuration corresponding to the lowest critical buckling load, the lowest load cannot be obtained from this procedure (nor would it be obtained from any other available procedure, such as methods involving minimum of energy). Such a situation would follow from the assumption of a deflection curve anti-symmetrical about the center line for the beam in Fig. 12. One would reject such a curve intuitively for this problem. Yet in an unusual case, it might be possible that a designer may reject, unthinkingly, the configuration that corresponds to the lowest buckling load. An example of such a case is shown subsequently in Fig. 20.

Ordinarily, convergence of several different sequences of computations involving different shapes of assumed deflection curves, to the same final shape, would be a sufficient indication that the designer had reached the configuration corresponding to the lowest critical load. In some cases, however, the convergence of a sequence of computations may be very slow; this will be so when the next higher critical load differs only slightly from the lowest critical load. Methods of handling such problems can be derived but are beyond the scope of the present paper.

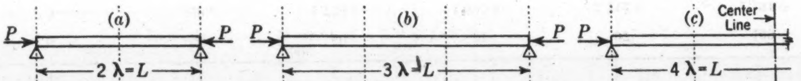
Determination of Maximum and Minimum Values for the Critical Load.—In general, the lowest critical buckling load must have a value between the limits defined by the smallest and largest values of the ratio of w_a to w'_a, when all values of w_a and w'_a are positive. One can reason as follows to justify this rule: If every point on the derived deflection curve lies outside of every point on the assumed deflection curve, the load must be greater than the load required to produce neutral equilibrium, since the bar is tending to deflect even farther away from its original straight configuration than assumed. This means that the initial straight configuration is now an unstable position of equilibrium. On the other hand, if every point on the derived deflection curve lies between the original straight configuration and the assumed configuration, then the load must be less than the load required to produce neutral equilibrium. Evidently, in this case, the undeflected position is a position of stable equilibrium; but the two conditions described correspond to the maximum and the minimum values of the ratio of w_a to w'_a. Therefore the critical buckling load must be between these limits. The rule is important for practical purposes; the designer can readily detect between what limits the buckling load must lie.

In using this rule to bound the value of the critical load, it must be remembered that the structure set up for analysis is not exactly the same as the structure it represents, although with a reasonably large number of divisions the two are closely similar. The process of dividing the bar into segments is equivalent to substituting for it a slightly different structure. This becomes evident if the buckling load is computed for a bar divided into only two segments, as in Fig. 14(*a*).

In certain cases the foregoing rule is inapplicable. Care must be taken in using it when axial loads are applied other than at the ends of a bar. Also, the rule would be misleading in such cases where the lowest critical load corresponds to a deflection curve that has both positive and negative deflections, whereas the next higher critical load might correspond to a deflection curve with only positive ordinates.

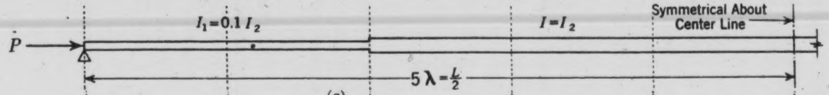
Illustrative Problems for Pure Buckling.—The problems shown in Figs. 14, 15, and 16 illustrate further uses of the procedure for computing the critical load for a beam subjected to pure buckling.

The effect of taking different numbers of segments in the length of the bar is illustrated in Fig. 14, for a simply supported bar of uniform section subjected to end thrust. The error in the buckling load computed by the approximate procedure described herein, compared with the exact buckling load, is 2.74% for two segments, 0.52% for three segments, and 0.16% for four segments, in the full length of the bar.



			(a)					(b)							(c)			Common Factors
Assumed Deflections, w_a	0		2		0	0		1		1		0	0		707		1,000	
Distributed Angle Changes	0		−2		0	0		−1		−1		0	0		−707		−1,000	$\frac{P}{EI}$
Average Slope, Part (1)		1		−1			1		0		−1			1,207		500		$\frac{P\lambda}{EI}$
Deflection, Part (1)	0		1		0	0		1		1		0	0		1,207		1,707	$\frac{P\lambda^2}{EI}$
Deflection, Part (2)	0		$-\frac{2}{12}$		0	0		$-\frac{1}{12}$		$-\frac{1}{12}$		0	0		−59		−83	$\frac{P\lambda^2}{EI}$
Resultant Deflection, w'_a	0		$\frac{10}{12}$		0	0		$\frac{11}{12}$		$\frac{11}{12}$		0	0		1,148		1,624	$\frac{P\lambda^2}{EI}$
Ratio, $\frac{w_a}{w'_a}$			2.40					1.091		1.091					0.6159		0.6158	$\frac{EI}{P\lambda^2}$
P_{cr}			$2.40\frac{EI}{\lambda^2} = 9.60\frac{EI}{L^2}$					$1.091\frac{EI}{\lambda^2} = 9.819\frac{EI}{L^2}$							$0.6159\frac{EI}{\lambda^2} = 9.854\frac{EI}{L^2}$			
Percentage Error in Terms of Exact Value, $P_{cr} = 9.870\frac{EI}{L^2}$,			2.74					0.52							0.16			

FIG. 14.—EFFECT ON BUCKLING LOAD OF TAKING DIFFERENT NUMBERS OF SEGMENTS FOR BAR OF CONSTANT CROSS SECTION



(a)

												Common Factors
Assumed Deflection, w_a	0		51.35		80.36		91.10		97.75		100.00	
Distributed Angle Changes	0		−513.5		−803.6 ¦ −80.36		−91.10		−97.75		−100.00	$P/E\,I_2$
Equivalent Concentrated Angle Changes			−494.88		−404.90		−90.76		−97.38		−99.63	$P\,\lambda/E\,I_2$
Average Slope		1,137.74		642.86		237.96		147.20		49.82		$P\,\lambda/E\,I_2$
Resultant Deflection, w'_a	0		1,137.74		1,780.60		2,018.56		2,165.76		2,215.58	$P\,\lambda^2/E\,I_2$
Ratio, $\frac{w_a}{w'_a}$			0.04513		0.04513		0.04513		0.04513		0.04513	$E\,I_2/P\,\lambda^2$

$$P_{cr} = 0.04513\,\frac{E\,I_2}{\lambda^2} = 4.513\,\frac{E\,I_2}{L^2}$$

(b)

												Common Factors
Assumed Deflection, w_a	0		36		64		84		96		100	
Distributed Angle Changes	0		−360		−640 ¦ −64		−84		−96		−100	$P/E\,I_2$
Equivalent Concentrated Angle Changes			−353.3		−312.3		−83.3		−95.3		−99.3	$P\,\lambda/E\,I_2$
Average Slopes		893.9		540.6		228.3		145.0		49.7		$P\,\lambda/E\,I_2$
Resultant Deflection, w'_a	0		893.9		1,434.5		1,662.8		1,807.8		1,857.5	$P\,\lambda^2/E\,I_2$
Ratio, $\frac{w_a}{w'_a}$			0.0403		0.0446		0.0505		0.0531		0.0538	$E\,I_2/P\,\lambda^2$

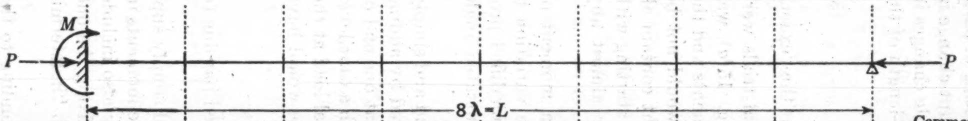
$$\sim P_{cr} = \frac{\Sigma\, w_a}{\Sigma\, w'_a} = 4.91\,\frac{E\,I_2}{L^2}$$

(c)

							Common Factors
Assumed Deflection, w_a	0	50	80	91	97	100	
Resultant Deflection, w'_a	0	1,120.8	1,758.3	1,995.5	2,142.1	2,191.9	$P\,\lambda^2/E\,I_2$
Ratio, $\frac{w_a}{w'_a}$		0.0446	0.0455	0.0456	0.0453	0.0456	$E\,I_2/P\,\lambda^2$

$$\sim P_{cr} = \frac{\Sigma\, w_a}{\Sigma\, w'_a} = 4.54\,\frac{E\,I_2}{L^2}$$

Fig. 15.—Buckling of a Bar with Change in Section



(*a*) Deflection Due to Moment $M = 240$ at Left End

																		Common Factors
Moments	240		210		180		150		120		90		60		30		0	
Distributed Angle Changes	−240		−210		−180		−150		−120		−90		−60		−30		0	$1/E\,I$
Assumed Average Slope, Part (1)		500		290		110		−40		−160		−250		−310		−340		$\lambda/E\,I$
Trial Deflection, Parts (1) and (2)	0		500		790		900		860		700		450		140		−200	$\lambda^2/E\,I$
Linear Correction to Deflection	0		25		50		75		100		125		150		175		200	$\lambda^2/E\,I$
Resultant Deflection, w_s	0		525		840		975		960		825		600		315		0	$\lambda^2/E\,I$

$$\text{Slope at left end} = (525 + 115)\frac{\lambda}{E\,I} = 640\frac{\lambda}{E\,I}$$

(*b*) Deflections Due to End Loads with Assumed Deflections

Assumed Deflection, w_a	0		32		115		212		286		307		259		148		0	a
Distributed Angle Changes	0		−32		−115		−212		−286		−307		−259		−148		0	$P\,a/E\,I$
Assumed Average Slope, Part (1)		585		553		438		226		−60		−367		−626		−774		$P\,\lambda\,a/E\,I$
Trial Deflection, Part (1)	0		585		1,138		1,576		1,802		1,742		1,375		749		−25	$P\,\lambda^2\,a/E\,I$
Deflection, Part (2)	0		−3		−10		−18		−24		−26		−22		−12		0	$P\,\lambda^2\,a/E\,I$
Linear Correction to Deflection	0		3		6		9		13		16		19		22		25	$P\,\lambda^2\,a/E\,I$
Intermediate Deflection, w_b	0		585		1,134		1,567		1,791		1,732		1,372		759		0	$P\,\lambda^2\,a/E\,I$

$$\text{End slope} = (585 + 3)\frac{P\,\lambda\,a}{E\,I} = 588\frac{P\,\lambda\,a}{E\,I}$$

(*c*) Effect of Making Slope at Left End Zero, Moment at Left End $= -\dfrac{588\dfrac{P\,\lambda\,a}{E\,I}}{640\dfrac{\lambda}{E\,I}}(240) = -220.5\,P\,a$

$-\dfrac{220.5\,P\,a}{240}w_s$	0	−482	−772	−896	−882	−758	−551	−289	0	$P\,\lambda^2\,a/E\,I$
Resultant Deflection, w'_a	0	103	362	671	909	974	821	470	0	$P\,\lambda^2\,a/E\,I$
Ratio, $\dfrac{w_a}{w'_a}$		0.310	0.318	0.316	0.314	0.315	0.315	0.315		$E\,I/P\,\lambda^2$

$$P_{cr} = 0.315\frac{E\,I}{\lambda^2} = 20.16\frac{E\,I}{L^2}$$

FIG. 16.—BUCKLING OF BAR FIXED AT ONE END

In Fig. 15(*a*), a solution is given for the buckling of a bar consisting of parts of constant but different moment of inertia. Because of the abrupt change in moment of inertia there is a discontinuity in the values of the angle changes in the bar. The result obtained with only five divisions in the half-length of the bar is

$$P_{cr} = 4.51 \frac{E I_2}{L^2} \dots\dots\dots\dots\dots\dots\dots (3)$$

which compares with $4.50 \frac{E I_2}{L^2}$ given by Professor Timoshenko[15] as the "exact" value of the critical load. It should be remembered that several trials were necessary before as uniform a set of ratios as is shown in Fig. 15(*a*) was obtained; but the intermediate work can be done without refinements and the final result obtained fairly rapidly. For practical purposes it would not be necessary to go so far. For example, Figs. 15(*b*) and 15(*c*) might contain all the calculations required in most cases, where even the first step, starting with an assumed parabolic deflection curve, would be adequate for almost any practical problem.

In a similar manner, other problems involving variations in moment of inertia along the length of the bar may be solved. Where the variation is smooth (that is, without abrupt changes) the relatively simple modified procedure which does not require calculation of "equivalent" concentrated angle changes may be used.

The solution of the problem of buckling of a bar fixed at one end and simply supported at the other is shown in Fig. 16. The problem is solved by adding to a simply supported bar an end moment to annul the rotation at one end of the bar. The problem might also have been solved by dealing with a cantilever beam acted on by a direct thrust, and adding the effect of a lateral load at the end in order to make the deflection at the end zero. The results would have been exactly the same.

The procedure used in Fig. 16 may be outlined as follows:

(*a*) Find the deflections and end rotation of a simply supported bar due to a moment applied at one end. Denote the deflections by w_s.

(*b*) Assume a deflection curve for the bar fixed at one end and simply supported at the other. Denote the deflections by w_a. Compute the moments in the bar due to the direct loads and the deflections w_a. One may also include assumed moments to account in some measure for the effect of fixing the one end of the bar. In general, it would be desirable to include such "indeterminate" moments, although in Fig. 16 they were omitted.

(*c*) Compute the deflections w_b and the end rotation corresponding to the moments in step (*b*). If the end rotation is not zero, add such a moment as would be required to make it zero. This involves adding deflections also, proportional to w_s. Denote the resultant deflections by the symbol w'_a.

(*d*) Compare w_a and w'_a, as in the procedure described previously for determining buckling loads for statically determinate bars. If w_a and w'_a are similar, one has the correct shape of the deflection curve, and one can obtain the critical load. If w_a and w'_a are not similar, one may repeat steps (*b*) to (*d*) as many times as necessary, until one obtains a sufficiently good value of the critical load.

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[15] "Theory of Elastic Stability," by S. Timoshenko, New York, N. Y., 1936, pp. 128–131.

A procedure similar to the foregoing may be developed for other statically indeterminate beams or columns.

Note that in Fig. 16(*a*) the moment diagram and the angle-change diagram are linear; therefore it was not necessary to compute part (2) of the deflections, as explained in section II of this paper. In Fig. 16(*b*) a common factor *a* is indicated for the deflections in order to make it clear that the end moment in Fig. 16(*c*) depends on the deflections. The final value of the critical load is practically exact.[16]

Illustrative Problems, Combined Axial and Lateral Loads.—When lateral loads act on a beam together with an end thrust, the effect of the end thrust is to produce additional deflections and additional moments beyond those produced by the lateral loads alone. The additional deflections are governed by the deflection due to the lateral load alone, and by the ratio of the axial loads to the critical value of the axial loads.

For the first step in the general procedure of solving such problems it is necessary to assume a set of values of the additional deflection, w_a. As a convenient approximation for the first trial value of w_a it is desirable to take w_a as follows:

$$w_a = \frac{1}{\frac{P_{cr}}{P} - 1} w_i \quad \text{.........................} \quad (4)$$

in which P_{cr} is the magnitude of the critical buckling load, P is the magnitude of the actual load, and w_i is the sum of the initial deflection and the deflection due to the lateral load alone. When w_i is of the same shape as the deflection curve corresponding to the lowest critical buckling load, the value of w_a given by Eq. 4 will be exact.[17] In other cases, it hastens the convergence toward the correct value of w_a if w_a is assumed as suggested.

The calculations for a simply supported bar subjected to end thrusts and uniform lateral load are shown in Fig. 17. The values of w_i for the uniform load are computed first. The value of P_{cr} for the bar can be taken from Fig. 12. Then with the given load, $P = 0.02 \frac{E\,I}{\lambda^2}$, and the critical load, $P_{cr} = 0.0987 \frac{E\,I}{\lambda^2}$, one finds from Eq. 4 the following result:

$$w_a = 0.254\, w_i \quad \text{.........................} \quad (5)$$

With this value of w_a, the computations in Fig. 17(*b*) lead to a set of values of w'_a which are practically equal to those assumed. If further refinement is desired one can repeat the calculations. One may also deal with additions to the values of w_a already assumed and obtain additions or corrections to w'_a; but in this problem no further computations appear to be necessary, and one may conclude that under the given conditions the effect of the axial load is to cause an apparent increase in the maximum moment due to the lateral loads alone of about 26%.

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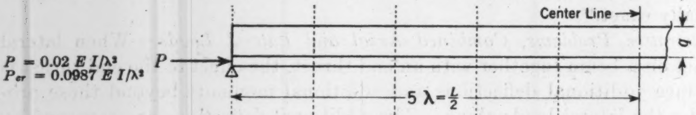
[16] "Theory of Elastic Stability," by S. Timoshenko, New York, N. Y., 1936, pp. 88–89.

[17] See, for example, "Buckling of Elastic Structures," by H. M. Westergaard, *Transactions*, Am. Soc. C. E., Vol. LXXXV (1922), pp. 576–676, especially pp. 618–619. Note difference in notation, however.

Similar calculations are shown in Fig. 18 for a bar subjected to a moment at one end combined with direct thrust. Here the first approximation is not nearly so close to the final answer since the deflection curve due to the moment alone differs considerably from the configuration corresponding to the critical buckling load.

The procedure described here is applicable also to problems in which P is negative; that is, where axial tensions instead of compressions act on the bar. In such cases, however, the effect of the end tension is generally to reduce the deflections due to the lateral loads or initial eccentricities only. The same general procedure may be used. The value of w_a suggested in Eq. 4 will be negative, since, if the axial tensions are denoted by T, one has the result

$$w_a = -\frac{1}{\frac{P_{cr}}{T} + 1} w_i \quad \text{.........................} (6)$$


$P = 0.02\ E\,I/\lambda^2$
$P_{cr} = 0.0987\ E\,I/\lambda^2$
P
Center Line
q
$5\ \lambda = \frac{L}{2}$

(a)

												Common Factors
Moments Due to Lateral Load	0		36		64		84		96		100	$q\lambda^2/8$
Distributed Angle Changes	0		−36		−64		−84		−96		−100	$q\lambda^2/8EI$
Average Slope, Part (1)		330		294		230		146		50		$q\lambda^3/8EI$
Deflection, Part (1)	0		330		624		854		1,000		1,050	$q\lambda^4/8EI$
Deflection, Part (2)	0		−3		−5		−7		−8		−8	$q\lambda^4/8EI$
Deflection, w_i	0		327		619		847		992		1,042	$q\lambda^4/8EI$

(b)

Assumed Deflection, w_a from Eq. 4	0		83		157		216		252		265	$q\lambda^4/8EI$
$w_0 = w_a + w_i$	0		410		776		1,063		1,244		1,307	$q\lambda^4/8EI$
Moments Due to P	0		8.20		15.52		21.26		24.88		26.14	$q\lambda^2/8$
Distributed Angle Changes	0		−8.20		−15.52		−21.26		−24.88		−26.14	$q\lambda^2/8EI$
Average Slope, Part (1)		82.93		74.73		59.21		37.95		13.07		$q\lambda^3/8EI$
Deflection, Part (1)	0		82.93		157.66		216.87		254.82		267.89	$q\lambda^4/8EI$
Deflection, Part (2)	0		−0.68		−1.29		−1.77		−2.07		−2.18	$q\lambda^4/8EI$
Resultant Deflection, w'_a	0		82		156		215		253		266	$q\lambda^4/8EI$

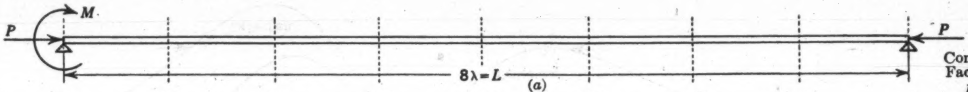
FIG. 17.—DEFLECTION OF A BAR SUBJECTED TO UNIFORM LOAD AND END THRUST

A difficulty arises in problems where T is numerically greater than the value of the lowest critical buckling load. In such cases the sequence of computations will oscillate, and will not converge. In general, each assumed value of w_a will lead to a value of w'_a which will be farther from the true configuration than w_a if w_a is not correctly chosen equal to its true value. Methods of solving such problems can be developed, however, and in general one can arrive eventually at reasonably good results since the effect of the end tensions can never be to produce greater deflections than w_i except at a few points. Further discussion of problems such as these will not be given in the present paper.

Buckling Due to Axial Loads Applied at Intermediate Points Along the Length of a Bar.—The problems previously treated herein concern axial loads applied at the ends of a bar. Bars with axial loads applied at interior points are considered in Figs. 19 and 20. In Fig. 19, the left part of the bar is in compression

$P = 0.10\, EI/\lambda^2$

$P_{cr} = 9.87\, EI/L^2$
$= 0.1542\, EI/\lambda^2$



(a)

										Common Factors
w_i from Fig. 16(a)	0	525	840	975	960	825	600	315	0	$\frac{M\lambda^2}{240\,EI}$

(b)

Assumed Deflection, w_a from Eq. 4	0	969	1,550	1,799	1,771	1,522	1,107	581	0	$\frac{M\lambda^2}{240\,EI}$
$w_s = w_a + w_i$	0	1,494	2,390	2,774	2,731	2,347	1,707	896	0	$\frac{M\lambda^2}{240\,EI}$
Moments due to P	0	149.4	239.0	277.4	273.1	234.7	170.7	89.6	0	$\frac{M}{240}$
Distributed Angle Changes	0	−149.4	−239.0	−277.4	−273.1	−234.7	−170.7	−89.6	0	$\frac{M}{240\,EI}$
Assumed Average Slope, Part (1)		750.0	600.6	361.6	84.2	−188.9	−423.6	−594.3	−683.9	$\frac{M\lambda}{240\,EI}$
Trial Deflection, Part (1)	0	750.0	1,350.6	1,712.2	1,796.4	1,607.5	1,183.9	589.6	−94.3	$\frac{M\lambda^2}{240\,EI}$
Deflection, Part (2)	0	−12.5	−19.9	−23.1	−22.8	−19.6	−14.2	−7.5	0	$\frac{M\lambda^2}{240\,EI}$
Linear Correction to Deflection	0	11.8	23.6	35.4	47.2	58.9	70.7	82.5	94.3	$\frac{M\lambda^2}{240\,EI}$
Resultant Deflection, w'_s	0	749.3	1,354.3	1,724.5	1,820.8	1,646.8	1,240.4	664.6	0	$\frac{M\lambda^2}{240\,EI}$

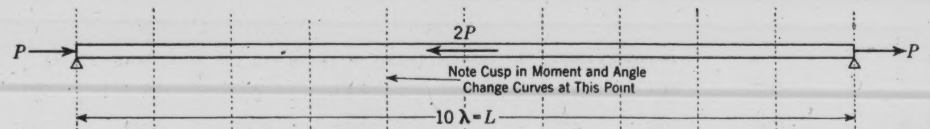
(c) Final Deflections w_s which Lead to Same Deflections w'_s

	0	720	1,319	1,705	1,826	1,672	1,271	685	0	$\frac{M\lambda^2}{240\,EI}$

(d) Moments in Beam

Due to M Alone	240	210	180	150	120	90	60	30	0	$\frac{M}{240}$
Due to Thrust	0	124.5	215.9	268.0	278.6	249.7	187.1	100.0	0	$\frac{M}{240}$
Total Moment	240.0	334.5	395.9	418.0	398.6	339.7	247.1	130.0	0	$\frac{M}{240}$
Total Moment	1.000	1.394	1.650	1.742	1.661	1.415	1.030	0.542	0	M

FIG. 18.—DEFLECTION OF A BAR SUBJECTED TO END MOMENT AND DIRECT COMPRESSION



(*a*) Assumed Parabolic Deflection Curve

												Common Factors
Assumed Deflection, w_a	0	375	667	875	1,000	1,042	1,000	875	667	375	0	a
Moment	0	175	267	275	200	−42	−200	−275	−267	−175	0	$P a$
Angle Changes	0	−175	−267	−275	−200	+42	+200	+275	+267	+175	0	$P a/E I$
Equivalent Concentrated Angle Changes	0	−168	−260	−268	−172	35	193	268	260	168	0	$P a \lambda/E I$
Assumed Average Slope		404	236	−24	−292	−464	−429	−236	32	292	460	$P a \lambda/E I$
Trial Deflection	0	404	640	616	324	−140	−569	−805	−773	−481	−21	$P a \lambda^2/E I$
Linear Correction to Deflection	0	2	4	6	8	11	13	15	17	19	+21	$P a \lambda^2/E I$
Resultant Deflection, w'_a	0	406	644	622	332	−129	−556	−790	−756	−462	0	$P a \lambda^2/E I$

(*b*) Final Deflection Curve Leading to a Negative Value of P_{cr}

w_a	0	226	460	710	1,000	1,323	1,547	1,544	1,255	698	0	a
Moment	0	26	60	110	200	−323	−747	−944	−855	−498	0	$P a$
w'_a	0	732	1,491	2,312	3,245	4,271	4,981	4,964	4,027	2,256	0	$-P a \lambda^2/E I$
Ratio $\frac{w_a}{w'_a}$		0.309	0.308	0.307	0.308	0.310	0.311	0.311	0.312	0.309		$-E I/P \lambda^2$

$$P_{cr} = -0.310\frac{E I}{\lambda^2} = -31.0\frac{E I}{L^2}$$

(*c*) Final Deflection Curve Leading to a Positive Value of P_{cr}

w_a	0	480	839	1,009	1,000	899	749	570	388	197	0	a
Moment	0	280	439	409	200	101	51	30	12	3	0	$P a$
w'_a	0	1,078	1,885	2,270	2,260	2,027	1,688	1,296	874	439	0	$P a \lambda^2/E I$
Ratio $\frac{w_a}{w'_a}$		0.445	0.445	0.444	0.442	0.444	0.444	0.440	0.444	0.449		$E I/P \lambda^2$

$$P_{cr} = 0.444\frac{E I}{\lambda^2} = 44.4\frac{E I}{L^2}$$

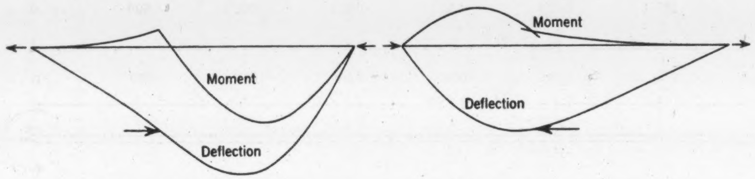
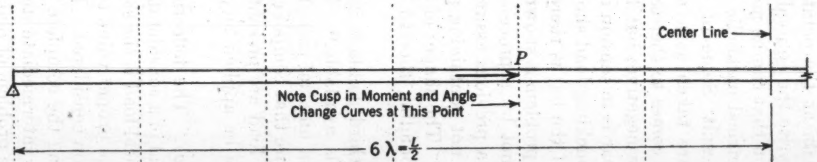

(*d*) Deflections and Moments for Fig. 19(*b*)   (*e*) Deflections and Moments for Fig. 19(*c*)

FIG. 19.—BUCKLING OF A BAR SUBJECTED TO AXIAL TENSION AND COMPRESSION



(a) Symmetrical Configuration

														Common Factors
Assumed Deflection, w_a	0		..		..		..		c		$c+1{,}414$		$c+2{,}000$	a
Moment	0		0		0		0		0		1,414		2,000	$P\,a$
Angle Changes	0		0		0		0		0		−1,414		−2,000	$P\,a/E\,I$
Equivalent Concentrated Angle Changes	0		0		0		0		−270		−1,345		−1,902	$P\,a\,\lambda/E\,I$
Average Slope		2,566		2,566		2,566		2,566		2,296		951		$P\,a\,\lambda/E\,I$
Resultant Deflection, w'_a									c'		$c'+2{,}296$		$c'+3{,}247$	$P\,a\,\lambda^2/E\,I$
Ratio, $\frac{w_a}{w'_a}$, if $\frac{c}{c'}$ has same value as ratio:											0.6159		0.6160	$E\,I/P\,\lambda^2$

$$P_{cr} = 0.6159\,\frac{E\,I}{\lambda^2}$$

Deflection, w'_a	0	2,566	5,132	7,698	10,264	12,560	13,511	$P\,a\,\lambda^2/E\,I$

(b) Antisymmetrical Configuration

Assumed Deflection, w_a	0		...		....		....		3,948		2,424		0	a
Moment	0		658		1,316		1,974		2,632		1,766		0	$P\,a$
Angle Changes	0		−658		−1,316		−1,974		−2,632		−1,766		0	$P\,a/E\,I$
Equivalent Concentrated Angle Changes			−658		−1,316		−1,974		−2,415.5		−1,691		0	$P\,a\,\lambda/E\,I$
Assumed Average Slope		3,500		2,842		1,526		−448		−2,863.5		−4,554.5		$P\,a\,\lambda/E\,I$
Trial Deflection	0		3,500		6,342		7,868		7,420		4,556.5		2.0	$P\,a\,\lambda^2/E\,I$
Linear Correction to Deflection	0		−0.3		−0.7		−1.0		−1.3		−1.7		−2.0	$P\,a\,\lambda^2/E\,I$
Resultant Deflection, w'_a	0		3,500		6,341		7,867		7,419		4,555		0	$P\,a\,\lambda^2/E\,I$
Ratio, $\frac{w_a}{w'_a}$									0.5321		0.5322			$E\,I/P\,\lambda^2$

$$P_{cr} = 0.5322\,\frac{E\,I}{\lambda^2}$$

FIG. 20.—BUCKLING OF A BAR WITH COMPRESSION IN MIDDLE THIRD OF LENGTH

and the right part in tension. The point of application of the interior load is assumed to deflect with the bar; consequently if the load point deflects shears must be applied at the ends of the bar for equilibrium. Since there will be a cusp, or discontinuity in slope of the angle-change diagram at the point of application of the interior load, the procedure used is to write the equivalent concentrated angle changes instead of making the correction that can be made for a smooth angle-change curve. In Fig. 19(*a*) a symmetrical parabolic deflection curve is assumed first. One finds a peculiar result: Some of the resulting deflections are negative. If these deflections are taken as a new deflection curve, and the process repeated, eventually one comes to the result shown in Fig. 19(*b*) where, apparently, the critical load is negative; but this merely indicates a situation in which the left part of the bar is in tension and the right part in compression. It is reasonable that the buckling load should be less for this arrangement of loads since a longer part of the bar is thereby subjected to compression. The final result for the original problem is shown in Fig. 19(*c*). It may be obtained by repeated trials, but not by a process in which each new configuration is the result obtained from a previous assumed configuration, unless the starting point is a configuration not containing any appreciable component of the type obtained in Fig. 19(*b*). The shapes of the final deflection curves and the moment diagrams corresponding thereto are shown in Fig. 19(*d*).

A bar subjected to two opposing loads applied at the third points is illustrated by Fig. 20. An exact solution for this problem is available.[18] The problem is given not only to illustrate the procedure for an unusual case, but also to show what can happen when care is not taken to insure that components of deflection corresponding to the lowest critical buckling load are present in the assumed deflection curve. The loads are assumed to be applied on the axis of the bar even when the bar deflects.

A symmetrical deflection of the bar is shown in Fig. 20(*a*). The deflections outside of the region subjected to compression are immaterial in a consideration of the critical buckling load. It will be noted that the critical load is the same as in Fig. 14(*c*); but some care is necessary in obtaining the proper value of c, the unknown constant part of all the deflections in the region considered. For the final deflection curve c can be obtained easily by taking the complete deflection curve for w'_a and repeating the calculations; but for intermediate steps, c can be chosen as having any value, which complicates the problem of placing a limit on the critical load. Obviously there should be no distortion in the region outside of the central part of the bar, however, and therefore one can always make a fair estimate of the situation in this case.

In Fig. 20(*b*) an antisymmetrical deflection is assumed, and the corresponding critical load is calculated. Here again, the deflections outside of the region subject to compression do not enter into the finding of the critical load. It is of interest and importance that the critical load corresponding to the antisymmetrical deflection is lower than that corresponding to the symmetrical configuration for the arrangement of loads chosen. The bar would actually tend to buckle by more or less of a rotation of the central section. However, this would not have been discovered if only symmetrical deflection curves had been assumed.

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[18] "Über die Knickung eines Balkens durch Längskräfte," by O. Blumenthal, *Zeitschrift für angewandte Mathematik und Mechanik*, Vol. 17, 1937, pp. 232–244, especially pp. 234–239.

PART IV.—CONCLUDING REMARKS

Treatment of Large Deflections.—In all of the problems discussed herein the fundamental relation between deformation and moment has been implicitly assumed to be of the following type:

$$\frac{d^2y}{dx^2} = -\frac{M}{E\,I} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

in which y is the deflection, positive downward. Where deflections are large, Eq. 7 is only approximately correct. One should replace it by the relation:

$$\frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} = -\frac{M}{E\,I} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (8a)$$

or

$$\frac{d^2y}{dx^2} = -\frac{M}{E\,I}\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (8b)$$

The use of the exact relation offers no serious difficulties. In computing deflections from it one must assume the values of the deflections first, and determine the "angle changes" from modified values of $-\frac{M}{E\,I}$ (by multiplying by a function of the slopes at various points along the bar). One computes the deflections by a series of successive approximations, in which each step is similar to the various procedures outlined in the paper. However, it is not often necessary to consider such refinements.

Further Applications.—The procedure described herein is applicable to many other problems, since it permits a relatively simple and accurate numerical integration of a class of differential equations.

For example, the problem of a beam on elastic supports can be solved by first assuming a set of deflections, then determining the forces acting on the beam, with the consequent moments and angle changes. From the angle changes, the deflections can be computed. If these are the same as the assumed deflections, the problem is solved. If they are different, the process must be repeated.

The general procedure may also be modified so as to solve the problem of determining the natural period of vibration of a beam, or the critical speed of a shaft. Problems of this kind have been solved previously by similar procedures.[5,6] The use of the present modification is to produce a more accurate solution with generally less effort.

Conclusion.—The numerical procedure described herein permits a simple and rapid calculation of deflections of beams and columns and of critical buckling loads for columns with a high degree of accuracy. The method can be extended to other problems of the same mathematical nature.

ACKNOWLEDGMENT

The method of analysis described herein was developed as part of the writer's work in the Engineering Experiment Station of the University of

Illinois, Urbana, Ill. Particular acknowledgment is due Harold Crate, Research Graduate Assistant in Civil Engineering, for assistance in making and checking the calculations, and for extended studies of the procedure.

APPENDIX

DERIVATION OF FORMULAS FOR EQUIVALENT CONCENTRATIONS

In Fig. 5(*a*) the origin of coordinates is at a, with x positive to the right, and q represents the magnitude of the load at any point, positive upward. Let $z = \frac{x}{\lambda}$, in order to obtain a dimensionless coordinate, and consider the curve of loading with ordinates a, b, c at $z = 0, 1, 2$, respectively, to be a second-degree function of z or of x.

It can be readily verified that Eq. 9 represents a general second-degree function of z having the required values $q = a, b, c$ at $z = 0, 1, 2$, respectively:

$$q = \tfrac{1}{2}\, a\, (z - 1)\, (z - 2) - b\, z\, (z - 2) + \tfrac{1}{2}\, c\, z\, (z - 1) \quad \ldots\ldots\ldots (9)$$

Then, from statics the equivalent concentrated loads R_{ab} and R_{ba} are determined by the equations:

$$R_{ab} + R_{ba} = \lambda \int_0^1 q\, dz \quad \ldots\ldots\ldots (10a)$$

and

$$R_{ba} = \lambda \int_0^1 z\, q\, dz \quad \ldots\ldots\ldots (10b)$$

Evaluation of the integrals yields the results

$$R_{ba} = \frac{\lambda}{24}\, (3\, a + 10\, b - c) \quad \ldots\ldots\ldots (11a)$$

and

$$R_{ab} = \frac{\lambda}{24}\, (7\, a + 6\, b - c) \quad \ldots\ldots\ldots (11b)$$

By analogy, one finds the value of R_{bc}:

$$R_{bc} = \frac{\lambda}{24}\, (3\, c + 10\, b - a) \quad \ldots\ldots\ldots (12)$$

from which is readily obtained the value of R_b:

$$R_b = R_{ba} + R_{bc} = \frac{\lambda}{12}\, (a + 10\, b + c) \quad \ldots\ldots\ldots (13)$$

Similar formulas can be written for a loading curve of higher degree in x or z in terms of ordinates at more points. One may also derive expressions for R in terms of differences or of central differences by expressing q in such terms. Furthermore, one may develop corresponding equations when the segments into which the loading curve is divided are not of equal length. However, for practical purposes Eqs. 11 and 13 are all that are generally needed.

DISCUSSION

Bruce Johnston,[19] Assoc. M. Am. Soc. C. E.—The numerical procedure presented by Professor Newmark has advantages of simplicity, accuracy, and speed that make application to actual design work particularly effective. In an extension course given at Lehigh University, in Bethlehem, Pa., the writer has had the opportunity of presenting the method in detail to a number of engineers. Several of these engineers have found the procedure superior to other similar methods. The procedure was recently applied in connection with the analysis and design of several unusual mill building frames that are now (May, 1942) under construction.

As stated by the author (see "Synopsis"), "The essential features of the procedure are not new"—they are based on the well-known relations between the geometry of a bent beam and its moment-stiffness ratio. The importance of the procedure is not its newness, but its usability in actual design. It reduces the analysis of bending and buckling of struts to a systematic and accurate procedure of arithmetic, with a minimum chance of computational errors, and is exact enough for most applications. In actual structural members the moment of inertia frequently varies in a manner that makes actual integration of the fundamental differential equations exceedingly complex, if not impossible. Simple numerical procedures such as the author's deserve relatively more attention in structural engineering literature than they now have.

The practical usefulness of the procedure in continuous frame analysis will be increased if a summary is made of its relation to the slope-deflection and moment-distribution procedures for obtaining terminal moments of members in framed structures. In Fig. 21 is shown a rotation notation for the angle changes due to unit positive moments applied at either end of a simply supported member. Moments are assumed as positive when they apply a clockwise couple to the end of the beam. The angles of rotation of the end tangents of the beam axis are also considered positive when clockwise. The first subscript indicates the location of the angle change and the second subscript indicates the location of the applied unit moment—that is, ϕ_{AB} = angle change at A due to the unit moment at B.

By the law of reciprocal deflections, $\phi_{AB} = \phi_{BA}$. The three independent angle changes ϕ_{AA}, ϕ_{BB}, and ϕ_{AB} may be determined by model analysis or by two applications of the simple numerical procedure described by the author and illustrated in Figs. 10 and 11. In the symmetrical member $\phi_{AA} = \phi_{BB}$, and only one application would be necessary. The angle changes ϕ'_A and ϕ'_B, due to any applied load (also shown in Fig. 21), are determined easily by one additional application of the author's numerical procedure. Note the difference in the sign convention for the terminal moments M_A and M_B; but the sign of the angle changes will be the same as that of the end slopes in the author's paper. These five angle changes determine any or all of the coefficients needed in a generalized solution either by slope deflection or moment distribution. The effect of direct load upon the bending stiffness could be included, but is usually neglected in bridge and building frame analysis.

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[19] Senior Engr., Johns Hopkins Laboratory of Applied Physics, Silver Spring, Md.

The positive rotation notation for moments, shears, angle changes, and lateral translation of the ends of any member in a loaded frame is shown in Fig. 22. In the case of the member with uniform cross section, the "slope-deflection" equations are written:

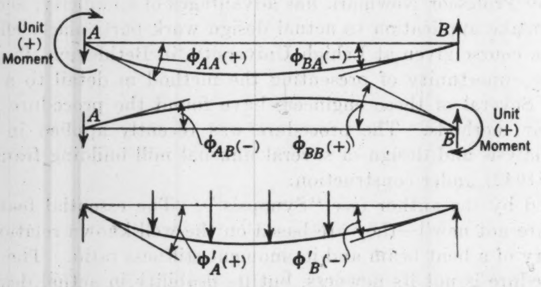


FIG. 21.—END-ANGLE CHANGES DETERMINED BY NUMERICAL PROCEDURE

$$M_{AB} = \frac{2\,E\,I}{l}\left(2\,\theta_A + \theta_B - \frac{3\,\Delta}{l}\right) \pm M_{FA} \qquad \text{(14a)}$$

and

$$M_{BA} = \frac{2\,E\,I}{l}\left(2\,\theta_B + \theta_A - \frac{3\,\Delta}{l}\right) \pm M_{FB} \qquad \text{(14b)}$$

in which M_{FA} and M_{FB} are "fixed-end" moments due to loads on the beam span. For downward loads on a horizontal member, M_{FA} is negative and M_{FB} is positive.

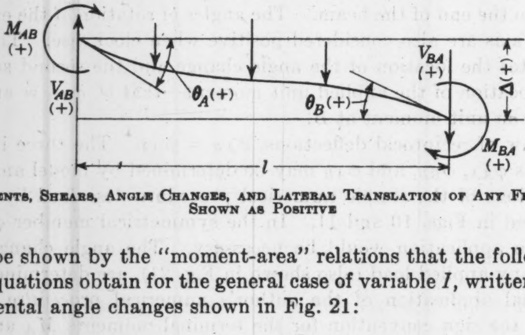
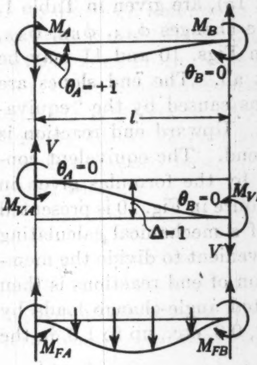
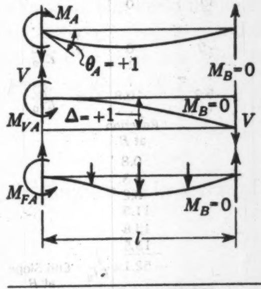
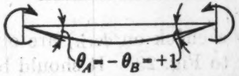
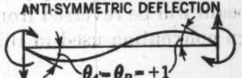


FIG. 22.—MOMENTS, SHEARS, ANGLE CHANGES, AND LATERAL TRANSLATION OF ANY FRAMED MEMBER, SHOWN AS POSITIVE

It may be shown by the "moment-area" relations that the following slope-deflection equations obtain for the general case of variable *I*, written in terms of the fundamental angle changes shown in Fig. 21:

$$M_{AB} = \frac{1}{\phi_{AA}\,\phi_{BB} - \phi^2_{AB}}\left[\phi_{BB}\,\theta_A - \phi_{AB}\,\theta_B + (\phi_{AB} - \phi_{BB})\,\frac{\Delta}{l} + \phi_{AB}\,\phi'_B - \phi_{BB}\,\phi'_A\right] \qquad \text{(15a)}$$

TABLE 1.—MOMENT DISTRIBUTION FACTORS FOR END *A* OF ANY MEMBER *AB*

			UNIFORM SECTION
STANDARD CASE-FAR END HELD FIXED  M_A M_B $\theta_B=0$ $\theta_A=+1$ l V $\theta_A=0$ $\theta_B=0$ M_{VB} M_{VA} $\Delta=1$ V M_{FA} M_{FB} Note - Factors at End *B* May Be Obtained by Interchanging Subscripts *A* and *B*	CARRY-OVER FACTOR $r_{AB}=\left\|\frac{M_B}{M_A}\right\| \begin{matrix}\theta_A=+1\\ \theta_B=0\\ \Delta=0\end{matrix}$	$r_{AB}=-\frac{\phi_{AB}}{\phi_{BB}}$	$\frac{1}{2}$
	MOMENT STIFFNESS $S_{MAB}=\left\|M_A\right\| \begin{matrix}\theta_A=+1\\ \theta_B=0\\ \Delta=0\end{matrix}$	$S_{MAB}=\frac{1}{\phi_{AA}+r_{AB}\phi_{AB}}$	$\frac{4EI}{l}$
	SHEAR STIFFNESS $S_{VAB}=\left\|V\right\| \begin{matrix}\theta_A=0\\ \theta_B=0\\ \Delta=+1\end{matrix}$	$S_{VAB}=\frac{1+2r_{AB}+\frac{r_{AB}}{r_{BA}}}{l^2(\phi_{AA}+r_{AB}\phi_{AB})}$	$\frac{12EI}{l^3}$
	MOMENT DUE TO UNIT SIDESWAY $M_{VA}=\left\|M_A\right\| \begin{matrix}\theta_A=0\\ \theta_B=0\\ \Delta=+1\end{matrix}$	$M_{VA}=\frac{-(1+r_{AB})}{l(\phi_{AA}+r_{AB}\phi_{AB})}$	$\frac{6EI}{l^2}$
	FIXED-END MOMENT $M_{FA}=\left\|M_A\right\| \begin{matrix}\theta_A=0\\ \theta_B=0\\ \Delta=0\end{matrix}$	$M_{FA}=\frac{-(r_{AB}\phi'_B+\phi'_A)}{\phi_{AA}+r_{AB}\phi_{AB}}$	$-\frac{4EI}{l}\times\left(\phi'_A+\frac{\phi'_B}{2}\right)$
SPECIAL CASE, FAR END SIMPLY SUPPORTED  M_A $\theta_A=+1$ $M_B=0$ V $\Delta=+1$ $M_B=0$ M_{VA} V M_{FA} $M_B=0$ l	MOMENT STIFFNESS $S_{MAB}=\left\|M_A\right\| \begin{matrix}\theta_A=+1\\ M_B=0\\ \Delta=0\end{matrix}$	$S_{MAB}=\frac{1}{\phi_{AA}}$	$\frac{3EI}{l}$
	SHEAR STIFFNESS $S_{VAB}=\left\|\frac{-M_A}{l}\right\| \begin{matrix}\theta_A=0\\ M_B=0\\ \Delta=+1\end{matrix}$	$S_{VAB}=\frac{1}{\phi_{AA}l^2}$	$\frac{3EI}{l^3}$
	FIXED-END MOMENT $M_{FA}=\left\|M_A\right\| \begin{matrix}\theta_A=0\\ M_B=0\\ \Delta=0\end{matrix}$	$M_{FA}=\frac{-\phi'_A}{\phi_{AA}}$	$-\frac{3EI\phi'}{l}$
SYMMETRICAL MEMBER SYMMETRIC DEFLECTION 	MOMENT STIFFNESS $S_{MAB}=\left\|M_A\right\| \begin{matrix}\theta_A=+1\\ \theta_B=-1\\ \Delta=0\end{matrix}$	$S_{MAB}=\frac{1-r_{AB}}{\phi_{AA}+r_{AB}\phi_{AB}}$ $S_{MAB}=S_{MBA}$ (Symmetrical Member)	$\frac{2EI}{l}$
SYMMETRICAL MEMBER ANTI-SYMMETRIC DEFLECTION 	MOMENT STIFFNESS $S_{MAB}=\left\|M_A\right\| \begin{matrix}\theta_A=+1\\ \theta_B=+1\\ \Delta=0\end{matrix}$	$S_{MAB}=\frac{1+r_{AB}}{\phi_{AA}+r_{AB}\phi_{AB}}$ $S_{MAB}=S_{MBA}$ (Symmetrical Member)	$\frac{6EI}{l}$

and

$$M_{BA} = \frac{1}{\phi_{AA}\,\phi_{BB} - \phi^2_{AB}} \left[\phi_{AA}\,\theta_B - \phi_{AB}\,\theta_A + (\phi_{AB} - \phi_{AA})\,\frac{\Delta}{l} + \phi_{AB}\,\phi'_A - \phi_{AA}\,\phi'_B \right] \qquad (15b)$$

The factors commonly used in the moment-distribution procedure, as defined by the general slope-deflection equations (Eq. 15), are given in Table 1.

In the actual calculation of the basic end-angle changes ϕ_{AA}, ϕ_{BB}, ϕ_{AB}, ϕ'_A, and ϕ'_B, the procedure used by the author in Figs. 10 and 11 may be simplified. Deflections need not be calculated at all. The end slopes are simply equal to the "conjugate" beam-end reactions caused by the "equivalent concentrated angle changes" treated as loads. Upward end reaction is positive on the left end and negative on the right end. The equivalent concentrated angle-change loads should be calculated by the formulas given in Figs. 3 and 5, and the alternate to the author's procedure in Fig. 10 is presented in Fig. 23. The results check those of Fig. 10. If a mechanical calculating machine is used in the computation, it would be convenient to divide the member into either five or ten segments. The calculation of end reactions is then obtained by multiplying the equivalent concentrated angle-change loads by successive decimal fractions, 0.2, 0.4, etc., or 0.1, 0.2, 0.3, etc., up to 1.0, in the case of five or ten segments, respectively.

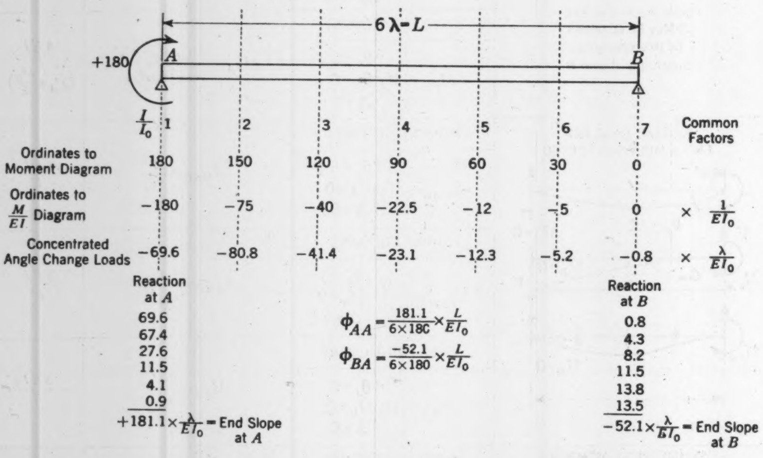

6λ=L
+180
A
B
I/I₀
1
2
3
4
5
6
7
Common Factors
Ordinates to Moment Diagram
180
150
120
90
60
30
0
Ordinates to M/EI Diagram
−180
−75
−40
−22.5
−12
−5
0
× 1/EI₀
Concentrated Angle Change Loads
−69.6
−80.8
−41.4
−23.1
−12.3
−5.2
−0.8
× λ/EI₀
Reaction at A
69.6
67.4
27.6
11.5
4.1
0.9
+181.1 × λ/EI₀ = End Slope at A
φ_AA = 181.1/(6×180) × L/EI₀
φ_BA = −52.1/(6×180) × L/EI₀
Reaction at B
0.8
4.3
8.2
11.5
13.8
13.5
−52.1 × λ/EI₀ = End Slope at B

FIG. 23.—ALTERNATE PROCEDURE TO FIG. 10 FOR CALCULATION OF END SLOPES ONLY

The other angle changes ϕ_{BB}, ϕ'_A, and ϕ'_B (also a check on ϕ_{AB}) are obtained by two additional sets of computation similar to Fig. 23. It should be noted that in computing ϕ_{BB} and ϕ_{AB} ($= \phi_{BA}$) the signs of the results and the sense of the applied moment at the right end of the beam will be reversed from that shown in Fig. 11 to conform to the rotation sign convention used in this discussion. After the basic angle changes are obtained, they may be substitued in Eq. 15 or in Table 1 to provide the necessary basis for analysis either by the slope-deflection or moment-distribution procedures, respectively. These details will be obvious to one already familiar with structural frame analysis.

The writer has discussed the application of the paper to structural frame analysis in cases where direct stress in a member may be neglected in so far as its effect on bending is concerned. The author's procedure is particularly adapted to the computation of critical buckling loads under direct stress for cases of nonuniform cross section. The method is clearly outlined by the author. The procedure furnishes the engineer with a simple method involving only the processes of arithmetic, and thereby bears a relation to the elastic stability theory of bars similar to that which the Hardy Cross moment-distribution method bears to older and more cumbersome methods of structural frame analysis.

M. S. KETCHUM, JR.,[20] ASSOC. M. AM. SOC. C. E.—The author's paper is another contribution to a distinctly American tradition of structural analysis founded on the work of Hardy Cross. This tradition is characterized by a nonmathematical solution of quite difficult cases in such a manner that the structural behavior is always evident in quantitative terms.

The writer would like to suggest a simple and convenient approximation that he has found to be useful in a large variety of buckling problems, including the lateral buckling of crane runway girders. The total deflection, w_0, in the strut analyzed in Fig. 17 may be estimated as soon as the deflection, w_i, due to the lateral load, is calculated and before the value of the critical load, P_{cr}, is determined. The assumption may be made that the shape of the bending-moment diagram for the moments, $P\,w_0$, due to the column load, is of the same shape as the bending moments due to lateral loads.

If M_L is represented as the moment due to the lateral load and w_i as the deflection due to this load, then the deflection, w_a, due to a moment, $P\,w_0 = P\,(w_i + w_a)$, is:

$$w_a = \frac{w_i\,P\,(w_i + w_a)}{M_L} \qquad \text{(16)}$$

Solving for w_0 in terms of w_i:

$$w_0 = \frac{w_i}{1 - \dfrac{w_i\,P}{M_L}} \qquad \text{(17)}$$

A substitution of the values of w_i, P, and M_L at the center of the strut gives:

$\frac{w_i\,P}{M_L} = \frac{1{,}042 \times 0.02}{100} = 0.2084$; and

$$w_0 = \frac{1{,}042\,q\,\lambda^4}{8\,EI}\left(\frac{1}{1 - 0.2084}\right) = \frac{1{,}320\,q\,\lambda^4}{8\,EI} \qquad \text{(18)}$$

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[20] Asst. Prof., Structural Eng., Case School of Applied Science, Cleveland, Ohio.

The corresponding value from Fig. 17 has a coefficient of 1,306 so that the approximation is satisfactory for most problems.

The critical buckling load for this strut determined in Fig. 12 also may be estimated by the condition that $w_i = w_0$, and

$$w_0 = \frac{w_i \, P \, w_0}{M_L} \qquad (19a)$$

and

$$P_{cr} = \frac{M_L}{w_i} \qquad (19b)$$

Substitution of values from Fig. 17 gives:

$$P_{cr} = \frac{100 \, E \, I}{1{,}042 \, \lambda^2} = \frac{9.60 \, E \, I}{L^2} \qquad (20)$$

The corresponding coefficient from Fig. 12 is 9.87 so that the approximation is in fair agreement.

The author is to be commended for his simple yet thorough and rigorous treatment of deflection and buckling. This paper should dispel much of the mystery that surrounds all but the most simple column problems.

John B. Wilbur,[21] Assoc. M. Am. Soc. C. E.—Professor Newmark's paper has been stimulating to the writer, particularly since it places emphasis on mathematically simple approaches to problems in elastic stability. Although Professor Newmark shows that simultaneous equations may be used to insure that the assumed deflection curve of a compression strut is essentially proportional to its shape as computed elastically on the basis of the assumed deflection curve, his detailed treatment is devoted to the substitution of a method of successive approximations for the solution of simultaneous equations.

The writer solved the problem illustrated by Fig. 13 on the basis of five simultaneous equations, and obtained substantially the same results. This solution was relatively straightforward, although the form of the simultaneous equations was such that it was convenient to adopt a method of successive approximations in their solution. Having successfully completed this solution, it was then decided to investigate the accuracy of simpler direct solutions, based on two simultaneous equations only. Symmetrical cases only were considered; otherwise three equations would have been necessary. The results of these relatively simple solutions were quite satisfactory, as is illustrated in the following discussion.

Consider first the case of Fig. 13. Assuming that the deflections of the strut at the center and quarter points are δ and $a \, \delta$, respectively, the equivalent concentrated $\frac{M}{E\,I}$-loads at the center and quarter points are computed from the relations for parabolic loading given in Fig. 5(*b*). These loads, together with the resultant net reaction, are shown in Fig. 24. By the moment area method:

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[21] Associate Prof., Structural Eng., Mass. Inst. Tech., Cambridge, Mass.

$$a\,\delta = \frac{P\,L\,\delta}{48\,E\,I}\,(11\,a + 6)\,\frac{L}{4} = \frac{P\,L^2\,\delta}{192\,E\,I}\,(11\,a + 6) \ldots\ldots\ldots (21a)$$

and

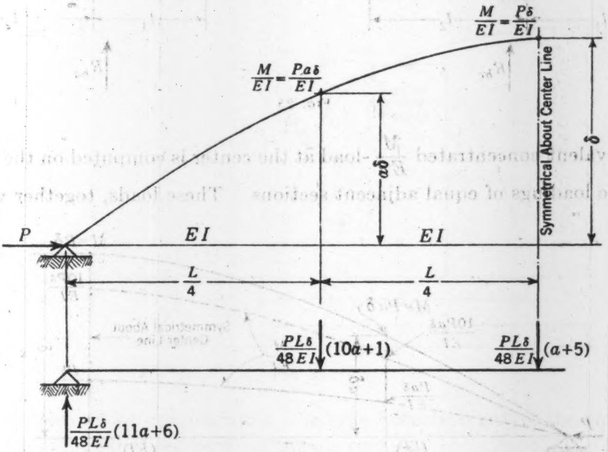
$$\delta = \frac{P\,L\,\delta}{48\,E\,I}\,(11\,a + 6)\,\frac{2\,L}{4} - \frac{P\,L\,\delta}{48\,E\,I}\,(10\,a + 1)\,\frac{L}{4} = \frac{P\,L^2\,\delta}{192\,E\,I}\,(12\,a + 11)\ .(21b)$$

Let $A = \frac{P\,L^2}{192\,E\,I}$ and Eqs. 21 become

$$a = A\,(11\,a + 6) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22a)$$

and

$$1 = A\,(12\,a + 11) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (22b)$$



$\frac{M}{EI} = \frac{P\delta}{EI}$
$\frac{M}{EI} = \frac{Pa\delta}{EI}$
Symmetrical About Center Line
$a\delta$
δ
P
EI
EI
$\frac{L}{4}$
$\frac{L}{4}$
$\frac{PL\delta}{48EI}(10a+1)$
$\frac{PL\delta}{48EI}(a+5)$
$\frac{PL\delta}{48EI}(11a+6)$

FIG. 24

Solving simultaneously: $a = 0.707$; $A = \frac{1}{19.48} = \frac{P\,L^2}{192\,E\,I}$. Hence, $P_{cr} = \frac{192}{19.48}\,\frac{E\,I}{L^2} = \frac{9.86\,E\,I}{L^2}$ vs. $\frac{\pi^2\,E\,I}{L^2}$ (exactly).

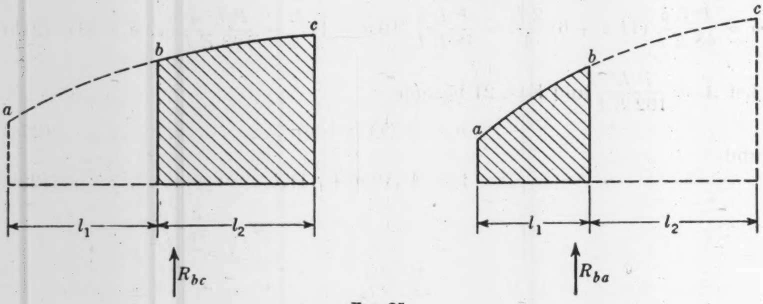
Before applying this procedure to the case illustrated by Professor Newmark in Fig. 15, where I is not constant throughout the length of the strut, it was convenient to develop equations for equivalent concentrated loads for parabolic loading curves for the case in which the spans of the two adjacent segments are not equal. Referring to Fig. 25, these equations are as follows:

$$R_{bc} = \frac{l_2}{12\,l_1\,(l_1 + l_2)}\,[a\,(-\,l^2_2) + b\,(4\,l_1 + l_2)(l_1 + l_2) + c\,l_1\,(2\,l_1 + l_2)] \ldots (23a)$$

and

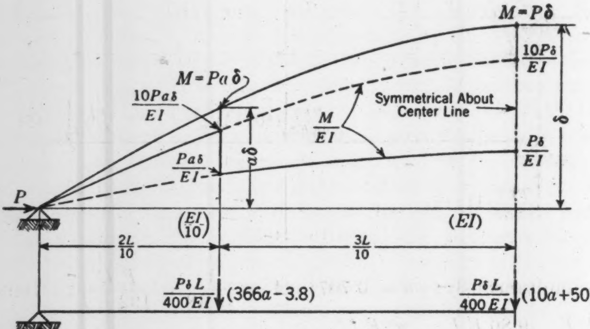
$$R_{ba} = \frac{l_1}{12\,l_2\,(l_1 + l_2)}\,[a\,l_2\,(l_1 + 2\,l_2) + b\,(l_1 + 4\,l_2)(l_1 + l_2) + c\,(-\,l^2_1)] \ldots (23b)$$

Assuming that the deflections of the strut at the center and the 0.2-points are δ and $a\,\delta$, respectively, the equivalent concentrated $\frac{M}{E\,I}$-load at the 0.2-point is computed from the foregoing relations for parabolic loadings, whereas the equivalent concentrated $\frac{M}{E\,I}$-load at the center is computed on the basis of parabolic loadings of equal adjacent sections. These loads, together with the resultant net reaction, are shown in Fig. 26. By the moment-area method,



a
b
c
l_1
l_2
R_{bc}
a
b
c
l_1
l_2
R_{ba}

FIG. 25



$M = P\delta$
$\frac{10P\delta}{EI}$
$M = Pa\delta$
$\frac{10Pa\delta}{EI}$
Symmetrical About Center Line
$\frac{M}{EI}$
δ
$a\delta$
$\frac{Pa\delta}{EI}$
$\frac{P\delta}{EI}$
P
$\left(\frac{EI}{10}\right)$
(EI)
$\frac{2L}{10}$
$\frac{3L}{10}$
$\frac{P\delta L}{400EI}(366a-3.8)$
$\frac{P\delta L}{400EI}(10a+50)$
$\frac{P\delta L}{400EI}(376a+46.2)$

FIG. 26

$$a\,\delta = \frac{P\,\delta\,L}{400\,E\,I}\,(376\,a + 46.2)\,\frac{2\,L}{10} = \frac{P\,L^2\,\delta}{4{,}000\,E\,I}\,(752\,a + 92.4)\ldots(24a)$$

and

$$\delta = \frac{P\,\delta\,L}{400\,E\,I}\,(376\,a + 46.2)\,\frac{5\,L}{10} - \frac{P\,\delta\,L}{400\,E\,I}\,(366\,a - 3.8)\,\frac{3\,L}{10}$$

$$= \frac{P\,L^2\,\delta}{4{,}000\,E\,I}\,(782\,a + 219.6)\ldots\ldots\ldots\ldots\ldots\ldots(24b)$$

Let $A = \dfrac{P\,L^2}{4{,}000\,E\,I}$, and Eqs. 24 become

$$a = A\,(752\,a + 92.4)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25a)$$

and

$$1 = A\,(782\,a + 219.6)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25b)$$

Solving simultaneously: $a = 0.823$; $A = \dfrac{1}{863} = \dfrac{P\,L^2}{4{,}000\,E\,I}$. Hence, $P_{cr} = \dfrac{4{,}000}{863}\,\dfrac{L^2}{E\,I} = 4.64\,\dfrac{L^2}{E\,I}$ vs. $\dfrac{4.5\,L^2}{E\,I}$ (exactly).

Since, for a strut composed of sections of constant $E\,I$, the deflection curve must be compounded of sine curves, an interesting exact solution to the foregoing problem may be made as follows, with reference to Fig. 27.

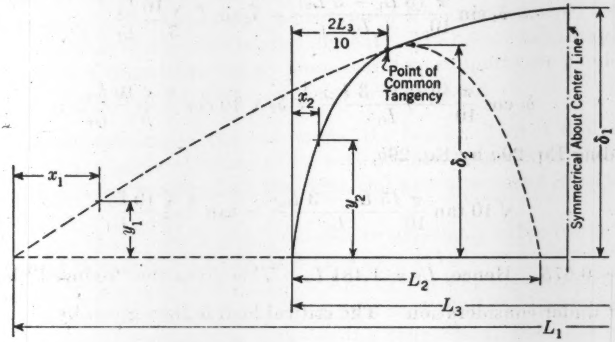


FIG. 27

For the case under consideration, the true deflection curve, shown by the full line, is composed of segments of the curves

$$y_1 = \delta_1 \sin\frac{\pi\,x_1}{L_1}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26a)$$

and

$$y_2 = \delta_2 \sin\frac{\pi\,x_2}{L_2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26b)$$

At the point of common tangency, where

$$x_1 = \frac{L_1 - L_3}{2} + \frac{2\,L_3}{10} = \frac{1}{10}\,(5\,L_1 - 3\,L_3)\ldots\ldots\ldots\ldots(27a)$$

and

$$x_2 = \frac{L_3}{5}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27b)$$

the following relations hold:

$$y_1 = y_2;\ \frac{dy_1}{dx_1} = \frac{dy_2}{dx_2};\ \text{and}\ \frac{d^2y_1}{dx^2_1} = \frac{1}{10}\,\frac{d^2y_2}{dx^2_2},\ \text{since}\ I_1 = 10\,I_2.$$

These relations reduce to

$$\delta_1 \sin \frac{\pi}{10} \frac{(5\, L_1 - 3\, L_3)}{L_1} = \delta_2 \sin \frac{\pi}{5} \frac{L_3}{L_2} \ldots\ldots\ldots\ldots (28a)$$

$$\frac{\delta_1}{L_1} \cos \frac{\pi}{10} \frac{(5\, L_1 - 3\, L_3)}{L_1} = \frac{\delta_2}{L_2} \cos \frac{\pi}{5} \frac{L_3}{L_2} \ldots\ldots\ldots\ldots (28b)$$

and

$$\frac{\delta_1}{L^2_1} \sin \frac{\pi}{10} \frac{(5\, L_1 - 3\, L_3)}{L_1} = \frac{\delta_2}{10\, L^2_2} \sin \frac{\pi}{5} \frac{L_3}{L_2} \ldots\ldots\ldots\ldots (28c)$$

Dividing Eq. 28c by Eq. 28a, $\frac{1}{L^2_1} = \frac{1}{10\, L^2_2}$; whence $L_2 = \frac{L_1}{\sqrt{10}}$. Eqs. 28a and 28b become

$$\delta_1 \sin \frac{\pi}{10} \frac{(5\, L_1 - 3\, L_3)}{L_1} = \delta_2 \sin \frac{\pi \sqrt{10}}{5} \frac{L_3}{L_1} \ldots\ldots\ldots\ldots (29a)$$

and

$$\delta_1 \cos \frac{\pi}{10} \frac{(5\, L_1 - 3\, L_3)}{L_1} = \delta_2 \sqrt{10} \cos \frac{\pi \sqrt{10}}{5} \frac{L_3}{L_1} \ldots\ldots\ldots\ldots (29b)$$

Dividing Eq. 29a by Eq. 29b,

$$\sqrt{10} \tan \frac{\pi}{10} \frac{(5\, L_1 - 3\, L_3)}{L_1} = \tan \frac{\pi \sqrt{10}}{5} \frac{L_3}{L_1} \ldots\ldots\ldots\ldots (30)$$

and $\frac{L_3}{L_1} = 0.675$. Hence, $L_1 = 1.481\, L_3$. This gives the "reduced" length of the strut under consideration. The critical load is then given by

$$P_{cr} = \frac{\pi^2\, E\, I}{(1.481\, L_3)^2} = \frac{4.50\, E\, I}{L^2_3} \ldots\ldots\ldots\ldots (31)$$

In the opinion of the writer, Professor Newmark has performed a definite service since he has dissociated one type of problem in elastic stability from the shroud of the formal solution of differential equations. It is to be hoped that relatively simple methods of solution will eventually be developed for other types of problems in elastic stability.

Ralph W. Stewart,[22] M. Am. Soc. C. E.—Effective procedures for determining the elastic curves of beams are presented in compact form by the author. The paper has value in an engineer's reference library since the analyses demonstrated have heretofore been scattered in different treatises. The use of progressive load, shear, and moment increments to establish deflections and the alinement of elastic curves is familiar in the analysis of arches. An excellent demonstration of the use of the device termed "linear correction to moments" (which, by analogy, is the same as "linear correction to deflections") has been presented by A. W. Buel, M. Am. Soc. C. E., and C. S. Hill.[23]

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[22] Engr. of Bridge and Structural Design, City of Los Angeles, Los Angeles, Calif.

[23] "Reinforced Concrete," by A. W. Buel and C. S. Hill, The Engineering News Publishing Co., New York, N. Y., 1906, Fig. 43, p. 140.

It appears to the writer that a part of the paper dealing with beams of variable section can be improved in analytical procedure and also that a rather important deficiency in the author's illustration of the use of his computations justifies a rewriting of this part of the paper.

To clarify and justify this opinion, a statement regarding the basic constants of beam flexure is necessary. A basic constant is defined as being either a simple constant quantity or a constant ratio between two variable quantities. A constant that is more complex than a basic constant will be referred to as a derived constant because it is derived by the use of two or more basic constants.

The basic constants of beam flexure are well illustrated by the dimensions governing railway curves. A circular railway curve is completely determined by the length of a tangent and the angle of intersection of the tangents. The transition spiral often used at the end of a circular curve is fully determined by the length of its tangents and their angle of intersection. If the over-all length of the spiral curve is known (as the span of a beam is known), then one tangent and the angle of intersection are sufficient. The basic constants governing the flexure of a beam that has no more than two supports are similar and are as follows, the distance between supports being known:

(1) The basic stiffness of the left end of the beam. This is the ratio of the moment to the angle of intersection between the tangents to the elastic curve when the beam is hinged at the right end and a moment is applied at the left end.

(2) The ratio of the length of one tangent to the elastic curve to the length of the beam for the same condition of flexure as in (1).

(3) and (4) These are the same as (1) and (2), except that the left end of the beam is taken as hinged and the moment is applied at the right end.

(5) The angle of intersection between the tangents of the elastic curve when the beam acts as a simply supported beam subjected to its loads.

(6) The ratio of the length of one tangent of the elastic curve to the length of the beam for the simply supported condition.

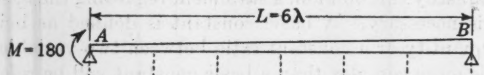
The first four constants are "beam constants," which are independent of the loading. The last two are "load constants," which depend on the loads.

Of the six basic constants of flexure, only five are independent, since the principle of Maxwell's theorem of reciprocal deflections (angular) will enable any missing beam constant to be computed from the others. It is understood in beam flexure that the tangents may be taken as equal to their projected length in the unsprung beam.

An end slope is not a basic flexure constant. This can be seen from the fact that in a cantilever beam the slope at the fixed end is known to be zero; but this knowledge, combined with the length of a tangent to the elastic curve, is insufficient to determine the curve. If, however, the length of a tangent and the intersection angle between tangents are known, the curve is determined.

The Hardy Cross stiffness factor is not a basic constant because it is a function of an end slope and also two basic stiffnesses, one at each end of the beam.

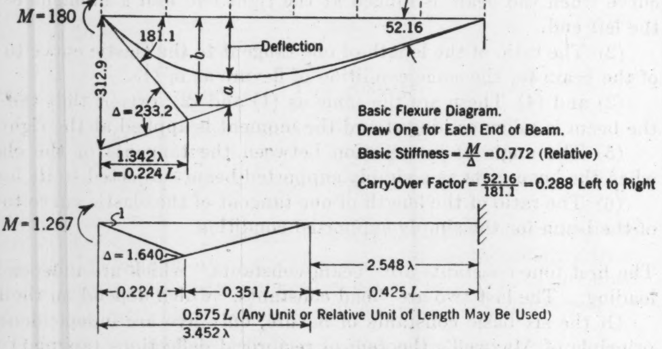
Much valuable time has been lost and is still being lost by engineering designers through injudicious selection of flexure constants in analyzing the various types of structures. The author's treatment of the beam in Figs. 10 and 11 would tend to encourage rather than correct this tendency. A revised computation of the properties of this beam is therefore offered in Fig. 28, and the justification for this revised computation is illustrated by Fig. 29. The columns in Fig. 29 are the same members whose properties are determined in Fig. 28. The deck is of constant section with relative stiffness as shown. The sideway moments and the appurtenant lateral force as shown in Fig. 29 are computed by the following consecutive steps:



Moments	180	150	120	90	60	30	0	
$\frac{I}{I_0}$	1	2	3	4	5	6	7	
Ordinates to $\frac{M}{EI}$ Diagram	180	75	40	22.5	12	5	0	$\times \frac{1}{EI_0}$
Concentrated Angle Changes	69.6	80.8	41.4	23.1	12.3	5.2	0.8	$\times \frac{\lambda}{EI_0}$ $\Sigma = 233.2 = \Delta$
Moments of Angle Change Loads About Left End	0	80.8	82.8	69.3	49.2	26	4.8	$\Sigma = 312.9$

Distance of Δ from Left End $= \frac{312.9}{233.2} = 1.342\lambda$ See Diagram Below.

Tangent Offset $= a =$	312.9	149.3	66.5	25.1	6.8	0.8	0	
Tangent Spread $= b =$	312.9	260.8	208.6	156.5	104.3	52.16	0	
Deflection $= b - a =$	0	111.5	141.1	131.4	97.5	51.3	0	$\times \frac{\lambda^2}{EI_0}$



Basic Flexure Diagram.
Draw One for Each End of Beam.
Basic Stiffness $= \frac{M}{\Delta} = 0.772$ (Relative)
Carry-Over Factor $= \frac{52.16}{181.1} = 0.288$ Left to Right

$0.575L$ (Any Unit or Relative Unit of Length May Be Used)

Cross Stiffness Factor $= 1.640 \times$ Basic Stiffness $= 1.267$
(Carry-Over Factor Right to Left $= 2.548 \div 3.452 = 0.738$)

FIG. 28

Write in deck traverse angles 1, 2, and 3. Multiply stiffness, 2.97, by angle value, 3, to get moment 8.91; divide moment 8.91 by stiffness 1.485 to

get column traverse angle 6. Add angles 1 and 6 to get bottom column traverse angle 7; multiply angle 7 by stiffness 0.772 to get moment 5.404. Add all column moments and divide by frame height to get lateral force. The effect of any other lateral force will be in direct proportion. The lateral deflection of the structure follows as a direct by-product.

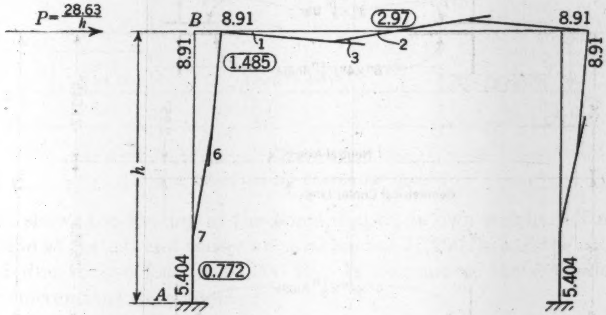


FIG. 29

It can be seen that the foregoing computation of sidesway moments, using basic constants, can be done in less than one minute, whereas computing these moments, with the derived constants, which are the only ones mentioned by the author, would take much longer, would give a less accurate result, and would fail to give the deflection of the structure as a direct by-product. The same situation would prevail with vertical loads.

The usefulness of the computation of deflections and elastic properties of an individual beam is obviously increased by bringing the results into a form that can be used for the quick computation of the deflections and moments of a structure in which the beam becomes a member.

STEFAN J. FRAENKEL,[24] **JUN. AM. SOC. C. E.**—Recently the writer was called upon to investigate the stresses in a 125-ft derrick boom. Since such long and slender members are subjected to considerable deformation which, in turn, influences the stresses, it was necessary to determine the deflection of this boom. The methods outlined in Professor Newmark's able paper lent themselves well to this investigation.

The boom (see Fig. 30) consisted of four angles—two 6-in. by 4-in. by ½-in. angles, which formed the top chord, and two 4-in. by 4-in. by ½-in. angles, which constituted the bottom chord. A preliminary investigation indicated that the strengthening of the middle section (which was 59 ft long) with 3-in. by ½-in. bars at the top and 3-in. by ⅜-in. bars at the bottom was advisable. That is the section shown in Fig. 30, which was also used in the investigation related herein. The section of the boom varied considerably outside the middle part, as is shown by the variations in the values of I and y.

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[24] With Eng. Dept., Pittsburgh-Des Moines Steel Co., Pittsburgh, Pa.

A general view of the boom is shown in Fig. 31. The boom will be analyzed in a horizontal position. Because of the fact that its cross section is unsymmetrical, the neutral axis does not coincide with the geometrical center line, the distance between them at any point being denoted by y. The line of action of the axial loads is determined at the right end of the boom by the location of the pin, and at the left end by the intersection of the boom line and the vertical line of action of the load P. (It should be understood that P includes both live load and the left reaction of the boom due to its own weight.) This intersection (which, of course, is imaginary) occurs for the case of the

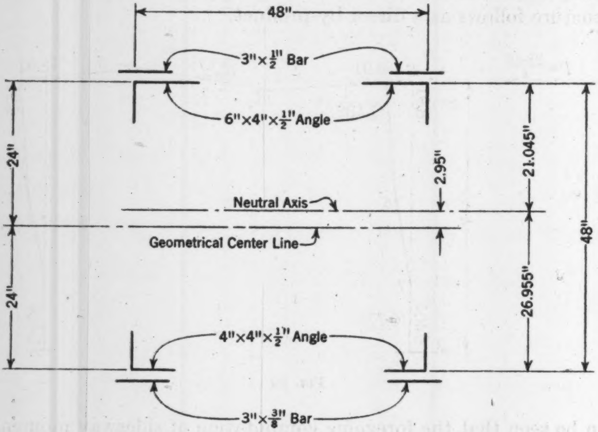

48"
3"×1/2" Bar
6"×4"×1/2" Angle
24"
2.95"
21.045"
Neutral Axis
Geometrical Center Line
48"
24"
26.955"
4"×4"×1/2" Angle
3"×3/8" Bar

FIG. 30.—SECTION IN CENTER PART OF BOOM

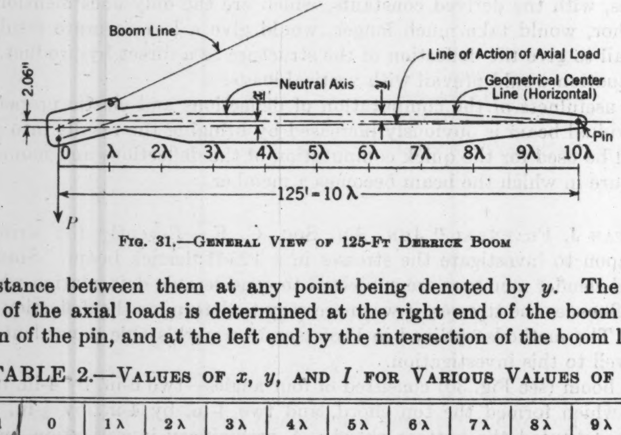

Boom Line
Line of Action of Axial Load
2.06"
Neutral Axis
Geometrical Center Line (Horizontal)
Pin
0
λ
2λ
3λ
4λ
5λ
6λ
7λ
8λ
9λ
10λ
125' = 10λ
P

FIG. 31.—GENERAL VIEW OF 125-FT DERRICK BOOM

TABLE 2.—VALUES OF x, y, AND I FOR VARIOUS VALUES OF λ

Symbol	0	1 λ	2 λ	3 λ	4 λ	5 λ	6 λ	7 λ	8 λ	9 λ	10 λ
x	2.06	1.86	1.65	1.45	1.24	1.03	0.83	0.62	0.41	0.21	0.00
y	−8.39	1.13	2.27	2.95	2.95	2.95	2.95	2.95	2.27	1.76	0.00
I	42,057	5,363	7,098	11,843	11,843	11,843	11,843	11,843	7,098	3,387	5,071
I	12.4 I_o	1.59 I_o	2.09 I_o	3.49 I_o	3.49 I_o	3.49 I_o	3.49 I_o	3.49 I_o	2.09 I_o	1.00 I_o	1.50 I_o

flat boom 2.06 in. below the geometrical center line. The distance at any point between the geometrical center line and the line of action of the axial load will be denoted by x. A summary of the values of x, y, and I for various values of λ is given in Table 2.

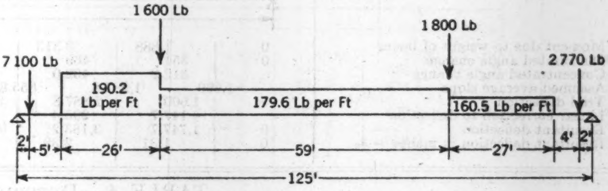


Fig. 32.—Loading Sketch of Boom

Fig. 32 shows the loading of the boom due to its own weight. The dead-load reaction at the left end causes an axial load of 51,250 lb, and the maximum axial load due to live load is 64,950 lb. In computing the deflections the following conventions were adopted:

(1) Downward deflection is positive;

(2) Inch and pound units are employed in the computations;

(3) The modulus of elasticity, E, is taken as 30 by 10^6 lb per sq in.; and

(4) The boom is divided into ten parts, so that $10\ \lambda = 125$ ft.

Three deflections are:

(a) δ_w = deflection caused by bending moment due to the weight of the boom;

(b) δ_a = deflection due to axial dead load; and

(c) δ = deflection due to axial live load.

The following relations hold:

$$\delta_w + \delta_a = \Delta_D \qquad (32a)$$

and

$$\delta_w + \delta = \Delta \qquad (32b)$$

in which, Δ_D is the total deflection under the dead load, and Δ is the total deflection under the dead load plus live load. The difference between Δ_D and Δ represents the effect which the live load has on the deflection.

Determination of δ_w.—The necessary computations are recorded in Table 3, and no further explanations are required. Of course, it would not have been necessary to compute the equivalent concentrated angle changes, since there were no discontinuities in the $\frac{M}{E\,I}$-diagram, and the simplified method mentioned in Part II of the paper could have been used.

Determination of δ_a.—This component of the deflection is the upward movement caused by the moment that the axial dead load has about the neutral

TABLE 3.—Determination of Downward Deflection

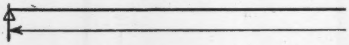
Line	Description	Tenth				
		0	1	2	3	
	Moment of inertia I (in.^{4})	42,057	5,363	7,098	11,834	
						
1	Moment due to weight of boom	0	1.888	3.313	4.313	
2	Distributed angle change	0	351	469	365	
3	Concentrated angle change		312.2	432.0	388.8	
4	Assumed average slope		1,600	1,287.8	855.8	467.0
5	Trial deflection		1,600	2,887.8	3,743.6	
6	Linear correction to deflection		+147.7	295.4	443.1	
7	Resultant deflection	0	1,747.7	3,183.2	4,186.7	
8	Resultant deflection in inches = δ_w	0	1.31	2.39	3.14	

TABLE 4.—Determination

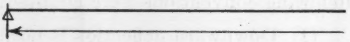
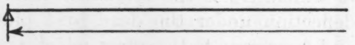
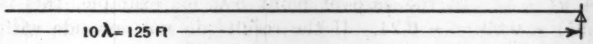
Line	Description	Tenth				
		0	1	2	3	
						
	Moment of inertia I (in.^{4})	42,057	5,363	7,098	11,834	
1	Assumed deflection due to axial dead load, δ_a	0	−0.02	−0.03	−0.04	
2	Moment arm $[\delta_w - (x + y) - \delta_a]$	+6.33	−1.70	−1.56	−1.27	
3	Moment (51,250 times moment arm)	+325	−87.2	−80.0	−65.2	
4	Distributed angle change	−7.71	+16.25	+11.26	+5.50	
5	Assumed average slope		−40.00	−23.75	−12.49	−6.99
6	Trial deflection (1)	+82.90	+42.90	+19.15	+6.66	
7	Trial deflection (2)	−0.64	+1.35	+0.94	+0.46	
8	Linear correction to deflection	−82.26	−74.03	−65.80	−57.58	
9	Resultant deflection, δ_a	0	−29.78	−45.71	−50.46	
10	Resultant deflection, δ_a	0	−0.0223	−0.0342	−0.0378	
		Total Dead Load Deflection $\Delta_D = \delta_w + \delta_a$				
11	....	0	1.29	2.36	3.10	

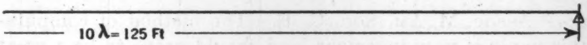
TABLE 5.—Determination

Line	Description	Tenth				
		0	1	2	3	
	Moment of inertia I (in.^{4})	42,057	5,363	7,098	11,834	
						
1	Assumed deflection due to axial D.L. & L.L., δ	0	−0.05	−0.08	−0.09	
2	Moment arm $[\delta_w - (x + y) - \delta]$	+6.33	−1.73	−1.61	−1.32	
3	Moment [(51,250 + 64,950) × moment arm]	+735	−201	−187	−154	
4	Distributed angle change	−17.5	+37.5	+26.4	+13.0	
5	Assumed average slope		−50.00	−12.5	+13.9	+26.9
6	Trial deflection (1)	−247.85	−297.85	−310.35	−296.45	
7	Trial deflection (2)	−1.46	+3.13	+2.20	+1.08	
8	Linear correction to deflection	+249.31	+224.37	+199.44	+174.51	
9	Resultant deflection, δ	0	−70.35	−108.71	−120.86	
10	Resultant deflection, δ	0	−0.0528	−0.0814	−0.0906	
		Total Dead and Live Load Deflection $\Delta = \delta_w + \delta$				
11	....	0	1.26	2.31	3.05	

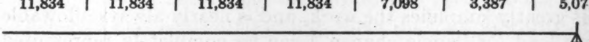
CAUSED BY BENDING MOMENT DUE TO WEIGHT OF BOOM (δ_w)

POINTS: 4	5	6	7	8	9	10	Common factor
11,834	11,834	11,834	11,834	7,098	3,387	5,071	....
 10 λ = 125 Ft							....
4.777	4.925	4.741	4.237	3.188	1.760	0	10^3
404	416	400	358	449	520	0	$1/E$
399.5	411.3	395.7	380.2	445.7	421.5		λ/E
67.5	−343.8	−739.5	−1,119.7	−1,565.4	−1,986.9		λ/E
4,210.6	4,278.1	3,934.3	3,194.8	2,075.1	509.7	−1,477.2	λ^3/E
590.8	738.5	886.2	1,033.9	1,181.6	1,329.3	+1,477.2	λ^3/E
4,801.4	5,016.6	4,820.5	4,228.7	3,256.7	1,839.0	0	λ^3/E
3.60	3.77	3.62	3.17	2.44	1.38	0	

OF δ_a AND Δ_D

POINTS: 4	5	6	7	8	9	10	Common factor
 10 λ = 125 Ft							
11,834	11,834	11,834	11,834	7,098	3,387	5,071	
−0.04	−0.03	−0.03	−0.03	−0.03	−0.02	0	
−0.63	−0.24	−0.19	−0.43	−0.26	−0.60	0	
−32.3	−12.3	−9.75	−22.1	−13.3	−30.7	0	10^3
+2.72	+1.04	+0.83	+1.87	+1.87	+9.08	0	$1/E$
−4.27	−3.23	−2.40	−0.53	+1.34	+10.42		λ/E
−0.33	−4.60	−7.83	−10.23	−11.76	−10.42	0	λ^3/E
+0.22	+0.09	+0.07	+1.56	+1.56	+0.76	0	λ^3/E
−49.35	−41.13	−32.90	−24.68	−16.45	−8.22	0	λ^3/E
−49.46	−45.64	−40.66	−33.35	−26.65	−17.90	0	λ^3/E
−0.0371	−0.0342	−0.0305	−0.0250	−0.0200	−0.0134	0	
(LINE 8, TABLE 3 + LINE 10, TABLE 4)							
3.56	3.74	3.59	3.14	2.44	1.37	0	

OF δ AND Δ.

POINTS: 4	5	6	7	8	9	10	Common factor
11,834	11,834	11,834	11,834	7,098	3,387	5,071	
 10 λ = 125 Ft							
−0.09	−0.08	−0.08	−0.06	−0.05	−0.03	0	
−0.68	−0.29	−0.24	−0.46	−0.29	−0.62	0	
−79.0	−33.5	−28.0	−53.5	−34.0	−72.1	0	10^3
+6.68	+2.83	+2.37	+4.52	+4.79	+21.3	0	$1/E$
+33.58	+36.41	+38.78	+43.30	+48.09	+69.39		λ/E
−269.55	−235.97	−199.56	−160.78	−117.48	−69.39	0	λ^3/E
+0.56	+0.24	+0.20	+0.38	+0.40	+1.77	0	λ^3/E
+149.58	+124.65	+99.72	+74.79	+49.86	+24.93	0	λ^3/E
−119.41	−111.08	−99.64	−85.61	−67.22	−42.69	0	λ^3/E
−0.0895	−0.0834	−0.0746	−0.0642	−0.0505	−0.0321	0	
(LINE 8, TABLE 3 + LINE 10, TABLE 5)							
3.51	3.69	3.54	3.11	2.39	1.35	0	

axis. That this movement is upward can be ascertained from the fact that, except at the left end, the line of action of the axial load is below the neutral axis. The moment arm is $\delta_w - (x + y)$ prior to the upward deflection δ_a, for which assumed values are given in line 1 of Table 4. The final moment arm is then $\delta_w - (x + y) - \delta_a$. In the case of point 5λ, for example, this is $3.77 - (1.03 + 2.95) - 0.03 = -0.24$. If the results do not coincide with the assumed values of δ_a, the procedure is repeated with corrected values. At the bottom of Table 4, in line 11, the deflection $\Delta_D = \delta_w + \delta_a$ of the boom under dead load is given.

Determination of δ.—This is the deflection due to both dead and live axial loads. Its computation follows the same pattern as that of δ_a. The force acting is 51,250 + 64,950 = 116,200 lb, and a set of values of δ is assumed and given in line 1 of Table 5. Again, if the results do not coincide with the assumed values of δ, corrected values are substituted. The values of Δ (which is the deflection of the boom due to live and dead load) are given in line 11, Table 5. Comparison with corresponding values of Δ_D shows that the live load reduces the deflection by a small amount, namely $\Delta_D - \Delta$.

ALFRED S. NILES,[25] ASSOC. M. AM. SOC. C. E.—The method of computation described in this paper is very ingenious, and should prove to be a great timesaver in the solution of many types of problems. Although the author has shown applications to both beams and columns of single span, he has failed to warn the reader that his method is not directly applicable to continuous members, for which the bending moments over the supports must be obtained by the use of the three-moment equation, the method of moment distribution, or an equivalent method. Perhaps the most awkward member of this type, from the point of view of the stress analyst, is a continuous beam of nonuniform section that is subjected to combined bending and compression. This is handled most readily by the method of moment distribution, proper allowance being made for the effect of the axial load when computing fixed-end moments and the carry-over and stiffness factors. The computation of these quantities for members of nonuniform section by previously published methods is a slow and tedious procedure. The methods proposed by the author appear more convenient for this work than any other that has yet been suggested, including that of the writer and J. S. Newell.[7]

In his numerical examples the author divides the beam into segments of equal length. This greatly simplifies the work, and is nearly always allowable when the transverse load (or "angle change") can be completely represented by a smooth curve. If this is not allowable, however, as when unequally spaced concentrated loads are present, much of the advantage is lost. In fact, the author's method becomes practically the same as that described,[26] the only differences being in the method of recording the computations, and in the more accurate, although more time-consuming, method of allowing for the curvature of the loading (or "angle change") diagram, which the writer treats as composed

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[25] Prof., Aeronautic Eng., Leland Stanford Junior Univ., Aero. Laboratory, Stanford Univ., Stanford University, Calif.

[26] "Airplane Structures," by A. S. Niles and J. S. Newell, 2d Ed., New York, N. Y., 1938, Vol. I, pp. 58–60.

of straight segments. In most practical problems, however, the value of the greater accuracy of the author's method of allowing for this factor is questionable.

In computing "equivalent concentrated angle changes," the author uses formulas from Figs. 3 and 5 that require division by 6, 12, or 24, depending on the shape of the angle-change curve. It would seem simpler to include the factor 6, 12, or 24 in the "common factor" by which the moments or deflections are to be multiplied to get the final results. This would result in most of the values in the tabulated computations being 6, 12, or 24 times as large as those obtained in the method as described, but would not affect the final results. Care would have to be taken, however, to use the same factor throughout the span when part of the loading curve was straight and part curved, but in any given problem it would be easy to multiply the formulas of Figs. 3 and 5 by $\frac{2}{2}$ or $\frac{4}{4}$ in order to obtain a common denominator for all the formulas used.

In some of his examples (as in Fig. 1(*d*)) the author computes the actual value of the shear at the left end of the beam before computing the shears at other points, whereas in others (as in Fig. 1(*e*)) he starts the shear computations from an arbitrarily assumed figure and makes a final correction to the bending moments, if necessary. The writer has two objections to the latter practice, although it seems to be preferred by the author. The first is that it is often necessary to know the shears at various points along the span, and, in the latter practice, it would be necessary to remember to correct the values originally found to obtain the true ones. This could be done easily, and the objection would be unimportant if it stood alone. The more serious objection is that the practice eliminates a valuable internal check on the computations. If the actual shear at the left end is first computed, then the moment at the right end, computed by summation of the shears along the span, should be the same as that stated in the formulation of the problem. If the two values are not in substantial agreement, an error has been made. In the method of Fig. 1(*e*), one does not know whether the necessary moment correction is due solely to the difference between the assumed and actual shears at the left end, or whether it is partly due to a numerical error of computation. Since the actual shear at the left end can be computed quite easily, the check obtained justifies the little additional work involved in using it.

The writer notes that the author has reversed the usual convention and considers that loads are positive when they act upward. He heartily indorses this practice. He wonders, however, why the same change was not made in the conventions for slope and deflection. That would have involved the elimination of the minus sign from the definition of "angle change" as $-\frac{M}{E\,I}$; but that sign is not essential. It is there only to reconcile some independently assumed conventions which proved to be lacking in logical consistency. It is really much simpler to assume upward loads and deflections as positive. Then one can differentiate the equation of the elastic curve four times, obtaining successively the slope, bending moment, shear, and loading, without having to remember to reverse signs arbitrarily at various steps.

A related point is in connection with nomenclature. In his paper the author terms the quantity, $-\frac{M}{E\,I}$, the "angle change." Actually, as he states, he is using that expression as an allowable approximation for what mathematicians call the "curvature," $\frac{1}{r}$, in which r is the radius of curvature. Since the mathematicians already have given the quantity a name, why was it necessary to rechristen it? It might be considered awkward to speak of a "curvature curve," although the expression should be quite as clear as "angle-change curve"; and "concentrated curvature" should be as clear as "concentrated angle change." What the author has termed "angle change" is really "rate of slope change," and the latter term would really be preferable to the former, if "curvature" is to be replaced by something else.

In studying the numerical examples, the writer was unable to verify one of the author's figures. In Fig. 15(a) the equivalent concentrated angle change at the section of change in moment of inertia is shown as −404.90. This appears to be a quantity to be obtained by use of the formulas of Fig. 5(a), assuming the distributed angle-change curves produced to have ordinates either one tenth of, or ten times, those of the actual curve, in the adjacent segments of the beam. On this basis, the concentrated angle change in question would appear to be $-(3 \times 513.5 + 10 \times 803.6 - 1 \times 911.0 + 3 \times 91.10 + 10 \times 80.36 - 1 \times 51.35) \times \frac{1}{24} = -403.79$. The actual difference between this value and the author's is of no practical consequence, but it would be interesting to learn whether the figure in the text was computed by some other method.

Although the author's paper is subject to the foregoing minor criticisms, he deserves much credit for developing a valuable new tool for the use of the structural engineer.

CAMILLO WEISS,[27] M. AM. SOC. C. E.—The method outlined will undoubtedly be found useful in many types of problems other than those discussed by Professor Newmark, and the determination of ordinates to influence lines is one of these. It is readily applicable because influence lines can be considered as ratios between corresponding deformations. Furthermore, because only ratios are required, the various "common factors" may be disregarded, and scales may be adopted and changed to suit convenience at any step in the consecutive computations, provided relative scales remain the same. The moment diagrams are bounded by straight lines; therefore the results are accurate for straight-line or parabolic variations of moments of inertia. For other variations satisfactory approximations may be obtained.

The writer has computed influence lines for three typical cases, shown in Figs. 33, 34, and 35, and a study of these calculations will show readily the relative ease of the work required. The conventional calculation methods involve the same steps, but by applying the author's method the amount of laborious arithmetical work is greatly reduced.

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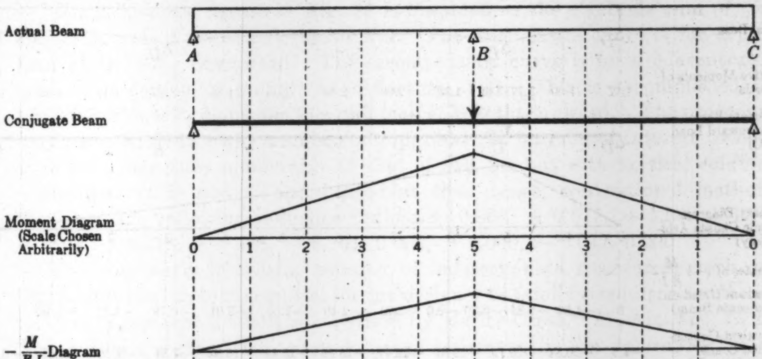
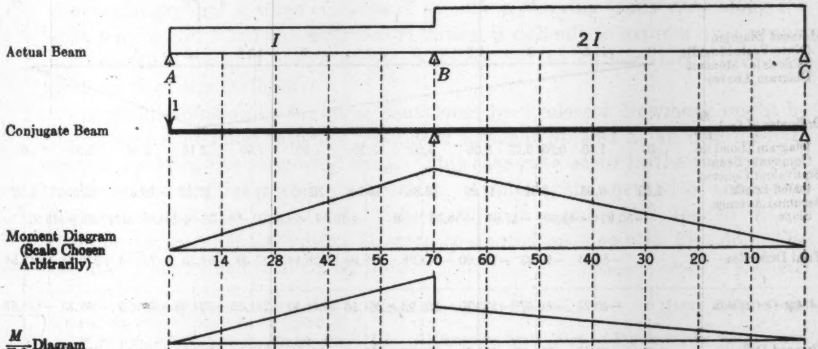
[27] Designer, Bethlehem Steel Co., Fabricated Steel Constr., Eng. Dept., Bethlehem, Pa.


Actual Beam
A
B
C
1
Conjugate Beam
Moment Diagram (Scale Chosen Arbitrarily)
0
1
2
3
4
5
4
3
2
1
0
− M/EI-Diagram

	0		1		2		3		4		5		4		3		2		1		0
(Load on Conjugate Beam)	0		−1		−2		−3		−4		−5		−4		−3		−2		−1		0
Equivalent Concentrated Loads	−1		−6		−12		−18		−24		−28		−24		−18		−12		−6		−1
Assumed Average Slope (Assumed Symmetrical)		74		68		56		38		14		−14		−38		−56		−68		−74	
Trial Deflection (=Actual Deflection)	0		74		142		198		236		250		236		198		142		74		0
Influence Line for Reaction at B	0		0.296		0.568		0.792		0.944		1.000		0.944		0.792		0.568		0.296		0

FIG. 33.—INFLUENCE LINE FOR REACTION AT CENTER SUPPORT OF BEAM CONTINUOUS OVER THREE SUPPORTS (SPANS EQUAL; MOMENT OF INERTIA CONSTANT)


Actual Beam
I
2 I
A
B
C
1
Conjugate Beam
Moment Diagram (Scale Chosen Arbitrarily)
0
14
28
42
56
70
60
50
40
30
20
10
0
M/EI-Diagram

	0		14		28		42		56		70		60		50		40		30		20		10		0
(Load on Conjugate Beam)	0		−14		−28		−42		−56		−70 −35		−30		−25		−20		−15		−10		−5		0
Equivalent Concentrated Loads	−14		−84		−168		−252		−336		−296		−180		−150		−120		−90		−60		−30		−5
Assumed Average Slope		1,226		1,142		974		722		386		90		−90		−240		−360		−450		−510		−540	
Trial Deflection	−4,450		−3,224		−2,082		−1,108		−386		0		90		0		−240		−600		−1,050		−1,560		−2,100
Linear Correction	−1,500		−1,200		−900		−600		−300		0		300		600		900		1,200		1,500		1,800		+2,100
Actual Deflection	−5,950		−4,424		−2,982		−1,708		−686		0		390		600		660		600		450		240		0
Influence Line for Reaction at A	1.000		0.744		0.501		0.287		0.115		0		−0.066		−0.101		−0.111		−0.101		−0.076		−0.040		0

FIG. 34.—INFLUENCE LINE FOR REACTION AT END SUPPORT A OF BEAM CONTINUOUS OVER THREE SUPPORTS (SPANS UNEQUAL; MOMENTS OF INERTIA DIFFER FOR SPANS BUT ARE CONSTANT WITHIN SPAN LENGTH)

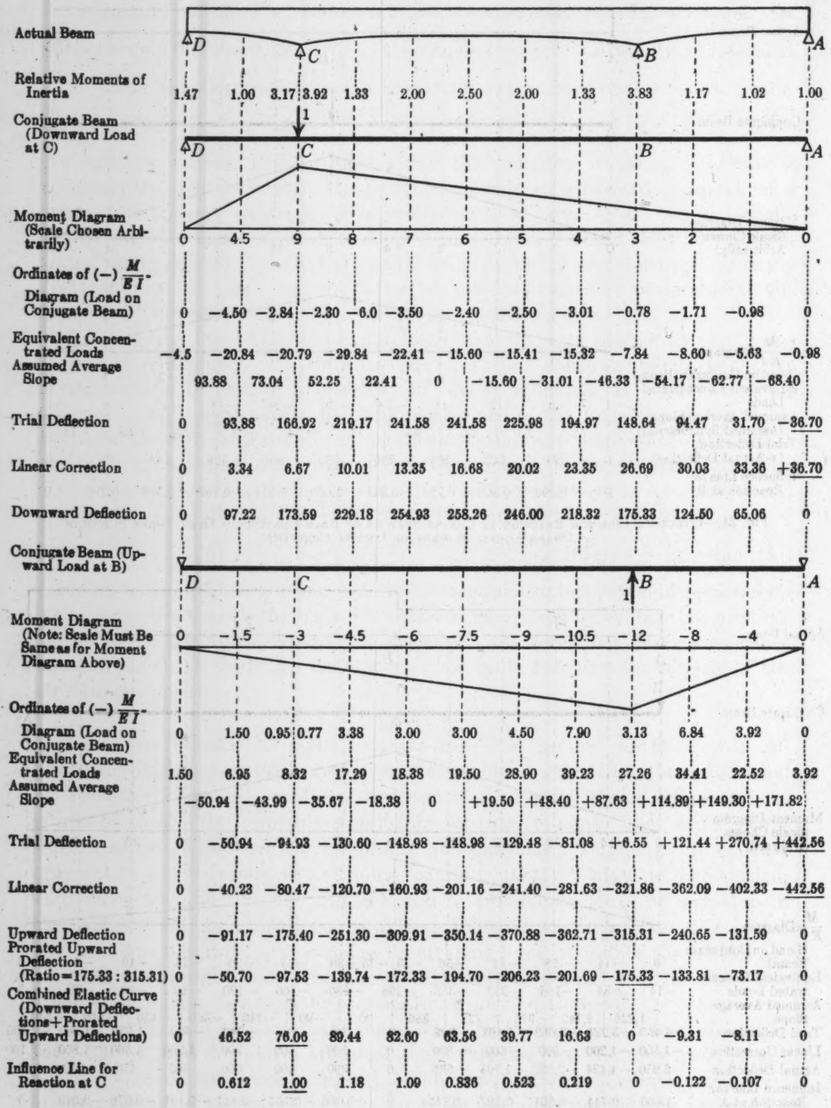

Actual Beam
D
C
B
A
Relative Moments of Inertia 1.47 1.00 3.17 3.92 1.33 2.00 2.50 2.00 1.33 3.83 1.17 1.02 1.00
Conjugate Beam (Downward Load at C)
1
D
C
B
A
Moment Diagram (Scale Chosen Arbitrarily) 0 4.5 9 8 7 6 5 4 3 2 1 0
Ordinates of (−) M/EI-Diagram (Load on Conjugate Beam) 0 −4.50 −2.84 −2.30 −6.0 −3.50 −2.40 −2.50 −3.01 −0.78 −1.71 −0.98 0
Equivalent Concentrated Loads −4.5 −20.84 −20.79 −29.84 −22.41 −15.60 −15.41 −15.32 −7.84 −8.60 −5.63 −0.98
Assumed Average Slope 93.88 73.04 52.25 22.41 0 −15.60 −31.01 −46.33 −54.17 −62.77 −68.40
Trial Deflection 0 93.88 166.92 219.17 241.58 241.58 225.98 194.97 148.64 94.47 31.70 −36.70
Linear Correction 0 3.34 6.67 10.01 13.35 16.68 20.02 23.35 26.69 30.03 33.36 +36.70
Downward Deflection 0 97.22 173.59 229.18 254.93 258.26 246.00 218.32 175.33 124.50 65.06 0
Conjugate Beam (Upward Load at B)
D
C
B
1
A
Moment Diagram (Note: Scale Must Be Same as for Moment Diagram Above) 0 −1.5 −3 −4.5 −6 −7.5 −9 −10.5 −12 −8 −4 0
Ordinates of (−) M/EI-Diagram (Load on Conjugate Beam) 0 1.50 0.95 0.77 3.38 3.00 3.00 4.50 7.90 3.13 6.84 3.92 0
Equivalent Concentrated Loads 1.50 6.95 8.32 17.29 18.38 19.50 28.90 39.23 27.26 34.41 22.52 3.92
Assumed Average Slope −50.94 −43.99 −35.67 −18.38 0 +19.50 +48.40 +87.63 +114.89 +149.30 +171.82
Trial Deflection 0 −50.94 −94.93 −130.60 −148.98 −148.98 −129.48 −81.08 +6.55 +121.44 +270.74 +442.56
Linear Correction 0 −40.23 −80.47 −120.70 −160.93 −201.16 −241.40 −281.63 −321.86 −362.09 −402.33 −442.56
Upward Deflection 0 −91.17 −175.40 −251.30 −309.91 −350.14 −370.88 −362.71 −315.31 −240.65 −131.59 0
Prorated Upward Deflection (Ratio = 175.33 : 315.31) 0 −50.70 −97.53 −139.74 −172.33 −194.70 −206.23 −201.69 −175.33 −133.81 −73.17 0
Combined Elastic Curve (Downward Deflections + Prorated Upward Deflections) 0 46.52 76.06 89.44 82.60 63.56 39.77 16.63 0 −9.31 −8.11 0
Influence Line for Reaction at C 0 0.612 1.00 1.18 1.09 0.836 0.523 0.219 0 −0.122 −0.107 0

FIG. 35.—INFLUENCE LINE FOR REACTION AT SUPPORT C OF A BEAM CONTINUOUS OVER FOUR SUPPORTS (SPANS UNEQUAL; MOMENTS OF INERTIA VARIABLE; THE VARIATION ASSUMED TO BE OF LINEAR DIMENSION)

The influence diagram in Fig. 35 is obtained as the algebraic sum of two elastic curves of the simple beam AD. The first elastic curve is for a unit load at C acting downward. The second elastic curve is for a concentrated load at B chosen to nullify the deflection at B. This is accomplished by prorating the deflections due to a unit load at B acting upward. The procedure may be extended to any number of supports. It may be of interest to note that for a structure comparable to that of Fig. 35, but with vertical columns continuous at B and C and hinged at their bases, conventional methods furnished the following influence ordinates: 0.000, + 0.617, + 1.00, + 1.16, + 1.04, + 0.775, + 0.448, + 0.161, 0.000, − 0.090, − 0.077, 0.000.

The constantly increasing number of indeterminate structures which are being built has made it essential for the designer to familiarize himself with the method of moment distribution devised by Hardy Cross,[28] M. Am. Soc. C. E. The author's method will provide great assistance in the determination of stiffness and carry-over factors, and in other less obvious ways.

Professor Newmark is to be congratulated for having produced a useful and well-presented paper, which is a definite contribution to engineering design methods.

A. A. EREMIN,[29] ASSOC. M. AM. SOC. C. E.—An interesting method of making successive approximations for the computation of stresses and deformations is described in this paper. The method of successive approximations is exceedingly useful when sections of members carrying loads vary along the span length and when the sectional variation is difficult to express by a simple mathematical formula. The problems solved by the author clarify the practical value of the method.

A useful addition to the cases considered by Professor Newmark might be the case of a member resisting a bending moment applied at an intermediate section between the supported ends. This case may occur in the column that receives load applied through a bracket. Applying the method of successive approximations, the stresses and deformations in member AB, loaded with a bending moment M at section C, were computed as shown in Fig. 36. The member was divided into six sections.

(Figure 36 follows on page 1212)

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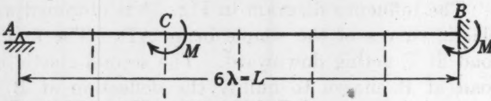
[28] "Analysis of Continuous Frames by Distributing Fixed-End Moments," by Hardy Cross, *Transactions*, Am. Soc. C. E., Vol. 96 (1932), p. 1.

[29] Associate Bridge Engr., Bridge Dept., Div. of Highways, State Dept. of Public Works, Sacramento, Calif.



(a) Deflection Due to Moment $M_B = 100$ at Right End

Moments	0		17		33		50		67		83		100
Distributed Angle Changes			−17		−33		−50		−67		−83		−47
Assumed Average Slope, Part (1)		−17		0		33		83		150		233	
Trial Deflection, Parts (1) and (2)	482		499		499		466		383		233		
Linear Correction to Deflection	482		402		320		240		160		80		0
Resultant Deflection	0		97		179		226		223		153		0

Slope at right end $= (153 + 47)\frac{\lambda}{E\,I} = 200\frac{\lambda}{E\,I}$

(b) Deflection Due to Moment $M = 100$

Moments	0		−17		−33 | +67		+50		+33		+17		0
Distributed Angle Changes			+17		−17		−50		−33		−17		−3
Assumed Average Slope, Part (1)		−37		−54		−37		+13		+46		+63	
Trial Deflection, Parts (1) and (2)	−6		31		85		122		109		63		
Linear Correction to Deflection	6		5		4		3		2		1		
Resultant Deflection	0		36		89		125		111		64		

Slope at right end $= (64 + 3)\frac{\lambda}{E\,I} = 67\frac{\lambda}{E\,I}$

Moment at left end $= M_B = \frac{67 \times 100}{200} = 33$

FIG. 36.—RESTRAINING MOMENT PRODUCED BY MOMENT APPLIED AT A SECTION BETWEEN BEAM SUPPORTS

MYRON L. GOSSARD,[30] JUN. AM. SOC. C. E.—The numerical procedure for finding deflections and slopes of beams combines fundamental concepts of beam elasticity and geometry, simplicity and accuracy of method, and clearness as to beam action under load. These qualities, plus the fact that the procedure follows closely that of computing shear and moment diagrams, certainly will make it valuable to both structural engineers and engineering students. The application of the method is particularly effective in the analysis of continuous frameworks where it is necessary to determine certain beam constants and load constants. The author's procedure seems to possess all the advantages of the method of the column analogy when applied to beams, and gives a clearer picture of the beam action.[31] Also, the column analogy does not give the deflections directly as does the numerical procedure.

The writer has used a special application of the fundamental Newmark method to arrive at a procedure which, it is believed, is somewhat more convenient for finding stiffnesses, carry-over factors, fixed-end moments, and deflections of members of continuous frames—especially unsymmetrical members

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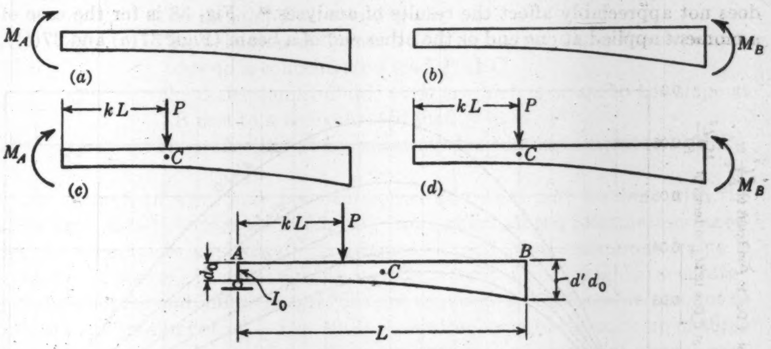
[30] Stress Analyst, Airplane Div., Curtiss-Wright Corp., Louisville, Ky.

[31] "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, New York, N. Y., 1932, pp. 46–47.

that may be subdivided into segments that can be approximated by "standard" beam forms. The "standard" beam forms are those shown in Fig. 37, where the haunch curves may be either straight lines, half quadratic parabolas, or



(e) TYPICAL NOTATION

FIG. 37.—STANDARD BEAM FORMS (BEAM OUTLINES ONLY), WITH LOAD TYPES

half cubic parabolas, with vertexes at the shallow end of the beam. Following Professor Newmark's numerical procedure, sets of curves similar to Figs. 38

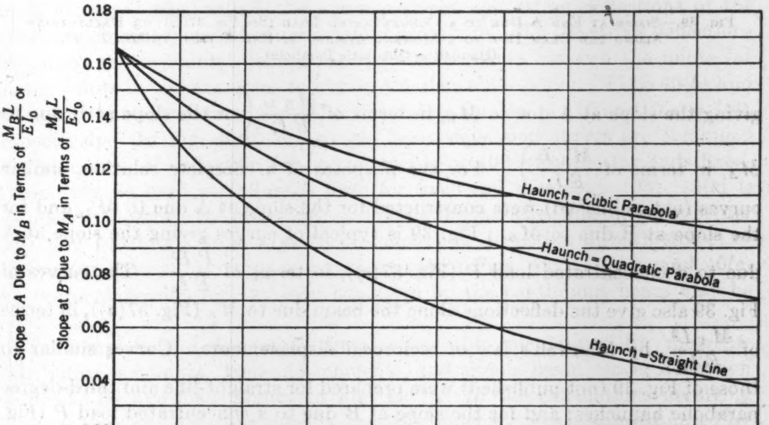


FIG. 38.—SLOPE AT ONE END OF A BEAM DUE TO A MOMENT APPLIED AT THE OTHER END (See Figs. 37(a) and 37(b))

and 39 were drawn for each case, using twelve equal divisions of beam length, and based on the assumption that, at any section, the moment of inertia varies as the $\frac{5}{2}$ power of the beam depth. This assumption seems to the writer to be valid and sufficiently accurate for both structural steel (plate girder or I-beam) and reinforced concrete (rectangular or T-beam) construction, inasmuch as the exponent must lie between 2 and 3 for these types and some error in this respect does not appreciably affect the results of analyses.[32] Fig. 38 is for the case of a moment applied at one end or the other end of a beam (Figs. 37(*a*) and 37(*b*)),

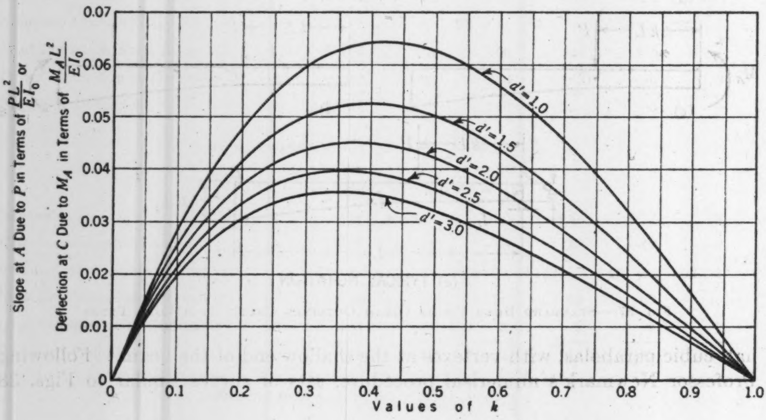


Fig. 39.—Slope at End A Due to a Concentrated Load (See Fig. 37(*c*)); or Deflections Along the Beam Due to a Moment Applied at End A (See Fig. 37(*a*)) (Haunch = Quadratic Parabola)

giving the slope at A due to M_B, in terms of $\frac{M_B\,L}{E\,I_o}\Big($ or the slope at B due to M_A, in terms of $\frac{M_A\,L}{E\,I_o}\Big)$. For the purposes of a complete solution, similar curves (not published) were constructed for the slope at A due to M_A, and for the slope at B due to M_B. Fig. 39 is typical of curves giving the slope at A due to a concentrated load P (Fig. 37(*c*)), in terms of $\frac{P\,L^2}{E\,I_o}$. The curves of Fig. 39 also give the deflections along the beam due to M_A (Fig. 37(*a*)), in terms of $\frac{M_A\,L^2}{E\,I_o}$, by Maxwell's law of reciprocal displacements. Curves similar to those of Fig. 39 (not published) were prepared for straight-line and third-degree parabolic haunches; and for the slope at B due to a concentrated load P (Fig. 37(*c*)), the latter set of curves also giving the deflections along the beam due to M_B (Fig. 37(*b*)). Briefly, six sets of curves are needed for a complete solution of a case of the type in Fig. 40, representing:

---

[32] "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, New York, N. Y 1932, pp. 3–5 and 169–171.

Case	Description
(*a*)	The slope at A due to M_A
(*b*)	The slope at B due to M_A, which is equal to the slope at A due to M_B
(*c*)	The slope at B due to M_B
(*d*)	The deflection at C due to M_A, which is equal to the slope at A due to a concentrated load P at C
(*e*)	The deflection at C due to M_B, which is equal to the slope at B due to a concentrated load P at C
(*f*)	The deflection at beam-center due to a concentrated load P

An example of what may be encountered and what may be done with the aid of such data is illustrated in Fig. 40, from which all the information necessary for a complete analysis of the beam as a part of a continuous frame is derived. However, if deflection is not important or if "highly accurate" deflection curves and influence lines are not required, a considerable amount of the work can be eliminated. Fig. 40(*a*) shows the beam to be made up of three elastic segments, L_1, L_2, and L_3, with an assumed inelastic segment at each end. The trapezoidal moment diagram for each elastic segment as a part of the whole is divided into two triangular diagrams; then end slopes and deflections for each segment due to each of its triangular moment diagrams are taken from the appropriate curves of the types of Figs. 38 and 39. The end slopes are added at each segmental junction or "joint" to become the "equivalent concentrated angle changes" from which the average slopes and string deflections of the joints are obtained as in the fundamental procedure. To the string deflections are added the segmental deflections at selected points between the joints (at center points in the example) to obtain the deflection curve. Figs. 40(*b*) and 40(*c*) show the calculations and deflection curves for $M_A = 100$ and $M_B = 100$, respectively. In Fig. 40(*d*) the simple beam deflection curves are combined to give curves of deflection for end moments of 100 with the far ends fixed These may be used as influence lines for fixed-end moments. Fig. 40(*e*) is included to illustrate the procedure for finding the influence line for beam-center deflection which is always close to the maximum deflection in both simple or continuous beams. By proper combinations of Figs. 40(*b*), 40(*c*), and 40(*e*) the center deflection for any load condition on the continuous beam may be found.

To complete the discussion there follow the calculations for moment-distribution constants, fixed-end moments, and simple beam center deflection for dead and live loads shown in Fig. 40(*f*). In the calculations involving influence lines and distributed loads, the area under a curve is approximated by finding the area under a parabola passing through three points, which area (A) is given by

$$A = \frac{\lambda}{6} (a + 4\,b + c) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (33)$$

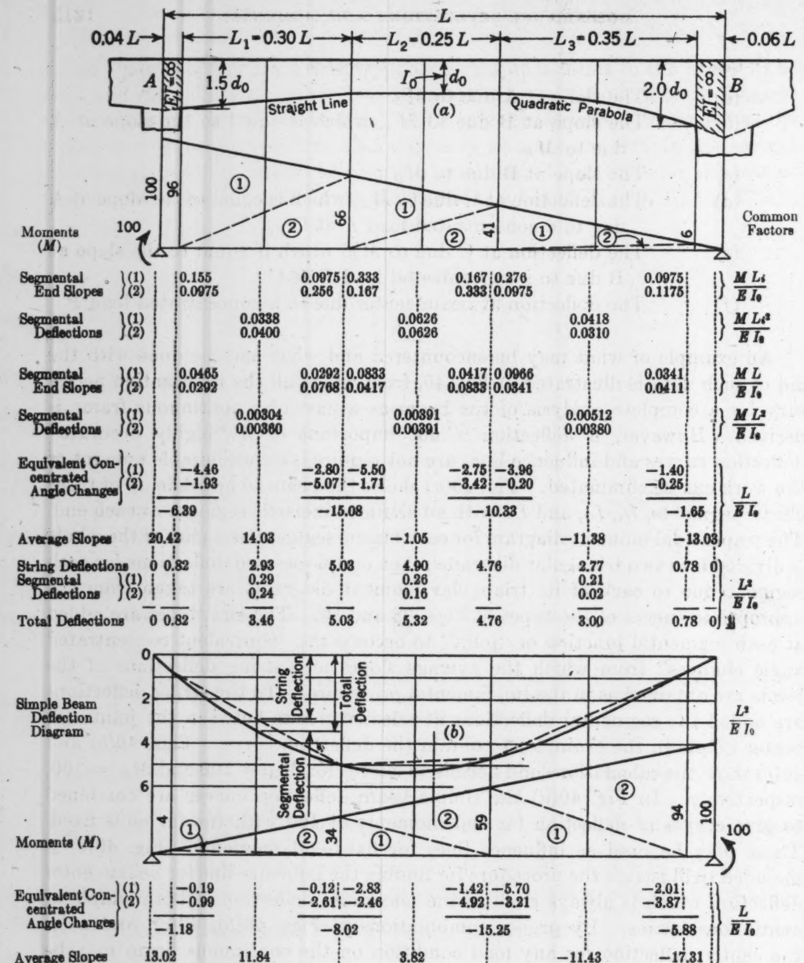
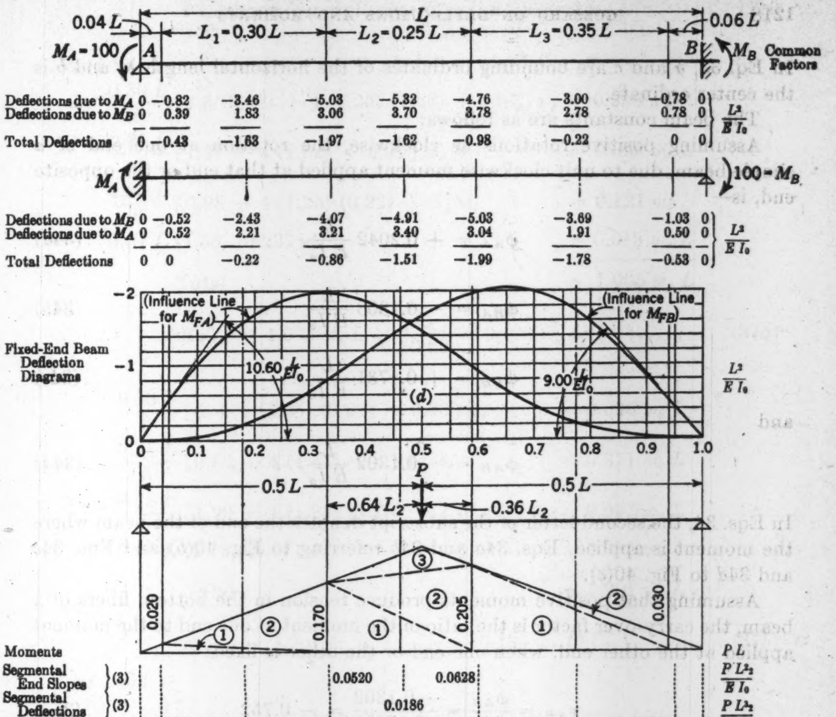

L
0.04 L
$L_1 = 0.30\,L$
$L_2 = 0.25\,L$
$L_3 = 0.35\,L$
0.06 L
A
B
$EI = \infty$
$1.5\,d_0$
I_0
d_0
$2.0\,d_0$
Straight Line
(a)
Quadratic Parabola
100
96
66
41
6
①
②
Moments (M)
100
Common Factors
Segmental End Slopes (1) 0.155 | 0.0975 0.333 | 0.167 0.276 | 0.0975
Segmental End Slopes (2) 0.0975 | 0.256 0.167 | 0.333 0.0975 | 0.1175
$\frac{M L_i}{E I_0}$
Segmental Deflections (1) 0.0338 | 0.0626 | 0.0418
Segmental Deflections (2) 0.0400 | 0.0626 | 0.0310
$\frac{M L_i^2}{E I_0}$
Segmental End Slopes (1) 0.0465 | 0.0292 0.0833 | 0.0417 0.0966 | 0.0341
Segmental End Slopes (2) 0.0292 | 0.0768 0.0417 | 0.0833 0.0341 | 0.0411
$\frac{M L}{E I_0}$
Segmental Deflections (1) 0.00304 | 0.00391 | 0.00512
Segmental Deflections (2) 0.00360 | 0.00391 | 0.00380
$\frac{M L^2}{E I_0}$
Equivalent Concentrated Angle Changes (1) −4.46 | −2.80 −5.50 | −2.75 −3.96 | −1.40
Equivalent Concentrated Angle Changes (2) −1.93 | −5.07 −1.71 | −3.42 −0.20 | −0.25
−6.39 | −15.08 | −10.33 | −1.65
$\frac{L}{E I_0}$
Average Slopes 20.42 | 14.03 | −1.05 | −11.38 | −13.03
String Deflections 0 0.82 | 2.93 | 5.03 | 4.90 | 4.76 | 2.77 | 0.78 0
Segmental Deflections (1) 0.29 | 0.26 | 0.21
Segmental Deflections (2) 0.24 | 0.16 | 0.02
Total Deflections 0 0.82 | 3.46 | 5.03 | 5.32 | 4.76 | 3.00 | 0.78 0
$\frac{L^2}{E I_0}$
Simple Beam Deflection Diagram
0
2
4
6
String Deflection
Total Deflection
Segmental Deflection
(b)
$\frac{L^2}{E I_0}$
Moments (M)
4
34
59
94
100
100
①
②
Equivalent Concentrated Angle Changes (1) −0.19 | −0.12 −2.83 | −1.42 −5.70 | −2.01
Equivalent Concentrated Angle Changes (2) −0.99 | −2.61 −2.46 | −4.92 −3.21 | −3.87
−1.18 | −8.02 | −15.25 | −5.88
$\frac{L}{E I_0}$
Average Slopes 13.02 | 11.84 | 3.82 | −11.43 | −17.31
String Deflections 0 0.52 | 2.30 | 4.07 | 4.55 | 5.03 | 3.03 | 1.03 0
Segmental Deflections (1) 0.01 | 0.13 | 0.30
Segmental Deflections (2) 0.12 | 0.23 | 0.36
Total Deflections 0 0.52 | 2.43 | 4.07 | 4.91 | 5.03 | 3.69 | 1.03 0
$\frac{L^2}{E I_0}$
Simple Beam Deflection Diagram
0
2
4
6
(c)
$\frac{L^2}{E I_0}$

FIG. 40.—COMPUTATIONS FOR AN UNSYMMETRICAL BEAM


0.04 L
L
L₁ = 0.30 L
L₂ = 0.25 L
L₃ = 0.35 L
0.06 L
M_A = 100
A
B
M_B
Common Factors
M_A
100 − M_B
Fixed-End Beam Deflection Diagrams
−2
−1
0
(Influence Line for M_FA)
(Influence Line for M_FB)
10.60 I/EI₀
9.00 I/EI₀
(d)
L²/EI₀
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
P
0.5 L
0.5 L
0.64 L₂
0.36 L₂
0.020
0.170
0.205
0.030
①
②
③
Simple Beam Deflection Diagram
0
0.01
0.02
Influence Line for δ_C (Simple Beam)
(e)
PL³/EI₀
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
Dead Load
M_FA
M_FB
1.57 w₀
1.5 w₀
1.25 w₀
w₀
w₀
1.25 w₀
2.0 w₀
2.37 w₀
(f)
Live Load
M_FA
M_FB
0.22 L
0.465 L
0.59 L
0.695 L
0.25 wL
1.75 wL
0.835 L
w
2 w

Deflections due to M_A 0 −0.82 −3.46 −5.03 −5.32 −4.76 −3.00 −0.78 0
Deflections due to M_B 0 0.39 1.83 3.06 3.70 3.78 2.78 0.78 0
Total Deflections 0 −0.43 −1.63 −1.97 −1.62 −0.98 −0.22 0 0 $\frac{L^2}{EI_0}$

Deflections due to M_B 0 −0.52 −2.43 −4.07 −4.91 −5.03 −3.69 −1.03 0
Deflections due to M_A 0 0.52 2.21 3.21 3.40 3.04 1.91 0.50 0
Total Deflections 0 0 −0.22 −0.86 −1.51 −1.99 −1.78 −0.53 0 $\frac{L^2}{EI_0}$

Moments PL
Segmental End Slopes (3) 0.0520 0.0628 $\frac{PL_2^2}{EI_0}$
Segmental Deflections (3) 0.0186 $\frac{PL_2^3}{EI_0}$
Equivalent Concentrated Angle Changes
(1) −0.00093 −0.00058 −0.01418 −0.00709 −0.0198 −0.0070
(2) −0.00496 −0.01307 −0.00855 −0.01710 −0.0010 −0.0012
(3) −0.00325 −0.00392
−0.0059 −0.0396 −0.0489 −0.0082 $\frac{PL^2}{EI_0}$
Average Slopes 0.0524 0.0465 0.0069 −0.0420 −0.0502
String Deflections 0 0.0021 0.0091 0.0161 0.01695 0.0178 0.01045 0.0031 0
Segmental Deflections
(1) 0.00006 0.00066 0.00105
(2) 0.00061 0.00080 0.00011
(3) 0.00029
Total Deflections 0 0.0021 0.0098 0.0161 0.0187 0.0178 0.0116 0.0031 0 $\frac{PL^3}{EI_0}$

FIG. 40.—COMPUTATIONS FOR AN UNSYMMETRICAL BEAM (*Continued*)

In Eq. 33, a and c are bounding ordinates of the horizontal length λ, and b is the center ordinate.

The beam constants are as follows:

Assuming positive rotations as clockwise, the rotation at one end of a simple beam, due to unit clockwise moment applied at that end or the opposite end, is—

$$\phi_{AA} = +\,0.2042\,\frac{L}{E\,I_o} \qquad (34a)$$

$$\phi_{BA} = -\,0.1303\,\frac{L}{E\,I_o} \qquad (34b)$$

$$\phi_{BB} = +\,0.1731\,\frac{L}{E\,I_o} \qquad (34c)$$

and

$$\phi_{AB} = -\,0.1302\,\frac{L}{E\,I_o} \qquad (34d)$$

In Eqs. 34, the second letter of the subscript denotes the end of the beam where the moment is applied, Eqs. 34a and 34b referring to Fig. 40(b) and Eqs. 34c and 34d to Fig. 40(c).

Assuming that positive moments produce tension in the bottom fibers of a beam, the carry-over factor is the ratio of the moment at one end to the moment applied at the other end, when one end or the other is fixed:

$$r_{AB} = \frac{\phi_{AB}}{\phi_{BB}} = \frac{-0.1302}{0.1731} = -\,0.752 \qquad (35a)$$

and

$$r_{BA} = \frac{\phi_{AB}}{\phi_{AA}} = \frac{-0.1302}{0.2042} = -\,0.637 \qquad (35b)$$

For example, Eq. 35a yields the carry-over factor at end A when end B is fixed, and Eq. 35b, the carry-over factor at end B when end A is fixed. For the moment stiffness S_M, the moment at end A required to produce unit rotation at end A when end B is fixed, is

$$(S_M)_A = \frac{1}{\phi_{AA} - r_{AB}\,\phi_{AB}} = 9.40\,\frac{E\,I_o}{L} \qquad (36a)$$

Similarly for the moment at end B required to produce unit rotation at end B when end A is fixed—

$$(S_M)_B = \frac{1}{\phi_{BB} - r_{BA}\,\phi_{AB}} = 11.1\,\frac{E\,I_o}{L} \qquad (36b)$$

From Figs. 40(d) and 40(f), the fixed-end moments (M_F) are computed as follows.

Dead Load.—

$$\frac{0.30\,L}{6}\,[1.5\,(0.43) + 4\,(1.25)\,(1.63) + 1.97]\,w_o = 0.538\,w_o\,L$$

$$\frac{0.25\,L}{6}\,[1.97 + 4\,(1.62) + 0.98]\,w_o = 0.393\,w_o\,L$$

$$\frac{0.35\,L}{6}\,[0.98 + 4\,(1.25)\,(0.22) + 0]\,w_o = 0.121\,w_o\,L$$

$$0.04\,L\,(1.53)\,(0.22)\,w_o = 0.013\,w_o\,L$$

$$\text{Total} = 1.065\,w_o\,L$$

$$(M_F)_A = 1.065\,w_o\,L \times \frac{L}{10.60} = 0.100\,w_o\,L^2\,(-) \ldots\ldots\ldots (37a)$$

and

$$\frac{0.30\,L}{6}\,[0 + 4\,(1.25)\,(0.22) + 0.86]\,w_o = 0.098\,w_o\,L$$

$$\frac{0.25\,L}{6}\,[0.86 + 4\,(1.51) + 1.99]\,w_o = 0.371\,w_o\,L$$

$$\frac{0.35\,L}{6}\,[1.99 + 4\,(1.25)\,(1.78) + 2\,(0.53)]\,w_o = 0.697\,w_o\,L$$

$$0.06\,L\,(2.18)\,(0.27)\,w_o = 0.035\,w_o\,L$$

$$\text{Total} = 1.201\,w_o\,L$$

$$(M_F)_B = 1.201\,w_o\,L \times \frac{L}{9.00} = 0.134\,w_o\,L^2\,(-) \ldots\ldots\ldots (37b)$$

Live Load.—

$$\frac{0.37\,L}{6}\,[1.75 + 4\,(1.83) + 0.98]\,w = 0.620\,w\,L$$

$$\frac{0.245\,L}{6}\,[0.98 + 4\,(0.40) + 0.05]\,2\,w = 0.215\,w\,L$$

$$0.25\,w\,L\,(1.62) = 0.405\,w\,L$$

$$1.75\,w\,L\,(0.46) = 0.805\,w\,L$$

$$\text{Total} = 2.045\,w\,L$$

$$(M_F)_A = 2.045\,w\,L \times \frac{L}{10.60} = 0.193\,w\,L^2\,(-) \ldots\ldots\ldots (38a)$$

and

$$\frac{0.37\,L}{6}\,[0.31 + 4\,(1.18) + 1.99]\,w = 0.433\,w\,L$$

$$\frac{0.245\,L}{6}\,[1.99 + 4\,(1.96) + 1.34]\,2\,w = 0.913\,w\,L$$

$$0.25\,w\,L\,(1.51) = 0.378\,w\,L$$

$$1.75\,w\,L\,(2.0) = 3.50\,w\,L$$

$$\text{Total} = 5.224\,w\,L$$

$$(M_F)_B = 5.224\,w\,L \times \frac{L}{9.00} = 0.580\,w\,L^2\,(-) \ldots\ldots\ldots (38b)$$

From Figs. 40(*e*) and 40(*f*) the simple beam center deflections are computed as follows.

Dead Load.—

$$\begin{aligned}
&\frac{0.30\,L}{6}\,[1.5\,(0.0021) + 4\,(1.25)\,(0.0098) + 0.0161]\,w_o &&= 0.00341\,w_o\,L\\
&\frac{0.25\,L}{6}\,[0.0161 + 4\,(0.0187) + 0.0178]\,w_o &&= 0.00453\,w_o\,L\\
&\frac{0.35\,L}{6}\,[0.0178 + 4\,(1.25)\,(0.0116) + 2\,(0.0031)]\,w_o &&= 0.00478\,w_o\,L\\
&0.04\,L\,(1.53)\,(0.0011)\,w_o &&= 0.00007\,w_o\,L\\
&0.06\,L\,(2.18)\,(0.0016)\,w_o &&= \underline{0.00021\,w_o\,L}\\
&\qquad\text{Total} &&= 0.01300\,w_o\,L
\end{aligned}$$

$$\delta_c = 0.01300\,w_o\,L \times \frac{L^3}{E\,I_o} = 0.0130\,\frac{w_o\,L^4}{E\,I_o}\,(+)\ldots\ldots\ldots\ldots(39)$$

Live Load.—

$$\begin{aligned}
&\frac{0.37\,L}{6}\,[0.0112 + 4\,(0.0179) + 0.0178]\,w &&= 0.00620\,w\,L\\
&\frac{0.245\,L}{6}\,[0.0178 + 4\,(0.0138) + 0.008]\,2\,w &&= 0.00662\,w\,L\\
&0.25\,w\,L\,(0.0187) &&= 0.00467\,w\,L\\
&1.75\,w\,L\,(0.0145) &&= \underline{0.0254\ \ w\,L}\\
&\qquad\text{Total} &&= 0.0429\ \ w\,L
\end{aligned}$$

$$\delta_c = 0.0429\,w\,L \times \frac{L^3}{E\,I_o} = 0.0429\,\frac{w\,L^4}{E\,I_o}\,(+)\ldots\ldots\ldots\ldots(40)$$

Lack of space prevents the inclusion of the various curves prepared for use in the discussion of Fig. 40. In other respects, however, the writer has entered into more detail than is necessary in most applications of the numerical procedure because the emphasis of the Newmark paper is on a technique. Technique may rightfully assume prominence in problems such as this.

Robert A. Williamson,[33] Jun. Am. Soc. C. E.—In members subjected to combined axial and bending loads (commonly called beam columns) the effect of secondary moments caused by end thrusts cannot be ignored safely when the compressive load is any appreciable percentage of the critical buckling load. This is plainly demonstrated by the results of Fig. 17.

When the beam column is one of a series of members comprising a continuous structure, the values of the elastic constants and fixed-end moments required for the usual moment-distribution analysis depend, in part, on the magnitude and sign of the axial load.

For the case of variable section, evaluation of these quantities is greatly facilitated by the use of Professor Newmark's procedure, details of which are shown in Figs. 41 to 44, inclusive, using as an example the beam of Figs. 10 and 11 subjected to an axial compressive load.

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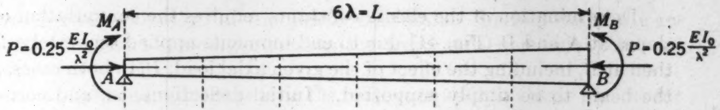
[33] Stress Analyst, Vega Aircraft Corp., Burbank, Calif.


6λ=L
M_A
M_B
$P=0.25\frac{EI_0}{\lambda^2}$
$P=0.25\frac{EI_0}{\lambda^2}$
A
B

I/I_0

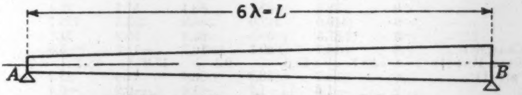
(a) Deflections and End Slopes for $M_A = 180$, $P = 0.25\,\frac{E\,I_0}{\lambda^2}$

	1		2		3		4		5		6		7	Common Factors
w_i for $M_A = 180$ From Fig. 10(c)	0		110.6		141.4		130.8		97.1		51.1		0	$\lambda^2/(E\,I_0)$
Assumed w_a	0		34.0		47.0		44.0		36.0		19.0		0	$\lambda^2/(E\,I_0)$
$w_0 = w_i + w_a$	0		144.6		188.4		174.8		133.1		70.1		0	$\lambda^2/(E\,I_0)$
Moments due to P	0		36.2		47.2		43.6		33.3		17.5		0	1
Distributed Angle Changes	0		−18.1		−15.7		−10.9		−6.7		−2.9			$1/(E\,I_0)$
Assumed Average Slope, Part (1)		38.1		20.0		4.3		−6.6		−13.3		−16.2		$\lambda/(E\,I_0)$
Trial Deflection, Part (1)	0		38.1		58.1		62.4		55.8		42.5		26.3	$\lambda^2/(E\,I_0)$
Deflection, Part (2)	0		−1.5		−1.3		−0.9		−0.6		−0.2		0	$\lambda^2/(E\,I_0)$
Linear Correction to Deflection	0		−4.4		−8.8		−13.2		−17.5		−21.9		−26.3	$\lambda^2/(E\,I_0)$
Resultant Deflection, w'_a	0		32.2		48.0		48.3		37.7		20.4		0	$\lambda^2/(E\,I_0)$
Final Deflections w_a Giving Same Deflections w'_a	0		32.4		48.4		48.9		38.1		20.6		0	$\lambda^2/(E\,I_0)$
Final w_0	0		143.0		189.8		179.7		135.2		71.7		0	$\lambda^2/(E\,I_0)$
Final Moments	180		185.7		167.4		134.9		93.8		47.9		0	1
Final End Slopes	212.9												−73.0	$\lambda/(E\,I_0)$

(b) Deflections and End Slopes for $M_B = 180$, $P = 0.25\,\frac{E\,I_0}{\lambda^2}$

	1	2	3	4	5	6	7	Common Factors
Final Deflections w_a Giving Same Deflections w'_a	0	19.1	30.4	32.4	26.4	14.7	0	$\lambda^2/(E\,I_0)$
Final w_0	0	67.6	113.1	120.4	115.4	71.7	0	$\lambda^2/(E\,I_0)$
Final Moments	0	46.9	88.2	122.3	148.8	167.9	180	1
Final End Slopes	72.7						−84.9	$\lambda/(E\,I_0)$

FIG. 41.—DEFLECTIONS AND BENDING MOMENTS FOR END MOMENTS COMBINED WITH AXIAL COMPRESSION



(a) Stiffness and Carry-Over Factors for Left End

A		B	Common Factors
212.9	End Slopes, $M_A = 180$	−73.0	$\lambda/(E\,I_0)$
−62.5	End Slopes, $M_B = -\,0.861 \times 180$	73.0	$\lambda/(E\,I_0)$
150.4	Total Slope, $M_A = 180$, $M_B = -\,155$	0	$\lambda/(E\,I_0)$

Carry-Over Factor $= -\,0.861 = C_A$

$$\text{Stiffness} = \frac{180}{150.4}\,\frac{E\,I_0}{\lambda} = 1.195\,\frac{E\,I_0}{\lambda} = K_A$$

(b) Stiffness and Carry-Over Factors for Right End

A		B	Common Factors
72.7	End Slopes, $M_B = 180$	−84.9	$\lambda/(E\,I_0)$
−72.7	End Slopes, $M_A = -\,0.341 \times 180$	24.9	$\lambda/(E\,I_0)$
0	Total Slope, $M_B = 180$, $M_A = -\,61.5$	−60.0	$\lambda/(E\,I_0)$

Carry-Over Factor $= -\,0.341 = C_B$

$$\text{Stiffness} = \frac{180}{60.0}\,\frac{E\,I_0}{\lambda} = 3.00\,\frac{E\,I_0}{\lambda} = K_B$$

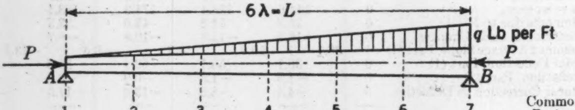
(c) Rigidity Factor
End Moments from Settlement, Δ, of End B,
A and B Restrained from Rotation

$$-\,0.371\,\frac{E\,I_0\,\Delta}{\lambda^2} = K_A(1 + C_A)\,\frac{\Delta}{6\,\lambda^2} \qquad K_B(1 + C_B)\,\frac{\Delta}{6\,\lambda^2} = 0.671\,\frac{E\,I_0\,\Delta}{\lambda^2}$$

$$\text{Rigidity} = \text{Shear when } \Delta = 1 = \frac{(0.371 + 0.671)}{6\,\lambda}\,\frac{E\,I_0}{\lambda^2} = 0.1739\,\frac{E\,I_0}{\lambda^2}$$

FIG. 42.—DETERMINATION OF ELASTIC CONSTANTS

Determination of the elastic constants requires the computation of the end slopes at A and B (Fig. 41) due to end moments applied separately, first at A, then at B, including the effect of the given axial load, P, in both cases, assuming the beam to be simply supported. Initial deflections, w_i, and corresponding end slopes were obtained from Figs. 10(c) and 11(b). The first trial is shown in detail for end moment M_A, the intermediate trials being omitted. Final results only are shown for end moment M_B.



I/I_o	1		2		3		4		5		6		7	Common Factors
(a) Deflections for Triangular Load with $P = 0$														
Ordinates to Load Diagram	0		−30		−60		−90		−120		−150		−180	$q/180$
Assumed Average Shear, Part (1)		180		150		90		0		−120		−270		$q\lambda/180$
Trial Moments, Parts (1) and (2)	0		180		330		420		420		300		30	$q\lambda^2/180$
Linear Correction to Moments	0		−5		−10		−15		−20		−25		−30	$q\lambda^2/180$
Ordinates of Moment Diagram	0		175		320		405		400		275		0	$q\lambda^2/180$
Ordinates to Angle-Change Diagram	0		−87.5		−106.7		−101.3		−80.0		−45.8		0	$q\lambda^2/(180\,E\,I_o)$
Assumed Average Slope, Part (1)		237.5		150.0		43.3		−58.0		−138.0		−183.8		$q\lambda^3/(180\,E\,I_o)$
Trial Deflections, Part (1)	0		237.5		387.5		430.8		372.8		234.8		51.0	$q\lambda^4/(180\,E\,I_o)$
Deflections, Part (2)	0		−7.3		−8.9		−8.4		−6.7		−3.8		0	$q\lambda^4/(180\,E\,I_o)$
Linear Correction to Deflections	0		−8.5		−17.0		−25.5		−34.0		−42.5		−51.0	$q\lambda^4/(180\,E\,I_o)$
Deflections, w_i	0		221.7		361.6		396.9		332.1		188.5		0	$q\lambda^4/(180\,E\,I_o)$
End Slopes	239.1 = 221.7 + 17.4												−188.5 − 8.1 = −196.6	$q\lambda^3/(180\,E\,I_o)$
(b) Deflections for Triangular Load with $P = 0.25\,\frac{E\,I_o}{\lambda^2}$														
Deflections, w_i	0		110.9		180.8		198.5		166.1		94.3		0	$q\lambda^4/(90\,E\,I_o)$
Assumed w_a	0		38.1		62.1		68.4		57.2		32.4		0	$q\lambda^4/(90\,E\,I_o)$
$w_o = w_i + w_a$	0		149.0		242.9		266.9		223.3		126.7		0	$q\lambda^4/(90\,E\,I_o)$
Moments due to P	0		37.3		60.8		66.9		55.9		31.7			$q\lambda^2/90$
Distributed Angle Changes	0		−18.7		−20.3		−16.7		−11.2		−5.3		0	$q\lambda^2/(90\,E\,I_o)$
Assumed Average Slope, Part (1)		38.7		20.0		−0.3		−17.0		−28.2		−33.5		$q\lambda^3/(90\,E\,I_o)$
Trial Deflections, Part (1)	0		38.7		58.7		58.4		41.4		13.2		−20.3	$q\lambda^4/(90\,E\,I_o)$
Deflections, Part (2)	0		−1.6		−1.7		−1.4		−0.9		−0.4		0	$q\lambda^4/(90\,E\,I_o)$
Linear Correction to Deflections	0		3.4		6.8		10.2		13.5		16.9		20.3	$q\lambda^4/(90\,E\,I_o)$
Deflections, w_a	0		40.5		63.8		67.2		54.0		29.7		0	$q\lambda^4/(90\,E\,I_o)$
Final Deflections w_a Giving Same Deflections w'_a	0		40.7		63.8		67.0		53.8		29.5		0	$q\lambda^4/(90\,E\,I_o)$
Final w_o	0		151.6		244.6		265.5		219.9		123.8		0	$q\lambda^4/(90\,E\,I_o)$
Final Moments	0		125.4		221.3		269.0		255.0		168.4		0	$q\lambda^4/90$
Final End Slopes	164.2												−128.6	$q\lambda^3/(90\,E\,I)$

FIG. 43 —DEFLECTIONS AND BENDING MOMENTS FOR TRIANGULAR LOAD COMBINED WITH AXIAL COMPRESSION

Fig. 42 illustrates the determination of stiffness and carry-over factors and rigidity from the results of Fig. 41 by a method similar to that of Fig. 11(c). (The formula for end moments in Fig. 42(c) is easily derived.[34])

For determining fixed-end moments the additional quantities needed are the end slopes of the beam, assumed simply supported and subjected to the given lateral and axial load. Fig. 43 shows the calculation of these values for a triangular lateral loading condition, intermediate trials being omitted.

The computation of fixed-end moments, shown in Fig. 44(a), utilizes the data of Figs. 42(a), 42(b), and 43 and the same general principles previously applied in the determination of the elastic constants. In Fig. 44(*b*) the resultant deflections and bending moments for fixed ends and axial load are tabulated.

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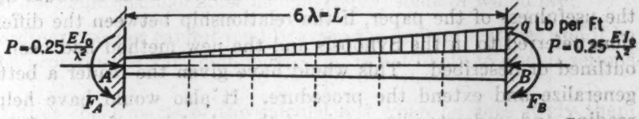
[34] "One Story Frames Analyzed by Moment Distribution," *Concrete Information Bulletin No. S T 42*, Portland Cement Assn., April, 1941, pp. 8 and 9.

As a check, the bending moments of Fig. 44(*b*) are used to obtain the results of Fig. 44(*c*), the computed deflections and end slopes differing from those of Fig. 44(*b*) by a maximum of about 2%. Much of the work was done with a 5-in. slide rule, the remainder with a 10-in. slide rule.



	(*a*) Fixed-End Moments		**Common Factors**
	End Slopes, Beam Simply Supported (Fig. 43(*b*))		
1.824		−1.427	$q\lambda^3/(EI_0)$
	End Rotation Producing Fixity at A, with B Clamped		
−1.824		0	$q\lambda^3/(EI_0)$
	Resulting End Moments		
$-2.180\,q\lambda^2 = 1.195\,\frac{EI_0}{\lambda}\left(-1.824\,\frac{q\lambda^3}{EI_0}\right)$		$-2.180\,q\lambda^2(-0.861) = 1.879\,q\lambda^2$	
	End Rotation Producing Fixity at B, with A Clamped		
0		1.427	$q\lambda^3/(EI_0)$
	Resulting End Moments		
$1.460\,q\lambda^2 = -4.281\,q\lambda^2(-0.341)$		$3.00\,\frac{EI_0}{\lambda}\left(1.427\,\frac{\lambda^3}{EI_0}\right) = -4.281\,q\lambda^2$	
	Fixed-End Moments		
$-0.720\,q\lambda^2 = F_A$		$F_B = -2.402\,q\lambda^2$	

(*b*) Resultant Deflections and Bending Moments
$P = 0.25\,EI_0/\lambda^2$ in All Cases

Deflections due to F_A	0	−103	−137	−130	−98	−52	0	$q\lambda^4/(180\,EI_0)$
Deflections due to F_B	0	−163	−272	−311	−278	−173	0	$q\lambda^4/(180\,EI_0)$
Deflections due to $F_A + F_B$	0	−266	−409	−441	−376	−225	0	$q\lambda^4/(180\,EI_0)$
Deflections due to Triangular Load	0	303	489	531	440	248	0	$q\lambda^4/(180\,EI_0)$
Resultant Deflections	0	37	80	90	64	23	0	$q\lambda^4/(180\,EI_0)$
Resultant End Slopes	0						0	$q\lambda^3/(180\,EI_0)$
Moments due to F_A	−130	−134	−121	−98	−68	−35	0	$q\lambda^2/180$
Moments due to F_B	0	−113	−212	−294	−358	−404	−433	$q\lambda^2/180$
Moments due to $F_A + F_B$	0	−248	−333	−392	−426	−440	−433	$q\lambda^2/180$
Moments due to Triangular Load	0	250	443	538	510	337	0	$q\lambda^2/180$
Resultant Moments	−130	2	110	146	84	−103	−433	$q\lambda^2/180$

(*c*) Check Values of Deflections and End Slopes
Obtained from Above Moments

Deflections	0	36.2	79.4	88.8	63.5	22.6	0	$q\lambda^4/(180\,EI_0)$
End Slopes	−3.0						−0.9	$q\lambda^3/(180\,EI_0)$

FIG. 44.—FIXED-END MOMENTS, DEFLECTIONS, AND BENDING MOMENTS FOR TRIANGULAR LOAD COMBINED WITH AXIAL COMPRESSION

In all cases where the value of w_a for the first trial was computed from Eq. 4, a total of two trials gave a value of w'_a so close to the value of w'_a from a third trial that the extra trial was unnecessary. The effect of omitting deflections w_a, computed from Eq. 4, was to increase the required number of trials to three.

In comparison with other procedures devised for this type of problem, Professor Newmark's method affects savings in time and labor, gives equally good results, and is more easily checked.

I. OESTERBLOM,[35] M. AM. SOC. C. E.—The "numerical procedure," presented by Professor Newmark, for computing a variety of important elements in structural design constitutes quite an important tool in the workshop of the practicing engineer. Fundamentally there is nothing new in his basic idea, but it has taken both vision and a live imagination to see how neglected this idea was and how extensively it might be put to work.

As very often is the case, the father of a new method knows his child so well that he can describe it only indifferently. Thus it would have added to the usefulness of the paper, if the relationship between the differential equations referred to in the Synopsis and the new method had been more clearly outlined or described. This would have given the reader a better chance to generalize and extend the procedure. It also would have helped to make reading and understanding easier if there had been fewer: "* * * the calculations are self-explanatory." The procedure can be discovered after some work; but if new ideas are to be spread and not forgotten they must first be promoted. To use an old sales slogan: "A new thing will not sell itself; it requires a good sales talk." At least one illustration, perhaps two or three, should have been explained, point for point, with nothing omitted. Then the remainder would have followed easily, and new problems also could have been set up by the novice.

The ultimate service and grace of the method are the same either way, but enthusiasm would have been greater if the invitation to a delightful program would have been more convincing. A delightful program it seems to be, if one may judge from the many fields to which the method has been extended—moments, slopes, deflections, point loads, uniform loads, variable loads, axial loads, critical loads, and buckling loads. What more can mere man desire—in one single paper!

The reaction formulas for the variable loadings by Nádai and Southwell are good to have; and better yet it is good to be shown how they may be used to advantage by the Newmark method. The buckling formulas and how they are to be used are equally interesting.

The writer has not yet had a chance to apply Professor Newmark's method to any commercial problems, but he can well remember many problems from his past experience for which he would have been grateful to have this new information; and he feels certain that many of the younger engineers will be equally grateful when they are faced with similar problems.

C. W. DUNHAM,[36] M. AM. SOC. C. E.—The method of solving certain problems of deflections and buckling loads by successive approximations, presented in this paper, is based on the obvious and sensible idea that moments and loads cause deflections and deformations of the members subjected to them, and, as so often stated by Hardy Cross, M. Am. Soc. C. E., and others, both statics and geometry must be satisfied. In other words, the deformations and the causes of those deformations must be consistent with each other. If a shape of the deflected member is assumed, but moments accompanying those deflections are not the same as the moments acting on the member, then of course the assumed shape is wrong. However, it is obvious that sufficient data may be found from the first trial to make a better guess next time, thus arriving at a shape that is more nearly correct.

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[35] Charleston, W. Va.

[36] Associate Prof., Civ. Eng., Yale Univ., New Haven, Conn.

As for the practicability and importance of the author's method, the writer has tried to look upon it as a tool for the designer to use and has tried to see it through the eyes of the ordinary man in a typical engineering office.

If it is desired to introduce a new method of analysis, or a modification of an established one, many things must be considered, some of which are:

1. If the ordinary reader cannot readily grasp the general features and if he thinks that the procedure is very complicated, he is not likely to give it serious study.
2. If he cannot check parts of the calculations when he does study them seriously, he will not be sure that the method can be trusted in new creative work of his own; hence he will not use it.
3. If he is to learn to use a method as a tool in his work, it must be simple, easy to understand, general in its application, and easy to remember and to apply.
4. If he is to develop facility in handling the procedure, he must apply it. This he will seldom do unless the method, as a tool, is applicable for solving problems with which he frequently comes in contact and which he must solve.

The writer will try to state his reactions to Professor Newmark's proposed method by discussing how it seems to meet the four preceding requirements:

1. A study of the paper gives the impression that the presentation makes the method appear more complicated than it really is. It is natural for any one who writes to believe that what is obvious to him is also obvious to the reader. However, such is not the case in many instances. It is unfortunate that the author did not give a little more explanation of the tie-up between his use of the "fictitious stringers" in computing moments in beams and the application of this procedure to the calculations of slopes and deflections when angle changes are used as loads. Of course, the explanation given by Professor Cross and N. D. Morgan, M. Am. Soc. C. E., is cited, but one illustrative problem showing this basic method more in detail would help to make the paper more complete in itself although risking repetition to the expert theorist. By its very nature, the paper is one that cannot be read casually and yet be appreciated. If the difficulties which it avoids were called to the reader's attention more forcefully, it might encourage him to give the paper the serious study which it deserves.

2. The writer decided to check most of the detailed work in the illustrative problems. In doing so, he found many "hops, skips, and jumps" which are likely to be confusing to a student—or, at least, hurdles which may handicap him in developing complete understanding of the work and confidence in his own ability to apply it independently. Some points which may be useful to others are listed below:

(*a*) In Fig. 6, the symmetry of shape and the fact that the tangent to the neutral curve is horizontal at the center of the span enables one to split the concentrated angle change of − 46 into two equal parts, the left one being positive because the tangent in that section slopes downward toward the right whereas the other side is the opposite. In Fig. 6, also, it might be advisable, as a first case, to label the last two lines of the calculations for deflection as follows: Average Slope (Shear); Deflection (Moment).

(*b*) A more detailed solution would be helpful in Fig. 7.

(*c*) In Fig. 10(*a*), the value 69.6 in the computation of the total end slope is derived by using Fig. 5(*a*) with values of the ordinates to the angle-change diagram.

(*d*) In Fig. 18(*a*), it might be well to give the reader some idea of how to make his first guess of the assumed average slope, Part 1. Here a good assumption would be about half of the sum of the distributed angle changes. However, in Fig. 19(*a*), trial appears to be the only way to determine a suitable starting value.

(*e*) The force 2 P in Fig. 19(*a*) should be shown clearly to be applied at the fourth division point. It would also be helpful to show the following for the computation of moments, as for the value 175 at the first division point: The rotational moment due to the deflection of the point of application of 2 P is 2 P (1,000 a) clockwise; the end reaction (vertical) due to this moment is $2\,P\,(1{,}000\,a) \div 10\,\lambda = 200\,\frac{P\,a}{\lambda}$ acting counter-clockwise; and the moment at the first division

$$= P\,(375\,a) - \left(200\,\frac{P\,a}{\lambda}\right)\lambda = 175\,P\,a.$$

(*f*) It would be helpful if the author used specific numerical cases for Figs. 19 and 20, showing how to derive a scale for the value of a.

3. The general method shown in the paper is really simple and readily understandable. However, it seems desirable to adopt one standard set of details of procedure rather than to show special short-cut methods for particular cases. A thorough understanding of the basic method is desirable. If it can be mastered thoroughly, one need seldom worry about the fact that a modification of it may be more efficient in a special case. It is wise to have the men in an office able to use basic procedures correctly and to be able to check each others' work without undue disputes as to refinement of methods. Furthermore, the assumptions that must be made in practical work regarding magnitudes of loads, their directions, their points of application, conditions of end restraint, original straightness of members, span lengths, and the properties of the materials introduce so many approximations that it seems unnecessary in most cases to refine the calculations for curvature of loading diagrams when the straight-line approximations may be far more correct than the basic data from which the computations are started. Would it be more beneficial to confine the method to the use of substitute straight-line moment diagrams and to illustrate its application more fully by numerical cases? This might make it very easy to remember and to apply. The arrangement of the calculations (bookkeeping) is very simple and satisfactory.

4. It appears that the proposed method is useful in determining the deflected shapes of members under certain assumptions. However, this is not generally important except in special cases. As for its use in designing columns, it seems to the writer that the buckling loads are seldom of great interest to the designer of ordinary structures although they may be important in the design of machinery, airplanes, and similar structures requiring special refinement and care in their design. In ordinary structural work, the designer generally selects a tentative member because of various practical reasons which make it seem desirable, or he chooses one by using various approximations, and then he analyzes it to prove that it is satisfactory for the loading conditions under which it must act. Generally, he is not interested in its ultimate buckling load. Therefore, the field of usefulness of the proposed method may be rather limited.

However, the author is to be congratulated upon developing such a simple approximation for the solution of problems. When they do arise and the designer must meet them, he will need such a handy tool and he will need it badly.

N. M. NEWMARK,[37] ASSOC. M. AM. SOC. C. E.—The suggestions, criticisms, and examples of the application of the numerical procedure to specific problems that have been given by the discussers of the paper are appreciated.

The use of the procedure to obtain elastic constants for beams is illustrated by Mr. Johnston. His suggested procedure is applicable to problems in which deflections are not desired, but where end slopes are required and are determined from the equivalent concentrated angle-change loads. In this way the procedure can be used to determine relatively accurate values for elastic constants with a comparatively small number of segments in the length of the beam.

Influence lines for fixed-end moment may be obtained from the deflection curve for a beam with a unit rotation at the end, by use of the so-called Müller-Breslau principle. Consequently, it may be convenient often to compute deflections even in problems such as those considered by Mr. Johnston.

Professor Ketchum has suggested a way of estimating the additional deflection due to the end thrust when a bar is subjected to lateral load and end thrust. His procedure amounts to assuming that the buckling configuration of the bar is the same in shape as the deflection curve due to the lateral load. For more or less uniform distributions of loading his approximation is reasonably good and leads to fairly accurate results.

In order to correlate Professor Ketchum's procedure with that suggested by the writer, one can determine an approximate value for the critical load from Eq. 19*b*, as the average ratio between the moment due to the lateral load and the deflection due to the lateral load. In order to estimate the additional deflection to be used in the numerical procedure, Eq. 4 can be used with the approximate value of the critical load determined from Eq. 19*b*. In many cases this will permit a problem to be solved without going through the routine of determining the critical load first, and therefore the suggestion is an important addition to the paper.

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[37] Research Asst. Prof., Civ. Eng., Univ. of Illinois, Urbana, Ill.

The use of simultaneous equations in solving the numerical problem was mentioned by Professor Wilbur. Although simultaneous equations can be written for the calculation of the deflections in a bar loaded with lateral loads and end thrusts, in general it is neither convenient nor desirable to write these equations. In the few cases where the numerical procedure does not converge, it may be desirable to have recourse to the equations. The following procedure will illustrate the way in which the equations may be written.

Let $m - 1$, m, and $m + 1$ be three neighboring points on the bar, at a distance λ apart. The following notation is used, in which the subscripts refer to the particular point on the bar:

$(w_i)_m$ = deflection due to the lateral load and the initial configuration, at point m;
$(w_a)_m$ = additional deflection at point m due to the axial loads; and
$(E\,I)_m$ = value of the product $E\,I$ at the point m.

The change in slope at point m is equal to the equivalent concentrated angle change at m; but the added change in slope is the quantity $\frac{(w_a)_{m+1} - 2(w_a)_m + (w_a)_{m-1}}{\lambda}$. If the moment at point m due to the axial load P is $P[(w_i)_m + (w_a)_m]$, and if the angle-change curve is a smooth curve, the equivalent concentrated angle change at point m is the quantity

$$-\frac{P\,\lambda}{12}\left[\frac{(w_a)_{m+1} + (w_i)_{m+1}}{(E\,I)_{m+1}} + 10\,\frac{(w_a)_m + (w_i)_m}{(E\,I)_m} + \frac{(w_a)_{m-1} + (w_i)_{m-1}}{(E\,I)_{m-1}}\right]$$

For axial loads, applied at other points than at the end of the bar, the expression for moment is changed, and, where the angle-change curve is not a smooth curve, the equivalent concentrated angle changes are somewhat different in form. The procedure for such cases is not essentially different from that described herein.

Equating the change in slope and the equivalent concentrated angle change at each point on the bar leads to a set of linear equations for the unknowns w_a for each point, since all the other quantities are known.

The equations are of the following form:

$$-\left[1 + \frac{P\,\lambda^2}{12(E\,I)_{m+1}}\right](w_a)_{m+1} + \left[2 - \frac{10\,P\,\lambda^2}{12(E\,I)_m}\right](w_a)_m - \left[1 + \frac{P\,\lambda^2}{12(E\,I)_{m-1}}\right]$$
$$\times\,(w_a)_{m-1} = \frac{P\,\lambda^2}{12(E\,I)_{m+1}}(w_i)_{m+1} + \frac{10\,P\,\lambda^2}{12(E\,I)_m}(w_i)_m + \frac{P\,\lambda^2}{12(E\,I)_{m-1}}(w_i)_{m-1} \quad \ldots (41)$$

Since there are as many equations as unknowns, Eqs. 41 can be solved for the additional deflections w_a.

When the values of w_i are zero at all points, there is a problem of pure buckling, and the equations are homogeneous equations which involve only the unknowns w_a and P, with the constant terms being zero. In order that the set of equations may have a solution different from the obvious one with all the quantities w_a being zero, the determinant of the coefficients must vanish. This leads to an algebraic equation for P in which P appears to some power equal to the number of points on the bar that can deflect. The solution of the equation yields the critical values of P corresponding to the different modes of buckling. The lowest value of P is generally the only one of interest.

From this discussion the reader may be able to see why it is desirable to use the method of successive approximations which avoids dealing with the equations.

Professor Wilbur has given formulas for the equivalent concentrated load when the loading curve is divided into segments unequal in length. These formulas are expressed by Eqs. 23*a* and 23*b* and refer to Fig. 25. In some applications it is desirable to have a formula for the magnitude of R_{ab}, which was not given by Professor Wilbur. This quantity is defined by the following equation, using the same notation that was used by Professor Wilbur.

$$R_{ab} = \frac{l_1}{12\, l_2(l_1 + l_2)} \left[a\, l_2(3\, l_1 + 4\, l_2) + b(l_1 + l_2)(l_1 + 2\, l_2) - c\, l^2_1\right] \, . \, . \, (42a)$$

The equivalent concentration at point b, namely, R_b, which is equal to $R_{bc} + R_{ba}$ may be simplified to the following form:

$$R_b = \frac{l_1 + l_2}{12}(a + 4\, b + c) + \frac{l^2_2}{12\, l_1}(b - a) + \frac{l^2_1}{12\, l_2}(b - c) \, . \, . \, . \, . \, (42b)$$

In general, it is not convenient to use different lengths of segments in the same problem although in some cases it may be desirable to use segments of one particular length for part of the beam and of another length for the remainder. In such cases, one can use the formulas that apply to segments of constant length in order to obtain the equivalent concentrations, by working from both sides at the point where the segments change in length.

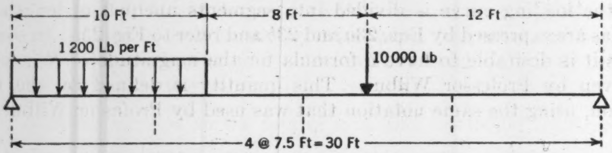
The use of segments of different length always can be avoided by the simple expedient of calculating the proper equivalent concentrations at the chosen division points to account for the actual conditions in the segments between division points. A simple example of such a procedure is shown in Fig. 45, in which moments are calculated for a given distribution of loading. The same type of procedure is used for calculating deflections from angle changes.

Regarding the calculation of the critical load for a bar composed of segments of different moments of inertia, Professor Wilbur has given a correct analysis by means of the usual formal solution of the differential equation. However, it is precisely such an analysis that the writer sought to avoid with the numerical procedure. In a relatively simple problem it is not too complicated a matter to solve the differential equation. For complicated variations in the moment of inertia, however, the formal solution of the differential equation may not be convenient. Even in this problem, the determination of the critical load requires the solution of a transcendental equation which may take considerable time for an engineer unaccustomed to solving such problems. For practical purposes the results so obtained are not of any greater accuracy than those given by the numerical procedure.

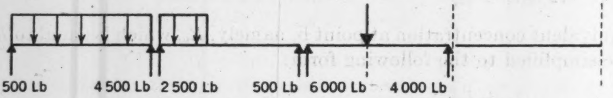
To summarize: Special problems can be solved by use of procedures such as those illustrated by Professor Wilbur, but a different technique is required in each case. With the numerical procedure only one technique is required and it is applicable to all problems.

Mr. Stewart illustrates the use of the numerical procedure in computing constants to be used in his particular method of frame analysis. The writer cannot concur with his statement concerning "basic" constants of beam flexure. The distinction between "basic" and "derived" constants depends entirely upon the point of view of the person making the analysis. In one procedure certain constants may be "basic" and in another procedure entirely different constants may be "basic." From the writer's point of view, the stiffness and carry-over factors in Professor Cross' moment distribution procedure are fundamental and basic constants from which any other constants used in other methods of analysis can be derived by simple equations. One can determine these constants by experiment or by analysis—by experiment, preferably, in cases where there are marked departures from Hooke's law or where there is slip at joints or other phenomena not readily amenable to analytical treatment.

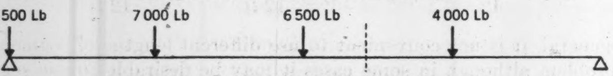
(*a*) Simply-Supported Beam with Loads


10 Ft
8 Ft
12 Ft
1 200 Lb per Ft
10 000 Lb
4 @ 7.5 Ft = 30 Ft

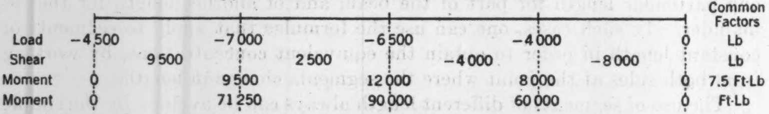
(*b*) Reactions on Sub-stringers


4 500 Lb
4 500 Lb
2 500 Lb
500 Lb
6 000 Lb
4 000 Lb

(*c*) Equivalent Concentrated Loads on Beam


4 500 Lb
7 000 Lb
6 500 Lb
4 000 Lb

(*d*) Calculation of Moment at Division Points


Common Factors
Load −4 500 −7 000 −6 500 −4 000 0 Lb
Shear 9 500 2 500 −4 000 −8 000 Lb
Moment 0 9 500 12 000 8 000 0 7.5 Ft-Lb
Moment 0 71 250 90 000 60 000 0 Ft-Lb

(*e*) Calculation of Moment under Concentrated Load

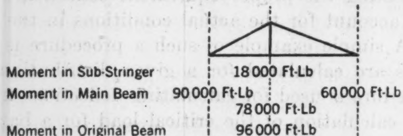

Moment in Sub-Stringer 18 000 Ft-Lb
Moment in Main Beam 90 000 Ft-Lb 60 000 Ft-Lb
78 000 Ft-Lb
Moment in Original Beam 96 000 Ft-Lb

FIG. 45.—TREATMENT OF LOADINGS WITH DISCONTINUITIES BETWEEN DIVISION POINTS

However, it was not the object of the paper to discuss the merits of different methods of analysis of frames. The procedure can be used, as Mr. Stewart himself has shown, to compute the beam constants for any of the methods of analysis.

Mr. Fraenkel's use of the numerical procedure in his analysis of a derrick boom is interesting. It can be seen from his analysis that the effect of the longitudinal force in the boom on its deflection is relatively negligible. This might be estimated in advance, as a very rough preliminary estimate of the critical load for the boom could have been made by assuming some equivalent constant moment of inertia, say, that in the central section. Since the ratio of the actual end thrust to the critical end thrust is very small, the effect of the end thrust on the deflection must be small also.

In connection with Mr. Fraenkel's discussion, it should be noted that the effect of longitudinal thrust cannot be superposed on the effect of lateral loads. Moments must be computed for each complete loading. That is, the effect of live loading cannot be determined from the live loading alone, but must be determined by computing the difference between the effect of the combined dead and live load and the effect of the dead load alone.

The writer wishes to thank Professor Niles for his kind remarks concerning the numerical procedure. As Professor Niles states, for continuous beams the numerical procedure is not directly applicable without some additional work. However, one can compute constants for use in any of the standard methods of analysis for continuous beams—for example, moment distribution, the three moment equation, or the slope deflection method. Then, by use of any of these methods, one can compute the bending moments over the supports. When the end moments are known, the deflection curve for each span can be obtained if it is desirable to do so. When the members are subjected to axial thrusts as well as lateral loads, the calculation of the beam constants is facilitated greatly by the use of the numerical procedure. Mr. Williamson has indicated ably an application to such a problem.

It is always possible to use segments of constant length in the analysis, even when unequally spaced concentrated loads are present. This is illustrated by the problem solved in Fig. 45. Where discontinuities in section occur between division points, one usually can estimate reasonably well the equivalent concentration to use at the division point. When formulas are desired, they can be derived readily by consideration of the sub-stringer between division points. If one finds the equivalent concentration on the sub-stringer at the point of discontinuity (usually by means of the equations applying to segments of unequal length), one then can determine the sub-stringer reactions due to the equivalent interior concentration and proceed from that point with segments of equal length. However, in most cases, taking short segments may make it unnecessary to consider the discontinuities between the division points.

In most practical problems one can use the short-cut procedure for making the correction to the moment diagram without serious error. It is only when a relatively small number of divisions are used that it is necessary to compute the equivalent concentrations and to use them instead of the ordinates to the loading or angle-change curve.

As Professor Niles has indicated, the formulas for equivalent concentrations require division by certain factors. In his numerical problems the writer has taken this factor as a part of the "common factor." In many cases it is very convenient to do so; in others it is worth while to compute the actual equivalent concentration merely to prove to one's self that this quantity does not differ materially from the ordinate to the actual distribution curve.

The writer has no liking for any particular convention of signs. In his own work he uses different conventions at different times. Since deflections are considered positive downward in most engineering literature, they were considered positive downward in the paper. It is simple enough to change the sign convention to one that seems better for a specific purpose if one wishes to do so.

Professor Niles objects to the use of the term "angle change." However, it seems to be descriptive enough of what is meant and it avoids somewhat cacophonous terminology, as for example, "curvature" curve. The term "angle change" is used among structural engineers, and the writer can claim no credit nor take any blame for developing the name. It would be just as acceptable to call the quantity the "rate of slope change."

The numerical value that Professor Niles refers to in Fig. 15(*a*) was computed by means of the following formula: $-404.90 = -\frac{1}{24}(7 \times 803.6 + 6 \times 513.5 - 1 \times 0 + 7 \times 80.36 + 6 \times 91.10 - 1 \times 97.75)$. The writer used this formula rather than the one that Professor Niles used since there is a discontinuity in section of the beam at this particular point. It will be noted that Professor Niles' formula requires multiplying or dividing certain ordinates to the angle-change curve by 10, whereas the foregoing expression uses only the actual ordinates without modification. As Professor Niles has indicated, the difference is of no practical consequence.

The use of the procedure to compute influence lines is illustrated ably by Mr. Weiss. Since deflections of a structure subjected to certain distortions are the influence values desired, the procedure leads directly to influence ordinates. In this connection the influence for moment at a fixed end of a beam can be obtained directly from the calculations of the elastic constants. For example, in Fig. 10 the influence for moment at the left end of the beam, when the left end is fixed and the right end is simply supported, is obtained by dividing the deflections of the various points by the slope at the end of the beam. That is, the influence ordinate for moment at the left end due to a unit load at the center is $\frac{131.3\,\lambda}{181.0} = 0.725\,\lambda$.

When computing influence lines for continuous beams, the writer generally prefers the procedure of introducing the required discontinuity in the span considered, finding the fixed-end moments, and distributing these moments to obtain the moments over each support. Then the deflections of the structure in each span can be determined readily. In certain instances, however, it might be more convenient to use the procedure given by Mr. Weiss, although the combination of the deflections due to the reactions at the various points may lead to results which are the differences between large quantities and which, therefore, may be relatively inaccurate unless intermediate calculations are carried to a large number of significant figures.

Mr. Eremin has illustrated the use of the numerical procedure for a problem in which a bending moment is applied at an intermediate point in a beam.

With a beam of constant moment of inertia, such as Mr. Eremin has considered, the numerical values should be exact since the moment diagram is made up of straight-line parts. It should be exact even if the moment diagram were made up of parabolic segments.

Mr. Gossard uses the procedure to obtain constants for haunched beams with various types of haunches at the ends. His application of the procedure is interesting. However, in view of the complications involved in such a procedure, it seems preferable to the writer to compute the constants for the beam by the numerical procedure directly, rather than to fit the different parts together. Whatever is done will depend upon the designer's personal preference. Mr. Gossard undoubtedly will find extensive use for the tables and curves he has computed.

Mr. Williamson's able discussion is appreciated. His illustration of the calculation of elastic constants for a member carrying axial thrust is a valuable addition to the paper and suggests further applications in the field of aircraft stress analysis. It is not difficult to use the procedure to make analyses of beams and columns in which the stresses go beyond the elastic limit. One determines a relationship between moment and "angle change" for any specified thrust and any given moment, either by means of trial or by a systematic set of calculations which lead to graphical relationships in the form of curves. With these data known, the analysis proceeds in the customary manner, using the relationship between angle change and moment determined from the magnitude of the moment.

With regard to Mr. Oesterblom's comments, the writer would like to state that using the procedure—that is, the actual numerical computation of a problem—is the only way in which the procedure can be learned. That it can be learned in such a way the writer has verified in his teaching.

Professor Dunham has made a careful study of the paper and concludes, from the viewpoint of the ordinary man in a typical engineering office, that the field of usefulness of the method is limited. He would prefer, apparently, to have one basic procedure, without even minor modifications, to fit every case that might arise, although he anticipates use of the method only for structures requiring special refinement and care in their design.

The writer is glad to have the comments of one familiar with the problems of the designer. However, he feels that the average designer does not need all his work laid out for him in such a way that all he has to do is to fill out a form. In any case, the paper is written also for the man who decides what forms are to be filled out. There are many considerations of immense practical importance, besides the technique of analysis, involved in any design problem that would require calculations of the type contemplated in the paper. The average designer, capable of taking these things into account, is also capable of deciding whether to consider straight-line or curved loading diagrams or whether to use merely the ordinates at the division points without referring to any equivalent concentration at all. He has the formulas for any of these possibilities available to him. Exactly what he should do is merely a matter of technique. Professor Dunham may decide on a special technique for the problems that he or his organization ordinarily encounters. Neither the writer nor Professor Dunham can decide on a scheme that also would be best in an application to a problem arising in the design of an airplane, for example.

Regarding specific comments of Professor Dunham's, the writer would like to suggest the following:

It is certainly not obvious that one may find from his first trial deflection curve sufficient data to make a better guess next time. It is extremely fortunate, for most practical cases, that the derived deflection curve is a better approximation to the true curve than the first guess. However, such is not always the case, as is demonstrated by the problems shown in Figs. 19 and 20. Two entirely different possibilities need to be considered if a procedure of successive approximations does not converge to the correct result. First, the procedure may not converge at all, which will be an obvious warning to the designer that he must find an alternative method for his study of the problem; but the other possibility is that the procedure converges to the wrong answer, and the designer has no obvious way of knowing when this is the case. It is the writer's earnest hope that the types of problem in which such eventualities occur are illustrated sufficiently in the text.

It is unfortunate that the writer's presentation makes the method "appear more complicated than it really is." Professor Dunham's comments may help to alleviate matters somewhat. In considering the presentation of the material, the writer attempted to take into account the fact that similar procedures have been presented before and have been used by many people in numerous problems. Furthermore, the simple basis for the procedure appeared to be common knowledge. Only departures from more or less standard routines were described in detail.

Referring to Professor Dunham's comments under 2(*f*), the value of *a* is entirely immaterial. It can be taken as one unit, as in the other illustrative problems. It is taken as a general quantity in the figures referred to in order that the moments, which depend on *a*, should have the proper dimensions. It might have been more consistent to have used such a factor in all the problems where a deflection had to be assumed.

In closing, the writer again wishes to thank all of the discussers for their efforts to add to the value of the paper. His only contribution has been more or less a bookkeeping procedure for using a method of analysis that is relatively old and that has been revised successively by various writers up to the present time. There are countless variations of similar numerical procedures, many of which the writer studied in detail before he arrived at the conclusion that the particular procedure given in the paper led to greater accuracy with less numerical work than any of the others that came to his attention. Even when one does not use the part of the procedure that calls for a calculation of equivalent concentrated loads or angle changes, the results of the calculation are reliable if one takes a somewhat greater number of segments, say, ten to twelve, in the length of a beam. It is hoped that those who have occasion to deal with problems of the flexure of members subjected to direct stresses and lateral load will find the procedure useful and time saving.

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FUNDAMENTAL ASPECTS OF THE DEPRECIATION PROBLEM
A SYMPOSIUM

PAGE

Foreword.
By Maurice R. Scharff, M. Am. Soc. C. E. 1236

Relationship to Regulated Industry.
By J. A. Walls, M. Am. Soc. C. E. 1237

Relationship to Competitive Industry.
By Eugene L. Grant, M. Am. Soc. C. E. 1242

Relationship to Public Works and Government Finance.
By Roy W. Crum and Robley Winfrey, Members, Am. Soc. C. E. 1255

With Discussion by Messrs. Edwin F. Wendt and L. T. Fleming, Anson Marston, William G. Atwood, George E. Goldthwaite, Nathan B. Jacobs, H. J. Flagg, Nelson Lee Smith, C. Beverley Benson, Paul T. Norton, Jr., H. L. Ripley, Carroll A. Farwell, Thomas R. Agg, Conde B. McCullough, W. V. Burnell, Roger L. Morrison, Wallace B. Carr, W. L. Waters, A. G. Mott, David A. Kosh, E. G. Walker, K. Lee Hyder, James T. Ryan, John C. Page, John S. Worley, John F. Miller, Fred Asa Barnes, Robert F. Legget and Erwin E. Hart, Kernan Robson, J. A. Walls, and Eugene L. Grant.

Note.—Published in November, 1941, *Proceedings*.

FOREWORD

BY MAURICE R. SCHARFF,[1] M. AM. SOC. C. E.

The Committee of the Power Division on Depreciation began its work in 1938 as a Sub-Committee of the Committee on Power Cost. It soon became apparent that the subject of depreciation was of interest to more than the Power Division, and eight Divisions became affiliated with it to create a Joint Committee on Depreciation. This Committee was created under the auspices of the Power Division, with the writer as general chairman, and includes in its membership the following Members of the Society:

C. A. Farwell (City Planning Division)
H. O. Locher (Construction Division)
Edwin F. Wendt (Engineering Economics Division)
Roy W. Crum (Highway Division), Robley Winfrey (Alternate)
E. L. Myers (Irrigation Division)
Maurice R. Scharff (Power Division)
E. B. Black (Sanitary Engineering Division)
S. S. Steinberg (Surveying and Mapping Division)
William G. Atwood (Waterways Division)

From time to time the Committee has undertaken to advance its objectives by the preparation of a number of technical papers on the depreciation problem. These have been circulated among members of the Committee; but a conviction has been growing that to advance a program of detailed, technical, and complex discussion is of little value unless the basic and simple definitions underlying it have been established previously; that is, until a common language has been outlined and an agreement has been reached as to its fundamental characteristics.

That is the central purpose of this Symposium: To avoid emphasis on technical details and make a new beginning toward formulating a preliminary agreement on fundamentals, upon which, later, can be built a sturdier "superstructure."

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[1] Lt. Col., U. S. Army, Washington, D. C.

RELATIONSHIP TO REGULATED INDUSTRY

By J. A. Walls,[2] M. Am. Soc. C. E.

Synopsis

Ever since the advent of the active regulation of public utilities, the problem of depreciation has been and now is in a very confused state. In an attempt to find some satisfactory solution to the problem, a great deal of effort has been spent by independent writers, public utilities, and regulatory authorities. Numerous treatises, reports, and books have been written on the subject. Numerous concepts and methods of depreciation have been advanced, and many different solutions advocated. The present situation might appear to be a mere battle of words, or a battle over theories and definitions, which not only fails to clarify the situation but, on the contrary, obscures the basic factors that control the problem.

As a part of the Symposium on Depreciation devoted to a consideration of the fundamental aspects of the problem, this paper approaches the problem from the viewpoint of the actualities of depreciation and it is free of theoretical formulas and definitions. Two current views on the problem are examined and their relative merits are discussed. One of these views is based on the so-called age-life theory and definitions resulting therefrom that involve assumptions as to the future or remaining life of property. The other view is that depreciation cannot be made to suit a definition, but rather the definition must truly express the actualities of depreciation. This view states that depreciation in property as of any time is an accomplished fact, and, to determine that fact, engineering and technical knowledge should be used in the same way as it is employed in connection with the safety standards, replacements of property, and selections of equipment. The writer favors the latter view.

The basic purpose of this Symposium on depreciation has wisely been limited to a consideration of the general aspects of the problem with the aim of exploring the true character of depreciation and the possibility of its measurement by engineering and scientific processes. Engineers are interested as to the rôle that the profession should play in helping to determine the proper treatment of this much discussed problem.

The most difficult aspects of the depreciation problem confronting public utilities today are those in connection with estimating depreciation for accounting and with determining depreciation for rate-making purposes. These difficulties arise from certain existing facts, circumstances, and contentions, the more troublesome of which may be stated as follows:

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[2] Pres., Pennsylvania Water & Power Co., Baltimore, Md.

(*a*) The earnings should be based upon fair return on the fair value of property devoted to public service.

(*b*) It has been suggested that depreciation taking place within any period of time is the difference in fair value at the beginning and end of the period.

(*c*) Some advocate specifically that public utility earnings should be limited to a "fair return" on the original cost less the balance in the depreciation reserve resulting from either the actual charges made or charges that should have been made in the past against operating expense on an age-life method of accounting for depreciation.

(*d*) Most enterprises, including public utilities, pass through three stages during their economic life—namely, (1) the development stage; (2) the normal productive life stage; and (3) the declining stage. The actual capacity of enterprises to make substantial provision for depreciation necessarily will often be more or less limited and difficult in the first and third stages and sometimes, by general business conditions, in the second.

(*e*) The desire is often expressed that the sums, however arrived at, that are currently set aside for depreciation and charged as operating expense should serve the purposes of accounting and rate making.

In attempts to satisfy these various facts, circumstances, and contentions, numerous definitions of depreciation have been formulated, many of which are mere combinations of words that do not contribute sufficiently toward a clarification of the problem. As a consequence, numerous "concepts" have grown up which seem to have for their purpose the conveyance of some factual or real substance to the various definitions of depreciation. One of the present problems is the formulation of a conception of depreciation that will afford a common understanding as to the true scope and character of the problem. Such a concept should recognize that depreciation cannot be made to suit a definition, but rather the definition must truly express the actualities of depreciation.

The approach to a common understanding of depreciation must, of necessity, give recognition to the manner in which, and the purposes for which, the result will be used. For example, depreciation related to reproduction cost new of a replica of an old item of property might properly include certain elements, or the effects of certain causes, that would not properly be included in depreciation related to reproduction cost of old facilities in terms of modern property; and, similarly, depreciation related to original cost may properly consist of still another group of elements.

It has been contended that depreciation at any given time cannot be measured by engineering or scientific methods with reasonable accuracy, and that future life can be estimated with greater accuracy either by deducting from a reasonable original estimate of life expectancy the elapsed life of the property, or by some other method. On the other hand, it has been claimed that the effect of forces that cause depreciation are generally measurable by engineering and scientific methods at any given time with greater accuracy and reliability than remaining life can be estimated.

If the assumptions on which the first group of concepts are founded were correct, then in so far as proponents of this group of concepts are concerned, the problem resolves into an estimate of total life (present age plus remaining life) and the application of mere arithmetical processes in the treatment of depreciation. Since the present age ordinarily may be ascertained, and since it is asserted that the necessary arithmetical processes are available, the indeterminate part of the problem simply reduces to an estimate of the remaining life. Therefore, particular interest is attached to the methods by which, and accuracy and reliability with which, such life can be estimated.

The future life of any element of property will be controlled by the combined action of those forces that cause depreciation—namely, physical (wear and tear, decay, action of the elements, etc.); and functional (inadequacy, obsolescence, changes in the art, changes in demand, and requirement of public authorities, etc.). The forces that cause functional depreciation are essentially economic in character, some of the more important of which are:

(1) The price of economically feasible new facilities, which prices will be controlled by manufacturing methods and general price level;

(2) Greater efficiency, capacity, reliability, better operating characteristics, less operating costs and hazards, or any other similar characteristics that may contribute to the justification of the replacement of facilities; and

(3) The rate of growth of the particular enterprise and of the territory or area in which products are sold or service is rendered.

Do not the same forces which control remaining life also control the amount of depreciation at any given time? However, is there not a very fundamental difference between the problem of measuring depreciation and estimating future life at a given time? In measuring the depreciation existing at a given time, comparatively reliable bases are available or obtainable as to the extent to which the forces causing depreciation have already acted. In estimating future life, however, there is little to serve as a guide as to when, and to what extent, these forces may or may not act. All estimates of future life involve uncertainties and are assumptions, and the farther these estimates are extended into the future, the less reliable they become.

The assertion is frequently heard that future life of public utility property can be estimated on an actuarial or statistical basis in the manner that insurance companies treat human lives. The correctness of such a conclusion seems open to question. Insurance companies deal with large numbers of humans whose life spans, by nature, vary comparatively slowly from generation to generation. They also segregate individuals by occupations, races, sexes, etc., into groups such that the risks may be appraised and made as nearly uniform for each group as possible. The extreme caution with which insurance companies use even their large volume of reliable statistics as a guide to future life is clearly shown by the care they use when they insure life against new and unknown risks.

The forces that control human-life characteristics are obviously not comparable to those that govern life characteristics of most utility property, and for very sound and obvious reasons. Consider, for example, the electric utility industry, which has grown from relative insignificance to its present status during the past several decades. Coincident with, and inseparably related to, this growth is the fact that the facilities that comprise the industry's property have been continually improved. Many of these improvements have resulted in functional depreciation, which has been one of the predominant causes of retirement of property in the past. Other improvements have resulted in a change in life span of facilities.

At the same time, the improvements in most property have made it impossible for the life of one generation of property to serve as a reliable guide to the life of a succeeding generation of property. Therefore, it may be true that in the case of most electric utility facilities, statistical treatment will not be possible until statistics are available of lives that have not been affected by improvements.

In the case of a relatively small proportion of the facilities of most utilities, proper statistical data may be a reliable guide to future life. However, with the possible exception of communication utilities, this proportion susceptible of statistical treatment is not a large percentage of the total property and thus does not carry much weight in the problem as a whole.

A general conclusion as to estimates of future life for most electric utility property might be stated briefly. Such estimates, even for the near future, have limited accuracy and reliability, and, when extended into the more distant future, they would appear to have even less accuracy and reliability. Even if future life could be determined with absolute accuracy, it is not, of itself, the measure of depreciation. An estimate of future life is merely an element to be considered to the extent that it may be a measure of future usefulness, but it should be recognized that future or remaining life is not necessarily synonymous with future usefulness.

The problem of measuring depreciation at a given time is fundamentally different from that of estimating future life in years. The factors that have caused depreciation are well known to engineers, for these are constantly applied by them in selections and replacements of facilities. Measurements of depreciation, like the selection and replacement of property, are essentially comparative procedures in which an item of existing property is compared in a proper manner with an available modern item of property. In such a comparison, the various points of difference such as efficiency, reliability, etc., are evaluated. Technical data and experience are available that can be utilized in making such valuations. There are also numerous technical and scientific data, tests, and procedures that are valuable aids in the determination of physical condition. An evaluation of the foregoing having been arrived at, the result must be expressed in terms appropriate to the use for which the determination is made.

The ascertainment of depreciation may well be a subject for engineering investigation and careful exercise of judgment rather than something that can be treated by those not familiar with engineering matters or something that can be solved by using perhaps the simplest arithmetic. Replacements and selections of property often require the exercise of sound engineering judgment, and decisions are not always easy to make; but such applications of judgment are not avoided through the adoption of simple methods that may lead to unsound results. The increasing complexity of modern manufacturing processes requires the development of engineering ability to solve the problems involved. Cannot this same engineering ability be devoted to the more accurate determination of the condition of equipment and depreciation? Therefore, is there a justification for the conclusion often hastily drawn that measurements of depreciation cannot, with study, be made with the needed accuracy and reliability?

There would seem to be no adequate justification for treating the matter of depreciation in a superficial manner merely for the purpose of attaining alleged simplicity. In estimating the rate at which depreciation may be expected to progress in the future, it seems proper that the estimated rate for the immediate future should be guided by the rate at which depreciation has progressed in the immediate past, modified by judgment as to the conditions and circumstances expected to prevail. This would seem a more accurate and reliable guide than reliance on estimates of future life based on assumptions and fragmentary statistics from the distant past when it is known that the prevailing conditions are not comparable therewith.

The problem that lies before engineers seems to be rather definite and is certainly pressing: First, a sound basic concept or common understanding of the true meaning and scope of depreciation must be developed; second, an orderly and basically sound procedure for measuring depreciation must be formulated that will use the vast volume of technical and scientific data and methods available, and, above all, one that will yield comparable results under comparable conditions and circumstances; and third, an orderly and basically sound procedure of deriving estimated rates of depreciation for the future from measurements of past depreciation must be formulated. None of these suggestions is intended to carry the implication that engineers must or can reduce the treatment of depreciation to a simple arithmetical process which will eliminate or even reduce the quality or amount of sound judgment required in the treatment of the problem.

RELATIONSHIP TO COMPETITIVE INDUSTRY

BY EUGENE L. GRANT,[3] M. AM. SOC. C. E.

SYNOPSIS

Writers on depreciation seem to agree on nothing except that other writers on the subject are somewhat confused. Many apparent differences of opinion on this controversial subject might be reconciled by recognition of the following two points:

1. A clear differentiation should be made between three fundamentally different depreciation concepts. These might be described briefly as the popular concept, the accounting concept, and the appraisal concept.

2. Depreciation estimates will enter in different ways into different kinds of business decisions; it is necessary to examine the specific alternatives presented in any case to see what depreciation estimates are relevant. All business decisions are between alternatives for the future; the only possible differences between any such alternatives are future differences. What is past is relevant only because of its possible aid in forecasting the future.

This paper illustrates these two points by brief discussions of the relationship of depreciation to five types of problems existing in competitive industry, as follows:

(*a*) Engineering economy studies relative to a proposed new investment;

(*b*) Engineering economy studies relative to proposed retirements of machines and structures which are still capable of rendering service;

(*c*) Establishing selling prices;

(*d*) Making appraisals based on replacement cost new; and

(*e*) Shaping public tax policy to recognize the national interest (particularly the national defense interest) in private decisions on plant expansion and plant retirement.

THREE FUNDAMENTAL DEPRECIATION CONCEPTS

A prerequisite for any discussion of depreciation principles should be a reading of the explanation, in the treatise by J. C. Bonbright entitled "Valuation of Property,"[4] of the different concepts attached to the word "depreciation." One concept involves the popular meaning of the word; two others involve special technical meanings. Failure to realize the fundamental differences between these three meanings has been responsible for much confusion in the literature of this subject. The concepts are: Decrease in value, amortized cost, and difference in value between an existing old asset and an hypothetical new asset taken as a standard of comparison. (Bonbright gives a fourth basic concept of depreciation as "impaired serviceableness." As he states, this is not a value concept at all although it has sometimes been used to confuse the issue in value determinations. "As good as new" is not the same as "as valuable as ever." Impaired serviceableness is only one of many possible reasons for fall in value.)

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[3] Prof. of Economics of Eng., Stanford Univ., Dept. of Civ. Eng., Stanford University, Calif.

[4] "Valuation of Property," by J. C. Bonbright, McGraw-Hill Book Co., Inc., 1937. See particularly Chapter X.

Decrease in Value.—This implies that value is somehow determined at two different dates. The value at the later date subtracted from the value at the earlier date is the "depreciation" regardless of what combination of causes may have been responsible for the value change. When "depreciation" is used in everyday speech, this is the meaning usually implied; it is also implied by most dictionary definitions. It should be noted that this meaning may be applied to various concepts of value. As explained by Bonbright, the most useful concepts of value are "market value" and "value to the owner."

Amortized Cost.—This is the concept of depreciation implied by the usual accounting practice. From the viewpoint of accounting, the cost of an asset is a prepaid expense to be apportioned among the years of its life by some more or less systematic procedure. The controversial questions here are what the estimated life should be and how the apportionment should be made. It should be emphasized that it is cost, not value, which is apportioned in orthodox accounting. Although the phrase "book value" is commonly used to describe the difference between the cost of an asset and the total of the depreciation charges made to date against the asset, this difference is more accurately described as "unamortized cost." Confused reasoning may result from a literal interpretation of those classifications of accounts which define depreciation as "loss in value." Only if value is defined in what might be called a neutral sense, as any money amount that may be associated with property, can it be said that the accounts really record value.

Difference in Value Between an Existing Old Asset and an Hypothetical New Asset Taken as a Standard of Comparison.—This is the appraisal concept of depreciation. An upper limit on the value of an old asset to its owner may be determined by considering the cost of reproducing the service which the old asset is expected to perform with the most economical new asset available for performing the same service. The present cost new of this hypothetical new asset is the depreciation base. The depreciation deducted from this to determine the value of the old asset is a measure in money terms of the inferiority of the old asset to the hypothetical new one; the old asset may be inferior for a number of reasons such as shorter life expectancy, higher prospective annual disbursements to get the service performed, and lower prospective annual receipts. The writer believes that it is in this field of measurement in appraisal depreciation that engineers can make their greatest contribution to theory and practice. Such measurements, which are primarily an engineering function, can shed light on comparisons of alternative methods of depreciation accounting. Moreover, the appraisals of which they are a part often may serve as the only criteria by which decrease in value (depreciation in the popular sense) may be estimated.

Despite the frequent confusion among them, all three of these depreciation concepts are useful. Any suggestion that two of them be given different names would doubtless be futile. Throughout the remainder of this paper they are called "popular depreciation," "accounting depreciation," and "appraisal depreciation," respectively. Some fundamental differences among these three concepts are revealed in Example 1.

Example 1.—A man bought a new home in 1932 for $6,000. In 1940 he sold it for $6,000. In the intervening years he rented it to a tenant for part of the time; and in reporting his taxable income for this rental period to the government, he made an annual depreciation deduction of $125. This was based on a $1,000 lot on which no depreciation was charged and a $5,000 house on which straight-line depreciation was calculated over an estimated 40-yr life. The purchaser arrived at his $6,000 bid by estimating the cost of buying a similar lot at $1,000 and the cost of building a comparable house in 1940 at $6,500; from this total of $7,500 he deducted $1,500 because of the shorter life expectancy, higher early maintenance costs, and less modern design of an 8-yr-old house.

In this incident the popular depreciation was evidently zero, as the market value in 1940 was the same as in 1932. The accounting depreciation was $1,000—that is, $\frac{8}{40}$ of $5,000. The appraisal depreciation was $1,500.

Increase in residence construction costs between 1932 and 1940 had compensated for the causes making for decline in value so that there was no popular depreciation. Changes in price levels had no effect on accounting depreciation, as the accepted principles of accounting call for basing depreciation charges on cost. Price level changes did not affect appraisal depreciation for quite a different reason; they had already been taken into account by using the present cost of a new asset as the depreciation base.

ENGINEERING ECONOMY STUDIES RELATIVE TO A PROPOSED NEW INVESTMENT

Answering the question "Will it pay?" with respect to a proposed plant investment may involve the conversion of a first cost, P, an estimated life, n, and a salvage value, L, to an equivalent uniform annual cost. This requires the use of an interest rate, i, which should be the minimum rate of return that will make the investment attractive. This minimum attractive return should depend on both the prospective return obtainable from other investments and the apparent risk of the particular investment in question.

With zero salvage value, the equivalent uniform annual cost is obtained by multiplying the first cost by the capital recovery factor, $\frac{i\,(1+i)^n}{(1+i)^n-1}$. With a salvage value, the annual cost of capital recovery is

$$C = (P - L)\,\frac{i\,(1+i)^n}{(1+i)^n - 1} + L\,i \qquad (1)$$

It may be shown readily that this is equal to the sum of interest on first cost and a "sinking-fund depreciation" annuity, provided the interest rate used on first cost is also used for the sinking fund. Thus the annual cost of capital recovery may be described as interest plus depreciation.

As the two preceding paragraphs describe conventional theory in engineering economy, they require no detailed explanation. However, a point not always understood should be emphasized—namely, that this equivalence calculation simply states: "Here is a uniform series of payments which, with the final salvage value, would serve to pay back the first cost in n years with the minimum attractive rate of return i." The calculation is not concerned with the central problem of depreciation accounting, the allotting of cost among a number of different years. The equivalence holds good regardless of what plan of accounting is to be used and has no relation to any actual or hypothetical sinking fund for the replacement of invested capital.

Although such a capital recovery calculation may be thought of as the amortization of cost with interest and thus appears to combine a recognition of the time value of money with the viewpoint of accounting depreciation, it is really the viewpoint of popular depreciation—prospective fall in value regardless of cause—which is relevant. For this reason, estimates of life and salvage value in economy studies to determine whether a proposed plant investment will pay may properly differ from the estimates of life and salvage value to be used in the depreciation accounting if the investment actually is made.

Example 2.—For instance, imagine that the home owner in Example 1 had anticipated that a house purchased in 1932 would be salable a number of years later for the same price. Obviously this forecast of no popular depreciation would have been relevant in an economy study comparing home ownership and renting. If price level changes are forecast as likely to influence possible resale value of a property, this forecast should influence the annual cost of capital recovery used in a study to determine the economy of acquiring that property. However, the forecast of price level changes has no place in depreciation accounting as it is ruled out by the conventions of accounting.

More important than this in many economy studies is the forecast that long before the date of its retirement the value to the owner of a proposed machine or plant will be very low. This forecast may be based on past experience that improvements in design render plant obsolete in a short time, or on an expectation of the termination of the present demand for the product, or on other reasoning. It is a common policy, particularly in the manufacturing industries, to require that new machinery "pay for itself in three years" or in some other short time from one to five years. It is also common for the machine, installed on the basis that it must pay for itself in three years, to have its depreciation accounting based on a 10-yr life. Past experience might show that similar machines had served fifteen years before retirement. Recent regulations of the Bureau of Internal Revenue, U. S. Department of the Treasury, have had the effect of bringing the accounting depreciation—at least for income tax purposes—more closely into line with past evidence of total realized life, and have thus increased the discrepancy between the life assumed in the economy study and that assumed in the subsequent depreciation accounting.

ECONOMY STUDIES FOR PROPOSED RETIREMENTS

Perhaps most property retirements in modern industry are retirements of machines and structures which are still capable of rendering service. Such retirements are made because of some one's decision that it pays to make them. Unless such decisions are made merely on a "hunch" basis, they require an estimate of the cost of extending the service of the old asset for some specified future time.

It should be obvious that this "cost" is a money difference between disposing of the asset at once and disposing of it at the specified future date. The only "depreciation" involved is the difference between the net amount realizable from the immediate sale of the asset and the net amount realizable from its future sale. This is depreciation in the popular sense of decrease in value; the relevant "value" is net price in a second-hand or scrap market, or value to the owner for some other purpose. Accounting depreciation based on the past cost of the asset is irrelevant; so also is the difference between the unamortized cost ("book value") and the present net realizable value. The money originally spent for the asset has been spent regardless of what decision is now made about continuing it in service.

Although this should be obvious, unfortunately it is not. The literature of replacement economy is full of formulas which, as part of the attempt to determine whether a proposed replacement will pay, add to the first cost of the proposed new asset the "loss" which the books would record on the disposal of the old asset—that is, the difference between the "book value" and the net realizable value. (Although under individual unit depreciation accounting, if the "book value" of a retired asset is greater than its net realizable market value, the difference must be written off on the books [either as a charge to current expense or in unusual cases as a charge directly to surplus]; the usual description of this difference as a "loss" which occurs at the time of the retirement is misleading. The so-called "loss" is simply a portion of a prepaid expense which would have been allotted against past accounting periods if correct forecasts of life and salvage value had been made; because it is impracticable to go back and correct previous accounting statements, this must be charged off at the time of the retirement. Under group depreciation accounting, no such "loss" entry is made as, presumably, a retirement short of the estimated average life of a group of assets will be balanced by another at more than the average life. This viewpoint is implied in recent regulations of the Bureau of Internal Revenue; "losses" on retirements made sooner than the estimated average life are no longer allowable as tax deductions except in extraordinary circumstances.)

The most celebrated of these formulas[5] was first published in 1923 and has been copied year after year since that time in textbooks and handbooks; it still (1941) seems to be "going strong" after eighteen years. One proposed formula[6] went so far as to charge against the proposed new asset not only the "loss" on the current retirement but also a portion of the "losses" on previous retirements of a series of predecessor assets which performed the same service!

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[5] "Formulas for Computing Economics of Labor-Saving Equipment," *Mechanical Engineering*, Vol. 45, September, 1923, p. 564.

It is the writer's impression that confused reasoning on this matter is not so common in industry as would be suggested by the high percentage of the replacement formulas which contain this mistake. Nevertheless, it is common enough; there is no question that high book value of old assets often operates as a deterrent to economical retirement. The writer believes that this error is more common in large concerns than in small ones. The control mechanism of a large corporation often depends on accounting figures to such an extent that its employees impart a fictitious reality to the apparent evidence of the accounts. The owner-manager of a small concern is likely to be more conscious of his competitive position and of the actual differences between alternatives.

Modern writers on semantics have emphasized how readily one may be misled by words or phrases that have unfavorable associations. Two such phrases may be partly responsible for some of the confusion that exists about depreciation. One is the definition of accounting depreciation as "loss" in value; the other is the common description of unamortized cost as "unrecovered investment." "Loss" has an unpleasant sound which somehow makes depreciation seem worse than other operating expenses. "Unrecovered investment," despite its sanction by the Bureau of Internal Revenue, is an inaccurate description of "unamortized cost." In some cases the "investment" in a machine actually may be "recovered" in its first few months of service, and in other cases it may never be recovered; neither fact has any relation to the depreciation that has been recorded on the books. (Even if "book value" really were unrecovered investment, it would not be relevant in a replacement economy study.)

DEPRECIATION IN PRICING POLICY

In a competitive market, prices are a result of the bargaining of possible buyers and possible sellers. Depreciation as one element of cost enters into the bidding of the possible sellers. According to economic theory, under free competition prices tend to reflect costs to the marginal producer—that is, the highest cost producer who is actually able to make a sale. The supply side of the market depends on decisions by individual possible sellers as to whether or not they will sell at some given price. In practice, the "cost" to any possible seller which should be relevant in his decision whether or not to sell at any given price depends on whether or not he has the necessary plant to produce the product. Thus two types of decisions involving depreciation enter into the bidding of possible sellers.

One is a decision whether prospective prices are high enough to justify the provision of new plant facilities. As already pointed out, in many competitive industries new construction is not undertaken unless there is the prospect of capital recovery within a much shorter period than the estimated total service life of the plant. A price higher than production cost by the most economical new method, including in production cost a much larger depreciation allowance than will subsequently be included in the cost accounts, tends to attract new capacity into the market. In this manner, improvements in design which make it possible to produce more economically tend to reduce the return to the owners of older, less economical plant.

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[6] "Economic Life of Equipment," by H. O. Vorlander and F. E. Raymond, *Transactions*, A.S.M.E., Vol. 54 (1932), RP-54-2.

The other decision is whether or not a given market price justifies continuing to operate an existing plant. If the owners of an old plant are faced with a choice between the alternatives (*a*) to sell at slightly more than the increment cost of production and (*b*) to shut down the plant, it is to their advantage to choose alternative (*a*). The only depreciation properly included in this increment cost is the difference between the decrease in value, that is, depreciation in the popular sense, which will take place if the plant is operated and if it is shut down. In the common instances in which decrease in value is practically independent of operation, this increment of depreciation entering into pricing policy will be practically nothing.

The danger that excess productive capacity in an industry may result in "cut-throat" competition in which prices are forced down to increment cost of production has its influence on economy studies relative to proposed new capacity. The "pay-for-itself-in-three-years" requirement is often partly attributable to this hazard.

Under the imperfect competition that actually exists in most competitive industries, prices in an excess capacity industry may not actually fall to the increment cost to the marginal producer. Producers may prefer a partial shutdown rather than a sale close to increment cost of production. If total allocated book cost determines the point at which this shutdown occurs, it is accounting depreciation which actually is relevant. It should be emphasized that it is what each prospective seller thinks at the time that his relevant costs are, not what subsequent events prove them to have been, that determines his pricing policy.

Because depreciation may enter into competitive prices in such different ways, it is desirable that contracts in which the determination of price involves the determination of "cost" should not depend on a competitive price analogy for the establishment of depreciation cost. Such contracts should provide definite money amounts for depreciation (or a means of readily determining the money amounts) and thus avoid the necessity for adjudicating disputes which inherently are extremely difficult to adjudicate.

DEPRECIATION IN APPRAISALS BASED ON REPLACEMENT COST

An appraisal of an asset on the basis of its replacement cost should give an answer to the question, "What price for this asset would provide its future services at the same annual cost as would be obtainable from the most economical new asset now available to reproduce the service?" The depreciation deduction from the price of the hypothetical new asset should be a money measure of the inferiority of the existing old asset to the most economical new one. The old asset is usually (although not always) inferior in having shorter life expectancy; it may also be inferior in requiring greater annual disbursements to obtain the service performed or in having the prospect of lower annual receipts.

The theory that should govern this depreciation measurement is the same theory that applies to a somewhat comparable problem in engineering economy—that of determining the investment justified to secure an increase in life or a decrease in annual disbursements. Consider, for instance, the following problem.

Example 3.—A structure, A, has a first cost of \$1,000, a prospective life of ten years, and involves estimated annual disbursements of \$500. An alternative structure, B, has a prospective life of twenty years with estimated annual disbursements of \$500. With interest at 7%, what is the maximum amount that could be paid for B without realizing higher annual costs than A?

This problem may be solved by letting x = the first cost of B which would just make annual costs "break even." Each investment may be converted into an equivalent annual cost by multiplying by the appropriate capital recovery factor. Equating the annual costs for A and B: $\$1{,}000\ (0.14238) + \$500 = x\ (0.09439) + \$500$. That is, $x = \$1{,}508$.

Because of the time value of money, in this problem a 100% increase in life justifies only a 50% increase in investment. Only with 0% interest would it pay to spend twice as much for a 20-yr service as for a 10-yr service. The interest rate used in this calculation should be the minimum attractive rate of return on investment, considering the risk and the return available from other investments. Obviously this return should be greater than the bare cost of borrowed money. The higher the minimum attractive return, the lower the valuation placed on additional life expectancy.

A prospective saving in annual disbursements may justify a much higher investment. For instance, if structure B had the added advantage of annual disbursements of \$300 instead of \$500, the equation would be: $\$1{,}000\ (0.14238) + \$500 = x\ (0.09439) + \$300$; or $x = \$3{,}627$.

Example 4.—In applying this viewpoint to the appraisal of existing old assets by comparing them to hypothetical new ones, it is the value of the shorter-lived asset that is the unknown in the equation. For example, the most economical new asset that might be used to perform the service of an existing old one has a first cost of \$1,000, an estimated life of twenty years, and estimated annual disbursements of \$500. The old asset has a remaining life expectancy of ten years and estimated annual disbursements of \$500. With interest at 7%, how much could be paid for the old asset to just "break even" on the annual costs with the new one?

Let x = the value of the old asset to "break even"; $x\ (0.14238) + \$500 = \$1{,}000\ (0.09439) + \$500$; and $x = \$663$.

If the new asset involves lower prospective annual disbursements, this decreases the value of the old asset. For instance, if the annual disbursements with the new asset are \$450 instead of \$500, $x\ (0.14238) + \$500 = \$1{,}000\ (0.09439) + \$450$; and $x = \$312$.

If the annual disbursements for the new asset are \$405 or less, the equation shows that the old asset has no value in comparison with the new one. That is, on the basis of a 20-yr life for the new asset it would pay to make an immediate replacement.

The preceding paragraphs explain a theory on which depreciation estimates in replacement cost appraisals might reasonably be made. The writer believes that so-called reproduction-cost appraisals in competitive industry have little meaning unless they are really based on the cost of replacing the service. He realizes, of course, that few appraisals are actually made in this manner.

The elements in such an equal-annual-cost appraisal are: (1) The expected service to be performed; (2) the selection of the hypothetical new asset to perform it and the estimate of its cost; (3) the estimation of the remaining life expectancy of the old asset; (4) the estimation of the life expectancy of the new asset; (5) the selection of an interest rate or minimum attractive return; and (6) the estimation of the saving in annual disbursements or the increase in annual receipts to be expected from the new asset, or both.

Each of these elements deserves a brief comment:

(1) The service should be the one that is to be performed in the future by the old asset. The result of the appraisal must be recognized as relevant only to the value, to the owner, of the asset for this particular service. For some other service a different conclusion regarding value might be reached.

(2) The reason for the relevance of replacement cost in an appraisal is its relation to the cost of replacing a service. This implies that the replacement cost used should be that of a new asset which would provide the contemplated service most economically. If there have been improvements in design or changes in the service performed, the new asset will not be physically identical with the old one; often it will be entirely different.

(3) The estimate of remaining life expectancy does not necessarily depend on age. As stated at the beginning of this paper, the only possible differences between any two alternatives for the future are future differences. It is the look forward that is relevant in the appraisal; the look backward (at age) is only useful if it helps in the look forward (at life expectancy). Sometimes—if the only event limiting the life of either an old or new asset is the prospective termination of the service—the life expectancy of the old asset may be the same as that of any new one that might replace it. In such cases, if there is no difference in prospective annual disbursements or receipts, the old asset has no value-inferiority to the hypothetical new one and the appraisal depreciation is zero. Sometimes it may seem likely that it will pay to retire the old asset very soon. It is the prospective date when it would pay to retire the asset from this particular service that is relevant in estimating its remaining life, rather than the prospective date of termination of its physical life. Where the estimate is being made from a mortality curve, the curve will usually indicate the remaining life expectancy of an old asset to be greater than the difference between the average life indicated by the curve and the present age of the asset.

(4) If a mortality curve based on past experience is used as a basis for the estimate, the average life indicated by the curve should appropriately be used as the expected life of the hypothetical new asset. On the other hand, if the prospect of obsolescence of a new asset is so great that an economy study rela-

tive to its proposed installation might reasonably be based on the requirement that it must "pay for itself" in a short time, it would seem appropriate to use that same short time as the expected life of the hypothetical new asset.

(5) The recognition of the time value of money is simply the recognition that an extra investment to secure longer life or lower operating disbursements or increased receipts is not justified without the prospect of a return. The interest rate appropriate to use should be the minimum attractive return which will justify an investment considering the risk. Usually this will be higher than either the cost of borrowed money or the interest rate obtainable on a conservatively invested sinking fund.

(6) It is the prospective saving in annual disbursements from the most economical replacement asset that is often given inadequate consideration in commercial appraisals. This saving may result from a combination of causes which might be described as wearing out, obsolescence (or obsoleteness), and inadequacy. Even if the new asset selected as a standard of comparison is identical with the old asset, the new asset is likely to require lower annual operation and maintenance expenditures because of its newness and the lack of wear and tear. If the new asset is superior because of improvements in design, it will have lower annual disbursements for operation. If the service to be performed has changed since the old asset was installed, a new asset will be better adapted to the present service and will cost less to operate. If all of these influences are present, it should be emphasized that they combine to create a value-inferiority of the old asset to the hypothetical new one.

An apparent paradox here is that the less the contempleted service, the less will be this value-inferiority of an existing old asset to a hypothetical new one. If an old plant is to be used merely for stand-by purposes, no saving at all in annual disbursements might be possible from the substitution of a new plant. However, in this connection it should be emphasized that the appropriate substitute plant for stand-by service may be quite different from that for regular operation; the price of similar machinery in the second-hand market might provide the relevant "replacement cost" from which no depreciation deduction would need to be made. It is also true that the alternative of abandoning the stand-by service may place an upper limit on the value of that service.

Subject to certain very important limitations, the preceding discussion of appraisal depreciation on an equal-annual-cost basis may be described as an advocacy of sinking-fund depreciation in appraisals—or present worth depreciation as it is somewhat more accurately called.[7] These important limitations are:

(*a*) An interest rate equal to the minimum attractive return sufficient to justify an extra investment is usually higher than the rate commonly used in sinking-fund depreciation. This tends toward higher valuations than usually are obtained by the sinking-fund method.

(*b*) A recognition that it is the comparison of the remaining life expectancy of the old asset with the life expectancy of the new asset will often tend to give higher valuations than the sinking-fund method based on the present age of the old asset.

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[7] "Engineering Valuation," by Anson Marston and T. R. Agg, 1st Ed., 1936, McGraw-Hill Book Co., Inc., New York, N. Y.

(*c*) The use of the cost of the most economical new plant to provide the service as the depreciation base rather than the cost of reproduction of a plant identical with the old one, and consideration of the prospective saving in annual disbursements from such a plant, will generally tend toward much lower valuations than will be found in conventional appraisals by the sinking-fund method.

The writer believes that in competitive industry this last limitation is likely to be the most important one, and that appraisals made on a true equal annual cost basis are likely to give values that not only are below reproduction cost appraisals on a sinking-fund basis but also are below such appraisals on a straight-line basis.

A word of caution is necessary regarding the addition of the appraised values of separate machines and structures in a manufacturing plant to reach an appraised value for the plant as a whole. This will often give an overvaluation; consideration of the alternative of replacing the plant as a whole with one of quite different design may give a much lower equal-annual-cost value for the entire plant than consideration of separate alternatives of replacing parts of the plant without changing its basic design or arrangement.

THE NATIONAL INTEREST IN THE EFFECT OF DEPRECIATION TAX POLICY ON PRIVATE DECISIONS REGARDING PLANT EXPANSION AND RETIREMENT

Defense needs have made every one conscious that there is a national interest in the nation having the best possible plant for the production of goods and services. This means the most economical plant for the bulk of production, and stand-by capacity of less economical plant to meet emergency needs. Tax policy with regard to depreciation may have two undesirable effects: It may handicap plant expansion and the substitution of more economical plant for less economical plant; and it may make it pay to retire plant which might subsequently be useful.

If the decrease in value of an asset is much greater than the allowable tax deduction for depreciation, what appears for tax purposes to be income may actually be recovered capital. Income and excess profits taxes may take as much as 62% of this recovered capital. The prospect of this can, and does, operate as a deterrent to plant expansion. This fact was given belated recognition in the excess profits law which permitted amortization of new plant for tax purposes in five years when it was certified by the military services to be necessary for defense needs.

Property taxes may make it pay to retire plant that would otherwise be kept to permit expansion of production during emergencies. The policy of basing allowable depreciation rates for tax purposes on the evidence of actual retirements may have a similar effect.

There is serious need for a study of the effect of depreciation tax policy on business decisions in private industry.

A SUGGESTION REGARDING DEPRECIATION ACCOUNTING

Many of the matters covered in this paper do not bear on the question of choice of the best of the various possible methods of making the time distribution, on the books of account, of the prepaid expense of fixed assets. This paper, rather, is concentrated on a discussion of many depreciation problems in which no scheme of depreciation accounting has any relevance. Nevertheless certain points brought out in this paper bear definitely on the selection of a depreciation accounting policy. These points may be summarized as follows:

1. In an industry in which production methods are rapidly improving, appraisals based on the cost of reproducing the service may show that fixed assets have a very low value to their owners within a short time after their installation even though it may be many years before their probable retirement.
2. The prospect that this will occur is commonly recognized in competitive industry by a refusal to purchase new fixed assets unless it seems likely that they will "pay for themselves" in a much shorter time than will be used as their lives in subsequent depreciation accounting.
3. Because of this, the limit on competitive prices which exists due to the possibility of production at the "cost" obtainable in a new plant of the most economical design practically depends on a "cost" which includes a greater depreciation allowance than is commonly included in depreciation accounting.
4. In contrast to the influence on competitive prices of this high prospective depreciation of a proposed new plant, the influence of depreciation of an old plant on competitive prices is likely to be relatively small.
5. High book values for old assets operate to discourage (irrationally, to be sure) replacements that are really economical.
6. Low depreciation rates allowable in income tax returns for new assets operate as an obstacle (an entirely rational one) to the installation of new plant.

The merits of alternative possible depreciation accounting methods may be compared by various standards. There is no inherently correct way to apportion depreciation on the books of account; it all depends on the objective being sought.

Maurice Scharff,[8] M. Am. Soc. C. E., has suggested that accounting depreciation in regulated industry be made consistent with decrease in value—that the accounting depreciation assigned to any period be based on the difference between the appraised values at the beginning and end of the period. In competitive industry the irregular operation of the forces influencing value changes would doubtless result in this method showing erratic fluctuations in annual depreciation. As some of the uses of accounting make it desirable that there be reasonable regularity in the annual depreciation charge, and that this charge be not subject to the suspicion of manipulation which would surely exist if appraisals were controlled by the management, this suggestion has obvious disadvantages.

Nevertheless it has much merit as a basis for judging the adequacy of the depreciation charges made under some regular systematic scheme of depreciation accounting. If the "decrease in value" shown by an appraisal based on the cost of reproducing the service with the most economical modern plant should be much greater than the accounting depreciation, one might reasonably infer that the accounting depreciation was inadequate.

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[8] *Transactions*, Am. Soc. C. E., Vol. 104 (1939), p. 1132; also "Public Utility Depreciation," by Maurice Scharff, *Columbia Law Review*, June, 1938.

It is the writer's guess that engineering studies in the manufacturing industries of the value-inferiority of the existing old plant to the most economical new plant would indicate that the decrease in value in the early years of life of a plant was much greater than the accounting depreciation in most of the cases where straight-line depreciation accounting is based on actual retirement practice. This guess is consistent with the usual practice of manufacturers of requiring the likelihood of a short capital recovery period as a prerequisite to the installation of new machinery.

Changes in depreciation accounting practice in the United States have seemed to follow criticisms of current methods made on theoretical grounds. As writers on accounting theory began to point out that the distinction between a capital and a revenue expenditure was one of degree rather than of fundamental character, depreciation accounting began to develop. When the arbitrary and irregular depreciation charges made by some corporations were criticized, a trend developed toward the adoption in each industry of standard rates for each class of assets used in the industry. A current theoretical criticism is that rates are not based on actual retirement practices, and the suggestion is made that they should be based on the available evidence of actual realized lives such as might be obtained by mortality studies using the annual rate method or turnover method. Recent policies of the Bureau of Internal Revenue with regard to depreciation deduction in income tax returns are based on the viewpoint involved in this criticism. The effect of these policies has been to make substantial reductions in the annual depreciation expense shown on the books of many taxpayers, particularly those in the manufacturing industries.

A first impression might be that, because this is a trend to substitute depreciation rates based on the facts of realized lives of property for arbitrarily guessed higher rates, it is desirable. The writer holds a contrary view, particularly with respect to the manufacturing industries. That is, he believes that the decrease in value in the early years of life of much manufacturing equipment is so great that straight-line depreciation based on actual retirement experience is inadequate in those years. A more realistic change in current depreciation accounting policies would be to use a high rate in the early years of life of an asset followed by a lower rate in later years. This would bring depreciation accounting more closely into line with decrease in value, and would bring the accounting estimate of profit more closely into correspondence with the estimates made in the economy study on which the plant investment was justified in the first place.

Factual studies in various industries by engineers whose familiarity with the most economical production methods now available enabled them to estimate the value-inferiority (that is, the appraisal depreciation) of existing plants would permit determination of the past decrease in value—the popular depreciation. Such studies would permit judgments on the adequacy of the accounting depreciation which would serve to confirm or disprove the writer's present views.

RELATIONSHIP TO PUBLIC WORKS AND GOVERNMENT FINANCE

BY ROY W. CRUM[9] AND ROBLEY WINFREY,[10] MEMBERS, AM. SOC. C. E.

SYNOPSIS

Physical properties depreciate regardless of whether they are privately or publicly owned or whether operated for a profit or solely for nonpecuniary purposes. It cannot be escaped, but generally public accounting methods and policies have given it no attention. Accounting for depreciation and the use of depreciation estimates in management are the possible points on which procedures involving public works and government finance may differ from those followed in privately owned industries and business. The object of this paper is to illustrate how depreciation should be used in the management of public works and enterprises operated by governments.

The administration and management of publicly owned utilities, which are supported entirely by user charges, such as water works and light and power plants, should involve in general the same principles and financial procedures as privately owned properties, and will not be discussed herein. It is true, however, that many of them are not operated under full adherence to these principles.

In the management of certain other publicly owned and operated services, of which the highway systems are perhaps the most prominent example, depreciation of the various property units is an important economic factor; but it is one which bears quite a different emphasis than in the case first cited.

In this instance knowledge about the rate at which the property units must be replaced is important to the administrator in order to enable him to plan future development as well as financial programs. The emphasis herein is upon planning for the future and upon the development of sound and adequate financial and economic programs.

INTRODUCTION

Typical of the accounting practices of most highway departments in the past is the example of an expensive piece of road machinery purchased out of current funds, and with its entire cost charged to the current year's maintenance expense, although it must continue in use for many years. Such a practice undoubtedly accounts for the money, which may be all that the law requires, but it does not give the administrative official a true picture of his operations nor any help in planning to have the necessary funds available when it becomes necessary to replace the machine.

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[9] Director, Highway Research Board, National Research Council, Washington, D. C.

[10] Capt., Air Corps, U. S. Army, Ames, Iowa.

For this reason, responsible highway officials have been giving much study during the past several years to developing systems that will enable them to account for the depreciation of the highway and other property in their charge. The same situations exist in other types of governmental service. The writers are relying principally on the highway illustration of their thought, since that is the business with which they are familiar, but they emphasize that the illustration is applicable to other activities of government.

Funds for the operation of such public services are secured through various tax imposts and direct charges to users; and in making his recommendations for financial support, the administrator is not concerned with making a return upon the investment but with providing adequate service in accordance with the wishes of the voters.

It is a fact that public accounting for governmental service does not usually distinguish between running expenses and capital investment, but nevertheless some breakdown into cost factors is necessary for the highway planner. Depreciation is the most significant of these.

It is desirable that public policy include some recognition of the depreciation of public property as a means of effecting better financial management. Knowledge of the length of service lives and of salvage values, which are the basis of the depreciation rate, could be used in the management of public services in many ways. A few of these are as follows:

(1) By application of the probable rates of retirement of property to the existing property, it is possible to estimate the retirements to be made in future years.

(2) The replacement program, estimated as in item (1), may be combined with a program of extension and betterment and may be used to formulate a financial program for the future.

(3) True annual cost can be shown only when the expense of depreciation is included. Since annual cost is the basic unit used by engineers in determining economic justification of selection of projects, as well as total revenue needed, depreciation is an important element on which complete knowledge should be had.

(4) Correct depreciation rates are necessary when making comparisons of highway costs with other modes of transportation.

In the case of highways the correct rates of depreciation to use can be determined by study and analysis of past experience. This work, along with other planning studies, is now (January, 1941) being done by forty-six state highway departments in cooperation with the U. S. Public Roads Administration. As reports of these studies become available, reliable information will be at hand on which to base estimates of depreciation and forecasts of probable future retirements.

VALUE AND DEPRECIATION

Although length of life and not money value is the factor of greater importance in operating a publicly owned business composed of thousands of items of physical property, it is necessary to take account of the property items by ordinary bookkeeping methods. On that account the public official must have a clear understanding of the nature of value and depreciation and of depreciation estimating and accounting.

Definition of Value.—Once attention is called to the point, it is quite easy to understand that it is only the anticipated worth of future services that gives value to any property. Past services and earnings are just so much "water gone by the mill." Every estimate of present value in reality is a forecast of future happenings.

The fundamental basis of value is present worth of probable future service. The fundamental basis of the fair value of each specific property item is the present worth to its owner of the probable future services to be rendered by the item during its future service life.

Manifestly, neither the number of future years of service nor the yearly return for services can be forecast with certainty or exactness; nevertheless, every sale of property and every estimate of value are based more or less consciously on the present worth of such anticipated future services as judgment indicates to be most probable. Reliable estimates of the values of industrial property units can be made only on the basis of the sound judgments of valuators, well qualified by training and experience, who give careful consideration and such weights as are just and right to every factor in each case.

Depreciation is negative value, to be determined at every date in the service life of a property unit by the sound judgment of well-qualified, impartial experts as to its probable future service life and its probable future economic efficiency.

Definition of Depreciation.—The depreciation of any physical property unit, at any service age, is its loss of value, since its service started, due to decrease in the present worth of its probable future earned operation returns, below what such present worth would be if the unit were still new.

Thus, depreciation is just a question of values at different dates based on forecasted (probable) numbers of years of future service and probable values of the yearly services. Such forecasts can be made correctly only by the sound judgments of impartial persons who are qualified to understand the mortality and other technical characteristics of different kinds of physical property units, and who personally examine the particular units whose depreciations are being determined.

Property units should be reexamined by qualified persons, from time to time throughout their service lives, and the estimates of the depreciations at different dates should be kept constantly in accord with sound current forecasts of the numbers of years of future service and of the yearly operation returns for such service.

DISTINCTION BETWEEN DEPRECIATION AND MAINTENANCE

Maintenance expenses are those for keeping the property units in good condition for use. Examples are the costs of inspecting, painting, and repairing structures and equipment. Maintenance expenses are paid from yearly income, and constitute part of the yearly operation costs. Maintenance cost accountancy, therefore, consists simply in making properly classified accounting records of the yearly maintenance expenditures.

Depreciation losses of value, on the other hand, are something which cannot be made good in a physical way until new units are substituted for the depreciated units at the ends of their service lives. Depreciation expense accountancy, therefore, is a comparatively complicated process by which annual depreciation expense allowances are made each year out of income and used (usually reinvestment in new property units for replacements and additions) to create a depreciation reserve, which should be at all times just equal to the total accrued actual depreciations of all existing property units. In this way the correct annual cost is known and the depreciation reserve account shows the true loss in value of existing property, from which, of course, the present true value of existing property is determined.

USE OF DEPRECIATION IN PUBLIC ACCOUNTING

Because of the time-honored custom in governmental accounting of staying strictly in the cash accounts, depreciation of public properties simply has not been accounted for. Even for municipalities it is probably safe to state that only in recent years has there been any attempt to account for depreciation of their electric, water, and other utilities. In those few cases where cities have made allowances for depreciation in their accounting, there has been little attempt to determine accurate rates.

When, therefore, mention is made of the depreciation of other public properties, particularly those which do not produce direct revenues, immediate objection may be raised on the ground that the government is not in business for profit, is not compelled to operate its functions and businesses solely from user revenues, and that there is no need of knowing the relations between investment, expenditures, annual costs, and revenues. Nevertheless, if economy in government is to be attained, sound principles of business management must be followed. As governments more and more take over and enter into operations that might be regarded as business enterprises, either competitive or noncompetitive with private interests, it is essential to adopt accounting practices, equal to those that the government prescribes for regulated utilities. This will include the accounting for physical properties, depreciation, operating revenues, operating expenses, and the compilation of balance sheets and operating statements.

In many government-operated functions, such as printing shops, warehouses, central purchasing bureaus, state garages, and municipal utilities (not all, of course), it is probable that if all the costs were collected and reported it could be shown that such activities handled by the government could be purchased at less cost from private companies. Where there is, and where there is not, legitimate opportunity for the government in business can best be determined by accurate cost analysis and scientific evaluation of the worth of the services. Depreciation is a large item of cost in the management of any business that requires large investments in buildings and equipment; yet it is the one that government agents and the public alike are prone to forget.

Engineers in government service are responsible for the operation of huge public works. Their future and the future of public works depend upon the showing they make. Yet generally they are deprived of the opportunity of making an actual financial accounting of their enterprises because of laws prescribing archaic accounting methods or because those in charge of government accounts permit nothing but "cash and carry" accounting methods. The engineer can do much to improve the accounting of governmental activities, particularly for those which involve his works.

ACCOUNTING FOR DEPRECIATION OF HIGHWAYS

Within the length of this paper there is opportunity to present just a suggestion of what might be done for a state highway system—in fact, what has been done, because the State of Kansas in 1941 completed a set of property investment and depreciation ledgers for its primary system of about 9,200 miles.

Obviously, before depreciation can be estimated it is necessary to account for the physical properties that depreciate. For highways, as well as for other properties, it is necessary to adopt accounting units that will permit accounting for the properties by location and kind. If any system of highways is laid out into a number of route sections with termini at more or less fixed geographical landmarks, a number of basic units are obtained, complex though they be, which will serve for accounting operations as well as for engineering and construction operations. These route sections should be governed in length by municipal, county, and state boundaries, route intersections, stream crossings, abrupt changes in topography, character of traffic, and convenient construction project termini.

Within the route section it is necessary to provide for investment accounts on the basis of the function and character of the construction, with due regard to the retirement characteristics and depreciation rates. Suggested classifications are those construction accounts tentatively recommended by the Committee on Accounting of the American Association of State Highway Officials. This classification results in the following investment accounts for each route section:

Right of way
Engineering
Roadway and drainage grading and earthwork
Drainage structures and roadway earthwork protective structures
Roadway surface and base (by roadway surface type)
Improved shoulders and approach surfacing
Bridges, viaducts, and tunnels (by individual structure)
Traffic and pedestrian services
Roadside development

The accounting unit, as will be observed from the foregoing classification, is best chosen as the dollar, with a control in miles or square yards for roadway surfacing, cubic yards for earthwork, and other suitable construction units for other items. It is difficult to identify most of the physical parts of the highway, so the average costs and the "first-in, first-out" theory must be used in accounting for certain retirements. Supporting the accounts of each route section should be construction cost reports made up from the final payment estimate and engineer's final report.

The process of estimating depreciation for the highway system need be no different from that used for other types of properties. Perhaps there is not available as much factual information and well-informed opinion as to depreciation and the factors causing depreciation of highways as there are for certain types of industrial properties, but the same basic principles and methods should be used. As the highway systems are rebuilt, expanded, and relocated, data become available that can be analyzed by accepted statistical methods to determine probable average lives and life expectancies. Again, these methods are the same as are being used by many utilities and other industries.

To account for depreciation of public properties on a correct basis is new; depreciation reserve accounts and depreciation expense in public highway accountancy until recently were wholly foreign elements. Because of a diversity of ages, qualities of construction, service conditions, types of materials, and variety of design, highway property lends itself to a modified form of group depreciation accountancy. If each route section is considered separately, all the variables can be investigated and the annual depreciation calculated for each investment account for each route section. With this breakdown, total depreciation can then be obtained for any complete route, area, and account.

To complete the accounting picture, maintenance costs, administrative costs, and other items of annual cost should be allocated to the same route sections so that the true annual cost may be obtained for comparison of types of construction, of the effects of volume and character of traffic, and for comparison of annual costs with annual revenue.

SUMMARY

It is not intended that the foregoing treatment of depreciation in relation to public works and public finance should set out any new principles of depreciation or accountancy. Instead, a suggestion is offered that depreciation accountancy does have a place in government accounting procedures and finance. The full meal in any business of government is not paid for until capital costs are properly accounted for. This is particularly true of those functions or activities which produce a definite revenue, or which are in substitution for services or goods that may be, or logically could be, supplied by private sources. Until financial reports take full account of all items of annual costs, and account for investment in physical properties, they must be regarded as only partial truths.

In recent years there has been much talk of budgeting, long-range planning, and public spending to increase the level of business activity. Knowledge of the depreciation of public properties is essential to a wisely made plan involving any one of these three ideas. A government spending or tax budget, and a building program, could be arranged somewhat on an even keel over a period of years if attention were given to the rates of depreciation of the several properties of the public and the probable future needs. All too frequently, once money is spent for a public works or a public building, it is considered that no future obligation will exist. Government buildings, no less the same as for private buildings, decay, become functionally obsolete, and suffer for need of modernization. Depreciation studies will bring out these needs, determine rates of losses in values, and become the basis of estimating annual sums necessary to provide the citizens with physical properties necessary to the proper functioning of their government. Why, for instance, could not every state legislature, school board, and city council provide about a constant sum annually for building replacements and extensions? In the end the costs would be less and the burdens more evenly spread over generation to generation. Depreciation is inevitable; a good government must recognize and must budget for it in the same manner as for other annual expense.

ACKNOWLEDGMENT

The definitions of value and depreciation, under the heading "Value and Depreciation," were written by Anson Marston, Past-President and Hon. M. Am. Soc. C. E.

DISCUSSION

Edwin F. Wendt,[11] **M. Am. Soc. C. E., and L. T. Fleming,**[12] **Esq.**—In 1911 the Board of Direction appointed a Special Committee to Formulate Principles and Methods for the Valuation of Railroad Property and Other Public Utilities composed of the following eminent engineers, all deceased: Frederic P. Stearns, Chairman, Leonard Metcalf, Secretary, Thomas H. Johnson, Alfred Noble, William G. Raymond, and Jonathan P. Snow, Members, Am. Soc. C. E. In 1914 Henry E. Riggs, Past-President, Am. Soc. C. E., William J. Wilgus, Hon. M. Am. Soc. C. E., and the late Charles S. Churchill, M. Am. Soc. C. E., were appointed to the committee to fill vacancies. The committee reported on every phase of the appraisal of public utilities, outlining the principles and discussing the fundamental aspects of the depreciation problem. Their report of 1917[13] was a classic of its time and so far as fundamentals are concerned there has been no significant change in the succeeding twenty-five years.

However, developments in procedure have resulted from the activities of the many regulatory agencies concerned with the subject. The Interstate Commerce Commission, charged by the Valuation Act of 1913 with the duty of making valuations of the railroads and other common carriers, has given considerable time and thought to the depreciation question, and its record on that subject consists of more than 10,000 pages of testimony and argument. The Commission appointed William D. Pence, John S. Worley, Edwin F. Wendt, and the late Robert A. Thompson and the late Howard M. Jones, Members, Am. Soc. C. E., to an Engineering Board which outlined the procedure that was followed in the field and office. During the progress of this work on the railroad valuations, from 1913 to 1921, there was developed what is known as the federal method of valuation, which is now very generally followed in public utility valuation work.

The 1917 report of the Valuation Committee[13] gave exhaustive treatment to depreciation theory and its application in the field of railroad and public utility valuation. This Symposium extends that report, in its fundamental aspects, to new fields—competitive industry, and public works and government finance.

As the leading thesis of his paper, Mr. Walls compares the determination of the depreciation in property as of any time by the method, which he favors, of using engineering and technical knowledge in the same way as it is used in connection with safety standards, replacements of property, and selections of equipment, and by methods based on age-life theory. A question is raised as to the correctness of the conclusion that future life of public utility property can be estimated on an actuarial or statistical basis in the manner that insurance companies treat human lives, with the possible exception of communication facilities and a relatively small proportion of other utilities.

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[11] Cons. Engr., Pittsburgh, Pa.

[12] Cons. Engr., Chicago, Ill.

[13] *Transactions*, Am. Soc. C. E., Vol. LXXXI (December), 1917, p. 1311.

Most valuations of regulated industry which require estimates of depreciation are for rate base determinations. Here, the broad concept of constitutional value based on the expectation of income from property and largely independent of the depreciation in the property as of the date of valuation must be disregarded. The rate base is an artificial element—it is not necessarily, or often, the value of the property, and the principal problem in these cases is the determination of fair return, toward which the estimation of existing depreciation is but a part. Fair return means what is right and just to all parties, and it can never be more than an approximation to the ideal. Throughout the life of each element of a property, the ideal would provide a uniform rate of return on the unamortized part of the investment, uniform annual depreciation charges, and a uniform total annual cost of return, depreciation, and operating expenses, which is manifestly an impossibility to achieve.

The estimates of depreciation must not be subject to unreasonable error, and it is highly desirable that the cost of making the estimates be kept to a minimum consistent with the necessary accuracy, and also that the method of estimation used be not too complex for clear exposition before commissions and courts. The age-life method is advantageous in this respect, especially in the case of properties of considerable size in the valuation of which the engineer who is to introduce and support the valuation can do but a small part of the work. Furthermore, age-life estimates are required for the purpose of setting, or substantiating, the estimated future annual depreciation charges to operating expenses which come ahead of fair return; and it is difficult to refute the argument that the same age and life estimates should determine the depreciation existing at the date of valuation.

The Interstate Commerce Commission has made valuations, for rate-making purposes, of the properties of more than 1,800 steam railroads, most of the field work being done between 1913 and 1922, inclusive. Depreciation was estimated on the straight-line basis, life expectancies being based, in general, on established average lives for the different types of property and not on a true actuarial basis. Although no publication has been made of a comparison of the actual subsequent mortality of the railroad property inventoried with the estimates of the Bureau of Valuation, it is believed that the estimates have not been discredited by the actual mortality experience.

Obviously, in estimating the depreciation applicable to a specific item of property as of a given date, all relevant facts pertaining thereto should be given consideration by the appraiser whatever depreciation theory is being followed. Under a mathematical basis using ages and lives, the size of the sample from which the life estimates were derived and the existing and prospective changes in conditions affecting those lives should govern the reliance placed on the experience of the past, whereas under no basis can the estimated remaining life of the unit under consideration be ignored. The experienced and careful appraiser will insist on the application of judgment after an inspection of the property and examination of the replacement and maintenance records, whatever the basis; and he will reach substantially the same result by any method. In short, the thought in Mr. Wall's closing sentence may be succinctly reiterated: There is no substitute for sound judgment.

Professor Grant opens up a subject which heretofore has not been reported on by committees of the Society, and which has had but scant recognition in papers published in *Transactions*. The subject is an important one to the many engineers engaged in industrial work, and the sections on engineering economy studies relative to a proposed new investment, and economy studies for proposed retirements, are equally applicable to regulated industry.

In the matter of the so-called losses on retirements of property depreciated under the group method, while the regulations of the Bureau of Internal Revenue preclude the allowance of such losses, for tax purposes, at the time of the retirement in the case of property retired before the estimated average service life of the group has been reached, such losses may often be allowable as deductions when the last unit of the group is retired, if the investment in the group has not been made up through the depreciation charges.

Although Professor Grant undoubtedly is correct in his opinion that, in general, the decrease in value of manufacturing equipment is greater in the early years of the life of the equipment than in the later years, such decrease in value is not, as he points out earlier in the paper, the same thing as accounting depreciation, and there are valid arguments against reflecting such differences in the rates of depreciation used in determining the annual depreciation charges, although they must be taken into account in estimating product costs and in reaching decisions relative to replacements of equipment.

Professor Grant's paper is excellent and is a permanent contribution to engineering literature.

The paper by Messrs. Crum and Winfrey contains a statement of sound public policy. The engineer is an important person in the preparation of budgets and the spending of public moneys: He must account for the expenditures. Depreciation is an element of cost and provision should be made for proper governmental accounting to aid the administrator and the engineer in their public work. These authors have outlined the elements of the problem and have clearly illustrated how depreciation should be used in the management of public works and enterprises operated by governments. They have proved their thesis under "Summary" that

> "depreciation accountancy does have a place in government accounting procedures and finance. * * * Until financial reports take full account of all items of annual costs, and account for investment in physical properties, they must be regarded as only partial truths. * * * Depreciation is inevitable; a good government must recognize and must budget for it in the same manner as for other annual expense."

ANSON MARSTON,[14] PAST-PRESIDENT AND HON. M. AM. SOC. C. E.—(1) There has been, and still is, much confusion of thought among writers on fundamental depreciation principles. The writer confesses that his own ideas have changed (he hopes progressed) during some thirty odd years of study and considerable writing on this subject, and he reserves the right to make further progress regardless of what he has already written. Perhaps this confession may win him pardon for disagreeing with some of the views expressed in the Symposium papers.

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[14] Dean Emeritus of Eng., Iowa State College, Ames, Iowa.

(2) Apparently the confusion of ideas about depreciation, referred to by Mr. Walls and Professor Grant, is due mainly to the seeming inability of most writers on depreciation to realize the fact, and the implications of the fact, that in every one of the many billions of purchases and sales, the world over, each year, the exchange value is determined by the present worth of the forecasted (not the recorded), probable (not the actual), future (not the past or the present) annual operation return values of the services yet to be rendered by the item purchased. The result is confused references to "physical deteriorations," "wear and tear," "obsolescence," "inadequacy," and "supersession" as if they were agencies whose effects on depreciation could be calculated in some mysterious way other than by judging their probable effects upon probable future service lives (expectancies) and upon probable ratios of future to average annual operation return values. There has been a "blind spot" here in thinking about value (including the negative value called depreciation). That this "blind spot" is not entirely a universal affliction is shown by Professor Grant's lucid statements in the "Synopsis" and "Example 4," Element (3), that, "It is the look forward that is relevant in the appraisal; the look backward (at age) is only useful if it helps in the look forward (at life expectancy)."

(3) Their mental "blind spot" as to the vital rôle of forecasts in determining value and depreciation has left most writers on depreciation in much the same situation as the old Roman water supply engineers who were unable to conceive a compound unit of measurement of flow of water that included velocity in addition to the cross section of the stream. So Frontinus, in his famous book (translated by the late Clemens Herschel, Past-President and Hon. M. Am. Soc. C. E.), had a hard time trying to explain why the sum of the cross sections of the aqueduct flows was not equal to the sum of the "quinaria" units of area of deliveries from the aqueducts, just as writers on depreciation become confused about its fundamental aspects.

(4) A widespread, unfortunate misconception about the rôle of forecasts in determining depreciation and value is the entirely mistaken idea that if the actual facts of future service lives prove (as they almost always do) to be different from the forecasts, the depreciations and values based on the forecasts were wrong. The truth is that it is not the actual future service lives and services, but their forecasts which determine the true depreciations and the true values at the time the forecasts were made. In estimating values and depreciations of values the engineer is concerned primarily with exchange values, what willing purchasers will pay and willing sellers take, in mediums of exchange or goods, for the units of property bought and sold. The other shades of meaning sometimes given by usage to the term "value" (and discussed by Bonbright in the two volumes to which Professor Grant refers[4]) need consideration in connection with depreciation only as they can be expressed in terms of their relationships to basic exchange values. Tucked away rather inconspicuously on a page in Bonbright's work[15] will be found a quotation from the decision written by the well-beloved late Justice Oliver Wendell Holmes of the United States Supreme Court in an inheritance division case, as follows:

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[15] "Valuation of Property," by J. C. Bonbright, McGraw-Hill Book Co., Inc., 1937, p. 743.

> "The value of property as of a certain date, like all values, as the word is used in law, * * * depends largely on more or less certain prophecies of the future, and the value is no less real at that time if later the prophecy turns out to be false than when it comes out true."[16]

(5) Another widespread, unfortunate misconception about the rôle of forecasts is the idea that once made they should ever afterward be adhered to regardless of actual later service life facts. On the contrary, re-examinations of the then existing property units and the conditions under which they are serving should be repeated and new forecasts made of expectancies and future annual operation return value ratios frequently enough to keep the forecasts constantly corrected to agree with the facts of actual service life. Industrial properties are not inert. They are changing constantly, partly from changes in management policies and business conditions, and partly from retirements, replacements, and additions of physical property units. The replacement items often differ greatly from those retired. The imminence of inadequacy, obsolescence, and supersession functional retirements must always be considered in making the repeated new forecasts. Upon completing each new forecast that conflicts with the preceding, the current depreciated values of the current existing property items, or groups of items, should be taken as new depreciation bases. In this manner, the annual actual depreciations are distributed throughout service lives in accordance with the actual cumulative growth of depreciation.

(6) The writer agrees with Mr. Walls' view (see "Synopsis") that "depreciation in property as of any time is an accomplished fact"; and that, "to determine that fact, engineering and technical knowledge should be used * * * [of course, in the most correct, simple, direct way]."

(7) These "accomplished fact" depreciations are the same for privately owned and publicly owned properties; they are the same also for public utilities and for private industrial enterprise properties, except in so far as these different classes of property differ as to original costs, costs of capital, and rates of fair return.

(8) It is the writer's view, apparently somewhat different from that of Mr. Walls, that the correct way to apply engineering and technical knowledge in estimating depreciation is that described in Paragraph 5, herein; and that industrial property mortality curves can be of great value in the work. Hundreds of such mortality curves have been prepared from actual mortality data of different kinds of physical property and it has been found that some eighteen or twenty types of curves may be used to represent all classes of these properties. In the technique of using them, these types of curves serve simply as aids to the judgment of forecasters who have examined the units personally.

(9) The writer has been greatly interested in the paper by Professor Grant, especially in his exposition of the economic principles that should govern judgments in deciding when to make retirements of existing industrial property units. The writer concurs almost completely in the views expressed in this exposition.

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[16] Ithaca Trust Co. vs. United States, 279 U. S. 151 (1929).

(10) Concerning concepts of depreciation the writer's own idea of the correct concept is stated in the "Definition of Depreciation" given by Messrs. Crum and Winfrey (see "Value and Depreciation"). It may be termed the "present worth" concept. In the technique of its use, the effects of changes in price levels are provided for by using different depreciation bases, equal respectively to:

(*a*) The original cost new, less salvage value—for customary depreciation accountancy;

(*b*) Fair value new, less salvage value—for the Baltimore Street Railway Case decision,[17] legal public utility depreciation—and for engineering valuation depreciation; and

(*c*) Reproduction cost new, less salvage value—for insurance damage depreciations.

(11) The "condition per cent" multipliers are the same for all three depreciation bases. For Professor Grant's Example 1, the salvage value is 0, the original cost base is $5,000, the reproduction cost base is $6,500; and the "straight line" condition percentage is 80%, the original cost depreciation is $1,000, and the reproduction cost depreciation is $1,300. (The $200 difference from Professor Grant's reproduction cost depreciation is his adopted additional deduction for the less modern design and assumed higher early maintenance costs of the 8-yr-old house.)

(12) The writer indorses almost completely Professor Grant's views about "Economic Studies for Proposed Retirements" and "Depreciation in Pricing Policies." Concerning their application in Example 3, it may be remarked that the longer the expectancy of the existing property "old asset," the less certainty there can be that its retirement would be wise on the basis of a calculation involving so many factors that can merely be estimated. The result of the calculation, as Professor Grant stresses, can be changed greatly at will by changing the assumed rate of fair return, or the estimated service life, and especially by changing the estimated "annual disbursements." Professor Grant's calculations show that a 40% decrease in the estimated annual disbursements for the substitute unit will increase the calculated wise cost payment for the substitute 245%. Is it not likely that this situation is a main reason for adopting a management rule that new units substituted for present units "must pay for themselves in a short time"? Nevertheless, such computations as those in Example 3 are better helps to judgment than mere hunches in deciding when retirements are wise. In his own practice, the writer has been restricting the use of wise retirement mathematics to cases in which the life expectancies of the "old assets" are nearly 0 without material disbursements for "extraordinary repairs."

(13) Example 4 furnishes a good illustration of what a little difference of opinion about the "annual disbursements" on a substitute unit would do to a valuation of the "old asset" on the basis of wise retirement mathematics.

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[17] United Railways and Electric Co. of Baltimore vs. West, chairman, *et al.* 280 U. S. 234, 247, 50 Sup. Ct. Rep. 123. U. S. Sup. Ct., Jan. 6, 1930.

Decreasing the estimated annual disbursements on the substitute unit only 10% cuts the appraisal value of the old asset 53% (from $663 to $312).

(14) The writer finds himself unable to agree with Professor Grant that it would be feasible, or desirable, or even correct, to substitute wise retirement mathematics for the present well-tried and established methods of making engineering valuations of industrial properties. The trend of progress in valuation methods is in a quite different direction. The courts, federal tax officials, and valuation engineers are shying more and more away from theoretical formulas to the solid bases of: (*a*) Continuous-inventory property ledger accounts of the actual original costs of all physical property units (or groups of units); (*b*) total and annual depreciation allowances that check closely with complete mortality records of physical property retirements, and which create and maintain depreciation reserves just equal to the total actual depreciation; (*c*) infrequent changes in the values new and corresponding depreciations not of the total but of individual old property units to allow for the effects in present values of changes in price levels; (*d*) inclusion of only actually disbursed overhead capital-cost expenditures on the values, new; and (*e*) only very conservative allowances for intangible values.

(15) Although the writer is in accord almost completely with the ideas expressed in the Symposium paper on "Relationship to Public Works and Governmental Finance" of the fundamental aspects of the depreciation problem, he rules himself disqualified, for obvious reasons, to comment on this able paper by his former students and present co-workers, Director Crum and Professor Winfrey.

WILLIAM G. ATWOOD,[18] M. AM. SOC. C. E.—The three papers presented in this Symposium show, as is realized by all students of the problem, that a precise definition for the term "depreciation" is the first requisite for a study of the subject. Mr. Walls makes this clear in his paper and suggests an approach. There is a great mass of literature on the subject, most of which has been written by authors with a limited class of problems in mind. After studying much of this material the reader cannot "see the forest for the trees."

Depreciation is a physical and economic fact which, in most cases, must finally be measured in dollars. In order to determine the number of dollars it is generally first necessary to find the percentage of value that remains. The dollars are merely markers. The use of the term "depreciation" in accounting is one of the causes of the difficulty in defining the term correctly. If instead of "depreciation" the account were called "amortized cost," as suggested by Professor Grant, one of the difficulties would be removed.

"Depreciation" charges in accounting are generally based on the "straight-line" or "age-life" theory. The writer agrees with Mr. Walls that this method does not show the facts correctly. Perhaps, among the reasons for its popularity is that it is simple and arithmetical and that it can be readily understood by the courts and commissions. It is probably necessary to use this method for accounting because of the difficulty and expense in finding the actual depreciation on all units of a large property at short enough intervals of time to make it useful in accounts. If the account were not called "depreciation" this necessity would be removed.

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[18] Civ. Engr., Corps of Engrs., U. S. Army, New York, N. Y.

The inaccuracy in this method may be illustrated readily. The original Interstate Commerce rules specified a 50-yr life for frame buildings. The writer happens to know of two frame houses in good condition which have been occupied for more than 250 years. A stationary engine or machine is said to have a life of 25 years. The writer remembers an engine used for driving machinery in a roundhouse that operated for more than 70 years. Other instances can be found easily where the actual useful life was less than the theoretical life.

When the accounting concept is removed from consideration, depreciation becomes a loss in value caused by age and wear, changes in the demand on the unit, and obsolescence. The determination of the value of each of these factors and their combination into a percentage which represents the loss in the value of the unit is a matter for trained engineering judgment aided by all pertinent records available.

There has been much discussion, and many books and articles have been written about how to depreciate property when the results are to be used for one of many possible purposes. The general tenor of much of this writing is that the sum used for estimating whether it is economical to replace a unit is entirely different from that to be reported to a commission as depreciation in a rate case. This does not seem to the writer to be good engineering thought although it may appear to be good legal strategy. A decision of the U. S. Supreme Court on April 30, 1934,[19] defined depreciation as follows:

> "Broadly speaking, depreciation is the loss, not restored by current maintenance, which is due to the factors causing ultimate retirement of the property. These factors embrace wear and tear, decay, inadequacy, and obsolescence."

This would seem to be a proper definition. Why should not the engineering profession adopt such a definition, cease to refer to the "straight-line" accounting method as depreciation, use the term "amortized cost" for accounting purposes, and eliminate the idea that the amount of depreciation varies with the purpose for which the data are to be used?

With this done the Joint Committee on Depreciation (under whose auspices the Symposium is presented) could collect, analyze, and publish a great amount of data that would aid the engineer in his judgment when establishing depreciation percentages in any case under consideration.

GEORGE E. GOLDTHWAITE,[20] ESQ.—The unwillingness of public utility representatives to face the depreciation problem is reflected in the paper by Mr. Walls. This unwillingness is understandable in view of past history, but it has now become extremely dangerous to public utility interests.

After the close of the "cutthroat" competitive era and before effective regulation had been imposed, most utility owners and many railroads were in the position of possessing monopoly franchises under which profits could be realized far in excess of the present conception of a fair return. Such franchises were considered things of great and permanent value, to which the physical properties were merely attachments, so to speak, which could be readily maintained or replaced without any particular care as to amortization procedure; or in fact without any amortization whatever. The owners did not consider it necessary to face any depreciation problem, and when conditions changed they were naturally reluctant to admit that such a problem existed. Even after the introduction of regulation, the long period of rapid business growth and the era of court-supported reproduction-cost valuations combined to make possible further postponement to a realization of the problem. This past history has colored the views of nearly all utility managements, and has inhibited their representatives from frank statements regarding depreciation.

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[19] Lindheimer *et al.* vs. Illinois Bell Telephone Co, 292 U. S. 151 (1934), p. 167.

[20] Cons. Engr., New York, N. Y.

A capital charge with regard to any item of perishable or depreciable property is merely a deferred operating expense. When a company has spent money for such property and charged it to capital account, it cannot issue an honest statement of earnings for any subsequent year without including in its expenses a due proportion of the cost of that capital item. It is immaterial as to whether it is called amortization, depreciation, or even retirement expense. If it is a due proportion of the cost of the capital asset, it must be based on an estimate of life. It is perfectly futile to argue that a reliable estimate of life cannot be made; an estimate is nevertheless absolutely necessary, because otherwise the earning statement is misleading if it is not actually fraudulent.

As a corollary to the foregoing, the value of a depreciable asset cannot be greater than the amount fairly chargeable against operating expenses within its remaining life. This conclusion also follows from the general premise that the value of a depreciable capital asset resides solely in its capacity for giving service in the future. Without postulating such capacity, there would be no excuse for charging it to capital account. Unless the service capacity is assumed to extend very far into the future with no impairment of efficiency, its value must be largely dependent on the remaining length of life.

The management that bought and installed a piece of electric generating machinery in 1925 had it charged to the capital account in the well founded hope or expectation that it would be useful for a considerable period of future years. None of the officials expected it to last forever; they all knew it would have to be retired within a very limited period of years for some reason or other, and so they should have faced, immediately, the accounting problem of distributing its cost as equitably as possible among the consumers benefited and within the period when it would be useful to them. What excuse did they have for not facing this problem, and how could they purport to issue an honest earning statement without making this effort?

The answer is that they felt relieved from any economic necessity for showing costs because they depended on their monopoly power, supported by the vague phrasing of the court-made law, to enable them to reimburse themselves for their unamortized investment after the asset had disappeared.

They object to the word depreciation because it carries unpleasant implications as to loss in value. Suppose, however, that it is merely called amoritization, and a plan is devised which seems financially sound. If (as the writer has tried to show) the limit to the value of a piece of machinery is the amount that can be recovered from its output to cover amortization charges during its remaining life, in addition to all other operating expenses and return or profit, then the unamortized cost is inherently an estimate of remaining value. Hence, an intelligently planned amortization program is inherently a depreciation program, and accrued depreciation is best measured by the balance in a properly accrued reserve, subject only to substantial alterations in life expectancy taking place since the program was initiated. Additional value can be realized only through the monopoly power of the utility company, when supported by legal rights, to collect revenues greater than can be economically justified by cost of service.

Professor Grant's excellent paper displays the realistic approach to the depreciation problem that is to be expected from representatives of competitive industry, who have no monopoly power backed by "due process" law to "bail out the water" accumulated by past neglect of financial "leaks." His attack on the spurious plausibility of sinking-fund accounting is good. The practical effect of sinking-fund accounting is to postpone amortization, which is one reason for its popularity in certain circles; but anything that postpones amortization is an invitation to financial suicide.

The writer would like to warn against tying up the replacement problem directly with the depreciation problem. Considerable confusion has resulted from assumptions that a utility plant is maintained in practically 100% condition by periodical replacements of parts as they become worn or obsolete, and from that the false conclusion is reached that the outstanding purpose of depreciation accounting is to build up a fund whereby replacement may be made. Actually the depreciation problem exists with regard to any perishable item of property whether or not it is replaced, and to the same degree. Once a charge has been made to the investment account for the acquisition of a perishable plant item, it is economically necessary to arrange for writing off that investment before the item is retired, and the theoretically correct use of the money is not merely to replace it (because it may never be replaced), but to retire the debt or other obligation incurred in connection with its purchase. It is best, for clear thought on the subject, to avoid as far as possible the conception that the plant item will be replaced by another at the end of its useful life. Actually the replacements are usually indirect. A new power plant is built to take care of additional load; the use of the old is gradually reduced until it becomes only a standby; and finally it is retired and dismantled. There has not necessarily been a direct replacement at any time, but the indirect replacement has been made long before the retirement of the old item. Other examples could be cited in which it is difficult even to find an indirect replacement, such as the discontinuance of service in a certain area due to the removal of industry or shift in population. In any case, it is economically necessary to amortize the investment during the service life, regardless of how or whether replacement is made.

Data available for the statistical study of the service life of a utility property are often of such nature as to justify skepticism as to the results. Nevertheless, numerous statistical studies have been made on a strictly dollar basis, wholly without reference to the physical items (which some may say are quite heterogeneous within the given accounting classification), and the writer has obtained results of remarkable consistency as between various properties owned by different companies in widely different areas. It is proper to say that the results should be examined with caution, and should not be used unless they seem plausible by other tests. At the very least, however, they represent, in certain property groups, the best methods now available for statistical study of property life under existing and past conditions. They are certainly better guides to estimates of life expectancy than pure judgment devoid of statistical support.

The paper by Messrs. Crum and Winfrey has so little bearing on the problem of public utility accounting that the writer does not believe it is pertinent to make comment from the regulatory standpoint. If he understands correctly, these authors recommend in effect that a system of accounting which is sound for private enterprise would aid the government in determining the economic status of many of its projects; although there is no direct financial occasion for depreciation accounting on such projects.

NATHAN B. JACOBS,[21] M. AM. SOC. C. E.—Although depreciation is an element that an engineer must consider in the design of any public improvement, it does not necessarily follow that the elements of the depreciation assumption which he uses in selecting a given project are the same as an accountant would consider in setting up annual or accrued depreciation on the books and records of the public authority; nor is either of these to be confused with the depreciation that would be fixed by an appraisal engineer in setting up the deduction which should be made for accrued depreciation in determining present value. In selecting a given project from a group of alternates the engineer computes the probable annual cost, to determine which is the most desirable and economical. In doing this he will use his judgment as to the probable length of service which the project will undergo. If this is a new project the engineer will be guided solely by his judgment, in comparing one project with the others. His judgment, of course, will be tempered by his past experience, but it is his estimate of the future. In accounting for depreciation annually the comptroller will have available and can use the past experience with the particular unit or project, and will continually have the advantage of actual observation to determine how much of the life of the unit, or how much of its value, has gone.

As stated by the authors, the setting up of depreciation in public works accounting and finance is the exception rather than the rule. In fact, clear and careful distinction between capital expenditures and operating or maintenance expenditures is seldom found.

Some years ago officials of the City of Pittsburgh, Pa., desired to exempt, from the limits of municipal indebtedness, bonds which were issued for water-works purposes[22] on the grounds that "the net revenue derived from said property * * * shall have been sufficient to pay interest and sinking fund charges." At that time municipal water-works utilities in the Commonwealth of Pennsylvania were required, by the Public Service Company Law, to keep their records according to a Uniform System of Accounts, which required setting up depreciation as an operating expense. The city controller called the attention of the referee appointed by the court to the fact that if depreciation were set up as an expense, the net revenue would not be sufficient to pay the interest and sinking-fund charges. The referee found:[23]

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[21] Pres., Morris Knowles, Inc., Pittsburgh, Pa.

[22] Section 15, Article 9, of the Constitution of the Commonwealth of Pennsylvania.

> "It seems reasonable, therefore, that a depreciation reserve should be set up in some form or other to keep the water plant in a functioning condition and the inclusion of a proper amount for this purpose is a proper item of operating expense. It is also clear that it would not be proper to compel the City in operating a public utility to charge as an item of operating expense both a depreciation reserve and the amount required to meet interest and sinking-fund charges on debt incurred for plant construction. It is immaterial, however, as to the form that a depreciation reserve takes, so long as it is in fact taken care of, and so that the public utility will be continued as a functioning plant * * * since the principal payments for water debt reduction exceeded the requirements for a depreciation reserve * * * no further charge need be made as an operating expense in determining the net revenue of the City from the operation of its water plant * * *."

Although the finding by the referee may be sound practice in public works financing, and the setting up of a reserve in public works, either by the retirement of bonds or by the creation of a sinking-fund, is equivalent to taking depreciation, it is not in accordance with a sound accounting system for a public authority to neglect to charge depreciation, since generally there is no relation whatever between the rates of depreciation and those for bond retirement. The annual debt service for the retirement of bonds or for a sinking-fund as provided by a municipal or public authority in administering its public works will be greater than the amount which an accountant or engineer would place as representing the annual reduction in value which is taking place.

In private business it was always considered to be good policy to "get out of debt" as soon as possible. This is probably more important in public works, where it is not unusual for any given project to be bonded for 100% of its cost, so that it is highly desirable to create an equity for the owner as rapidly as possible. Following this principle, advocates of governmental economy are recommending pay-as-you-go practice in an attempt to get municipalities and others on to a basis where debt will be continually reduced and the item in the budget for debt service can be ultimately eliminated. In many cities today the debt service for interest and retirement or sinking-fund bonds over a period of recent years has been in excess of the amount spent each year on new improvements. The advocates of this pay-as-you-go principle, therefore, are recommending a return to a receipts and expenditures basis, thus eliminating the need for depreciation in municipal accounting.

Although these considerations may indicate that it is not necessary for depreciation actually to be set up on the books for accounting purposes in public works, nevertheless, depreciation must be taken into consideration by engineers in recommending public works, and if a capital account or capital budget is kept by the public authority, depreciation should be taken into consideration. The use of debt service in lieu of depreciation is dangerous and open to considerable error. In the first place, long-term bonds are often issued in payment of the construction of short-life property, so that the debt service will continue far beyond the physical service of the works constructed. The usual example is the issuance of 30-yr bonds to pay for short-life street paving and re-paving. In public works accounting there is often not a clear distinction between maintenance and replacement. Sometimes bonds are issued for contemplated construction which may not have been done. Recently in one large city 20-yr bonds, issued for subway construction, which subway was never built, were refunded and extended for another period of years. Depreciation should be included in public works accounting and finance, so that proper statements of assets and liabilities can be compiled.

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[23] Hoffman vs. Kline *et al.* 300 Pa. 485.

H. J. Flagg,[24] M. Am. Soc. C. E.—For several years the writer has had the privilege of membership on the Committee on Depreciation of the National Association of Railroad and Utilities Commissioners (NARUC). This Committee is composed of men trained and experienced in one or more of the fields of economics, accountancy, engineering, and management. The function of the Committee has been, and is, to study the various aspects of depreciation. A report is in preparation which will present the results of the Committee's labors. The contacts not only with the members of this Committee but with other students of the subject have disclosed such a variety of opinions that the writer is reminded of the allegorical poem by John Godfrey Saxe entitled, "Blind Men and the Elephant." The poem is too long to quote in full, but its essence may be expressed by abbreviating all verses except the first and the last:

"It was six men of Indostan to learning much inclined
Who went to see the elephant, though all of them were blind,
That each by observation might satisfy his mind.
The first (side) 'is very like a wall';
The second (tusk) 'is very like a spear';
The third (trunk) 'is very like a snake';
The fourth (knee) 'is very like a tree';
The fifth (ear) 'is mighty like a fan';
The sixth (tail) 'is very like a rope';
And so these men of Indostan disputed loud and long,
Each in his own opinion exceeding stiff and strong,
Though each was partly in the right and all were in the wrong."

The writer does not intend to suggest that those in the various fields of professional endeavor who are studying depreciation are either blind or all wrong. The obvious point of the poem is that the nature of the contact an individual has with any subject is certain to influence his views and conclusions. There is a real need for the enlightenment that will flow from an exchange of views such as this Symposium affords.

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[24] Chf. Engr., Board of Public Utility Commrs. of New Jersey, Newark, N. J.

Mr. Walls has referred to the problem of estimating remaining life of property and has pointed out the forces that both cause depreciation and control remaining life. He correctly states that,

> "All estimates of future life involve uncertainties and are assumptions, and the farther these estimates are extended into the future, the less reliable they become."

Most engineers will readily admit the practical impossibility, in most instances, of forecasting with certainty the remaining life of machines or facilities. With respect to those items of plant that waste away because of wear and tear, corrosion, abrasion, or erosion, there is considerable factual data available which may enable a qualified person to make a reasonable forecast of remaining life in a given situation. The main difficulty arises with respect to those items of plant which are greatly affected by forces whose incidence is unpredictable. It is true that statistical and actuarial methods are in use which undertake to project past experience as a guide into the future. This is done regardless of any question concerning the preponderant and variable influence of unpredictable future events on the retirement of property. These methods appear to give answers which satisfy the estimators, but it may be questioned whether the apparent refinement of the process is worth the cost and effort of the special research that is involved.

One of the projects of the NARUC Committee on Depreciation is to accumulate, with the help of the utility industries concerned, factual data with respect to the causes of retirement of utility property. These causes have been classified in three groups, namely: "Physical causes" such as wear and tear, action of the elements, etc., "functional causes" such as inadequacy and obsolescence, and "other causes" such as orders of public authorities, civic improvements, etc.

It is probably true that different companies will apply the classification differently, although a serious effort has been made to maintain uniformity. Nevertheless it is believed that the diversity of the returns is such that any irregularities in the application of the classification are reasonably well compensated.

Twenty-nine companies for which returns are available retired an aggregate of slightly more than $349,261,000 of plant in the most recent 10-yr period. These companies represent more than $3,300,000,000 in operating fixed capital and more than $678,000,000 in annual operating revenues. Of the total retired it is found that approximately 18.1% was due to physical causes, approximately 72.2% was due to functional causes, and approximately 9.7% was due to other causes. With functional causes having such a dominant influence in the retirement process the difficulty of making dependable estimates of future lives is obvious.

The present-day concepts of depreciation of public utilities appear to have had a slow evolution from the early days when depreciation was not even recognized as a factor in rate making. The present concentration of interest in the subject dates back only to 1934 or 1935. Much progress has been made in a relatively short period, but in view of the intensified attention to the subject, even greater progress can be expected in the future. It is to be hoped that the time is not far distant when a common basis will be found on which accountants, economists, and engineers can reconcile their presently divergent views. Regardless of what may be accomplished in this direction, it will always be true, as Mr. Walls states, that the treatment of depreciation cannot be reduced "to a simple arithmetical process that will eliminate, or even reduce, the quality or amount of sound judgment required in the treatment of the problem."

NELSON LEE SMITH,[25] ESQ.—Referring to the view of Professor Grant, the writer wishes to re-emphasize that much of the confusion that has existed, and still persists, regarding the much-discussed subject of depreciation in relation both to competitive industry and to the public utility field is due to a failure to differentiate sharply between different concepts and to keep clearly in mind the setting of the problem, the purposes of the study, and the uses to which conclusions are to be put.

Mr. Walls has recognized specifically the necessity of this realistic type of approach; and, therefore, he doubtless would agree that, in considering depreciation, regulatory authorities must refer constantly back to the relationships between that phenomenon and the objectives and processes of public control in general. The point which Mr. Walls emphasizes suggests that he is concerned chiefly with depreciation in rate-making and particularly with the accuracy of the deduction made from the rate base on this account. From the viewpoint of the regulatory commission it is equally important that the current charges or accruals for depreciation be determined properly, for they enter directly into the costs, which must be reflected in the rates for service. Since, in a rate case, these two determinations—the rate base deduction and the annual charge—are merely intermediate steps in an integral process (that is, the establishment of rates), it is not surprising that the commissions in their renewed interest in and more concentrated attention upon this problem are emphasizing the proposition that consistency requires the use of the same basic data and assumptions in each.

The regulatory agencies generally would agree with Mr. Walls that depreciation, as he expresses it, cannot be made to suit a definition but, rather, that the definition must truly express the actualities of depreciation; but they would go on to argue that what are alleged to be the facts and realities of depreciation for one step of rate-making must be accepted as having equal validity for other parts of the same process.

Mr. Walls appears to contend that there is a very fundamental difference between the problem of measuring depreciation at any given time and that of estimating future service life. He contrasts the reliability of estimates of the extent to which causes of retirement have already acted with the uncertainties of their operation in the future. At first glance, the difficulty—if it be a real difficulty—of predicting the future will seem to support his position; yet, the distinction cannot be accepted when dealing with depreciation for regulatory purposes, because this process requires the measurement and expression of depreciation in economic terms.

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[25] Chairman, New Hampshire Public Service Commission, Concord, N. H.

It may be feasible, through the use of engineering techniques, to determine the extent to which physical deterioration has occurred. At the same time, however, forces of obsolescence may be in operation. Although it may be that the ultimate impact and effect of such factors on service life cannot be foretold exactly, the fact that they are acting is currently and necessarily affecting the total service life, and this is just as important in its relation to the present depreciation of the property as is its mere physical condition.

Of course, this is simply another way of stating that the result can be no more accurate than any one of the factors upon which it depends and, hence, that no greater reliance can be placed on the estimate of the depreciation that has occurred in the past than on the prediction of remaining life in the future.

From his paper, the writer cannot be sure just how much significance is attached by Mr. Walls to the distinction which he seeks to make in this connection. Presumably he does not mean to suggest that data and methods should be used in establishing a basis for the current depreciation charges which are different from those utilized in determining the depreciation already suffered. In such event and for the reasons already suggested, the writer is certain that commissions would be likely to hold that the estimates must be consistent with each other and that greater weight cannot be given to one than to the other. Perhaps Mr. Walls intends, by his emphasis upon this distinction, simply to summarize or emphasize his conclusion that depreciation should not be treated in a superficial or oversimplified manner. With this conclusion the regulatory commissions generally would be in substantial agreement. They, and their representatives on the committee dealing with depreciation for the National Association of Railroad and Utilities Commissioners, certainly concur in the view that the estimated rate for the future should not be a matter merely of calculation based on arbitrary assumptions or fragmentary statistics. They would agree that it should reflect judgment as to what the future is likely to hold. Even so, they would hold that statistical studies of past experience are useful in showing what has happened and what, therefore, other things being equal, may be expected to happen in the future.

If it is proposed by Mr. Walls that such studies are to be rejected or abandoned despite the extent to which they have been utilized voluntarily by utility concerns in their attempts to meet their own problems (which may create some presumption as to their usefulness), a question which may be fairly asked is: What shall be substituted as a means of forming judgments? Is it to be a "medical examination," and if so, what is that "medical examination" to involve?

The element of judgment may enter most appropriately into this process by influencing the determination of the nature and extent of the statistical studies which shall be undertaken and by indicating what studies of this character can be justified. Are group or unit methods better adapted to dealing with particular types of property; and in what manner shall the results, and can the results, properly be applied?

Consequently it appears that the argument for the exercise of judgment is not an argument for substituting that judgment entirely for the statistical approach, but that a better argument can be made for the usefulness of the statistical approach as a means of informing the judgment.

C. Beverley Benson,[26] M. Am. Soc. C. E.—The principal conclusions to be drawn from Mr. Walls' paper are: (*a*) That he considers depreciation and future life as entirely separate matters rather than that one is, in part at least, a function of the other; (*b*) that the forces which cause depreciation can be measured with reasonable accuracy but that the depreciation they cause cannot be so measured; (*c*) that remaining service life cannot be estimated with any degree of accuracy; and (*d*) that the estimate would be worthless if it could be made. Of course, competitive industry dares not adopt so vague a plan for depreciation; but perhaps utilities, protected in their monopoly position by the courts, can rely upon the regulatory bodies to save them in the event that their depreciation policies prove inadequate.

Granting the difficulties inherent in any objective plan for estimating depreciation, the writer believes that a more definite procedure than the vague plan outlined by Mr. Walls is necessary. Remaining service life is the most important single criterion to be considered in estimating the value of any economic item or group of items. In ordinary activities the amount one will pay for such an item depends upon his conscious or unconscious estimate of the length of time it probably can be used. It is at this point that one finds the distinction between deterioration as an engineering concept of efficiency and depreciation as an economic concept of value. A 4-in. candle will give as much light at the moment as an 8-in. candle of the same diameter, but the value of the 4-in. candle is less because of the fewer hours it can continue to give light. Similarly, improvements to a power plant may improve its coal economy or reduce maintenance expense or improve reliability, but the ultimate value of the improvements is a function of the remaining life of the plant itself. Professor Grant's brilliant paper gives added emphasis to this point, because estimated remaining life forms the basis of much of his extremely interesting treatment.

Although the importance of remaining life as a criterion of value is enormous and (in the writer's opinion) is controlling, it is nonetheless true that the difficulties of estimating remaining life are real and that precision is probably not possible. Therefore, the wise executive will adopt a plan which permits flexibility.

However, there are many reasons for thinking that estimates of remaining life of physical property can be made with reasonable accuracy. In the first place, in a property of any considerable size sudden changes in life expectancy of the group as a whole are not likely, since the inertia of the mass acts as a stabilizer. For example, if a company has 20,000 electric meters and is currently replacing 400 of them per year, it will take a long time for the introduction of a new type to change the average life expectancy of the group.

The objection is often made that changes in the arts cause unpredictable changes in service life; but a properly organized actuarial study contains the effect of past improvements in the arts. There is little evidence to show that the rate of such improvement varies greatly. Consider, for example, improvements in steam locomotives—the water-tube boiler, the compound cylinder, the outside valve gear, the feed-water heater, higher and higher boiler pressures, superheaters, return to simple cylinders with short cutoff, uniflow boilers, mechanical stokers, etc. Who can say that improvements in locomotives have ended or will continue at an accelerated pace?

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[26] Prin. Statistician (Eng.), New York Public Service Comm., New York, N. Y.

It is sometimes said that the effect of obsolescence makes life estimates of little value. It is probably true that the preponderance of retirements of utility property is due to obsolescence and inadequacy, but there is little evidence that these factors are unrelated to age. This point is illustrated by Fig. 1. If the retirement rate is constant (that is, no greater tendency to retirement at one age than at another), the resulting life table will have the shape of a compound discount curve, as shown in Fig. 1(*b*). On the other hand, if retirements are an increasing function of age, the life table will have the shape shown in Fig. 1(*d*).

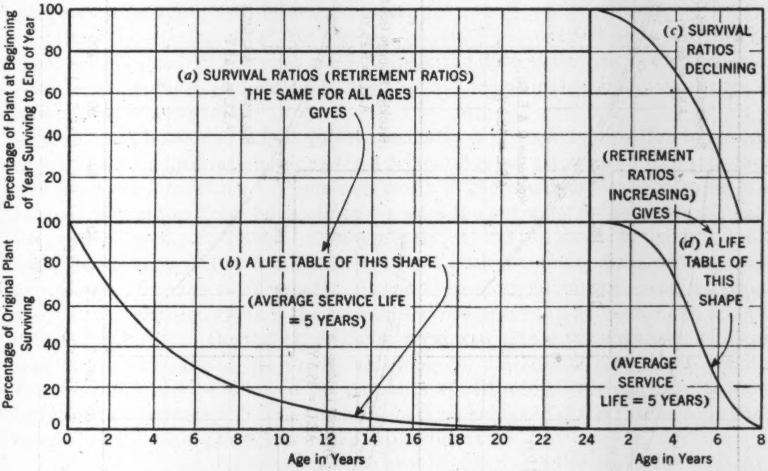

FIG. 1

Engineers of the New York Public Service Commission have constructed more than 300 life tables representing more than $400,000,000 worth of utility property. Since all the life tables they have been able to construct are of the "normal" shape, the conclusion is inescapable that obsolescence is a function of age and, consequently, is no less predictable than many other variable factors.

Another strong reason for having confidence in life estimates lies in the close similarity of results for different companies. Fig. 2 illustrates this similarity for a few groups of property. The most interesting point shown by these graphs is in the similarity of the shapes of the life expectancy curves, although

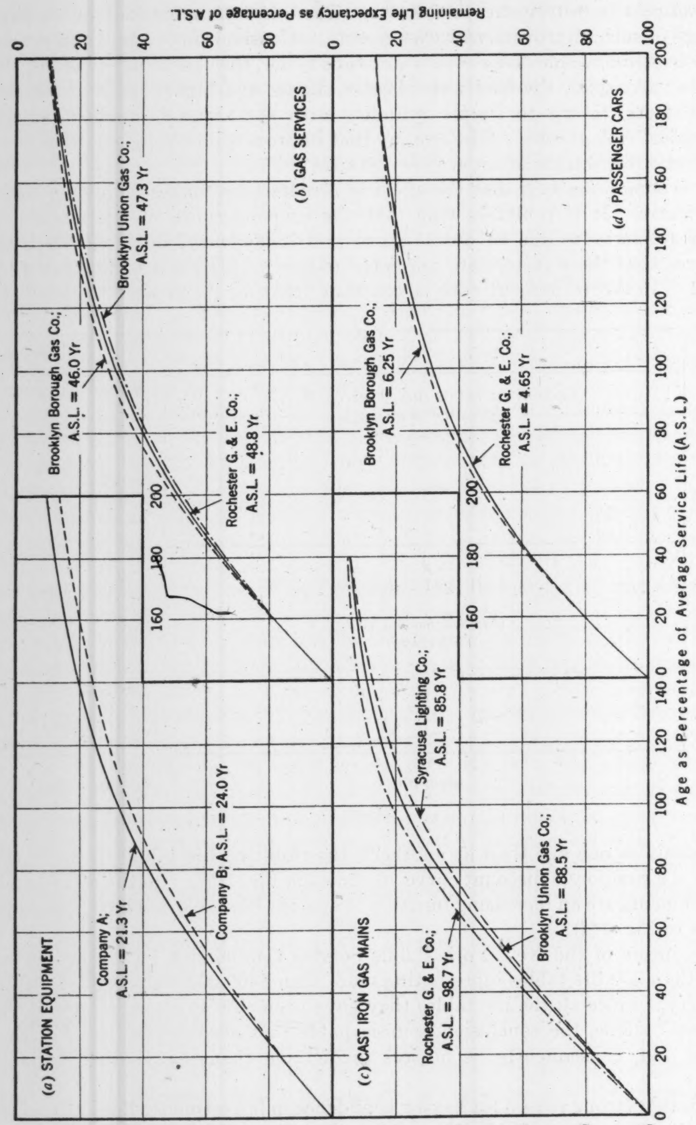
(a) STATION EQUIPMENT
Company A; A.S.L. = 21.3 Yr
Company B; A.S.L. = 24.0 Yr
(b) GAS SERVICES
Brooklyn Borough Gas Co.; A.S.L. = 46.0 Yr
Brooklyn Union Gas Co.; A.S.L. = 47.3 Yr
Rochester G. & E. Co.; A.S.L. = 48.8 Yr
(c) CAST IRON GAS MAINS
Rochester G. & E. Co.; A.S.L. = 98.7 Yr
Syracuse Lighting Co.; A.S.L. = 85.8 Yr
Brooklyn Union Gas Co.; A.S.L. = 88.5 Yr
(d) PASSENGER CARS
Brooklyn Borough Gas Co.; A.S.L. = 6.25 Yr
Rochester G. & E. Co.; A.S.L. = 4.65 Yr
Accrued Depreciation Based on Life Table
100 80 60 40 20 100 80 60 40 20 0
Remaining Life Expectancy as Percentage of A.S.L.
0 20 40 60 80 0 20 40 60 80 100
Age as Percentage of Average Service Life (A.S.L.)
0 20 40 60 80 100 120 140 0 20 40 60 80 100 120 140 160 180 200
160 180 200
160 180 200

FIG. 2

there is considerable variation in the life estimation of similar property for different companies.

However, the writer wishes to emphasize the importance of having such statistical estimates made by engineers. An engineering basis should be used in choosing the type of mathematical analysis; but statistical estimates should be used only as a guide to engineering judgment in connection with a detailed knowledge of the physical plant under consideration.

It is important that actuarial analysis be confined to fairly recent experience, although the width of the experience band will have to vary somewhat with the life expectancy of the type of property being investigated. There seems to be little question that present or very recent conditions should be reflected in any estimate for the immediate future. It must be remembered that the rate at which old property is being retired now is some indication of the rate at which property now young will be retired later.

The cost of statistical analysis of physical property is not prohibitive. Once set up, mortality studies can be kept up to date at very little expense and, with recently developed methods, the cost of the original setup, even for small companies, is not great.

It cannot be denied that regulated industry is in a special position regarding provision for depreciation, since the law requires an attempt to fix rates that will lead to a fair return. Therefore, many of the usual concepts of value in a competitive group must be modified because the ordinary forces of supply and demand cannot act freely. Consequently, a device must be adopted for regulated industry that will tend to protect both owners and consumers from the effects of errors in forecasts. Items once charged in operating expenses must not be charged again if the estimate of life proves too short (if the annual rate is too large). On the other hand, owners must be recompensed in some way if the estimate of life proves too long (if the annual rate is too small). Return must not be allowed on amounts contributed by customers toward ultimate retirements. These aims can best be achieved if the rate base is considered as the total plant less depreciation reserve.

The required flexibility can best be provided through the use of the straight-line group basis for both accrued and annual depreciation. In other words, the calculation of annual charges for depreciation must be based upon the same set of factors as the calculation of the accrued depreciation. Straight-line depreciation on a group basis does not mean equal annual charges in dollars but means a constant annual percentage of surviving dollars. The result of the use of the group basis is that a deficient reserve on property which lasts less than average life expectancy is made up by an excess reserve on property which lasts longer than average life expectancy.

By the use of the fixed capital less depreciation reserve as a rate base, errors in life estimates tend to be offset by inverse errors in return. That is, if the life estimate is too great, the annual charge is too small, the reserve is too small, the rate base is too large, and the owners get a greater return. On the other hand, if the life estimate is too small, the annual charge is too great, the reserve is too great, the rate base is too small, and the owners get less return.

Paul T. Norton, Jr.,[27] Esq.—There is so much to discuss in these three papers that it is difficult to find a place to start. Professor Grant has discussed prevalent fallacies in depreciation thinking,[28] but formulas containing these fallacies are still appearing in new textbooks and handbooks, as well as in articles in engineering and trade periodicals. The writer agrees with Professor Grant that this confused reasoning is probably less common in industry than in the published formulas, but this fortunate condition may not continue indefinitely. The next generation of industrial managers will include a much larger percentage of graduates of schools of engineering and business administration, who will probably not be nearly so unwilling to use mathematical formulas as were their predecessors.

Mr. Walls poses the question: "* * * is there not a very fundamental difference between the problem of measuring depreciation and estimating future life at a given time?" The two sentences that follow this quotation would seem to indicate a belief that the depreciation existing at a given time can be measured with comparative accuracy and reliability, but that estimates of future life are much less reliable. One also gets the impression that Mr. Walls believes that measuring depreciation existing at a given time and estimating future life are two separate and distinct problems. Depreciation means so many different things to different persons that such statements are difficult to discuss with assurance, but the writer believes that for most practical purposes these two things are merely parts of the single problem of determining present value, and also that the problem ordinarily cannot be solved without using both parts.

Messrs. Crum and Winfrey have given many reasons why there should be a better handling of government finance in relation to public works, but engineers should not forget that there are real dangers in the use of the capital-budget method for handling expenditures for nonliquidating public works. Many persons, including (unfortunately) a number of engineers and accountants whose duties include a consideration of the depreciation problem, seem to be obsessed with the idea that, if an expense is capitalized on the books and then amortized through depreciation charges, the cost is somehow transferred into the future. What an opportunity to befuddle the public! If it were only understood by every one that an investment in depreciating assets is a fully prepaid operating expense, this fallacy would be immediately apparent to every one who wanted to have a true picture of what really happens when so-called "capital investments" are made. That this is not merely an idle apprehension of the writer was proved in 1934[29] when it was proposed in Congress that depreciation rates on tax returns be reduced temporarily. This assumption, it was claimed, would not injure any one because very little equipment would probably be replaced during the period when the reduced rates were to be in effect. The idea that depreciation is actually paid for either when depreciation charges are made on the books, or when the asset is replaced, is obviously wrong; but this idea is held by many persons who claim to know more about depreciation than do the congressmen who could not see through the fallacy in the proposal just mentioned.

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[27] Prof., Virginia Polytechnic Inst., Blacksburg, Va.

[28] "Principles of Engineering Economy," by Eugene L. Grant, Ronald Press, New York, N. Y., 1938.

[29] See "Depreciation Under the Revenue Act of 1934," by Maurice E. Peloubet, *Journal of Accountancy*, Vol. 58, September, 1934, pp. 169–197.

In making economy studies for all kinds of public works, liquidating or nonliquidating, engineers and economists should try, of course, to determine which projects will yield the greatest returns—sometimes merely a nonfinancial social return. In comparing alternatives it is probably best to use a method that converts the initial cost into an equivalent uniform annual cost of capital recovery with a return; but this can be done just as easily, and perhaps more safely, under present cash accounting methods. It should be remembered that the so-called "depreciation" that is used when making economy studies is entirely different from the depreciation charges made later on the books when investments are capitalized. In the case of public projects like power plants, where it is expected that all costs will be returned through charges for the service, it would seem that the same accounting methods should be used as are used in similar privately owned utilities, but one should not forget that the true financial results depend on conditions as they really are, and not necessarily on what the books make them seem to be.

Professor Grant refers all too briefly to the effects of the current depreciation tax policy on private decisions regarding plant expansion and retirement. In nearly all cases these effects seem to be bad when judged from the viewpoint of society as a whole and particularly from the viewpoint of National Defense. (It may be noted in passing that local property taxes tend to have similar bad effects.) As long as corporation tax rates were comparatively low, the depreciation tax policy, although of questionable soundness, was of no great consequence. Until recently it could be stated without fear of contradiction that the economic aspects of depreciation were much more important than the tax aspects. It is not so certain that this is true today. Under present tax laws and regulations, it is not at all easy to secure approval of adequate depreciation charges on corporation income tax returns. This is a serious matter because every dollar by which depreciation charges are inadequate means an additional dollar of unreal but taxable profits, which are taxed in the highest bracket applicable to the particular company. That involves a rate as high as 72.4% under the Revenue Act of 1941, and every one expects that the rates will again be increased in 1942. In other words, many a corporation will be merely getting its investment back, but, because of inadequate depreciation charges, will be paying up to 72.4% of this recovered capital to the federal government. One would think that every top industrial executive in the United States would be trying to avoid this extreme penalty by making certain that his depreciation claims were adequate and were backed by such conclusive evidence that the U. S. Bureau of Internal Revenue would be certain to approve his claims. From having asked the question of many such executives, the writer believes that the majority do not even know that there is such a penalty.

From much that was printed and said when the special amortization provisions of the Second Revenue Act of 1940 were being considered in Congress, one would have thought that industry was asking for something for nothing. It would surprise many persons to learn that, even under these special provisions, industrial concerns in some cases may have to pay the 72.4% penalty on some of the money that is merely recovered investment and not really profit at all. In no case can a company do more than recover its investment, free of tax on such recovered capital; all real profits are fully taxable even under these special provisions.

Under present tax regulations a concern can scarcely hope both to get adequate depreciation allowances and to keep equipment in reserve for possible emergencies. Stand-by equipment is likely to be so costly in federal taxes that the average concern will hesitate to keep it for what may be possible but not very probable uses in the future. This particular tax penalty has probably not had much effect on equipment retirements to the present time (1942) because industrial executives are only beginning to realize that such a penalty exists; but the problem will be extremely serious in the future if the present tax policies are not modified.

H. L. Ripley,[30] M. Am. Soc. C. E.—One statement in the Symposium that is beyond dispute is the one appearing in the "Synopsis" of the paper by Professor Grant—namely, that "Writers on depreciation seem to agree on nothing except that other writers on the subject are somewhat confused." He might well have left out the "somewhat."

The writer agrees with the point raised by Professor Grant, to the effect that a clear definition should be had. This is easier to say than to do, and has not been done yet. He agrees also that depreciation estimates will enter in different ways into any proper solution of a concrete case, dependent upon the method of approach and the purpose and use to be made of the value after it is obtained.

It is being realized more and more that this word "depreciation," standing alone, has little or no fixed meaning that is recognized in common parlance, and the companion word "value" is in the same category; or, to state it another way, both words have many meanings dependent upon the concept, purpose, or intent of the user, and require a modifier to give them any clear standing in the mind of the reader. This last statement is borne out by the experience of the Interstate Commerce Commission, which found it necessary to make an interpretation of these two words when construing the language of the railroad valuation law, in which the bare words were used. In order to comply at all, the Commission coined such phrases as "valuation for rate-making purposes" (which phrase involved the matter of depreciation as a major factor) and "depreciation" (which was defined or described to fit the use made of it at the time it was introduced).

The position taken by the carriers' organization at that time (of which organization the writer was a member and so must share in the onus of any faulty presentation of principles or recommended practices) was confined also to a reiteration of a didactic assertion, and so perhaps was not very helpful.

Accounting practice is based upon the cost of a unit of property, expressed in dollars expended at the time of purchase or construction, without regard to the changing power in exchange of that dollar during the life in service of the unit of property which it represents, and the writer takes no issue with the propriety of such handling by accountants. When it comes to a question of finding a value at a specific date, however, it should be recognized that cost and value are not synonymous terms, but may be very different. When it comes to computing depreciation in obtaining present worth, that worth or power in exchange of the dollar is just as vital a factor as is the dollar itself. Both its power to purchase a replacement for the unit that has not yet reached the end of its useful life and the time that will elapse between the date selected for the appraisal and the day in the future when that unit will have reached the end of its useful life, and will, or should be, replaced, makes the date when the new dollar will have to be supplied a vital part of the problem. Because a dollar has earning power just as truly as does a machine or other piece of property that it represents, so the time at which the money must be provided as contrasted with the present date is a highly important factor, especially in connection with long-lived units, as present value is the objective of the inquiry. This is a truism that seems plain enough but is often ignored, as was done by the Interstate Commerce Commission. The Society recognized this fundamental principle in the Report of its Committee of 1917,[13] whereas the Commission handled it by the simple process of ignoring it in formulating its rules of procedure to direct its Bureau of Valuation.

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[30] Contract Agt., N. Y., N. H. & H. R. R. (Retired), New Haven, Conn.

All the foregoing means that there is a fundamental difference in the propriety of a method to be used in accountancy and of a method to be used in finding the present worth in a going concern at a designated date for rate making, for purchase and sale, for condemnation of property, and perhaps for other purposes, as noted by other contributors to this Symposium.

Obsolescence and inadequacy frequently are more important factors in fixing the day when a piece of property is, or should be, taken out of service than wear or decay, although there are some exceptions such as railroad ties and the like. Obsolescence "happens"; it does not "accrue" in a straight line or any other line, and the same is true of "inadequacy" to a lesser extent. It is brought about by new inventions, suddenly, or by changed conditions, and so cannot be anticipated for any long period of future time with any close approach to precision. Faced with this stubborn fact, the appraiser, whatever method he may use, through study of past experiences, age-life expectancy tables, trained observation, and good judgment, or whatever tool or process he uses to guide him, must do the best he can and make the best estimate he can.

If he uses well-considered life tables as a tool to aid his judgment, he can proceed in an orderly manner, handling the problem step by step, and he can review his work, check his conclusions, and have some tangible basis for his final judgment. If he attempts to base his appraisal only upon general impressions gathered while viewing the great mass of unlike units that go to make up a railroad or a large industrial enterprise, no matter how experienced he may be, or how carefully he may study, or how sincere he may be, he is attempting that which is beyond the power of the finite mind. Any conclusions so reached will be little better than a "hunch," depending upon the experience and intelligence of the appraiser—or so it seems to the writer, who has tried it.

When a new appraisal of a later date is required, any circumstances that have developed to change the propriety of the previous forecast of life expectancy can, and should, then be studied and the forecast corrected. If the original appraisal is made by the step-by-step method, that can be done, but if it was made wholesale, by the general impression method, the writer doubts if a proper weight can be envisioned and applied to its effect upon the property as a whole in a way that would be consistent or convincing. Therefore, the writer agrees that age-life expectancy tables constitute a proper tool to be used in the solution of the problem.

If the step-by-step and age-life expectancy method is used in both appraisals, then the depreciation in the units as expressed in dollars can be transposed into its effect upon present worth by a simple application of proper factors. The Society's Special Committee on Railway Valuation has shown how to do it in its "Final Report." [31]

A table can be compiled readily at the rate of interest that money will earn at the time of the inquiry. Such a table will greatly facilitate the computations, if the computer will remember that the answer can be only an approximation at best. The answer is not mathematically precise from its very essence, because of the uncertainty inherent in the factors used in the solution by any process, and the fallibility of human judgment.

The important thing is to show the facts—all of them, and not just a part of them—and to handle them rationally in accordance with established principles that take into consideration all their essential implications and limitations. This must be done in a way that is susceptible of demonstration and analysis and can be checked for substantial accuracy. It should make no difference to the appraiser whether the property is publicly owned, privately owned, privately owned but under public control, or whatever its status, in computing the depreciation in physical property or the equivalent dollars that represent it.

The writer realizes that he has not solved the much discussed and controversial problem, and perhaps has not helped it, and that he has qualified many of his assertions because he is today much less prone to say "I know" than he was forty years ago, when he knew even less than he does now.

CARROLL A. FARWELL,[32] **M. AM. SOC. C. E.**—The writer wishes to reinforce Mr. Scharff's prefatory statement to the effect that it is important, early in the proposed program of study of the depreciation problem, to establish basic and simple definitions. The paper by Messrs. Crum and Winfrey is valuable in this particular, in addition to its emphasis on the desirability of the use of sound public accounting methods.

Why should there be given to the word "depreciation," when used without qualification, any meaning other than "decrease in value"? This has been called the "popular" or "dictionary" definition. Admittedly, under this definition, the terms "depreciation" and "value" are indeterminate unless qualified.

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[31] *Transactions*, Am. Soc. C. E., Vol. LXXXI (December), 1917, p. 1464 *et seq.*, and Appendix I.

[32] Cons. Engr. (Fay, Spofford & Thorndike), Boston, Mass.

Depreciation, or a rate of depreciation, may be estimated by an engineer or appraiser, or it may be established by the courts or other authority for a particular purpose. An accountant may make an allowance for depreciation on his books or in his financial statements; however, none of these actions determine depreciation but, instead, make an approximation to serve a particular purpose. The phrases, "estimated depreciation," "established depreciation," and "allowed depreciation," are not particularly cumbersome, and they tell the story without subterfuge. Such phrases as "ascertainment of depreciation," "correct depreciation," and "depreciation concept" will not be used if such an idea is followed.

Engineers sometimes hesitate to qualify their statements accurately as preciseness is often ridiculed, but the writer submits that "depreciation" (as well as "value") is indeterminate and, in using a definite figure to represent it, the engineer should make clear the relationship between the definite figure and the indeterminate quantity. If it is agreed that the unqualified word "depreciation" will be used only for the indeterminate quantity, the next step may well be an attempt to standardize methods of estimating, establishing, or allowing for, depreciation.

Most of the papers in the Symposium deal with the estimating or establishing of depreciation or depreciation rates and as such are valuable. However, it is important, first, that engineers agree as nearly as may be expected upon the definition of depreciation and go on from there to set up a correct and clear nomenclature.

Thomas R. Agg,[33] **M. Am. Soc. C. E.**—The several papers presented in the Symposium on depreciation give the impression that one may consider that there are several depreciations existing in a property at any time and that one may select the particular depreciation that is applicable to the purposes of a particular analysis. This is a fundamental misconception that has retarded greatly the correct handling of the loss of value of physical assets through age and use in valuation and accounting.

Perhaps the confusion that is so apparent has its origin in the fact that the word "depreciation" (*de* = down + *pretium* = price) has a definitely qualitative meaning in the strict dictionary sense but over the years has gradually come to be used rather in a quantitative sense. One of the first steps toward clarification of the subject would seem to be to adopt suitable modifying terms to use in conjunction with the word when it is used in a quantitative sense to indicate the dollar equivalent of the decrease in future usefulness due to age and use, including functional depreciation in so far as it can be estimated. The following may serve to indicate a step toward the clarification of the use of terms.

Actual depreciation (annual or total) is the true loss of value for the period considered, determined by estimating the fair value at the beginning and end of the period. This is the one true figure on depreciation and the one that must be used when the correct annual or total depreciation reserve accounts are being set up and when deductions are being claimed for tax purposes. It is the figure that represents the true loss in value of the property from all predictable causes and, were it possible to overcome the irrational use of the term "depreciation," it might be wise to adopt this conception as the definition of the term for the purpose of engineering economics.

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[33] Dean of Eng., Iowa State College, Ames, Iowa.

Theoretical depreciation (annual or total) is an estimate of the loss of value, during the period considered, which is made according to some formula that does not require the determination of fair value at the beginning and end of the period. Theoretical depreciation may be less or greater than actual depreciation. It might, by merest accident, be equal to actual depreciation in certain cases. The use of theoretical depreciation only, in the management of properties, may lead eventually to financial difficulties.

A point has been made of the difficulty, in a year of low business activity, of providing depreciation appropriations from earnings that will be equal to the true depreciation of the property during that year. Good management would seem to require that the true actual depreciation be known and recognized even then. In fact it must be known if the proper deductions are to be used in tax returns and in fixing the true value of the property. The annual depreciation appropriation can be made according to the exigencies of the business situation, except for businesses whose rates are under public regulation, in which case an annual depreciation appropriation equal to the annual actual depreciation should be made before calculating and distributing the net return.

The several papers make frequent reference to the difficulty of forecasting the future service life of a unit of industrial property because of the impact of obsolescence, inadequacy, and similar elements of functional depreciation. This is a bogey that has long confronted the appraiser, but in reality such factors, when actually unforeseeable by qualified experts, have nothing whatever to do with the problem.

The value of an asset as of today is the present worth of its future operation returns. Since they lie in the future, these returns can only be forecast according to the best judgment of qualified persons; but once such a forecast is made and accepted by the management of an enterprise, it becomes the present value. Those things that cannot be foreseen cannot affect the value until the future has become the present and the now unforeseeable is actually foreseen.

The same confusion exists in connection with estimates of the probable future service life of units of industrial property. The probable future service life is estimated on the basis of various factors, such as:

(*a*) The present physical condition of the unit;

(*b*) The nature of the future service so far as that can be foreseen; and

(*c*) The mortality characteristics of similar property if they have been established.

Functional depreciation that cannot be foreseen at the time has nothing to do with the estimates of future service life; but established mortality data should, and usually do, include the effect of every factor that causes retirements, such as accidents, changes in the art, inadequacy, supersession, and all like factors—that is, present value is an estimate and probable future service life is an estimate. The actual service life cannot be known until it is too late for the information to be of any use in dealing with the actual depreciation of that particular unit of property.

If several well-qualified persons make estimates of the actual depreciation of a property on the basis of their best judgment, and without bias, it is extremely unlikely that they will arrive at approximately identical figures. Try having three civil engineers make independent estimates of the value of a used transit, for example. There is nothing disconcerting in this situation, and the courts have long recognized it to be inevitable. All values rest on the forecasts of the future, and these must always be subject to the uncertainties that characterize the human mind when it seeks to penetrate the future. In engineering economics, however, there is no alternative to the use of the best estimates that can be made by those responsible, and such estimates of fair value and actual depreciation must be accepted for all administrative purposes until the shadows of coming events provide a basis for a re-estimate of the situation.

There are various philosophies about the economic considerations that are applicable to public expenditures for such things as highways, canals, harbors, power plants, dams, and the like. One is that such expenditures may be considered a governmental function just like those required for the courts, fire protection, sanitation, police protection, and similar familiar governmental activities. If that is accepted, including the principle that no particular consideration need be given to the cost of the service so long as there is no misappropriation of funds, then engineering economics has no place in the establishment of policies; but on that basis the costs will inevitably be very much higher than really required. If it is desired to furnish service at the lowest cost, all alternative methods of producing the service must be studied on the basis of the true cost, and this will inevitably require consideration of the actual depreciation to be expected. Such things as annual depreciation appropriations and depreciation reserves will not be required, however.

If the public utility theory of such improvements is preferred, the several facilities will be operated by the government on a nonprofit basis, but revenues in the form of taxes will be exacted in an amount equal to the annual cost of the service. Here again the principles of engineering economics, and especially those relating to depreciation, must be considered if the services are to be furnished at minimum cost.

In the long run the public will be served best if governmental agencies keep, and utilize intelligently, records that will enable them to determine correctly the annual cost of the service of the utility they manage for the public and to make periodic reports summarizing these data.

Conde B. McCullough,[34] **M. Am. Soc. C. E.**—The presentation by Messrs. Crum and Winfrey is timely and valuable. All too frequently have engineers failed to account correctly for depreciation of public properties.

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[34] Asst. Chf. Engr., State Highway Dept., Salem, Ore.

Furthermore, as these authors so aptly point out, there is apparent a regrettable lack of understanding on the part of many engineering administrators in regard to the real nature of depreciation as an element of cost.

Messrs. Crum and Winfrey state (see heading "Value and Depreciation: Definition of Depreciation"):

> "The depreciation of any physical property unit, at any service age, is its loss of value, since its service started, due to decrease in the present worth of its probable future earned operation returns, below what such present worth would be if the unit were still new."

This is a clear-cut definition and should serve to eliminate any misapprehension as to the nature of depreciation itself. It might be enlightening, however, to pursue this definition a little further in order to discover where it leads when reduced to mathematical terms.

Consider a given highway property unit whose first cost is C, and whose assumed service life is n years. Assume that the annual income from this unit will be uniform throughout its service life, and let such annual income be represented by the term I_a.

The "present worth of the probable future earned operation returns" of this property unit, as of the date when it was first put into service, is given by the expression

$$P_{no} = I_a \left[\frac{(1+r)^n - 1}{r\,(1+r)^n} \right] \cdots\cdots\cdots\cdots (2)$$

in which r is the rate of interest or net return. At the end of m years the present worth of the probable future returns from this same unit is lessened or depreciated by virtue of the fact that it has only $n - m$ years of service left. Applying the same formula, the present worth of these probable future returns at the end of the m-yr period will be given by the expression

$$P_{nm} = I_a \left[\frac{(1+r)^{n-m} - 1}{r\,(1+r)^{n-m}} \right] \cdots\cdots\cdots\cdots (3)$$

The depreciated value of this unit at the end of m years obviously may be obtained by multiplying its original cost C by the ratio, $\frac{\text{Eq. 3}}{\text{Eq. 2}}$, or

$$V_{Dm} = C \left[\frac{(1+r)^{n-m} - 1}{r\,(1+r)^{n-m}} \right] \left[\frac{r\,(1+r)^n}{(1+r)^n - 1} \right]$$
$$= C \left[\frac{(1+r)^n - (1+r)^m}{(1+r)^n - 1} \right] \cdots\cdots\cdots\cdots (4)$$

in which V_{Dm} represents the depreciated value at the end of the mth year. If m is made equal to unity the expression for the first year's depreciation is

$$D_1 = C \left[\frac{r}{(1+r)^n - 1} \right] \cdots\cdots\cdots\cdots (5)$$

The depreciation during any succeeding year (say the mth year) can be obtained from Eq. 5 by the formula

$$D_m = D_1 (1 + r)^{m-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Clearly, therefore, the utilization of the definition of depreciation offered by Messrs. Crum and Winfrey (which, incidentally, appears to be the only logical and tenable one) develops a parabolic curve exactly coincident with the well-known curve for constant increment amortization over the same period. It will be observed that the annual depreciation varies from year to year. However, the sum of this depreciation and the interest on the depreciated balance does not vary but remains constant throughout the entire service period. It is given by the formula,

$$C_{ac} = C\left[r + \frac{r}{(1 + r)^n - 1}\right] \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

This is a very important fact in that it indicates a method whereby the total capital cost (C_{ac}) of any project may be calculated by a single simple formula (Eq. 7) once the service life is determined and the interest rate established.

As shown by Messrs. Crum and Winfrey, the determination of the probable service life expectancy n of any individual highway property unit presents a problem difficult of exact solution. However, the work being done in connection with the various statewide highway planning surveys will eventually yield data from which this value can be estimated with some degree of certainty.

The term r is usually defined as the "fair rate of net return." Messrs. Marston and Agg define[7] this as "that rate which is just large enough to secure what capital is really needed for investment in enterprises of the particular character." In view of this definition the rate r is easily established as identical with the net interest rate fixed by the highway bond market as of the time of construction. With the foregoing data, therefore, it is only necessary to multiply the first cost of any highway property unit by the term $\left[r + \frac{r}{(1 + r)^n - 1}\right]$ in order to derive the annual capital cost—that is, the value of annual depreciation plus interest on undepreciated balances.

The foregoing derivation is based upon the hypothesis that the net rate of return will be uniform throughout the service life of the property unit in question. Such an assumption is contrary to fact in the case of many highway property units because of the possibility of changes in traffic density, or of a modification of rate structure by legislative fiat, and for other reasons. For the valuation of industrial property units it is often possible to take into account any modification in future operating return ratios; but for highway property units there appears to be insufficient data at present to support such a refinement in analysis.

The foregoing discussion is predicated upon the hypothesis of complete depreciation. In the case of highway property units, it is often possible to salvage a certain percentage of the original value for utilization in the reconstruction. If it is possible to estimate the salvageable component V_s of any unit, the depreciation curve may be adjusted to take such fact into consideration, as indicated in Fig. 3. An inspection of this figure will indicate that, if

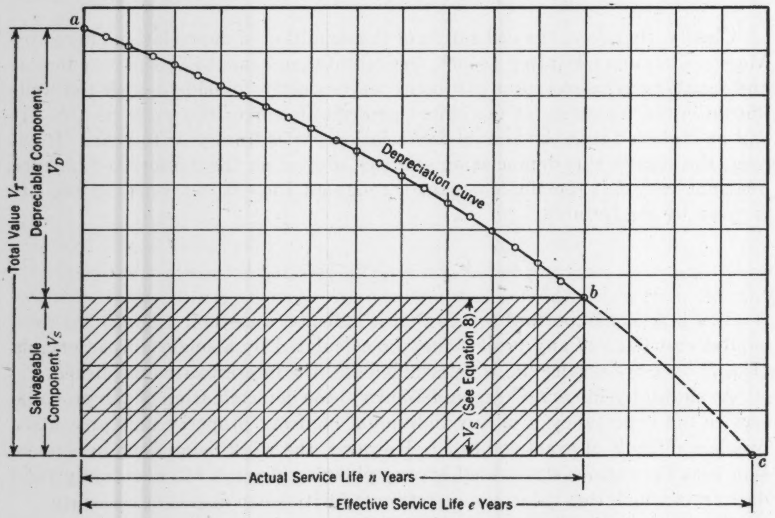
a
Total Value V_T
Depreciable Component, V_D
Salvageable Component, V_S
Depreciation Curve
b
V_S (See Equation 8)
c
Actual Service Life n Years
Effective Service Life e Years

FIG. 3

the depreciation curve is prolonged beyond the point b to intersect the base line at point c, a new life value e is determined. From the relationships previously demonstrated for depreciation curves, in general, one may then write:

$$V_s = V_T \left[\frac{(1+r)^e - (1+r)^n}{(1+r)^e - 1} \right] \qquad \text{(8)}$$

from which:

$$(1+r)^e = \left[\frac{(1+r)^n - \dfrac{V_s}{V_T}}{1 - \dfrac{V_s}{V_T}} \right] \qquad \text{(9a)}$$

Whence:

$$e \log (1+r) = \log \left[\frac{(1+r)^n - \dfrac{V_s}{V_T}}{1 - \dfrac{V_s}{V_T}} \right] \qquad \text{(9b)}$$

and

$$e = \frac{\log \left[\dfrac{(1+r)^n - \dfrac{V_s}{V_T}}{1 - \dfrac{V_s}{V_T}} \right]}{\log (1+r)} \qquad \text{(10)}$$

The computation of capital costs can then be handled by substituting the effective service life *e* for the actual service life *n* in Eq. 7.

The purpose of the foregoing discussion is simply to translate the Crum-Winfrey definition of depreciation into mathematical terms and to indicate that, if it is combined with the interest charge on undepreciated balances to form a total (which may be termed the capital cost), it can be computed by a single simple formula and will remain as a uniform charge throughout the years.

As stated, the only unknown quantity is the service life expectancy *n* or its equivalent *e*. In regard to highway property units, these data are being obtained in connection with the various state-wide planning surveys so that as the years pass it will be possible to make closer approximations for the initial setup. Moreover, since it is possible by means of Eqs. 2 to 10 to calculate residual values at the end of any given year, the value of *n* may be revised at any time during the service life of the property unit and depreciation schedules modified accordingly. Even though the initial assumptions are somewhat in error, it is possible, therefore, by means of frequent inspections and appraisals, to modify depreciation schedules so that they may be brought into closer coincidence with actual depreciation facts.

W. V. BURNELL,[35] ESQ.—The question of what to do about depreciation of public utility property, although it has been dealt with at great length and by many people, still remains a question. Is it necessary to do anything?

Irrespective of any requirement that has been or may be imposed by regulation, the fundamental answer is simply this: Prudence in business management requires that provision be made in advance for anticipated future losses—losses that occur when the productive usefulness of a property is lowered, and that culminate in the complete retirement of property from service when its usefulness has finally terminated. The making of such a provision is a process by which certain assets are earmarked as a reserve for this one purpose and no other—a process generally referred to as either retirement or depreciation accounting.

During the useful lifetime of a steam boiler, for example, the boiler itself is consumed, as definitely as the fuel that is burned in it; and this consumption of capital property is as truly a cost of producing steam as the cost of the fuel. If this process of retirement or depreciation accounting is to serve its purpose, the total cost of the boiler thus will eventually appear on the books of account as an expense of doing business.

At this early stage in the approach to the subject, a much debated question arises: "By what particular mathematical program should these retirement or depreciation charges be made?" Aside from the factor of regulatory compulsion, the question can never be answered categorically for all cases; each situation requires individual consideration in the light of its own surrounding conditions and circumstances.

Over the active life of a property a reserve must be built up, say from 0% at *A* (beginning of life) to 100% at *B* (end of life), as a cost of service. The best method of laying out the path from *A* to *B*, or of allocating this cost of service, cannot be determined without consideration of the price at which the service can be sold. In a particular situation the path from *A* to *B* may best be that of a straight line; in another, a convex or a concave curve; in still another an irregular course, all depending upon the economics of the particular situation and the relative ability of the "traffic" to bear the charge at one time as compared with another. However, it always must be remembered that the total cost must be "set off" sooner or later against the total revenue. Where the responsibility for decision is left with those who have been entrusted with the management of a property, it may be suggested, as a general principle, that this cost of service may better be collected from the customer a little too early than too late—when the customer may decline to pay it.

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[35] Stone & Webster Eng. Corp., New York, N. Y.

No one, however, may properly conclude that business prudence invariably requires the rapid "writing off" of plant investment during the early years of its use. To do so, in many cases, would stifle expansion of the business by showing "losses" not in fact realized and, therefore, would be most imprudent. To have followed the "rapid-write-off" policy in the electric power industry, for example, would have resulted in a much higher "cost of service," with attendant higher rates, and thus would have discouraged the development of that industry.

From the viewpoint of basic principles, the foregoing statement is perhaps sufficient for present purposes because management is confronted with a condition, rather than a theory, with the artificial factor of regulatory requirements, and with certain extremely important corollary implications. This regulatory development, it is safe to say, is predicated on the underlying contention that management, or those who have been in control of management, in many cases have set aside insufficient reserves and, therefore, have failed to fulfil their responsibilities; that the reduction in the net book value of the property has not been enough to compensate for the depreciation actually suffered; and that the management has thus abused its privilege.

There can be no doubt that in many instances the accrued reserves are clearly too small and that dividend policies have often been too liberal, at the expense of the reserve. This is not to say, by any means, that a failure of the reserve to equal the amount determined by a particular accounting formula is sufficient justification for a conclusion as to the present adequacy of the reserve.

Management is now confronted, however, not so much by a theory as by a condition that imposes upon the custodians of industry an urgent obligation to contend vigorously and constructively for an equitable solution of the problem, which will definitely preclude the possibility of a confiscation of property. Otherwise, the advocates of a most unfavorable formula may succeed in carrying its application to the rate base and then "prove" that they are right by being in a position to enforce artificially a reduction in value which, as a matter of economics, has not occurred in fact.

Unless the management has labored long under a grievous misapprehension, the books of account of a public utility, like those of any business enterprise, do not purport to account for value, but for cost. In particular reference to the depreciation reserve, if the management is correct, it is not within the province of the accountant to pass judgment as to the propriety of its size on the assumption that it is a function of value; but rather the accountant should regard it as a provision for eventual loss in terms of cost, and point out, purely as a matter of fact, the amount which the management has set aside, in anticipation of the loss.

A concept of depreciation accounting has been suggested, under which the reserve and the current appropriations to it would be determined on the basis of periodical engineering studies of losses actually realized, as opposed to the concept that the reserve should contain also a provision for future losses. Underlying this concept of accounting only for losses realized is the evident desire to harmonize the reserve with the amount of depreciation claimed in a rate case. Although this objective of "consistency," in the equity of the transaction as between the investor and the customer, is entitled to emphatic approval, there is a more practical and prudent way in which that objective can be attained.

If the contribution to the reserve actually made by the customer, as a recognized factor in the rates charged to him, is in excess of the deduction, if any, which is made in arriving at the rate base, let his current burden be reduced by an interest credit on the difference, at an interest rate commensurate with the characteristics of the reserve as a senior obligation. Thus, the requirement of "consistency" would be satisfied simply and effectually.

The next move is clearly up to the public utility industry. It must defend itself to be sure; but that is not all. It is not enough to oppose a program that is patently unfair. Its job is to propose a better one. Surely it is not constructive to attack all age-life programs on the mere ground that no one can predict, reliably, the life expectancy of utility property, for the most ardent proponents of such a program will admit the point and the corresponding propriety of a periodical adjustment of the estimates. Is it not a waste of valuable time, also, to argue for a return to the previously approved policy of retirement accounting, where a transition is now required to depreciation accounting? Would it not be a definite and constructive contribution to the solution of a serious problem if industry would adapt itself, as best it could, to regulatory preferences, so long as the principle of equity is not violated?

Any requirement that a reserve which was adequate under the previously authorized principles of accounting be suddenly built up out of surplus to meet the so-called deficiency should be "fought to the last ditch" and without compromise of principle, on the ground of the inequitable nature of such a requirement. From here on, however, why not meet the issue realistically? Develop an age-life program that will include a provision for the normal current requirements, plus an increment to recoup, gradually, the theoretical "deficiency" or, if the case requires, a minus increment to reduce an over-accrual, so as to bring the reserve to the theoretical requirement at some reasonable time in the future.

Management has available a wide range of latitude within which a specific program can be developed that will be the most suitable to the circumstances, and also within the means, of almost any individual situation—all in accordance with the technical requirements of regulation and, which is more important, will accomplish the necessary transition in a manner that will be completely equitable to all the parties who have a legitimate interest in the problem.

ROGER L. MORRISON,[36] **M. AM. SOC. C. E.**—During the period when road construction activities were confined primarily to making main highways passable under all weather conditions, the economic advantages of such construction were too obvious to require much study. Under present conditions, however, with large expenditures being made and projected for the further improvement of roads that are already surfaced, economic factors must receive careful consideration if the funds are to be spent wisely. Depreciation, of course, is one of the most important of these economic factors and, as Messrs. Crum and Winfrey point out, it is one that has been largely ignored in the past, at least in accounting.

The definition of value used in this paper is "anticipated worth of future services" or "present worth of probable future service" (see heading "Value and Depreciation: Definition of Value"). One definition given for depreciation is "negative value." This does not seem to be quite consistent, because zero value would mean that future service is worth nothing at present, and negative value would seem to be a condition under which future service would be actually detrimental, as, for example, the present worth of the future "service" of termites working on a building. It is possible, of course, that an acceptable definition of "negative value" would make it synonymous with "loss of value." The point is of little importance anyway.

After the statement that "Depreciation is negative value * * *" appears the following (see heading "Value and Depreciation: Definition of Depreciation"):

> "The depreciation of any physical property unit, at any service age, is its loss of value, since its service started, due to decrease in the present worth of its probable future earned operation returns, below what such present worth would be if the unit were still new."

This covers total depreciation over any period from the beginning of service but omits a definition of depreciation in general. For instance, the depreciation of a property during its second five years of use could not be defined as "its loss of value since its service started" or as its loss of present worth, at the end of the ten years, compared to what it "would be if the unit were still new." A possible way to make the definition more general would be to substitute the clause "during any period of time" for "since its service started" and to substitute "below what its value was at the beginning of the period" for "below what such present worth would be if the unit were still new." Such a change would tend to complicate the definition, but the original does not seem to be exactly a "definition of depreciation."

The expression "probable future earned operation returns" is open to various interpretations when applied to a highway, and its interpretation is still more difficult when applied to a schoolhouse or other public building. Perhaps in the latter case the earned operation returns would be the rent that would have to be paid for similar accommodations in a privately owned building.

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[36] Prof. of Highway Eng. and Highway Transport, Univ. of Michigan, Ann Arbor, Mich.

In the case of highways, there is a tendency to divide the total vehicle taxes paid in a year by the total vehicle-miles traveled to determine the average tax per vehicle-mile, and then to compute the "earned operation returns" on any given road by multiplying the annual vehicle-miles traveled on the road by the average vehicle-mile tax. The writer objects to this because the motorists, or other taxpayers, finance road improvements in order to get returns in the form of reduced vehicle operating costs, or otherwise, and not to produce tax money for the states, counties, or other government units of highway administration. Considering the purposes for which the improvements are made, it would seem that their "earnings" are their returns to the motorists, or others, who pay for building them. The paper does not include an interpretation of earned operation returns.

The authors ask the question (see "Summary"), "Why * * * could not every state legislature, school board, and city council provide about a constant sum annually for building replacements and extensions?" One answer is that the needs are seldom constant. Population in any given locality may increase rapidly during one period and then increase slowly, or even decrease, during another period. Also, it is obviously much easier to raise funds for such purposes during prosperous times than during financial depressions.

Messrs. Crum and Winfrey have given extensive study to problems of highway economics and have done much to guide the economic thinking of highway engineers along sound lines. Their paper calls attention to points in regard to depreciation that will be of increasing importance if large sums are borrowed after the war and spent on highways primarily to furnish employment. It is under such circumstances that sound economics is most likely to give way to other considerations.

WALLACE B. CARR,[37] M. AM. SOC. C. E.—It is significant that the "Foreword" by Mr. Scharff and the papers by Mr. Walls and Professor Grant stress the state of confused thinking with which the question of depreciation continues to be surrounded. The Symposium is a decidedly progressive attempt to clarify this confusion.

Depreciation of property is essentially a term related to worth or capital represented by the property. This is indicated by its root derivation as well as by Professor Grant's three concepts of its meaning—"Decrease in Value," "Amortized Cost," and (the third one rephrased) Difference in Value of Two Assets Capable of the Same Service. This conception is also the basis of the definition written by Dean Marston in the paper by Messrs. Crum and Winfrey (see heading "Acknowledgment"). In all cases, however, depreciation should be determined factually or measured by the physical, functional, or economic ability to perform a prescribed or required service. The future anticipated period of such service is an important factor in the process of measurement.

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[37] Engr. in Chg., Special Studies, Buffalo, Niagara & Eastern Power Corp., Buffalo, N. Y.

This twofold significance of the term is responsible for much of the existing confusion. The factual measurement of depreciation, when expressed in terms of worth or capital, becomes a measure of the actual integrity of such capital. Accounting for depreciation, however, is generally treated as an amortization process, the rate of amortization being related to one of several methods of mathematical calculation and, in most cases, not to the facts of the case. Unfortunately, however, whatever process of mathematical calculation may be used, it is sometimes assumed that the results express depreciation correctly. Even though the period of amortization may be exactly equal to the useful life of the property, it does not follow that the factual depreciation has been equal, at all times, to that calculated by some mathematical law.

In discussing the problem of "Economy Studies for Proposed Retirements" in a competitive (unregulated) industry, Professor Grant states that "The only 'depreciation' involved is the difference between the net amount realizable from the immediate sale of the asset and the net amount realizable from its future sale." This statement may be open to question from a certain viewpoint. For instance, assuming a capitalistic economy, the objective or function of depreciation accounting with which this argument is really concerned could be regarded as the protection of the integrity of the invested capital. Another principal objective in this type of economy can be regarded similarly as the earning of a proper wage of capital while it is usefully employed. The accumulated accomplishment of the first objective would be reflected in the depreciation reserve, whereas that of the second objective would be reflected in the earned surplus account plus the accumulated payments of interest and dividends to the investors. Any deficiency of the reserve (which, according to Professor Grant's argument, should be charged to surplus) could not alter the fact that the reason for the charge would be depreciation. The procedure to record the facts properly would be to transfer the required amount from surplus to depreciation reserve and then to charge the retirement loss to the reserve. From this viewpoint, the answer would be simply that the reserve and surplus accounts had been stated improperly, rather than that the retirement loss should not be regarded as depreciation.

Carrying this viewpoint to a regulated industry, or to an industry subject for some period of time to excess profits taxes, the surplus might represent nothing more than the residue (after payments to investors), of a minimum, or even less than a fair, return. If in such a case the deficiency of the reserve for a retirement loss would not be chargeable as a depreciation expense in the future (which admittedly would mean a charge against the new asset), or if as a charge against present or future net income it would reduce that below a fair level, then one or the other of the fundamental objectives of a capitalistic economy would have been abandoned. Although it is true that in group-depreciation accounting this problem might not be of serious moment for the reason that under-accruals on short-lived items will be offset by over-accruals on long-lived items, nevertheless this situation should not obscure the fact that the objective of proper depreciation accounting should be the complete protection of the integrity of the invested capital.

Professor Grant's discussion of replacement valuation in appraisal matters is interesting. Theoretical economic support of this theory can be advanced, of course. It simply means that services at any time should be worth not more than the cost of producing such services by the most economical method. Practically, it may not be possible to achieve this ideal objective in all cases and, therefore, due regard to such a practical situation should be observed in the application of the theory in appraisals of existing property. If this theory were applied consistently to all enterprises in a given industry, and the market price of the product were based fairly upon the most economical production cost, the plants of higher production cost would meet this competition through the medium of depreciated plant values. However, it is questionable whether such a rigid rule can be invoked in the fixing of market prices or whether it is economically wise that it should be done. The question might be presented as to how to value, by this theory, a new plant of higher production cost than the most economical existing plant, the new plant being required to supply the market demand. Surely the new plant must not immediately, or even a year after completion, be depreciated to an "equal cost level" with the old plant. Rather, it might be that all the old plants would be appreciated to an "equal cost level" with the required new plant. Here again, as in all questions of value and depreciation, is a broad field for the application of sound engineering and business judgment. The effective leverage exerted by a small variation in the assumptions or estimates upon the final answer is well illustrated in Professor Grant's examples and should be a warning that judgment should be used generously in reaching a decision.

No discussion of depreciation can escape dealing with such terms as service lives, mortality curves or tables, etc. Mr. Walls discusses this phase of the problem very ably and emphasizes the character of the forces that control the useful life of physical property—in other words, the forces that cause depreciation. He states correctly that depreciation is of two kinds, physical and functional. The forces causing physical depreciation are wear and tear, decay, and normal action of the elements, all of which are controlled largely by natural laws and, therefore, operate quite continuously and with reasonable uniformity. Unlike these physical forces, those causing functional depreciation—inadequacy, obsolescence, changes in the art, and changes in demand and requirement of public authority—are essentially economic in character. They have no such natural uniform control, but rather they reflect the activities of mankind, such as his accomplishments in the field of invention, cycles and growth of business, movements and habits of people, etc. The incidence of their effect is erratic, therefore, and cannot be predicted correctly far in advance. It is folly to assume that mortality studies of historical data of capital retirements can always, or even generally, be used as a guide to future life with any hope of reasonable accuracy in the prediction. In the first place, mortality data of physical property generally reflect the composite effect of all causes of retirement, whether they are physical, functional, or a combination of both. Future retirements likewise will be made for various causes, some physical and others functional. In the past, functional causes have been a predominant element in by far the largest number of retirements of property in regulated industry. The lives indicated by statistical analysis of mortality data as available from the past, therefore, are largely "functional" lives reflecting such economic or industrial conditions as were at work during the period of the data. To the extent that such statistically determined lives reflect the effect of these artificial forces of depreciation, the projection of these lives into the future is surrounded with a broad assumption of the continuity of largely artificial conditions.

If the causes of depreciation are thus differentiated, it then becomes apparent that physical property should likewise be classified as to its susceptibility to these various causes. This fundamental aspect of the problem has been brought out only indirectly in the Symposium. For instance, a large unit of property may have physically depreciable parts of such proportion and probable service life as to be equivalent from the viewpoint of turnover of capital to the complete depreciation of the entire unit in some period of time. If these physically depreciable parts are replaced continuously, as necessary, does physical depreciation proceed in the other parts of the unit in accordance with the equivalent over-all life, or are those parts immune to physical depreciation? Do they not await only the incidence of functional depreciation as, when, and if, it approaches in accordance with actual knowledge? If such a situation is true, progressive depreciation, calculated by some over-all formula or simply charged against property on a "hunch," would not represent factual depreciation because it would not be measured by a known impaired ability to perform a prescribed or required service. Rather, the calculation would represent amoritzation.

In general, the Symposium is sufficiently replete with warnings that judgment must be a very generous ingredient in any method of estimating or determining depreciation. The writer agrees heartily with Mr. Walls' inference that there is no substitute for sound engineering and business judgment.

W. L. Waters,[38] **M. Am. Soc. C. E.**—Depreciation is the result of (1) wear and tear, and (2) obsolescence. In some cases, such as shoes, wear and tear is all-important. In others, such as women's hats, obsolescence is everything. Frequently, the average rate of wear and tear can be roughly approximated; but obsolescence is usually a pure guess, as it depends on the future progress of the art and the development of the property using the facilities. Therefore, the average over-all rate of depreciation to be allowed in any case usually cannot be anything more than an educated guess.

The Interstate Commerce Commission has had fifty years of experience with depreciation rates, so that its opinions can be considered authoritative. Its instructions are:[39]

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[38] Cons. Engr. (Bury & Waters), New York, N. Y.

[39] This quotation is from Section 8 of Special Instructions entitled "Depreciation of Fixed Improvements," on p. 33 of the Issue of 1914 of the I.C.C. "Classification of Operating Revenues and Operating Expenses of Steam Roads," to which Section reference is made in the various depreciation items in that Classification Order. This wording is repeated in Section 24, entitled "Depreciation Accounting," on p. 4 of the I.C.C. Order of June 13, 1934, entitled "In the Matter of Modifying the Uniform System of Accounts for Steam Railroads."

> "Depreciation charges shall be based in each instance upon the percentage of the original cost (estimated if not known), ledger value, or purchase price of the property determined to be equitable by the carrier's experience and the best sources of information as to the average current loss from depreciation."

This opinion indicates that the annual depreciation rate to be allowed is only an educated guess. It would be well for engineers to remember this, and not delude themselves with long-winded discussions that may only result in more muddle-headed concepts of the subject.

A. G. MOTT,[40] M. AM. SOC. C. E.—Professor Grant has contributed a remarkably well-stated paper on the subject of depreciation. It represents much progress of thought since the Final Report of the Special Committee of the Society to Formulate Principles and Methods for the Valuation of Railroad Property and Other Public Utilities published in 1917.[41]

In its broadest sense (and this is designated by Professor Grant as the "popular depreciation" concept), depreciation in its relation to value may be defined as loss of value from any cause. (Increase of value, or appreciation, may be properly included in this definition if recognized as negative depreciation.) To apply depreciation properly in the art of engineering appraisal, this broad definition requires a classification of causes. The various causes of depreciation appear to be fairly well agreed upon in the literature on the subject, and for convenience are listed herewith:

1. Physical causes, including wear, tear, decay, action of the elements, and the effect of casualties or accidents that have actually occurred prior to the time of appraisal (it will be noted that casualties or accidents that have not yet occurred cannot be included as a contributing cause of loss of value. In accounting practice, however, it is customary to determine the rate at which capital costs shall be spread over the period of use from an experience study of the average lives of the property. The effect of casualties is included in this experience. Depreciation reserves so accrued are in fact also insurance reserves);
2. Functional causes, including inadequacy, non-adaptability, changes of design, and other functional reasons for supersession;
3. Economic causes including the effect of legislative enactments, population changes, changes in habits, changes in the prices or supply of raw materials and labor, and changes in the market for the output; and
4. Monetary causes including the effect of changes in price levels, and other factors affecting costs of reproducing the property being appraised.

Since, by definition, depreciation is a loss of value, a proper application of depreciation can be made only after an understanding of the processes of arriving at value. If value can be ascertained directly at the beginning and at the end of the two periods for which depreciation is being considered, the difference between the two values gives, directly, the amount of the depreciation; but, for appraisal purposes, it is a sum that has no significance, since the result (value) has already been obtained directly for the two periods. The usefulness of depreciation so obtained is to serve as a guide in predicting economic life or in forecasting future depreciation. Ordinarily, depreciation is merely one of the tools by which the appraiser finally arrives at value; and one of the misapplications of the use of this tool arises out of some confusion as to the nature of value itself. One of the most common errors in using the concept of depreciation is the attempt to use it as a device to convert, directly, some form of a cost figure to a value figure. This misuse of depreciation is excellently illustrated by Professor Grant's example of the $6,000 home bought in 1932 and sold in 1940 at the same price. One party to the transaction, translating original cost to present value, found no depreciation, whereas the other party, translating current reproduction cost to present value, found a $1,500 depreciation. (For the purposes of making this point, it is assumed that the sales price in this illustration is identical with the present value. The accrued depreciation found by the accounting process is $1,000, but this sum could not be subtracted from either original cost or reproduction cost to obtain value. The accounting process was for the purpose of spreading original cost over the life of the depreciable asset—a very different purpose from that of finding value during the useful life of the asset.)

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[40] Director, Valuation Div., State Board of Equalization, Sacramento, Calif.

[41] *Transactions*, Am. Soc. C. E., Vol. LXXXI (December), 1917, p. 1448.

In general, where the property being appraised is of a type that is being bought and sold frequently in a free and competitive market, the prices of such transactions may be conclusive evidence of value, and no determination of depreciation is necessary to arrive at value. The types of property which engineers are ordinarily called upon to appraise, however, are more commonly properties for which so few actual transactions take place that other tests must be applied to determine an appraisal which may be considered a fair substitute for market value.

The determination of value then depends upon the sound application of the appraiser's judgment to all the evidence of value that he is able to accumulate. Among the evidences of value more commonly considered are such factors as: Original cost of the property, the cost of reproducing the property new, the value indicated by a capitalization of earnings, and (for corporate properties) a value indicated by an appraisal of the stocks and bonds of the corporation. In a proper use of these or other factors, care must be taken so that any depreciation which properly applies to the particular factor is taken into account. (Depreciation is here used in a restricted sense; for example, cost figures are modified to express a comparison between the existing property and a similar new property to reflect physical and functional differences from a cost point of view; also the depreciation in the earning value is expressed by the difference in the present worth of the future estimated earnings for the existing property as compared with a new property.) By this means the appraiser can be more certain that he has in fact included somewhere in his appraisal all elements of depreciation in their proper perspective.

Of the factors mentioned two are definitely cost factors; namely, the original cost and the cost of reproduction. When depreciation is applied to these cost factors, they remain nothing more or less than depreciated cost factors. (Professor Grant has well stated that those classifications of accounts which have defined depreciation in accounting as "loss in value" have contributed to confused thinking on the subject.) They have not become by that process a statement of value in themselves. The factor of earning value should have an allowance for depreciation. The proper earnings to capitalize are those that may be reasonably expected to be earned in the future after having allowed for all anticipated operating expenses, including the depreciation that will continue to accrue over the period for which the earnings are estimated. Obviously, the factor based upon the appraisal of the securities, properly arrived at in the first instance, would have reflected in it all forms of depreciation.

Another cause of confusion in the use of depreciation for appraisal purposes arises out of two different, but somewhat related, concepts for the term. These two concepts are depreciation as affecting value, and depreciation as a means of spreading the cost occasioned by the ultimate replacement or retirement of the property over the period during which the property was used.

The use of depreciation for value purposes is primarily an engineering and economic phenomenon, whereas the spreading of depreciation to current expense is partly engineering and partly economic, but largely an accounting and managerial function.

The application of age-life concepts of depreciation for value purposes should exclude the lives of those particular units which have already been retired and are not included in the property actually being appraised. On the other hand, in setting up rates of depreciation to include in operating expense, it is proper that the cost incurred because of the retirement of the short-lived items of property also be included. Since much of both the theory and practice of depreciation is related to time (age-life, for example), it follows that the time value of money (interest) must be a necessary element in its measurement. The value concept is based primarily upon an appraisal of the future usefulness of the property, whereas the depreciation expense concept is based upon a requirement to make good the total consumption of the depreciable assets, including those already retired.

The foregoing differences indicate some of the reasons why a properly accumulated reserve for depreciation (referred to by Professor Grant as "amortized cost") will not represent, accurately, the amount of depreciation that should be deducted, even from a cost figure, to reflect a properly depreciated cost figure for consideration as evidence of value.

David A. Kosh,[42] **Jun. Am. Soc. C. E.**—Depreciation is an economic problem, and only secondarily a technological one; and unless this is realized, much confusion may, and usually does, arise.

Messrs. Crum and Winfrey define depreciation of a unit (see "Value and Depreciation: Definition of Depreciation") as "its loss in value, since its service started, due to decrease in the present worth of its probable future earned operation returns, below what such present worth would be if the unit were still new." This definition is very restrictive since it places major, if not sole emphasis on the technological aspect of depreciation. It implies that if the equipment were new, there would be no depreciation! A vital distinction must be made between "Maintaining the Plant," and "Maintaining the Investment." The former is primarily a technological problem—that of keeping the plant at its optimum condition of newness; but providing for depreciation must do far more than that. The value of the investment must be maintained, and not merely its physical manifestations. "Until financial reports take full account of all items of annual costs," write Messrs. Crum and Winfrey in the "Summary," "and account for investment in physical properties, they must be regarded as only partial truths." This is quite true, but "investment in physical properties" is not synonymous with "physical properties," and these concepts may not be used interchangeably. A further difficulty arises when the service life of a piece of equipment exceeds the duration of the accounting period. Hence, the appraiser must make some allocation to this "annual cost" on account of depreciation—the annual provision for depreciation. The basis for such allocation is the crux of the problem. Here Mr. Wall's suggestion in the last paragraph is most important:

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[42] Research Asst., Public Utilities Dept., New York Univ., New York, N. Y.

> "The problem that lies before engineers seems to be rather definite and is certainly pressing: First, a sound basic concept or common understanding of the true meaning and scope of depreciation must be developed; second, an orderly and basically sound procedure for measuring depreciation must be formulated * * *."

(The writer however sees no reason why the problem belongs primarily to the engineer.)

Although the following discussion is centered about depreciation in the public utility industries, the principles apply to depreciation in industry in general, both privately and publicly owned. The generally accepted scope of depreciation (as well as a listing of the major causes thereof) has been stated as follows:[43]

> "Depreciation, as applied to electric plant, means the net loss in service value not restored by current maintenance, incurred in connection with the consumption or prospective retirement of electric plant in the course of service from causes which are known to be in current operation and against which the utility is not protected by insurance. Among the causes to be given consideration are wear and tear, decay, action of the elements, inadequacy, obsolescence, changes in the art, changes in demand and requirements of public authorities."

This is substantially the same as the definition used by the Federal Power Commission; and, although it was formally applied to electric utilities, it is entirely appropriate in the same sense to all types of plant. In the light of the foregoing definition, Messrs. Crum and Winfrey elaborate on only a minor part of depreciation, neglecting the vital portion. They state (see "Value and Depreciation: Definition of Depreciation"), "Thus, depreciation is just a question of values at different dates based on forecasted (probable) numbers of years of future service and probable values of the yearly services." The "probable values" depend on strictly non-technological factors; hence, the writer cannot agree with the contention (under the same heading) that "Such forecasts can be made correctly only by the sound judgments of impartial persons who are qualified to understand the mortality and other technical characteristics of different kinds of physical property units, and who personally examine the particular units whose depreciations are being determined." Messrs. Crum and Winfrey are concerned with technological depreciation only —wear, tear, decay, and possibly depreciation due to changes in the arts. It must be realized that it is obsolescence, and its usual antecedent, change in demand, that is responsible for from 70% to 75% of all electric utility depreciation. It was not technological depreciation that so worried the railroad industry but a short time ago; it was not the lack of newness of its equipment. It was rather the "economic depreciation" engendered by the business cycle, and the reduction in demand resulting from motor truck and pipe-line competition. A change in demand will leave a brand new trolley quite depreciated. Inflation might reduce, by a considerable extent, the previously calculated depreciation on an old hydro-plant.

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[43] "Uniform System of Accounts for Electric Corporations," New York Public Service Comm., p. 2.

So much for the annual provision for depreciation necessary to show the real annual costs. What of total accrued depreciation? Messrs. Crum and Winfrey state quite correctly (see "Distinction Between Depreciation and Maintenance") that "Depreciation losses of value * * * are something which cannot be made good in a physical way until new units are substituted for the depreciated units at the ends of their service lives." How do they propose to make good "losses of value" that cannot be made good in a "physical way"? They propose "* * * to create a depreciation reserve, which should be at all times just equal to the total accrued actual depreciations of all existing property units. In this way the correct annual cost is known and the depreciation reserve account shows the true loss in value of existing property, from which, of course, the present true value of existing property is determined." There are major objections to this proposal:

(1) The authors consider only physical property units, and further assume, quite correctly, that the sum of the depreciation of individual units is the "total accrued actual depreciation" in the plant. A new fleet of trolley cars with no market should, on the foregoing basis, have no depreciation reserve, should show no "loss in value," and should have a true value of presumably the cost of the new cars.

(2) If the words "total accrued actual depreciation" are not intended to restrict the reserve to physical depreciation, then it is submitted that a depreciation reserve adequate to maintain the investment can never, except by accident, measure the "true loss in value." It cannot be denied that physical depreciation is the more certain, and more easily predicted and measured, and that depreciation resulting from obsolescence and loss in demand cannot even be predicted with any kind of statistical certainty. Such depreciation may not even be classified appropriately as a risk, but rather as an uncertainty.

Safety factors are nothing new to the engineer. Hence, such a depreciation reserve must carry in it an item very like insurance, dependent not on precalculated risk probabilities, as is insurance, but on true uncertainties. Thus such a reserve would exceed the accrued depreciation by this "insurance" amount. This item would be in the nature of a prepayment by the customers for economic depreciation not yet matured, but as sure to mature as physical depreciation. Such a reserve certainly could not be construed as a proper deduction from any figure to yield present value. This prepayment, by consumers of the service, may in no case be considered as a disinvestment of the plant owners. However, the consumer would be entitled to a return on this prepayment, until the contingency which it anticipates actually matures. Thus a reserve, as proposed by Messrs. Crum and Winfrey, is inadequate to maintain the investment, whereas a reserve adequate to accomplish that purpose is not a suitable deduction for present value determination. This is the reason for the insistence that a clear distinction must be made between maintaining plant and maintaining investment.

Although some of the foregoing details are predicated on the assumption that the users of the service should pay its entire cost, the major portions of the thesis would apply equally well to such services as highways or public water systems. In the latter instances, the community as a whole is the consumer and hence must bear the cost. An unused road is just as depreciated as an unused manufactured gas plant. The depreciation has occurred and has been paid for in advance; and an accomplished fact must be recognized.

Professor Grant is correct in holding that, "If the 'decrease in value' shown by an appraisal based on the cost of reproducing the service with the most economical modern plant should be much greater than the accounting depreciation, one might reasonably infer that the accounting depreciation was inadequate" (see "A Suggestion Regarding Depreciation Accounting"). Since he considers the service, and not the original physical plant, the economic, as well as the technological causes of depreciation, have been included. Hence, Professor Grant's method constitutes a good check on how much depreciation has occurred; it is a good historical milepost. However, it is an inadequate solution for the real problem—how much depreciation is occurring, and (which is of even greater importance) how much will occur in the future. It is usually economically impracticable to collect past uncollected depreciation costs. When such an attempt is made in an active market, the development of the market is stifled. When depreciation, resulting from a reduced demand, is felt, it is then impossible to add extra depreciation charges without further diminishing a collapsing market. As a rule, a large portion of the depreciation reserve must be in the nature of a prepayment if the reserve is to serve its purpose. The sad effects of inadequate depreciation reserves, caused by a misunderstanding of the rôle of economic factors in depreciation, are clearly seen in the street railway industry, the interurban lines, and the domestic market of the manufactured gas industry—to mention but a few.

If it is argued that change in demand and obsolescence do not belong in the list of causes of depreciation, that the Federal Power Commission and the New

York Public Service Commission, among others, are in error in so listing them, that change in demand is a risk to be taken by the investor and hence not to be insured by the consumer, that the inclusion of that type of depreciation results in "unsocial pricing"—then one is on the threshold of a very wide field of discussion. May it merely be stated that the consumer ultimately must pay for full "depreciation" in any case—either in operating expenses above the line, or in an increased return to the investor below the line. Again the field is only collateral to the issue of depreciation, and will not be developed further. However, regulatory commissions do list loss in demand and obsolescence as causes of depreciation and, working, as they are, within the institutional framework of a defined economy, engineers and appraisers must also do so. A discussion of possible changes in this framework is clearly beyond the scope of this discussion.

To recapitulate, the major causes of depreciation, as now conceived, are economic; hence they cannot be detected by tools (accurate as they may be) designed to observe physical depreciation only. The annual cost of depreciation, and the reserve accumulated therefrom, must reflect these major economic causes. Since a sizable portion of such a reserve would represent a prepayment in anticipation of economic depreciation, the deduction of this reserve for purposes of rate-base determination is incorrect. Present value reflects past and matured depreciation; a future reserve must, of necessity, consider future, and as yet unmatured, depreciation.

E. G. Walker,[44] M. Am. Soc. C. E.—Consideration of depreciation and of amortization and obsolescence which, although they involve separate ideas, are related intimately to it, enter—or, perhaps it is more accurate to write, should enter—into a very wide range of activities. Possibly because of this condition, much confusion of thought has arisen. It is well, therefore, before attempting detailed studies of standard methods of practical application, to concentrate attention upon an endeavor to formulate the fundamental conditions from which the practice of making financial provision for conservation of capital sunk in wasting assets originates.

It is implied in the Symposium that attention is directed exclusively to such assets as are met with in engineering practice and constructional and manufacturing industries. The three papers of the Symposium give a good idea of how wide even this restricted range is. They are well selected to show that, as Mr. Walls states, "depreciation cannot be made to suit a definition, but rather the definition must truly express the actualities of depreciation" (see "Synopsis"). These actualities may differ widely in their incidence.

A primary difficulty is the laxity with which the terms depreciation, amortization, and obsolescence are used among men of affairs, and the confusion that exists between these terms and maintenance. Only quite recently the writer, in connection with certain problems of assessment of values of an industrial plant, was met with the objection that certain provisions for depreciation were unnecessary because the practice of the directors of the plant in question had always been to spend considerable sums on maintaining it in the best possible condition. This is the confusion referred to in Professor Grant's paper between "as good as new" and "as valuable as ever." The difference between the two is obvious to most people, although it does not appear always to be clear to the so-called "businessmen" whose influences figure largely in the direction of many engineering and manufacturing concerns. It may be well to emphasize this difference to an even greater extent than it is in the papers under discussion.

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[44] Prin., Maxted and Knott, Putney, London, S.W. 15, England.

Experience also shows that similar confusion exists as to the meanings of the terms amortization and obsolescence, although there should be no difficulty in appreciating clearly the differences between them and the term depreciation. For the benefit of those who confuse these terms it is suggested that a somewhat greater reference to them would be appropriate in any report of the Committee on Depreciation than has been made in the Symposium.

The authors give, in some detail, their opinions as to what constitutes depreciation. They differ from one another slightly but are united in making provision for flexibility in treatment. Thus, Professor Grant distinguishes three kinds of depreciation and applies them according to the financial point of view of the party concerned with the assessment. He uses the terms "popular depreciation," "accounting depreciation," and "appraisal depreciation," and explains clearly in his paper the differences between them. Are they all rightly defined as depreciation, however? The home builder in Example 1 sold his home, after eight years, at the price he paid for it. He lost no money on the deal; but, even if he had maintained the property well, its life had been reduced by eight years and there had been a depreciation. The accountant and the appraiser take note of these facts, but the layman does not, so that, in the end, the author offers three sums—nothing, $1,000, and $1,500, respectively—on an asset originally valued at $6,000. An aim of the discussion is to determine the possibilities of measuring depreciation by engineering and scientific processes. In this simple example there are three sums differing widely in their relation to the original price. They are all referred to as depreciations, and the methods of assessing them are all based on the logical reasonings set forth in the paper. The question arises, if this line of argument is accepted, whether the appraiser is following engineering and scientific processes or is being led away from them along a course of opportunism—of adapting the argument to the circumstances—which is in essence nonscientific and unsuited to bring about the conditions set as the objective.

The nature of depreciation prevents the possibility of reaching a state in which it can be calculated with the exactness with which interests and discounts are computed. However, it may be possible to obtain for general adoption some more clear-cut procedure to enable any particular case to be examined and a fair judgment expressed as to the correctness and adequacy of the depreciation provision. Dean Marston's statements under the heading "Value and Depreciation" in Messrs. Crum and Winfrey's paper are progressive steps to this end, and establish fundamental principles. It is generally accepted that the fair price of any article or service is that paid in an unrestricted transaction in the open market by a willing buyer to a willing seller. In such a deal the buyer naturally can have regard only for what profitable use he can make of his purchase, or, in the terms of the paper, the "present worth of probable future service." The difference between the present worths at the beginning and end of any period must be the true appreciation or depreciation of the asset over the period. This fundamental principle provides a true measure that can be applied to any case at any time to ascertain whether adequate financial provision has been made to meet the changes arising from the conditions—some very direct, some quite remote—to which the asset has been subjected.

The application of the principle is not simple because it may involve much that is not susceptible of straightforward calculation. It implies in most instances exercise of skilled judgment and the enunciation of considered opinion, and thereby introduces possibilities of considerable divergence. One has only to think of the numbers of technical cases that are argued daily before the courts of law to realize the extreme divergence of opinions put forward by men of equal experience and judgment on matters that they approach from opposed viewpoints. If engineers are to arrive at a generally acceptable basis for estimating depreciation, they must rid themselves of this condition. Some sacrifice of ideals may be necessary to achieve practical results.

If true application of the principle were possible, determination of depreciation would be largely a matter of periodical revaluation, and the variation of the periodical appropriations would be very different in different cases. Applied to the finances of certain classes of heavy civil engineering works, such as those which Messrs. Crum and Winfrey claim, rightly, should be—but by no means always are—subjected to adjustment from this point of view, there would be many cases in which little variation would be found from year to year. In others, such, for example, as plant installed in new and rapidly developing industries, there would be large and irregular variations, and the influence of possible obsolescence would be much greater. (This factor is not absent even from cases of the most uniform depreciation.) A man well versed in the technicalities of an industry is well equipped for assessing the probable future life of an item of plant in that industry, and is better able to estimate the chances of drastic changes in processes, markets, etc.; but even he cannot tell whether or not some new invention or idea may be developed at any moment that will alter completely the capabilities of his plant for effective manufacture and competition. This obsolescence is a vital and incalculable factor in depreciation. Calculations based on data as to realized or estimated lives must be adjusted to allow for these incalculable possibilities if effective financial provision is to be made. When obsolescence is an important factor, preliminary estimates and economy studies may be stultified and an ample basis afforded for writing off values quickly, irrespective of actual real depreciation.

The accountant's approach to the problem is not concerned with physical factors. He must see that adequate provision is made so that money may be available for replacements or reconstruction when it is needed. He must maintain a fund that is adequate for meeting any demands likely to be made on it. As long as there are resources out of which to meet demands arising from depreciation, he has done all that is required of him. If he is an auditor, he must see that depreciation provisions are reasonable. From this outlook has developed the idea of Professor Grant's "accounting depreciation." It might be described not as a depreciation but as a provision for depreciation. It is derived usually by an arithmetical process based on original cost and deductions therefrom at fixed or varying rates. On a single item, such as the simple one used in Professor Grant's Example 1, it may result in a wide difference between the actual depreciation (the "appraisal depreciation") and the provision for depreciation (the "accounting depreciation"), but over the large number of items that combine to form a complete engineering entity there is a much greater possibility of averaging. Book figures for depreciation provisions should be considered in this manner. Individually they may not give an accurate view of the actual value of the item, but in the aggregate, if the finances are controlled properly, the discrepancy between the depreciation and the provision made in the books to meet it should be far less, proportionately. Obviously, for sound finance the provision must exceed the depreciation.

The writer believes there is much in favor of a standardization of the term "depreciation" to mean one thing and one thing only. The idea of depreciation should be based on the physical conceptions that determine the future possibilities of the working life of the asset. The reduction in the present worth of these physical conceptions over a period is the depreciation over that period, whether resulting from the normal shortening of the working life or from extraneous causes. How the depreciation is to be met is a separate problem. As, in general, periods of numbers of years are involved, in ordinary cases there is no urgent necessity to keep strict mathematical equality between depreciation and the provision made to meet it, although obviously the difference between them must be kept within proper limits. The former may be irregular in its increments and the latter (usually) regular. It is essential, however, that during the greater part of the time the provision be ahead of the depreciation if a sound financial position is to be maintained.

In Professor Grant's examples he introduces a point that has not been elaborated in the Symposium—namely, the increase of value of an asset over a period in spite of its physical deterioration. The circumstance, although unusual, arises sometimes in connection with items of engineering plant. It is likely to do so increasingly with the rising demands for such items, which are a feature of the present time as the result of war conditions. When a machine, purchased some years ago at a lower price, must be replaced at a price that may be several time that paid originally, it scarcely can be expected that the provision for depreciation made over a number of years, when there was little or no indication of this increase, will be adequate to meet more than a fraction of the new cost. The corresponding problem of insurance can be dealt with by an increase of coverage, but in extreme conditions the greatest practicable increase in the rate of provision may be far from sufficient to write up the fund to the new price. It would be of interest to have the authors' views on this problem.

K. Lee Hyder,[45] M. Am. Soc. C. E.—Writers on the subject of "depreciation" almost invariably preface their remarks with comments as to the state of confusion that appears to exist in the minds of those called upon to deal with the matter, either practically or academically. The several contributors to this Symposium support this observation in the presentation of their papers. They proceed, however, to make notable contributions which, in turn, should be of inestimable value in clarifying the basic thinking. By inference, at least, if not specifically or intentionally, almost the entire scope of controversial material has been brought forward in one or more of these papers.

Nevertheless, after a thorough review of these papers, the writer feels that there still remains a tendency upon the part of each to merge discussions of "concepts" with those of "procedures." The emphasis throughout seems to be directed primarily to the methods and to the variations in such methods for differing purposes. Perhaps a rearrangement of the material would go a long way toward clearing up the confusion. Professor Grant alone has introduced his treatment of the subject by a section entitled "Three Fundamental Depreciation Concepts," in which he clearly differentiates the three fundamental or foundational approaches, and makes reference to Professor Bonbright's great work, "Valuation of Property,"[4] in presenting these fundamentals as "concepts." In the writer's opinion, if these are to be so accepted (and he makes no point to the contrary), then they should in turn be predicated upon a still deeper foundational premise—that is, the character of the property subject to depreciation.

There is no fundamental difference in the causes that bring about the depreciation of assets, whether such assets are used in regulated industry, competitive industry, or in public works. Certainly there is no difference in the life expectancy of a boiler or generator, as such, simply because it may be a part of an electric-light system under regulation, a private manufacturing enterprise, or a municipally owned plant. It may be concluded, therefore, that the basic segregation in the Symposium is made in order to recognize the differing treatments to be accorded in the case of (1) a business monopoly, (2) a competitive operation, or (3) a politically controlled budget.

If the appraiser is really to get down to fundamentals, the property itself should first be subjected to clarification. To illustrate: Assume the extremes as reflected in five types of depreciable assets (or "properties"), each of which—under the realistic approach—obviously must be accorded a different treatment, as follows: (*a*) A leased fee in a parcel of improved real estate; (*b*) a complex industrial "plant"; (*c*) an automobile; (*d*) an oil well; and (*e*) a motion picture.

Any one of the foregoing constitutes a "property" and a "depreciable asset"; that is, it will at some future date be expected to reach the end of its useful life. Now, the character of each of the foregoing properties (or assets) is different. The "leased fee" (type *a*), for instance, has a known or agreed upon future life. The "property" itself is in reality a right to receive a stated sum of money for a definite number of years. In this property, depreciation actually accrues at a rate that may be charted in the form of a parabola. Very little will accrue the first year, and the sum will continue to increase each year thereafter to the end of the period. The accruals may be determined by simply taking the difference in the "present value" (or "present worth") of the annual income as of any date and the original "present value," using ordinary interest or discount tables for such computations.

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[45] Vice-Pres., The Am. Appraisal Co., Chicago, Ill.

In the "industrial plant" (type *b*) the appraiser is dealing with a complex assembly of items comprising the depreciable plant, each of which is subject to an individual life expectancy from both physical and functional factors. However, the enterprise as a whole is further subject to all of the risks and hazards of competition, economic and social changes, etc., and hence such factors may enter into the depreciation problem. The wide variation in the life expectancy of the component parts, for reasons which need not be discussed here, leads to the usual adoption of the "straight-line" basis as more realistically applicable.

The economic life of "an automobile" (type *c*), by both nature and custom, is heavily influenced by the factors of obsolescence. This influence starts quickly, and it is generally realized that the moment the new car is driven out of the salesroom it must be considered to have taken on a substantial "depreciation." Therefore, the "curve" of type *a* is reversed in character, and again—realistically—the "heavier" sums accrue during the earlier years.

The economic life of "an oil well" (type *d*) and its "curve" of realistic depreciation are similar to that of the automobile, although perhaps for more basic reasons. The gas pressure when a well is first opened serves to bring out the highest production of petroleum during the first year; and, as the gas pressure (and volume of oil) declines, the flow is less, and "depreciation" (depletion) is reduced accordingly. (The analogy here, of course, is that, as the deposit is depleted, the investment is proportionately depreciated.) The curve can be rather closely defined by the experience of other wells, and the engineer may make his estimates accordingly. Finally, the "motion picture" (type *e*) is also a "property" and an asset subject to depreciation. Keeping to actuality, the value (or cost, if the purposes so dictate) is found by experience to depreciate at perhaps 85% the first year, 10% the second year, and 5% the third year. Whatever is left after three years may be viewed in the same manner as "salvage" in other types of property after the end of the useful life.

With this initial distinction in the character of property defined, it would appear that the points raised and discussed in the papers could be developed with greater clarity. To a large extent (except for certain illustrations), the entire scope of the material presented by Mr. Walls, Professor Grant, and Mr. Crum and Professor Winfrey may be considered to be directed toward the classes of assets similar to type *b*—namely, the complex industrial plant.

Under such assumption, it must then be concluded that the discussion to follow should be so limited. If this is true, it would support the opinion that the material is to be considered primarily as directed toward the questions of procedure under the hypotheses of (*A*) controlled industry, (*B*) competitive industry, and (*C*) public works.

Carried a step further, all of the authors, in one manner or another, present the causes of depreciation which the writer, for discussion purposes, elects to summarize as physical deterioration, functional obsolescence, and economic obsolescence. There appears to be no clear-cut distinction made between "functional obsolescence" and "economic obsolescence" and—again—if such were recognized it might well tend toward clarification. Realistically, the factors embraced in such causes differ in character and hence are subject to segregation. Practically, it may be difficult to draw a definite line of demarcation. A theory may be set up, however, and the appraiser may proceed logically from that point to establish the proof. In the regulated industry, all plant items are subject to "physical deterioration" in exactly the same manner as in the competitive industry or the municipally owned plant.

In like manner (although perhaps not to the same degree) this may be said of "functional obsolescence." (A power plant becomes outmoded, irrespective of the type of operating control.) However, the obsolescence of a plant occasioned by changing economic conditions is usually present to a much less degree in regulated than in competitive industry. The possession of a franchise, granting monopolistic rights to perform given services in a community, places the control of retirements much more strongly in the hands of the public utility operator than in the case of the competitive enterprise. Of course, the artificial gas company faces the threat of the encroachment of natural gas, but nevertheless the hazard is greatly circumscribed.

Furthermore, the municipal plant is rarely faced with economic obsolescence to any great degree, except as public opinion may bring pressure to bear upon the political unit in demanding major changes in entire systems. Mr. Crum and Professor Winfrey's excellent paper admits the inadequate recognition of depreciation in the case of public works—even as they so ably set forth the need for a system and an accounting practice no less complete than that essential to other types of ownership.

Analyzing the papers from a slightly different standpoint, one may clarify the subject by bringing the various procedures suggested under what is termed the "two-property concept," by which is meant the classification of the complex property as:

(1) Tangible assets (the physical facilities or "tools" devoted to the conducting of the enterprise); and
(2) Intangible assets (the business itself).

Now, the problem may be narrowed for any particular discussion. The nature of the business (property (2)) and its operation under or outside of public regulation, as the case may be, would be reflected in the intangible assets, but in reality would present factors that might well limit the use and life expectancy (and hence the amount of, and provision for, depreciation) of the tangible assets. It is this influence that might be embraced, under the theory suggested, in the category of "economic obsolescence."

If the foregoing discussion is a correct interpretation of the papers, then the entire problem presented in the Symposium might be summed up as follows:

> "What are the proper procedures in estimating accrued depreciation and providing for future depreciation for the depreciable tangible assets of complex industrial enterprises which are (1) under public regulation, (2) of competitive character, and (3) public owned?"

With a thoroughly developed exposition of the underlying concepts of "property" and "depreciation" as a foundational section, then the treatment of the principles and methods—whether for value estimates, rate making, or accounting, and each under various types of control—might be more effectively presented.

James T. Ryan,[46] **Assoc. M. Am. Soc. C. E.**—This Symposium reveals quite clearly both the extent and the sources of the confusion in current thought relative to the subject. The stresses to which language has been subjected in the processes of adaptation to the need of the machine age for agencies of description and expression have been at least as severe as those entailed by the machine upon metal, timber, and stone. By the same token, the genius that created the new industrial culture has had no harder task than those involved in defining the principles underlying it and explaining their meaning and scope. To do this it has been necessary virtually to create a new language in which a vast number of words and phrases in previous use have been adapted to purposes for which, in many instances, they were but poorly suited. The word "depreciate" and its derivatives, for example, originally applied solely to value, and dictionaries still define them as related to conditions that are measurable in terms of money.

With these definitions, of course, the writer has no contention. The engineer's trouble arises from an extension of the terms to conditions not contemplated in their original definition. With the growing importance of problems in physics, in chemistry, and in mechanics, arising from the decay of timber, the corrosion of metals, the disintegration of masonry, and the attrition of wearing surfaces, there came a need for a general term that would characterize all such processes in the same way that "depreciation" covered all influences that adversely affected values. Unfortunately, no new word seems to have been available, and the old one was appropriated for the purpose and has come into general use.

At the same time the accountants needed a word for a similar purpose. They were not interested in depreciation as the economists applied the term to declining values or as the engineers had begun to use it with reference to impaired physical integrity or functional efficiency. They were concerned with the problem of compensation for the consumption of capital resulting from use, regardless of how effected or indicated; and they too (probably also because no other term seemed to be available for the purpose) appropriated "depreciation" for the purpose, and gave to that word a new meaning consistent with the new uses to which they adapted it.

From these premises it seems pertinent to suggest that Professor Grant's "three fundamentally different depreciation concepts" may well be restated, perhaps as follows:

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[46] San Francisco, Calif.

> First (and oldest), the economic concept, in which depreciation is related solely to loss in value, from any and all causes;
>
> Second, the engineering concept, in which it is essentially related to impaired physical integrity or functional efficiency, irrespective of their effects on values; and
>
> Third, the accounting concept, in which conclusions reached under the first and second concepts are applied to the measurement of capital lost or consumed in current operations.

Obviously, these concepts may lead to entirely different and unrelated conclusions.

Mr. Walls apparently prefers to limit consideration of the subject to the engineering concept, although he fully recognizes the engineer's responsibility to both accountant and economist. Professor Grant virtually ignores the engineer's concept—or, at least, subordinates it entirely to those of accountant and economist.

Messrs. Crum and Winfrey present an additional concept which, although primarily based upon economic considerations, is confused by adapting the definitions quoted by them toward an objectivity that seems to be related to political expediency rather than principle. It seems incredible that they should be able to deduce from their definition of depreciation (see heading "Value of Depreciation: Definition of Depreciation") the procedure for determining, from subsequent accounting records, the "true value" of a property outlined under the heading "Distinction Between Depreciation and Maintenance." The accountant cannot keep a record that will at any time exhibit such a result. His entries begin with a statement of cost. This cost may or may not be equivalent to value. The periodical entries recording the subsequent consumption of this original capital, either as depreciation, depletion, or amortization, may or may not be equivalent to concurrent "loss in value" or "decline in present worth" applicable to the same property. The accountant is not in a position to make such a determination and cannot assume responsibility for it. His records are kept primarily for an entirely different purpose, and their usefulness for that purpose would be impaired if not destroyed if he undertook to reflect in them current or periodical adjustments of invested capital to conform with "present true value." The accountant cannot assume this function of the economist; nor can the extent of physical or functional impairment, either as a periodical occurrence or as an accumulated total, be determined by him. That is the engineer's job. If he performs it properly and reports the results intelligently, the accountant will find his conclusions to be of inestimable value in maintaining the accuracy of accounting records, but neither of the two should be mistaken for the other.

It might be suggested that Professor Grant's admirable paper is faulty to the extent that it subordinates engineering analysis to economic considerations. His Examples 3 and 4 are calculations showing comparisons between the present values of specified annuities with different time factors, and are related to values rather than to a determination of depreciation as an engineering concept. Such calculations are of importance in many respects, but do not seem relevant to the questions under discussion, unless it is the intention of Professor Grant to suggest that the depreciation applicable to a given property may be measured properly by its earnings. This may have been one purpose of his Example 1, in which the accountant has a record of depreciation, and the engineer will doubtless discover convincing indications of it from an inspection of the structure, whereas the economist finds no evidence of loss in value and, therefore, no depreciation.

Under Messrs. Crum and Winfrey's suggested procedure, this sequence is reversed. If the accountant's records show charges against earnings for depreciation, a corresponding decrease in value is established, regardless of evidence to the contrary, or the subsequent discovery of error in the bookkeeper's estimates. The fact of value, although supported by conclusive evidence, is to be disregarded if inconsistent with accounting records. The obsession that a theory is so beautiful that it must be protected from injury by the distortion or suppression of contradictory facts is not new in engineering, literature, or law, but seems to be out of place here.

It should be emphasized that depreciation is not the only influence that limits the useful life or the earnings of structures or machinery, and that a decline in either usefulness or earnings does not progress uniformly with the passage of time. Any great industrial enterprise requires the coordination of various kinds of equipment to operate in each of its major processes. The depreciation of any given structure or machine, or any component element therein, may entail injurious effects upon other elements in the same structure or machine, or upon other structures or machines in the same plant or system. The replacement of one unit or element in such a plant or system frequently entails the necessity for the replacement of others that may be in excellent condition for further service. A change of detail in its products may involve extensive remodeling or discarding of plant equipment and facilities.

Physical condition is not, therefore, a safe basis for a forecast of future usefulness; nor is an estimate of life expectancy a satisfactory basis for determining physical condition. The suggestion, so frequently implied in this Symposium, that an accurate determination of depreciation may be arrived at by reference to the calendar (for example, from estimates of probable life expectancies) is fundamentally unsound.

The engineer's perspective of depreciation embraces both causes and effects, and involves the consideration of measures calculated to abort causative influences and to minimize or avert resultant damages. Accountants and economists alike have profited by making full use of his judgment and experience.

John C. Page,[47] M. Am. Soc. C. E.—In analyzing "Fundamental Aspects of the Depreciation Problem: Relationship to Public Works and Government Finance," Messrs. Crum and Winfrey have brought forth for inspection some of the differences between governmental and private accounting procedures. Emphasizing the lack of consideration of depreciation in governmental administration, the authors suggest that depreciation should receive a proper place in this field if accounting of public funds is to be other than a partial truth.

Federal officials charged with the administration of public works must observe, of course, the laws of Congress and the rules promulgated by regulatory bodies deriving their authority from the Congress. These laws and rules may be sufficiently restrictive to prevent the accumulation of reserve funds for replacements. This is not invariably the case, however, and, even in cases where maintenance of a reserve account is not practicable, recognition can be given to depreciation of property as a factor to be considered in planning future operations. The Bureau of Reclamation, for example, is engaged in the construction of large engineering projects, the operation and maintenance of major power plants, and the operation and maintenance of extensive irrigation systems. In each of these three primary activities, the principles of depreciation are recognized in the attendant accounting procedures. As an illustration of the applicability of these principles, and in the hope that something of value may be contributed to the discussion, a résumé of the accounting system of the Bureau of Reclamation is submitted.

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[47] Commr., Bureau of Reclamation, U. S. Dept. of the Interior, Washington, D. C.

The Bureau of Reclamation of the U. S. Department of the Interior since 1902 has been engaged in the construction, operation, and maintenance of engineering works designed to reclaim for agriculture the arid and semi-arid valleys of the Western States, to produce power, and to provide flood control and related benefits. Varying in size from the large-scale Grand Coulee Dam (Washington), Boulder Dam (Arizona-Nevada), and Shasta Dam (California) to the relatively small Great Plains projects, the Bureau has had the responsibility for expending more than $690,000,000 on engineering construction alone since 1902. The accurate accounting of this large sum has been in accordance with standardized regulations.

Before continuing the discussion of the subject, a general outline of the Bureau's financial structure may be helpful. The federal reclamation of arid lands in the West, concerned with the conservation of water, the production of electrical energy, and the created potentialities for the establishment of independent farm homes, is essentially a business enterprise. The capital is provided by a revolving fund known as the Reclamation Fund, which is supplemented by occasional special appropriations. The money expended is contracted for repayment by the users of the finished product. The cash collected by the government as reimbursement for construction of one project becomes capital for the construction of another project.

The repayment plan or amortization of the construction cost of the completed project is based on an exhaustive study of crop production, income from crop yields, and use of electrical energy. The total repayable obligation must be within the ability of the consumer to pay, and the repayment must be completed within maximum periods fixed by statute. At the end of the amortization period the project becomes in effect the property of the users, although ratification of the transfer by Congress is necessary. The anticipated costs after construction are routine operation and maintenance costs and occasional major replacements, which are made by the users' organization or by the Bureau acting as their representative.

In so far as its power developments are concerned, the Bureau of Reclamation recently has adopted the Federal Power Commission's uniform system of accounts for public utilities and licensees where practicable. The Boulder Canyon Project Adjustment Act of July 19, 1940, definitely provided for a rate structure that included, in addition to operation and maintenance, amortization at 3% interest of moneys advanced by the Treasury of the United States for financing its construction and the establishment of an adequate reserve for replacements. The principle of depreciation is also recognized in the establishment of rates for the sale of electrical energy from its other power developments and in the establishment of rates for the sale of water from its irrigation facilities for purposes other than irrigation.

Depreciation as used by the Bureau of Reclamation in its power-plant operations is considered to be the reduction over an estimated period of time of the original cost or value of a structure, facility, or unit of equipment, due to its deterioration by wear or tear, or otherwise. It is so recognized for the purpose of insuring accurate and proper fund accounting.

The Bureau further has recognized the advisability of depreciation accounting for its irrigation operations; however, existing law and contractual commitments do not permit charging depreciation, as such, as an item of cost of operation because annual fixed charges are collected from irrigation beneficiaries for the amortization of the fixed capital indebtedness, in addition to operation and maintenance charges for irrigation service.

In order that the consumer shall repay only the actual cost of construction to the government, the cost-accounting system used during the construction period allocates all costs to the principal and physical features of the project. This system employs a commercial practice of depreciation for all nonexpendable property during construction, allowing a credit to the project costs for the salvage value of unused plant equipment and supplies on completion of construction. The records accumulated in this fashion, in addition to determining the exact sum that must be repaid by the water users or power consumers, have served to provide a reliable basis for estimating the costs of new projects or parts of a project, or the cost of replacement of structures on operating (completed) projects.

Costs during construction are maintained by principal features, of which a single reclamation project usually contains two or more. Principal features represent a general division of the project into its primary parts or units; the nature or function of each feature is usually implied by its name, such as examination and surveys, storage works, canal systems, etc. Principal features are further broken down into physical features. A breakdown of principal features into physical features is made for the purpose of collecting exact costs on separate and distinct jobs or units or divisions of a project. They represent some tangible section of a project when construction is completed, such as a group of water-control structures, a tunnel, etc.

Supervision, labor, and materials comprise the bulk of construction costs. Obviously, an accurate distribution of these costs to units can be made with no unusual difficulty. However, many items of equipment are not fully "deteriorized" in the construction of a single unit and may be used in a number of units under varying circumstances. Such equipment is termed "nonexpendable equipment," and the method of distributing its cost to individual units is through charges based on depreciation of the item.

The costs of equipment and its erection into a working unit are accumulated into what is known as a "plant account." The costs of the equipment are absorbed into the cost of features upon which they are used by the monthly transfer from the plant account of an amount that is proportioned to the work accomplished during the month. The transfer of these costs involves two elements: (1) "Plant" charges proper, which cover the direct costs of installation and erection; and (2) equipment charges, which cover the depreciation to be charged to the feature for the use of nonexpendable equipment. The transfer of plant charges proper presents no unusual problem. The transfer of equipment charges through depreciation is described in the following paragraphs.

Based on the Bureau's experience, different rates of depreciation for the various kinds and classes of equipment are established. The rates for motor vehicles are based on a computation of percentage of original cost against units of mileage. For instance, motor-truck costs are usually charged off on a basis of 40,000 miles total travel; the charge for the first 10,000 miles is estimated at 40% of the original cost, for the second 10,000 miles at 30%, for the third 10,000 miles at 20%, and for the fourth 10,000 miles at 10%. The reason for this graduated scale is to decrease the depreciation charge as the operation cost increases. The mileage basis of rates varies, depending on the nature of the work to be performed by the vehicle, but the foregoing rates are generally used for the Bureau's field operations.

Small tools are charged out on a monthly rate to the plant or work on which they are used. Charges usually are made to each feature based on a percentage of the value of tools in use on the feature at a rate of 8% monthly or they are based on a charge that is a percentage of the monthly cost of the feature, whichever is most appropriate. Depreciation on small tools varies greatly in the different classes of work and, in case the percentage selected is found to be either too high or too low, it is changed to agree as nearly as practicable with actual conditions.

Other articles of major equipment are also grouped into "plants" upon which monthly depreciation charges are made as follows:

Name of plant	Percentage rate per month
Camp construction plant equipment	5
Construction plant equipment	4
Corral plant equipment	4
Engineering, field plant equipment	2
Hospital plant equipment	3
Mercantile store plant equipment	3
Miscellaneous equipment and tools	8
Office plant equipment	2
Preliminary investigations and testing plant equipment	4
Railroad plant equipment	3
Sawmill plant equipment	5
Shop plant equipment	3
Storehouse plant equipment	3
Telephone system (temporary plant) equipment	3

These charges are made on appraised value, determined annually, and not on original cost, except in the case of the first year for new equipment. The charge is made for the entire period the equipment is retained for use on the feature to which it is charged, regardless of whether or not its use is continuous during that period, except that no charge will be made when the performance of the class of work on which the equipment is used is suspended for one month or more. If the equipment is used more than an average of one shift per day, the charge is increased accordingly. No charge is accrued on equipment in inventory, if properly housed and protected.

The plant accounts are charged with all extensive equipment repairs in order to insure distribution of cost over a longer period than the month in which extensive repairs are made, thus arriving at a better unit cost of work in progress. The cost of such repairs is distributed in the same manner, and at the same rates per month, as the charges for articles of equipment to which the repairs are made. Minor or ordinary maintenance repairs, however, are charged directly to the work.

Under the Bureau's procedure for depreciating nonexpendable property during construction of a project, many years might be required to depreciate each item fully—certainly many years more than it takes to construct a project. Some provision must be made for disposition of the balances for undepreciated values that remain on the books at the close of the construction period. Upon completion of the work for which equipment or plant has been used (and this may occur at any time during the construction period, not necessarily at the completion of construction), a salvage value is fixed by a board of survey. The equipment is then transferred to another project, or sold, using the appraised value. The appraisal is based on the actual value as represented by the physical condition and the current market value of similar equipment. Adjustments are necessary to reconcile differences between the appraised values and the undepreciated balances.

The procedure just outlined applies to the construction activities of the Bureau of Reclamation. It enables the Bureau to determine precisely the cost of the entire project and all of its major subdivisions. The costs repaid by the users are no more or no less than are required by the governing statutes.

As previously mentioned, the Bureau is in the unique position of not only constructing projects, but frequently remaining as the operating agency following completion of the works. The handling of depreciation on power projects is controlled by rules of fixed procedure that are too complex in character to explain in this discussion. Essentially, it is that a certain portion of power revenues are set aside each year in a reserve fund for replacement of the major items of the plant. The accumulation of these reserves has a direct relation to the estimated service lives of the items of the plant.

The depreciation problems in connection with operating and maintaining completed irrigation projects (as distinguished from power projects) are similar to those encountered in any operating enterprise. Operation and maintenance funds for reclamation projects are either advanced by the water users at the beginning of each year or are appropriated by Congress and repaid by the water users at the end of each year. It would be desirable to set aside in a reserve account, each year, from these appropriations or advances, a sum that could be deposited or invested in order that at the end of a fixed period of years adequate capital would have been amassed to provide for replacement of major structures with a limited service life. As a matter of policy, the Bureau favors the establishment of such reserve accounts. Their establishment requires the approval of the water users, and it is expected that the maintenance of larger reserve accounts will become more widespread as their value is more fully appreciated. Whether additional legislation would be helpful in providing an answer to the depreciation problem for federal public works cannot be known without further study, but certainly this Symposium has been helpful in revealing the weaknesses of the subject and the need for uniformity in handling it.

JOHN S. WORLEY,[48] M. AM. SOC. C. E.—The subject of this Symposium divides itself into two categories—one, the nature of depreciation; the other, the treatment to be given to it in the various situations having to do with private and public businesses and public utilities. In ascertaining the nature of depreciation one deals with a specific and tangible fact as of some instant of time, whereas the matter of its treatment is solely one of public or private policy, which, in some cases, must be made effective by legislative action.

Every business property, when acquired by purchase or construction, is for the purpose of providing service. That is, during a period of time, with normal maintenance, it is capable of manufacturing a certain quantity of products, thousands of barrels of flour, gross of shoes, kilowatts of electricity, ton-miles of goods-carrying, locomotive miles, car miles, passenger miles, cubic feet of gas, and all other products of industries. As a plant is operated these units of service are consumed, but as maintenance, repairs, and renewals are made these units are in part replenished. It seems logical and reasonable to define depreciation as the loss of service units.

There are many types of property, the normal maintenance, repairs, and renewals of which will keep it in first-class operating condition. When a property reaches this stage the annual maintenance, repairs, and renewals just equal the average losses and there is no further accumulation of depreciation. This condition is found in the tie system of the American railroad which is receiving its regular maintenance, repairs, and renewals. Where the annual use is fairly uniform and the maintenance, repairs, and renewals become uniform, the annual losses are "made good" currently and it may be said that there has been no depreciation for that year. When the losses are due to wear and tear from use, the annual depreciation is in a direct proportion to the use. When the losses are from deterioration due to weather the depreciation is uniform from year to year.

Obsolescence, as of a specific time, or during a certain year, is more difficult to determine than any other kind of depreciation. Where there is a uniform change in an art from year to year, and where, at the end of a certain period, a machine is found to be obsolete and inadequate, it would seem reasonable to say the obsolescence had occurred in a uniform manner each year. On the other hand, where a machine becomes obsolete and inadequate because of an invention, it would seem that the major part of the obsolescence had occurred in the year of the invention and its adoption. However, it can be reasoned that from past experience these new inventions are to be expected and therefore obsolescence does occur annually.

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[48] Prof. of Transportation Eng., Univ. of Michigan, Ann Arbor, Mich.

A splendid example of the irregularity with which units of service may be used up is found in the heavy artillery of the United States Army. Through long experience it has been found that each piece of artillery, depending upon its type, can be fired a certain number of rounds, after which it is no longer accurate in its range nor safe from explosion. As a result, each piece is fired only the determined number of rounds before it is retired from service. Now if a new gun is held in reserve for two years, and one half of the units of service are fired within one week, there has been no depreciation for the two years, and 50% of its value is depreciated within one week. However, if the gun had been fired once a day the depreciation would accumulate from day to day until all the units of service had been used up, at which time the gun would have been retired.

As a rule, the engineer is called upon to determine depreciation as a fact, as of a particular date. At no time should the engineer lose sight of the fact that depreciation cannot be effected by subtle metaphysical reasoning or by any other process. Whenever the engineer deals with this subject the quality of his work will be in direct proportion to the degree with which he ascertains and records the facts. The work of the accountant varies from that of the engineer in that his estimates must be made in advance of the fact. He estimates what the probable depreciation will be so that he may make proper debits against annual income.

With the knowledge of the nature of depreciation, a policy with reference to its treatment can be adopted intelligently. In the administration of a policy either an engineer or an accountant, or both, are the parties usually employed to participate in implementing the policy adopted. This requires the determination of the depreciation in certain units, which is translated into the medium of exchange. This work is one of professional abilities and skills and there should be little or no difference in the results determined by either engineers or accountants, and such differences as may exist should be readily reconciled.

It is the writer's belief that much of the difficulty which has been experienced in dealing with depreciation has been the attempt to have it controlled by a predetermined definition or thought of some social school or economic order. He also believes that the solution of this problem, like many other social problems, should be predicated upon as complete a knowledge of the nature of the subject as possible, from which a policy of treatment is determined.

JOHN F. MILLER,[49] ESQ.—Theories of analysis and deduction are valueless unless actuated by sound methods. In none of the papers of this Symposium does there appear any information as to a practical method of collecting requisite data to determine a sound and true depreciation rate. The most important and dominant element in valuation is depreciation, and the most important elements in depreciation, as applied to all progressive industrial organizations in the United States, are (1) obsolescence and (2) inadequacy. Other elements—

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[49] Cons. Engr., Washington, D. C.

wear and tear, maintenance and deterioriation—are of rather secondary importance.

The industrial developments in the United States have been so rapid since the beginning of the twentieth century that mechanically sound machinery and equipment often become useless because industry is finding new and larger, or different equipment so valuable that it has been profitable to scrap units which yesterday were considered the best. To decide the economy of such retirement, however, requires a knowledge of local conditions and a general understanding of the industry as a whole which no outside valuation expert, however proficient he may be, possesses to a degree to qualify him, alone, to establish a sound depreciation rate.

When he starts such work in a new plant, the writer has made it a practice to become acquainted with the foreman and operators of the various machines and equipment. In an informal manner he undertakes to acquaint them with the problem and to start them thinking along the lines of what they see, surmise, or dream, for the future development of the machines and equipment which they handle. For example, what improvements, and along what lines, do there seem to be indications of changes either in method, or need of manufacture, of the articles now produced? The writer explains, in layman's language, the problem of obsolescence and inadequacy (not using these terms at first without a clear understandable definition).

At first the response was negative, but as the work progressed, acquaintance with the men grew, and in the process of reviewing other similar problems the writer was gratified to note the development in the thinking of the shopmen, and the sound conclusions formed as to the future value of their machines and equipment. Men who direct the operations and operate machines for any length of time come to know the limitations of those machines and to dream of improvement.

The collection of these developed opinions, based on sober continued thought during the course of the work—their assimilation, balancing and comparing with other data of other plants—never failed to produce invaluable information which, to a rather large extent, determined the depreciation factor to be used in establishing a sound and true "present value" of the equipment of the plant.

The innumerable tables of straight-line, and other depreciation rates extant, served as a crude check, but such values were many times considerably, and sometimes radically, changed by this collection of first-hand knowledge and opinions of those whose work and experience peculiarly qualify them to have a voice in rate making. This procedure requires time and patience—patience to picture the problem to men who are unfamiliar with valuation work, particularly obsolescence (a word that rather stopped their thinking temporarily). When the picture was perfectly clear, however, the writer found a ready, intelligent response, which gave a much clearer vision of the value now existing than the most skilful experts working alone could possibly have obtained.

FRED ASA BARNES,[50] M. AM. SOC. C. E.—The central purposes of this Symposium—"To avoid emphasis on technical details and make a new beginning toward formulating a preliminary agreement on fundamentals," as stated by Mr. Scharff in his "Foreword"—seem to have miscarried badly; there are plenty of technical details in the second and third papers and nothing particularly new in any of them.

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[50] Prof., Railroad Eng., School of Civ. Eng., Cornell Univ., Ithaca, N. Y.

Why continue this battle of words about little or nothing? All agree that there is such a thing as depreciation, but not on how to determine it. All might agree, however, that the first essential to its determination with a fair and reasonable degree of accuracy is an honest intention to do so. A company that sets up depreciation reserves three or four times larger than necessary to meet requirements according to its own replacement records is not attempting to determine depreciation accurately but is attempting to "put something over" on the authorities and the public. A company that uses one set of depreciation figures for the Bureau of Internal Revenue and an entirely different set for the authority that governs its rates is also scarcely honest. Although these and other irregularities have caused some of the existing confusion, it is becoming increasingly safe to leave them in the hands of the proper authorities whose duty it is to detect them; but much difference of opinion will still be found in regard to the main problem—how to determine depreciation.

An example to support the foregoing is as follows: Assume a machine with a first cost of $10,000, a rated life of six years, a salvage value of $400, a service capacity of 2,000 hr per yr, and annual repair costs as given in Table 1. The differences between the results determined by the straight-line and the annuity methods are seen to be comparatively small. Furthermore, the effect of the heavy repair costs in the later years is not obscured as it is with the Matheson or the constant-percentage method.

TABLE 1.—COMPARISON OF THE COSTS OF REPAIRS AND CARRYING CHARGES BY DIFFERENT DEPRECIATION METHODS
(Interest Rate, 7%)

Year	Hours	Repair costs[a] (dollars)	Dollars per Hour of Service			
			Straight line	Annuity	Matheson	Equalized
1	2,000	2,100	2.20	2.07	3.48	2.524
2	4,000	1,000	1.59	1.52	1.92	2.524
3	6,000	3,330	2.70	2.69	2.49	2.524
4	8,000	3,570	2.77	2.81	2.27	2.524
5	10,000	4,000	2.93	3.02	2.28	2.524
6	12,000	4,800	3.27	3.42	2.57	2.524

[a] Substantially as given in "Construction Planning and Plant," by A. J. Ackerman and C. H. Locher, 1st Ed., McGraw-Hill Book Co., Inc., New York, N. Y., and London, England, 1940; see Fig. 22, p. 70.

The equalized results in the last column of Table 1 were computed by using the annuity equivalent to the repair costs; but they could be approximated by using the average annual repair cost, $3,167, discounted by 5% to 7% because of the larger values in the early years due to the use of the average. The yearly depreciations or "write-offs" by this method are as follows:

Age	Depreciation (dollars)
1	2,248
2	3,505
3	1,421
4	1,280
5	940
6	206

These are quite variable, of course, but none is as large as that for the first year by the Matheson method ($4,150).

The writer firmly believes that slavishness to method is one of the major difficulties, and there is absolutely no reason for conforming to any particular method; for it can be proved easily, if any proof is necessary, that there is no financial difference between the different methods, provided that interest is paid or properly accounted for, except for the case of the sinking-fund method when the rate of interest paid on deposits in the fund differs from the rate paid on the debt. That is, the sum of the present worths of the payments actually made, whatever the method or lack of method, will be the debt, and why should it be anything other than the debt?

Thus, in many cases where life histories are available (including repair and overhauling records), it would be possible to adjust carrying charges so that the total cost per unit of product would remain nearly constant throughout the life of the machine or structure, which would be an ideal result from the standpoint of fairness and equity. Of course, obsolescence might "upset the applecart," and in the case of construction machinery the annual or seasonal use would be most difficult to predict. In any case, "adjusting entries" might be necessary from time to time. They are used quite freely for other purposes, however, so why should they not be used for correcting depreciation estimates?

As to a definition of depreciation, the writer suggests the one formulated by Chief Justice Charles Evans Hughes on April 30, 1934, in the so-called Chicago Telephone Case. It is:

> "Broadly speaking, depreciation is the loss, not restored by current maintenance, which is due to all the factors causing the ultimate retirement of the property. These factors embrace wear and tear, decay, inadequacy and obsolescence."

Aside from the wilful and intentional misrepresentations by utilities through their lawyers and accountants, some writers on these subjects have contributed their quota to the existing confusion of thought. One very prominent author calls an annuity a depreciation payment, although the context indicates plainly his knowledge that an annuity consists of interest and depreciation. He states also that the straight-line method is the most straightforward (whatever that may mean) but that the diminishing-balance method (diminishing-value or Matheson method, to engineers) has the advantage of simplicity since no complex calculations or elaborate records are necessary. If the percentage loss is assumed (which seems probable since he gives no method of computing it), this method would still require more computations than the straight-line method, as both the depreciation and the interest change from year to year.

Professor Grant, in Example 1, complicates the problem by introducing a change in price level, and the writer does not agree that there is no "popular" depreciation. Certainly the $6,000 received for the property has less purchasing power than the $6,000 paid for it. Furthermore, the owner would have a difficult time convincing the Bureau of Internal Revenue that he was not subject to a capital gains tax on the total depreciation he had written off while the property was leased.

Another example of confused thought is a recent book in which the sinking fund is called an annuity, and the sinking-fund method is lauded because the fund accumulates at compound interest while the "accumulation of carrying charges is at simple interest." Of course, there is no "accumulation of carrying charges" if interest is paid or accounted for annually, as it should be; furthermore, the amount in the sinking fund is only a small fraction of the debt at first and reaches the amount of the debt only at the very end of the period.

Finally, the writer has no patience with those who maintain that there is a conflict between theory and practice, or actualities. Correct theory and correct practice support and sustain each other, and one or the other is wrong if there is a conflict between them.

Robert F. Legget,[51] **Assoc. M. Am. Soc. C. E., and Erwin E. Hart,**[52] **Esq.**—In the practice of engineering, depreciation is not the only subject to be much misunderstood due to the lack of a clear definition of meaning. It is among the most important of such matters, however, and one that inevitably will demand an increasing amount of attention. The appearance of this Symposium is therefore to be welcomed, especially since it is intended—in the words of Mr. Scharff—"to * * * make a new beginning toward formulating a preliminary agreement on fundamentals."

The three papers which constitute the Symposium make somewhat unequal contributions toward the objective set by Mr. Scharff. Although the writers agree with many of the individual statements of Mr. Walls, the general effect of his paper is but to confirm the complexity of the subject and the divergent views that are possible upon so many aspects of it. It will be generally agreed that "depreciation may well be a subject for engineering investigation and careful exercise of judgment." The fact that the Symposium is presented by the Society is tribute to this view; but it must also be admitted that factors other than those which can be classed strictly as engineering matters affect depreciation to a marked degree. Therefore, although engineers have a definite contribution to make to what may be called "the depreciation problem," theirs is not the only contribution, as Mr. Walls' paper would seem to infer.

The view expressed by Mr. Walls that "depreciation * * * is an accomplished fact" is fundamental. It contrasts somewhat strangely with the suggestion under item (*d*) that the capacity of enterprises to make provision for depreciation will vary. Since depreciation is an accomplished fact at any one time, the provision for it is not, therefore, connected with the profitability of the enterprise as a whole. Provision for the latter is, rather, the contingency. Correspondingly, since depreciation is a fact, then "statistics" (using this term generally, as does Mr. Walls) will serve as an indication of improvement in the art currently taking place, and will be a guide at least to estimates of future life.

The writers are in complete agreement with the views expressed by Messrs. Crum and Winfrey. Their only regret is that, due to the lack of interest on the part of the average citizen in all governmental financial matters other than

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[51] Asst. Prof., Dept. of Civ. Eng., The Univ. of Toronto, Toronto, 5, Ont., Canada.

[52] Toronto, Ont., Canada.

"reducing his taxes," they see little hope of the wide adoption in the near future of the sound principles advocated in this third paper. The appearance of the paper, and the indications it gives of the study now being given to this important aspect of public finance, give much encouragement to all who have at heart the interests of good government. One factor only does not appear to receive full consideration in this paper—namely, the fact that there are now, and there will be in increasing numbers, public works for which the benefits cannot be expressed wholly in financial terms. For such cases, some modification of general depreciation practice will have to be evolved.

Professor Grant's paper is interesting, provocative, and challenging. His examples are well chosen to illustrate the points he makes and the definitions he suggests. If the writers question certain features of his paper, as they must, they wish to preface such questionings with this genuine tribute.

Professor Grant starts his paper with three explanations of terms in common use in relation to depreciation—"popular depreciation," or decrease in value; "accounting depreciation," or amortized cost; and "appraisal depreciation," or the difference in value between the existing old asset and a hypothetical new asset. He proceeds to illustrate the use of these terms in the interesting cases that he cites in his paper. Example 1 is a particularly good illustration of the use of the three terms; it is referred to in the following note.

The writers wish to suggest that applying the word depreciation to these three values is somewhat confusing and represents one of the difficulties that must be cleared up before the matter can be brought to a proper basis. The values cited for popular depreciation and appraisal depreciation differ fundamentally from the value given as that determined by accounting depreciation, in that both of them are calculated at the actual time that the purchaser is considering buying the house. There is, therefore, little uncertainty about them except in so far as the appraisal depreciation makes a deduction for shorter life expectancy, etc. On the other hand, the value given for accounting depreciation was determined on the basis of calculations made in the year 1932 when no one could tell with certainty what the conditions with respect to property would be in the year 1940.

The writers believe that to compare these three values on an equal footing is scarcely a fair procedure. They suggest that in all discussions of depreciation a clear distinction should be made between actual depreciations that can be calculated with respect to past time and estimates of depreciation that have to be made with respect to future time.

It seems to the writers that a fundamental understanding of depreciation is obtained by looking at the trend of annual or unit costs which the most efficient and modern machine designed for use in some particular field provides, using the lowest attractive interest rate that the class of investment involved may direct. In general, two things will be observed for most products:

(1) The price obtainable for the product will decline continuously—that is, the amount available from the sale of the product or service will decline from year to year, and thus the sum of: Operations (O) + maintenance (M) + depreciation (D) + interest (I) = total (T) must decline. This means that

it is the sum of the last three $(M + D + I)$ which must decline, as the first item may be taken to be constant.

(2) The trend of maintenance costs will be upward as the asset becomes older.

Thus there are two limiting conditions:

(*a*) Due to the declining allowable total T, there will come a time when the allowable T will equal the total of $O + M$, regardless of how slowly M may rise. At this point the asset will cease to have value as an operating unit because its operating costs are relatively too high, and it thus has become "obsolete."

(*b*) Due to the gradually rising M, there will come a time when the $O + M$ total alone will equal T, regardless of how slowly T may fall. This may be defined as a statement of the extent of dilapidation.

Now any formula or arrangement for estimating either value or depreciation allowances must satisfy the series of figures associated with the two trends (1) and (2) and the two limits associated with (*a*) and (*b*). This yields a basic rule that the total of the charges $O + M + D + I$ must in general be steady, or decline from year to year, and must never rise.

Now apply this law to Professor Grant's formula under "Engineering Economy Studies Relative to a Proposed New Investment." Although, as Professor Grant indicates, it may be possible to arrange a series of annual payments to a depreciation account, which would end in establishing the same capital value as is given by this formula (Eq. 1), and not involve uniform annual payments, such a series would be extremely limited as to form and not at all likely to fit very many cases found in practice. Eq. 1 is thus essentially bound up with the idea of equal annual payments. The aforementioned rule thus limits its application to those cases where, for practical purposes, the curve of T is a straight, horizontal line.

If Eq. 1 is used with an asset having other than a horizontal, straight-line relationship for T, then the totals of T must be averaged, in which case the result will be only approximate. If the curve of T is not horizontal and also not a straight line, the averaged result may be considerably in error. The latter condition is frequently found. In such a calculation it thus will be essential to determine the curve of M and take an average value of that too. This average will be open to the same risks as that of T itself. Professor Grant has apparently taken M into his value for annual disbursements in Examples 3 and 4. This is an equal and presumably averaged figure. In general, this formula would seem to be applicable to those assets having a horizontal, straight-line relationship for T, and at the same time having M equal to zero, or forming a horizontal straight line. It is thus not generally applicable as an accurate formula.

Table 2 is an example of a case in which a decreasing series for T is required (operating costs, being constant, have been omitted), and in which an irregular trend of maintenance costs is encountered.[53] It will be noted that this series will support a valuation of $2,000. For $n = 12$ and $i = 6\%$ in Eq. 1, the capital recovery factor is 8.576. The average annual net total of $216 if interest is taken on the investment at the beginning of the year, or $211 if interest is taken on the average investment during the year, will support a valuation of

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[53] "Depreciation and Its Influence on Unit Costs," by E. E. Hart, undergraduate thesis (unpublished) presented to the Faculty of Applied Science and Engineering, Univ. of Toronto, Toronto, Ont., Canada, 1939.

TABLE 2.—EFFECT OF IRREGULAR MAINTENANCE TREND AND DECREASING ALLOWABLE COSTS (DOLLARS) ON THE ACCURACY OF EQ. 1

COST AT:			Depreciation	Maintenance	Interest	Total
Beginning of year	End of year	Average				
2,000	1,600	1,800	400	0	108	508
1,600	1,300	1,450	300	75	87	462
1,300	1,100	1,250	200	150	75	425
1,100	950	1,025	150	175	62	387
950	800	875	150	175	53	378
800	650	725	150	175	44	369
650	525	588	125	200	35	360
525	400	463	125	200	28	353
400	275	338	125	200	20	340
275	175	225	100	225	14	339
175	75	125	100	225	8	332
75	0	38	75	250	2	327
Total	....	....	2,000	2,050	....	4,580
Annual	....	....	....	171	....	382
Maintenance deducted ($171)						211
Sum added for interest to the beginning of the year						5
Net cost as of the beginning of the year						216

($C = 216 \times 8.576 =$) $1,850 in the former case or ($C = 211 \times 8.576 =$) $1,810 in the latter. These values are 7.5% and 9.5%, respectively, below the actual worth of the asset. The formula assumes the former case. It thus would seem possible to obtain errors of as much as 10% or more with this formula. It will be noted that this is an error inherent in the formula, and consequently additional to any errors in the basic data, upon which the calculations were made. The writers realize that Eq. 1 provides a most valuable means of determining the value of many items of property. The object of this criticism is to indicate that errors are possible, nevertheless.

KERNAN ROBSON,[54] ESQ.—In the Symposium, the treatment of the depreciation problem seems concerned mainly with plant and equipment. From the point of view of the average realtor who makes depreciation deductions from tens of billions of dollars of depreciable improvements on real estate, there are many other factors to be weighed in addition to those proposed by the authors.

Although the writer will not discuss the mathematics of these factors, he wishes to emphasize the immensely important tax problems involved. These problems will be greatly accentuated by the 5-yr depreciation program applied to billions of dollars worth of war structures.

One of these problems may be called the "Problem of Annual Low Brackets Against Total High Brackets When Profit or Loss is Ascertained."

An important inequity resulting from the present treatment of depreciation and depletion reserves in adjusting, annually, the cost or other base of property is the following:

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[54] Real Estate and Investments, San Francisco, Calif.

The deduction from net income for such reserves as the taxpayer is allowed from year to year is in the lower brackets; but upon sale or other disposition of these assets, resulting in gain or loss, the total of depreciation or depletion reserves theretofore taken is segregated into one sum which brings the taxpayer into a higher bracket than those in which his income put him in the years these reserves were taken. An example used as far back as 1938 began with the assumptions that:

(*a*) A taxpayer has a plant which he bought for $100,000;
(*b*) The allowed depreciation was 10% per year for ten years;
(*c*) The taxpayer had no other net taxable income at the end of each of the ten years;
(*d*) The tax rate remained the same throughout the 10-yr period; and
(*e*) At the end of the ten years, the taxpayer sold the property for the original cost price ($100,000).

He would then have an accumulated depreciation reserve of $100,000 on which the total taxes—federal and State of California (1938)—would be $41,950. However, the total tax savings (and thus the total sum withheld and due the government by reason thereof) would have been $8,500. That is all he really owes the government by reason of his withdrawal of the allowed or allowable annual depreciation from annual income. The government would thus make a profit of $33,450.

Again, if the taxpayer were to pay someone, at the end of the 10-yr period, $100,000 to take the property off his hands, the $100,000 would not be a loss but would actually be matched by the accumulated depreciation.

The case of Kennedy Laundry Company vs. the City of San Francisco (Case 46 BTA, No. 9, January 14, 1942) is one of many that set forth amongst other matters the principle as to the adjustment of the cost basis by depreciation or depletion allowed or allowable. In substance this principle may be stated as follows (46 BTA No. 9, Section 19.113 (b) (1) 1):

> The cost or other basis of property value to which depreciation or depletion is applicable shall be decreased annually, in addition to all the other deductions, by an amount allowed or allowable due to the exhaustion, wear and tear, obsoleteness or depletion sustained by said property.

It is the condition defined in the general rule that brings about the inequity to which the writer refers herein.

Under that rule such deductions result in the taxpayer's being in the lower brackets when those deductions are made, but when property is disposed of and gain or loss is ascertained, the taxpayer is brought into the upper brackets. Thus, whether the taxpayer makes a gain or a loss, the government gains more (in many instances excessively more) than it is entitled to gain.

This entire doctrine that cost or the adjusted basis shall be reduced annually by the allowable or allowed withdrawals from net income of depreciation or depletion reserves is based on both a mistaken business theory and a mistaken accounting theory.

From a business standpoint, the cost price or sales price of property is entirely different from the reserves that may be set up by reason of depreciation or depletion. Correct accounting which seeks to reflect the true business process could scarcely support a practice in which a debtor, at the end of a transaction, receives a larger sum than has been withheld from him or than he could show, to that time, as belonging to him.

The writer therefore suggests, as a means of correcting the inequity herein set forth, that the taxpayer's books should be set up as follows:

Every year in which depreciation or depletion is allowed or allowable, there should be set up what might be denominated a "suspended reserves account."

Into this account would be entered the difference between the amount the taxpayer paid in income taxes and the amount he should have paid, had no deductions from net income for these reserves been made. When the property is finally sold, or otherwise disposed of, so that gain or loss is ascertained, these annual entries adjusted by charges against them for repairs and replacements (not maintenance or upkeep, which goes to operating costs, and additions or betterments which go to capital costs), added together would constitute the sum of money due the government. The difference between the base of the cost of the property plus or less other proper adjustments (not depreciation or depletion deductions) and the selling price is a profit or a loss in that transaction.

This is an entirely different transaction from the setting up of annual reserves and the fixing of the contingent or final interest of the government therein.

The profit or loss to the taxpayer in the first instance should not be confused or mixed up in any way with the gains to the government which would be set forth in this "suspended reserves account."

The writer considers that "obsolescence" is probably the proper term for the continuous process, and "obsoleteness" for its completion. The Bureau of Internal Revenue, U. S. Treasury Department, defines obsolescence as:

That impairment of depreciable property brought about

(1) By progress of the arts;
(2) Effluxion of time; and
(3) Change of use or useful life.

Of these factors, Item (3) is vastly the most important. The writer knows of no all-embracing rule that can be formulated to treat it, either for equipment, plant, or structure.

J. A. WALLS,[55] M. AM. SOC. C. E.—The numerous discussions of papers comprising the Symposium on depreciation show clearly a great divergency of thought on the subject. The writer does not believe that this divergency results from a confusion of thought, as is suggested by several contributors, but thinks rather that it represents a basic difference as to the concept of the field covered by the general use of the unqualified term, "depreciation." The several contributions to the discussion can be classified generally into those favoring the concept of depreciation which enters into the determination of fair value and those favoring the concept of cost amortization on the basis of age-life assumptions.

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[55] Pres., Pennsylvania Water & Power Co., Baltimore, Md.

The widely divergent views expressed in the discussions show amply the need of developing that particular phase of the problem defined as the scope of the Symposium. Worth-while progress on a satisfactory solution of the problem cannot be expected until there is agreement as to the basic character and scope of what ultimately will be called "depreciation" or, for that matter, some other name. The writer believes that much of the divergency of view will be eliminated when such an agreement has been reached. Unfortunately, much of the discussion was not limited to the scope of the Symposium but was directed more toward methods and details of a yet undefined problem. Of necessity, the following comments will deal largely with the subject matter of the discussions rather than with the original scope of the Symposium.

The writer agrees that, if the amortization of cost is made on an arbitrary age-life basis, the result of that process should be carefully distinguished from depreciation actually existing. Perhaps the term "amortized cost," suggested by several contributors, would be an appropriate clarifying distinction. Unhappily, those who are concerned with the affairs of regulated enterprises are helped but little by the mere drawing of a distinction between different concepts of depreciation. As a general proposition, the writer does not agree that the amortization of cost should proceed on any arbitrary age-life basis. At the same time, he does not disagree with the use of an age-life method for certain types of property where such a method may reasonably approximate the progress of loss of value.

It has appeared to the writer that the valuation concept of depreciation is generally well understood by students of valuation. There have been, of course, those who contended that the valuation concept of depreciation should contain the effects of physical causes only. Others have urged that it should contain the effect of intangible causes as well—that is, obsolescence, requirement of public authority, inadequacy, etc. The writer believes that these two different scopes of the valuation concept of depreciation are purely a matter of the basis for reproduction costs to which they are to be related. Either may be proper if logically applied to estimates of reproduction cost on appropriate bases—that is, cost of a modern substitute or cost to reproduce a new replica. In both cases, the obsolescence resulting from the availability of modern facilities will be correctly reflected if the facts are properly treated.

Since he is primarily concerned with the affairs of regulated industries, the writer cannot dismiss the process of cost amortization without serious consideration. Even in the treatment of amortized cost, he believes that the rate of amortization, in the case of regulated industries, should be in harmony with those forces which bring about a reduction in usefulness of property. The importance of the use to which the results of an amortization might be applied in the future is so great that it should not be based upon fragmentary and immature statistics that are now and, for an indefinite time, will be the only statistics available for most classes of property. There is every reason to believe that depreciation provisions created by any arbitrary process of amortization of cost will have an adverse effect upon regulated industries in the future. It is of far-reaching importance to these enterprises, therefore, that the rate of such amortization be controlled by realistic measurements of the effect of all those forces which justify the amortization.

In several instances, the general use of the age-life theory in the treatment of depreciation, as well as in cost amortization, was urged because of its alleged simplicity. In this connection, the writer wishes to point out that the end of usefulness of plant facilities (abandonment) is not determined by any such superficial theory. Therefore, he sees no justification for predicating earnings upon such a basis during the period of usefulness of the facilities. One suggestion offered was that the simpler age-life method should be used in order that the courts may understand depreciation. The writer disagrees with this as a reason for using this arbitrary method, not only because it is arbitrary but, also, because he doubts that there is any lack of understanding of the problem on the part of the courts.

The writer has never been able to follow the line of reasoning that age-life ratios measure the true change in usefulness of property. Except by the merest chance, this cannot be true, particularly where obsolescence is a cause. The fallacy of such a line of reasoning seems to be beyond question if one considers that a single item of property, having a single estimated life, could have almost numberless different amounts of "depreciation" merely by the application of different theories. Furthermore, the results will vary over a wide range. This variation is particularly striking in the difference between the straight-line and compound-interest or sinking-fund methods of computation.

There are some classes of property, including a minor part of the property of electric utilities, which are reasonably susceptible of statistical treatment. Almost without exception, the lives of these classes of property are determined by physical causes. The writer has no intention of dissuading any one from the proper use of statistics where they afford a reasonably reliable measure of the reduction of usefulness. However, as stated in the discussion, a study made of a large volume of utility property retirements showed that less than 20% of these retirements were due to the effects of physical causes. Since not even 20% of utility property is susceptible of statistical treatment, the inescapable conclusion is that such treatment can offer no proper solution to the problem.

The same analysis showed that more than 80% of utility property retirements have been due primarily to non-physical causes—that is, obsolescence, requirement of public authority, inadequacy, etc. The writer believes that a very large proportion of these retirements was due primarily to obsolescence. Much utility property is long-lived and, as a consequence, the property already retired would be, predominantly, that installed during the infancy of the industry before any substantial development in the art had taken place. For this reason alone, the life of such property can be no reliable guide to the life expectancy of later and more modern equipment.

Of course, there is nothing in the nature or character of obsolescence which relates its incidence to the passage of time. However, entirely aside from this fact, there are other important reasons why statistical treatment of this form of depreciation cannot be reliable. Statistical or actuarial treatment of any problem presumes the existence of a large number of homogeneous units and a reasonably predictable repetition of forces or influences acting upon them. In the case of the far greater proportion of electric utility property, neither of these requirements is met even approximately. It would seem, therefore, that if any statistical approach were found to coincide reasonably with facts, such results would be conditioned by coincidence only, rather than by the logic or soundness of the methods used.

The writer does not agree with the unqualified suggestion that depreciation from physical causes is more susceptible of technical measurements than depreciation from functional causes (obsolescence, inadequacy, etc.). It seems clearly evident that these intangible causes of depreciation, in a broad, general sense, are closely related to growth, scientific development, and even to changes of price level. No one can question that the ultimate effects of such forces are susceptible of technical determinations. In fact, such determinations are continually made in connection with replacements of property. Therefore, it appears conclusive that the effect of these causes must also be susceptible of interim measurements.

The sense of certain discussions conveys the impression that utility companies are suspected of being reluctant to make adequate provision for depreciation; but the writer does not believe such a suspicion accords with the facts. The management of any enterprise, regulated or unregulated, will not continue long unless it exercises due discretion in the protection of the interests of investors. Indeed, it is the writer's impression that many utility companies have been widely suspected of having protected the investors' interests on an overgenerous scale in the past. Even in regard to the provision for depreciation, several state regulatory agencies, in the past, have fixed arbitrary limits above which depreciation reserves could not be maintained by public utilities.

The writer disagrees with those who profess to believe that a public utility enterprise is entitled to no more than a rate of return allegedly established as fair in the "market place," applied to original cash investment, less almost any kind of a deduction that may be made in the name of depreciation. Such a basis, in the writer's opinion, does not afford a reasonable chance for survival so long as enterprises which are not subject to regulation continue to earn a fair return on fair value. It would not be of serious moment if all other industries were limited to the same basis of earnings as advocated for regulated industry.

The problem of depreciation in regulated enterprises cannot be isolated from the effects of competitive forces in the general economic structure. Every one is aware that the economic system of free enterprise functions on the basis of a fair return on fair value. Transactions in the "market place" maintain this basis beyond any present means of effective artificial control. Since the public utility industry is a part of this economic structure, it is obvious that it must obtain its compensation on a comparable basis if it is to survive. On any less favorable basis of compensation, the utility enterprise will not long be able to attract either required capital or competent personnel and, therefore, cannot expect to survive as a private enterprise.

Eugene L. Grant,[56] **M. Am. Soc. C. E.**—The large volume of discussion on this Symposium reflects the importance of the subject and the widespread interest in it; the diversity of the discussion serves to illustrate many of the points of controversy that exist. The writer wishes to express appreciation of the contributions of the many individuals taking part in the discussion.

One basic point of controversy relates to definitions and fundamental concepts. Because of Dean Marston's distinguished contributions to the field of valuation, it seems particularly important to comment at length on his discussion of these matters, including also his definitions quoted in the original paper by Messrs. Crum and Winfrey. Dean Marston states that "* * * in every one of the many billions of purchases and sales, the world over, each year, the exchange value is determined by the present worth * * *" of the prospective future services which the asset will render to the purchaser. This statement considers only the demand side of the determination of market price—that is, the estimates by prospective purchasers of the amounts they think they can afford to pay for assets—and completely neglects the supply side. It takes two to make a bargain.

The supply side of the market is likely to have the greater influence in the determination of market prices in transactions in new machines and structures used by industry. The estimates of possible producers of any type of asset of the prices at which they can afford to sell constitute the supply side of the market. Each possible buyer for any type of asset will have his own valuation of the present worth of its prospective future services, depending on the use he proposes to make of it. The composite of these estimates constitutes the demand side of the market. If the producers' estimates, presumably based on cost of production, are close together, and, if buyers' estimates, based on a great diversity of anticipated uses, are far apart, the market price will be influenced more by cost of production than by present worth of prospective services.

Thus Dean Marston's statement about determination of market value is somewhat misleading. It is not in market value, but rather in determinations of value to the owner that present worth of prospective net money receipts logically plays the major rôle. If practicable, this should be the basis of all determinations of value to the owner. A leased fee in a parcel of improved real estate, an oil well, and a motion picture—Mr. Hyder's items (*a*), (*d*), and (*e*)—are examples for which it is likely to be possible to estimate prospective net money receipts. Unfortunately, from the standpoint of making valuations, in estimating value to the owner of individual assets or groups of assets which are part of a complex industrial plant, it is seldom practicable to isolate the prospective receipts which are attributable to particular assets. Because of this difficulty, the principle of substitution comes into play.

As clearly pointed out by Bonbright,[57] the relevance of replacement cost in the determination of value of an asset to the owner arises out of this principle of substitution. If an owner were deprived of the services of an existing old asset, he presumably could substitute a new one for it. This fact generally places an upper limit on the value to him of the existing old asset; hence the need for measurement of the value-inferiority of the existing old asset to the substitute new one.

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[56] Prof. of Economics of Eng., Stanford Univ., Dept. of Civ. Eng., Stanford Univ., Calif.

[57] "Valuation of Property," by J. C. Bonbright, McGraw-Hill Book Co., Inc., 1937, see particularly Chapter IX.

Dean Marston states that he is unable to agree with the writer "that it would be feasible, or desirable, or even correct, to substitute wise retirement mathematics for the present well-tried and established methods of making engineering valuations of industrial properties." The writer recognizes that the appraisal of industrial properties is an art in which the appraiser must weigh all of the relevant evidence. He does insist that, where replacement cost is considered to be part of this evidence, the principle of substitution implies that this should be the cost of the most appropriate substitute asset. Sometimes well-tried and established methods may be based on false reasoning. This is true of those common reproduction-cost appraisal methods which give major weight to the cost of a new substitute asset identical in design to the existing old asset, regardless of the obsolescence of this old asset or its unsuitableness to the prospective service required of it. In describing his custom of obtaining opinions of foremen and machine operators regarding matters related to obsolescence and inadequacy, Mr. Miller gives a good method of securing part of the evidence necessary to determine the value-inferiority of the existing old assets to the most economical new substitute assets.

As Mr. Hyder states, the Bonbright classification of depreciation concepts applies to regulated industry and public works just as it applies to competitive industry. The writer cannot emphasize too strongly his view that a recognition of these concepts and their relationships to one another is a starting point for any clear thinking about depreciation. This view is well reinforced by the excellent comments of Messrs. Smith and Ripley. A contrasting view, voiced by Dean Agg and others, is that there is "one true figure on depreciation" and that all other figures are necessarily false. This is really an argument whether a word should be used in several meanings or in only one meaning.

Mr. Farwell attempts to settle this question by his suggestion that the usage of the word "depreciation" be limited to the popular sense of decrease in value. This would require the substitution of some other word or phrase for the accounting and appraisal concepts of depreciation. In somewhat similar vein, Mr. Atwood suggests that accounting depreciation always should be designated as "amortized cost," and Mr. Walker states "there is much in favor of a standardization of the term 'depreciation' to mean one thing and one thing only."

Although there can be no doubt that clarity would be promoted if all who use the word "depreciation" would agree to follow these suggestions, there is no practical possibility of securing such agreement. The word has been used by accountants, the courts, regulatory bodies, tax authorities, and the business world in general with these different meanings for so long that any attempt by engineers to persuade every one else to standardize terminology would be futile. What is necessary is to face the fact that "depreciation"—like many other words in the language—is a word which has acquired several meanings through usage, and it is likely to keep them.

It follows that an attempt to write a multi-purpose definition somehow embracing all of the concepts can only result in confusion. Thus Dean Marston's definition with its reference to "loss in value * * * below what such present worth would be if the asset were still new" seems to confuse the popular concept with the appraisal concept. The writer believes the points made by Professor Morrison and Mr. Kosh in criticism of this definition are well taken.

Depreciation definitions should relate to a particular usage. For instance, a definition quoted in a classification of accounts (such as the New York Public Service Commission definition quoted by Mr. Kosh from the "Uniform System of Accounts for Electric Corporations") should be interpreted as saying "For the purposes of this classification of accounts, this is what depreciation shall mean." Such a definition may be judged as to its appropriateness for the particular purpose intended.

The writer emphasized three of the four depreciation concepts and gave only brief mention to a fourth concept, "impaired serviceableness." In his restatement of the three concepts emphasized, Mr. Ryan appears to have left intact the popular and accounting concepts but to have eliminated the appraisal concept. In its place, he has placed impaired serviceableness as the "engineering concept."

Unquestionably "impaired serviceableness" is one of the meanings which the word "depreciation" has acquired through usage. The writer regretfully concedes that engineers have been largely responsible for this usage.

In an attempt to clarify the situation which exists when a word has developed several meanings which get confused with one another, either one may advise the elimination of all but one of the meanings, or one may advise a clear differentiation among them with an attempt, whenever the word is used, to make clear which meaning is intended. In this Symposium, the writer takes the position that the former course is impracticable, and he, therefore, advocates the latter course. However, if the writer thought it were possible to eliminate one of the meanings of depreciation, his first choice for elimination would be this "engineering concept," impaired serviceableness. Engineers too often have misused it in valuation testimony. Statements have been all too common to the effect that because an old property has only 2% impairment of serviceableness, it is 98% as valuable as ever, or—even worse—that it is 98% as valuable as the present cost of replacement with an identical property regardless of the fact that any appropriate replacement would be with a quite different and more economical property.

It is unfortunate that some engineers have concentrated their attention on one engineering consideration—impaired serviceableness—to the exclusion of other equally important engineering considerations such as shorter life expectancy and the availability of better alternatives.

Mr. Mott's interesting discussion emphasizes the relationship of the appraisal function to the various depreciation concepts. Depreciation in the popular sense may be the end product of two appraisals, as it requires a valuation of a property on two different dates. As Mr. Mott points out, the need for the appraiser's services usually arises from the absence of a satisfactory valuation at the later date. Hence the appraiser gets little help from the popular definition of depreciation. He should be helped, however, by understanding that a "depreciation" deduction from replacement cost is logically the difference between the values of two assets: A hypothetical new replacement asset and an actual old asset at one date—the date of the appraisal—rather than the difference between the past and present values of one asset.

Mr. Goldthwaite warns against "tying up the replacement problem directly with the depreciation problem" on the ground, which he clearly illustrates, that the decrease in value of an asset is in no way influenced by whether, on retirement, it is or is not to be replaced. In this connection it should be noted that the depreciation accounting problem is not necessarily tied up even with the retirement problem. Because of obsolescence or changes in service requirements, an old asset assumed to have perpetual life might well have value-inferiority to the most economical new substitute asset. This depreciation in the appraisal sense might be reflected in depreciation in the popular sense which would make it sound policy to have the amortized part of the cost of the asset on the books of account.

Two of the discussions bear specifically on the compound interest calculations relative to appraisal depreciation. In his statement of annual depreciation corresponding to his equalized annual costs, Professor Barnes presents results that can be obtained by application of the writer's Example 4, given certain assumptions regarding the most economical substitute asset. Professor Legget and Mr. Hart, in Table 2, seem to be attempting something similar to Professor Barnes, but somehow have missed the point.

Professor Barnes' statement of yearly "write-off" for equalized annual costs can be used to determine the book value of his asset at each age:

Age	Annual depreciation	Depreciated book value
0	....	$10,000
1	$2,248	7,752
2	3,505	4,247
3	1,421	2,826
4	1,280	1,546
5	940	606
6	206	400

These equalized, annual-cost book values will correspond to the "value to the owner" of the old asset which would permit him to realize the same annual costs keeping the old asset as he would realize over the life of the most economical new asset, provided the assumption is made that the new asset will repeat the cost experience of the old one. This is true at all ages, and may be illustrated by the calculations to be made when the asset is 3 years old.

To equate annual costs for old and new assets, it is necessary to determine the equivalent uniform annual repair cost over the life of the new asset, and over the remaining life of the old asset.[58] The present worth of the repair costs for the new asset is $2,100 (0.9346) + $1,000 (0.8734) + $3,330 (0.8163) + $3,570 (0.7629) + $4,000 (0.7130) + $4,800 (0.6663) = $14,328. This is equivalent to a uniform annual series of $14,328 (0.20980) = $3,006. The present worth of the repair costs for the remaining 3 years of life of the old asset is \$3,570 (0.9346) + \$4,000 (0.8734) + \$4,800 (0.8163) = \$10,748. This is equivalent to a uniform annual series of \$10,748 (0.38105) = \$4,096.

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[58] Appraisal calculations involving variable annual disbursements are illustrated at greater length in "Depreciation Estimates in Appraisals of Manufacturing Equipment," by P. T. Norton, Jr., and E. L. Grant. *Transactions*, A. S. M. E., July, 1942, Vol. 64, pp. 509–520.

Let x = the value of the old asset to "break even." Equate annual costs for the old and new assets as follows: $(x - \$400)(0.38105) + \$400\ (0.07) + \$4{,}096 = (\$10{,}000 - \$400)(0.20980) + \$400\ (0.07) + \$3{,}006 = \$5{,}048$. Then $x = \$2{,}826$.

This calculation has given due weight to the shorter life expectancy and higher prospective annual repair costs of the old asset. It has not given weight to the possibility of improved alternatives. For instance, if the new asset still may be purchased for \$10,000, but is improved to the extent of having lower prospective repair costs, the 3-yr-old asset is worth less than \$2,826 to its owner, in light of the principle of substitution.

It may be noted from inspection of Table 1, presented by Professor Barnes, that the retirement of this type of asset at the age of 6 years is presumably not a matter of collapse like the "one-hoss shay" at this age, but rather a matter of too high prospective repair costs in the seventh year and thereafter. The equivalent annual cost over the prospective life of a new asset is \$5,048. If the sum of prospective seventh year repair costs, interest on salvage value, and prospective decline in salvage value in the seventh year is greater than \$5,048, and, if this sum appears likely to continue to increase in subsequent years, it obviously pays to replace at the end of the sixth year. The trend of repair costs in Table 1 indicates that these alone are likely to be considerably more than \$5,048 during the seventh year of life.

In contrast to this, Professor Legget and Mr. Hart appear to have chosen a "one-hoss shay" for their illustration in Table 2; at least, it is evident that the trend of maintenance costs does not appear to justify replacement in the twelfth year. Equivalent annual maintenance cost (calculated in the manner illustrated in the writer's comment on Professor Barnes' example) is \$159. Total annual costs over a 12-yr period (assuming no salvage value at the end) are \$159 + \$2,000 (0.11928) = \$398. Unless maintenance costs are expected to jump from \$250 in the twelfth year to \$399 or more in the thirteenth year and thereafter, it will not pay to replace with an asset which is expected to repeat the cost experience of the old one.

The writer sees no reason for the assumption by Professor Legget and Mr. Hart that "the price obtainable for the product will decline continuously." It might just as well be true that this price would increase or remain constant or vary in an irregular manner. Unless the quality of a product is influenced by the age of the machines which produce it, there is no reason to believe that the price will be influenced in one way or the other by the age of the machine.

The difficulties which Professor Legget and Mr. Hart experience with Table 2 arise from an error frequently observed in compound interest calculations. This error is the failure to make all conversions from one year to another by the use of the appropriate compound interest factor corresponding to the given interest rate, which in this case is 6%. By this error one may seem to prove that two equals one or get similar results much more disquieting than Professor Legget and Mr. Hart's demonstration that \$2,000 seems to equal \$1,850. The treatment of a simple arithmetic average of irregular money amounts occurring over a series of years as if this average were a uniform annual payment is, in effect, an equivalence conversion at 0% interest. The association of this with an equivalence conversion at 6% interest is certain to lead to inconsistent results.

In any case, the writer does not understand how Table 2 has any bearing on the validity of Eq. 1 which is conventional and unimpeachable compound interest mathematics. He wishes to reiterate his original point that Eq. 1 is merely an equivalence calculation to facilitate a comparison of irregular money-time series. It converts certain elements of the time series (the disbursement of first cost at one date and the receipt of salvage value at a later date) to an equivalent uniform annual series. Other irregular items in the series may need similar conversions, as was illustrated in the foregoing discussion of Table 1.

Comments on the income tax aspects of depreciation are made by Messrs. Wendt and Fleming, Professor Norton, and Mr. Robson. As pointed out by Professor Norton, this subject has acquired increased importance with the higher tax rates of recent years.

The income tax view of depreciation is the accounting view; that is, that the first cost of an asset (or more correctly the difference between the first cost and the prospective salvage value) is a prepaid operating expense to be prorated among the years of life in some systematic manner. Each such depreciation expense prorated against any year reduces taxable income for that year. If any one such expense is considered apart from all other expenses, it appears to reduce income applied to the highest bracket (or brackets) paid by the taxpayer for a particular year.

When replacement is made of an asset, such as the piece of real estate in Mr. Robson's example, which is not treated for tax purposes as a member of a group, the unamortized part of the cost is likely to be different from the price received for the asset. This difference is an increase or decrease in the taxable income of the year in which the retirement takes place. As shown by Messrs. Wendt and Fleming, this also applies to the retirement of the final unit of a group written off under the group method. Considered by itself, this addition or deduction from taxable income applies to the highest bracket (or brackets) paid by the taxpayer in the retirement year.

Each year has a new revenue act with possible new tax rates for individuals and corporations. Moreover, even if there were no changes in tax rates, the income of an individual or corporation varies from year to year; in some years it will reach higher tax brackets than others. As stated by Mr. Robson, the retirement of an asset may itself make a great change in the highest bracket reached.

If tax rates and maximum brackets did not change from year to year, the question of the amount of the depreciation allowances to be permitted in each year for income tax purposes would not be quite so troublesome as it is. In this situation, only the time of tax payments, and not their total amount, would be affected.

Actually, with any specified variation of tax rates and maximum brackets, a taxpayer's tax liability over the life of a group of assets will depend on the method used for apportioning the cost of the assets among the years of their lives. As demonstrated by Mr. Robson, different schemes of making depreciation charges may result in great differences in total tax payments. As Professor Norton so clearly explains, one aspect of this situation is the influence of legislation and Treasury Department policy regarding depreciation on private decisions as to plant expansion and retirements. Another aspect, illustrated by Mr. Robson's example, is the question of fair treatment of taxpayers.

This important question of depreciation allowances for tax purposes is particularly troublesome because the depreciation accounting problem is fundamentally indeterminate. It should be remarked that the reason for this indeterminateness of depreciation accounting is not always too clearly understood. It is not primarily (as some discussions seem to imply) because of the difficulty of estimating the prospective life of assets; in fact, as Mr. Benson so effectively shows, mortality studies of many groups of assets disclose a surprising stability of average realized lives within each group, despite the variety of causes motivating retirements.

The basic trouble is that, given a life, first cost, and salvage value, there is no one right method for distributing the difference between the first cost and salvage value among the years of life. In the literature of depreciation, many different criteria are advocated for judging the different ways of doing this. In the Symposium, the writer suggested as one appropriate criterion that depreciated book values in the accounts should not exceed value to the owner determined from the cost of the most economical substitute asset minus the value-inferiority of the existing old asset. Another criterion, frequently mentioned in the literature of depreciation accounting, is suggested by Professor Barnes—namely, that the sum of disbursements, the depreciation charge made in the accounts, and interest on depreciated book value, be constant from year to year.

Professor Legget and Mr. Hart suggest that this sum preferably should decrease each year throughout the life of an asset. Mr. Hyder's interesting discussion of his five types of depreciable assets seems to imply present worth of prospective earnings as his suggested criterion for (*a*) a leased fee in a parcel of improved real estate, (*d*) an oil well, and (*e*) a motion picture, and second-hand market value as his suggested criterion for (*c*) an automobile. Mr. Worley seems to advocate "loss of service units" as the reasonable criterion. These and other criteria recommended for evaluating systems of apportioning depreciation charges among years of life could be applied in any set of circumstances both to predetermined systems such as straight line, sinking fund, fixed percentage of diminishing balance, and 60-months amortization regardless of length of life, and to irregular systems based on periodic appraisals or other considerations. Nevertheless, the fact remains that it is impossible to state that one system is right and another wrong; one merely may say that they correspond well or badly to various criteria which may be established as a basis for their comparison and the respective merits of which are debatable.

This paradox—that, although there is no one correct method of depreciation accounting for any given asset, different methods may involve great difference in tax liability—suggests that federal legislation and Treasury Department regulations regarding depreciation deductions might well be based in part on consideration of the matter emphasized by Professor Norton, namely the probable influence on private decisions as to plant expansion and replacement. As this influence was negligible until recent years for all but a few individual taxpayers in the high income brackets, it is really a new consideration to enter into tax policy. Dealing as it does with the motivation of private capital investment, a mainspring of the economic system, it is an important matter for the postwar years. The writer knows of no adequate study of it which has been made. The following statements might serve as a starting point for such a study:

1. Those tax rates which will enter into such calculations will be prospective rates in the immediate future years. They will be the highest bracket rates expected to be paid by the taxpayer. Thus they will be different for different taxpayers at any given time.

2. Whenever the maximum tax rate to which a taxpayer seems likely to be subject in the next few years is very high, he cannot hope to "break even" on the purchase of assets that have a high prospective rate of decline in value to the owner and a much lower depreciation rate permissible in tax returns. As Professor Norton states, many corporation executives have not been fully conscious of this fact. It seems probable that, after they have had the experience of some years of very high tax rates during the war, it will have become much clearer to them as it affects decisions to be made in the immediate postwar period.

3. On the other hand, an opportunity to write off an asset very rapidly during a period of high tax rates may provide a motive for some otherwise uneconomical investments. Thus, some corporations in excess-profits tax brackets, which acquired plant under the 60-months-amortization provision, may turn out to have done so at a small net cost to them. This has the possibility of altering the competitive positions of various producers in some industries.

This Symposium serves as an excellent illustration of Mr. Flagg's statement that "the nature of the contact an individual has with any subject is certain to influence his views and conclusions" which he so happily illuminates with Mr. Saxe's poem. This applies not only to the different aspects of the subject presented by the thirty discussers but also to the apparent diversity of meanings which some of them apply without qualification to various terms. Throughout this closure, the writer has tried to relate his comments to this important matter of clarification of terminology by illustrating in various ways the need of a clear distinction among the depreciation concepts described at the start of his paper.

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Paper No. 2204

ENERGY LOSS AT THE BASE OF A FREE OVERFALL

By Walter L. Moore,[1] Jun. Am. Soc. C. E.

With Discussion by Messrs. Merit P. White, Boris A. Bakhmeteff and N. V. Feodoroff, Carl E. Kindsvater, J. E. Christiansen, L. Standish Hall, Hunter Rouse, and Walter L. Moore.

Synopsis

Experimental studies were made of a free overfall with a view to obtaining information that would be of value to designers of hydraulic structures. Detailed laboratory measurements showed that the energy losses at the base of a fall were of appreciable magnitude and hence must be considered in hydraulic design. These measured energy losses were applied in the development of a rational formula for calculating the height of the jump below a fall. Limited information was also obtained on the length characteristics of the jump and on the effect of submergence of the jump on energy dissipation. The presence of standing water behind the fall is explained, and its height is calculated by application of the momentum equation.

Introduction

Hydraulic model studies have been used with much success in the design of spillway structures and outlet works of all sizes and types. Ordinarily, they are used only after the more common and much cheaper analytical design methods, which are based on experience, research, and analysis, have been investigated and considered inadequate. The extension of the common design methods to cover problems which at present can be solved satisfactorily only by the use of models will come, along with an appreciable saving of time and expense, when the proper fundamental information is made available. The use of the analytical methods that have a sound basis not only saves the cost of the model studies but, in general, results in more reliable designs.

The experimental study reported herein was made at the Cooperative Laboratory of the Soil Conservation Service, U. S. Department of Agriculture, and the California Institute of Technology as part of a general program, the

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Note.—Published in November, 1941, *Proceedings*.

[1] Senior Structural Research Engr., Lockheed Aircraft Corp., Burbank, Calif. (formerly Junior Engr., SCS Laboratory, California Inst. of Technology, Pasadena, Calif.).

objective of which is to supply the engineer with basic information that can be applied in the design of hydraulic structures. Since the overfall is used frequently in soil conservation work, it was selected for study. Stress has been laid upon energy loss or energy dissipation not only because it is of importance in itself, but also because it can be used in calculating the flow characteristics, which in turn determine the design. The results are of special interest to technicians concerned with the design of small dams used in gully control work, as well as to engineers dealing with other hydraulic structures where the overfall is used to dissipate energy.

This investigation was conducted by the usual experimental method, which combines physical measurements with analysis. It was decided to study first a structure that had been simplified as much as possible in the conviction that the simple structure must be thoroughly understood before a more complicated one can be studied intelligently. Following this method the complications introduced by side contractions and expansions were eliminated in an effort to approach the condition of two-dimensional flow. Likewise, the effects of sills and pools were not included in the study.

Apparatus and Procedure

A free overfall model was constructed having a rectangular channel 11 in. wide, discharging into a similar channel at a lower elevation. The channel sides were carefully joined to form two smooth parallel planes with which the edges of the falling jet were in continuous contact. One side of the model was of plate glass and permitted direct observation of the flow pattern (see Fig. 1).

Fig. 1.—Free Overfall Model; Vertical Strings Are Spaced at 0.5 Ft

The lower channel could be raised and lowered to vary the height of fall to a maximum of 1.5 ft. Water was supplied by an electrically driven centrifugal pump discharging into a constant head tank from which the flow was conducted through an 8-in. pipe to the forebay of the model. The discharge was regulated with an 8-in. gate valve, and measured by a venturi meter installed in the supply line. The differential head on the venturi meter could be read to 0.001 ft on either a water or mercury-filled manometer, which made it possible to set the rate of flow to within 1% from 0.15 cu ft per sec to 1.85 cu ft per sec. The disturbance caused as the 8-in. pipe discharged into the forebay was stilled very effectively by passing the flow through a 1-ft wall of loosely piled 1½-in. rock. From the tranquil forebay a transition section led the flow into the model channel with a minimum of disturbance (see Fig. 2). Elevations were determined to 0.001 ft with a point gage mounted upon an instrument carriage that could be rolled alon rails above the channel. On the center line of the lower channel a brass strip with forty piezometer openings was installed in order to investigate the longitudinal variation of the bottom pressure. The depth of tailwater in the downstream channel was controlled by the wicket-type tail gate shown in Fig. 3. This gate, which caused minimum disturbance to the flow pattern upstream, was located 11 ft from the fall and 3 ft downstream from the last section at which measurements were taken.

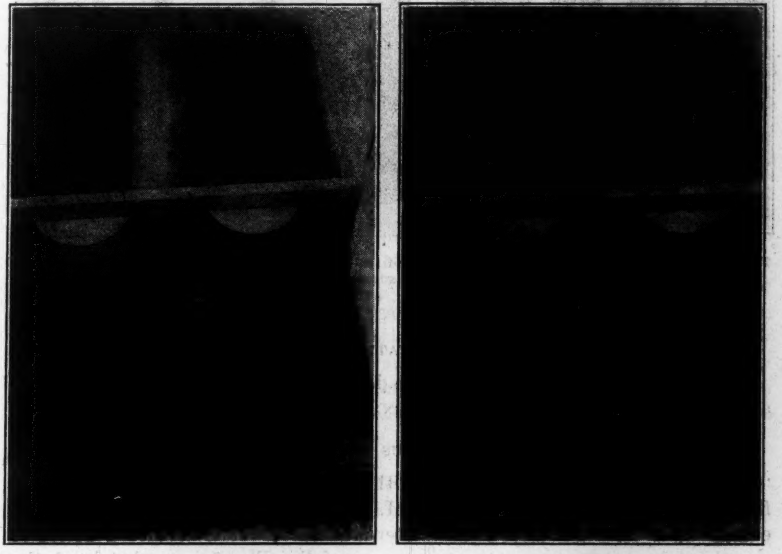
(a) (b)

FIG. 2.—VIEW OF MODEL FACING DOWNSTREAM ON FOREBAY: (a) TRANSITION SECTION IN EMPTY FOREBAY; (b) UNDISTURBED FLOW ENTERING TRANSITION

When the depth of flow would permit, velocities were measured with a standard Prandtl pitot-static tube that had a coefficient of 1.00 (Fig. 4(A)). In order to determine velocities in shallow flows, a hypodermic needle was used to measure the total head (Fig. 4(B)) and a piezometer was used to obtain the static head. By using hypodermic needles of three sizes (16, 18, and 19, Birmingham wire gage) the diameter of the measuring tube was kept within one tenth of the depth of flow. In regions where the bottom piezometer could not be relied upon for the static pressure, measurements were made with a hypodermic needle closed at the end and drilled with static holes according to the standards given by H. Peters[2] (Fig. 4(C)). This apparatus was well adapted to a detailed investigation of a two-dimensional drop and allowed particular attention to be given to the energy losses involved.

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[2] "Druckmessung," von H. Peters, *Handbuch der Experimentalphysik*, IV, I Teil, pp. 489–509.

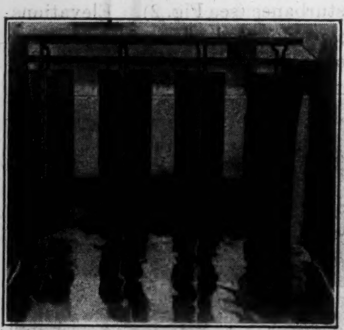

FIG. 3.—UPSTREAM FACE OF TAIL GATE WITH WICKETS PARTLY CLOSED TO RAISE TAILWATER

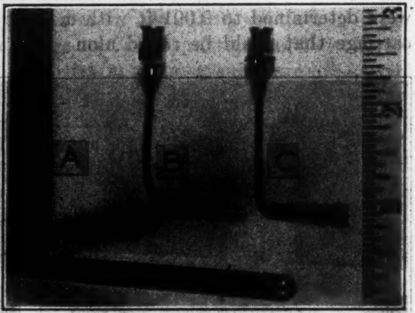

FIG. 4.—TUBES USED IN MAKING VELOCITY MEASUREMENTS: (A) PRANDTL PITOT-STATIC TUBE; (B) AND (C) HYPODERMIC NEEDLE FOR READING TOTAL AND STATIC HEAD, RESPECTIVELY

NOTATION

The letter symbols in this paper are defined where they first appear and are assembled for reference in the Appendix.

PREVIOUS STUDY

Very little study has been given to the problem of a free overfall striking a flat bottom. Boris A. Bakhmeteff,[3] M. Am. Soc. C. E., has shown that the energy equation may be applied to determine the flow conditions at the toe of a fall. Fig. 5 is a sketch of the water profile for a typical overfall with the downstream depth, d_1, unaffected by the tailwater in the channel—that is, with the flow in the downstream channel at supercritical velocity.

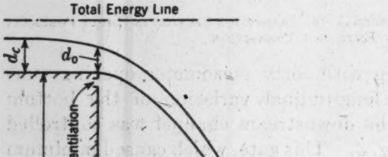

FIG. 5.—TWO-DIMENSIONAL FREE OVERFALL

Assuming hydrostatic pressure and uniform velocity distribution, the energy head E_0 at the upper level is

$$E_0 = h + \frac{3}{2} d_c \ldots\ldots (1)$$

in which d_c is the critical depth and h is the height of the fall. The velocity at section 1–1 may be expressed as

$$V_1 = \phi \sqrt{2\,g\,(E_0 - d_1)} \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

in which ϕ is a velocity coefficient, g is acceleration due to gravity, and d_1 is the depth at section 1–1. Transposing and introducing the expression for E_0 into Eq. 2, there follows

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[3] "Hydraulics of Open Channels," by Boris A. Bakhmeteff, McGraw-Hill Book Co., Inc., 1932, p. 295.

$$\frac{V^2_1}{2\,g} = \phi^2 \left(h + \frac{3}{2} d_c - d_1 \right) \quad (3)$$

Introducing the continuity relation, $V_1 d_1 = d_c \sqrt{g\,d_c}$, and rearranging Eq. 3 give

$$\left(\frac{d_c}{d_1}\right)^3 - \phi^2 \frac{d_c}{d_1}\left(\frac{2\,h}{d_c} + 3\right) + 2\,\phi^2 = 0 \quad (4)$$

Eq. 4 may be transformed to agree identically with the equation given by Professor Bakhmeteff. It is presented herein in a dimensionless form in which the ratio $\frac{d_c}{d_1}$ describes the flow conditions at the base of the fall, and $\frac{h}{d_c}$ represents the relative height of fall. Professor Bakhmeteff states that the coefficient ϕ may be taken equal to unity, which means there is no loss up to section 1–1, Fig. 5. P. M. Bobin,[4] who uses a similar method of attack, makes no statement as to the correct value of ϕ, but in one example uses a value of $\phi = 0.95$. So far as the writer knows, an experimental determination of the energy loss in a free overfall has not been made previously. Since the flow characteristics in the remainder of the structure are dependent upon the energy loss at the base of the fall, the first aim of this investigation was to determine this loss and thus make it possible to calculate the vertical elements of a simple two-dimensional drop structure.

Flow Conditions with Shooting Tailwater

When there was shooting flow in the lower channel, the energy loss at the base of the fall could be most easily investigated because the area being studied was unaffected by downstream disturbances.

Energy Loss at Base of Fall.—When these studies were begun, it was felt that the energy content in the lower channel could be determined by obtaining the value of d_1 and applying the relation

$$E_1 = d_1 + \frac{V^2_1}{2\,g} = d_1 + \frac{q^2}{2\,g\,d^2_1} = d_1 + \frac{d^3_c}{2\,d^2_1} \quad (5)$$

in which q is discharge per foot of width. It should be noted that E_1 is very sensitive to changes in the value of d_1. Obviously, then, the accurate measurement of d_1 was essential, but the presence of heavy spray near the base of the fall prevented this by introducing errors which, in the case of thin flows and high falls, exceeded 25%. Furthermore, Eq. 5 is based upon the assumption of uniform velocity. When an investigation revealed that the velocity profile below the fall was not uniform and, consequently, that the basic assumption was not valid under the existing circumstances, this approach to the problem was abandoned.

It was found necessary to evaluate the velocity energy directly by taking velocity measurements in the thin sheet of water shooting away from the toe of the fall. This was accomplished by using a hypodermic needle to obtain the total head and bottom piezometers to determine the static head. For points not located on the center line over the piezometers, an adjustment based on the transverse variation of depth was made in the static head reading. For each of three stations in the downstream channel, velocity profiles were taken at the lateral odd tenth-points, and plots were made showing values of velocity, V, and velocity cubed, V^3. The velocity energy at each station was computed from the summation, $\dfrac{\sum(V^3\,\Delta A)}{2\,g\sum(V\,\Delta A)}$, which gives the average kinetic energy per pound of fluid flowing across the section. Measurements also were taken of the surface elevation as given by the point gage, and of the bottom pressure as given by the piezometers. The data for each run were plotted as shown in Fig. 6, in which Z = the elevation of the plane of the horizontal velocity

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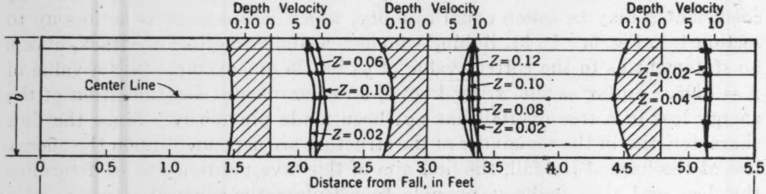
[4] "The Design of Stilling Basins," by P. M. Bobin, *Transactions*, Scientific Research Inst. of Hydrotechnics, Vol. XIII, 1934, Leningrad, pp. 79–123.



(*a*) SURFACE AND VELOCITY CROSS SECTIONS

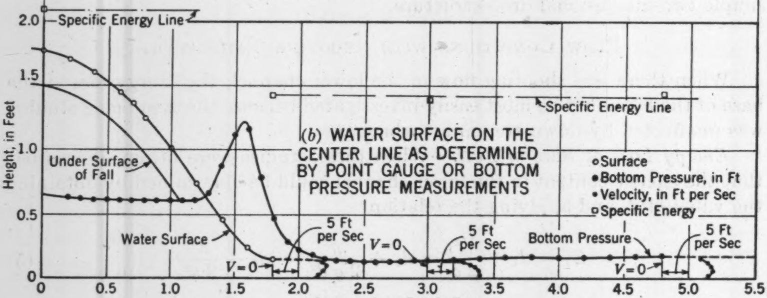


FIG. 6.—TYPICAL PLOT OF SURFACE PROFILE, PRESSURE, AND VELOCITY MEASUREMENTS THROUGH FALL

traverse; h_v = venturi meter reading = 0.42 in. of mercury; Q = 1.218 cu ft per sec; d_c = 0.380 ft; d_0 = 0.272 ft; d = 0.152 ft at 5.0 ft from the fall; and $\dfrac{\sum(V^3\,\Delta A)}{2\,g\sum(V\,\Delta A)} + d$ = specific energy. Six runs were made with the fall height of 1.5 ft covering a range of $\dfrac{h}{d_c}$ from 3.0 to 14.6. The value of d_c varied from 0.50 ft to 0.103 ft. The fall height was then decreased to 0.5 ft, and five runs were made that covered a range of ratios $\dfrac{h}{d_c}$ from 1.0 to 4.85. (A minimum value of 0.1 ft was set for d_c in order to avoid the range where viscosity and surface tension exert appreciable effects on the flow. Because extremely small depths were avoided, and because of the disturbed nature of the flow below the fall, negligible "scale effects" are to be expected, and the results can be applied to actual field cases without modification.) The energy E_1 at the base of the fall was taken as the sum of the integrated velocity energy and the average depth of flow at the section. The experimental values of $\frac{E_1}{d_c}$ are plotted against $\frac{h}{d_c}$ in curve (2), Fig. 7(*a*). In order to compare this curve with the values obtained from the energy equation, assuming no loss, curve (1), Fig. 7(*a*), was plotted from the equation:

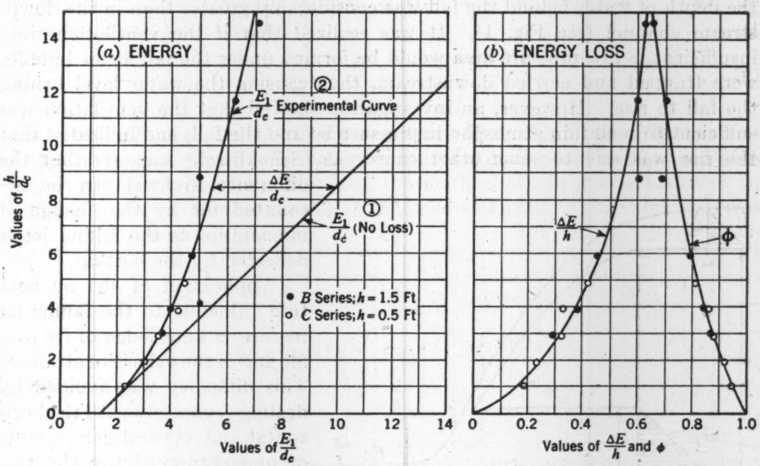


FIG. 7.—ENERGY AT THE BASE OF THE FALL

$$E_1 = h + \frac{3}{2} d_c \qquad \text{(6a)}$$

or

$$\frac{E_1}{d_c} = \frac{h}{d_c} + \frac{3}{2} \qquad \text{(6b)}$$

The difference between these two curves represents the energy loss divided by the critical depth, $\frac{\Delta E}{d_c}$. To show more clearly the effect of $\frac{h}{d_c}$ on the rate of energy loss, Fig. 7(*b*) was prepared, in which the energy loss is presented as a fraction of the fall height. Although this indicates that as much as two thirds of the energy released by the fall may be dissipated as the jet strikes the flat bottom, for a small value of $\frac{h}{d_c}$ (say 1.5) only one fifth of the energy is dissipated. With a given discharge and fall height, more effective dissipation may be achieved by increasing the width of the overfall section, and thereby reducing the value of d_c.

Values of the velocity coefficient ϕ were calculated from the observed energy loss and plotted in Fig. 7(b) as a function of $\frac{h}{d_c}$. Since the use of ϕ in Eq. 4 involves solving a cubic equation, this curve is of less practical value than curve (2), Fig. 7(a), which gives, directly, the energy content at the base of the fall.

Standing Water Behind Fall.—In all runs in this series it was noted that the depth of water behind the fall was considerably greater than in the downstream channel (see Fig. 1). It was realized that if the ventilation were insufficient, a low-pressure area would be formed under the fall as air bubbles were trapped and carried downstream, thus causing the water level behind the fall to rise. However, an investigation showed that the ventilation was sufficient to maintain atmospheric pressure behind the fall, and indicated that the rise was due to some other cause. A. Schoklitsch[5] suggests that the difference in level can be accounted for by the change of momentum as the falling jet is deflected by the bottom.

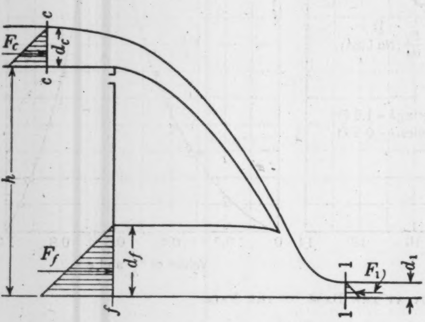


FIG. 8.—OVERFALL, SHOWING FORCES ACTING

Application of the momentum principle to the falling jet requires a knowledge of its profile and of the velocities attained. This difficulty was avoided by dealing separately with the horizontal and vertical components of momentum. Since the vertical force component resulting from the change in the vertical momentum must be taken by the bottom, the horizontal change of momentum may be dealt with in a manner similar to that used for the hydraulic jump. Referring to Fig. 8 and neglecting friction, the momentum equation gives

$$F_c + F_f - F_1 = \frac{\gamma}{g} q \, (V_1 - V_c) \ldots\ldots\ldots\ldots\ldots\ldots (7a)$$

in which the terms on the left side of the equation represent the hydrostatic forces and the right side gives the rate of change of momentum between sections c–c and 1–1. The hydrostatic forces are given by

$$F_c = \frac{\gamma \, d^2_c}{2}; \quad F_f = \frac{\gamma \, d^2_f}{2}; \quad \text{and} \quad F_1 = \frac{\gamma \, d^2_1}{2} \ldots\ldots (7b)$$

in which d_f is the depth of standing water behind the fall. Substituting Eq. 7b

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[5] "Hydraulic Structures," by A. Schoklitsch (translated by Samuel Shulits), A. S. M. E., Vol. I, 1937, pp. 118–119.

in Eq. 7*a*, introducing the continuity equation, and solving for $\left(\frac{d_f}{d_c}\right)^2$:

$$\left(\frac{d_f}{d_c}\right)^2 = \left(\frac{d_1}{d_c}\right)^2 + 2\left(\frac{d_c}{d_1}\right) - 3 \;\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

In attempting to check this relation against experimental values, difficulties were encountered because it was not possible to measure d_1 accurately for the reasons mentioned previously in dealing with losses at the base of a fall. To overcome this difficulty a value of d_1 was calculated from the experimental value of E_1, using Eq. 5, or:

$$\frac{E_1}{d_c} = \frac{d_1}{d_c} + \frac{1}{2}\left(\frac{d_c}{d_1}\right)^2 \;\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Inasmuch as E_1 was measured to an accuracy of ± 5%, it was felt that this method gave a more probable average value for d_1 than could be obtained with a point gage in spite of the fact that the method of calculation assumed an artificial condition of uniform velocity at the base of a fall. The value of d_f was obtained accurately from the piezometers. In Fig. 9(*a*) a plot of the momentum Eq. 8 is compared with experimental values of $\frac{d_1}{d_c}$ and $\frac{d_f}{d_c}$. This curve demonstrates that the presence of standing water behind the fall can be explained by the change in horizontal momentum as the water strikes the channel floor. A knowledge of the presence of standing water behind the fall and of its depth should be of value in making stability calculations of a drop structure.

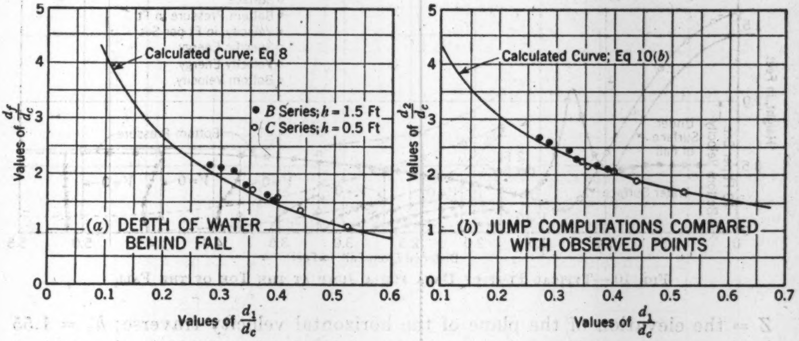

FIG. 9.—CURVES OF MOMENTUM EQUATIONS

FLOW CONDITIONS WITH TRANQUIL TAILWATER

The second part of the investigation checked the vertical elements of the jump below a fall against the momentum equation, and included an investigation of the longitudinal characteristics of the flow as influenced by the tailwater.

A series of runs was made with the tailwater adjusted to form a hydraulic jump at the toe of the fall. In addition to determining the surface profile and bottom pressure throughout the length of the jump, velocity profiles were taken on the center line of the channel. A good indication of the variation in velocity energy along the high-velocity jet that undershoots the "roller" of the jump was obtained from integrations based upon the velocity profiles. Fig. 10 presents a sample plot of these data, in which (similarly to Fig. 6):

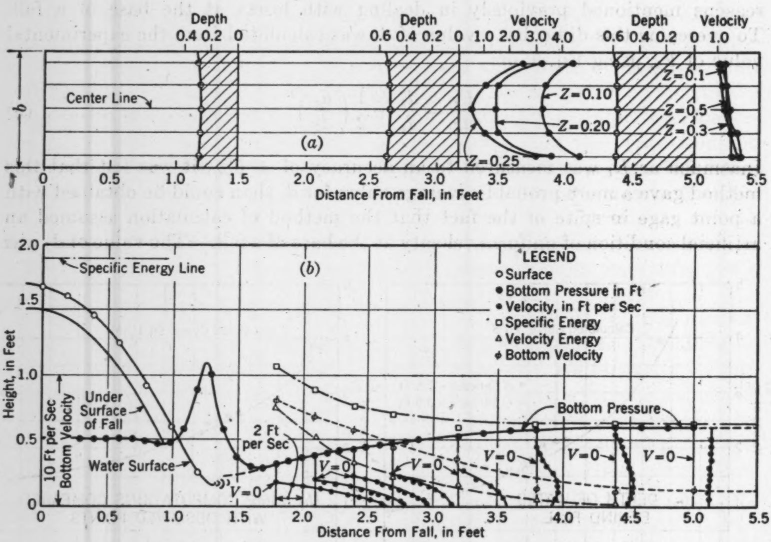

Fig. 10.—Typical Plot of Data for a Jump at the Toe of the Fall

Z = the elevation of the plane of the horizontal velocity traverse; $h_v = 1.55$ ft of water; $Q = 0.664$ cu ft per sec; $d_c = 0.255$ ft; $d_0 = 0.182$ ft; $d = 0.61$ at 7.0 ft from the fall; and $\frac{\sum(V^3\,\Delta A)}{2\,g\sum(V\,\Delta A)} + d$ is the specific energy. Longitudinal variations of the velocity head and the specific energy head are shown by the dash and dot-dash lines, respectively. It is interesting to note how rapidly energy is lost near the start of the jump. The dotted line shows the way in which the bottom velocity decreases in the downstream direction. The boundary of the undershooting jet was taken as the zero velocity point of the velocity profiles in the region of the jump.

Vertical Elements of Jump.—The vertical elements of the jump checked very closely with the momentum equation given by Professor Bakhmeteff:[6]

$$\frac{d_2}{d_1} = \frac{1}{2}\left[-1 + \sqrt{1 + 8\left(\frac{d_c}{d_1}\right)^3}\right] \ldots\ldots\ldots\ldots (10a)$$

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[6] "Hydraulics of Open Channels," by Boris A. Bakhmeteff, McGraw-Hill Book Co., Inc., 1932, p. 241.

or

$$\frac{d_2}{d_c} = \frac{d_1}{2\,d_c}\left[-1+\sqrt{1+8\left(\frac{d_c}{d_1}\right)^3}\,\right] \ldots\ldots\ldots\ldots\ldots (10b)$$

in which d_1 is the depth of water at the toe of the fall before the jump and d_2 is the depth of water after the jump.

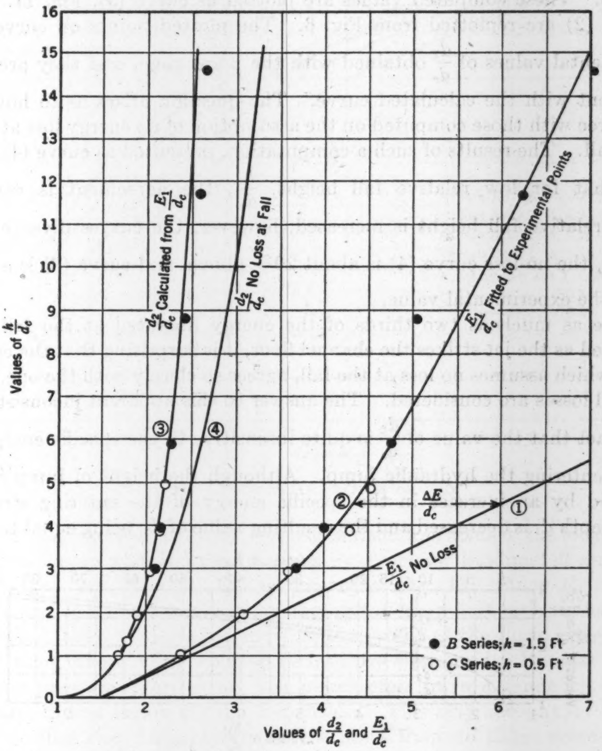

FIG. 11.—COMPUTED AND OBSERVED VALUES OF TAILWATER DEPTH REQUIRED TO FORM THE JUMP

The ratio $\frac{d_2}{d_c}$ was computed from experimental values of d_2 taken with a point gage, and $\frac{d_1}{d_c}$ was computed from experimental values of $\frac{E_1}{d_c}$ using Eq. 9. Fig. 9(b) shows that the values obtained from the momentum equation are in good agreement with those obtained experimentally. This indicates that $\frac{d_1}{d_c}$ computed from $\frac{E_1}{d_c}$ may be used in the calculation of the hydraulic jump.

It is now possible to combine the experimentally determined energy loss with the standard hydraulic jump equation to calculate the minimum depth of tailwater required to form the jump. The values of $\frac{E_1}{d_c}$ may be taken from curve (2), Fig. 7(*a*), and then the value of $\frac{d_2}{d_c}$ may be computed using Eqs. 9 and 10*b*. These computed values are plotted as curve (3), Fig. 11. Curves (1) and (2) are replotted from Fig. 6. The plotted points on curve (3) are experimental values of $\frac{d_2}{d_c}$ obtained with the point gage, and they are in good agreement with the calculated curve. The question arises as to how closely they agree with those computed on the assumption of no energy loss at the base of the fall. The results of such a computation, presented as curve (4), Fig. 11, show that for low relative fall height, $\frac{h}{d_c}$, the agreement is very good. As the relative fall height is increased, however, the curves diverge until at $\frac{h}{d_c} = 12$, the no-loss curve (4) is about 20% above, and curve (3) is about 5% below, the experimental value.

Since as much as two thirds of the energy liberated at the fall may be dissipated as the jet strikes the channel floor, it is surprising that the calculated curve, which assumes no loss at the fall, agrees so closely with the one obtained when all losses are considered. The answer to this apparent inconsistency lies in the fact that the value of $\frac{d_2}{d_c}$ is quite insensitive to the specific energy of the stream entering the hydraulic jump. Although the height of jump d_j will be increased by an increase in the specific energy of the entering stream, the initial depth d_1 is decreased and the resulting value of d_2, being equal to $d_j + d_1$, is only slightly increased. This relation is expressed by the curves of Fig. 12, in which the ordinate shows values of $\frac{E_2}{d_c}$ and $\frac{d_2}{d_c}$ after a jump and the abscissa represents values of $\frac{E_1}{d_c}$. A scale of $\left(\frac{d_c}{d_1}\right)^3 = \lambda_1$ (in which λ is the kinetic flow factor at section 1–1) is included for those who are familiar with Bakhmeteff's kinetic flow factor. It will be noted that $\frac{E_1}{d_c}$ may vary over a considerable range with only a small change in $\frac{d_2}{d_c}$.

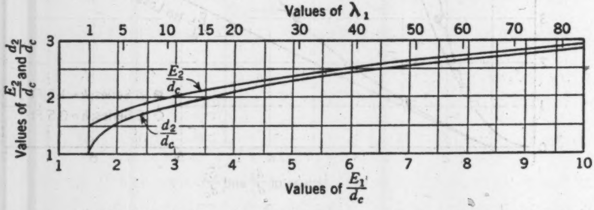

FIG. 12.—DEPTH AND ENERGY AFTER THE JUMP, PLOTTED AGAINST ENERGY BEFORE THE JUMP

When the energy loss at the base of the fall is neglected, the jump calculation may give a tailwater depth as much as 20% high. In some instances this will provide a desirable factor of safety against the jump being washed downstream. Any additional safety factor may cause submergence and impair the effectiveness of the jump as an energy dissipator.

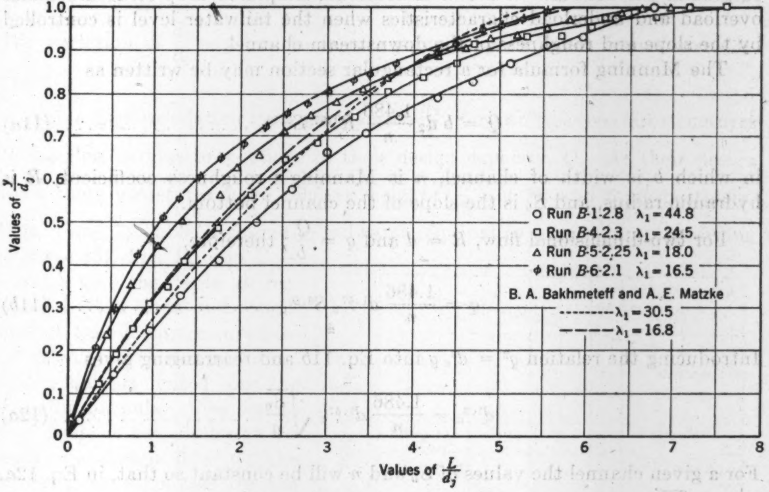

FIG. 13.—LONGITUDINAL PROFILES OF JUMP

Longitudinal Elements of Jump.—A comparison was made of the longitudinal profile of the jump with the results given by Professor Bakhmeteff and A. E. Matzke,[7] Jun. Am. Soc. C. E., as shown in Fig. 13. Since the bottom piezometers gave a good means of averaging the large fluctuations that occur in the surface, they were used to indicate the profile of the jump except at its very beginning where a few point-gage readings were used. Because of non-hydrostatic pressure distribution, the piezometers have a tendency to read slightly high in this region of rapid expansion. This may account for the fact that the profiles rise more rapidly at the start than do those presented by Messrs. Bakhmeteff and Matzke. The profiles given by the writer indicate that below a free overfall the jump is about 20% longer than is shown by the curves of Messrs. Bakhmeteff and Matzke. Since their jump was formed downstream from a regulating sluice, lack of agreement may be caused by a difference in the velocity profile of the shooting flow entering the jump.

The velocity measurements and the energy summations show that the kinetic energy in the undershooting jet is dissipated by the time the surface profile indicates maximum depth. This means that the end of the jump as indicated by the surface profile marks the end of excessive velocities in the undershooting jet. This statement applies to well-formed hydraulic jumps and may well apply to a submerged jump.

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[7] "The Hydraulic Jump in Terms of Dynamic Similarity," by Boris A. Bakhmeteff and Arthur E. Matzke, *Transactions*, Am. Soc. C. E., Vol. 101 (1936), p. 630.

Effect of Changing Discharge.—By using curve (3), Fig. 11, it is possible to obtain the minimum tailwater depth necessary to support the jump for any relative fall height. This relation may be combined with the Manning formula to predict the behavior of a two-dimensional simplified drop structure under overload and underload characteristics when the tailwater level is controlled by the slope and roughness of the downstream channel.

The Manning formula for a rectangular section may be written as

$$Q = b\, d_2 \frac{1.486}{n} R^{0.67} S^{0.50}{}_0 \qquad \text{(11a)}$$

in which b is width of channel; n is Manning's roughness coefficient, R is hydraulic radius, and S_0 is the slope of the channel bottom.

For two-dimensional flow, $R = d$ and $q = \frac{Q}{b}$; therefore,

$$q = \frac{1.486}{n} d^{1.67}{}_2\, S^{0.50}{}_0 \qquad \text{(11b)}$$

Introducing the relation $q^2 = d^3{}_c\, g$ into Eq. 11*b* and rearranging gives

$$d^{1.5}{}_c = \frac{1.486}{n} d^{1.67}{}_2 \sqrt{\frac{S_0}{g}} \qquad \text{(12a)}$$

For a given channel the values of S_0 and n will be constant so that, in Eq. 12*a*, $\frac{1.486}{n}\sqrt{\frac{S_0}{g}} = C =$ constant; and

$$d^{1.5}{}_c = C\, d^{1.67}{}_2 \qquad \text{(12b)}$$

TABLE 1.—RELATIVE TAILWATER DEPTHS SHOWING OVERLOAD AND UNDERLOAD CHARACTERISTICS FOR TWO STRUCTURES

d_c	Q (%)	DESIGN $\frac{h}{d_c} = 10$			DESIGN $\frac{h}{d_c} = 5$		
		$\frac{h}{d_c}$	$\frac{d_2}{d_c}$	$\frac{d_2}{d_c}$	$\frac{h}{d_c}$	$\frac{d_2}{d_c}$	$\frac{d_2}{d_c}$
(1)	(2)	(3)	(4)[a]	(5)[b]	(6)	(7)[a]	(8)[b]
0.5	35	20.0	2.60	2.57	10.00	2.39	2.34
0.6	46	16.7	2.55	2.53	8.33	2.33	2.29
0.7	59	14.3	2.50	2.48	7.15	2.28	2.26
0.8	71	12.5	2.45	2.46	6.25	2.24	2.23
0.9	85	11.1	2.42	2.42	5.55	2.20	2.20
1.0	100	10.0	2.40	2.40	5.00	2.18	2.18
1.1	115	9.1	2.35	2.38	4.55	2.15	2.16
1.2	131	8.3	2.32	2.35	4.16	2.13	2.14
1.3	148	7.7	2.30	2.34	3.84	2.11	2.13
1.4	165	7.15	2.28	2.32	3.57	2.09	2.11
1.5	184	6.66	2.26	2.30	3.33	2.07	2.09
2.0	283	5.00	2.18	2.24	2.50	1.97	2.03

[a] For the jump. [b] In the channel.

Rearranging Eq. 12*b* gives

$$d^{-0.1}{}_c = C^{0.6}\,\frac{d_2}{d_c} = C'\,\frac{d_2}{d_c}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12c)$$

Eq. 12*c* shows that when the tailwater is governed only by slope and roughness the value of $\frac{d_2}{d_c}$ will vary inversely as the tenth root of the critical depth. The variation of $\frac{d_2}{d_c}$ required to form a jump may be taken from curve (3), Fig. 11.

Table 1 gives relative tailwater depths, $\frac{d_2}{d_c}$, for two free overfall structures discharging various proportions of their design capacity, Q. At their design flows these structures have relative heights, $\frac{h}{d_c}$, of 10 and 5, respectively. Cols. 4 and 7, Table 1, show the tailwater required to form a jump according to curve (3), Fig. 11, and Cols. 5 and 8 give the depth as regulated by the slope and roughness of the downstream channel calculated according to Eq. 12*c*, which is based on the Manning formula. The results show that, as the discharge exceeds the design flow Q, the tailwater as regulated by the downstream channel will tend to exceed that necessary to form a jump. When the discharge is less than the design flow, the opposite tendency is indicated. Note that the differences between the values of $\frac{d_2}{d_c}$ required to support the jump and those supplied by the channel are very small. Therefore, if the design tailwater depth is chosen a little high, the structure should perform satisfactorily over a wide range of discharges.

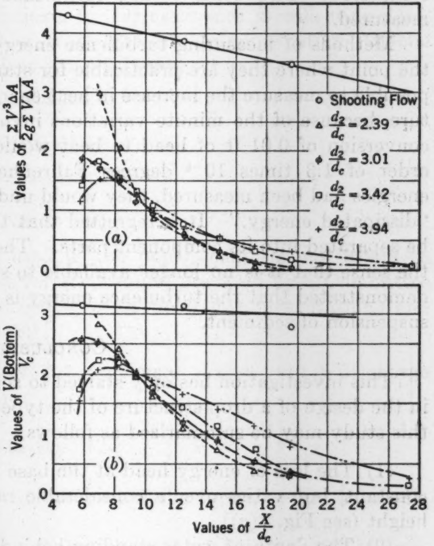

FIG. 14.—EFFECT OF SUBMERGENCE

Effect of Submergence.—An investigation was made to determine the effects of increasing the tailwater until the jump was completely submerged. Data similar to those shown in Fig. 10 were obtained for a single fall height ratio $\left(\frac{h}{d_c} = 5.88\right)$, with four tailwater depths $\left(\frac{d_2}{d_c} = 2.39, 3.01, 3.42, \text{and } 3.94\right)$. The first of these was just sufficient to hold the jump at the base of the fall and agrees with the value given by curve (3), Fig. 11—that is, for $\frac{h}{d_c} = 5.88$, $\frac{d_2}{d_c} = 2.3$. The higher values caused increasing submergence. By plotting the velocity energy in the high-velocity jet against the distance downstream as in Fig. 14(*a*), it was found that increasing submergence caused the high-velocity jet to persist for a greater distance. A plot of the bottom velocities (Fig. 14(*b*)) also showed higher values downstream as the tailwater depth was increased. This would indicate that oversubmergence of the jump may require bottom protection to be extended farther down the channel. It is felt that further study should be made of the effects of submergence on the rate of decay of the undershooting jet.

"DISSIPATED ENERGY"

It will be noted that in the energy summations made in connection with this investigation only elevation, pressure, and axial velocity energy were considered. Two other forms, turbulence and heat energy, could not be measured.

Methods of measuring turbulence energy have not yet been developed to the point where they are practicable for standard laboratory use. It was not possible to measure the increase in heat energy from the rise in water temperature, because of the minute variations involved. For instance, the complete conversion of 0.01 ft of head to heat would cause a temperature rise of the order of 1.3 times 10^{-5} degrees Fahrenheit. If the turbulence and heat energies had been measured, they would undoubtedly account for the so-called "dissipated energy." It is regretted that this "dissipated energy" could not be separated into its component parts. The heat energy is truly dissipated in the sense that it is no longer available to scour the channel, but it has been demonstrated that the turbulence energy is available for the entrainment and suspension of sediment.

CONCLUSION

This investigation has only started to solve the problems that are involved in the design of a drop structure of the type used in the field. The findings of this study may be summarized as follows:

(1) The loss of energy head at the base of the fall is neither negligible nor constant, but varies over a considerable range as a function of relative fall height (see Fig. 7(*a*)).

(2) The depth of water standing behind the fall is greater than the depth of water shooting from the base of the fall and may be computed from the momentum principle.

(3) The momentum equation for the hydraulic jump may be applied to the flow conditions at the base of the fall to determine the vertical elements of the jump.

(4) The longitudinal profile of the jump is in fair agreement with the profile obtained by Messrs. Bakhmeteff and Matzke. The length of jump below a fall appears to be about 20% longer than the jump below a sluice.

(5) The velocity energy of the undershooting jet is spent by the time the surface profile indicates the end of the jump.

(6) Oversubmergence of the jump will cause the undershooting jet to persist for a greater distance downstream.

(7) The free overfall two-dimensional drop structure will perform satisfactorily over a wide range of discharges.

This investigation has provided information sufficient to enable the hydraulic engineer to produce a rational design for a two-dimensional drop structure, the performance of which can be predicted accurately. Probably it is even more important that the designer can now obtain at least a first approximation of the performance of those more complicated structures that circumstances usually necessitate.

ACKNOWLEDGMENTS

Generous help from members of the staff of the Cooperative Laboratory, California Institute of Technology, is gratefully acknowledged. Ralph W. Koehnke deserves particular recognition for his assistance in collecting data. The work was done with the aid of Work Projects Administration (Official Project No. 65-2-07-58, Work Project No. PS-11496).

APPENDIX

NOTATION

The following letter symbols, used in the paper, conform essentially to American Standard Letter Symbols for Hydraulics, prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1942:[8]

A = cross-sectional area of flow; ΔA = element of area used in discharge and energy summations;

b = width of channel;

C = (C and C' are constants in Eqs. 12);

d = depth of water:

- d_0 = depth of flow at the crest of the overfall—that is, at section 0–0;
- d_1 = depth of water at the toe of the fall before the jump—that is, at section 1–1;
- d_2 = depth of water after the jump—that is, at section 2–2;
- d_c = hydrostatic critical depth = $\sqrt[3]{\frac{q^2}{g}}$;
- d_f = depth of standing water behind fall;
- d_j = height of jump = $d_2 - d_1$;

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[8] ASA—Z10.2—1942.

E = total energy head:

E_0, at the crest of the fall (section 0–0);
E_1, at the toe of the fall (section 1–1);
E_2, after the jump (section 2–2);
ΔE = loss from the crest to the toe of the fall;

F = hydrostatic force; F_1, F_c, F_f = forces at sections 1–1, c–c, and f, respectively (see Fig. 8);
g = acceleration of gravity = 32.2 ft per sec per sec;
h = height of the fall;
h_v = head across venturi meter;
L = length of jump measured from the beginning of the jump;
M = horizontal momentum; ΔM = change in horizontal momentum;
n = roughness factor in the Manning formula;
Q = discharge in cubic feet per second;
q = discharge in cubic feet per second per foot of width;
R = hydraulic radius;
S = slope; S_0 = slope of channel bottom;
V = velocity:

V_1 = velocity at the base of the fall (section 1–1);
V_c = critical velocity = $\sqrt{g\, d_c}$;

X = distance downstream from the fall, in feet;
y = height coordinate of the jump, measured above the initial depth d_1;
Z = elevation of the plane of the horizontal velocity traverse;
γ = specific weight of water (weight per unit volume);
λ = kinetic flow factor = $\frac{V^2}{g\, d} = \left(\frac{d_c}{d}\right)^3$;
ϕ = velocity coefficient to account for losses.

DISCUSSION

Merit P. White,[9] Assoc. M. Am. Soc. C. E.—In this paper the author points out that flow conditions below an overfall are completely determined, provided the amount of energy lost in the transition is known. The author measures this energy loss in a model and gives a curve (Fig. 7) showing the relation between $\frac{E_1}{d_c}$ and $\frac{h}{d_c}$, in which E_1 is the total flow energy below the fall (expressed as head), d_c is the critical depth for parallel flow corresponding to the given discharge, and h is the drop in the floor of the channel (Fig. 5). It is shown that for large drops the energy loss is considerable.

Actually, this loss of energy can be determined fairly accurately by theory. This has been done, and the results are given in Fig. 15, which also shows, for comparison, the author's experimental curve (2) taken from Fig. 7.

The standing water behind the fall, shown in Fig. 5, results from the fact that a jet of fluid striking a surface tends to disperse in all directions along the surface. Fig. 16 shows a sheet of water striking a flat surface, which makes the angle θ with the jet. In this case there is nothing to interfere with the flow of water away from the point of impact. Here the greater part of the water flows to the right and the remainder to the left, as indicated by the dimensions d_a and d_b on the jet. Note that d_a and d_b are also the thicknesses of the water sheets flowing along the plane—that is, the water velocity changes only in direction, not in magnitude. The energy loss is negligible in this case. The ratio between d_a and d_b can be found from the fact that equal and opposite horizontal forces (parallel to the plane) act on the two water sheets to turn them from the original direction—that is, since the plane can only exert forces normal to itself, the net change in momentum parallel to the plane must be zero. Fig. 17 shows the velocity vectors for the upper water sheet before and after impact. In this figure the broken line shows ΔV, the change in velocity, which is the difference between the two velocity vectors. The horizontal component of the velocity change equals $V(1 - \cos\theta)$. In the upper sheet the mass of fluid flowing per second in unit width is $\frac{\gamma}{g} V d_a$. Therefore, the change in the horizontal momentum in one second in unit width of channel must be $\frac{\gamma}{g} V^2 d_a (1 - \cos\theta)$ in the upper sheet, and $\frac{\gamma}{g} V^2 d_b (1 + \cos\theta)$ in the lower. Equating these gives

$$\frac{d_b}{d_a} = \frac{1 - \cos\theta}{1 + \cos\theta} \qquad (13)$$

Because of the water standing beneath the fall (Fig. 5), the actual situation is somewhat different from that shown in Fig. 16, in which nothing hinders the flow of water along the plane. The water in the lower sheet flows into the bottom of the standing pool, causing clockwise rotation, while comparatively still water passes from the pool into the jet at exactly the same rate. This mixing is responsible for the loss of energy.

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[9] Care, Palmer Physical Laboratory, Princeton Univ., Princeton, N. J.

Fig. 18 outlines the general method of analysis and should be referred to frequently in the following: In the jet, just above the surface of the pool, there flows a quantity of water per second, Q, at a velocity V. Below the level of the surface of the pool the jet acquires additional fluid from the pool, which reduces the velocity of the jet. The jet must thicken here on account of the increase in Q and the decrease in V. At the floor the jet must divide into sheets of thicknesses d_a and d_b whose ratio depends on the angle of approach θ, according to Eq. 13. Since the flow to the right, away from the fall, must be Q, the flow

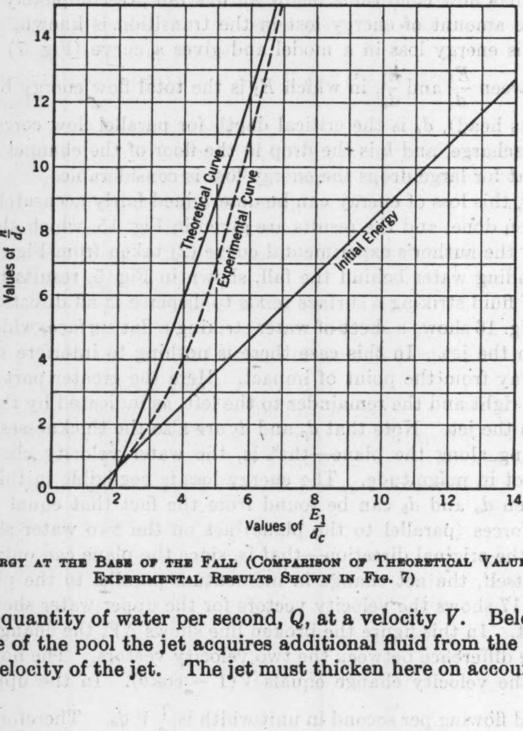

Fig. 15.—Energy at the Base of the Fall (Comparison of Theoretical Values with the Experimental Results Shown in Fig. 7)

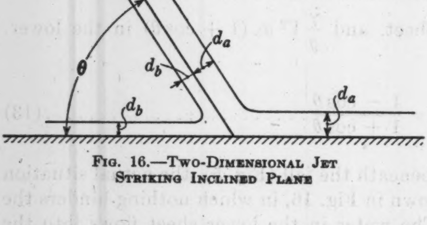

Fig. 16.—Two-Dimensional Jet Striking Inclined Plane

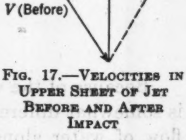

Fig. 17.—Velocities in Upper Sheet of Jet Before and After Impact

to the left into the pool, Q_f, is determined in terms of θ—that is,

$$Q_f = \frac{d_b}{d_a} Q = \frac{1 - \cos\theta}{1 + \cos\theta} Q \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

This is also the flow from the pool back into the jet. Assuming that this return flow has negligible momentum in the direction of the jet, the total momentum

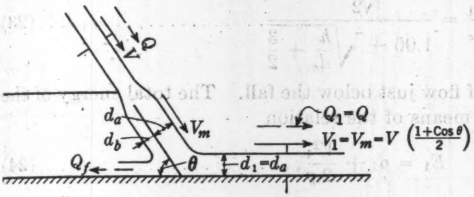

Fig. 18.—Inclined Jet with Standing Pool on One Side

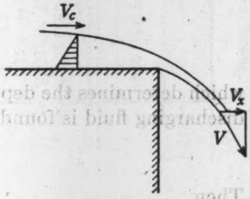

Fig. 19.—Determination of V_x

of the jet will not be changed in the mixing process. This determines V_m, the velocity after mixing. The momentum equation is

$$\frac{\gamma}{g} Q\, V = \frac{\gamma}{g}\left(Q + Q\,\frac{1 - \cos\theta}{1 + \cos\theta}\right) V_m \ldots\ldots\ldots\ldots (15)$$

which gives

$$V_m = \frac{V}{2}(1 + \cos\theta) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

Since V_m equals V_1 (the outflow velocity):

$$V_1 = \frac{V}{2}(1 + \cos\theta) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (17)$$

and the corresponding depth of flow is

$$d_1 = \frac{Q}{V_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18)$$

In these equations, V is the velocity and θ is the angle of inclination that the jet would have at the floor if there were no pool. The cosine of the angle θ is the ratio of the horizontal component of the jet velocity, V_x, to the total velocity V. Velocity V_x is found by equating the horizontal force at the critical section above the fall to the change in horizontal momentum between this point and any point in the free jet (Fig. 19)—that is,

$$\frac{\gamma}{2} d^2_c = \frac{\gamma}{g} Q\,(V_x - V_c) \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

Since $Q = V_c\, d_c$ and $V_c = \sqrt{g\, d_c}$, this gives

$$V_x = \frac{3}{2} V_c \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (20)$$

Velocity V, the velocity corresponding to the total fall, is obtained from the initial energy.

$$V = \sqrt{2\,g\left(h + \frac{3}{2} d_c\right)} \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

Then

$$\cos\theta = \frac{1.5\,V_c}{\sqrt{2\,g\left(h + \frac{3}{2}\,d_c\right)}} = \frac{1.06}{\sqrt{\frac{h}{d_c} + \frac{3}{2}}} \qquad (22)$$

Substitution of these values in Eq. 18 results in

$$\frac{d_1}{d_c} = \frac{\sqrt{2}}{1.06 + \sqrt{\frac{h}{d_c} + \frac{3}{2}}} \qquad (23)$$

which determines the depth of flow just below the fall. The total energy of the discharging fluid is found by means of the relation

$$E_1 = d_1 + \frac{V^2{}_1}{2\,g} \qquad (24)$$

Then

$$\frac{E_1}{d_c} = \frac{\sqrt{2}}{1.06 + \sqrt{\frac{h}{d_c} + \frac{3}{2}}} + \frac{\left(1.06 + \sqrt{\frac{h}{d_c} + \frac{3}{2}}\right)^2}{4} \qquad (25)$$

The solid line of Fig. 15 shows this relation, and the broken line gives the results of the author's experimental investigation. In view of the crudeness of the present analysis the agreement between theory and measurement is satisfactory.

BORIS A. BAKHMETEFF,[10] M. AM. SOC. C. E., AND N. V. FEODOROFF,[11] ESQ.—Mr. Moore's paper is a distinct contribution to engineering science. The subject is of great practical interest and the manner of presentation exemplifies well what modern "hydraulics" should be. Basic mechanical principles are comprehensively applied. Observed facts are reduced to generalized dimensionless curves. Above all, emphasis is laid on the physical aspects of the phenomena, recognizing the often baffling complexity of the ways in which water actually behaves, as compared with the simplified, antiquated notions.

In the course of its researches, the Fluid Mechanics Laboratory of Columbia University, in New York, N. Y., has gathered data bearing on the different topics presented in the paper. The experiments in part complement Mr. Moore's investigations in that they also embrace nonaerated nappes. Also, they help in further elucidating some of the physical features revealed in the paper, which at first sight may appear disconcerting.

An example in particular is the shape of the velocity profiles in the streaming below the fall as given in Fig. 6. The velocities shown are smallest at the surface and increase with the depth. There is no dynamical explanation for such "inverted" profiles in what otherwise appears to be more or less "parallel" streaming, and the source of the anomaly must be due to some undisclosed physical circumstance. In fact, the writers believe that the abnormalities are apparent, and are due to the fact that pitot-tube observations were made in the presence of air entrained by the falling nappe and gradually liberated in the subsequent streaming. In the course of escaping, air naturally concen-

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[10] Prof., Civ. Eng., Columbia Univ., New York, N. Y.

[11] Research Associate, Columbia Univ., New York, N. Y.

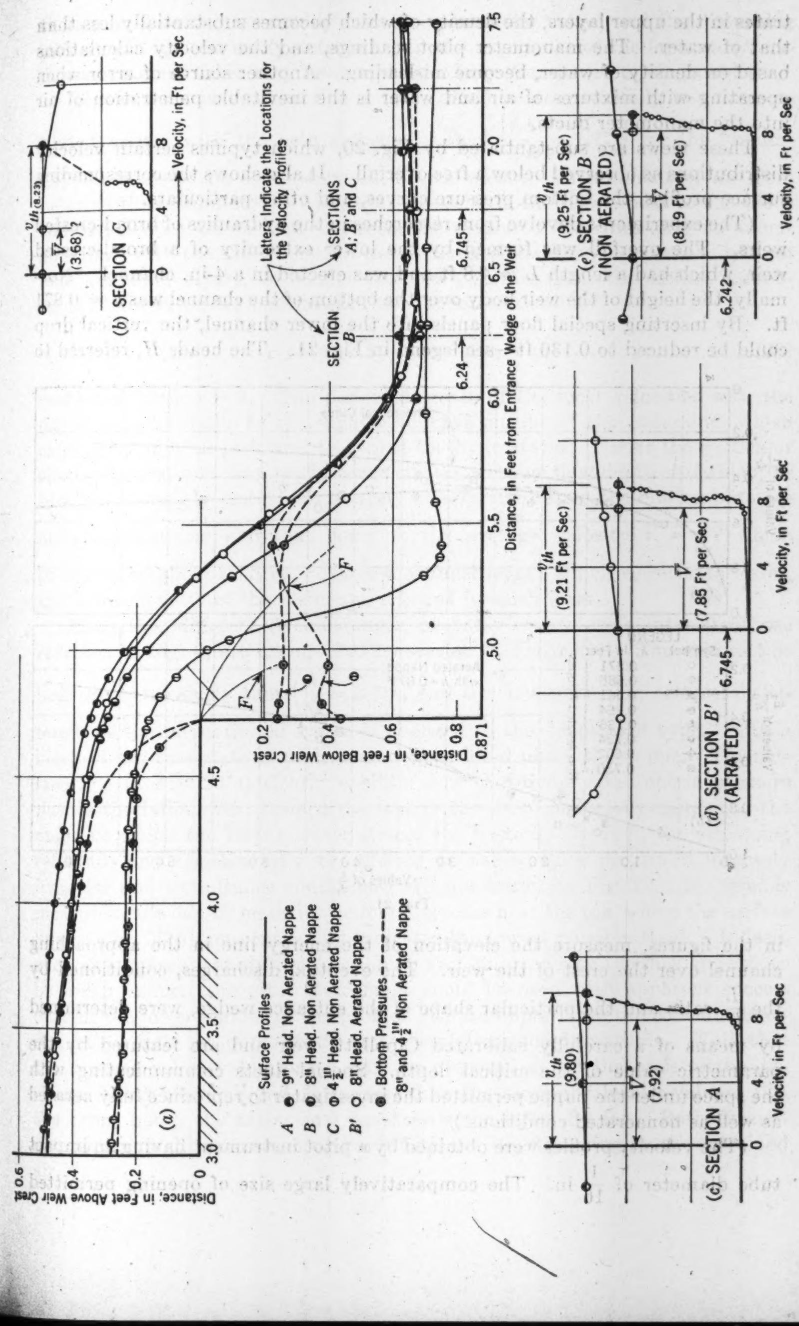

FIG. 20.—VELOCITY DISTRIBUTIONS OBSERVED BELOW A FREE OVERFALL

trates in the upper layers, the density of which becomes substantially less than that of water. The manometer pitot readings, and the velocity calculations based on density of water, become misleading. Another source of error when operating with mixtures of air and water is the inevitable penetration of air into the manometer ducts.

These views are substantiated by Fig. 20, which typifies certain velocity distributions as observed below a free overfall. It also shows the corresponding surface profiles, the bottom pressure curves, and other particulars.

(The experiments devolve from researches in the hydraulics of broad-crested weirs. The overfall was formed by the lower extremity of a broad-crested weir, which had a length $L = 4.8$ ft and was erected in a 4-in. channel. Normally, the height of the weir body over the bottom of the channel was $h = 0.871$ ft. By inserting special floor panels into the lower channel, the vertical drop could be reduced to 0.136 ft—see legend in Fig. 21. The heads H, referred to in the figures, measure the elevation of the energy line in the approaching channel over the crest of the weir. The eventual discharges, conditioned by the $\frac{L}{H}$-ratio and the particular shape of the entrance wedge, were determined by means of a carefully calibrated Cipolletti weir and are featured by the parametric value of the critical depth. Special ducts communicating with the space under the nappe permitted the investigator to reproduce fully aerated as well as nonaerated conditions.)

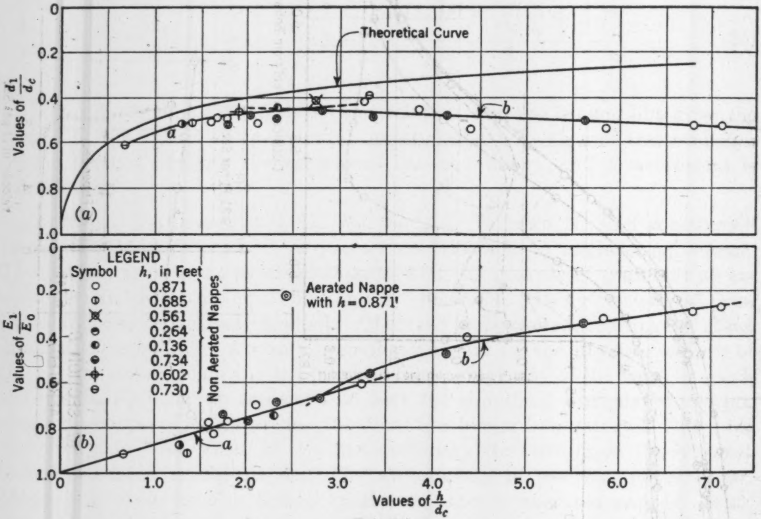

FIG. 21

(The velocity profiles were obtained by a pitot instrument having an impact tube diameter of $\frac{1}{16}$ in. The comparatively large size of opening permitted the air bubbles to pass readily into the manometer tubes, and the disturbing influence of air was easily detected. In the experience of the laboratory, undersized impact tubes may become dangerously "sensitive" to air trouble.)

The velocity curve at section *A* (Fig. 20(*c*)) refers specifically to an unventilated nappe, with the vein sufficiently thick $\left(\frac{h}{d_c} = 1.77\right)$ for the "under-nappe" space to be completely filled with water. Entrainment of air from below is thus eliminated. Also, the streaming in the region next to the toe is comparatively undisturbed. As seen, the measured velocities offer no anomalies and, with the exception of a clearly defined friction zone near the bottom (the boundary layer), are rather uniform. In fact, near the surface the observed local velocities tend toward the "theoretical value"

$$v_{th} = \phi \sqrt{2\,g\,(h + 1.5\,d_c - d_1)} \quad \text{.................} (26)$$

computed with $\phi = 1$. The partial decrease of the local velocities with the depth is attributable to the slightly concave outline of the streaming, which causes the internal pressures to exceed the hydrostatic. Under the particular circumstances, one may easily ascertain the average flow depth d_1 (which, as Mr. Moore rightly states, can scarcely be appraised when the motion becomes more tumultuous) and thus compute the average velocity $\bar{V} = \frac{q}{d_1}$, which proves to be about 0.85 v_{th}. Narrow channel forms, which accentuate losses, and the concavity of the filaments, account for such facts.

Altogether different circumstances, probably exceeding in intensity those reflected in Fig. 6, are exemplified by section *C*, Fig. 20(*b*). The vein in this instance is relatively thin $\left(\frac{h}{d_c} = 3.7\right)$, and the flow pattern is completely altered, as the unventilated nappe is changed to the "depressed type," with a block of air, in a state of underpressure, enclosed under the nappe. The surface line in Fig. 20(*a*) clearly exhibits how the excess atmospheric pressure pushes the falling vein toward the face of the drop, materially increasing the angle at which the falling water strikes the bottom. In fact, the streaming rebounds and splashes all over, offering on the whole a picture of intensely irregular and tumultuous conditions, well illustrated by Fig. 22. No reliable measurements can be made in the foaming mass near the toe, where the surface line in Fig. 20(*a*) conventionally indicates the lowest possible stage. Indeed, even at a subsequent station (section *C*, Fig. 20(*b*)), where the flow appears to be much calmer, the pitot instrument could be used with apparent success only in the lower regions of the profile. In the upper parts the continuous influx of air into the ducts made any reliable readings impossible. Note that the average velocity computed with the observed d_1, on the basis of the discharge being wholly liquid, is far less than the measured local *v*-values. On the other hand, the theoretical v_{th} exceeds the latter by far. Both circumstances are readily explained by the inflation of the streaming by the entrained air.

One would naturally assign air entraining action to the locus (curve *F*, Fig. 20(*a*)), where the lower surface of the falling vein "plunges" into the "under-nappe" roller. In fact, Fig. 1 very distinctly portrays the "mist" produced by the drawn-in air particles. If this were the only source of insufflation, nappes with blocks of air under them would behave very differently under all circumstances from their behavior when the under-nappe space is wholly filled with water. Observation shows that such is not the case, at least not entirely. Indeed, Figs. 20(*d*) and 20(*e*) exemplify velocity distributions obtained with the same discharge $\left(\frac{h}{d_c} = 2.01\right)$, but in one instance with an aerated under-nappe space (section *B'*, Fig. 20(*d*)), and in the other (section *B*, Fig. 20(*e*)) with the space entirely filled with water.

The writers are led to believe that, for high $\frac{h}{d_c}$-values, air entrainment is caused additionally by the splashing and the generally tumultuous conditions that the motion assumes when a thin vein strikes the bottom at a large angle (Fig. 22).

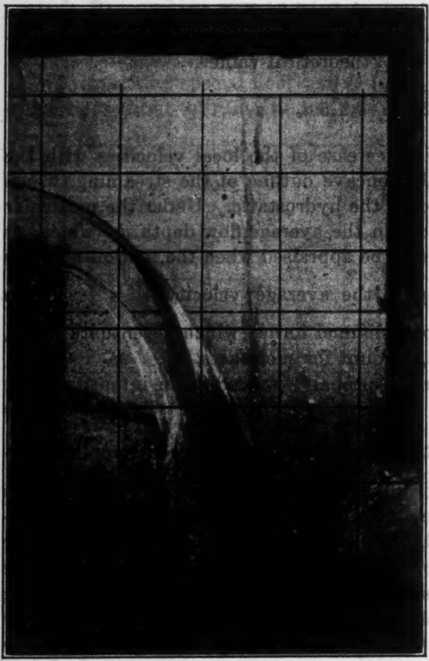

FIG. 22

Whatever the cause and effect, one is led to accept the evidence that, in the instance of flow at the base of an overfall, just as in so many other structures, hydraulic analysis may become seriously handicapped by the presence of a new physical factor in the form of insufflated air, a factor which unfortunately is not easy to appraise and which totally upsets computations based on a purely liquid basis. Among other features, air entrainment naturally upsets energy appraisals. Under the heading "Previous Study," Mr. Moore refers to a recommendation by the senior writer[3] to ignore eventual losses in the design of stilling basins below falls and to use $\phi = 1$ in Eq. 2. Naturally, the suggestion at the time was in the nature of a "marginal" rule, prompted by a complete absence of experimental data. Also, as evidenced by sections *A* and *B* (Fig. 20), for relatively low falls the local surface velocities are actually not far from the theoretical frictionless values.

For the phenomenon as a whole, however, and particularly for high $\frac{h}{d_c}$-values, the losses are very considerable, and probably should be ascribed to the intensive local turbulence engendered in the separation surfaces between the rebounding live vein and the underlying and the overlying rollers. (Most interesting flow patterns illustrating this case have been presented[12] by B. T. Morris, Jun. Am. Soc. C. E., and D. C. Johnson, Assoc. M. Am. Soc. C. E.) Under all circumstances it is interesting and useful to compare theoretical relations with those actually observed, even if the observations dealing with insufflated streaming must be accepted as necessarily approximate. As an addition to the valuable facts presented in Figs. 7 and 11, some of the Columbia observations are summarized dimensionlessly in Fig. 21. In Fig. 21(*a*), the ratio $\frac{d_1}{d_c}$, as observed, is compared with the theoretical values computed from

$$\frac{h}{d_c} = \frac{d_1}{d_c} + \frac{1}{2}\left(\frac{d_c}{d_1}\right)^2 - \frac{3}{2} \qquad (27)$$

which is equivalent to Eq. 4. In Fig. 21(*b*), the observed lower-channel specific energy

$$E_1 = d_1 + \frac{q^2}{2\,g\,d^2_1} \qquad (28a)$$

is compared with

$$E_0 = h + 1.5\,d_c \qquad (28b)$$

As Mr. Moore justly states, numerically the energy values (Fig. 21(*b*)) in the highly kinetic "shooting" state are extremely sensitive toward the d_1-values, and the latter by no means can be reliably appraised in the tumultuous current, crisscrossed by supersonic waves, etc., which obtains when $\frac{h}{d_c}$ becomes high; but, as stated, under such conditions the presence of entrained air inevitably baffles any attempts toward precision, and appraisals based on measuring the average depth may suffice for the purpose.

In the case of nonaerated nappes, the curves distinctly show two regimens. Curves *a* refer to relatively thick nappes $\left(\frac{h}{d_c} < 2.5\right)$, corresponding to flow forms where the nappe is wholly filled with water. Other experiments have demonstrated that the change from a fully filled under-nappe to the "depressed" form with an air block beneath it takes place in the region $2.2 < \frac{h}{d_c} < 2.7$, with an intermediary "unstable" reach between the indicated limits, where the one or eventually the other form may take place, depending on whether the flow is being raised or lowered.

Accordingly, the curves in the upper region (curves *b*) correspond to nonventilated flow with depressed nappes, with insufflation enhanced by the presence of air under the vein and by an increased angle of impact at the toe. As indicated in the legend (Fig. 21(*b*)), the measurements were made with largely

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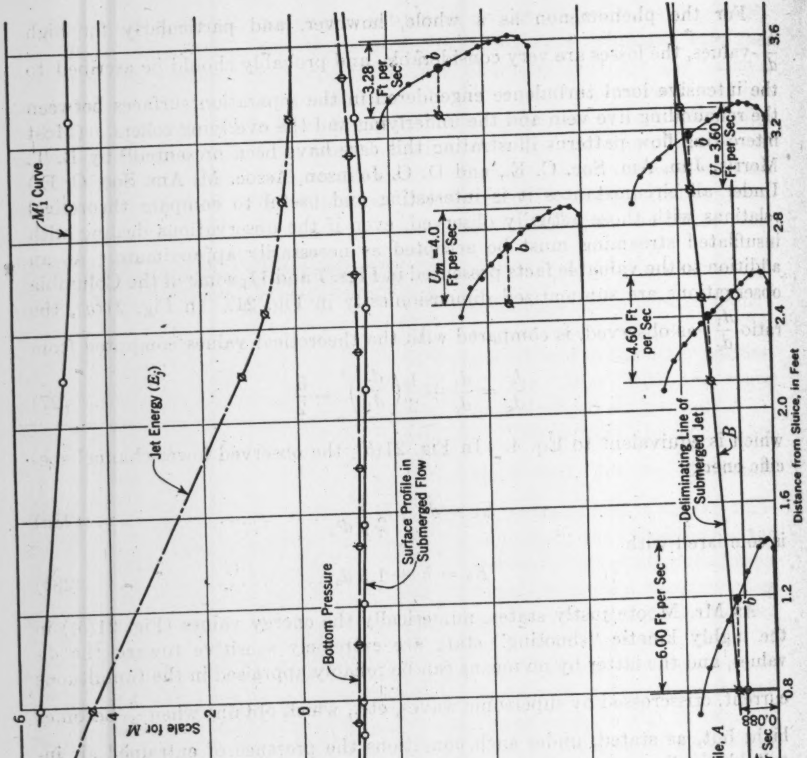
[12] *Transactions*, Am. Soc. C. E., Vol. 108 (1943), Figs. 5 and 6, pp. 896 and 897.


M'₁ Curve
Jet Energy (E_j)
Surface Profile in Submerged Flow
Bottom Pressure
Level for Submerged Flow
Level for Free Outflow
Scale for M
6
4
2
0
Distance, in Feet Above Weir Crest
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
3.28 Ft per Sec
b
$U_m = 4.0$ Ft per Sec
b
$U = 3.60$ Ft per Sec
b
4.60 Ft per Sec
Delimiting Line of Submerged Jet
B
6.00 Ft per Sec
b
0.088
Free Jet Profile, A
0.12
q = 0.472 Cu Ft per Sec
0
0.4
0.8
1.2
1.6
2.0
2.4
2.8
3.2
3.6
Distance from Sluice, in Feet

varying weir heights, so that the plotted points, particularly in the "reliable" low-drop zone, reflect a wide range of circumstances. The plottings also include observations relating to ventilated nappes with [illegible] = 0.841. Surprisingly, the points follow the same course closely.

Obviously, the presence of air tends to exaggerate the distortions between the observed and the theoretical curves in Figs. 20 and 21. In fact, the measured velocities used for determining E_2 (see "Flow Conditions with Tailwater: Energy Loss at Base of Fall") are less than the actual values, just as the observed values of d_2 used in curve c and in Fig. 21(a) are artificially inflated. Thus, in either case the apparent energy losses are exaggerated.

Perhaps, under such circumstances, the most reliable and practical procedure would be to use curves of the type of Nos. (3) and (4), Fig. 11. This is particularly true if one were to associate the E-values with the eventual depth of the tailwater, which will balance the jump unfolding in proximity with the toe. Observations of such tailwater are reliable, in that they refer to a comparatively calm region, beyond the uncertainties of the inflated zones. With regard to energy, an analogous role would be played by observed ratios of $\frac{E_2}{E_a}$.

A few closing remarks are due with regard to the very interesting observations relating to the effect of submergence, referred to under "Flow Conditions with Tranquil Tailwater: Effect of Submergence" as a continuation of the paper. Mr. Moore obviously has been dealing with a curious phenomenon, observed also on different occasions at Columbia University, and which in the parlance of the laboratory has been named the "drowned jump." The author's description may be complemented by Fig. 23, which relates to patterns observed in the flow from under a sluice. (The experiments, performed in a similar flume, were conducted by Fred Kibbler, who at the time (1937–1938) was research assistant at the Columbia Laboratory.) As seen, when the heretofore free outflow (curve J, Fig. 23) becomes submerged by tailwater that exceeds the upper stage of a jump, the drowned jet continues for quite a while under the roller without much apparent change, slowly expanding until it reaches a certain section, section J, Fig. 23, beyond which it begins to rise rapidly, reaching the surface of the tailwater T on a steep gradient. In the part between the vena contracta and section J the divergence is sufficiently low for the streamline to be treated as "parallel" in terms of the habitual Bernoulli equation, with the decreasing velocity adjusting itself to the increasing overpressure of the roller (s_p) and to the eventual energy losses. (The demarcation line B between the "jet" and the covering "roller" was found by determining in the successive velocity profiles the points b . . . b, which delimit the discharge as it flows through the sluice. Computations were based on midstream conditions, as shown in Fig. 23, and obviously are approximate.)

Between sections J and T, Fig. 23, the rapid expansion resembles the features of a jump. In fact, the adverse slope of the veil between sections J and T is not far from the customary surface gradient of an "open jump" surface. Obviously, circumstances in the zone suggest the use of the "momentum principle." Lower or higher tailwater will shift the toe of the submerged jump

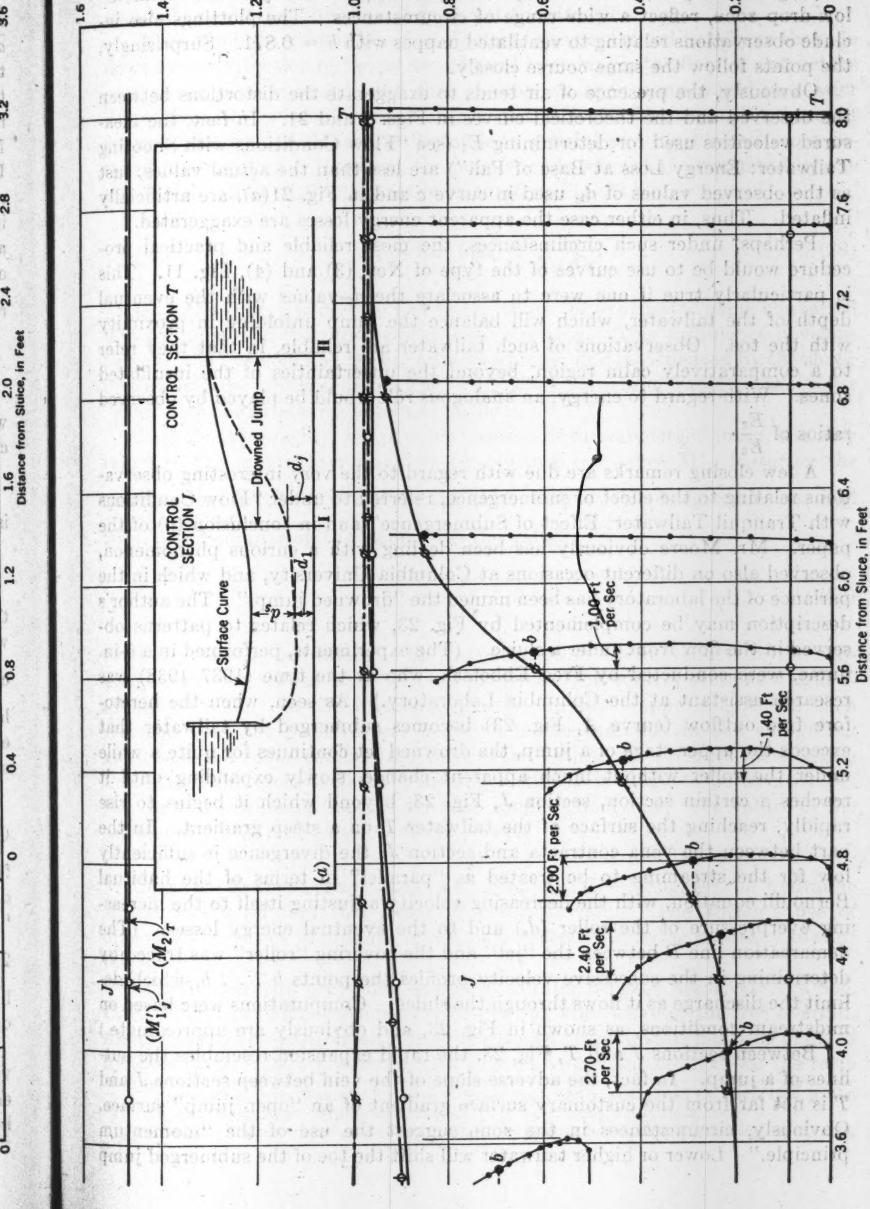


FIG. 23

varying weir heights, so that the plotted points, particularly in the "reliable" low-drop zone, reflect a wide range of circumstances. The plottings also include observations relating to ventilated nappes with $h = 0.871$. Surprisingly, the points follow the same course closely.

Obviously, the presence of air tends to exaggerate the distortions between the observed and the theoretical curves in Figs. 7 and 21. In fact, the measured velocities used for determining E_1 (see "Flow Conditions with Shooting Tailwater: Energy Loss at Base of Fall") are less than the actual values, just as the observed values of d_1, used in curve *c* and in Fig. 21(*a*), are artificially inflated. Thus, in either case the apparent energy losses are exaggerated.

Perhaps, under such circumstances, the most reliable and practical procedure would be to use curves of the type of Nos. (3) and (4), Fig. 11. This is particularly true if one were to associate the d_2-values with the eventual depth of the tailwater, which will balance the jump unfolding in proximity with the toe. Observations of such tailwater are reliable, in that they refer to a comparatively calm region, beyond the uncertainties of the insufflated zones. With regard to energy, an analogous rôle would be played by observed ratios of $\frac{E_2}{E_0}$.

A few closing remarks are due with regard to the very interesting observations relating to the effect of submergence, referred to under "Flow Conditions with Tranquil Tailwater: Effect of Submergence" and in conclusion (6) of the paper. Mr. Moore obviously has been dealing with a curious phenomenon, observed also on different occasions at Columbia University, and which in the parlance of the laboratory has been named the "drowned jump." The author's description may be complemented by Fig. 23, which relates to patterns observed in the flow from under a sluice. (The experiments, performed in a 6-in. flume, were conducted by Fred Ebbetsch, who at the time (1937–1938) was research assistant at the Columbia Laboratory.) As seen, when the heretofore free outflow (curve *A*, Fig. 23) becomes submerged by tailwater that exceeds the upper stage of a jump, the drowned jet continues for quite a while under the roller without much apparent change, slowly expanding until it reaches a certain section, section *J*, Fig. 23, beyond which it begins to rise rapidly, reaching the surface of the tailwater *T* on a steep gradient. In the part between the vena contracta and section *J*, the divergence is sufficiently low for the streaming to be treated as "parallel" in terms of the habitual Bernoulli equation, with the decreasing velocity adjusting itself to the increasing overpressure of the roller (d_s) and to the eventual energy losses. (The demarcation line *B* between the "jet" and the covering "roller" was traced by determining in the successive velocity profiles the points *b* . . . *b*, which delimit the discharge as it flows through the sluice. Computations were based on midstream conditions, as shown in Fig. 23, and obviously are approximate.)

Between sections *J* and *T*, Fig. 23, the rapid expansion resembles the outlines of a jump. In fact, the adverse slope of the vein between sections *J* and *T* is not far from the customary surface gradient of an "open jump" surface. Obviously, circumstances in the zone suggest the use of the "momentum principle." Lower or higher tailwater will shift the toe of the submerged jump (section J) away from the aperture, and vice versa, without, however, substantially modifying the general aspect of the situation.

When the phenomenon was first disclosed, it was natural to believe that the downstream extension of the jet would tend to endanger the floor of a structure over a considerable length, and that possibly an open jump would be a safer device from the point of view of energy dissipation. The argument, brought forward also by Mr. Moore, still stands in general and must await more detailed investigation. However, the poignancy is somewhat attenuated by the calculated outline of the jet energy featured in Fig. 23 by

$$E_j = (d_1 + d_s) + \frac{q^2}{2\,g\,d^2_1} \qquad \text{.................. (29)}$$

As seen, the energy is rapidly lost in the course of expansion, being converted into turbulence in the process of intensive mixing. This turbulence develops between the live vein and the superimposed roller and is evidenced by the shape of the velocity profiles, indicating with what surprising rapidity the roller "eats" into the originally potential outflow.

To estimate, roughly, the possible location of the submerged jump, one may resort to the momentum principle by plotting the curve M'_1, featuring the quantity $\frac{q^2}{g\,d_1} + 0.5\,(d_1 + d_s)^2$ for the expanding jet (for designations see Fig. 23(a)), and comparing the latter with the value

$$M_2 = \frac{q^2}{g\,d_2} + 0.5\,d^2_2 \qquad \text{...................... (30)}$$

computed for the tailwater.[13] To the exclusion of friction effects in the submerged jet, the values of $(M'_1)_J$ and $(M_2)_T$ for the control sections J and T should coincide. In the foregoing greatly simplified appraisal, it has been assumed that the positive and negative momentum flux in the circulating motion in the roller compensate each other. Also, the pressure distribution throughout the cross sections preceding section J is taken to be hydrostatic. It is indeed surprising that, under such rough "short cuts," the values of the calculated $(M'_1)_j$ and $(M_2)_T$ conform so well.

CARL E. KINDSVATER,[14] **JUN. AM. SOC. C. E.**—Designers of hydraulic structures should find this paper of practical value. The author's use of dimensionless ratios in presenting test data reflects a growing tendency toward this practice.

Coordinated with previous research by Hunter Rouse,[15] Assoc. M. Am. Soc. C. E., this paper adds considerably to existing knowledge of the free overfall. An interesting comparison with Professor Rouse's experiments has been omitted by the author. Data presented with Fig. 6 give a value of 0.716 for the ratio

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[13] "Hydraulics of Open Channels," by Boris A. Bakhmeteff, McGraw-Hill Book Co., Inc., 1932, p. 234.

[14] Associate Hydr. Engr., Hydroelectric Section, U. S. Engr. Office, Little Rock, Ark.

[15] "Discharge Characteristics of the Free Overfall," by Hunter Rouse, *Civil Engineering*, April, 1936, p. 257

$\frac{d_0}{d_c}$. Similarly, from Fig. 10, $\frac{d_0}{d_c} = 0.714$. The depth d_c, used by the author as a reference parameter, should be defined as the nominal critical depth in the region of parallel flow upstream from the overfall. True critical depth, corresponding to the point of minimum specific energy, occurs at the brink of the overfall and is equal to the author's depth d_0. Professor Rouse found an average value of $\frac{d_0}{d_c} = 0.715$, which happens to correspond to the coefficient of contraction for a weir of zero height. It would be of interest to see a plot of all the author's data on this ratio.

In the seventh line following Eq. 6*b*, the author states: "With a given discharge and fall height, more effective dissipation may be achieved by increasing the width of the overfall section, and thereby reducing the value of d_c." The effect of spillway width upon energy dissipation is an important consideration in the selection of gates for large overflow dams. Fig. 24 shows one method of facilitating this choice. For illustration, consider an overflow spillway of width *b*, with a drop *h* between reservoir and tailwater levels. For preliminary study, it might be assumed that a perfect hydraulic jump will be caused to form at the foot of the spillway and that the lower conjugate depth, or the depth before the jump, will be reasonably close to $d_1 = \frac{Q}{b\,V_1} = \frac{Q}{b\,\sqrt{2\,g\,h}}$. The depth d_2 below the jump could be computed from Eq. 10*a* or 10*b*, but the

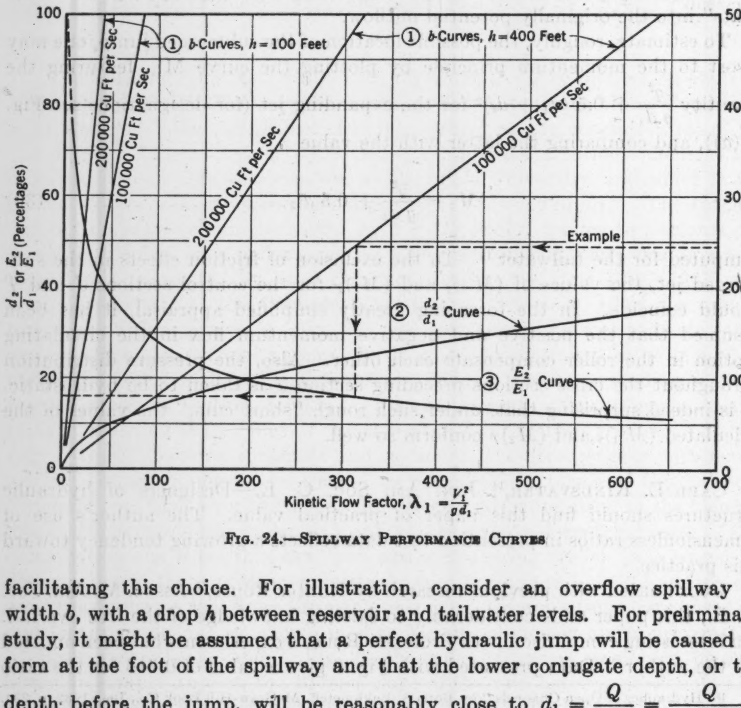

FIG. 24.—SPILLWAY PERFORMANCE CURVES

writer prefers to use an equation proposed by Messrs. Bakhmeteff and Matzke,[16]

$$\frac{d_2}{d_1} = \frac{1}{2}\left(-1 + \sqrt{1 + 8\lambda_1}\right) \qquad (31)$$

in which λ_1 is the "kinetic flow factor," analogous to Froude's number as a criterion for dynamic similitude. For the present case, the kinetic flow factor can be computed directly from the data given, or

$$\lambda_1 = \frac{V^2_1}{g\,d_1} = \frac{2\,b\,h^{3/2}\sqrt{2\,g}}{Q} \qquad (32)$$

Fig. 24 shows curves (1) in which λ_1 is plotted against b for values of h equal to 100 and 400 ft and values of Q equal to 100,000 and 200,000 cu ft per sec. Curve (2) in Fig. 24 is a curve of λ_1 against the ratio $\frac{d_2}{d_1}$ from Eq. 31. If, as defined by the author, E_1 and E_2 are the specific energies of flow before and after the jump, then $\frac{E_2}{E_1}$ represents the "efficiency" of the jump, a measure of energy dissipation below the dam. Curve (3) in Fig. 24 shows $\frac{E_2}{E_1}$ as a function of λ_1. Since curves (2) and (3) are applicable for all values of Q, b, and h, Fig. 24 demonstrates the practical advantages of the kinetic flow factor as a reference parameter.

The author's data showing a slight excess of pressure head over depth in the region of the toe of the jump are interesting. Close agreement or considerable disagreement between observers of such data may indicate only similarities or differences in laboratory technique. Tests on the hydraulic jump in a sloping channel might afford a better comparison with the author's data than tests on the level-floor jump. Messrs. Bakhmeteff and Matzke,[17] working with slopes as steep as 1 on 10, concluded, "* * * no appreciable difference can be observed between the depths and the local bottom pressures, except for a very limited zone near the toe of the jump * * *." Similar unpublished data taken by the late D. L. Yarnell, M. Am. Soc. C. E., for slopes from 1 on 6 to 1 on 1, generally show fair agreement between piezometer and point-gage profiles. Because of considerable admixture of air within the jump, it might be expected that the point-gage depths would exceed depths measured by piezometers.

J. E. CHRISTIANSEN,[18] ASSOC. M. AM. SOC. C. E.—In the opinion of the writer, the usefulness of this very interesting paper would be improved by plotting, or expressing, all of the dependent variables as functions of the one independent variable, $\frac{h}{d_c}$. For example, to determine the depth of standing water behind the fall, d_f, or the height of the jump, d_j, for any given or assumed

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[16] "The Hydraulic Jump in Terms of Dynamic Similarity," by Boris A. Bakhmeteff and Arthur E. Matzke, *Transactions*, Am. Soc. C. E., Vol. 101 (1936), p. 636.

[17] "The Hydraulic Jump in Sloped Channels," by Boris A. Bakhmeteff and Arthur E. Matzke, *Transactions*, A.S.M.E., Vol. 60, February, 1938, p. 115.

[18] Irrig. and Drainage Engr., U. S. Regional Salinity Laboratory, U. S. Dept. of Agriculture, Riverside, Calif.

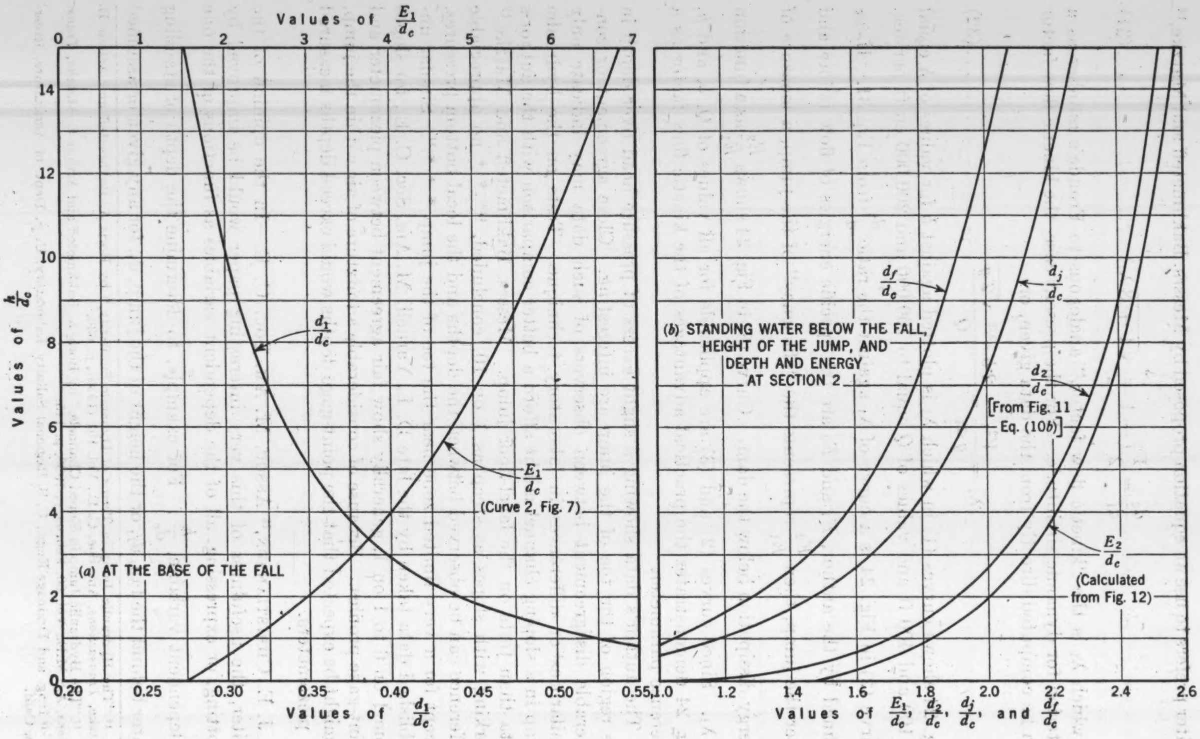

FIG. 25.—ENERGY AND DEPTHS BELOW THE FALL FOR VARIOUS RATIOS OF $\frac{h}{d_c}$

values of h and d_c, it is first necessary to determine d_1. To do this, one must select values of $\frac{E_1}{d_c}$ from curve (2), Fig. 7, and then solve a cubic equation (Eq. 9), which can be done graphically. Since $\frac{d_1}{d_c}$ is solely dependent upon $\frac{E_1}{d_c}$, which in turn is dependent upon $\frac{h}{d_c}$, why not plot $\frac{d_1}{d_c}$ against $\frac{h}{d_c}$? This has been done in Fig. 25(a), which also includes $\frac{E_1}{d_c}$, replotted from Fig. 7. The next step is to plot $\frac{d_f}{d_c}$ and $\frac{d_j}{d_c}$ directly against $\frac{h}{d_c}$. From Eqs. 8 and 9 and curve (2), Fig. 7(a), the relationship between $\frac{d_f}{d_c}$ and $\frac{h}{d_c}$ can be established. This has been plotted in Fig. 25(b). On this same figure, the values of $\frac{d_2}{d_c}$ have been replotted from Fig. 11. Values of $\frac{d_j}{d_c}$, being equal to $\frac{d_2}{d_c} - \frac{d_1}{d_c}$, are also shown. Likewise, for comparison, the relationship between $\frac{E_2}{d_c}$ and $\frac{h}{d_c}$ has been determined from Figs. 7(a) and 12, and is plotted in Fig. 25(b).

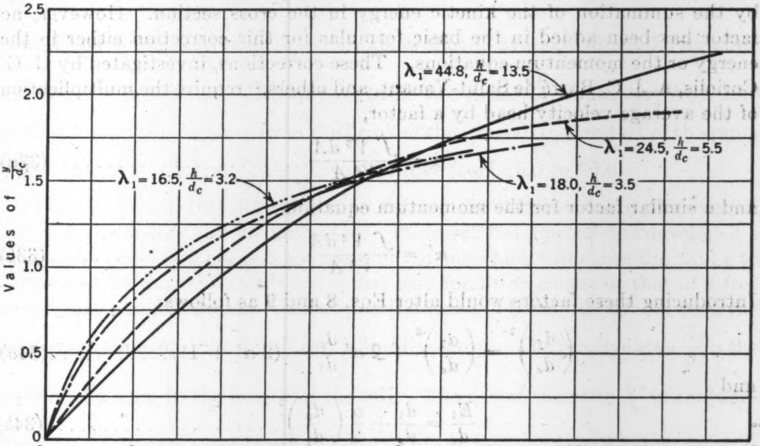

FIG. 26.—COMPARISON OF THE HEIGHTS AND PROFILES OF THE JUMP FOR VARIOUS RATIOS OF $\frac{h}{d_c}$

To compare the longitudinal profiles of the jump, for constant flow, and different values of h, the writer has plotted values of $\frac{y}{d_c}$ against $\frac{L}{d_c}$, as shown in Fig. 26. To do this it was necessary to calculate values of $\frac{d_1}{d_c}$, $\frac{h}{d_c}$, and $\frac{d_j}{d_c}$, from the given values of λ_1, then select points on the curves in Fig. 13, and calculate points for the new set of curves. It would be interesting to know how the computed values of $\frac{h}{d_c}$ as given in Fig. 26 compare with the actual values.

L. Standish Hall,[19] M. Am. Soc. C. E.—The studies by the author have added needed experimental data to the hydraulics of the energy loss at the base of a free overfall. Engineering literature has been notably lacking in definite information concerning these losses. Experiments of the type described in this paper are necessary to permit the design of safe types of stilling pools at the base of dams or other overfalls.

The determination of values of ϕ in Eq. 2 is particularly interesting, as it supplies data previously lacking, as stated by the author. A graph of the relation between $\frac{h}{d_c}$ and $\frac{d_1}{d_c}$ as expressed in Eq. 4 has not been supplied. Such a graph is helpful in showing that the ratio $\frac{d_1}{d_c}$ decreases rather slowly for values of $\frac{h}{d_c}$ greater than 3.

The author indicates that the velocity head determined from the mean velocity was not of sufficient accuracy, and he has determined the velocity head by the summation of the kinetic energy in the cross section. However, no factor has been added in the basic formulas for this correction either in the energy or the momentum equations. These corrections, investigated by G. G. Coriolis, A. J. C. Barré de Saint-Venant, and others,[20] require the multiplication of the average velocity head by a factor,

$$\alpha = \frac{\int V^3\, dA}{V^3 A} \qquad (33a)$$

and a similar factor for the momentum equation,

$$\alpha' = \frac{\int V^2\, dA}{V^2 A} \qquad (33b)$$

Introducing these factors would alter Eqs. 8 and 9 as follows:

$$\left(\frac{d_f}{d_c}\right)^2 = \left(\frac{d_1}{d_c}\right)^2 + 2\,\alpha'\,\frac{d_c}{d_1} - (2\,\alpha' + 1) \qquad (34a)$$

and

$$\frac{E_1}{d_c} = \frac{d_1}{d_c} + \frac{\alpha}{2}\left(\frac{d_c}{d_1}\right)^2 \qquad (34b)$$

The values of α and α' determined from the experiments would be of interest. The use of these factors adds to the complex problem of solving Eqs. 34, and their omission is usually allowed for by a factor of safety in the design.

The profiles in Fig. 13 indicate that the jump below a free overfall is about 20% longer than determined by Messrs. Bakhmeteff and Matzke downstream from a regulating sluice.[7] It is possible that the air entrained in the water by the overfall caused the phenomenon noted. This condition has been observed at the toe of many chute structures and, acting upon the writer's suggestion, laboratory experiments were made of the effect of entrained air upon the hydraulic jump below a sluice-gate by introducing air in the bottom of the flume just as the thin sheet of water emerged at high velocity from the sluice.[21] The following conclusions were reached:

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[19] Hydr. Engr., East Bay Municipal Utility Dist., Oakland, Calif.

[20] *Annales des Ponts et Chaussées*, 1836; see also, *Transactions*, A.S.M.E., Vol. 54, 1932, HYD-54-4, p. 57.

(1) The admittance of air into the water before the jump increases the velocity instead of the initial depth.

(2) The increase in velocity due to the entrainment of air, therefore, would tend to require a longer apron below a spillway or a longer stilling basin below a chute.

(3) For values of λ less than 12 the admission of air into the hydraulic jump decreased the comparative length of the jump, whereas for values of λ above 12 the comparative length of jump was increased. Since in the experiments made by the author the values of λ were greater than 12, the entrainment of air in the water would increase the length of the jump.

A free overfall for studying energy losses is very desirable, but from a practical standpoint this type of overfall is not economical, due to the great length of channel below the drop requiring protection. From Fig. 11 it is seen that values of $\frac{d_2}{d_c}$ normally range from 2.0 to 2.5. Messrs. Bakhmeteff and Matzke have found from their experiments that $\frac{L}{d_2}$ averages about 5.0 for quite a range of values of λ.[22] Multiplying these two ratios, $\frac{L}{d_c}$ would vary from 10.0 to 15.0, and, if the distance from the brink of the overfall to the start of the jump is designated as L', then $\frac{L'}{d_c} + \frac{L}{d_c}$ would range from 15.0 to 20.0.

The solution of this difficulty is to shorten the distance below the overfall in which the energy can be dissipated. Professor Bakhmeteff has developed a method by providing either a toe basin or barrier which will cause the nappe to be covered by the tailwater.[23] Whether this condition exists, or that of a free nappe with a repelled jump, depends on the value of the drop (Z_0) between the total energy line and the depth (d_2) below the jump. The relative step $\frac{Z_0}{h}$ is the ratio of the drop to the height of the fall. The use of the ratio $\frac{Z_0}{h}$ in conjunction with $\frac{d_c}{h}$ affords a direct relation between the loss above and below the fall. In the author's presentation this can be determined only by following through several equations and diagrams.

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[21] "Effect of Entrained Air on the Hydraulic Jump," by Emery H. Willes, thesis submitted to the College of Civ. Eng., Utah State Agricultural College, Logan, Utah, in 1937, in partial fulfilment of the requirements for the degree of Bachelor of Science.

[22] "The Hydraulic Jump in Terms of Dynamic Similarity," by Boris A. Bakhmeteff and Arthur E. Matzke, *Transactions*, Am. Soc. C. E., Vol. 101 (1936), Fig. 6, p. 643.

[23] "Hydraulics of Open Channels," by Boris A. Bakhmeteff, Engineering Societies Monograph, 1932, Chapter XXI (a study of this chapter, particularly the numerical examples, will clarify the following discussion by the writer).

Using the author's nomenclature with Professor Bakhmeteff's treatment, this ratio can be expressed by the formula

$$\frac{Z_0}{h} = 1 + \frac{3}{2}\frac{d_c}{h} - \frac{d_1}{2\,h}\left[-1 + \sqrt{1 + 16\,\phi^2\left(\frac{h}{d_1} + \frac{3}{2}\frac{d_c}{d_1} - 1\right)}\right] \ldots (35)$$

This equation contains too many variables for direct solution, but it can be solved by means of the author's Eq. 4, which also may be expressed in two different forms:

$$\frac{h}{d_c} = \frac{1}{2\,\phi^2}\left(\frac{d_c}{d_1}\right)^2 + \frac{d_1}{d_c} - \frac{3}{2} \ldots\ldots\ldots\ldots (36a)$$

and

$$\frac{h}{d_1} = \frac{1}{2\,\phi^2}\left(\frac{d_c}{d_1}\right)^3 - \frac{3}{2}\frac{d_c}{d_1} + 1 \ldots\ldots\ldots\ldots (36b)$$

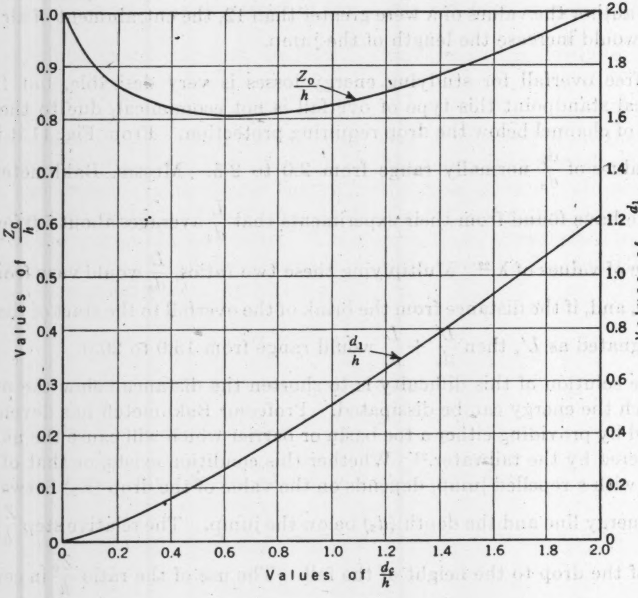

FIG. 27.—CURVES OF RELATIVE WATER SURFACE DROP $\left(\frac{Z_0}{h}\right)$ FOR LIMITING VALUES BETWEEN FREE AND COVERED TOE ROLLS AND RELATION BETWEEN $\frac{d_1}{h}$ AND $\frac{d_c}{h}$ FOR FREE TOE ROLLS BELOW AN OVERFALL

By using $\frac{d_c}{d_1}$ as an argument, Eq. 36a can be solved for $\frac{h}{d_c}$. Values of ϕ^2 must be known in advance and, as only a relation between ϕ^2 and $\frac{h}{d_c}$ can be obtained from Fig. 7(*b*), the solution must be by trial and error. A direct solution could be made more laboriously by using the cubic equation (Eq. 4). Since all of the ratios in Eqs. 36 are interdependent, the values of ϕ^2 corresponding to

equal values of $\frac{d_c}{d_1}$ from Eq. 36(*a*) can be substituted in Eq. 36*b* and values of $\frac{h}{d_1}$ calculated. Using the values of the several ratios and of ϕ^2 corresponding to values of $\frac{d_c}{d_1}$, the ratio $\frac{Z_0}{h}$ then can be computed from Eq. 35.

From Fig. 11, values of $\frac{Z_0}{h}$ can be derived more easily, since

$$Z_0 = h + \frac{3}{2} d_c - d_2 \ldots\ldots\ldots\ldots\ldots\ldots (37a)$$

Then

$$\frac{Z_0}{d_c} = \frac{h}{d_c} + \frac{3}{2} - \frac{d_2}{d_c} \ldots\ldots\ldots\ldots\ldots\ldots (37b)$$

The ratio $\frac{Z_0}{d_c}$ can be computed readily from values taken from curve (3) in Fig. 11; thus,

$$\frac{Z_0}{h} = \frac{Z_0}{d_c} \times \frac{d_c}{h} \ldots\ldots\ldots\ldots\ldots\ldots (38)$$

Values of $\frac{Z_0}{h}$ computed from Figs. 7(*b*) and 11 differ slightly, due to the difficulty of scaling accurately from the small scale of the graphs. The relation of $\frac{d_c}{h}$ to $\frac{d_1}{h}$ and $\frac{Z_0}{h}$ is shown in Fig. 27. The flow below the overfall will be submerged or it will be free, depending on the height of d_2. From Fig. 27, the value of d_1 can be determined first and then that value of Z_0, which separates the covered nappe from the free nappe, can be computed. A toe basin, a barrier, or a combination of both can be provided from these data as may be desired.

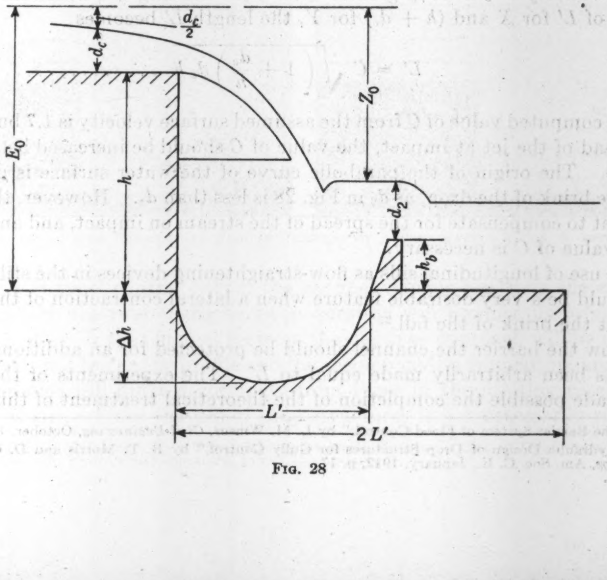

FIG. 28

Referring to Fig. 28, if a barrier is provided, then

$$Z_0 = h + \frac{3}{2} d_c - d_2 - h_b \ldots\ldots\ldots\ldots\ldots\ldots (39)$$

Since all of the values on the right-hand side of the equation are known, or can be assumed, Z_0 can be computed. If the ratio of $\frac{Z_0}{h}$ is less than the value determined from Fig. 27, the nappe will be submerged. If a toe basin is desired, the total height of fall is increased by the amount Δh.

Let $h' = h + \Delta h$, and assume a value of $\frac{Z_0}{h'}$ and compute h'. The computed value then can be compared with Fig. 27 and adjustments made if necessary. The use of a combination of the barrier and the toe basin usually provides an economy in excavation.

The combination of the toe basin and barrier as shown in Fig. 28 has the advantages of providing a pool in which to dissipate energy and of shortening the length of channel below the overfall requiring protection. This type of structure is not new but has been used effectively for flood and erosion control.[24] However, although the ratio $\frac{Z_0}{h}$ could be computed with reasonable accuracy from existing data, the writer has been unable to find any reference to the proper location of the barrier from the brink of the fall, or the distance L' in Fig. 28. This distance can be determined from the equation for the path of a trajectory, traveling at a horizontal velocity, V:

$$X^2 = \frac{2\, V^2}{g}\, Y \ldots\ldots\ldots\ldots\ldots\ldots (40)$$

The velocity at the critical depth $V_c = \sqrt{g\, d_c}$, and it may be assumed that the surface velocity V'_c is 20% greater than V_c. Making the additional substitutions of L' for X and $(h + d_c)$ for Y, the length L' becomes

$$L' = C \sqrt{\left(1 + \frac{d_c}{h}\right) d_c\, h} \ldots\ldots\ldots\ldots\ldots\ldots (41)$$

The computed value of C from the assumed surface velocity is 1.7 but, due to the spread of the jet at impact, the value of C should be increased between 1.7 and 2.0. The origin of the parabolic curve of the water surface is upstream from the brink of the drop, as d_0 in Fig. 28 is less than d_c. However, this is not sufficient to compensate for the spread of the stream on impact, and an increase in the value of C is necessary.

The use of longitudinal sills as flow-straightening devices in the stilling pool also would be a very desirable feature when a lateral contraction of the stream exists at the brink of the fall.[25]

Below the barrier the channel should be protected for an additional length that has been arbitrarily made equal to L'. The experiments of the author have made possible the completion of the theoretical treatment of this type of roller basin. Placing the barrier as near the falls as possible eliminates the erosion caused by the undershooting jet, such as would exist if the oversubmergence were created by a barrier located at a considerable distance downstream, as in the author's experiments.

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[24] "The Barrier System of Flood Control," by L. M. Winsor, *Civil Engineering*, October, 1938, p. 675.

[25] "Hydraulic Design of Drop Structures for Gully Control," by B. T. Morris and D. C. Johnson, *Proceedings*, Am. Soc. C. E., January, 1942, p. 17.

Hunter Rouse,[26] M. Am. Soc. C. E.—Since the writer was privileged to take part in the staff discussions that preceded the experimental study reported upon by Mr. Moore, he has been interested to note the extent to which the final investigation incorporated suggestions made at that time. The dimensionless representation of all data and the adept use of the momentum and energy principles to clarify the general problem are quite in keeping with present-day trends, and Mr. Moore is to be complimented upon an able presentation of his results.

The author evidently restricts his analysis to flow which approaches a drop structure at critical depth—conditions which are commonly considered to define the overfall phenomenon. It is all too seldom realized, however, that flow at critical depth represents only a limiting case of the general overfall problem. In other words, a drop of this nature at the end of a channel will serve as a channel control only if the channel slope is less than the critical—that at which the rate of discharge in question will produce uniform flow at critical depth. If the channel is steep, the depth of approach will be less than the critical, and the nappe profile and the accompanying momentum and energy characteristics will differ accordingly from those in the limiting case studied by Mr. Moore. Since the use of drop structures is not necessarily restricted

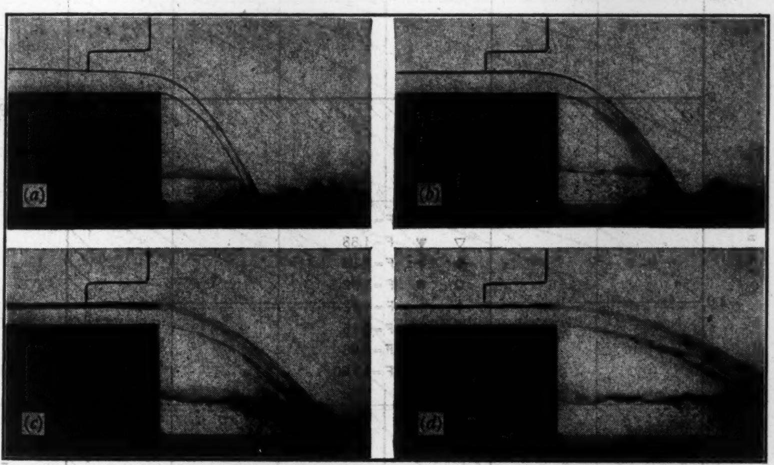

Fig. 29.—Nappe Profiles at Froude Numbers of Approximately 1, 2.5, 6, and 14

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[26] Prof., Fluid Mechanics, State Univ. of Iowa, Associate Director in Chg. of Laboratory, Iowa Inst. of Hydr. Research, Hydraulics Laboratory, Iowa City, Iowa.

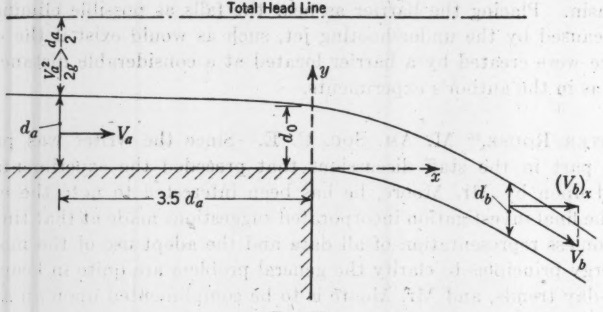

Total Head Line
$\frac{V_a^2}{2g} > \frac{d_a}{2}$
d_a
V_a
y
d_0
x
$3.5\ d_a$
d_b
$(V_b)_x$
V_b

FIG. 30

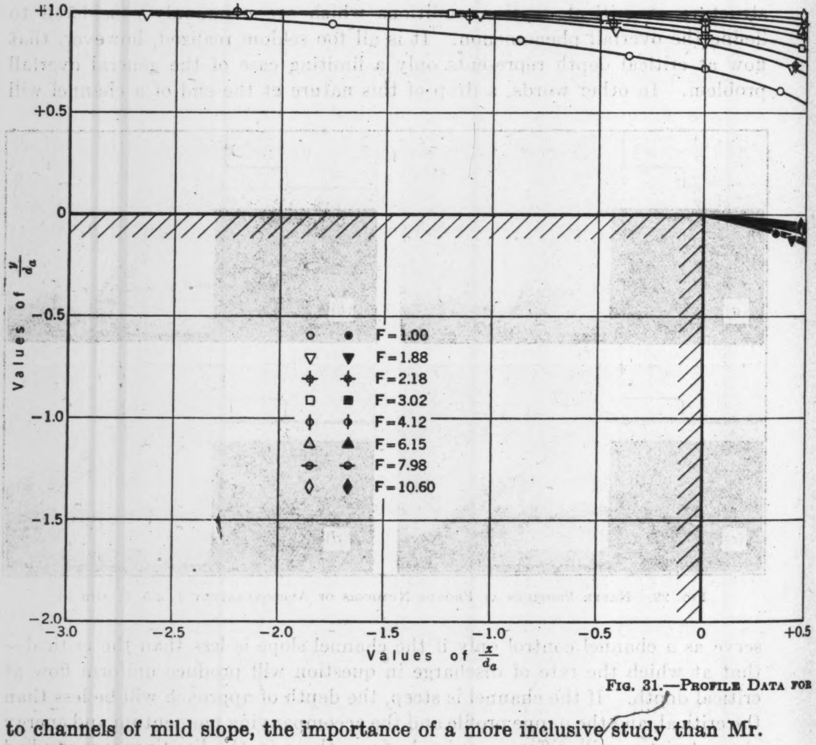
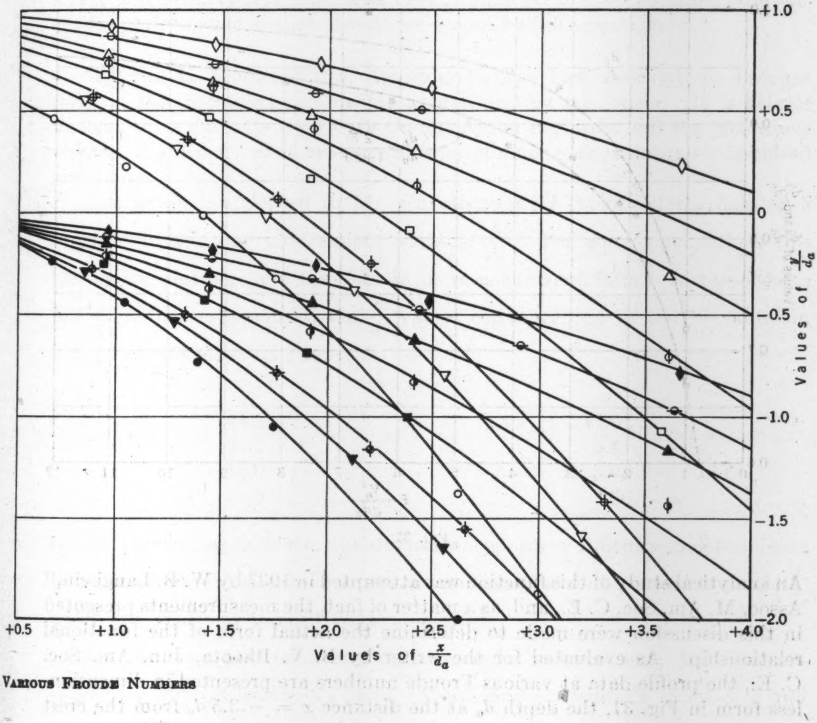


FIG. 31.—PROFILE DATA FOR

to channels of mild slope, the importance of a more inclusive study than Mr. Moore's is obvious.

To illustrate this point, the writer makes use of data obtained by A. F. Leder, Prof. J. F. Mangold, Assoc. M. Am. Soc. C. E., and Prof. G. W. Thomas in the Hydraulics Laboratory of the University of Iowa, in Iowa City, as a special project of the National Defense Course in Mechanics of Fluids conducted during the summer of 1941. As shown in Fig. 29, a drop structure 1 ft high was placed in a glass channel 1 ft wide, the depth of the approaching flow being held essentially constant as the velocity was varied by means of a sluicegate a short distance upstream. Although the channel floor was actually horizontal, the resulting profiles should not deviate appreciably from those for similar depths and velocities at the end of a channel which is hydraulically steep.

Through use of the momentum principle, it is at once possible to evaluate the ultimate vertical thickness of the nappe, which is approached asymptotically as the pressure intensity within the falling sheet approaches that of the atmosphere. With reference to Fig. 30, the rate of change of the horizontal component of momentum per unit width between sections a and b will be $\rho\{[(V_b)_x]^2\,d_b - (V_a)^2\,d_a\}$, and the force per unit width producing such change will be $\frac{\gamma\,(d_a)^2}{2}$. Equating these values,



VARIOUS FROUDE NUMBERS

$$\frac{\gamma\,(d_a)^2}{2} = \rho\,\{[(V_b)_x]^2\,d_b - (V_a)^2\,d_a\}\,\ldots\ldots\ldots\ldots\ldots\ldots(42)$$

Upon introducing the continuity relationship $V_a\, d_a = (V_b)_x\, d_b$ and the dimensionless ratio $\mathbf{F} = \frac{(V_a)^2}{g\, d_a}$, it will be found that

$$\frac{d_b}{d_a} = \frac{2\,\mathbf{F}}{1 + 2\,\mathbf{F}} \qquad (43)$$

The quantity **F** will be recognized as the Froude number of the approaching flow, to which Mr. Moore's "kinetic flow factor" is dimensionally equivalent.

For the particular case $\mathbf{F} = 1$, which is that investigated by Mr. Moore, $\frac{d_b}{d_a}$ attains the minimum value $\frac{d_b}{d_a} = \frac{2}{1 + 2} = 0.667$, which is sometimes erroneously assumed to be the vertical depth at the brink. The brink depth is known to attain the minimum value $d_0 = 0.715\, d_c$ for a Froude number of unity,[15] but its magnitude at higher values has not previously been measured.

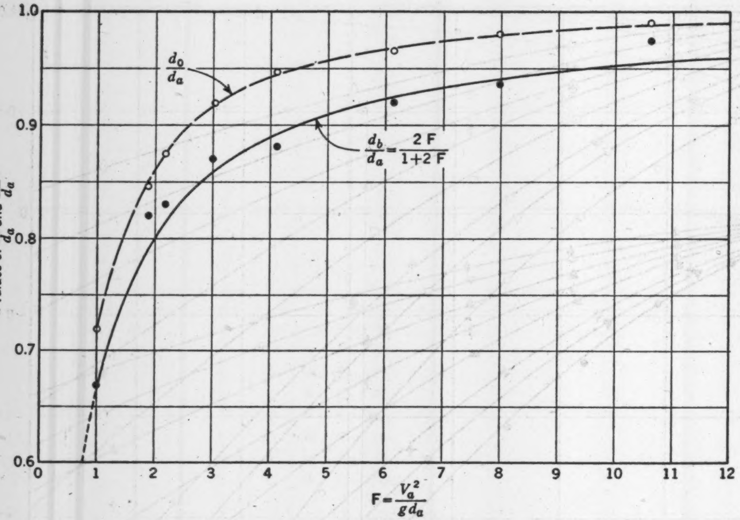


FIG. 32

An analytical study of this function was attempted in 1937 by W. B. Langbein,[27] Assoc. M. Am. Soc. C. E., and, as a matter of fact, the measurements presented in this discussion were made to determine the actual form of the functional relationship. As evaluated for the writer by B. V. Bhoota, Jun. Am. Soc. C. E., the profile data at various Froude numbers are presented in dimensionless form in Fig. 31, the depth d_a at the distance $x = -3.5\, d_a$ from the crest being used as reference. Values of the ratios $\frac{d_0}{d_a}$ and $\frac{d_b}{d_a}$ obtained from these profiles are plotted in Fig. 32 against the corresponding values of the Froude number. Values of $\frac{d_b}{d_a}$ follow closely the general trend of Eq. 43, approaching the limit of unity as the Froude number becomes large—that is, as the gravitational effect upon the nappe form steadily diminishes in relative magnitude. In agreement with earlier measurements by the writer, the ratio $\frac{d_0}{d_a}$ attains the limit 0.715 for a Froude number of unity (Mr. Moore's case); this ratio likewise approaches unity, but somewhat more rapidly, with increasing values of F.

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[27] *Civil Engineering*, May, 1937, pp. 349–350; see also July, 1937, p. 518.

The writer regrets that the backwater depth below the nappe and the subsequent rate of energy dissipation were not measured at the time these studies were made. However, through use of the parameters suggested by the writer, the methods of analysis presented by Mr. Moore could be adapted to all phases of the overfall problem, and the writer recommends their further experimental study to the Pasadena laboratory. Until designers of drop structures become aware of overfall characteristics at Froude numbers in excess of unity, the successful performance of such structures cannot be fully guaranteed.

Walter L. Moore,[28] **Jun. Am. Soc. C. E.**—The contributions brought forth in the discussions are deeply appreciated by the writer. It is helpful to have the comments from both the practicing engineers and the laboratory research engineers. Such evidence of interest among the membership is indeed gratifying.

The writer is indebted to Mr. Christiansen for replotting the data as a function of the one independent variable $\frac{h}{d_c}$. For some applications this method of plotting puts the data in more convenient form. In response to Mr. Christiansen's request, the $\frac{h}{d_c}$ values corresponding to the λ_1 values in Fig. 13 are given, as follows:

λ_1	$\frac{h}{d_c}$
44.8	14.6
24.5	5.8
18.0	3.95
16.5	3.00

It is not surprising that Mr. Christiansen encountered difficulty in determining $\frac{h}{d_c}$ values from the curves since it was necessary to make use of the ratio $\frac{d_1}{d_c}$ which is a relatively insensitive parameter.

Mr. Hall's request for expressing the results in terms of $\frac{d_1}{d_c}$ has been answered by Mr. Christiansen's plots. Expression of results in terms of ϕ and $\frac{d_1}{d_c}$, as in Eq. 4, was purposely subordinated in the paper since the objective of the study was to trace the flow of energy through the fall. Use of the parameter $\frac{d_1}{d_c}$ tends to obscure the energy values since $\frac{d_1}{d_c}$ is very insensitive to energy changes for values of $\frac{E_1}{d_c}$ greater than 4. Mr. Hall suggests changing the assumption of uniform velocity used in Eqs. 8 and 9 and including factors α and α' as shown in Eqs. 34*a* and 34*b* to account for the deviation from uniform velocity conditions. Since the value of α' is dependent upon the velocity profile, its value would be different at section *c–c* from that at section 1–1. Also its value at section 1–1 would change with changing values of $\frac{h}{d_c}$. Eq. 34*a* assumes an equal value of α' at sections *c–c* and 1–1. If Eq. 34*a* were modified to suit this condition it would appear as

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[28] Senior Structural Research Engr., Lockheed Aircraft Corp., Burbank, Calif.

$$\left(\frac{d_f}{d_c}\right)^2 = \left(\frac{d_1}{d_c}\right)^2 + 2\,\alpha'_1\,\frac{d_c}{d_1} - (2\,\alpha'_c - 1) \qquad (44)$$

in which α'_1 applies to section 1–1 and α'_c to section *c–c*.

The complications introduced by including the velocity factors are not warranted in this application as evidenced by the close agreement between the experimental points and the points calculated on the assumption of uniform flow (see Fig. 8).

In the discussion supporting the statement that a free overfall is not economical due to the great length of channel below the drop requiring protection, Mr. Hall has apparently confused the performance of the hydraulic jump with that of the overfall. A toe basin or barrier may be used at the end of an apron below a free overfall to shorten the length of channel requiring protection.[29]

Mr. Hall's application of the experimental data to the problem of holding a submerged jump on the apron below the fall should be of interest to the practicing engineer who is confronted with a specific structure. It should be remembered, however, that in the more complicated field structures many questions arise which, as yet, cannot be answered positively on the basis of experimental data.

Mr. Kindsvater suggests that data be presented on the value of the ratio $\frac{d_0}{d_c}$ which in previous publication[15] by Professor Rouse has been found to be equal to 0.715. During the course of the writer's experiments, Professor Rouse's value for the ratio $\frac{d_0}{d_c}$ was checked within the accuracy of measurement. At the time the paper was written it was felt that this value was sufficiently well established to need no further verification.

In Fig. 24 Mr. Kindsvater presents a number of curves entitled "Spillway Performance Curves." These curves are built up from the equation

$$d_1 = \frac{Q}{b\,V_1} = \frac{Q}{b\,\sqrt{2\,g\,h'}} \qquad (45)$$

which Mr. Kindsvater assumed for preliminary study. (Note: The writer has changed Mr. Kindsvater's h to h' since h, as used in the original paper, refers to the height from the apron floor to the crest of the fall, whereas Mr. Kindsvater refers to the difference between reservoir and tailwater levels. Therefore, in the nomenclature of the original paper $h' = h + \frac{3}{2} d_c - d_2$.) In Eq. 45 it is assumed that flow conditions at the base of the fall are defined by

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[29] "Hydraulic Design of Drop Structures for Gully Control," by B. T. Morris and D. C. Johnson, *Transactions*, Am. Soc. C. E., Vol. 108 (1943), p. 887.

$$V_1 = \sqrt{2\,g\,h'} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (46a)$$

or, in the original nomenclature,

$$V_1 = \sqrt{2\,g\left(h + \frac{3}{2} d_c - d_2\right)} \ldots\ldots\ldots\ldots\ldots\ldots (46b)$$

It will be seen that Eq. 46*b* makes no allowance for energy loss at the base of the fall, which the writer has shown to be appreciable, and that it makes flow conditions upstream from the jump dependent on the depth downstream from the jump (d_2), a fact which is contrary to the nature of the jump. As a matter of interest, Eq. 45 was transformed, using the relation $\frac{Q^2}{b^2\,g} = d^3{}_c$, into the form

$$\left(\frac{d_c}{d_1}\right)^2 - \left(2\,\frac{h}{d_c} + 3\right) + 2\,\frac{d_2}{d_c} = 0 \ldots\ldots\ldots\ldots\ldots (47)$$

and values of $\frac{d_1}{d_c}$ were compared with the experimental values, but no correlation was found.

In Fig. 24, the relations shown by curves (2) and (3) have a rational foundation and have been checked experimentally. They give a reliable indication of the performance of the hydraulic jump as a function of λ_1. Curves (1), however (which give values of λ_1, as determined by Eq. 45), do not agree with experimental values.

The comments of Professor Rouse were especially appreciated since his example in the classroom helped to kindle the writer's interest in hydraulics, and his guidance and inspiration during the first phases of the project were especially valuable.

The emphasis by Professor Rouse on the limitation of the original paper to conditions resulting from flow at the critical depth in the upstream channel is worth while. It is obviously true that, if the stream above the fall is flowing at greater than critical velocity, the nappe profile will differ from that of the present study. The jet would fall farther downstream and strike the apron at a different angle. This problem might be studied by breaking the fall into two parts. Making use of the parameters suggested by Professor Rouse, the angle and position of the impinging jet could be determined for various upstream flow conditions. The energy loss and flow conditions on the apron could be determined as a function of the angle of the jet and its distance from the upstream end of the pool. It appears that this method would facilitate the study since similar conditions might occur at the base of the fall for different combinations of $\frac{h}{d_c}$ and approach velocity.

Professor White's analytical determination of the energy loss by use of the momentum principle and the assumption of uniform velocity at all sections is very interesting. In view of the simplicity of some of the assumptions it is truly surprising that the analytical results agree so closely with the experimental curve. It may be helpful for the writer to enumerate three of the assumptions which he feels are questionable in the hope that it may lead to a clearer understanding of the mechanics of energy dissipation at the base of the fall:

1. The assumption of uniform velocity V_m in the thickened jet (see Fig. 18) is known to be in error. Velocity measurements taken in the inclined jet indicated a transition zone of considerable width in which the velocity increased from zero in the pool to the maximum in the jet. The deviation from a uniform velocity at this location is of greater importance than at d_a or d_b (see Fig. 16) since it would materially affect the distribution of flow between Q_f and Q_1.

2. The assumption that the presence of the pool does not affect the distribution of flow between the jet shooting downstream and the jet shooting upstream is doubtful. The presence of the pool results in a nonuniform velocity profile in the inclined jet as previously mentioned. The hydrostatic pressure of the water in the pool gives the falling jet a downstream acceleration before it reaches the apron floor and therefore tends to reduce the effective angle at the point of impingement. This effect may be noted in Fig. 1.

3. The assumption that all of the energy in the upstream part of the jet is dissipated in the pool after striking the apron is believed to be in error. It is the writer's belief that the region of maximum energy dissipation is in the transition zone between the pool and the jet where the velocity gradient has its maximum value.

The foregoing comments should not be interpreted as detracting in any way from the value of Professor White's analysis. They are presented merely to emphasize the fact that, although the analysis agrees strikingly well with the experimental results, it does not follow that the assumptions used give an accurate picture of the mechanics of energy dissipation at the base of an overfall.

Messrs. Bakhmeteff and Feodoroff raised the most serious question in regard to the accuracy of the data that occurred in any of the discussions. Judging from their experience in hydraulic laboratory study of similar flow patterns, and the appearance of the velocity profiles in Fig. 6, they concluded that the presence of entrained air in the flow streaming from the base of the fall resulted in abnormally low velocity readings and consequently an exaggeration of the measured energy losses. The reasoning leading to this conclusion and the data on which it was based were very well presented.

At the time the data for the paper were being taken the danger of trouble from entrained air was recognized, and precautions were taken to reduce such inaccuracies to a minimum. The manometer lines were of large size and were brought up vertically to a glass air trap so that any air entering the pitot tube would rise quickly and be detected. Provision was made for flushing air from the system efficiently by a quick connection to the domestic water supply. The manometer system was frequently flushed as a matter of procedure, and any velocity readings which appeared inconsistent were remeasured immediately after flushing. The writer feels certain that errors due to the presence of air in the pitot-tube manometer system were negligible.

Air entrained in the flow streaming from the base of the fall unquestionably would have the effect of reducing the fluid density and thereby result in artificially low velocities as measured by the water manometer. The writer believes that the magnitude of this effect is small and that certainly energy summations based on velocity measurements are less in error than those based on measurement of average depth and assumption of uniform velocity. Admittedly, in either case the tendency is to exaggerate the energy loss.

The uniform velocity profiles (*c*), (*d*), and (*e*), shown in Fig. 20, are taken from the Columbia experiments at low relative fall heights $\left(\frac{h}{d} = 1.77\right)$. Similar results were obtained by the writer for low fall heights. Profile (*b*), Fig. 20, is for a fall height of 3.7. The average velocity shown for this profile is considerably less than the velocity measurements shown near the floor. If the energy at the section is based on the average velocity, a lower value will be obtained than if a summation is taken which gives weight to the high velocities measured near the bottom.

It is the writer's belief that the flow conditions in the Pasadena experiments were not disturbed as violently as in the Columbia experiments. With one exception, the experiments at the Columbia Laboratory which are presented in the discussion were made on unventilated falls. In the writer's experience this type of fall produces two conditions which differ from the ventilated type: (1) The under pressure behind the fall causes the jet to curve more rapidly and strike the apron at a steeper angle giving the same effect as a higher value of $\frac{h}{d_c}$; and (2) the pressure behind the fall fluctuates causing the path of the falling vein to pulsate. These conditions result in more violently disturbed flow on the apron below an unventilated fall and may partly explain the difficulties encountered in making reliable measurements for the falls studied in the Columbia Laboratory. In the writer's opinion, the flow pattern below the fall at the Pasadena Laboratory was not sufficiently disturbed to make velocity measurement impractical.

The results of the Columbia experiments are compared with the writer's results in Fig. 33. Curves (1), (2), (3), and (4) are repeated from Fig. 11. Values of $\frac{E_1}{d_c}$ for the Columbia experiments were computed from the $\frac{d_1}{d_c}$ values of Fig. 21(*a*) using Eq. 9. These values appear as curve (5) of Fig. 33 and show approximately twice the energy loss observed by the writer. Using Fig. 12, the corresponding values of $\frac{d_2}{d_c}$ for the Columbia experiments were calculated and plotted as curve (6) in Fig. 33. These values are considerably less than the values of $\frac{d_2}{d_c}$ obtained by the writer. The difference in the results may be traced to two factors: (1) The energy values from the Columbia experiments were based on measured depth and average velocities rather than measured velocities; and (2) for the most part the Columbia experiments dealt with unventilated nappes whereas the results reported by the writer were all from ventilated nappes.

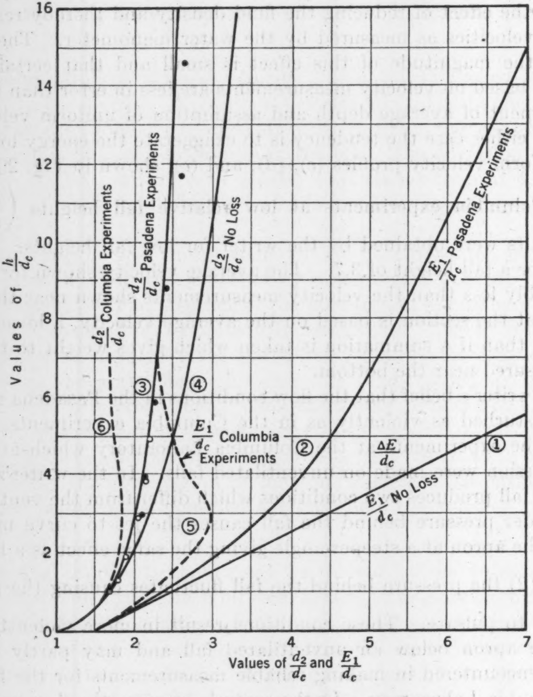

FIG. 33.—COMPARISON WITH COLUMBIA DATA

Referring again to Fig. 11 or Fig. 33, it will be noted that the plotted points adjacent to curve (3) are the experimental values of $\frac{d_2}{d_c}$ obtained with the hydraulic jump forming at the toe of the fall. The writer agrees with the statement made in the discussion by Messrs. Bakhmeteff and Feodoroff that experimental values of $\frac{d_2}{d_c}$, which may be measured in the tranquil region below the jump, form the best over-all check on the performance of the drop. Inasmuch as the experimental points are in such good agreement with the values calculated from curve (2), Fig. 11, the writer feels confident that the energy values represented by curve (2) are a reasonably accurate determination for the falls studied.

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ENTRAINMENT OF AIR IN FLOWING WATER

A SYMPOSIUM

PAGE

Open Channel Flow at High Velocities.
By L. Standish Hall, M. Am. Soc. C. E. 1394

Closed Conduit Flow.
By A. A. Kalinske, Assoc. M. Am. Soc. C. E., and James M. Robertson, Jun. Am. Soc. C. E. 1435

With Discussion by Messrs. Warren DeLapp, Karl R. Kennison, Robert T. Knapp, Carl E. Kindsvater, J. H. Douma, Joe W. Johnson, J. C. Stevens, D. C. McConaughy, T. J. Corwin, Jr., L. Standish Hall, and A. A. Kalinske and J. M. Robertson.

Note.—Published in September, 1942, *Proceedings*.

OPEN CHANNEL FLOW AT HIGH VELOCITIES

By L. Standish Hall,[1] M. Am. Soc. C. E.

Synopsis

One of the problems in the engineering design of chutes or steep spillway channels is the proper freeboard allowance to be made for the entrainment of air at high velocities. Calculations of the channel dimensions also must be modified with respect to the application of normal frictional resistance which is altered as the result of air entrainment. In all cases where water flowing in a steep chute is allowed to accelerate over a sufficient distance, air is mixed with the stream and the phenomenon of "white water" results. The same admixture of air with water can be observed in waterfalls or in natural stream channels where the velocity is sufficiently high to draw air into the water by its agitation.

Very few complete observations have been made for the measurement of the volume of air which will be mixed with the water under these conditions. Model tests are futile as sufficient velocity cannot be attained to create the phenomenon. To gather information as to the behavior of water flowing in steep channels and to measure the bulking of the water, some observations in actual spillway chutes were made in 1936 and 1937. The Pacific Gas and Electric Company granted permission for measurements to be made in three of their chutes and cooperated in maintaining uniform discharges during the period of the tests.

Incidental to the measurement of the bulking of the water, it was possible also to observe the performance of the stream while it flowed around vertical and horizontal curves in the spillway channels. The data gathered during these tests are presented in this paper, together with theoretical analyses leading to the development of a formula for computation of air bulking and general conclusions as to its application.

Field Procedure

Ordinary field methods cannot be used in measuring velocities ranging from 35 to 80 ft per sec. The initial test was made on Hat Creek No. 2 chute in a western state. The spillway channel was laid out for observation on July 11, 1936, and measurements were taken on the following day. To obtain more complete data on this chute a second set of observations was made on April 17, 1937.

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[1] Hydr. Engr., East Bay Municipal Utility Dist., Oakland, Calif.

The second test was made in Rapid Flume on July 20 and 21, 1936. The third test was made in the South Canal chute of another river project. The gages were located and levels run on April 22 to 24, 1937. The test was made on May 5.

The general field procedure was first to place gages along the chute channel for the purposes of measuring the velocity and cross-sectional area. The distances were measured along the actual slope of the chute rather than using the horizontal distances, stations being marked every 25 ft or 50 ft by scribing a paint mark and the station number on the side of the flume. For the concrete chutes, gages were stenciled on both inside surfaces of the channel at right angles to the bottom slope, using a prepared templet, the paint being a quick-drying black lacquer applied with a long-handled paint brush. Painted gages were not used in the wooden channel at Rapid Flume, but a measuring point was established on top of the flume and distances were measured down to the water surface. Levels were then run to each station and the elevation of the top of the chute wall recorded. At the same time the channel dimensions were checked.

The procedure for the actual observations, following this preliminary work, consisted of several steps: First, the discharge in the power canals above the chutes was gaged before and after the test. Four discharge measurements were usually made from a gaging bridge, using both the Price and Hoff current meters, two measurements being taken at the beginning of the test and two at the end. Observations were also taken at frequent intervals of a gage installed in the power canal to note any fluctuations in the water level. Then, after water had been turned into the chute and the flow had become steady, readings were taken on all gages along the channel. When the water is traveling at a high velocity with entrained air, particles of water leave the main body of the stream with the result that the water level is choppy and oscillates up and down. Therefore, observations were taken of the maximum and minimum levels at each station over a sufficient period of time to note the extreme range. A mean of these readings was then used as the average water level. If cross waves developed in the chute, due to curvature or entrance conditions, additional readings were taken across the channel by measuring down to the water surface from a board laid across the top of the chute.

Finally, velocity measurements were taken between various stations along the channel. Prior to making the measurements coloring matter was mixed, using either flourescein or potassium permanganate. Commercial flourescein was used, with lime first added to the mixing water to insure more complete solution of the crystals. About one teaspoonful of flourescein to a pint of water was used for each 240 cu ft per sec of discharge. For the potassium permanganate about ten teaspoonfuls to a pint of water were used for each 240 cu ft per sec of discharge. The colored solutions were poured into half-pint pasteboard containers prior to insertion in the stream channel. Before the test was begun, the observers were placed in their positions along the flume. Men with the dye solutions were stationed at the head of the flume or at the start of the course to be timed. A second observer with the revolver was located at the initial station, and the timers with stop watches were located at the lower station. At a given signal, the container with the dye was dropped into the head of the channel and the upstream observer noted the passing of the color at his station by firing the revolver into the air. The timers at the lower end of the stretch started their watches on this signal and stopped them when the center of the slug of color passed the lower station. Four to six observers with stop watches were used to time each color cloud, and the average of their observations was used in determining the mean time interval.

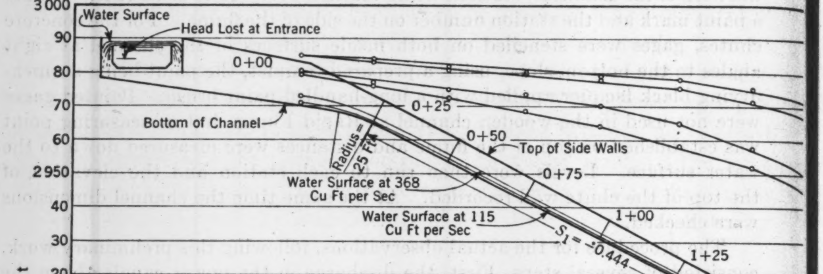
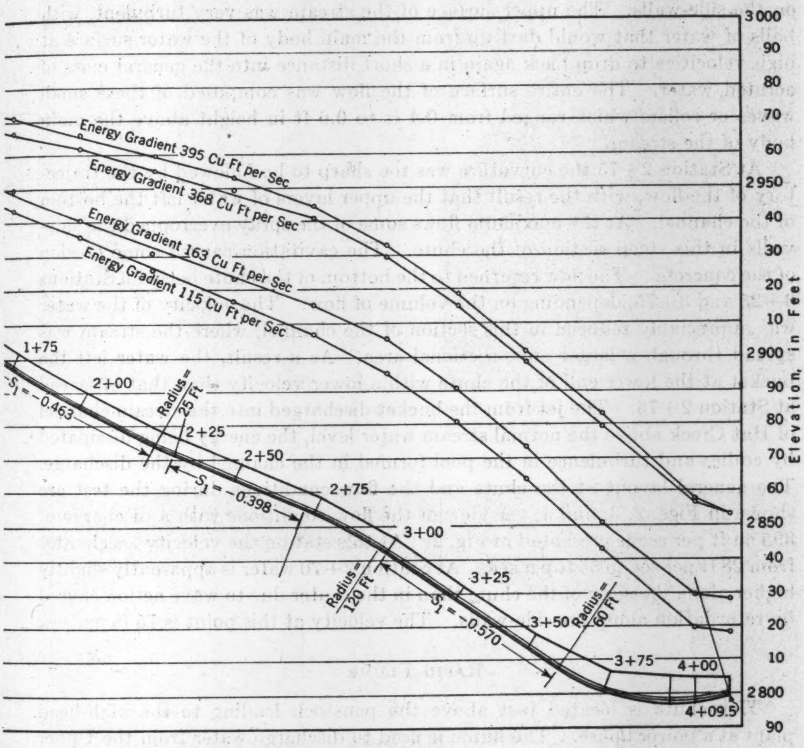


Fig. 1.—Observed Velocities, Longitudinal Wate

Conditions During Test

Hat Creek Chute.—The chute at the Hat Creek No. 2 power plant is located a short distance upstream from the penstock leading to the power house. This is a steep straight concrete channel having vertical curves and dropping 189 ft vertically in a distance of 409.5 ft measured along the channel. The waste water from the flume was discharged through side-channel spillway weirs located on both sides of the canal. After dropping over the weir lip the discharge was gathered into four collecting channels which unite at the head of the chute.

The concrete of the chute was rather rough, with pronounced form marks. The tie wires used in constructing the forms had not been clipped off from the concrete and these added to the roughness of the surface. The channel had a uniform width of 5.75 ft with side-walls 5 ft high except near the entrance and outlet where the height was increased.



SURFACE AND ENERGY PROFILES, HAT CREEK NO. 2 CHUTE

Due to two of the collecting channels entering the chute inlet at a right angle to the direction of flow, a cross wave was developed creating rather turbulent entrance conditions and also increasing the percentage of air in the water at this point. However, the cross wave and a portion of the entrained air had disappeared in about 100 ft. The profile of the chute is shown in Fig. 1. The first 275 ft of the chute consisted of tangents connected by short-radius vertical curves. Immediately below Station 2+75 there was a short convex vertical curve leading to a steep pitch having an inclination of 34° 45′. The lower part of the curve below Station 3+60 was a concave vertical curve forming the outlet bucket.

The flow in the chute accelerated to Station 2+75. In this section of the channel running waves formed along the side-wall and gave the appearance of higher water levels than in the center of the channel. However, a check made by measuring the depth to water in the center portion of the stream showed that the maximum height reached by the water was the same as that observed on the side-walls. The upper surface of the stream was very turbulent, with balls of water that would dart up from the main body of the water surface at high velocities to drop back again in a short distance into the general mass of aerated water. The entire surface of the flow was composed of these small waves or rollers which ranged from 0.4 ft to 0.6 ft in height above the main body of the stream.

At Station 2+75 the curvature was too sharp to be followed by the trajectory of the flow, with the result that the upper layers of water left the bottom of the channel. At the maximum flows some of the spray overtopped the side-walls in this steep section of the chute. The cavitation caused some erosion of the concrete. The flow returned to the bottom of the chute between Stations 3+25 and 3+75, depending on the volume of flow. The velocity of the water was appreciably reduced in this section of the channel, where the stream was spread through a larger cross-sectional area. As a result, the water left the bucket at the lower end of the chute with a lower velocity than that observed at Station 2+75. The jet from the bucket discharged into the stream channel of Hat Creek above the normal stream water level, the energy being dissipated by eddies and turbulence in the pool formed in the channel by the discharge. The general layout at the chute and the flow conditions during the test are shown in Figs. 2, 3, and 4. A view of the flow conditions with a discharge of 395 cu ft per sec is presented in Fig. 2. At this station the velocity accelerates from 28 ft per sec to 55 ft per sec. At Station 2+70 water is apparently slightly higher along the sides of the chute than in the center due to wave action caused by retardation along the side-walls. The velocity at this point is 75 ft per sec.

RAPID FLUME

This chute is located just above the penstock leading to the high-head plant at a power house. The flume is used to discharge water from the Upper Standard Canal down the slope into a natural stream channel leading to Lake Tabeaud, where it is used afterward through the low-head plant. It is a wooden chute having a horizontal curve to the right near the upper end and a concave vertical curve near the terminal end of the channel. The total vertical drop is 145 ft in a distance of 667 ft measured along the chute. The discharge of water from the Upper Standard Canal into the flume is controlled by a head gate. Water passes from the canal through a culvert pipe and drops into the tapered inlet of the flume, creating a considerable initial turbulence in the water. In addition to this tapered section the chute channel is built on a

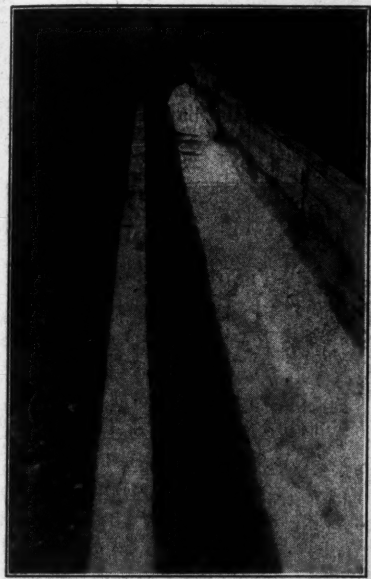


FIG. 2.—FACING UPSTREAM FROM STATION 1+25, HAT CREEK CHUTE; Q = 395 CU FT PER SEC

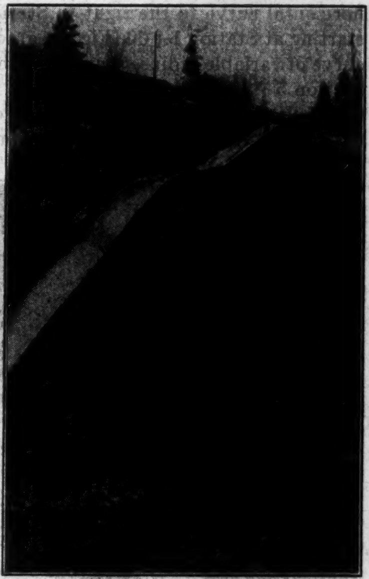


FIG. 3.—WATER LEAVES THE BOTTOM OF THE CHUTE AT AN ABRUPT CHANGE OF GRADE (HAT CREEK CHUTE; Q = 395 CU FT PER SEC)

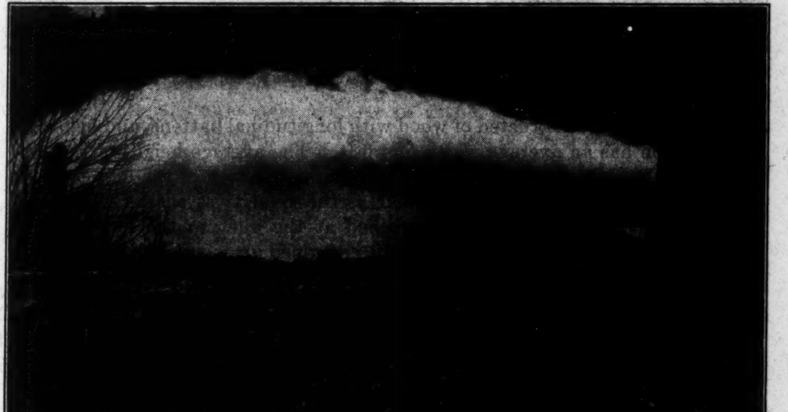


FIG. 4.—DISCHARGE OF 395 CU FT PER SEC FROM THE END OF HAT CREEK CHUTE; TERMINAL VELOCITY, 58 FT PER SEC

horizontal curve to the right to Station 2+00. A steep section of the chute, starting at Station 1+00 (Fig. 5), gradually merges into a long concave vertical curve of variable radius extending from Station 2+75 to Station 5+25. Below Station 5+25 the flume is built on a flatter grade down to the baffle box used to check the velocity. The water discharges from the baffle box into the natural drainage channel described previously.

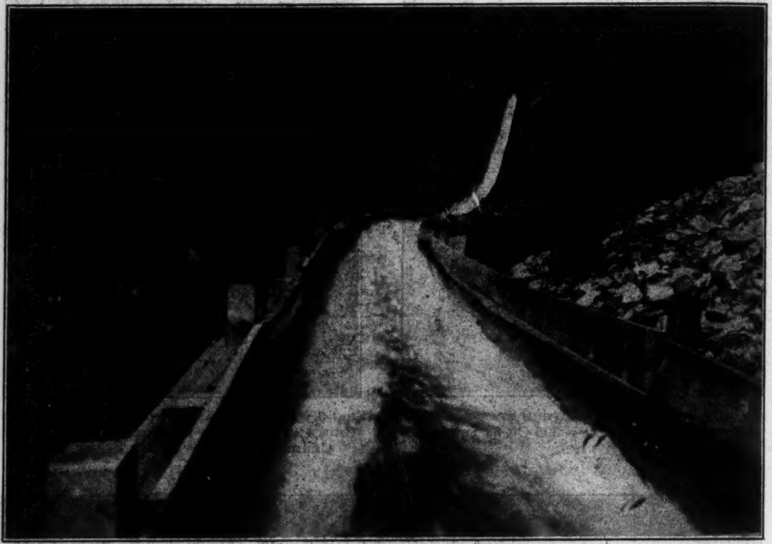


FIG. 5.—FACING DOWNSTREAM ALONG RAPID FLUME AT STATION 1+00; VELOCITY, 35 FT PER SEC

The channel is constructed of wood with longitudinal battens, and below the taper section has an average width of 4.6 ft. The average height of the side-walls is 2.4 ft. Cross waves develop in the upper section of the flume, due to the horizontal curve. These disappear as the velocity is accelerated on the steep straight section. The air entrained at the inlet to the flume due to entrance conditions is partly lost and the entrained air is again drawn into the water, the insufflation commencing along the side-walls of the flume until the entire cross section below Station 1+50 has become aerated. Where turbulent entrance conditions do not exist the entrainment of air commences along the side-wall, as described in 1936 by E. W. Lane,[2] M. Am. Soc. C. E.

The velocity in Rapid Flume accelerated to a maximum of 52 ft per sec at Station 3+75 but gradually diminished as the grade flattened until it was reduced to a velocity of 33 ft per sec as the water passed into the baffle box. The profile of the flume is shown in Fig. 6.

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[2] "Recent Studies of Flow Conditions in Steep Chutes," by E. W. Lane, *Engineering News-Record*, January 2, 1936, pp. 5–7.

SOUTH CANAL CHUTE

This chute is located at the end of the South Fork canal which picks up water below a power house. It is planned ultimately to use this water through Power Plant No. 6, but at the present time the flow is only used for irrigation. The water discharges down the chute into a natural channel. This channel is 3,535.25 ft long measured along the slope and has a total vertical drop of 140 ft. The discharge from the end of the South Canal is controlled by a radial gate set across the channel. Water can also discharge by a side channel spillway which drops the water back into the main chute channel within a distance of 100 ft below the radial gate. Water dropping into the chute from the side channel augmented the percentage of entrained air at the head of the chute. If no water had entered by this route, a much greater distance of channel would be traveled by the flow before the entire cross section was aerated.

The channel of the chute was constructed of concrete, the surface being rather rough and pitted, probably due to sand being carried down the chute at high velocities. On the bottom particularly, the cement and sand had been worn away, leaving the aggregate fragments exposed. From the head of the channel to Station 23+00 the width of the cross section was 10 ft. From Station 23+00 to Station 23+50 there was a tapered section and from the latter station to the end of the chute the channel had a width of 7 ft. The side-walls on the 10-ft section varied in height from 3.5 ft to 4.5 ft whereas on the 7-ft section the height of the side-walls varied from 3.5 ft to 5.25 ft.

There are thirteen horizontal curves of radii varying from 100 ft to 300 ft, the sharpest having an internal angle of 98° and a radius of 142 ft. The floor of the channel on these curves is not superelevated on the outer edge. An opportunity was afforded here to observe the flow of water around curves. The total angular deflection in the entire chute was 422° 32′. As stated, the entire length of channel was 3,535.25 ft, of which 1,328.8 ft was located on curves. There are numerous changes in grade, including three re-entrant grades inserted to decelerate the water at the toe of steep sections. Except for a steep pitch at the inlet, the steepest bottom slope is 0.100 for about 300 ft. The other principal grades are 700 ft at 0.055, 600 ft at 0.030, 200 ft at 0.022, 600 ft at 0.011, 715 ft at 0.025, and 170 ft in three short lengths at an average slope of 0.080.

A total time of 2 min 20 sec was required for a flow of 334 cu ft per sec to pass from the head of the chute to the outlet, the average velocity being about 25 ft per sec. Due to the many changes in grade, the velocity in various sections of the channel varied from 17 to 35 ft per sec. The profile of the channel is not shown, due to its great length.

As the channel bottom on the curves was not superelevated, the water piled up on the outside of each curve and produced standing cross waves in the tangent section below the curve (see Fig. 7), in addition to the choppy roller and surface waves elsewhere noted in normal chute flow. These cross waves lead to an error in computing the wetted area at some sections as only side gage readings were taken. Such errors are not of sufficient magnitude to affect the results for the entire chute. In the 10-ft width of channel the superelevation of the water surface at the outer side of the curve did not form a uniform slope across the entire cross section, due to a larger percentage of the flow following the outer wall. However, on the 7-ft channel widths the slope of the water surface was more or less uniform so that the difference in level between the gage readings on the outer and inner walls of the curves indicated the slope of the water surface across the stream section. On the curve of 142-ft radius, in the 7-ft section, constructed on a 10% grade, the flow was very turbulent (see Fig. 8). The discharge in this section tended to proceed in a series of chords with a result that water crowded at certain points on the outside of the curve, permitting spray to discharge over the top of the wall. Between these points the water surface was well below the top of the wall.

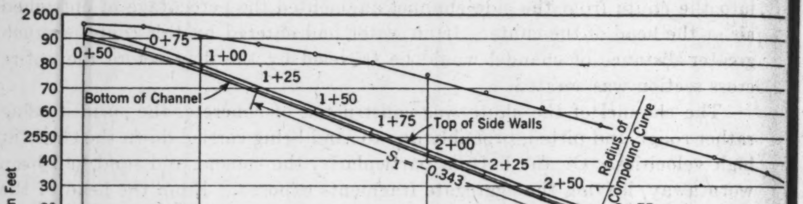


FIG. 6.—OBSERVED VELOCITY, LONGITUDINAL WATER SURFACE AND

NOTATION

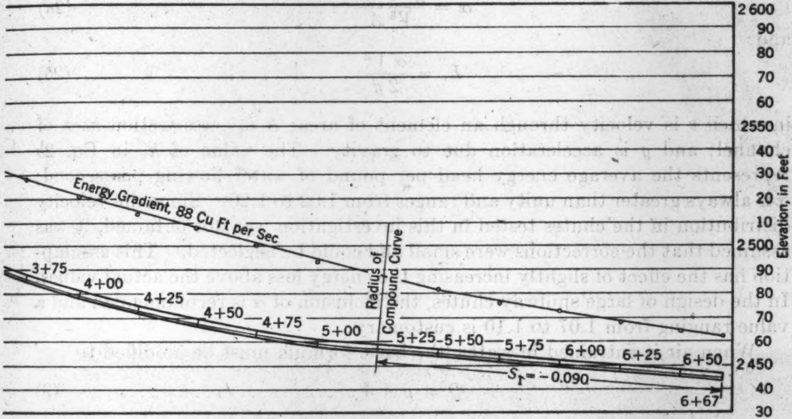
The letter symbols in this paper are defined where they first appear and are assembled for convenience of reference in the list of symbols appended to the Symposium.

THEORY OF FLOW IN CHUTES

The objective, desired to be gained by the tests, was the determination of a formula for the calculation of air entrainment in projected spillway channels of large capacity. To be of practical application, the method of procedure should satisfy three requirements: (1) It should conform as nearly as practicable to established methods of hydraulic calculations; (2) it should be of sufficient simplicity to be readily applied; and (3) it should yield results that are in reasonable conformity with the measured tests of air entrainment.

Meager data available indicated that with small concrete chutes, having velocities ranging from 15 to 40 ft per sec, the entrained air varied from 15% to 35% of the cross section.[3] Measurements of a masonry lined chute in Austria gave air entrainment ranging from 60% to 80%.[4] The use of such large percentages for air bulking in design would make large spillway chutes for the discharge of river floods very costly.

It was indicated that the velocity, the distance traveled, and the roughness of the channel had some bearing on the air entrainment. Personal observation had led to the conclusion that air bulking was reduced with larger cross sections and greater volumes of discharge.



Energy Profiles, Rapid Flume, Tabeaud Reservoir

In developing the theory of flow in chutes the following assumptions were made:

1. The value of the roughness coefficient *n* in the Manning or Kutter formula is constant for the particular type of material of which the chute is constructed;

2. The air in and above the water caused no additional loss in energy, the reduction in the specific gravity compensating for the added area;

3. For computation, a computed hydraulic radius can be used

$$R_c = \frac{Q}{V P_c} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

with a smaller value of *n* than normal; and

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[3] "The Determination of *n* in Kutter's Formula for Various Canals, Flumes, and Chutes on the Boise Project and Vicinity," by W. G. Steward, U. S. Bureau of Reclamation, Rept. on 2d Annual Conference of Operating Men held at Boise, Idaho, January, 1913.

[4] "Overflow Channels from Surge Tanks," by Oberbauret Dr. Ing. Karl Innerebner, World Power Conference, 1924, Vol. II, p. 481.

4. The velocity head (h_v) computed from the mean velocity can be used without substantial error.

In Eq. 1, Q is the discharge in cubic feet per second; V is the observed average velocity in a section, in feet per second; and P_e is the computed wetted perimeter of the section of water only, $A_e = \frac{Q}{V}$.

The true velocity head h_v is greater than that computed from the mean velocity, since the true kinetic energy is the summation of the product of the velocity head and the weight of water for elemental areas, a, in the cross section. Then

$$\alpha = \frac{\int v^3\, da}{V^3 A} \qquad (2a)$$

and

$$h_v = \frac{\alpha V^2}{2g} \qquad (2b)$$

in which v is velocity through an element of area; A is cross-section area of channel; and g is acceleration due to gravity. The value of h_v in Eq. 2b represents the average energy head per pound of water flowing per second; α is always greater than unity and ranges from 1.02 to 1.20. Since the velocity distribution in the chutes tested in this investigation was not obtained, it was assumed that the corrections were small and could be neglected. This assumption has the effect of slightly increasing the energy loss above the actual values. In the design of large spillway chutes, the inclusion of α is recommended and a value ranging from 1.07 to 1.10 is customary.

When air is entrained in water, the basic formula must be modified to

$$Q = \rho A V \qquad (3)$$

in which ρ = the ratio of water in a mixture of air and water or the relative density of the mixture ($1 - \rho$ = ratio of air). Assuming that the flow in a channel has reached a uniform velocity and neglecting internal friction of the water, the relation between the force represented by the weight of the water and the frictional resistance (f) of the channel walls for a cross-sectional area of elemental length can be expressed as $\rho A\, dl\, \omega \sin\theta = f P\, dl$, or

$$f = \frac{\rho A \omega \sin\theta}{P} \qquad (4)$$

in which l is the length measured along the channel; ω is the weight of 1 cu ft of water (= 62.5 lb); θ is the slope of the channel in degrees; and P is the wetted perimeter for a mixture of air and water.

When air is entrained in the water, the normal assumption given in most hydraulic textbooks as to the relation between the friction and the velocity for uniform flow must be modified as the entrained air reduces the density of the mixture and the friction correspondingly,[5]

$$f = k \rho V^2 \qquad (5)$$

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[5] "Applied Hydro- and Aero-Mechanics," by L. Prandtl and O. G. Tietjens, Engineering Societies Monographs, McGraw-Hill Book Co., Inc., 1934, p. 71.

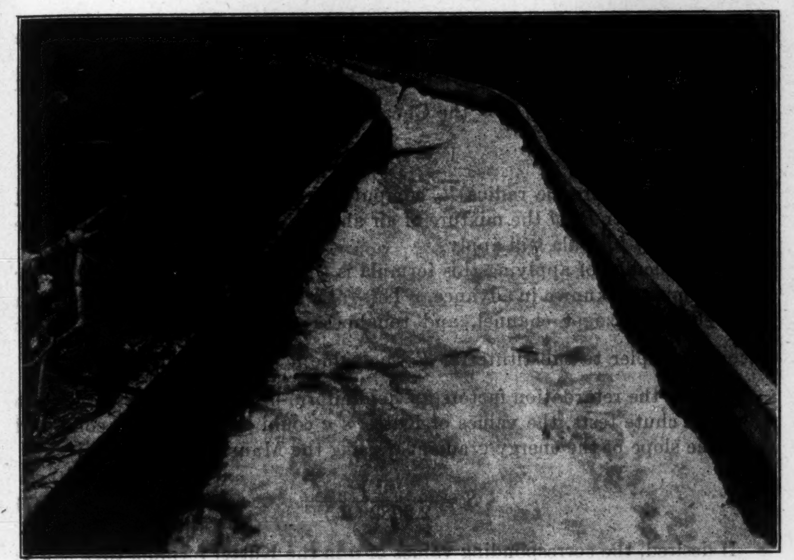

FIG. 7.—FACING UPSTREAM FROM STATION 7+75, SOUTH CANAL CHUTE

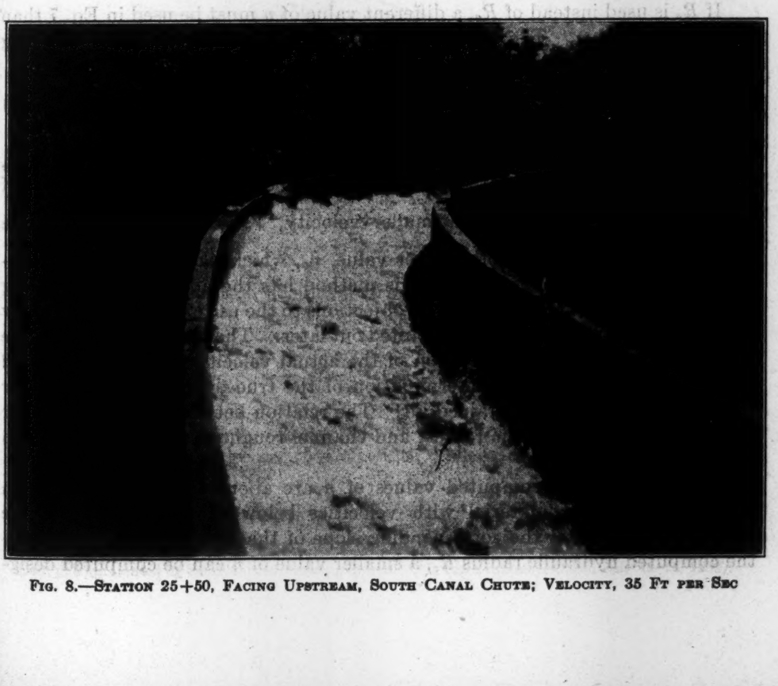

FIG. 8.—STATION 25+50, FACING UPSTREAM, SOUTH CANAL CHUTE; VELOCITY, 35 FT PER SEC

in which k is a coefficient. The substitution of $R = \frac{A}{P}$ and $S_1 = \sin \theta$, the sine of slope of the channel, can be made in Eq. 4; then $k \rho V^2 = \rho R \omega S_1$ and $V = \sqrt{\frac{\omega}{k} R S_1}$ but $\sqrt{\frac{\omega}{k}}$ = the Chézy C; hence

$$V = C \sqrt{R S_1} \qquad (6)$$

The observed hydraulic radius R_o computed from the observed area A_o and wetted perimeter P_o of the mixture of air and water must be used. Then, the normal Chézy formula will apply.

One difficulty of applying this formula is that the ratio of air in the water $(1 - \rho)$ must be known in advance. This ratio is not constant but varies with the velocity, size of channel, and roughness of the side-walls. Hence, it becomes simpler to substitute for R_o the value R_c, $A_c = \frac{Q}{V}$ and $R_c = \frac{A_c}{P_c}$.

To find the retardation factors for nonuniform flow with the data available from the chute tests, the values of Kutter's n could be most easily computed from the slope of the energy gradient S using the Manning formula as

$$S = \frac{n^2 V^2_o}{2.2082 \, (R_o)^{4/3}} \qquad (7)$$

If desired, the corresponding values of n in the Kutter formula then can be determined by the relation between Manning's n and Kutter's n for various values of R.

If R_c is used instead of R_o, a different value of n must be used in Eq. 7 than the nominal value. The relation between the values of n for the Manning formula would be

$$n_c = n \left(\frac{R_c}{R_o} \right)^{2/3} \qquad (8)$$

Since R_c is always less than R_o, n_c must be smaller than n. Values of n_c, therefore, were also calculated. Some writers have recommended the use of the observed area A_o with a smaller velocity, $V_c = \frac{Q}{A_o}$. This procedure would necessitate the use of a larger value, n_o, which likewise has been computed from the available data. This method has the advantage of utilizing the actual area, but there are several objections to the use of a fictitious velocity less than the actual which will be pointed out later. The most logical procedure would appear to be the calculation of the actual velocity from the calculated depth without air and the determination of the true depth of air plus water from the percentage of entrained air. The relation between the entrained air and the velocity, hydraulic radius, and channel roughness is determined from experimental data.

Using Eq. 7, the computed values of n are about the same as those in channels of similar material with velocities below critical. By using the observed velocity V_o, the corresponding slope of the energy gradient, S_o, and the computed hydraulic radius R_c, a smaller value of n can be computed designated here as n_c. On the other hand, by using the observed hydraulic radius R_o, the computed velocity V_c and the corresponding slope of the energy gradient S_c, a larger value of n can be computed, designated here as n_o:

$$n_c = 1.486 \frac{(R_c)^{2/3} (S_o)^{1/2}}{V_o} \ldots\ldots\ldots\ldots\ldots\ldots (9a)$$

and

$$n_o = 1.486 \frac{(R_o)^{2/3} (S_c)^{1/2}}{V_c} \ldots\ldots\ldots\ldots\ldots\ldots (9b)$$

The increment in velocity as flow progresses down the chute may be expressed as a function of the time. However, this leads to an equation involving hyperbolic functions that are awkward to use.[6]

A simpler solution can be derived by expressing the increment in velocity as a function of the distance traveled. Such an equation will be developed applying to a rectangular section of uniform width, b. Since many chutes are of this type, this formula will have wide application.

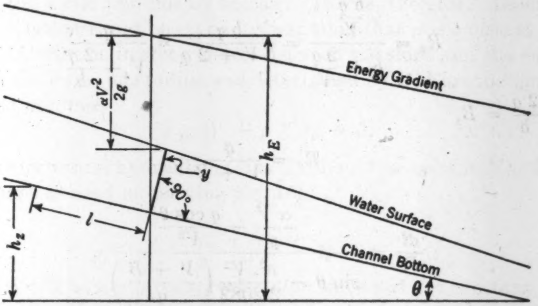

FIG. 9

Applying Bernoulli's theorem to Fig. 9,

$$h_E = h_z - l \sin\theta + y \cos\theta + \frac{\alpha V^2}{2 g} \ldots\ldots\ldots\ldots\ldots (10)$$

α being the velocity head coefficient from Eq. 2b.

Taking the derivative of h_E with respect to l,

$$\frac{dh_E}{dl} = -\sin\theta + \cos\theta \frac{dy}{dl} + \frac{\alpha V}{g} \frac{dV}{dl} \ldots\ldots\ldots\ldots\ldots (11)$$

In a rectangular channel, by substituting the computed depth $y_c = \rho\, y$ for the value of y, the discharge per foot of width q becomes $q = V y_c$; but

$$dy_c = -\frac{y_c\, dV}{V} = -\frac{q\, dV}{V^2} \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

---

[6] "The Flow of Water in Channels Under Steep Gradients," by W. F. Durand, *Transactions*, A.S.M.E., 1940, p. 9 (presented at Semi-Annual Meeting of A.S.M.E., San Francisco, Calif., July 10–15, 1939).

Substituting Eq. 12 in Eq. 11,

$$\frac{dh_E}{dl} = -\sin\theta - \frac{\cos\theta\, q}{V^2}\frac{dV}{dl} + \frac{\alpha\, V}{g}\frac{dV}{dl} \tag{13}$$

but $S = \frac{dh_E}{dl}$, and substituting the value of S as derived from Eq. 9*a*,

$$\frac{dh_E}{dl} = -\frac{n^2{}_c\, V^2}{2.2082\,(R_c)^{4/3}} \tag{14}$$

The negative sign is used in Eq. 14 since h_E decreases with an increase in l. Equating Eq. 13 and Eq. 14 and solving for $\frac{dl}{dV}$

$$\frac{dl}{dV} = \frac{\frac{\alpha\, V}{g} - \frac{q}{V^2}\cos\theta}{\sin\theta - \frac{n^2{}_c\, V^2}{2.2082\,(R_c)^{4/3}}} \tag{15}$$

Since R_c is a function of V, this value can be substituted in Eq. 15

$$R_c = \frac{\frac{b\, q}{V}}{b + \frac{2\, q}{V}} = \frac{b\, q}{b\, V + 2\, q} = \frac{q}{V + \frac{2\, q}{b}} \tag{16a}$$

or letting $\frac{2\, q}{b} = B$,

$$R_c = \frac{q}{V + B} \tag{16b}$$

Then

$$\frac{dl}{dV} = \frac{\frac{\alpha\, V}{g} - \frac{q\cos\theta}{V^2}}{\sin\theta - \frac{n^2{}_c\, V^2}{2.2082}\left(\frac{V + B}{q}\right)^{4/3}} \tag{17}$$

This is the basic equation for flow in steep rectangular chutes as derived from the simplified relation $q = V\, y_c$ rather than the more general equation

$$q = \rho\, V\, y_o \tag{18}$$

If the latter is used, three variables are introduced making the solution more difficult. Then,

$$\rho\, y_o\, dV + y_o\, V\, d\rho + \rho\, V\, dy_o = 0 \tag{19a}$$

$$dy_o = -\left(\frac{\rho\, y_o\, dV + y_o\, V\, d\rho}{\rho\, V}\right) \tag{19b}$$

A solution cannot be made unless $\rho = f(V)$. Then $d\rho = f_1(V)\, dV$.

In Eq. 19*b*, substitute $y_o = \frac{q}{\rho\, V}$ and $\rho = f(V)$:

$$dy = -\left[\frac{\rho\, q\, dV}{(\rho\, V)^2} + \frac{q\, V\, d\rho}{(\rho\, V)^2}\right] = -\left\{\frac{f(V)\, q\, dV}{[f(V)]^2\, V^2} + \frac{q\, V\, f_1(V)\, dV}{[f(V)]^2\, V^2}\right\}$$

$$= -\frac{q}{[f(V)]^2\, V^2}[f(V) + V\, f_1(V)]\, dV \tag{20}$$

Substitution of Eq. 20 in Eq. 11 gives

$$\frac{dl}{dV} = \frac{\dfrac{\alpha\, V}{g} - \dfrac{q \cos \theta}{[f(V)]^2\, V^2}\,[f(V) + V\, f_1(V)]}{\sin \theta - \dfrac{n^2\, V^2}{2.2082\, (R_o)^{4/3}}} \ldots\ldots\ldots\ldots (21)$$

The use of Eq. 21 would be difficult.

Returning to the basic Eq. 17, the solution for $\frac{dl}{dV}$ using a constant value of n_c is not exact, since, by Eq. 8, $n_c = n \left(\frac{R_c}{R_o}\right)^{2/3}$. Hence, it is seen that n_c is actually a function of ρ.

To determine an exact value of n for chute computations, several assumptions may be made. It may be assumed that the slope of the energy gradient may be separated into two components representing frictional resistance and air resistance using the computed hydraulic radius R_c.[7] This solution has a fictitious character since the air resistance is based on the area occupied presumably by the water and not by the air. It was, therefore, discarded.

Another solution could be made by assuming that ρ is constant for a given value of θ. (A relation in steep chutes between the slope and the entrained air varying with the hydraulic radius was determined by R. Ehrenberger in 1930.)[8] Then Eq. 19*a* becomes

$$\rho\, y\, dV + \rho\, V\, dy = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

In this case, the actual hydraulic radius R_o should be used in Eq. 16*a*. Then by similar steps as used in deriving Eq. 16*b*,

$$R_o = \frac{q}{\rho\, V + B} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (23)$$

This solution is at best an approximation since ρ is not constant for a given value of θ but varies with the velocity. In many steep chutes the velocity accelerates throughout the length of the channel.

The most satisfactory solution was obtained by using the Froude number. The air entrained in the water would appear to bear some relation to the Froude number, as air entrainment is essentially related to the kinetic energy of the stream. The proper application of the dimensionless parameter to hydraulic phenomena has previously been pointed out[9] by Boris A. Bakhmeteff, M. Am. Soc. C. E., and Arthur E. Matzke, Assoc. M. Am. Soc. C. E.

In the present case, the frictional resistance, in addition to the kinetic energy, has a definite bearing on the air entrainment. Hence, it is not incongruous to use the hydraulic radius R_c as the longitudinal parameter.

After some experimenting with the data, it was found that the best relation with the Froude number existed by using the ratio of the two ratios, air to water, in the mixture, or $\frac{1 - \rho}{\rho}$.

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[7] Unpublished report of U. S. Bureau of Reclamation.

[8] *Die Wasserwirtschaft*, Vol. 23, 1930, pp. 573 and 595.

[9] "The Hydraulic Jump in Terms of Dynamic Similarity," by Boris A. Bakhmeteff and Arthur E. Matzke, *Transactions*, Am. Soc. C. E., Vol. 101 (1936), p. 635.

Experimental data were plotted, as will be shown subsequently in the paper, using $\frac{1-\rho}{\rho}$ as ordinates and $\frac{V^2}{g R_c}$ as abscissa, and the following general relation was found to exist:

$$\frac{1-\rho}{\rho} = K \frac{V^2}{g R_c} - K_1 \qquad (24a)$$

in which K is a coefficient and K_1 is a constant for air entrainment. With much of the data it was found that $K_1 = 0$, but with smooth entrance conditions K_1 has a finite value. By transposition ρ can be determined, assuming $K_1 = 0$, as follows:

$$\rho = \frac{1}{1 + \frac{K V^2}{g R_c}} \qquad (24b)$$

or, substituting for R_c its value from Eq. 16*b*,

$$\rho = \frac{1}{1 + \frac{K V^2}{g q}(V + B)} \qquad (24c)$$

Dividing Eq. 16*b* by Eq. 23

$$\frac{R_c}{R_o} = \frac{\rho V + B}{V + B} \qquad (25)$$

Substituting the value of $\frac{R_c}{R_o}$ from Eq. 25 in Eq. 8 and replacing ρ by its value from Eq. 24*c*,

$$n_c = n \left[\frac{1 + \frac{B}{q}\frac{K V^2}{g}}{1 + \frac{K V^2}{g}\left(\frac{V + B}{q}\right)} \right]^{2/3} \qquad (26)$$

Eqs. 24*b* and 26 constitute basic formulas for the solution of problems of air entrainment in spillway chutes. The use of values of n_c, obtained from Eq. 26, in Eq. 17 would appear to afford the most rational theoretical solution for flow in rectangular chutes.

In the design of chutes having a trapezoidal section, or in fact any shape other than rectangular, and in transition sections of any form in which the width varies, Eq. 17 cannot be applied. Similar formulas can be developed, but they are too cumbersome for practical use. An easier solution is obtained by a trial-and-error method.

Starting with Bernoulli's theorem from Eq. 10, the loss of energy between two adjacent stations can be expressed using subscripts 1 and 2 to indicate upstream and downstream sections, respectively. Equating this to the slope of the energy gradient in the Manning formula, the following equation results:

$$\frac{n^2_c Q^2}{2.2082 A^2_c (R_c)^{4/3}} = \left(\frac{y_{c1} - y_{c2}}{l}\right) \cos\theta + \sin\theta + \frac{\alpha}{2 g l}(V^2_1 - V^2_2) \qquad (27)$$

In using Eq. 27, a value of y_{c2} is first assumed and the corresponding value of V_2 is calculated. The value of R_c to be used in this computation is the mean of the hydraulic radii at the two sections. If the two sides of the equation do not balance with the assumed value of y_{c2}, another value must be tried. Usually after two incorrect assumptions, the true value can be arrived at by interpolation. Eq. 27 can be used for rectangular sections, if desired. Calculations are facilitated by first preparing a series of curves of the several functions of each section for varying depths. These must be prepared for each discharge to be calculated. With the discharge Q fixed, values of A_c, R_c, P_c, and V can be determined readily. The ratio of water (ρ) can be calculated from Eq. 24a with the values of V and R_c for specific depths y_c, using the values of K and K_1 assumed for the design. From the ratio of water in the section and the computed area, the actual area of the cross section can be computed:

$$A_o = \frac{A_c}{\rho} \quad \text{(28a)}$$

For trapezoidal sections, the actual depth of air and water y_o can be found by the formula

$$y_o = \frac{-b_1 + \sqrt{b^2_1 + \dfrac{4\,s\,Q}{\rho\,V}}}{2\,s} \quad \text{(28b)}$$

in which b_1 = bottom width and s = side slopes in trapezoidal sections. Values of A_o, R_o, and P_o then can be readily determined. For use in Eq. 27 values of n_c are calculated from Eq. 8. For practical use, a curve of the function n_c can be plotted against the depth y_c. The curve can be marked off to indicate increments in the depth for changes in n_c to the nearest 0.0005. These values are sufficiently accurate and avoid too frequent change in the value of n_c, thereby facilitating the computations.

Friction loss computations in the Manning formula can be made most readily by first preparing a curve of the function 1.486 $A_c\,(R_c)^{2/3}$ plotted against the depth y_c. Once a value of y_{c2} is assumed the friction loss can be computed in the left-hand member of Eq. 27 from the values of n_c and of 1.486 $A_c\,(R_c)^{2/3}$ interpolated from the curves.

The computations of flow down the incline must be started from the head of the chute. With smooth entrance conditions, the computations can start from the critical section. When the velocity factor α is used, the formula for critical depth usually given in hydraulic texts must be modified by the introduction of this term:[10]

$$Q\sqrt{\frac{\alpha}{g}} = A_c\sqrt{\frac{A_c}{b_c}} \quad \text{(29)}$$

in which b_c = water surface width in trapezoidal sections.

An advantage of determining the calculated areas of solid water is realized in the design of jump basin, which in many cases forms the chute terminal.

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[10] Compare with formula for critical depth in "Hydraulics of Open Channels," by Boris A. Bakhmeteff, Engineering Societies Monographs, McGraw-Hill Book Co., Inc., 1932, p. 35.

Here, the values of y_c and A_c together with the actual water velocity V_o at the toe of the chute should be used in the energy and momentum equations of the hydraulic jump. If the velocity factor α has been used in the energy equation, a similar factor based on the summation of V^2 over the cross section must be applied to the momentum equation. This factor is smaller than α and values ranging from 1.04 to 1.07 are generally applied. (The correction to the average velocity applicable to the energy and momentum equations has been determined by Coriolis, De Saint Venant, and others.)[11]

TABLE 1.—GAGE READINGS, HAT CREEK NO. 2 CHUTE

Station	South Side		North Side		Mean	South Side		North Side		Mean
	High (1)	Low (2)	High (3)	Low (4)	(5)	High (1)	Low (2)	High (3)	Low (4)	(5)
Velocity	(a) 115 Cu Ft per Sec					(b) 163 Cu Ft per Sec				
0+00	1.2	1.0	2.0	0.6	1.20	2.0	1.6	3.2	0.6	1.85
0+25	1.7	1.5	1.6	1.0	1.45	2.2	1.8	2.0	1.4	1.85
0+50	1.4	1.1	0.6	0.4	0.88	1.3	1.0	1.0	0.6	0.97
0+75	0.9	0.6	1.2	0.9	0.90	1.1	0.9	1.4	1.1	1.13
1+00	0.7	0.4	1.0	0.8	0.72	1.0	0.7	1.2	0.9	0.95
1+25	0.8	0.6	0.9	0.7	0.75	1.1	0.8	1.2	0.9	1.00
1+50	1.0	0.8	0.9	0.6	0.82	1.1	0.8	1.1	0.8	0.95
1+75	1.2	0.9	0.9	0.7	0.92	1.1	0.8	1.3	1.0	1.05
2+00	1.1	0.8	1.0	0.8	0.92	1.0	0.8	1.3	0.9	1.00
2+25	0.9	0.7	1.1	0.9	0.90	1.1	0.9	1.2	0.9	1.02
2+50	0.9	0.7	1.1	0.9	0.90	1.1	0.9	1.4	1.0	1.10
2+75	1.0	0.8	1.0	0.7	0.88	1.1	0.9	1.4	1.0	1.10
3+00[a]	...	...	1.8	0.8	1.55	3.3	1.2	3.4	1.3	2.30
3+25[a]	...	...	1.1	0.8	1.08	2.9	1.0	3.2	1.2	2.08
3+50[a]	1.2	0.8	1.2	0.8	1.00	2.3	0.8	2.0	1.3	1.60
3+75	...	...	1.1	0.9	0.85	1.2	0.8	1.5	1.1	1.15
4+00	...	...	1.0	0.8	0.90	1.4	1.0	1.8	1.4	1.40
4+09.5	1.2	0.8	1.0	0.8	0.95	1.8	1.3	1.6	1.0	1.42
Velocity	(c) 368 Cu Ft per Sec					(d) 395 Cu Ft per Sec				
0+00	4.4	3.8	4.0	3.2	3.85	4.5	3.9	4.0	3.6	4.00
0+25	3.2	2.6	3.6	2.6	3.00	3.3	2.7	3.8	2.8	3.15
0+50	2.8	2.0	1.9	1.5	2.05	3.2	2.4	2.0	1.8	2.35
0+75	1.9	1.6	2.1	1.7	1.82	2.1	1.7	2.3	1.9	2.00
1+00	1.8	1.4	2.0	1.5	1.68	2.0	1.6	2.1	1.7	1.85
1+25	1.9	1.6	1.9	1.3	1.68	2.1	1.7	1.9	1.5	1.80
1+50	1.8	1.5	1.8	1.2	1.58	2.1	1.5	2.0	1.5	1.77
1+75	1.9	1.4	1.9	1.5	1.68	2.1	1.5	2.2	1.6	1.85
2+00	1.8	1.4	1.8	1.4	1.60	2.1	1.5	2.1	1.5	1.80
2+25	1.8	1.5	1.9	1.4	1.65	2.1	1.5	2.1	1.5	1.80
2+50	1.9	1.7	2.0	1.5	1.78	2.2	1.6	2.2	1.6	1.90
2+75	1.8	1.5	2.0	1.4	1.68	2.0	1.5	2.3	1.6	1.85
3+00[a]	...	...	3.2	2.4	2.90	4.2	2.6	4.4	2.9	3.40
3+25[a]	...	...	4.4	3.2	3.80	6.0	3.0	6.0	3.4	4.60
3+50[a]	...	...	2.4	1.6	2.00	4.8	2.0	5.0	2.2	3.50
3+75	1.7	1.5	1.9	1.7	1.70	2.1	1.6	2.3	1.9	1.85
4+00	1.7	1.5	1.9	1.7	1.70	2.2	1.6	2.2	1.9	1.90
4+09.5	1.5	1.1	1.9	1.7	1.55	2.4	2.0	2.4	1.9	2.20

[a] Between these stations the flow did not follow the bottom of the flume because the vertical curve was sharper than the trajectory of the flowing water.

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[11] See *Annales des Ponts et Chaussées*, 1836; also *Transactions*, A.S.M.E., Vol. 54, 1932, p. 57.

Calculation of the jump basin dimensions in this manner will avoid the inadequate lengths and depths encountered in many constructed designs.

Another of the problems in the design of chutes is the determination of the path of the water around vertical and horizontal curves. Concave vertical curves present no difficulty in design, except for stress created by the centrifugal force. Convex vertical curves must not be made sharper than the trajectory that would be followed by the high velocity water under the action of gravity. If the flume bottom is steeper than this trajectory, the water will leave the bottom and may rise above the top of the side-walls as observed at Hat Creek chute. The trajectory will be a parabola tangent to the slope of the chute.

TABLE 2.—CORRECTIONS TO OBSERVED TIMES ON FULL RUNS IN THE LOWER PART OF HAT CREEK NO. 2 CHUTE

TEST DATE, JULY 12, 1936				TEST DATE, APRIL 17, 1937								
Full Run Stations		Time (Sec)		Full Run Stations		Time (Sec)		Partial Run Stations		Time (Sec)		Ratio, $\frac{\text{Col. 11}}{\text{Col. 12}}$
Upper	Lower	Observed[a]	Corrected	Upper	Lower	Observed[a]	Corrected	Upper	Lower	Observed[a]	Comparative[b]	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
115 CU FT PER SEC				163 CU FT PER SEC				163 CU FT PER SEC				
0+25	1+25	2.48	2.48	0+25	1+25	2.27	2.27	1+25	2+25	1.91	2.02	95
0+25	2+25	4.65	4.52	0+25	2+25	4.29	4.17	1+25	2+75	2.70	2.865	94
0+25	2+75	5.52	5.32	0+25	2+75	5.135	4.95	....	......	.....	.....	..
0+25	3+75	7.60	7.07	0+25	3+75	6.963	6.48	2+75	4+09.5	2.065	2.695	77
0+25	4+09.5	8.58	7.87	0+25	4+09.5	7.83	7.19	....	......	.....	.....	..
368 CU FT PER SEC				395 CU FT PER SEC				395 CU FT PER SEC				
0+25	1+25	2.29	2.29	0+25	1+25	2.16	2.16	1+25	2+25	1.46	1.51	97
0+25	2+25	3.89	3.79	0+25	2+25	3.67	3.58	1+25	2+75	2.24	2.58	87
0+25	2+75	4.50	4.35	0+25	2+75	4.74	4.56	....	......	.....	.....	..
0+25	3+75	6.05	5.65	0+25	3+75	6.22	5.80	2+75	4+09.5	1.74	1.96	89
0+25	4+09.5	6.68	6.17	0+25	4+09.5	6.70	6.19	....	......	.....	.....	..

[a] Observed times have been corrected by factors determined for length of travel.
[b] Equivalent time on full run.

The equation of the curve, neglecting air resistance, is an adaptation of the usual equations of the path of a projectile given in many engineering texts.[12]

To have acquired a velocity equal to the vertical component, the water would have dropped a vertical distance

$$y_t = \frac{V^2 \sin^2 \theta}{2\,g} \qquad (30a)$$

In this same time the flow would cover a horizontal distance

$$x_t = \frac{V^2 \sin \theta \cos \theta}{g} \qquad (30b)$$

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[12] See *American Civil Engineers' Handbook*, 5th Ed., p. 119.

The point of tangency to the parabolic curve is at coordinates y_t, x_t from the point of origin. The equation of the curve is $x^2 = m\,y$, in which

$$m = \frac{2\,V^2 \cos^2 \theta}{g} \qquad (31)$$

In many constructed chutes, it is found that the water does not follow the bottom of the channel on convex vertical curves. This condition is probably due to the fact that the method of calculation did not yield the true velocity of the stream. Even if the true mean velocity has been determined by the method previously outlined, it is necessary to determine the maximum velocity in the channel which will occur near the surface of the center portion of the

TABLE 3.—HYDRAULIC PROPERTIES,

Station	DEPTHS (FT)		HYDRAULIC RADIUS (FT)		AREA (SQ FT)		PERCENTAGES		Ratio, $\frac{\text{Col. 8}}{\text{Col. 7}}$ (9)	Observed velocity adjusted V_o (10)	Velocity head[a] H_v (11)	$\frac{V^2}{g\,R_o}$ (12)
	Observed y_o (1)	Computed y_c (2)	Observed R_o (3)	Computed R_c (4)	Observed A_o (5)	Computed A_c (6)	Water (ρ) (7)	Air ($1-\rho$) (8)				
(a) 115 CU FT PER SEC												
0+00	1.20	0.92	0.88	0.71	6.46	4.90	75.7	24.3	0.32	23.5	8.6	24.2
0+25	1.45	0.83	1.00	0.66	7.89	4.41	55.9	44.1	0.79	26.1	10.6	32.1
0+50	0.88	0.65	0.67	0.53	4.62	3.37	72.9	27.1	0.37	34.2	18.2	68.6
0+75	0.90	0.55	0.70	0.45	4.73	2.82	59.7	40.3	0.68	40.8	25.8	115.0
1+00	0.72	0.50	0.58	0.41	3.70	2.51	67.9	32.1	0.47	45.8	32.6	159.0
1+25	0.75	0.47	0.60	0.39	3.87	2.33	60.2	39.8	0.66	49.4	37.9	194.0
1+50	0.82	0.45	0.65	0.38	4.27	2.21	51.8	48.2	0.93	52.0	43.1	227.0
1+75	0.92	0.43	0.72	0.36	4.85	2.11	43.5	56.5	1.30	54.5	46.2	257.0
2+00	0.92	0.41	0.72	0.35	4.85	2.04	42.1	57.9	1.38	56.4	49.5	283.0
2+25	0.90	0.41	0.70	0.35	4.73	2.01	42.5	57.5	1.35	57.4	51.2	293.0
2+50	0.90	0.40	0.70	0.34	4.73	1.98	41.9	58.1	1.39	58.0	52.3	308.0
2+75	0.88	0.40	0.69	0.34	4.61	1.98	42.9	57.1	1.33	58.1	52.5	308.0
3+00	1.55	0.40	1.05	0.34	8.47	1.98	23.4	76.6	3.27	58.1	52.5	308.0
3+25	1.08	0.40	0.81	0.34	5.77	1.98	34.3	65.7	1.92	58.1	52.5	308.0
3+50	1.00	0.41	0.76	0.35	5.30	2.03	38.3	61.7	1.61	56.7	50.0	285.0
3+75	0.85	0.45	0.67	0.38	4.44	2.22	50.0	50.0	1.00	51.8	41.6	219.0
4+00	0.90	0.61	0.70	0.49	4.73	3.12	66.0	34.0	0.52	36.9	21.2	87.0
4+09.5	0.95	0.64	0.73	0.54	5.02	3.35	66.8	33.2	0.50	34.3	18.3	68.0
(b) 163 CU FT PER SEC												
0+00	1.85	1.23	1.18	0.89	10.19	6.65	65.3	34.7	0.53	24.5	9.3	20.8
0+25	1.85	1.13	1.18	0.84	10.19	6.06	59.6	40.4	0.68	26.9	11.2	26.7
0+50	0.97	0.85	0.75	0.67	5.13	4.49	87.5	12.5	0.14	36.3	20.5	61.2
0+75	1.13	0.71	0.85	0.57	6.05	3.68	60.8	39.2	0.64	44.4	30.7	108.0
1+00	0.95	0.63	0.74	0.52	5.02	3.28	65.4	34.6	0.53	49.7	38.4	148.0
1+25	1.00	0.60	0.77	0.49	5.31	3.05	57.4	42.6	0.74	53.5	44.6	182.0
1+50	0.95	0.57	0.74	0.48	5.02	2.92	58.2	41.8	0.72	55.9	48.7	203.0
1+75	1.05	0.55	0.80	0.45	5.59	2.82	50.5	49.5	0.98	57.7	51.7	230.0
2+00	1.00	0.54	0.77	0.45	5.31	2.75	51.8	48.2	0.93	59.2	54.5	242.0
2+25	1.02	0.53	0.78	0.45	5.42	2.70	49.9	50.1	1.00	60.4	56.7	252.0
2+50	1.10	0.52	0.83	0.44	5.88	2.66	45.3	54.7	1.26	61.4	58.6	266.0
2+75	1.10	0.52	0.83	0.44	5.88	2.61	44.4	55.6	1.25	62.4	60.5	275.0
3+00	2.30	0.51	1.34	0.43	12.78	2.57	20.1	79.9	3.98	63.5	62.7	292.0
3+25	2.08	0.51	1.26	0.43	11.52	2.58	22.4	77.6	3.46	63.3	62.3	290.0
3+50	1.60	0.52	1.08	0.44	8.76	2.64	30.1	69.9	2.32	61.7	59.2	269.0
3+75	1.15	0.57	0.86	0.47	6.17	2.89	46.8	53.2	1.14	56.5	49.7	211.0
4+00	1.40	0.70	0.98	0.57	7.60	3.64	47.9	52.1	1.09	44.7	31.1	109.0
4+09.5	1.42	0.70	0.99	0.59	7.72	3.86	50.0	50.0	1.00	42.3	27.8	94.2

[a] Velocity head, H_v,

cross section. In short chutes this maximum velocity will be practically equal to the theoretical velocity resulting from the existing vertical drop. In longer chutes, the maximum velocity will be from 15% to 20% greater than the mean velocity.[13] Professor Lane found that the maximum velocity in the Uncompahgre chute was 15% greater than the mean.[14]

Aside from the possibility of damage to the concrete in the chute from cavitation, or from the tendency to lift the entire bottom slab, there appears to be no serious objection in allowing the stream to leave the floor of the chute on convex vertical curves, provided the side-walls are of sufficient height to prevent

HAT CREEK NO. 2 CHUTE

Station	DEPTHS (FT)		HYDRAULIC RADIUS (FT)		AREA (SQ FT)		PERCENTAGES		Ratio, $\frac{\text{Col. 8}}{\text{Col. 7}}$	Observed velocity adjusted V_o	Velocity head[a] H_o	$\frac{V^2}{g R_e}$
	Observed y_o	Computed y_e	Observed R_o	Computed R_e	Observed A_o	Computed A_e	Water (ρ)	Air ($1-\rho$)				
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
(*c*) 368 CU FT PER SEC												
0+00	3.85	2.32	1.71	1.34	21.70	12.90	59.4	40.6	0.68	28.5	12.6	18.8
0+25	3.00	1.99	1.53	1.23	16.80	11.02	65.6	34.4	0.52	33.4	17.3	28.1
0+50	2.05	1.63	1.25	1.08	11.34	9.03	79.5	20.5	0.26	40.8	25.9	48.0
0+75	1.82	1.40	1.20	0.98	10.01	7.65	76.5	23.5	0.31	48.1	35.9	73.2
1+00	1.68	1.25	1.11	0.90	9.21	6.79	73.6	26.4	0.36	54.2	45.7	101.0
1+25	1.68	1.16	1.11	0.85	9.21	6.23	67.7	32.3	0.48	59.1	54.3	128.0
1+50	1.58	1.08	1.07	0.81	8.63	5.84	67.6	32.4	0.48	63.0	61.7	152.0
1+75	1.68	1.03	1.11	0.78	9.21	5.52	59.8	40.2	0.67	66.6	69.1	177.0
2+00	1.60	0.99	1.08	0.76	8.75	5.27	60.3	39.7	0.66	69.7	75.6	199.0
2+25	1.65	0.96	1.10	0.74	9.04	5.10	56.4	43.6	0.77	72.1	80.9	219.0
2+50	1.78	0.94	1.15	0.72	9.79	4.97	50.8	49.2	0.97	74.0	85.1	236.0
2+75	1.68	0.92	1.11	0.71	9.21	4.93	53.4	46.6	0.87	74.6	86.5	244.0
3+00	2.90	0.92	1.51	0.71	16.23	4.93	30.3	69.7	2.30	74.6	86.5	244.0
3+25	3.80	0.92	1.70	0.71	21.40	4.93	23.0	77.0	3.35	74.6	86.5	244.0
3+50	2.00	0.94	1.24	0.71	11.05	4.99	45.2	54.8	1.21	73.8	84.7	244.0
3+75	1.70	0.97	1.12	0.81	9.33	5.20	55.8	44.2	0.79	70.8	78.0	193.0
4+00	1.70	1.12	1.12	0.86	9.33	6.05	64.9	35.1	0.54	60.9	57,7	134.0
4+09.5	1.55	1.17	1.09	0.91	8.46	6.36	75.2	24.8	0.33	57.9	52.1	115.0
(*d*) 395 CU FT PER SEC												
0+00	4.00	2.48	1.74	1.39	22.56	13.84	61.5	38.5	0.63	28.5	12.6	18.1
0+25	3.15	2.14	1.57	1.28	17.67	11.83	67.0	33.0	0.49	33.4	17.4	27.2
0+50	2.35	1.75	1.36	1.13	13.07	9.67	74.0	26.0	0.35	40.8	25.9	45.8
0+75	2.00	1.50	1.23	1.02	11.06	8.20	74.1	25.9	0.35	48.1	36.0	70.6
1+00	1.85	1.33	1.18	0.95	10.19	7.21	71.0	29.0	0.41	54.7	46.6	98.1
1+25	1.80	1.20	1.16	0.88	9.91	6.53	65.8	34.2	0.52	60.5	56.9	129.0
1+50	1.77	1.13	1.15	0.84	9.73	6.08	62.6	37.4	0.60	65.0	65.7	156.0
1+75	1.85	1.07	1.18	0.80	10.19	5.74	56.4	43.6	0.77	68.7	73.5	184.0
2+00	1.80	1.03	1.16	0.78	9.91	5.50	55.5	44.5	0.80	71.7	80.0	205.0
2+25	1.80	1.00	1.16	0.76	9.91	5.34	53.8	46.2	0.86	73.9	84.9	223.0
2+50	1.90	0.98	1.20	0.75	10.48	5.24	50.0	50.0	1.00	75.3	88.2	235.0
2+75	1.85	0.97	1.18	0.74	10.19	5.20	51.0	49.0	0.96	75.9	89.6	242.0
3+00	3.52	0.97	1.59	0.74	19.80	5.19	26.2	73.8	2.82	76.0	89.8	243.0
3+25	4.60	0.97	1.83	0.74	26.01	5.21	20.0	80.0	4.00	75.8	89.4	242.0
3+50	3.50	0.98	1.64	0.75	19.68	5.27	26.8	73.2	2.73	74.9	87.3	233.0
3+75	1.97	1.02	1.23	0.78	10.88	5.46	50.2	49.8	0.99	72.3	81.3	208.0
4+00	1.95	1.20	1.21	0.88	10.77	6.46	60.1	39.9	0.66	61.1	58.0	132.0
4+09.5	2.17	1.26	1.30	0.91	12.03	6.82	56.6	43.4	0.77	57.9	52.1	115.0

corresponding to the observed velocity.

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[13] "Flow of Water in Rivers and Other Channels," by E. Ganguillet and W. R. Kutter, John Wiley & Sons, Inc., 1901, p. 122.

[14] *Engineering News-Record*, January 2, 1936, p. 7.

TABLE 4.—GAGE READINGS AND HYDRAULIC PROPERTIES, RAPID FLUME

Station	(a) GAGE READINGS							(b) HYDRAULIC PROPERTIES AND PERCENTAGE OF AIR IN WATER (88 CU FT PER SEC)											
	South Side		North Side		Center		Mean (y)	Depths (Ft)		Hydraulic Radius (Ft)		Area (Sq Ft)		Percentages		Ratio, Col. 15 / Col. 14	Observed velocity adjusted, V_o	Velocity head[b] H_v	$\frac{V^2}{g R_e}$
	High	Low	High	Low	High	Low		Observed y_o	Computed y_e	Observed R_o	Computed R_e	Observed A_o	Computed A_e	Water (ρ)	Air $(1-\rho)$				
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
0+50	0.68	0.58	1.91	1.71	1.12	1.02	1.28[a]	1.28	0.69	0.94	0.76	8.45	4.54	53.6	46.4	0.87	19.4	5.9	15.5
0+75	1.50	1.30	0.55	0.45	1.32	1.12	1.08	1.08	0.74	0.78	0.59	5.23	3.58	68.5	31.5	0.46	24.6	9.4	31.9
1+00	0.99	0.89	0.75	0.55	0.83	0.78	0.80	0.80	0.64	0.59	0.50	3.67	2.94	80.2	19.8	0.25	29.9	13.9	55.6
1+25	1.15	1.05	0.65	0.45	0.80	0.55	0.75	0.75	0.55	0.55	0.44	3.44	2.51	73.0	27.0	0.37	35.0	19.0	86.3
1+50	0.76	0.66	0.88	0.68	0.69	0.54	0.59	0.59	0.48	0.45	0.40	2.71	2.23	82.3	17.7	0.21	43.5	24.3	121.0
1+75	0.79	0.64	0.66	0.46	0.72	0.57	0.64	0.64	0.44	0.50	0.37	2.94	2.02	68.8	31.2	0.45	43.5	29.4	159.0
2+00	0.98	0.88	0.45	0.35	0.54	0.34	0.55	0.55	0.42	0.43	0.35	2.54	1.91	75.2	24.8	0.33	46.0	32.9	188.0
2+25	1.09	0.99	0.30	0.20	0.72	0.57	0.64	0.64	0.40	0.50	0.34	2.96	1.86	62.9	37.1	0.59	47.2	34.6	204.0
2+50	0.86	0.76	0.87	0.77	0.62	0.52	0.69	0.69	0.40	0.54	0.34	3.18	1.82	57.2	42.8	0.75	48.4	36.4	214.0
2+75	0.62	0.52	0.97	0.77	0.62	0.52	0.65	0.65	0.38	0.49	0.33	2.97	1.77	59.6	40.4	0.68	49.7	38.4	221.0
3+00	0.53	0.43	0.82	0.67	0.68	0.53	0.60	0.60	0.38	0.48	0.32	2.78	1.73	62.3	37.7	0.60	50.8	40.1	251.0
3+25	0.89	0.79	0.67	0.62	0.62	0.52	0.66	0.66	0.37	0.50	0.32	3.02	1.72	57.0	43.0	0.75	51.2	40.8	255.0
3+50	0.89	0.79	0.66	0.56	0.73	0.58	0.69	0.69	0.37	0.52	0.32	3.16	1.70	53.8	46.2	0.86	51.8	41.7	261.0
3+75	0.79	0.69	0.83	0.68	0.67	0.52	0.67	0.67	0.37	0.51	0.32	3.08	1.69	54.9	45.1	0.82	52.0	42.0	263.0
4+00	0.80	0.70	0.77	0.67	0.83	0.63	0.74	0.74	0.37	0.56	0.32	3.40	1.70	50.0	50.0	1.00	51.8	41.7	261.0
4+25	0.63	0.53	0.82	0.67	0.65	0.55	0.63	0.63	0.37	0.49	0.32	2.90	1.72	59.4	40.6	0.68	51.2	40.8	255.0
4+50	0.66	0.56	0.89	0.74	0.77	0.67	0.72	0.72	0.38	0.55	0.32	3.31	1.73	52.3	47.7	0.91	50.8	40.1	251.0
4+75	0.72	0.62	0.75	0.65	0.73	0.63	0.68	0.68	0.39	0.52	0.33	3.13	1.78	56.9	43.1	0.76	49.4	37.9	230.0
5+00	0.80	0.70	0.71	0.61	0.75	0.65	0.70	0.70	0.41	0.54	0.34	3.22	1.87	58.1	41.9	0.72	47.0	34.3	202.0
5+25	0.86	0.76	0.73	0.58	0.79	0.69	0.74	0.74	0.43	0.56	0.36	3.41	1.99	58.4	41.6	0.71	44.2	30.4	169.0
5+50	0.79	0.64	0.92	0.77	0.83	0.73	0.78	0.78	0.46	0.58	0.39	3.59	2.13	59.3	40.7	0.68	41.3	26.5	136.0
5+75	0.75	0.60	0.96	0.81	0.83	0.73	0.78	0.78	0.49	0.58	0.40	3.59	2.25	62.8	37.2	0.59	39.0	23.6	118.0
6+00	1.04	0.89	0.77	0.67	0.89	0.79	0.84	0.84	0.52	0.62	0.42	3.86	2.38	61.7	38.3	0.62	37.0	21.3	101.0
6+25	0.84	0.69	0.80	0.70	0.81	0.71	0.76	0.76	0.54	0.57	0.44	3.50	2.48	70.9	29.1	0.41	35.0	19.6	89.1
6+50	0.78	0.68	0.85	0.70	0.80	0.70	0.75	0.75	0.56	0.57	0.45	3.45	2.56	72.1	27.9	0.39	34.4	18.4	81.7
6+67	0.93	0.78	0.92	0.77	0.89	0.79	0.84	0.84	0.57	0.61	0.46	3.86	2.64	68.5	31.5	0.46	33.3	17.2	74.8

[a] Additional reading taken on this section to determine mean. [b] Velocity head, H_v, corresponding to the observed velocity.

overtopping by the spray. The observations at the Hat Creek chute indicate that the spreading of the jet over a greater cross section increased the friction and aided in the dissipation of the energy. This fact may be useful in design, where it is desirable to reduce the velocity of the stream before discharging the jet from the end of the chute.

Since water traveling in the chutes is always at less than the critical depth, changes in horizontal direction are more difficult to make and, if possible, should be avoided. If a horizontal curve becomes necessary, it should preferably be inserted near the head of the chute before the water has gathered much velocity. Model tests should be made to test the design prior to actual construction.

TABLE 5.—VELOCITY OBSERVATIONS, RAPID FLUME

FULL RUN STATIONS		TIME (SEC)		PARTIAL RUN STATIONS		TIME (SEC)		Ratio, Col. 7 / Col. 8	PARTIAL RUN STATIONS		TIME (SEC)		Ratio, Col. 7 / Col. 8
Upper (1)	Lower (2)	Observed[a] (3)	Corrected (4)	Upper (5)	Lower (6)	Observed (7)	Equivalent full run (8)	(9)	Upper (5)	Lower (6)	Observed (7)	Equivalent full run (8)	(9)
0+50	1+25	2.88	2.88	2+25	4+25	3.93	4.18	94.0	2+25	3+25	2.00	2.09	95.5
0+50	2+25	5.26	5.23	2+25	5+25	6.06	6.61	91.5	2+25	4+25	3.97	4.18	95.0
0+50	3+25	7.35	7.22	2+25	6+67	10.40	10.85	96.0	3+25	4+25	1.94	2.09	93.0
0+50	4+25	9.44	9.15	2+25	.....	....	....	....	3+25	5+25	3.98	4.52	88.0
0+50	5+25	11.87	11.15	3+25	4+25	1.84	2.09	88.0	4+25	5+25	2.00	2.43	82.0
0+50	6+67	16.11	14.94	3+25	5+25	4.08	4.52	90.0	4+25	6+67	5.90	6.67	88.5
.....	.....	....	....	3+25	6+67	8.20	8.76	93.5	5+25	6+67	3.72	4.24	88.0

[a] Observed times have been corrected by factors determined for length of travel.

The water surface across the channel of a chute constructed on a curve will be tangent to a parabola.[15] Of course, this is assuming that the channel is relatively narrow permitting the use of the radius of the center line r as the average curvature of the section. Then, the inclination of this tangent with the horizontal is

$$\tan \psi = \frac{V^2}{g\,r} \qquad \text{(32)}$$

Verification of this formula was possible on the South Canal chute, where the difference in elevation of the water surface between the inner and outer walls approximated the slope ψ. In model tests[16] conducted by the East Bay Municipal Utility District, Robert C. Kennedy, M. Am. Soc. C. E., has demonstrated that the flow of water around a horizontal curve is greatly improved if the bottom of the flume is superelevated on the outer wall in accordance with this formula.

CALCULATION OF RESULTS OF TESTS

The results of the depth measurements and velocity observations are given in Tables 1 to 11. The gage readings are recorded for both the highest and

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[15] *Text Book of Hydraulics*, by G. E. Russell, Henry Holt & Co., 4th Ed., p. 49.

[16] *Western Construction News*, April, 1942, p. 158.

TABLE 6.—Gage Readings, South Canal Chute

Station	Left Side		Right Side		Mean (y)
	High (1)	Low (2)	High (3)	Low (4)	(5)
363.5 Cu Ft per Sec					
0+00	2.0	1.9	2.1	1.9	1.98
0+50	...	...	1.8	1.6	1.74
1+00	1.7	1.5	1.8	1.5	1.63
1+50	1.6	1.4	1.8	1.5	1.59
2+00	1.6	1.4	1.7	1.3	1.50
2+50	1.5	1.3	1.8	1.4	1.50
3+00	1.1	0.9	2.3	1.9	1.55
3+50	1.2	1.0	2.1	1.5	1.45
4+00	1.2	0.8	2.3	1.9	1.50
4+50	1.7	1.4	1.5	1.1	1.43
5+00	1.4	1.2	1.7	1.3	1.40
5+50	1.6	1.3	1.7	1.3	1.47
6+00	1.8	1.5	1.7	1.3	1.58
6+50	1.9	1.5	1.6	1.1	1.52
7+00	1.9	1.6	1.5	0.9	1.48
7+50	1.85	1.5	1.8	1.4	1.64
8+00	1.8	1.4	2.3	1.7	1.80
8+50	1.7	1.4	1.6	1.1	1.45
9+00	2.0	1.7	1.7	1.3	1.68
9+50	1.8	1.5	1.9	1.4	1.65
10+00	1.7	1.4	2.0	1.5	1.65
10+50	1.8	1.5	2.1	1.6	1.70
11+00	1.9	1.7	1.8	1.4	1.70
11+50	1.9	1.7	1.8	1.4	1.70
12+00	1.9	1.5	2.0	1.7	1.78
12+50	1.9	1.6	1.9	1.5	1.72
13+00	1.9	1.6	1.8	1.4	1.68
13+50	2.0	1.7	1.8	1.4	1.72
14+00	1.9	1.5	1.9	1.5	1.70
14+50	2.0	1.6	2.1	1.7	1.85
15+00	2.2	1.8	1.8	1.4	1.80
15+50	2.2	1.8	1.9	1.4	1.82
15+75	1.7	1.4	1.6	1.3	1.50
16+00	2.2	1.9	2.0	1.7	1.95
334 Cu Ft per Sec					
0+00	...	...	1.9	1.7	1.78
0+50	...	...	1.6	1.5	1.70
1+00	1.6	1.4	1.5	1.3	1.45
1+50	1.6	1.3	1.6	1.3	1.45
2+00	1.7	1.4	1.5	1.2	1.45
2+50	1.6	1.2	1.6	1.3	1.42
3+00	1.1	0.8	2.1	1.8	1.45
3+50	1.4	1.0	1.8	1.4	1.40
4+00	1.2	0.8	2.1	1.7	1.45
4+50	1.7	1.4	1.4	1.1	1.40
5+00	1.4	1.1	1.5	1.2	1.30
5+50	1.6	1.3	1.5	1.2	1.40
6+00	1.8	1.4	1.5	1.2	1.48
6+50	1.9	1.5	1.5	1.1	1.50
7+00	2.2	1.7	1.3	0.9	1.52
7+50	2.0	1.5	1.7	1.3	1.62
8+00	1.9	1.4	2.1	1.6	1.75
8+50	1.7	1.3	1.6	1.1	1.42
9+00	1.9	1.6	1.7	1.3	1.62
9+50	1.7	1.4	1.7	1.3	1.52
10+00	1.7	1.4	1.9	1.5	1.62
10+50	1.7	1.4	1.9	1.6	1.65
11+00	1.9	1.6	1.7	1.3	1.62
11+50	1.9	1.6	1.65	1.3	1.62
12+00	1.9	1.6	1.8	1.5	1.70
12+50	1.8	1.4	1.8	1.4	1.60
13+00	1.8	1.5	1.7	1.4	1.60
13+50	1.8	1.5	1.7	1.3	1.58
14+00	1.7	1.3	1.7	1.4	1.52
14+50	1.7	1.4	1.8	1.4	1.58
15+00	1.9	1.6	1.6	1.3	1.60
15+50	1.9	1.6	1.7	1.4	1.65
15+75	1.4	1.0	1.4	1.1	1.22
16+00	1.9	1.6	1.8	1.4	1.68
16+50	2.0	1.7	2.0	1.7	1.85
17+00	2.1	1.8	2.0	1.7	1.90
17+50	2.2	1.9	2.2	1.9	2.05
18+00	2.4	2.1	2.2	1.9	2.15
18+50	2.6	2.3	2.1	1.9	2.22
19+00	2.3	2.1	2.3	2.1	2.20
19+50	2.4	2.2	2.3	2.1	2.25
20+00	2.3	2.0	2.6	2.3	2.30
20+50	2.6	2.3	2.3	2.1	2.32
21+00	2.2	2.0	2.7	2.5	2.35
21+50	2.9	2.7	1.9	1.7	2.30
22+00	2.7	2.4	2.4	2.2	2.42
22+50	2.7	2.4	2.1	1.9	2.27
23+00	2.0	1.75	2.9	2.6	2.31
23+25	2.3	2.0	2.5	2.3	2.28
23+50	1.7	1.5	3.0	2.8	2.25
23+75	1.8	1.6	2.7	2.5	2.15
24+00	1.5	1.3	2.8	2.4	2.00
24+25	1.5	1.3	2.7	2.4	1.98
24+50	1.3	1.1	2.9	2.6	1.98
24+75	1.3	1.1	2.6	2.4	1.85
25+00	0.8	0.6	3.7	3.2	2.08
25+25	0.8	0.6	3.5	3.2	2.03
25+50	1.0	0.8	3.1	2.7	1.95
26+00	1.7	1.4	2.6	2.2	1.98
26+50	2.2	1.9	2.4	2.2	2.18
27+00	2.8	2.3	1.7	1.2	2.00
27+50	2.4	2.1	2.1	1.7	2.08
28+00	2.4	2.0	2.2	1.9	2.13
28+50	2.5	2.2	2.3	2.0	2.25
29+00	2.1	1.8	2.5	2.2	2.15
29+50	2.1	1.8	2.6	2.3	2.20
30+00	2.1	1.8	2.7	2.3	2.22
30+50	2.4	2.1	2.3	2.0	2.20
31+00	2.6	2.3	2.3	2.0	2.30
31+50	2.5	2.2	2.3	2.0	2.25
32+00	2.7	2.4	2.1	1.8	2.25
32+50	2.5	2.2	2.0	1.7	2.10
32+60	2.4	2.2	2.0	1.7	2.08
33+20	2.6	2.3	1.9	1.6	2.10
33+50	2.2	1.9	2.4	2.1	2.15
34+00	2.0	1.8	2.7	2.3	2.20
34+50	2.4	2.2	2.1	1.8	2.12
35+00	2.6	2.3	1.8	1.5	2.08
35+20	2.9	2.5	1.5	1.2	2.03
35+35	2.0	1.6	2.3	2.0	1.97

lowest point observed at each station, together with the mean depth calculated from these observations. In the case of Rapid Flume, readings are given for depth in the center of the flume for the upper part of the channel where cross waves developed due to the curvature of the flume. The results for Hat Creek chute are given in Table 1, for Rapid Flume in Table 4(*a*), and for the South Canal chute in Table 6.

TABLE 7.—VELOCITY OBSERVATIONS, FULL RUN STATIONS, SOUTH CANAL CHUTE

Stations		Time (Sec)	
Upper (1)	Lower (2)	Observed[a] (3)	Corrected (4)
1 +00	2 +65	5.12	5.12
1 +00	3 +95	9.41	9.24
1 +00	6 +20	16.41	15.58
1 +00	8 +00	22.93	20.70
8 +00	9 +94	7.02	7.02
8 +00	10 +65	9.75	9.61
8 +00	14 +00	23.30	22.01
10 +65	17 +00	28.56	26.82
17 +00	19 +50	13.22	13.22
17 +00	21 +50	26.02	24.85
17 +00	23 +00	36.36	33.39
23 +00	25 +00	7.65	7.65
25 +00	27 +00	5.93	5.93
25 +00	28 +00	9.40	9.21
28 +00	30 +00	7.58	7.58
28 +00	32 +00	15.93	15.46
30 +00	32 +00	7.84	7.84
30 +00	33 +50	14.07	13.72
32 +00	35 +35	12.47	12.22

[a] Observed times have been corrected by factors determined for length of travel.

In making velocity tests, it was better to time the color over a rather short stretch of flume, because, if the measuring course were too long, the color cloud spread out over such a long distance that it was difficult to estimate the time of passing of the center of the color band. On the first series of observations color was added at the head of the chute and the time of travel was observed over courses of increasing length by having the timers moved progressively to stations farther down the chute. An analysis of these results indicated that the observed time at the lower stations was too great. On the latter tests the color was dropped in at various places along the chute and the velocity was timed over courses not exceeding 100 ft in length where conditions would permit. On the South Canal chute the great length of channel and slower velocities indicated that an increased length of course could be used without error. The results obtained from separate courses on these tests were totaled so as to give the entire time for the water to travel from the inlet to the outlet of the chute. In all cases the sum of the times taken over the short courses was less than the observed time over the total course. This anomalous condition can be explained readily.

The color cloud was retarded along the side-walls as it proceeded down the chute and became attenuated with increasing distances of travel. In observing the velocities, the time was taken to the approximate center of the color cloud as closely as it could be judged by eye. Along a 400-ft course the color cloud extended between 75 and 100 ft. Analyses of the results indicate that probably the greatest concentration of color was at the head of the slug; hence, the time lapse recorded to the center of the cloud was too long. On shorter courses the color cloud stayed together better, and less error, therefore, was introduced by estimating the location of the center by eye. The results of these observations showing the variation in time over long and short courses have been tabulated for Hat Creek chute in Table 2, Rapid Flume in Table 5, and for the South Canal chute in Tables 7 and 7*A*. These observations on short courses were used to obtain factors applied to correct inaccuracies in the observed time over the longer courses. Since for calculation purposes the depth readings needed to be recorded for the perpendicular distance, corrections were made by multiplying the observed depth taken perpendicular to the inclination of the chute by the cosine of the angle of the slope.

TABLE 7*A*.—VELOCITY OBSERVATIONS, PARTIAL RUN STATIONS, SOUTH CANAL SHUTE

STATIONS		TIME (SEC)		Ratio, $\frac{\text{Col. 3}}{\text{Col. 4}}$
Upper (1)	Lower (2)	Observed (3)	Equivalent full run (4)	(5)
3+95	6+20	6.79	7.00	97.0
3+95	8+00	13.30	13.52	98.5
10+65	14+00	13.29	13.55	98.5
21+50	23+00	8.57	10.34	83.0
30+00	32+00	7.84	8.35	94.0
32+00	33+50	5.28	6.23	85.0

The rapid acceleration of the velocity, particularly near the head of the chutes, required a segregation of the time over shorter courses than was practical for timing. Therefore, an estimated subdivision of the time from the observations for purposes of subsequent computation was made for 25-ft intervals on Hat Creek chute and Rapid Flume and for 50-ft intervals on the South Canal chute. In making this subdivision, the times on the various tests were compared graphically and increments for the 25-ft or 50-ft intervals estimated from the elapsed time over the longer courses.

A comparison of the recorded observations of depth and velocity illustrates forcibly the difficulty in obtaining precise results in measuring flows at high velocities where air is entrained with the water.

The velocity head for the adjusted mean velocity in each 25-ft or 50-ft section was plotted on the profile of the respective chutes (see Figs. 1 and 6 for Hat Creek chute and Rapid Flume), and the energy gradient was drawn. From this gradient line the velocity head at each station was scaled and the corresponding velocity calculated. These velocities were used in conjunction with the observed areas to compute the coefficient of retardation. In addition, the observed velocity and energy gradient were used in conjunction with the area computed by the relation $A_c = \frac{Q}{V_o}$ and the corresponding hydraulic radii to calculate retardation factors. The velocity at each station was also computed by the equation $V_c = \frac{Q}{A_o}$, using for the area the values computed from the observed depths, including water plus entrained air. These velocity heads, which were smaller than the observed values, were plotted on graphs and the corresponding energy slopes determined. Necessarily these slopes were greater in the zone of accelerated flow than those obtained by use of observed velocity heads. These three methods of computation were used by Fred C. Scobey, M. Am. Soc. C. E., in his study of flow of water in chutes.[17]

In his bulletin the computation of the retardation factor has been made by taking the energy gradient parallel to the bottom slope. This may be correct in many instances, but in the tests described in this paper it was found that the energy slope was seldom parallel to the bottom slope, as can be seen by an examination of Figs. 1 and 6.

These elements of the water prism obtained in the manner just described were used for the determination of the coefficients of the flow in the Chézy formula and of the retardation factors in Kutter's and Manning's formulas by each of the three methods described; namely, by using (1) the observed velocities and the observed area, (2) the observed velocity and areas computed as $\frac{Q}{V_o}$, and (3) the observed areas and velocities computed as $\frac{Q}{A_o}$.

The hydraulic properties of the water prism in the three chutes tested together with the percentage of air in the water and the value of the parameter $\frac{V^2}{g\,R_c}$ were tabulated for each flow observed. These results are given in Table 3 for Hat Creek chute, in Table 4(*b*) for Rapid Flume, and in Table 8 for South Canal chute.

Bureau of Reclamation Tests on Kittitas Chute.—In 1938 the Bureau of Reclamation made tests of air entrainment on the wasteway from the Kittitas Main Canal located on the Yakima Project in a western state. This chute discharges water from the main canal into the Yakima River. The flow of water from the canal into the inlet of the chute is controlled by a radial gate. The total length of the chute is 1,301 ft measured on the slope, and the total drop from the canal to the bottom of the bucket at the Yakima River is 340 ft. The first 753 ft of the chute are on a 10° 12′ slope and the initial 318 ft are built on a taper section. At the end of this slope there is a convex vertical curve about 90 ft in length connecting with a steep slope of 33° 10′, having a length of 289 ft. This joins with a concave vertical curve about 90 ft in length, joining an almost horizontal tangent with a slight downward slope of 0° 04′ for a distance of 79 ft to the end of the chute. The jet at the end of the incline discharges into the Yakima River above the maximum high-water level.

The concrete of the chute is smooth on the side-walls, but judging from photographs, the bottom appears to be roughened slightly, probably due to sand carried down the chute with the discharge. The width of the channel decreases from 20 ft at the inlet to 8 ft at the end of the taper section. From this point a uniform width of 8 ft is carried for the remaining length of the chute. The side-walls are 7 ft high, except at the inlet and near the outlet.

Discharge measurements were taken with a current meter in the canal above the chute. Measurements of the depth to the water surface in the incline were taken with a point gage, reading the distance down from the top of the side-walls. A traverse of the channel was made, taking observations at the side-walls and at every foot across the section. The first point gage station was located at a distance of 343 ft from the inlet, a short distance below the end of the taper section. Depth readings were taken at approximately 100-ft intervals for a distance of 410 ft along the 10° 12′ slope, for 210 ft along the 33° 10′ slope, and for 50 ft along the 0° 04′ slope. The depth and area at each of the ten observation stations were calculated from these measurements.

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[17] "The Flow of Water in Flumes," by Fred C. Scobey, *Technical Bulletin No. 393*, U. S. Dept. of Agriculture, pp. 89–91.

Velocity observations were taken electrically, using salt as a conductor, balls of moistened flour salt being dropped into the chute for each test run. The velocity was recorded on an oscillograph through electrodes installed in the walls of the chute. The oscillograph consisted essentially of three moving coil galvanometers, an optical system, and a motor-driven camera. Two of the galvanometers recorded the passage of the salt cloud past the upper and lower stations. The third was connected to a 60-cycle alternating current and recorded a time scale on the film.

This method has the advantage of affording a permanent record of the test. Nevertheless, it was difficult to determine the center of the gravity of the salt cloud as it passed each station. Also, it was noted on the film that the flow, particularly on the steep grade, occurred in surges, with the result

TABLE 8.—HYDRAULIC PROPERTIES, SOUTH CANAL CHUTE

Station	HYDRAULIC PROPERTIES AND PERCENTAGE OF AIR IN WATER											
	Depths (Ft)		Hydraulic Radius (Ft)		Area (Sq Ft)		Percentages		Ratio, Col. 8 / Col. 7	Observed velocity V_o	Velocity head[a] H_v	$\frac{V^2}{g R_c}$
	Observed y_o	Computed y_c	Observed R_o	Computed R_c	Observed A_o	Computed A_c	Water (ρ)	Air $(1-\rho)$				
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
(a) DISCHARGE 363.5 CU FT PER SEC												
0+00	1.98	2.12	1.46	1.55	19.36	20.7	106.9	−06.9	....	17.6	4.8	6.2
0+50	1.74	1.33	1.34	1.08	16.96	12.9	76.1	23.9	0.31	28.2	12.4	23.0
1+00	1.63	1.24	1.27	1.02	15.86	11.95	75.3	24.7	0.33	30.4	14.4	28.2
1+50	1.59	1.17	1.25	0.98	15.46	11.3	73.1	26.3	0.36	32.1	16.0	32.7
2+00	1.50	1.16	1.19	0.97	14.56	11.15	76.6	23.4	0.31	32.6	16.5	34.0
2+50	1.50	1.16	1.19	0.97	14.56	11.15	76.6	23.4	0.31	32.6	16.5	34.0
3+00	1.55	1.16	1.22	0.97	15.06	11.15	74.0	26.0	0.35	32.6	16.5	34.0
3+50	1.45	1.16	1.16	0.97	14.06	11.15	79.3	20.7	0.26	32.6	16.5	34.0
4+00	1.50	1.16	1.19	0.97	14.56	11.15	76.6	23.4	0.31	32.6	16.5	34.0
4+50	1.43	1.16	1.15	0.97	13.86	11.15	80.4	19.6	0.24	32.6	16.5	34.0
5+00	1.40	1.16	1.13	0.97	13.56	11.15	82.2	17.8	0.22	32.6	16.5	34.0
5+50	1.47	1.16	1.17	0.97	14.26	11.15	78.2	21.8	0.28	32.6	16.5	34.0
6+00	1.58	1.17	1.24	0.98	15.36	11.25	73.2	26.8	0.36	31.4	15.3	31.2
6+50	1.52	1.20	1.20	1.00	14.76	11.6	78.6	21.4	0.27	31.3	15.2	30.4
7+00	1.48	1.24	1.18	1.03	14.36	12.0	83.6	16.4	0.20	30.3	14.3	27.8
7+50	1.64	1.29	1.28	1.06	15.96	12.5	78.3	21.7	0.28	29.1	13.2	24.9
8+00	1.80	1.36	1.37	1.10	17.56	13.15	74.9	25.1	0.34	27.6	11.8	21.5
8+50	1.45	1.36	1.16	1.10	14.06	13.15	93.5	6.5	0.07	27.6	11.8	21.5
9+00	1.68	1.36	1.30	1.10	16.36	13.15	80.4	19.6	0.24	27.6	11.8	21.5
9+50	1.65	1.36	1.28	1.10	16.06	13.15	81.9	18.1	0.22	27.6	11.8	21.5
10+00	1.65	1.36	1.28	1.10	16.06	13.15	81.9	18.1	0.22	27.6	11.8	21.5
10+50	1.70	1.37	1.31	1.11	16.56	13.3	80.3	19.7	0.25	27.3	11.6	20.9
11+00	1.70	1.38	1.31	1.12	16.56	13.4	80.9	19.1	0.24	27.1	11.4	20.4
11+50	1.70	1.38	1.31	1.12	16.56	13.4	80.9	19.1	0.24	27.1	11.4	20.4
12+00	1.78	1.40	1.36	1.13	17.36	13.55	78.1	21.9	0.28	26.8	11.2	19.8
12+50	1.72	1.40	1.32	1.13	16.76	13.6	81.1	18.9	0.23	26.7	11.1	19.6
13+00	1.68	1.40	1.30	1.13	16.36	13.6	83.1	16.9	0.20	26.7	11.1	19.6
13+50	1.72	1.41	1.32	1.14	16.76	13.65	81.4	18.6	0.23	26.6	10.8	18.9
14+00	1.70	1.42	1.31	1.14	16.56	13.75	83.0	17.0	0.20	26.4	10.8	18.9

TABLE 8—(*Continued*)

Station	Hydraulic Properties and Percentage of Air in Water											
	Depths (Ft)		Hydraulic Radius (Ft)		Area (Sq Ft)		Percentages		Ratio, Col. 8 / Col. 7	Observed velocity V_e	Velocity head[a] H_e	$\frac{V^2}{gR_e}$
	Observed y_o	Computed y_e	Observed R_o	Computed R_e	Observed A_o	Computed A_e	Water (ρ)	Air $(1-\rho)$				
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
(*b*) Discharge 334 Cu Ft per Sec												
10+50	1.65	1.36	1.28	1.10	16.06	13.15	81.9	18.1	0.22	25.4	10.0	18.1
11+00	1.62	1.36	1.26	1.10	15.76	13.15	83.4	16.6	0.20	25.4	10.0	18.1
11+50	1.62	1.37	1.27	1.11	15.76	13.25	84.1	15.9	0.19	25.2	9.9	17.8
12+00	1.70	1.38	1.31	1.12	16.56	13.35	80.0	20.0	0.25	25.0	9.7	17.3
12+50	1.60	1.38	1.25	1.12	15.56	13.35	85.8	14.2	0.17	25.0	9.7	17.3
13+00	1.60	1.39	1.25	1.12	15.56	13.45	86.4	13.6	0.16	24.8	9.6	17.1
13+50	1.57	1.39	1.23	1.12	15.26	13.5	88.5	11.5	0.13	24.7	9.5	17.0
14+00	1.53	1.40	1.21	1.13	14.86	13.55	91.2	8.8	0.10	24.6	9.4	16.6
14+50	1.58	1.48	1.24	1.18	15.36	14.4	93.8	6.2	0.06	23.2	8.4	14.2
15+00	1.60	1.51	1.25	1.20	15.56	14.7	94.5	5.5	0.06	22.7	8.0	13.3
15+50	1.65	1.53	1.28	1.21	16.06	14.9	92.8	7.2	0.08	22.4	7.8	12.9
16+00	1.68	1.57	1.30	1.24	16.36	15.3	93.5	6.5	0.07	21.8	7.4	11.9
16+50	1.85	1.64	1.40	1.28	18.06	16.0	88.6	11.4	0.13	20.9	6.8	10.6
17+00	1.90	1.80	1.43	1.37	18.56	17.6	94.8	5.2	0.05	19.0	5.6	8.2
17+50	2.05	1.80	1.51	1.37	20.06	17.6	87.7	12.3	0.14	19.0	5.6	8.2
18+00	2.15	1.80	1.56	1.37	21.06	17.6	83.6	16.4	0.20	19.0	5.6	8.2
18+50	2.22	1.80	1.60	1.37	21.76	17.6	80.9	19.1	0.24	19.0	5.6	8.2
19+00	2.20	1.80	1.59	1.37	21.56	17.6	81.6	18.4	0.23	19.0	5.6	8.2
19+50	2.25	1.80	1.61	1.37	22.06	17.6	79.8	20.2	0.25	19.0	5.6	8.2
20+00	2.30	1.91	1.64	1.43	22.56	18.65	82.7	17.3	0.21	17.9	5.0	7.0
20+50	2.32	1.91	1.65	1.43	22.76	18.65	81.9	18.1	0.22	17.9	5.0	7.0
21+00	2.35	1.98	1.66	1.46	23.06	19.4	84.1	15.9	0.19	17.2	4.6	6.3
21+50	2.30	2.01	1.64	1.49	22.56	19.65	87.1	12.9	0.15	17.0	4.5	6.0
22+00	2.42	2.01	1.69	1.49	23.76	19.65	82.7	16.3	0.20	17.0	4.5	6.0
22+50	2.27	2.05	1.62	1.51	22.26	20.10	91.1	8.9	0.10	16.6	4.3	5.7
23+00	2.31	2.05	1.64	1.51	20.06	20.10	92.8	7.2	0.08	16.6	4.3	5.7
23+50[b]	2.25	2.26	1.43	1.43	15.31	15.40	100.0	0	0.00	21.7	7.3	10.2
24+00	2.00	1.72	1.33	1.20	13.56	11.60	85.5	14.5	0.17	28.8	12.9	21.5
24+50	1.98	1.63	1.32	1.16	13.42	10.95	81.6	18.4	0.23	30.5	14.5	25.0
25+00	2.08	1.49	1.36	1.09	14.12	9.97	70.6	29.4	0.42	33.5	17.4	31.9
25+50	1.90	1.42	1.28	1.05	12.86	9.52	74.0	26.0	0.35	35.1	19.2	36.6
26+00	1.98	1.39	1.32	1.03	13.42	9.30	69.4	30.6	0.44	35.9	20.0	38.8
26+50	2.18	1.46	1.40	1.07	14.82	9.80	66.1	33.9	0.51	34.1	18.2	34.0
27+00	2.00	1.59	1.33	1.16	13.56	10.70	78.9	21.1	0.27	31.2	15.1	26.0
27+50	2.08	1.72	1.36	1.20	14.12	11.60	82.2	17.8	0.22	28.7	12.8	21.3
28+00	2.13	1.87	1.38	1.27	14.47	12.65	87.4	12.6	0.14	26.4	10.8	17.0
28+50	2.25	1.87	1.43	1.27	15.31	12.65	82.6	17.4	0.21	26.4	10.8	17.0
29+00	2.15	1.87	1.39	1.27	14.61	12.65	86.6	13.4	0.16	26.4	10.8	17.0
29+50	2.20	1.87	1.41	1.27	14.96	12.65	84.6	15.4	0.18	26.4	10.8	17.0
30+00	2.22	1.90	1.42	1.28	15.10	12.85	85.1	14.9	0.17	26.0	10.5	17.0
30+50	2.20	1.90	1.41	1.28	14.96	12.85	85.9	14.1	0.16	26.0	10.5	16.4
31+00	2.30	1.94	1.45	1.30	15.66	13.10	83.6	16.4	0.20	25.5	10.1	16.4
31+50	2.25	1.94	1.43	1.30	15.31	13.10	85.6	14.4	0.17	25.5	10.1	15.5
32+00	2.25	1.94	1.43	1.30	15.31	13.10	85.6	14.4	0.17	25.5	10.1	15.5
32+50	2.12	1.78	1.38	1.23	14.40	12.00	83.3	16.7	0.20	27.9	12.1	19.7
33+00	2.06	1.71	1.35	1.19	14.02	11.50	82.1	17.9	0.22	29.0	13.1	22.1
33+50	2.15	1.79	1.39	1.23	14.61	12.10	82.8	17.2	0.21	27.6	11.8	19.2
34+00	2.20	1.89	1.41	1.28	14.96	12.75	85.2	14.8	0.17	26.2	10.7	16.7
34+50	2.12	1.90	1.38	1.28	14.40	12.85	89.2	10.8	0.12	26.0	10.5	16.4
35+00	2.08	1.70	1.36	1.19	14.12	11.45	81.1	18.9	0.23	29.2	13.3	17.9
35+35	1.97	1.59	1.31	1.14	13.35	10.70	80.1	19.9	0.25	31.3	15.2	26.7

[a] Velocity head, H_e, corresponding to the observed velocity. [b] Width changes from 10.0 ft to 7.0 ft between Station 23+00 and Station 23+50.

that the velocity varied on the same course between several tests. Hence, it is doubtful if this method yields much greater accuracy than the color timing method.

Observations of velocity were recorded for ten flows, ranging from 89 cu ft per sec to 1,005 cu ft per sec.

A cross wave developed at the head of the chute, due to entrance conditions, and was reflected from side-wall to side-wall, but this had practically disappeared before the first point gage station had been reached. On the convex vertical curve the flow left the bottom of the chute and caused a spray to overtop the side-walls. This was particularly true of the larger flows. This spray prevented point gage readings at the first station on the steep slope for all but

TABLE 9.—HYDRAULIC PROPERTIES, KITTITAS CHUTE

Q, cu ft per sec	HYDRAULIC PROPERTIES AND PERCENTAGE OF AIR IN WATER												
	Station	DEPTHS (FT)		HYDRAULIC RADIUS (FT)		AREA (SQ FT)		PERCENTAGES		Ratio, $\frac{\text{Col. 8}}{\text{Col. 7}}$ (9)	Observed velocity V_o (10)	Velocity head[a] H_v (11)	$\frac{V^2}{g R_e}$ (12)
		Observed y_o (1)	Computed y_e (2)	Observed R_o (3)	Computed R_e (4)	Observed A_o (5)	Computed A_e (6)	Water (ρ) (7)	Air ($1 - \rho$) (8)				
	(a) SLOPE 10° 12′												
89	1–3	0.49	0.38	0.44	0.35	3.88	3.11	79	21	0.27	28.6	12.72	72.6
	3–5	0.50	0.41	0.45	0.37	3.96	3.25	82	18	0.22	27.4	11.67	63.1
193	1–3	0.80	0.79	0.68	0.66	6.35	6.33	99	1	0.01	30.5	14.63	44.3
	3–5	0.81	0.81	0.68	0.68	6.44	6.48	100	0	0.00	29.8	13.81	40.6
	5–7	0.83	0.79	0.70	0.66	6.55	6.33	97	3	0.03	30.5	14.46	43.1
	5–9	0.82	0.76	0.69	0.64	6.48	6.11	94	6	0.06	31.6	15.53	48.5
362	1–3	1.25	1.12	0.96	0.87	9.90	9.00	91	9	0.40	40.2	25.13	57.8
	3–5	1.25	1.07	0.96	0.84	9.95	8.54	86	14	0.16	42.4	27.95	66.5
	5–8	1.24	1.14	0.96	0.89	9.85	9.10	92	8	0.09	39.8	24.63	55.3
	8–9	1.24	0.91	0.96	0.74	9.84	7.31	74	26	0.35	49.5	38.1	103.0
401	1–3	1.26	1.26	0.98	0.98	10.06	10.66	100	0	0.00	37.6	21.98	44.8
	3–5	1.26	1.07	0.97	0.86	9.98	9.52	95	5	0.05	42.1	27.56	64.1
491	1–3	1.54	1.48	1.14	1.075	12.37	11.66	94	6	0.06	42.1	27.56	51.3
	3–5	1.55	1.48	1.13	1.075	12.30	11.69	96	4	0.04	42.0	27.42	51.1
	5–8	1.56	1.38	1.14	1.02	12.36	11.01	89	11	0.12	44.6	30.93	60.6
	5–9	1.56	1.33	1.15	0.995	12.43	10.63	86	14	0.16	46.2	33.19	66.7
587	1–3	1.80	1.65	1.26	1.165	14.32	13.19	92	8	0.09	44.5	30.79	56.4
	3–5	1.80	1.61	1.26	1.145	14.33	13.98	91	9	0.10	45.6	32.33	53.2
	5–7	1.79	1.55	1.26	1.11	14.28	12.41	87	13	0.15	47.3	34.78	62.7
	7–9	1.80	1.42	1.26	1.04	14.36	11.33	93	7	0.08	51.8	41.7	80.1
719	1–3	2.02	1.93	1.36	1.30	16.11	15.43	96	4	0.04	46.6	33.76	51.9
	3–5	2.03	1.95	1.37	1.31	16.12	15.63	97	3	0.03	46.0	32.90	50.2
	5–7	2.05	1.82	1.38	1.25	16.33	14.55	89	11	0.12	49.4	37.94	57.9
	7–9	2.07	1.67	1.39	1.175	16.49	13.36	81	19	0.23	53.8	45.00	76.6
777	1–3	2.24	1.86	1.45	1.26	17.87	14.91	83	17	0.21	52.1	42.2	67.0
	3–5	2.22	1.90	1.45	1.29	17.68	15.24	87	13	0.15	51.0	40.4	62.6
	5–7	2.21	1.91	1.44	1.29	17.58	15.27	87	13	0.15	50.9	40.3	62.5
	7–9	2.26	1.89	1.46	1.28	17.97	15.12	84	16	0.19	51.4	41.1	64.2
922	1–3	2.52	2.14	1.56	1.40	20.02	17.14	86	14	0.16	53.8	45.0	64.3
	3–5	2.49	2.14	1.55	1.40	19.79	17.14	87	13	0.15	53.8	45.0	64.3
	5–7	2.50	2.26	1.55	1.44	19.84	18.08	92	8	0.09	51.0	40.4	56.1
	7–9	2.51	2.13	1.56	1.39	19.96	17.04	86	14	0.16	54.1	45.5	65.5
1,005	1–3	2.70	2.13	1.63	1.39	21.50	17.03	79	21	0.27	59.0	54.1	77.8
	3–5	2.65	2.31	1.62	1.46	21.11	18.49	88	12	0.14	54.4	46.0	63.0
	5–7	2.66	2.25	1.62	1.44	21.15	18.01	86	14	0.16	55.8	48.4	67.2
	7–9	2.66	2.37	1.62	1.49	21.19	18.93	84	16	0.19	53.1	43.8	58.8

TABLE 9.—(*Continued*)

HYDRAULIC PROPERTIES AND PERCENTAGE OF AIR IN WATER

Q, cu ft per sec	Station	Depths (Ft)		Hydraulic Radius (Ft)		Area (Sq Ft)		Percentages		Ratio, Col. 8 / Col. 7	Observed velocity V_o	Velocity head[a] H_v	$\frac{V^2}{g R_o}$
		Observed y_o	Computed y_c	Observed R_o	Computed R_c	Observed A_o	Computed A_c	Water (ρ)	Air ($1-\rho$)				
		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
(*b*) SLOPE 33° 10′													
193	12–14	1.02	0.58	0.83	0.50	8.07	4.34	55	45	0.82	44.0	30.10	120.0
	14–16	1.14	0.45	0.90	0.40	9.04	3.61	40	60	1.50	53.4	44.3	222.0
	14–15	1.14	0.45	0.90	0.40	9.04	3.57	40	60	1.50	54.0	45.3	226.0
362	11–14	1.50	0.77	1.11	0.65	11.83	6.13	52	48	0.92	59.1	54.3	167.0
	12–14	1.50	0.79	1.11	0.67	11.83	6.33	53	47	0.89	57.2	50.9	152.0
	14–15	1.63	0.80	1.17	0.67	12.88	6.41	50	50	1.00	56.5	49.6	148.0
	14–16	1.63	0.86	1.17	0.705	12.88	6.87	53	47	0.89	52.7	43.2	123.0
401	11–13	1.49	0.84	1.11	0.70	11.75	6.68	57	43	0.75	60.0	56.0	160.0
	13–16	1.51	0.80	1.12	0.67	12.08	6.43	52	48	0.92	62.4	60.5	181.0
491	12–13	1.94	1.04	1.32	0.83	15.32	8.32	55	45	0.82	59.0	54.1	130.0
	11–13	1.94	0.96	1.32	0.77	15.32	7.70	51	49	0.96	63.8	63.5	164.0
	12–14	1.94	0.99	1.32	0.80	15.32	7.91	55	45	0.82	62.1	60.0	150.0
	13–15	1.90	0.93	1.30	0.75	15.02	7.46	50	50	1.00	65.8	67.3	179.0
	14–16	1.90	1.00	1.30	0.80	15.02	7.98	53	47	0.89	61.5	58.8	147.0
587	11–13	2.05[b]	1.15	1.37	0.89	16.23[b]	9.17	57	43	0.75	64.0	63.4	142.0
	13–15	2.05	1.10	1.37	0.86	16.16	8.80	55	45	0.82	66.7	69.2	161.0
	13–16	2.05	1.09	1.37	0.86	16.16	8.70	53	47	0.89	67.5	70.8	165.0
719	11–13	2.18[b]	1.21	1.43	0.925	17.26[b]	9.70	56	44	0.79	74.1	85.4	185.0
	13–15	2.20	1.36	1.43	1.01	17.42	10.84	62	38	0.61	66.3	68.3	135.0
	13–16	2.20	1.28	1.43	0.97	17.42	10.24	58	42	0.72	70.2	76.7	158.0
777	11–13	2.35[b]	1.47	1.50	1.07	18.61[b]	11.79	63	37	0.59	65.9	67.5	126.0
	13–16	2.40	1.08	1.51	0.85	19.00	8.64	46	54	1.17	89.9	125.6	296.0
922	12–13	2.63[b]	1.58	1.60	1.13	20.83[b]	12.61	61	39	0.64	73.1	83.1	147.0
	12–14	2.63[b]	1.63	1.60	1.15	20.83[b]	13.08	63	37	0.59	70.5	77.3	134.0
	13–15	2.68	1.54	1.61	1.11	21.28	12.29	58	42	0.72	75.0	87.5	158.0
	14–16	2.68	1.68	1.61	1.18	21.28	13.44	63	37	0.59	68.6	73.1	124.0
1,005	12–15	....	1.51	....	1.09	.....	12.09	..	..	....	83.1	107.3	197.0
	13–15	....	1.53	....	1.10	.....	12.22	..	..	....	82.2	105.0	191.0
	14–16	2.94[b]	1.69	1.71	1.185	23.20[b]	13.56	58	42	0.72	74.1	85.4	144.0
(*c*) SLOPE 0° 04′													
362	18–20	1.52	1.31	1.12	0.98	12.03	10.49	87	13	0.15	34.5	18.50	37.7
491	17–20	1.82	1.14	1.06	0.89	14.41	9.13	63	37	0.59	53.8	45.0	101.0
	18–20	1.82	1.35	1.06	1.01	14.41	10.81	74	26	0.35	45.4	32.05	63.5
587	17–19	2.14	2.18	1.41	1.40	16.95	17.42	100	0	0.00	33.7	17.67	25.2
	17–20	2.14	1.50	1.41	1.09	16.95	12.00	71	29	0.41	48.9	37.18	68.2
719	17–19	2.36	1.56	1.50	1.11	18.69	12.50	67	33	0.49	57.5	51.4	92.6
	17–20	2.36	1.41	1.50	1.14	18.69	11.27	61	39	0.64	63.8	63.3	111.0
777	17–19	2.61	1.80	1.58	1.24	20.76	14.36	70	30	0.43	54.1	45.5	73.3
922	17–20	2.91	1.97	1.68	1.32	22.93	15.73	68	32	0.47	58.6	53.4	80.9
	18–20	2.91	2.06	1.68	1.35	22.93	16.46	71	29	0.41	56.0	48.8	72.3
1,005	17–20	2.90	2.02	1.68	1.34	22.98	16.16	71	29	0.41	62.2	60.2	89.9
	18–20	2.90	2.27	1.68	1.45	22.98	18.17	80	20	0.25	55.3	47.6	65.7

[a] Velocity head, H_v, corresponding to the observed velocity. [b] Point gage readings made at the end of the section because of spray.

TABLE 10.—TRAJECTORY OF WATER SURFACE AT CONVEX VERTICAL CURVE, HAT CREEK CHUTE

(395 Cu Ft per Sec)

Station	Mean velocity (ft per sec)	DEPTHS, IN FEET				ELEVATIONS, IN FEET					
		Mean		Maximum		Bottom	Surface		Maximum Computed		
		Observed	Corrected[a]	Observed	Corrected[a]		Mean	Maximum	76 ft per sec	84 ft per sec	91 ft per sec
m	....	....	....	....	....	......	......	......	302.11	369.04	433.09
y_t	....	....	....	....	....	......	......	......	14.28	17.44	20.46
x_t	....	....	....	....	....	......	......	......	65.68	80.23	94.13
2+75	75.9[b]	1.85	1.73	2.15	2.11	2,853.7	2,855.4	2,855.8	2,855.8	2,855.8	2,855.8
3+00	76.0	3.52	2.92	4.2	3.49	2,841.9	2,844.8	2,845.4	2,844.6	2,844.9	2,845.1
3+25	75.8	4.60	3.82	6.0	4.98	2,827.9	2,831.7	2,832.9	2,831.2	2,832.3	2,833.0
3+50	74.9	3.50	2.90	4.9	4.06	2,813.4	2,816.3	2,817.5	2,814.8	2,817.6	2,819.2
3+75	72.3	1.97	1.70	2.2	1.91	2,800.5	2,802.2	2,802.4	2,795.4	2,799.7	2,802.6

[a] Corrected for slope. [b] Total drop at Station 2+75 is 129.3 ft. The theoretical velocity equals 91.3 ft per sec.

the low flow; and even at the second station readings were prevented on the highest flows.

From the data furnished by the U. S. Bureau of Reclamation, calculations of the retardation factor have been made by the three methods previously described from observations for the 10° 12′ slope and for the 33° 10′ slope. Velocity observations were not taken on the vertical curves or in the taper section, which prevents the calculation of a complete energy profile for this chute.

TABLE 11.—SUPERELEVATION OF WATER SURFACE, SOUTH CANAL CHUTE; ANOTHER PROJECT

(Radius of Curve, 142 Ft)

Station	Observed velocity V_o	Tan ψ	GAGE READINGS		Difference in elevation (ft)	Slope[a] in a 7-ft width (tan ψ)
			Left side	Right side		
23+00	16.6	0.060	1.88	2.75	0.87	0.124
23+25	19.2	0.081	2.15	2.40	0.25	0.036
23+50	21.7	0.103	1.60	2.90	1.30	0.186
23+75	25.2	0.139	1.70	2.60	0.90	0.129
24+00	28.8	0.181	1.40	2.60	1.20	0.172
24+25	29.6	0.192	1.40	2.55	1.15	0.164
24+50	30.5	0.204	1.20	2.75	1.55	0.222
24+75	32.0	0.224	1.20	2.50	1.30	0.186
25+00	33.5	0.245	0.70	3.45	2.75	0.394
25+25	34.3	0.257	0.70	3.35	2.65	0.379
25+50	35.1	0.269	0.90	2.90	2.00	0.286

[a] The computed tangent agrees reasonably well with the observed tangent. The superelevation of Station 23+00 or the P.C. of the curve is due to cross waves resulting from the curvature of the channel upstream.

The hydraulic properties of the water prism, together with percentage of air in the water and the values of the parameter $\frac{V^2}{g\,R_c}$, were tabulated for each of the observations in Table 9.

ANALYSES OF RESULTS

The values of the retardation factor conform, in general, with the range discussed under the heading, "Theory of Flow in Chutes." Exceptions were noted near the inlets where high values were recorded as a result of entrance conditions. Similar high values occurred below the convex vertical curve on the Hat Creek chute due to the spreading of the cross section as the water left the bottom of the chute. Also, on the sharp horizontal curve in the South Canal chute an increased friction loss was noted. In each chute the most representative values in both the Manning and Kutter formulas have been tabulated, leaving the exceptions for separate comment.

The retardation factors obtained from the observed areas and observed velocities are equivalent to those observed in normal channels. This value, therefore, has been designated as n. The retardation factor obtained by using the observed velocity and the area computed by $\frac{Q}{V_o}$ is designated as n_c, and the values obtained by using the observed areas and the velocity computed by $\frac{Q}{A_o}$ has been designated as n_o. The use of observed velocities and observed areas gives results for the retardation factor comparable with those obtained in wood and concrete channels under normal flow conditions as contrasted with the shooting flow being tested. The values of n_c are less than normal values and correspond to those which could be computed by Eq. 26. The value of n_o is considerably larger than the results obtained by the first two methods.

The results obtained for n_o on all chutes (see Table 12) are the most inconsistent of the three methods used, largely due to the vagaries of the computed energy gradient. Therefore, this method has little to commend it either in theory or in practice.

TABLE 12.—COMPARISON OF RETARDATION FACTORS

Formula	HAT CREEK		RAPID FLUME		SO. CANAL CHUTE		KITTITAS CHUTE			
							Slope, 10° 12′		Slope, 33° 10′	
	From:	To:	From:	To:	From:	To:	From:	To:	From:	To:
n—										
Manning	0.010	0.013	0.010	0.012	0.012	0.013	0.015	0.016	0.019	0.021
Kutter	0.010	0.013	0.011	0.012	0.012	0.013	0.015	0.016	0.019	0.021
n_c—										
Manning	0.008	0.010	0.008	0.010	0.011	0.012	0.014	0.015	0.014	0.015
Kutter	0.008	0.010	0.008	0.010	0.011	0.012	0.014	0.015	0.014	0.015
n_o—										
Manning	0.024	0.031	0.014	0.019	0.015	0.016	0.016	0.018	0.030	0.055
Kutter	0.023	0.030	0.014	0.018	0.015	0.016	0.016	0.018	0.030	0.045

The values for the Hat Creek chute include the tests for 115, 163, 368, and 395 cu ft per sec as observed on the straight section of the channel between Stations 0+50 and 2+75. The retardation factor materially increased on the steep grade below the convex vertical curve and in the first portion of the bucket. Using the observed values of velocity and area, the values of Kutter's and Manning's n average about 0.020 in this section of the channel. The values of n_c and n_o correspondingly increase over values previously cited by the other methods of computation. The increase in the retardation factor in this section of the channel is due to the increased friction resulting from the path taken by the trajectory of the flow around the vertical curve. This trajectory has been calculated for $Q = 395$ cu ft per sec by means of Eq. 31, using in the calculations the mean velocity and also velocities 10% and 20% above the mean velocity. These results are given in Table 10.

The mean velocity at the start of the vertical curve was 76 ft per sec, but a comparison of the results in Table 10 indicates that the water surface follows a trajectory corresponding to a velocity of 91 ft per sec. This is equal to the theoretical velocity of the water falling from the inlet of the chute to the start of the vertical curve. In a short channel the water surface in the center of the chute, therefore, is not retarded by friction, but travels at the theoretical velocity. In compensation, the velocity at the side and bottom of the incline necessarily must travel at less than the mean velocity, with the result that on vertical curves the bottom layers should adhere to the floor of the chute, whereas the upper layers separate from the water prism, throwing the stream into a spray. Air resistance ultimately slows the surface velocity as a comparison of the calculations with the observations shows that the surface curve drops suddenly at Station 3+25 after following the trajectory for a velocity of 91 ft per sec for a distance of 50 ft. The increase in the retardation factor in this section of the chute indicates, as previously mentioned, that there exists a possibility of dissipating energy by this means. The same principle is utilized to some extent in discharging water from a chute bucket above the water surface of the discharge pool. The throwing of the jet into a spray assists in dissipating the energy, in addition to a further dissipation of energy by eddies and turbulence in the pool itself.

The results on Rapid Flume correspond in a large measure to those obtained from the tests on the Hat Creek chute. In this case the convex vertical curve afforded a uniform retardation factor throughout the length of the chute after the turbulent conditions at the inlet had been passed.

On the South Canal chute the computed retardation factors are more erratic. Some of the apparent variation is probably due to faulty observations of area or velocities. The average results for the entire length of channel are reasonable, although when the roughness of the bottom is considered, a value of n between 0.014 and 0.015 would have been expected, rather than the lower value actually obtained.

On the steep section of the chute constructed on the 142-ft radius, located between Station 23+00 and Station 25+00, the value of n_o averaged 0.021. This was due to the fact that the increased insufflation of air in this portion of the channel increased the cross section and correspondingly reduced the computed values of the velocity.

The superelevation of the water surface around the 142-ft radius curve has been tabulated and compared with the theoretical values computed according to Eq. 32. These results are given in Table 11. The agreement between the actual and theoretical results is sufficiently close to indicate that it is evident that the flow around this curve would have been improved by the superelevation of the bottom of the channel in accordance with the theoretical values.

The flow, thereby, would have been kept within the channel walls and the overtopping of the outer wall by spray would have been avoided. At the other curves along the channel the radius of curvature was not sufficiently sharp to cause overtopping of the wall. Hence, the necessity for superelevation of the bottom is not so essential.

On the Kittitas chute the slope of the energy gradient could be determined only from velocity observations taken over relatively short detached portions of the chute. The energy gradient computed from these data proved to be erratic, but as the average of the results approximated the bottom slope of the chute, this slope was used for the energy gradient. The values of the retardation factor are larger than those obtained on the tests of the other three chutes, but the difference may be due in part to the fact that the slope of the energy gradient is actually less than the bottom slopes, or from unusual roughness of the bottom of the channel not apparent from the information available. The high value of n on the 33° 10′ slope based on observed areas and velocities is undoubtedly due in part to the stream having been thrown into spray on the convex vertical curve immediately above this incline (see Table 12). The same effect was noted on the test at the Hat Creek chute.

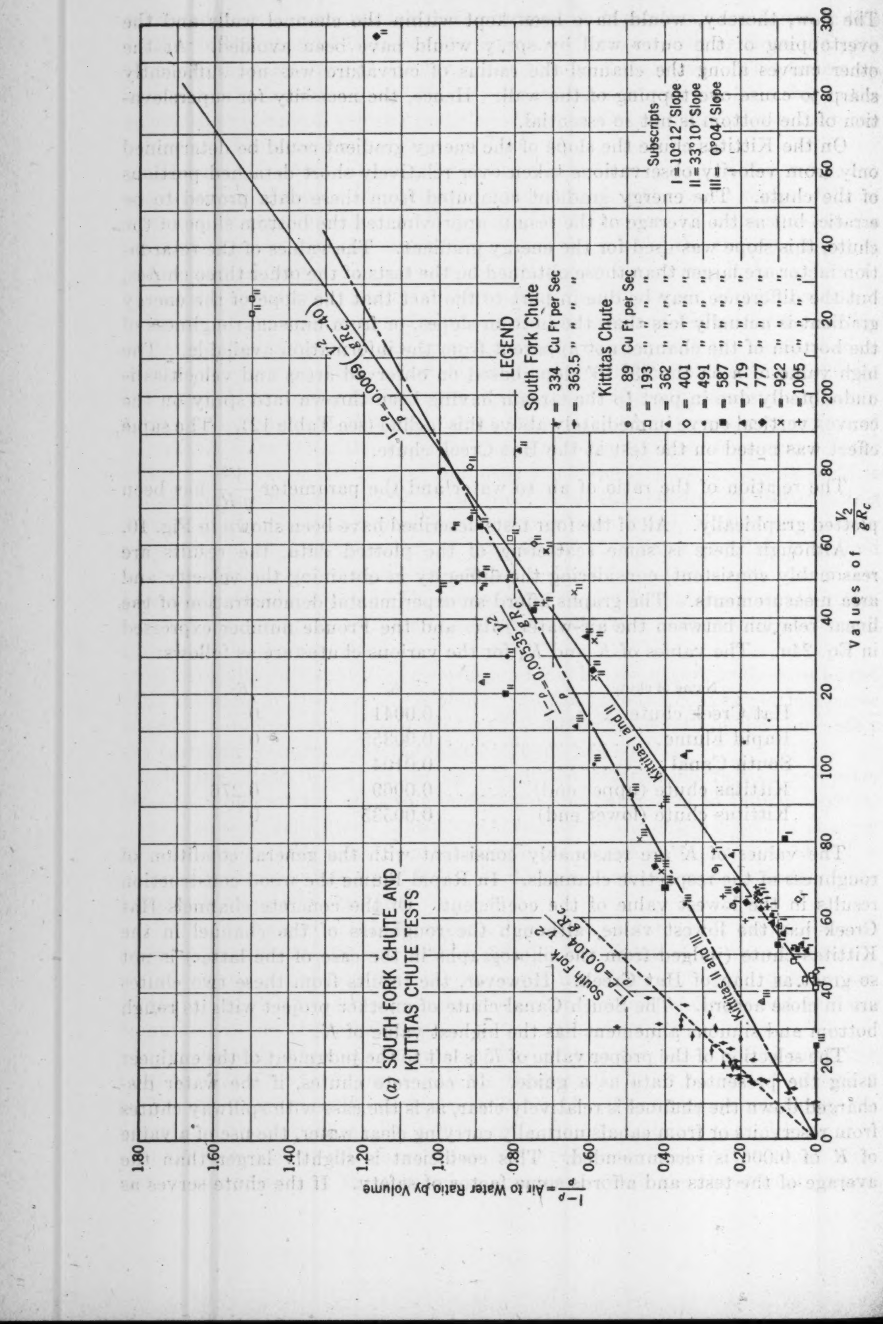
The relation of the ratio of air to water and the parameter $\frac{V^2}{g R_c}$ has been plotted graphically. All of the four tests described have been shown in Fig. 10.

Although there is some scattering of the plotted data, the results are reasonably consistent, considering the difficulty in obtaining the velocity and area measurements. The graphs afford an experimental demonstration of the linear relation between the air-water ratio and the Froude number expressed in Eq. 24*a*. The values of K and K_1 for the various chutes are as follows:

Name of chute	K	K_1
Hat Creek chute	0.0041	0
Rapid Flume	0.00355	0
South Canal	0.0104	0
Kittitas chute (upper end)	0.0069	0.276
Kittitas chute (lower end)	0.00535	0

The values of K are reasonably consistent with the general condition of roughness of the respective channels. In Rapid Flume the wood construction results in the lowest value of the coefficient. Of the concrete channels Hat Creek has the lowest value, although the roughness of the channel in the Kittitas chute (judged from the photographs in the case of the latter) is not so great as than of Hat Creek. However, the results from these two chutes are in close accord. The South Canal chute of another project with its rough bottom and sinuous alinement has the highest value of K.

The selection of the proper value of K is left to the judgment of the engineer using the presented data as a guide. In concrete chutes, if the water discharged down the channel is relatively clear, as is the case with spillway chutes from reservoirs or from canals normally carrying clear water, the use of a value of K of 0.006 is recommended. This coefficient is slightly larger than the average of the tests and affords some factor of safety. If the chute serves as


(a) SOUTH FORK CHUTE AND KITTITAS CHUTE TESTS
$\frac{1-\rho}{\rho}$ = Air to Water Ratio by Volume
1.80
1.60
1.40
1.20
1.00
0.80
0.60
0.40
0.20
0
Values of $\frac{V^2}{gR_c}$
0
20
40
60
80
100
20
40
60
80
200
20
40
60
80
300
$\frac{1-\rho}{\rho}=0.0069\left(\frac{V^2}{gR_c}\right)$
$\frac{1-\rho}{\rho}=0.00535\frac{V^2}{gR_c}$
$\frac{1-\rho}{\rho}=0.004\frac{V^2}{gR_c}$
Kittitas I and II
South Fork
Kittitas II and III
LEGEND
South Fork Chute
■ = 334 Cu Ft per Sec
● = 363.5 " " "
Kittitas Chute
■ = 89 Cu Ft per Sec
□ = 193 " " "
▲ = 362 " " "
◇ = 401 " " "
● = 491 " " "
■ = 587 " " "
▲ = 719 " " "
+ = 777 " " "
× = 922 " " "
+ = 1005 " " "
Subscripts
I = 10°12' Slope
II = 33°10' Slope
III = 0°04' Slope

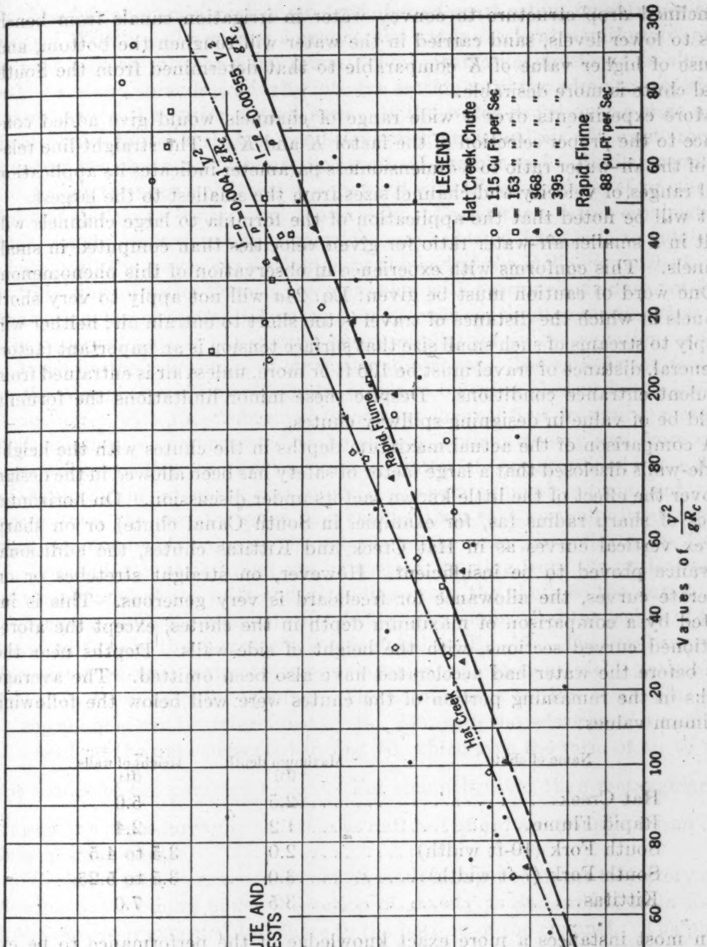


FIG. 10.—RELATION OF AIR-WATER RATIO TO $\frac{V^2}{gR_c}$

an inclined drop structure to convey water in irrigation canals from bench lands to lower levels, sand carried in the water will roughen the bottom, and the use of higher value of K comparable to that determined from the South Canal chute is more desirable.

More experiments over a wide range of channels would give added confidence to the proper selection of the factor K and K_1. The straight-line relation of the air-water ratio to a dimensionless parameter indicates its application to all ranges of velocity and channel sizes from the smallest to the largest.

It will be noted that the application of the formula to large channels will result in a smaller air-water ratio for given velocities than computed in small channels. This conforms with experience in observation of this phenomenon.

One word of caution must be given: Eq. 24*a* will not apply to very short channels in which the distance of travel is too short to entrain air; neither will it apply to streams of such small size that surface tension is an important factor. In general, distance of travel must be 100 ft or more, unless air is entrained from turbulent entrance conditions. Despite these minor limitations the formula should be of value in designing spillway chutes.

A comparison of the actual maximum depths in the chutes with the height of side-walls disclosed that a large factor of safety has been allowed in the design to cover the effect of the little known factors under discussion. On horizontal curves of sharp radius (as, for example, in South Canal chute) or on sharp convex vertical curves as in Hat Creek and Kittitas chutes, the additional allowance proved to be insufficient. However, on straight stretches or on moderate curves, the allowance for freeboard is very generous. This is indicated by a comparison of maximum depth in the chutes, except the aforementioned curved sections, with the height of side-walls. Depths near the inlet before the water had accelerated have also been omitted. The average depths in the remaining portion of the chutes were well below the following maximum values:

Name of chute	Maximum depth (ft)	Height of walls (ft)
Hat Creek	2.5	5.0
Rapid Flume	1.2	2.4
South Fork (10-ft width)	2.0	3.5 to 4.5
South Fork (7-ft width)	3.0	3.5 to 5.25
Kittitas	3.5	7.0

In most instances a more exact knowledge of the performance to be expected of the water at high velocities would have permitted a reduction in the freeboard allowance with a corresponding economy in the cost of the structure.

CONCLUSIONS

The flow of water on steep gradients is essentially different from that found in channels of ordinary slope. Velocities are greater than the critical, and range from a minimum value of 10 to 15 ft per sec to maximum values of from 80 to 100 ft per sec. The kinetic energy greatly exceeds the static pressure of the water prism.

The initial entrainment of air depends to a great extent on entrance conditions. With a well-designed entrance, the water accelerates uniformly, gradually drawing air into the water prism. This insufflation develops first along the side-walls and spreads gradually into the section until the admixture of air and water covers the entire wetted area. If the water drops over a weir at the inlet to the chute, turbulent conditions with entrained air will exist from the head of the chute throughout the length of the channel. Similar conditions exist when the approach is not in alinement with the chute channel, or when the inlet of the chute is narrowed too rapidly. Under these latter circumstances high, rough, cross waves develop which are reflected from side to side of the chute until the waves are dissipated or the end of the incline is reached.

The flow of water down a steep channel with entrained air does not follow the normal law of hydraulics, as expressed by the continuity equation $Q = A\ V$. The observed area of the stream, before application in the equation, must be reduced by the percentage of air in the admixture. The retardation factors n in the Manning or Kutter formula, obtained by using observed velocities and areas, agree with those obtained in normal channel flow in similar materials. This relation is demonstrated to be correct, both in theory and from actual observations. For practical purposes computations are simplified in the design of chutes by the use of the computed area, assuming no air entrainment. This necessitates the use of a value of n smaller than that normally used, and one that depends for its value on the velocity of the water, as indicated by Eq. 26.

Bernoulli's equation then can be written, starting from the critical section of the channel at the inlet and extending down the chute to the outlet. For rectangular chutes, increments of velocity are assumed and the length of travel required to gain this increment is calculated by Eq. 17. The resulting velocities are then true water velocities, but such calculations yield wetted areas less than those actually existing. The actual cross section of air plus water in the channel can be determined by Eq. 24*b*, with the selection of a coefficient K based on the data presented in Fig. 10, which give the ratio of air to water for values of the parameter $\frac{V^2}{g\,R_e}$. For channels other than rectangular, Eq. 27 and the procedure of calculation as outlined afford a rational solution of the flow in steep chutes.

Convex vertical curves must not be made sharper than the trajectory of the flowing water falling under the action of gravity as determined from Eq. 31, if the stream is expected to adhere to the bottom of the channel. The portion of the water prism near the surface and in the center of the channel travels at a velocity from 15% to 20% greater than the mean velocity in the cross section and this maximum velocity must be used in computing trajectories. In chutes of short length the maximum velocity will be found to be equal to the theoretical velocity, resulting from the vertical drop between the inlet and the point of vertical curvature. If the side-walls are of sufficient height to prevent overtopping, no serious consequence results from the jet leaving the bottom of the channel. The evidence indicates that more energy is dissipated when the stream cross section is expanded by leaving the floor of the chute than would otherwise be the case. Under extreme conditions damage to the concrete floor of the chutes may result from cavitation.

Flow around horizontal curves should be avoided whenever possible. If used, they should be installed at the head of the chute before the water has had opportunity to gain great momentum. On large channels they should never be used unless the performance of the water has been verified by model tests. Model tests are also desirable to check entrance conditions. The superelevations of the water surface on curves correspond with that computed by the centrifugal theory as indicated by Eq. 32. In cases where the channel is too wide with respect to the depth, a large percentage of the water will either follow around the outside of the curve, leaving the inner portion of the floor exposed, or in some instances the slope of the superelevation will not be uniform across the wetted section, due to the variation in velocities across the water prism. On sharp curves, if the computed superelevation of the water surface exceeds a value of $\tan \psi = 0.120$, the superelevation of the bottom is recommended.

More experimental data over a wider range of channel sizes and with varying degrees of roughness would add to the engineering knowledge of the phenomenon of flow at high velocities.

ACKNOWLEDGMENTS

The field work on these tests was done under the direction of the late C. E. Grunsky, Jr., M. Am. Soc. C. E. Acknowledgment is made for the cooperation of I. C. Steele, M. Am. Soc. C. E., and F. R. George of the Pacific Gas and Electric Company in arranging for the tests and for maintenance of uniform flow conditions during the observations. Credit for the collection of the data must be given to those employees of the East Bay Municipal Utility District attached to the Hydrographic Division, who took part in the tests and contributed to their success.

Indebtedness is acknowledged to B. T. Ploeger and R. E. Hall, Jun. Am. Soc. C. E., for valuable assistance in the calculations and the preparation of the drawings. The kindness of the U. S. Bureau of Reclamation in permitting access to the results of their tests on the Kittitas chute of the Yakima Project is appreciated.

The tabulations, figures, and photographs have been condensed in this paper, but the complete original manuscript has been placed on file for reference at Engineering Societies Library in New York, N. Y.

CLOSED CONDUIT FLOW

By A. A. Kalinske,[18] Assoc. M. Soc. C. E., and James M. Robertson,[19] Jun. Am. Soc. C. E.

Synopsis

The general problem treated in this paper is the study of air entrainment by flowing water in pipes, particularly in reference to the removal of air pockets from water supply lines. The formation of a hydraulic jump in a pipe is an important method by which air can be removed, and this paper presents data relating to the ability of a hydraulic jump to entrain air and have it carried away by the flowing water. Quantitative data are presented relating the characteristics of the jump with the rate of air removal, and also data regarding the hydraulic features of jumps in closed conduits.

The General Problem

When an air pocket is formed at the summit of a pipe line or beyond a partly open gate valve, the conduit flows only partly full, and at the end of the pocket a hydraulic jump usually occurs. The air may be removed, of course, by use of an air valve; but this is not feasible if the pipe line is under sub-atmospheric pressure. In any case, it is of interest to know under what conditions an air valve may be necessary or, if an air valve cannot be or is not used, whether the air can be removed by the flowing water.

Since the hydraulic jump at the end of an air pocket is a common phenomenon, it was decided that the first phase of this investigation should be concerned with the ability of a hydraulic jump to entrain air. This paper is specifically concerned with this particular item. Of course, the entrainment of air by the hydraulic jump is not the sole item to consider in the general problem of air removal from pipe lines; but it is an important feature of the general problem.

The violent eddies of a hydraulic jump at the end of an air pocket will entrain air and pump this air into the conduit beyond the jump. Whether all the air entrained by the jump will be carried along by the flowing water beyond the jump is another problem to be considered. Also the problem of the movement of finite bubbles of air in the pipe line beyond a hydraulic jump must be considered. This may be the entire problem involved if the conduit is at a very flat slope and no jump forms. K. R. Kennison,[20] M. Am. Soc. C. E., has indicated that if the slope of the conduit is not greater than the hydraulic gradient, such bubbles will be moved along without any difficulty. He also discusses the use of the hydraulic jump to remove air from conduits.

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[18] Asst. Prof., Univ. of Iowa, Iowa Inst. of Hydraulic Research, Iowa City, Iowa.

[19] Asst. Prof., Dept. of Eng. Mechanics, Pennsylvania State College, State College, Pa.; formerly Research Fellow in Hydraulics, Univ. of Iowa, Iowa City, Iowa.

[20] "The Design of Pipe Lines," by K. R. Kennison, *Journal*, N. E. W. W. A., Vol. 47, 1933, pp. 27–39.

EXPERIMENTAL APPARATUS AND PROCEDURE

To study the entrainment of air by a hydraulic jump in circular pipes, a pipe line made of 6-in. transparent "Lucite" was set up in the laboratory. The line was about 35 ft long with an inside diameter, D, of 0.49 ft. The line could be placed at any slope up to 30%. Water was discharged into it through a sliding gate, the head on which could be controlled; thus the depth and velocity could be varied independently. Fig. 11 is a photograph of the pipe line at a 30% slope. Note the hydraulic jump and the large volume of air beyond it, indicating the jump is pumping a large quantity of air.

FIG. 11.—GENERAL VIEW OF 6-IN. TRANSPARENT PIPE INSTALLATION

The rate of air pumped by the jump was measured by means of standard orifices. To maintain steady conditions air was supplied from an independent source to the pocket above the jump at a rate equal to the rate at which the jump was pumping air. At the lower end of the pipe line the water flowed into a rectangular vertical box open at the top to permit the escape of the air. A gate on this box controlled the water level in it and thereby the pressure on the pipe line. The rate of water flow was measured by calibrated orifices.

Studies were made with the line horizontal and at slopes of 0.2%, 2.0%, 5.0%, 10.0%, 20.0%, and 30.0%. For any particular slope various gate openings were used thus determining the depth of flow ahead of the jump. A number of different water discharges were run for each gate opening. Water depth and pressure readings were taken along the pipe line for each run. The water flow approaching the jump was always kept under atmospheric pressure. This item did not have any influence on the results since a higher or lower pressure could have been maintained simply by raising or lowering the pressure on the entire system.

NOTATION

The letter symbols used in this paper are defined where they first appear and are assembled for convenience of reference in the Appendix.

RESULTS ON AIR DEMAND OF JUMP

The results reported herein are for conditions in which all the air entrained by the jump is carried out of the pipe line. The general flow conditions obtained are typically illustrated in Fig. 12, where y_1 is the initial depth, Q_w is the water discharge, Q_a is the air discharge, and $\mathbf{F}_1$ is the Froude number of the approaching flow defined as $V_1/\sqrt{g\,y_e}$, where V_1 is the approach velocity, and y_e, the effective depth, is the water area divided by the surface width. The air pumped into the flowing water by the jump forms a large pocket beyond the jump which extends to the point where the air leaves the pipe line. For low water flows and especially at the low slopes a single long pocket did not exist, but rather a series of relatively large bubbles was present, which moved downstream. Where a single long pocket did occur the water flow in this pocket was at the normal depth for that particular slope. The air in this downstream pocket was under higher pressure than the air ahead of the jump, since the pressure head of the lower stage of the jump was always greater than the pipe diameter.

For any given initial depth of flow and slope of pipe, as the water discharge was decreased, a point was reached where the air pumped into the jump exceeded the rate at which air was carried along the pipe beyond the jump. Visually this critical point was quite apparent. The air pumped by the jump would form a large bubble just beyond the jump. This bubble did not extend to the end of the pipe, but some of the air from it would be carried downstream in smaller bubbles. However, periodically this large bubble would blow out over the jump causing the jump to drop downstream; then another jump would form and the process would be repeated. It appeared that for discharges below this critical value the average rate of air removal from the pipe line was controlled by the flow conditions below the jump and not by the jump itself.

Considering the rate of air entrainment by a hydraulic jump in a circular pipe, general analyses indicated that for geometrically similar conditions it should depend on the water discharge and the intensity of agitation or turbulence of the jump. The relative intensity or violence of the jump is dependent on the initial Froude number, $\mathbf{F}_1$, which in open channel flow is found to be related to the energy loss in the jump. The slope of the pipe line, S, might also affect the air entrainment by the jump; but this did not seem so likely. Dimensional considerations thus lead to the following functional relationship:

$$\frac{Q_a}{Q_w} = \text{function}\left(\mathbf{F}_1, \frac{y_1}{D}, S\right) \ldots\ldots\ldots\ldots\ldots\ldots (33)$$

A plotting of the data obtained did not reveal any particular variation of the air-water ratio with y_1/D or S. The only consistent variation was with $\mathbf{F}_1$ (see Fig. 13). The air-water ratio is plotted against $(\mathbf{F}_1 - 1)$, since no hydraulic jump would occur when $\mathbf{F}_1 = 1$. From this it appears that the functional relationship of Eq. 33 can be expressed by the equation

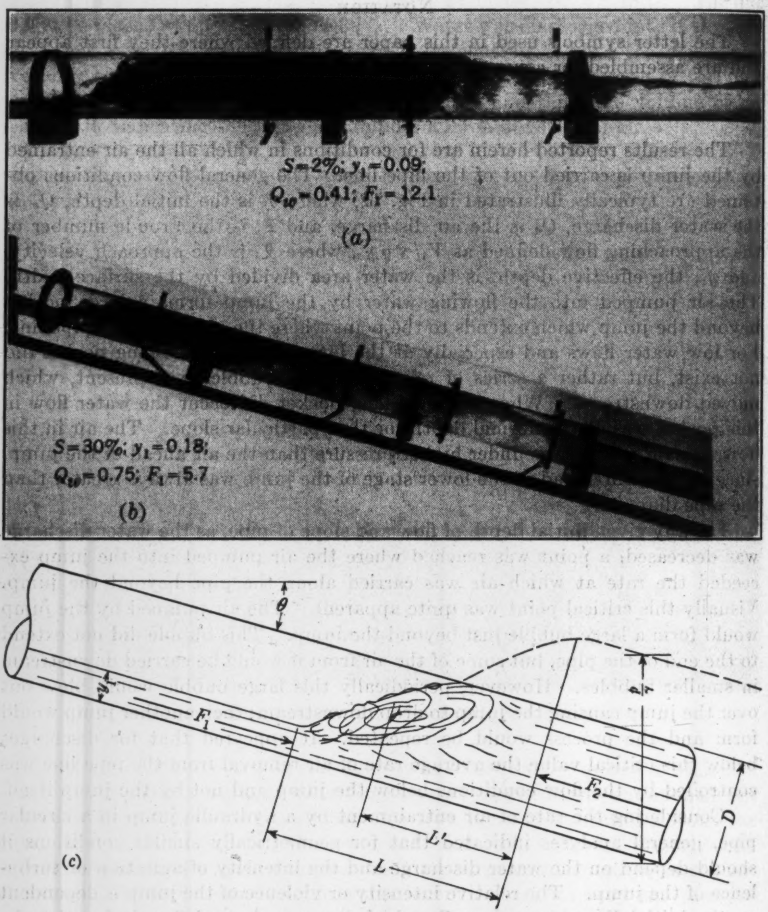
S=2%; y_1=0.09;
Q_w=0.41; F_1=12.1
(a)
S=30%; y_1=0.18;
Q_w=0.75; F_1=5.7
(b)
θ
y_1
F_1
y_2
F_2
L′
L
D
(c)

FIG. 12.—ELEMENTS OF HYDRAULIC JUMP IN SLOPING PIPE LINE

$$\beta = \frac{Q_a}{Q_w} = 0.0066\,(\mathbf{F}_1 - 1)^{1.4} \quad \text{.................} \quad (34)$$

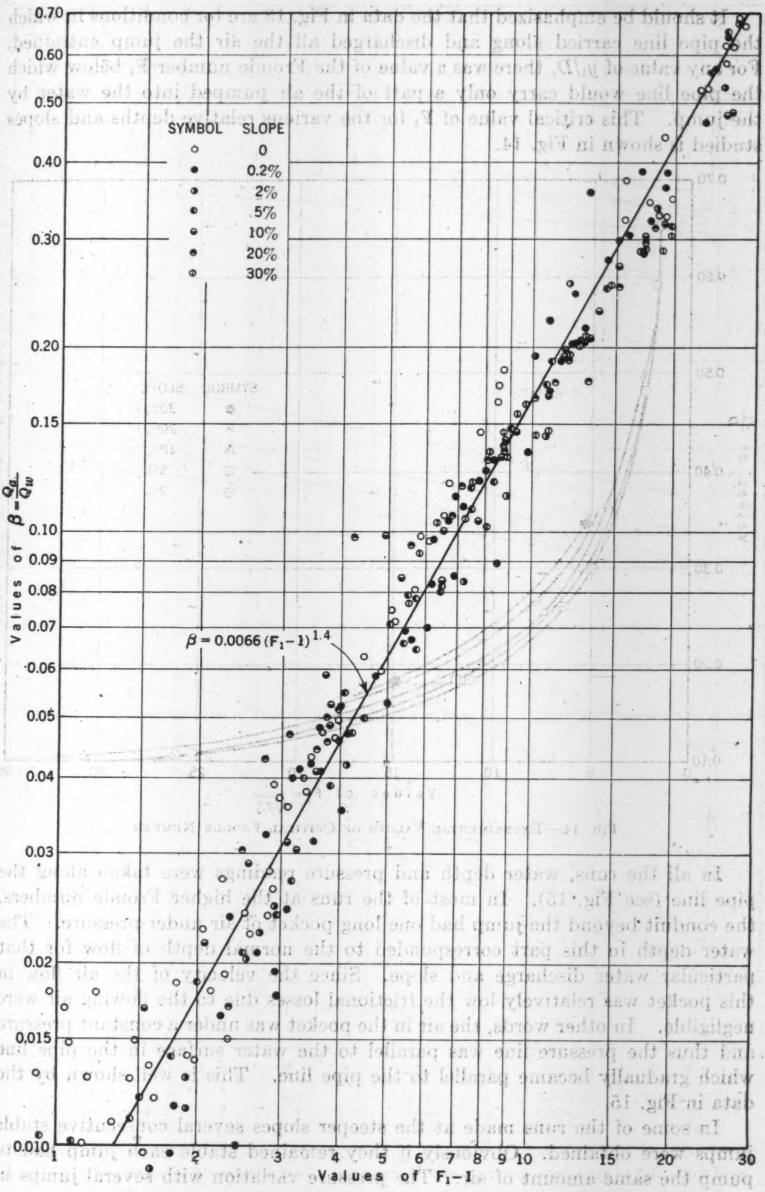

FIG. 13.—CORRELATION OF DATA ON RATE OF AIR ENTRAINMENT BY HYDRAULIC JUMP

It should be emphasized that the data in Fig. 13 are for conditions in which the pipe line carried along and discharged all the air the jump entrained. For any value of y_1/D, there was a value of the Froude number $\mathbf{F}_1$ below which the pipe line would carry only a part of the air pumped into the water by the jump. This critical value of $\mathbf{F}_1$ for the various relative depths and slopes studied is shown in Fig. 14.

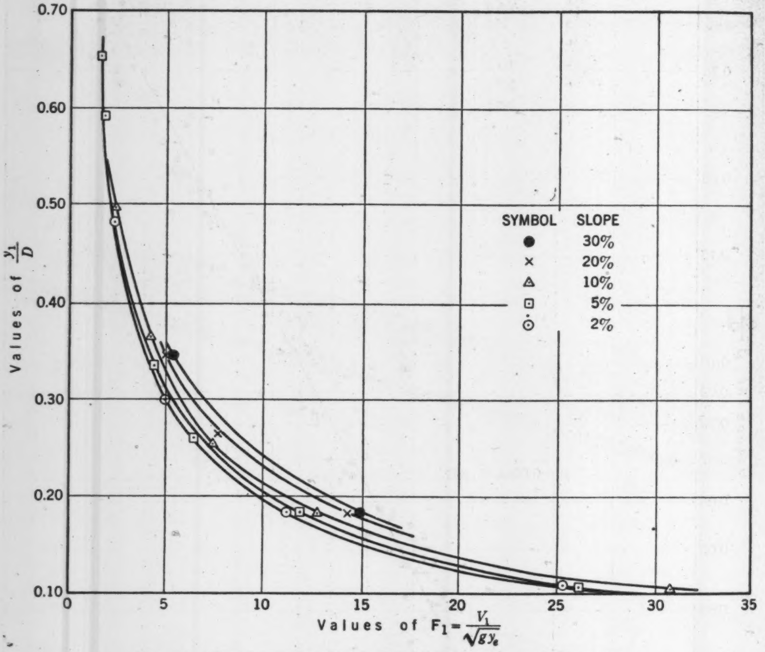

FIG. 14.—EXPERIMENTAL VALUES OF CRITICAL FROUDE NUMBER

In all the runs, water depth and pressure readings were taken along the pipe line (see Fig. 15). In most of the runs at the higher Froude numbers, the conduit beyond the jump had one long pocket of air under pressure. The water depth in this part corresponded to the normal depth of flow for that particular water discharge and slope. Since the velocity of the air flow in this pocket was relatively low the frictional losses due to the flowing air were negligible. In other words, the air in the pocket was under a constant pressure and thus the pressure line was parallel to the water surface in the pipe line which gradually became parallel to the pipe line. This is well shown by the data in Fig. 15.

In some of the runs made at the steeper slopes several consecutive stable jumps were obtained. Obviously if they remained stable each jump had to pump the same amount of air. The pressure variation with several jumps in

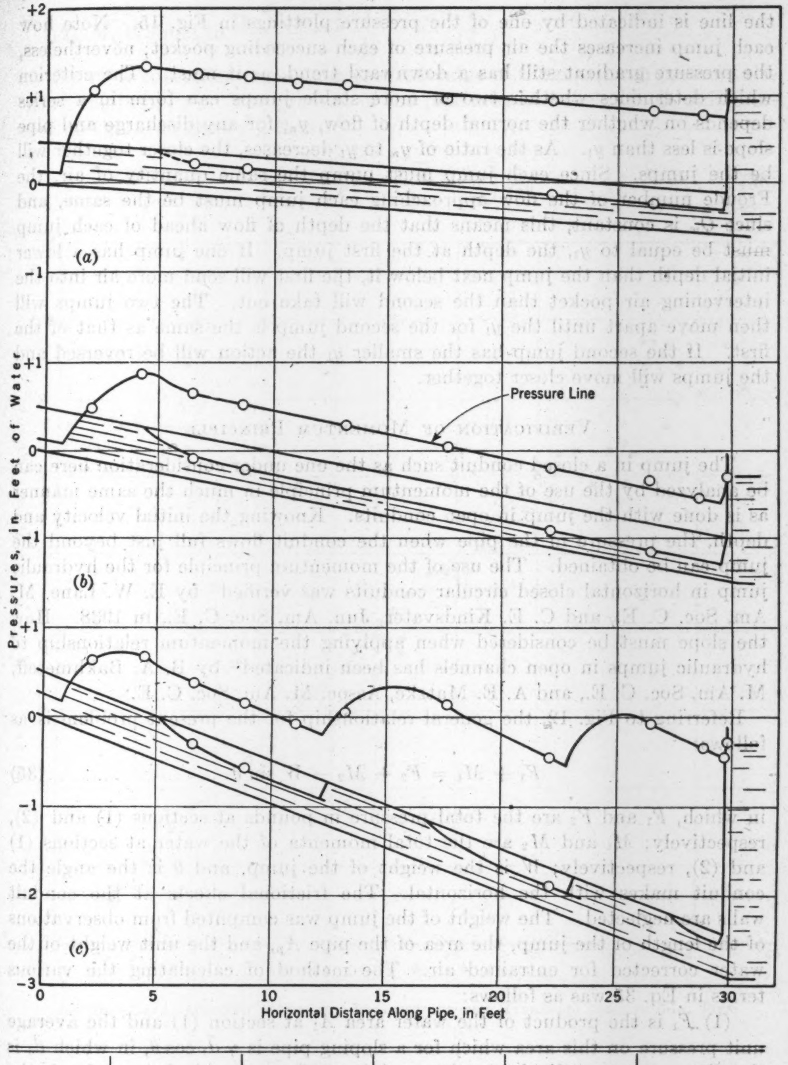

Fig.	y_1	F_1	Q_w	Q_a	S
(a)	0.147	8.2	0.720	0.060	2%
(b)	0.127	7.5	0.482	0.039	5%
(c)	0.290	2.6	0.848	0.014	10%

FIG. 15.—TYPICAL DATA ON WATER DEPTHS AND PRESSURES BEYOND HYDRAULIC JUMP IN 6-IN. PIPE LINE

the line is indicated by one of the pressure plottings in Fig. 15. Note how each jump increases the air pressure of each succeeding pocket; nevertheless, the pressure gradient still has a downward trend, as it must. The criterion which determines whether two or more stable jumps can form in a series depends on whether the normal depth of flow, y_n, for any discharge and pipe slope is less than y_1. As the ratio of y_n to y_1 decreases, the closer together will be the jumps. Since each jump must pump the same quantity of air, the Froude number of the flow approaching each jump must be the same, and since Q_w is constant, this means that the depth of flow ahead of each jump must be equal to y_1, the depth at the first jump. If one jump has a lower initial depth than the jump next below it, the first will send more air into the intervening air pocket than the second will take out. The two jumps will then move apart until the y_1 for the second jump is the same as that of the first. If the second jump has the smaller y_1 the action will be reversed and the jumps will move closer together.

VERIFICATION OF MOMENTUM PRINCIPLE

The jump in a closed conduit such as the one under consideration here can be analyzed by the use of the momentum principle in much the same manner as is done with the jump in open conduits. Knowing the initial velocity and depth, the pressure in the pipe when the conduit flows full just beyond the jump can be obtained. The use of the momentum principle for the hydraulic jump in horizontal closed circular conduits was verified[21] by E. W. Lane, M. Am. Soc. C. E., and C. E. Kindsvater, Jun. Am. Soc. C. E., in 1938. How the slope must be considered when applying the momentum relationship to hydraulic jumps in open channels has been indicated[22] by B. A. Bakhmeteff, M. Am. Soc. C. E., and A. E. Matzke, Assoc. M. Am. Soc. C. E.

Referring to Fig. 12, the general relationship for the present problem is as follows:

$$F_1 + M_1 = F_2 + M_2 - W \sin \theta \dots\dots\dots\dots\dots (35)$$

in which, F_1 and F_2 are the total pressure in pounds at sections (1) and (2), respectively; M_1 and M_2 are the total momenta of the water at sections (1) and (2), respectively; W is the weight of the jump, and θ is the angle the conduit makes with the horizontal. The frictional effects at the conduit walls are neglected. The weight of the jump was computed from observations of the length of the jump, the area of the pipe A_p, and the unit weight of the water corrected for entrained air. The method of calculating the various terms in Eq. 35 was as follows:

(1) F_1 is the product of the water area A_1 at section (1) and the average unit pressure on this area which for a sloping pipe is $\gamma \bar{d}_1 \cos \theta$, in which $\bar{d}_1$ is the distance, perpendicular to the conduit, to the centroid of A_1, and γ is the unit weight of water.

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[21] "Hydraulic Jump in Enclosed Conduits," by E. W. Lane and C. E. Kindsvater, *Engineering News-Record*, Vol. 121, 1938, pp. 815–817.

[22] "The Hydraulic Jump in Sloped Channels," by Boris A. Bakhmeteff and Arthur E. Matzke, *Transactions*, A. S. M. E., Vol. 60, 1938, pp. 111–118.

(2) M_1 is equal to $\frac{Q_w V_1 \gamma}{g}$.

(3) F_2 is computed from the measured value of the maximum pressure rise below the jump, p_2. The product of the pipe area and the average unit pressure on this area gives F_2, thus:

$$F_2 = \left(p_2 - \frac{\gamma D}{2} \cos \theta \right) A_p \ldots\ldots\ldots\ldots\ldots\ldots (36)$$

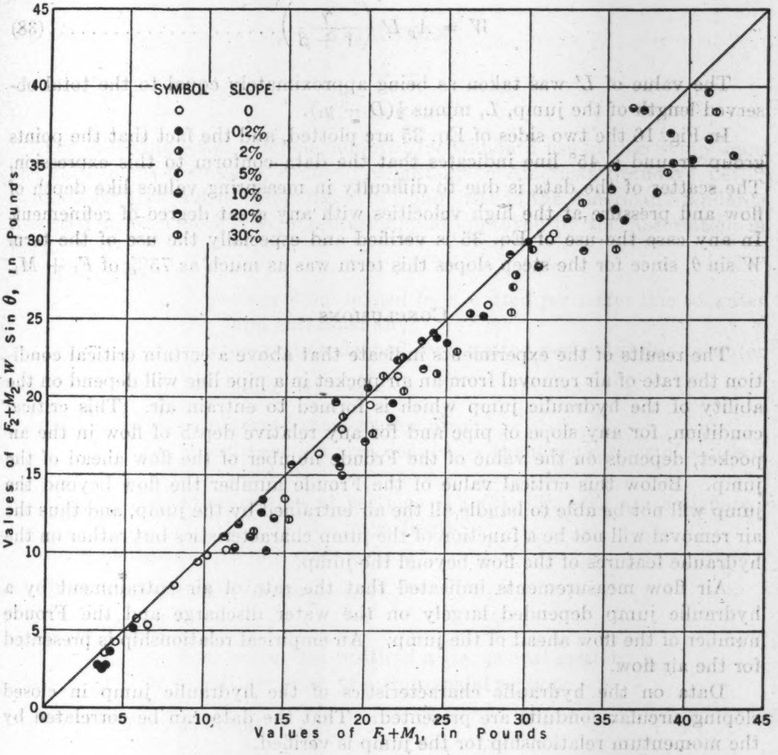

FIG. 16.—DATA CHECKING MOMENTUM RELATIONSHIP FOR HYDRAULIC JUMP IN 6-IN. PIPE

(4) M_2 is equal to $\frac{Q_w V_2 \gamma}{g}$. In calculating V_2, the fact that at section (2) the entire conduit is not filled with water should be taken into account; this can best be done by letting the area at section (2) be equal to $\frac{A_p}{1 + \beta}$, in which

β is the air-water flow (or volume) ratio. Therefore,

$$M_2 = \frac{\gamma Q^2_w (1 + \beta)}{g A_p} \qquad (37)$$

(5) W is equal to the product of the average length of the jump L', the specific weight of the air and water mixture which can be expressed as $\frac{\gamma}{1 + \beta}$ and the pipe area:

$$W = A_p L' \left(\frac{\gamma}{1 + \beta} \right) \qquad (38)$$

The value of L' was taken as being approximately equal to the total observed length of the jump, L, minus $\frac{1}{2}(D - y_1)$.

In Fig. 16 the two sides of Eq. 35 are plotted, and the fact that the points group around a 45° line indicates that the data conform to this expression. The scatter of the data is due to difficulty in measuring values like depth of flow and pressure at the high velocities with any great degree of refinement. In any case the use of Eq. 35 is verified and especially the use of the term $W \sin \theta$, since for the steep slopes this term was as much as 75% of $F_1 + M_1$.

Conclusions

The results of the experiments indicate that above a certain critical condition the rate of air removal from an air pocket in a pipe line will depend on the ability of the hydraulic jump which is formed to entrain air. This critical condition, for any slope of pipe and for any relative depth of flow in the air pocket, depends on the value of the Froude number of the flow ahead of the jump. Below this critical value of the Froude number the flow beyond the jump will not be able to handle all the air entrained by the jump, and thus the air removal will not be a function of the jump characteristics but rather on the hydraulic features of the flow beyond the jump.

Air flow measurements indicated that the rate of air entrainment by a hydraulic jump depended largely on the water discharge and the Froude number of the flow ahead of the jump. An empirical relationship is presented for the air flow.

Data on the hydraulic characteristics of the hydraulic jump in closed sloping circular conduits are presented. That the data can be correlated by the momentum relationship for the jump is verified.

Acknowledgments

This paper largely summarizes a dissertation entitled "Entrainment of Air in Pipes by Flowing Water," submitted in August, 1941, by the junior writer to the Graduate College, State University of Iowa, Iowa City, Iowa, in partial fulfilment of the degree of Doctor of Philosophy in Hydraulics. The investigation was conducted under the direction of the senior author, as a project of

the Society's J. Waldo Smith Fellowship Fund. The complete original manuscript has been placed on file for reference at Engineering Societies Library.

The authors gratefully acknowledge the advice and suggestions of F. M. Dawson, M. Am. Soc. C. E., Dean, College of Engineering, State University of Iowa, and Mr. Kennison, Chairman of the J. Waldo Smith Fellowship Committee, and the other members of this Committee.

APPENDIX

NOTATION

The following letter symbols, adopted for standard use in this Symposium and subsequent discussion, conform essentially to American Standard Letter Symbols for Hydraulics[23] prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1942.

A = area:

A_a = cross section defined by a wetted perimeter due to water and entrained air;

A_c = cross section defined by a wetted perimeter due to water alone; $A_c = \frac{Q}{V}$;

A_o = observed cross-section area;

A_p = cross-section area defined by the inner surface of a closed conduit;

A_1 cross section at sections (1), (2), (3), etc., are designated A_1, A_2, A_3, respectively;

a = elemental area of a cross section;

B = a substitution constant $= \frac{2q}{b}$;

b = width:

b_c = water-surface width of a trapezoidal section;

b_1 = bottom width of a trapezoidal section;

C = Chézy coefficient $= \sqrt{\frac{\omega}{k}}$;

D = inside diameter of conduit;

d = distance: $\bar{d}_1$ = distance perpendicular to a conduit, to the centroid of the area A_1 at section (1);

F = Froude's number, subscripts referring to given sections; thus $\mathbf{F}_1$ = the Froude number of the flow approaching section (1);

F = force; total pressure at a section denoted by appropriate subscripts;

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[23] ASA-Z10.2-1942.

f = frictional resistance of the channel walls for a cross-section area of elemental length; subscripts 1, 2, 3, etc., refer to the corresponding sections;
g = acceleration due to gravity;
h = head:
    h_o = observed head;
    h_v = velocity head, computed from the mean velocity;
    h_z = potential energy head;
K = coefficient of air entrainment: K_1 = a constant for air entrainment;
k = coefficient of frictional resistance;
L = total observed length of jump: L' = average length of the jump;
l = (see Fig. 9) length measured along channel bottom or closed conduit;
M = total momenta of the water, subscripts indicating the section;
m = parameter of parabola for trajectory of jet = $\frac{2\,V^2 \cos^2 \theta}{g}$;
n = Kutter's or Manning's roughness coefficient:
    n_c = computed roughness coefficient to be used with R_c and V_o;
    n_o = computed roughness coefficient to be used with R_o and V_c;
P = wetted perimeter for a mixture of air and water:
    P_c = computed wetted perimeter of a section of water only;
    P_o = observed wetted perimeter of a section of air and water;
p = unit pressure (see p_2 in Eq. 36);
Q = flow:
    Q_a = air discharge;
    Q_w = water discharge;
q = unit discharge; flow per foot of width;
R = hydraulic radius = A/P:
    R_c = computed hydraulic radius, $\frac{Q}{V\,P_c}$;
    R_o = observed hydraulic radius;
r = radius of the center line of a channel;
S = sine of angle between energy gradient and horizontal:
    S_c = sine of angle between energy gradient and horizontal for h_v computed from V_c;
    S_o = sine of angle between energy gradient and horizontal for h_v computed from V_o;
    S_1 = sine of the slope of the channel bed;
s = side slopes of a trapezoidal section;
V = velocity, average in a section:
    V_a = average observed velocity in a section;
    V_c = computed velocity, $\frac{Q}{A_o}$;
    V_o = observed velocity;
    V_1 = approach velocity, or velocity of water entering section (1) (subscript denoting appropriate section);
v = velocity through an element of area;

W = weight of the jump;

x = horizontal lengths referred to an xy plane: x_t = horizontal component of trajectory of jet;

y = depths, numerical subscript denoting section; vertical distances referred to an xy plane:

y_a = depth of a stream comprising a mixture of air and water;

y_c = computed depth;

y_e = effective depth;

y_n = normal depth of flow;

y_o = observed depth;

y_t = vertical component of trajectory of jet;

y_1 = initial depth = depth at section (1);

α = mean velocity-head coefficient used to obtain the average kinetic energy in a section;

β = air-water flow ratio;

γ = unit weight of water (ω is used by Mr. Hall);

θ = angle of channel bottom or pipe with horizontal, in degrees;

ρ = ratio of water in a mixture of air and water or the relative density of the mixture; $1 - \rho$ = ratio of air in a mixture of air and water; and

ψ = transverse slope of the water surface between the inner and outer walls of a channel.

DISCUSSION

Warren DeLapp,[24] Jun. Am. Soc. C. E.—It is safe to state that Mr. Hall's paper is by far the most comprehensive to have been published on this subject, and should prove a valuable addition to the present meager knowledge of the behavior of high velocity flow.

The apparent inconsistency notable in Fig. 10 may be caused by two factors in addition to errors in the data. The first and most important of these is Mr. Hall's attempt to relate the percentage of air in the flow to a parameter of questionable significance, $\frac{V^2}{g\,R_c}$, erroneously designated the Froude number. In addition to the objections that might be raised to the use of so fictitious a term as R_c, attention must be called to the fact that the Froude number $\frac{V}{\sqrt{g\,D}}$ is an indication of the ratio of the kinetic energy of the flow to its potential energy; it loses its significance completely if the linear dimension becomes a function of the channel width.

A second reason for the scattering of the points in Fig. 10 is the author's attempt to find a single equation generally applicable to the wide variety of flow conditions occurring in each channel. An examination of the data reveals that, for the same value of $\frac{V^2}{g\,R_c}$, the percentage of air is consistently greater for retarding flow than for accelerating flow. Flow behavior at changes in grade—vertical and horizontal curves—introduces new variables so that Eq. 24*a* (representing a mean of all these conditions when K and K_1 are evaluated for a given channel) can scarcely be more than a crude approximation when applied to any channel not similar in plan and profile to the one for which it was derived.

A function of perhaps greater significance may be developed by considering only points at which the flow was approximately in equilibrium, eliminating data taken when the flow was rapidly accelerating or retarding and observations near horizontal or vertical curves. These data are of little value in the solution of the fundamental problem, and tend only to obscure the true form of the function. All data from the Kittitas flumes are acceptable on this basis, although, obviously, normal flow was not attained on the slope of 0° 04′, since it was less than critical. However, the rate of retardation was apparently sufficiently low to have little effect on the air entrainment. Normal flow was approximated only near Station 2 + 50 and Station 2 + 75 in the Hat Creek chute, and in Rapid Flume from Station 3 + 50 to Station 4 + 00.

As Mr. Hall has implied (see heading, "Analyses of Results"), although the amount of air entrained in the water should depend primarily on the Froude number, it also should depend to some extent on the channel roughness, since the latter will affect the intensity of agitation of the flow for a given value of F.

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[24] Instr., Dept. of Math. and Mechanics, Univ. of Minnesota, Minneapolis, Minn.

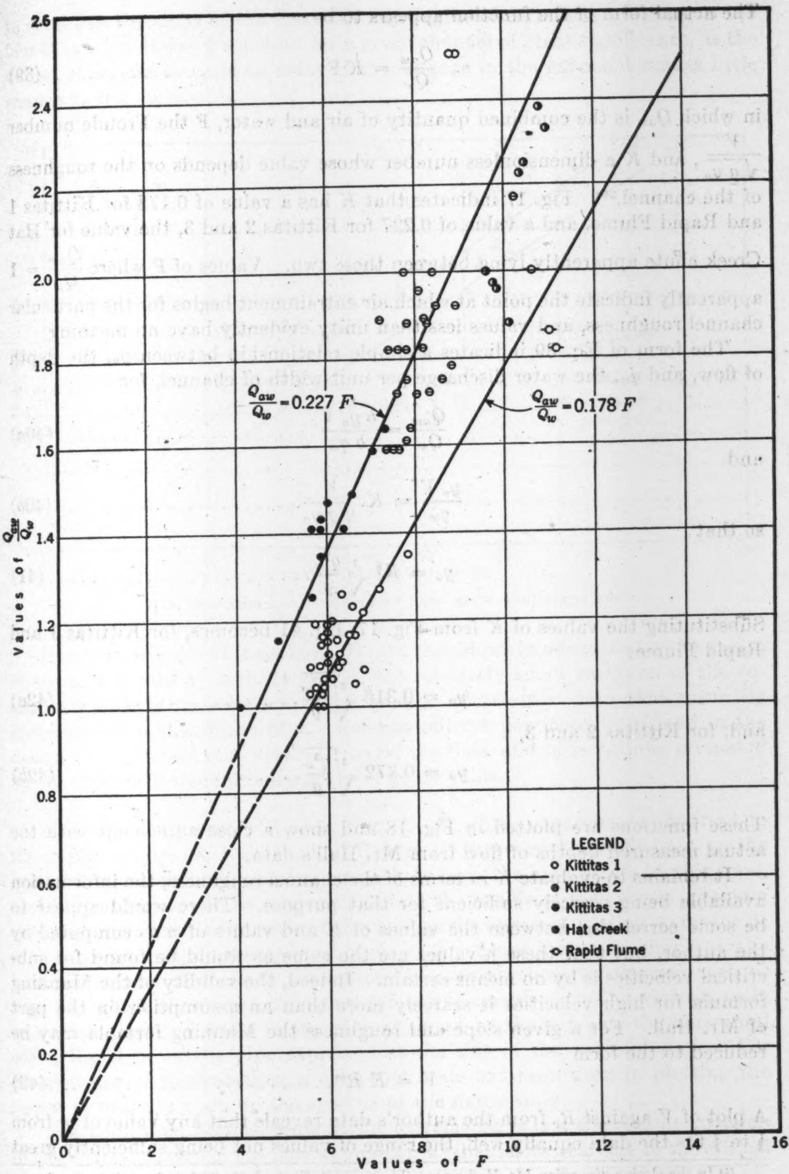

FIG. 17.—RELATION OF $\frac{Q_{aw}}{Q_w}$ TO F

The actual form of the function appears to be

$$\frac{Q_{aw}}{Q_w} = K\,\mathbf{F} \qquad (39)$$

in which Q_{aw} is the combined quantity of air and water, **F** the Froude number $\frac{V}{\sqrt{g\,y_o}}$, and K a dimensionless number whose value depends on the roughness of the channel.[24a] Fig. 17 indicates that K has a value of 0.178 for Kittitas 1 and Rapid Flume, and a value of 0.227 for Kittitas 2 and 3, the value for Hat Creek chute apparently lying between these two. Values of **F** where $\frac{Q_{aw}}{Q_w} = 1$ apparently indicate the point at which air entrainment begins for the particular channel roughness, and values less than unity evidently have no meaning.

The form of Eq. 39 indicates a simple relationship between y_o, the depth of flow, and q_w, the water discharge per unit width of channel, for

$$\frac{Q_{aw}}{Q_w} = \frac{b\,y_o\,V}{b\,q_w} \qquad (40a)$$

and

$$\frac{y_o\,V}{q_w} = K\,\frac{V}{\sqrt{g\,y_o}} \qquad (40b)$$

so that

$$y_o = K^{\frac{2}{3}}\sqrt[3]{\frac{q^2_w}{g}} \qquad (41)$$

Substituting the values of K from Fig. 17, Eq. 41 becomes, for Kittitas 1 and Rapid Flume:

$$y_o = 0.316\sqrt[3]{\frac{q^2_w}{g}} \qquad (42a)$$

and, for Kittitas 2 and 3,

$$y_o = 0.372\sqrt[3]{\frac{q^2_w}{g}} \qquad (42b)$$

These functions are plotted in Fig. 18 and show a close agreement with the actual measured depths of flow from Mr. Hall's data.

It remains to evaluate K in terms of the channel roughness, the information available being scarcely sufficient for that purpose. There would appear to be some correlation between the values of K and values of n as computed by the author, but that these n-values are the same as would be found for subcritical velocities is by no means certain. Indeed, the validity of the Manning formula for high velocities is scarcely more than an assumption on the part of Mr. Hall. For a given slope and roughness the Manning formula may be reduced to the form

$$V = K\,R^m \qquad (43)$$

A plot of V against R_o from the author's data reveals that any value of m from $\frac{1}{2}$ to $\frac{2}{3}$ fits the data equally well, the range of values not being sufficiently great to establish the slope of the line with any degree of certainty. Neither is the fact that *n* is relatively constant for a given channel of great significance, as the values of R_o are so close to unity that a change in the exponent makes little change in the value of *n*.

[24a] In the closing discussion Mr. Hall makes this constant K_L to distinguish it from K in Eq. 24a.

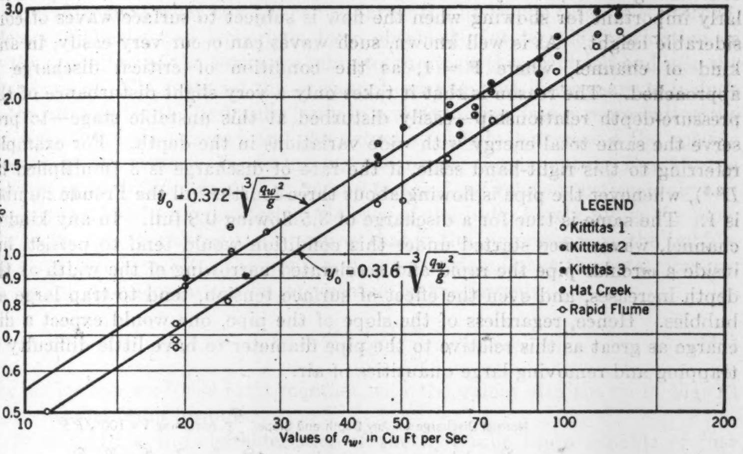

FIG. 18.—RELATION OF DEPTH TO RATE OF WATER DISCHARGE

However, in spite of these facts, Eq. 41 should prove adequate for the design of straight chutes at uniform grades, the relatively small variation of the coefficient making its selection of no greater uncertainty than that normally encountered in the choice of *n*. For nonuniform flow conditions, and in the design of horizontal and vertical curves, the data and observations presented by Mr. Hall will undoubtedly prove of great value.

KARL R. KENNISON,[25] M. AM. SOC. C. E.—The experiments reported in the paper by Messrs. Kalinske and Robertson throw considerable light on the behavior of water in eliminating air pockets from the high points in pipe lines, and the authors are to be congratulated on the results of their work. From a study of these results the writer believes that the action of flowing water inside a pipe, particularly in the removal of air by successive hydraulic jumps, can be best shown and most easily understood in a comprehensive diagram such as Fig. 19, which shows the condition for any diameter *D*. The vertical scale represents depth of water in the pipe as shown at the left, and the horizontal scale, the slope of the pipe expressed as the sine of the angle of slope. For convenience of presentation, a cube-root scale has been used in plotting the slopes in order to open up the spacing at the flatter ones.

The additional vertical scale at the extreme right is a convenient criterion of the critical nature of the flow at any depth; that is, its nearness to the critical discharge at which the Froude number, **F**, or $\frac{V_1}{\sqrt{g\,y_e}}$ in the Symposium nomenclature, equals 1. This scale of relation between depth, rate of discharge, and Froude number is independent of the slope or smoothness of the pipe. In considering the capability of the water to remove air, this criterion is particularly important for showing when the flow is subject to surface waves of considerable height. As is well known, such waves can occur very easily, in any kind of channel, where $\mathbf{F} = 1$, as the condition of critical discharge is approached. The reason is that it takes only a very slight disturbance of the pressure-depth relationship—easily disturbed at this unstable stage—to preserve the same total energy with wide variations in the depth. For example, referring to this right-hand scale, if the rate of discharge is 3 (multiplied by $D^{2.5}$), whenever the pipe is flowing about three-fourths full the Froude number is 1. The same is true for a discharge of 3.5 flowing 0.8 full. In any kind of channel, waves once started under this condition would tend to persist, but inside a circular pipe the rapid and accelerated narrowing of the width as the depth increases, and even the effect of surface tension, tend to trap large air bubbles. Hence, regardless of the slope of the pipe, one would expect a discharge as great as this relative to the pipe diameter to have little difficulty in trapping and removing large quantities of air.

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[25] Chf. Engr., Met. Dist. Water Supply Comm., Commonwealth of Massachusetts, Boston, Mass.

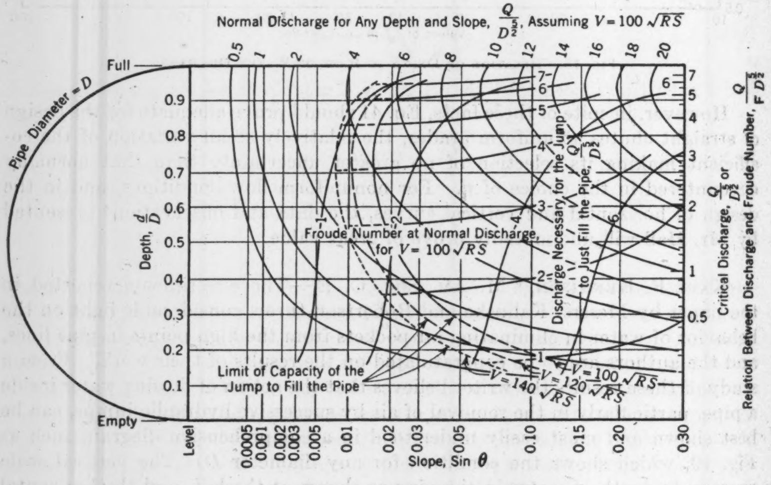

FIG. 19.—HYDRAULIC JUMP INSIDE A PIPE

The effect of slope is indicated by the lines running diagonally across the chart in Fig. 19, and identified by the scale at the top which shows the normal discharge for any depth and slope. The latter are computed for a value of the coefficient, in the formula $V = C\sqrt{R\,S}$, of 100. For smoother pipe, the lines would be moved a little to the left, and vice versa. As air accumulates for any cause at the high point in a pipe line, the surface as the water flows away from this high point cannot drop lower than is indicated by these lines. Introduction of more air merely increases the length of the bubble. For example, if the pipe is on a slope of 0.03 and the rate of discharge is 4 $D^{2.5}$, the chart indicates that, if $C = 100$, the depth cannot decrease to less than 0.55 D.

Consider next what can happen so far as the hydraulic jump is concerned. Messrs. Kalinske and Robertson have obtained considerable information as to the capability of the hydraulic jump to pump air through itself and into the next bubble. The fact that the air cannot escape makes it necessary to include in the formula for the hydraulic jump in a closed pipe a factor representing the amount of the air pumped and the extent to which it decreases the specific gravity of the water. In evaluating the terms of momentum Eq. 35, the authors have omitted from Eq. 36 the factor β, or relation between air and water quantities. In considering the case of the jump barely filling the pipe, the writer has considered the pressure, that is the air pressure in the bubble, the same before as after the jump and has considered that the total pressure on the full pipe section can be represented by the formula

$$F_2 = \frac{A_p \gamma D \cos \theta}{2 (1 + \beta)} \qquad (44)$$

The writer has used this form together with the values of β shown in Fig. 13 in solving the jump formula.

In computing the characteristics of the hydraulic jump capable of just filling the pipe, the writer has made an approximation (which, from a study of the situation, seems sufficiently accurate for practical purposes) that the length of the jump is twice its height and, using the Symposium nomenclature, the loss of momentum (divided by D^3) is

$$F_2 - F_1 - W \sin \theta = \frac{A_p \gamma D \cos \theta}{2 (1 + \beta)} - A_1 \gamma d_1 \cos \theta - \frac{2 (D - y_1) \text{ (average area) } \gamma \sin \theta}{1 + \beta} \qquad (45)$$

The reason for taking length of jump equal to 2 $(D - y_1)$ is to be on the safe side in making an air seal against the top of the pipe. Ordinarily, for example, to insure drowning out a jet from a spillway gate (rectangular channel), the writer would make the length three times the height. Note that the use of this multiplier 2 in the formula for the hydraulic jump in a sloping rectangular channel gives a value of the depth after the jump approximately 97% of that for the multiplier 3 on a slope of 0.05, and about 93% on a slope of 0.10.

The results are plotted in the series of almost horizontal curves running across Fig. 19, but a little higher at the right and identified by the vertical scale in the body of the diagram. For example, if the slope of the pipe equals 0.03 and the water is flowing at a depth of 0.55 D, it cannot jump to the top of the pipe unless the quantity flowing is at least 3 (multiplied by $D^{2.5}$). Thus, the intersection of the two sets of curves marks the limit of the capacity of the hydraulic jump to fill the pipe and hence to trap any air and carry it along. This limit is indicated by a line in Fig. 19 drawn through the intersections; and for convenience other similar lines are drawn for the case of somewhat smoother pipe; namely, three limiting lines for $C = 100$, 120, and 140. These show not only the inability of any jump whatever to form below a certain slope, but also the fact that, for a slope in the vicinity of 0.01, the depth at which a jump can form tends to be confined to a limited range, about two-thirds to three-fourths depth. These curves also show that, under no possible conditions of depth and rate of flow, can a hydraulic jump form inside a pipe if the slope of the pipe is materially less than 0.01.

Fig. 14 appears to be merely an expression of this capability of the jump to fill the pipe. For the low depths, it shows discharges so great that they must be produced artificially. Other curves in Fig. 19 running almost vertically, with their tops bending to the right, show the value of the Froude number, **F**, for the condition of normal discharge at any depth and slope. They are computed for a coefficient of roughness, C, equal to 100, and, as in the case of the lines for normal discharge, they would be moved somewhat to the left in case of a smoother coefficient, and vice versa. For example, these curves show that for any condition of normal discharge throughout a wide range of depths (say, from 0.3 D to 0.7 D), there can be no hydraulic jump in any event unless the slope is as much as 0.0045 and 0.007, respectively—a little less for the smoother pipe. Even though, at a slope somewhat greater than these, a jump does form, it cannot reach the top of the pipe unless the slope is enough greater to fall inside the lines marked "limit of ability of the jump to fill the pipe."

From a study of these and other laboratory test results, in spite of the theoretic considerations summarized in Fig. 19, there seems to be a limiting minimum depth beyond which the jump cannot fill the pipe, for the reason that the approaching jet of water is confined laterally by the shape of the pipe invert. In other words, to fill the pipe requires a much higher rate of discharge than the momentum equation of the jump would indicate. This is shown in Fig. 19 by a lower limit at a depth of about 0.25 D to 0.3 D. The additional loss of energy would appear to be in the piling up at the sides in a manner entirely different from that in a rectangular channel. Other than the important trapping of air bubbles due to surface waves lapping the top of the pipe as described herein for rates of discharge equal to 3 $D^{2.5}$ or 3.5 $D^{2.5}$, Fig. 19 indicates that in a very smooth pipe a rate of discharge only slightly more than 1 $D^{2.5}$, if the slope is at least as steep as 0.02, could successfully pump air from one large bubble to another with a drop in each bubble to a depth of about 0.3 D.

Fig. 19 also indicates that in a case of this kind, where each jump barely makes a seal at the top of the pipe, as soon as sufficient air is removed to decrease the thickness of the air bubble, the discharge ceases to be sufficient to jump to the top of the pipe, with the result that the seal is broken. Accordingly, until further data are available from tests, the writer concludes that (a point not emphasized by Messrs. Kalinske and Robertson) the hydraulic jump alone cannot be as effective in cutting down the head loss in a downward sloping stretch of pipe line as is the trapping of large bubbles by surface waves. In either case the air pressure in each individual bubble is greater than that in the one behind it, the difference being measured by the difference between the slope of the pipe and that of the hydraulic gradient. If this is not the case, then there is no saving of head loss and no point in designing to this end. The case of one long, continuous air bubble from the top to the bottom of the slope is the worst possible condition of air binding, because there is no difference between the slope of the pipe and that of the hydraulic gradient. In such a case all the air that reaches the high point of the line flows out at the bottom of the slope without the hydraulic jump or any other water action lending any material assistance in cutting down the head loss.

The approach of Messrs. Kalinske and Robertson to the problem has been to analyze the behavior of the hydraulic jump in the pipe and its power to pump air. Nevertheless, one should not lose sight of the fact that the ultimate aim of the designer in dealing with these matters is not just to remove air, but to remove it so as to make the most effective saving of head, or to prevent air binding. It would be most useful to have a graph showing the relationship between (1) the slope of the pipe compared with the slope of the hydraulic gradient, (2) the discharge compared with the diameter, or $\frac{Q}{D^{2.5}}$, and (3) the relative quantity of air removed.

The part played by the hydraulic jump in such removal has been most effectively analyzed by Messrs. Kalinske and Robertson.

ROBERT T. KNAPP,[26] M. AM. SOC. C. E.—The entrainment of air by high-velocity flows in open channels is an important phenomenon about which too little is known. Mr. Hall is to be congratulated for his contribution, both from the observational and analytical standpoints. He deals not only with the air entrainment problem, but also with the general problem of high-velocity flow around horizontal and vertical curves in open channels. The writer wishes to comment briefly upon, first, the flow around horizontal curves, and, second, the problem of air entrainment through the free surface of the flow.

High-Velocity Flow around Horizontal Curves.—Until quite recently, little attention has been given to curvilinear flow in high-velocity open channels. However, in the past few years considerable work has been done in this field.[27] Since Mr. Hall has not presented the concepts of this regimen of flow that have been found useful, a brief review seems to be in order.

When applied to the flow of a liquid in an open channel, the term, "high-velocity flow," means flow at a velocity greater than the critical, or the wave velocity for that particular depth. The velocity of a surface wave is the velocity of propagation of a disturbance; that is, a force through the flow. In flow around a horizontal curve in an open channel ("horizontal" is used loosely in the sense of being "in the plane of the undisturbed surface"), the walls of the channel force the adjacent layers of liquid to move in curved paths. In flow that is slower than the wave velocity, these forces are transmitted across and along the channel at a speed sufficient to permit equilibrium to be established between the centrifugal forces, resulting from the curved path, and the lateral pressure gradient which manifests itself as the banking of the free surface. This equilibrium is seen to be attained at a cross slope of $\tan \psi = \frac{V^2}{g r}$ (Eq. 32). Unfortunately, this equilibrium condition does not hold for conditions of supercritical flow. The reason for this is clear. In low-velocity curvilinear flow two forces act on each liquid element to cause it to travel in curved paths. They are an "over-force" from the superelevation on the outer wall, and an "under-force" from the depression on the inside wall. This is possible only because the wall effect travels faster than the flow. In supercritical flow this state of affairs no longer exists. The effect of the change of direction of the wall is no longer felt directly across the channel but propagates laterally only at the wave velocity C. Thus it crosses the channel at an angle

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[26] Associate Prof., Hydr. Eng., California Inst. of Technology, Pasadena, Calif.

[27] "Curvilinear Flow of Liquids with Free Surfaces at Velocities Above that of Wave Propagation," by Robert T. Knapp and Arthur T. Ippen, *Proceedings*, Fifth International Cong. for Applied Mechanics, 1938, p. 531.

$$\beta = \sin^{-1} \frac{C}{V} \qquad (46)$$

which means, if b is the width of the channel, that the effect reaches the opposite side first at a distance downstream of

$$L = \frac{b}{\tan \beta} \qquad (47)$$

For a channel whose mean radius of curvature is r, this corresponds to an angular distance

$$\theta_o = \tan^{-1} \frac{b}{\left(r + \frac{b}{2}\right) \tan \beta} \qquad (48)$$

In this distance all of the force required to cause the water to follow the curved path was supplied by one wall instead of by equal forces from both walls. Therefore, the pressure rise (the rise in the water surface at the outside wall) keeps increasing until it is about twice as high as it would be for a flow at the same velocity, but with a depth sufficient to cause the critical velocity to be above the velocity of flow. This maximum pressure rise corresponds to a cross slope, $\psi_{s,c}$, which is twice that given by Eq. 32:

$$\tan \psi_{s,c} = \frac{2\,V^2}{g\,r} \qquad (49)$$

However, below the point at which the effect from the opposite wall has succeeded in crossing the channel (Eqs. 47 and 48), the superelevation is obviously too great; so it decreases, leaving a maximum at this point. The subsequent history is easily visualized by considering that the behavior of the surface is thought of as a wave train oscillating about the equilibrium depth, given by Eq. 32 with the oscillation having a wave length of $2\,\theta_o$ and an amplitude of $\frac{V^2 b}{2\,g\,r}$. The alternate maxima and minima are formed on both the outer and inner walls. The disturbance train persists into the downstream tangent with a magnitude depending upon the relative position of the last maximum in the curve. This behavior in the curve is clearly shown in Fig. 8, in which the last maximum is on the outer wall, apparently near the end of curvature. In Fig. 7 the characteristic diagonal path of the oscillation is plainly visible. It can be traced twice across the channel, beginning on the right wall at the last maximum in the curve, crossing diagonally to the left wall of the downstream tangent, and being reflected back again to the right wall, which it is approaching at the bottom of the picture.

Since the effective cross slope ψ varies from zero at the beginning of curvature to twice the value given by Eq. 32 at the points of maximum on the outer wall, and actually may have a negative slope at the minimum points, it is not surprising that points were found in the South Canal chute that verified Eq. 32. Indeed, as previously stated, this is the equilibrium slope, but it would be disastrous to design the channel freeboard on this basis.

Arthur T. Ippen and the writer[27] have shown that this oscillating disturbance can be eliminated in channels of rectangular cross section by the use of compound curves, in which the central curve, irrespective of its length, is preceded and followed by circular transition curves, each one-half wave length long, having mean radii twice that of the central curve. These transition curves can be thought of as producing counter disturbances of the correct magnitude and phase to interfere with and eliminate the oscillations—first, in the curve itself, and second, in the downstream channel. The same results can also be obtained by transition curves accompanied by banking the channel bottom.

In many cases the use of these concepts and methods will eliminate the need for the model tests recommended by Mr. Hall. However, for complicated constructions involving divided channels and accelerating flow, model studies will certainly be helpful. In such cases the work must be done with extreme care to simulate both depth and velocity very accurately, since the wave pattern, and therefore the acceptability of the design, is so very sensitive to unsynchronized variations of these quantities.

Mechanism of Air Entrainment.—In studying the phenomenon of air entrainment in open-channel flow, the writer has found it convenient to analyze the physical nature of the phenomenon. Therefore, the following remarks are offered in amplification of Mr. Hall's description of the process.

Air entrainment through the free surface of a flowing liquid is essentially a wave phenomenon. More exactly, it is accomplished through the medium of breaking waves, which fold in air and carry it through the interface. The conditions under which waves will break were delineated by H. Helmholz[28] in the nineteenth century. The physical concept behind his analysis is that waves traveling as fast or faster than the relative velocity across the interface (namely, the differential velocity between the water and the air), lose energy, decrease in amplitude, and finally die out. However, waves that travel slower than this differential velocity pick up energy through the interfacial friction, build up their amplitude and finally break, entraining air as they do so. Thus air entrainment is a surface phenomenon and, because of its very nature, is restricted in general to velocities of flow above the critical.

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[28] "Wissenschaflishe Abhandlungen," by H. Helmholz, Leipzig, 1882–1895, Vol. 1, pp. 146–157.

Since the relative densities of air and water are about 1 to 800, a very powerful tendency for the air to separate from the water beginning with the moment of entrainment is always present. The mechanism that carries the air downward against this tendency to separate is the turbulent motion in the liquid. Thus no static equilibrium is possible—only a dynamic equilibrium—and the amount of air in the mixture is determined by the relative value of these opposing forces. There is a direct analogy between this transport of air and sediment transportation, with the air bubbles taking the places of the particles of sand and silt, and the tendency of the bubbles to rise to the surface replacing the tendency of the sediment to fall to the bottom. In the case of sediment transportation, the concentration is far from uniform in the vertical direction, even for laboratory conditions of uniform-sized particles. For coarse materials, little will be found above the bottom layers of the flow. On the other hand, a much more even, but still nonuniform distribution will be found for the transportation of fine silt and clay particles. Much the same conditions will be found in air transport. The depth of penetration of the air below the surface will depend on the size of the bubbles and the level of turbulence. Large bubbles will penetrate less than small ones. The air concentration must be greatest near the surface and may decrease to zero before the bottom is reached if the flow is sufficiently deep.

Since air entrainment occurs by mixing across the interface, help in understanding it can be obtained through the study of other types of interfacial mixing. This phenomenon is greatly exaggerated and presented in "slow motion" for ease in observation in the case of density currents in lakes and reservoirs. A recent review of the general behavior of density currents has been presented by H. S. Bell,[29] with whom the writer has been working for several years. A further discussion of this subject was presented by the writer at the Second Hydraulics Conference in Iowa City, Iowa, June, 1942.[30] Studies of interfacial mixing of density currents in the laboratory show that the mixing is largely restricted to the upper layers of the flow and the bottom layers are relatively undisturbed. Furthermore, the mixing zone soon forms a second definite density flow which travels on the surface of the first and moves at an intermediate velocity between the bottom current and the overlying liquid. This intermediate layer acts as a buffer to inhibit further mixing. It seems very probable that a similar situation exists with air entrainment in open channels. Here the layer of foam is the overlying flow. Since the density of this foam is still very great as compared to air, it would be expected to travel at a velocity much nearer to that of the water than to that of the air. Nevertheless, it is suspected that this velocity is definitely lower than that of the underlying flow.

The preceding considerations lead to numerous conclusions, some of which are outlined in the following items:

1. Because of the nature of the mechanisms of air entrainment and transportation, a nonuniform vertical distribution of air is to be expected. There-

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[29] "Density Currents as Agents for Transporting Sediments," by Hugh Stevens Bell, *Journal of Geology*, Vol. L, No. 5, July–August, 1942, pp. 512–547.

[30] *Proceedings*, Second Hydraulics Conference, Iowa City, Iowa, 1942 (publication pending).

fore, an assumption of uniform mixing is dangerous, although it may be tolerable for high velocities and low depths such as those reported by Mr. Hall.

2. Air entrainment is basically a phenomenon of breaking waves which obtain the energy for breaking from the relative velocity between the air and the water. Therefore, the amount of air entrained probably will be affected by the magnitude and velocity of the wind at the surface of the flow.

3. Waves will not be initiated simply by the high velocity of the flow. Initial disturbances are necessary to cause wave formation. Typical sources of such disturbances are (*a*) channel roughness, which affects the level of turbulence, (*b*) uncompensated curves in channel, (*c*) local irregularities, and (*d*) turbulent structure of wind if velocity is sufficiently high.

4. Neglecting local disturbances, channel roughness should be one of the most important factors in air entrainment and transportation since it determines the turbulence level, which affects not only the amount of surface disturbance but also the depth of penetration of the air.

5. Foam of upper layers probably travels appreciably slower than clear water or layers having lower air concentrations. Therefore, methods of velocity measurements that basically determine velocity of upper layers are probably unreliable and underestimate the true velocity. The color method used in the experiments quoted is subject to this question.

6. Since the entrainment and transportation of air and sediment have so many points in common, it appears that parallel treatments should yield many useful results.

It will be noted that not all of the foregoing conclusions are completely compatible with the assumptions and treatment used by Mr. Hall.

CARL E. KINDSVATER,[31] JUN. AM. SOC. C. E.—A considerable portion of the paper by Messrs. Kalinske and Robertson is devoted to a study of the hydraulic jump in enclosed conduits. Data obtained from jumps in a sloping pipe verify the application of the pressure-momentum analysis to that phenomenon. In an earlier study[21,32] by the writer, this method of analysis was substantiated for the hydraulic jump in horizontal enclosed conduits. It is interesting to note that the experimental data plotted in Fig. 16 show a slight deviation from the theoretical curve, almost identical with results obtained by the writer. Although this discrepancy is of negligible importance in practical applications, it is of interest to attempt an explanation of the small but definite tendency for the observed "force plus momentum" after the jump to be less than the sum of the initial pressure and momentum. Three explanations were offered by the writer:[21] First, because of the nonuniform distribution of velocities in section (2), the computed momentum, based on an assumption of uniform distribution of the average velocity, is less than the true momentum in the cross section; second, because of the admixture of air with water in section (2), the true velocity of the water is somewhat greater than the average velocity based on the total cross-sectional area of the conduit; and third, because of extremely turbulent, nonuniform flow conditions below the jump, it is impossible to compute accurately the loss of head due to wall friction between sections (1) and (2).

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[31] Associate Hydr. Engr., Hydroelectric Section, U. S. Engr. Office, Little Rock, Ark.

[32] "The Hydraulic Jump in Enclosed Conduits," by Carl E. Kindsvater, a thesis presented to the State University of Iowa, Iowa City, Iowa, in 1936, in partial fulfilment of the requirements for the degree of Master of Science.

The first explanation was subsequently shown to be of minor importance by a series of carefully conducted pitot-tube traverses in the region below the jump. It was shown that the nonuniformity of velocities in this section could be expected to account for an increase in M_2 of not more than 2% or 3%.

Now, based on the Kalinske-Robertson study of air entrainment, it is possible to evaluate the second explanation. Fig. 20 shows a comparison of the writer's original results with the same data corrected for the effect of air entrained in the water at section (2). Values of β were read from Fig. 13 and applied to the original data as indicated by Eq. 37. The corrected data show a considerably improved relation to the theoretical curve, but there is still a slight tendency for $F_2 + M_2$ to be somewhat less than $F_1 + M_1$.

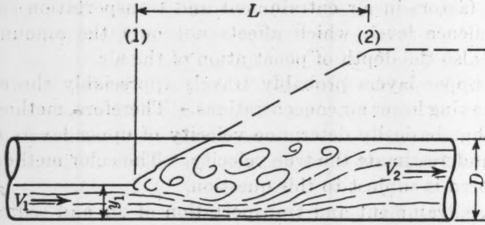

FIG. 20.—HYDRAULIC JUMP IN HORIZONTAL ENCLOSED CONDUIT

The third explanation for this discrepancy is extremely difficult to evaluate. In the writer's study, $\frac{p_2}{\gamma}$ was corrected for the loss of head due to boundary friction between the section where the pipe first flowed full and the section corresponding to the highest point on the pressure profile. However, this friction loss correction was based on Manning's formula for pipe flow, using measured values of n obtained from tests in a reach of fairly uniform flow conditions. Additional losses in the region of partial-depth flow and miscellaneous losses due to the concentration of velocities along various sections of the pipe are almost impossible to estimate.

It is quite true that the discrepancy shown in Figs. 16 and 21 is of small practical importance. It should be emphasized that the question is largely of academic interest and in no way should invalidate the application of the pressure-momentum analysis to practical problems.

It is gratifying to note the success with which Messrs. Kalinske and Robertson have applied a simplified treatment of the weight term in Eq. 35 for jumps in sloping conduits. In the sentence following Eq. 38, the average length L', used in computing the volume and weight of the jump, is defined as "being approximately equal to the total observed length of the jump, *L*, minus $\frac{1}{2}(D - y_1)$." Although this definition is sufficiently accurate for jumps having relatively large values of $\frac{p_2}{\gamma}$, apparently it would be quite inaccurate when $\frac{p_2}{\gamma}$ approached the magnitude of *D*. The length term is especially critical when, as pointed out by the authors, the slope is steep and *W* becomes a large factor in Eq. 35.

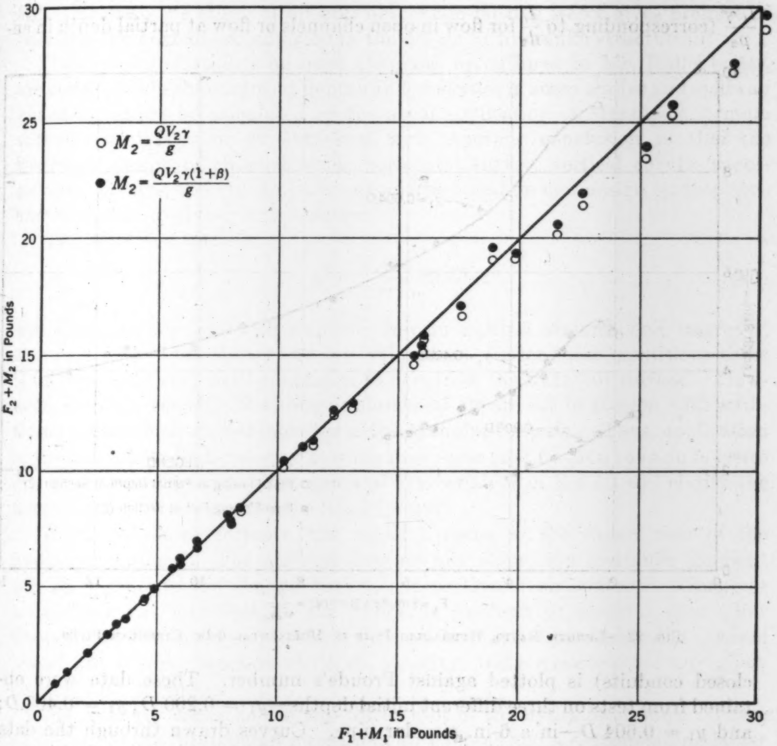

FIG. 21.—VERIFICATION OF EQ. 35 FROM TESTS ON HYDRAULIC JUMP IN HORIZONTAL 6-IN. CIRCULAR PIPE

Since a practicable treatment for the hydraulic jump in sloping channels, either open or enclosed, is dependent upon measured dimensions of the jump body, the analysis suggested by Messrs. Kalinske and Robertson will have practical significance only when data are available for computing the total length of the jump. Numerous studies[33] on the hydraulic jump in open channels have demonstrated that the length of the jump can be conveniently expressed as a function of Froude's number. A similar relation might have been developed for the jump in sloping enclosed conduits. It is of interest to investigate the approximate form of the latter relation by further reference to the writer's data on the hydraulic jump in a horizontal, circular conduit. Fig. 22 shows a plot of the limited data available, in which the length ratio $\frac{L\,\gamma}{p_2}$ (corresponding to $\frac{L}{d_2}$ for flow in open channels or flow at partial depth in enclosed conduits) is plotted against Froude's number. These data were obtained from tests on three different initial depths—$y_1 = 0.200\ D$; $y_1 = 0.408\ D$; and $y_1 = 0.604\ D$—in a 6-in. circular pipe. Curves drawn through the data are generally similar in shape to length-ratio curves for the hydraulic jump in open rectangular channels. However, apparently due to the circular cross section at section (1), the length ratio varies considerably more with y_1 than with Froude's number. Additional data for this study, for various slopes, initial depths, and Froude's numbers, would be essential for a complete treatment of the hydraulic jump in enclosed conduits.

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[33] "The Hydraulic Jump in Terms of Dynamic Similarity," by Boris A. Bakhmeteff and Arthur E. Matzke, *Transactions*, Am. Soc. C. E., Vol. 101 (1936), p. 630.

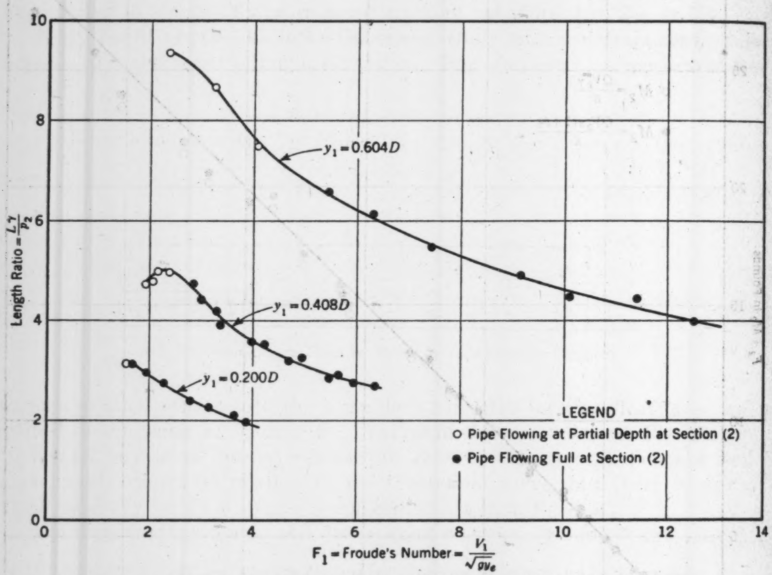

FIG. 22.—LENGTH RATIO, HYDRAULIC JUMP IN HORIZONTAL 6-IN. CIRCULAR PIPES

J. H. DOUMA,[34] JUN. AM. SOC. C. E.—The past half century has witnessed the construction of a number of major hydraulic structures in which the problem of air entrainment is important. In predicting the hydraulic flow characteristics of these large structures, engineers have relied upon engineering judgment, upon empirical formulas, and, in the past decade, upon hydraulic models. Engineering judgment has been developed largely through the tedious process of trial and error. Empirical formulas have been based on experimental measurements with comparatively low velocities and no entrained air. Model experiments of hydraulic structures have not involved air entrainment. Thus, although design methods with respect to air entrainment are still irrational, the study of air entrainment in flowing water gives promise of effecting the next major advance in the design of hydraulic structures.

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[34] Associate Hydr. Engr., U. S. Engr. Office, Los Angeles, Calif.

The results of studies on open channels, as outlined in Mr. Hall's paper, are conclusive in showing that depths and velocities in steep chutes and spillway channels cannot be calculated by the usual application of Manning's formula without consideration of entrained air. Another conclusion is that the hydraulic design of channel walls, horizontal curves, vertical curves, superelevated inverts, and stilling basins should be based on new design assumptions and formulas involving entrained air.

The Manning formula

$$V = \frac{1.486}{n} R^{\frac{2}{3}} S^{\frac{1}{2}} \tag{50}$$

was intended for use with uniform flow in natural streams and improved channels with small slopes and low velocities. Under these conditions there is undoubtedly very little resistance to flow from the water-air surface. However, for high-velocity flow, large volumes of air are set in motion with additional losses which are not included in the Manning formula. Thus, application of the formula to high-velocity flow requires some modification to include losses due to air entrainment. Discussion of the variables in Eq. 50 will clarify the development of a suggested modified formula.

In the Manning formula, the velocity varies as the square root of the hydraulic gradient. For uniform flow on any slope, the hydraulic gradient will be parallel to the channel invert, irrespective of air content. Thus, entrained air has no effect on the hydraulic gradient of uniform flow. For nonuniform flow, however, the hydraulic gradient is not parallel to the channel invert but is some function of depth, velocity, and air content for any given channel and discharge.

The velocity varies as the two-thirds root of the hydraulic radius. Then, according to the formula, an increase in air content and a corresponding larger hydraulic radius would result in a higher velocity. This result is quite the opposite from that experienced with aerated flow.

Values of Manning's roughness factor *n* have been determined for many channels with various kinds of boundary materials, but with low-velocity flows containing no entrained air. Under these conditions, the value of *n* is a measure of the channel-boundary roughness, as defined by resistance to the flowing water. Entirely new sets of *n*-values, which would include air resistance and other losses due to entrained air, could be determined for aerated flow. This procedure, however, would result in a variation in the *n*-value, depending on the air content, for any one channel roughness. It does not follow that a concrete channel lining, for instance, placed under similar conditions on a

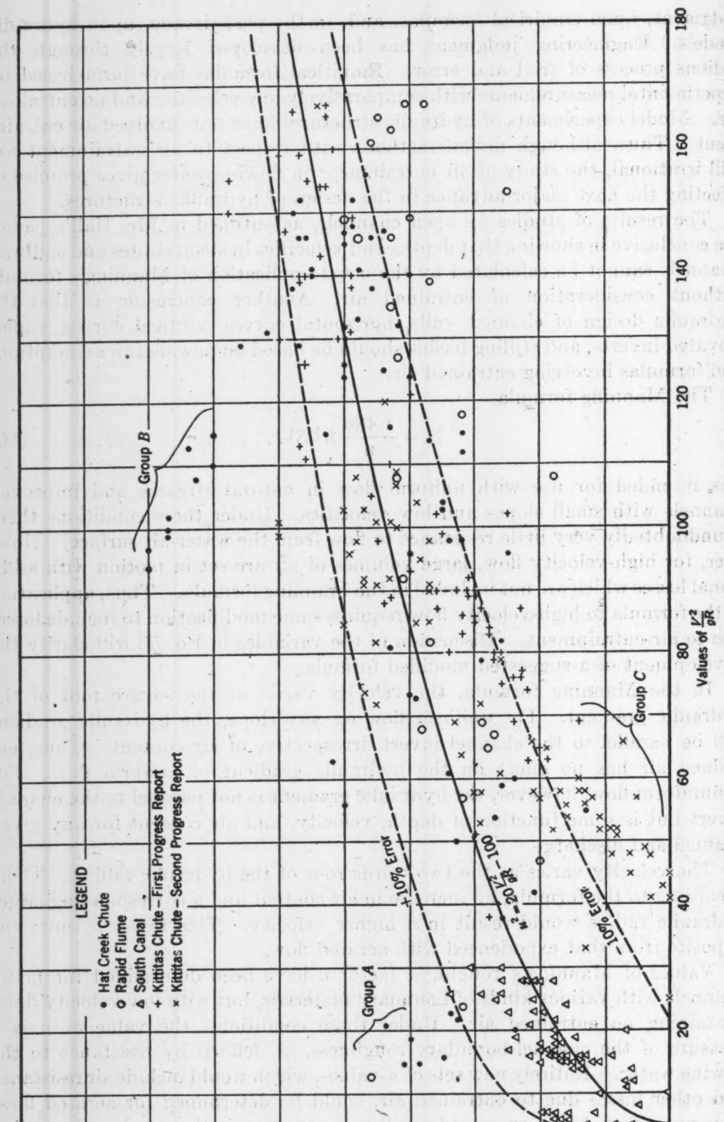

FIG. 23.—RELATION OF PERCENTAGE OF ENTRAINED AIR TO FROUDE'S NUMBER

steep slope, should have a roughness factor different from that placed on a flat slope.

The expression $\frac{1.486}{n}$ in the Manning formula was designed to make values of n correspond to the values of Kutter's n. For aerated flow the expression can be changed to $\frac{N}{n}$, in which values of n are the same as those established for nonaerated flow and N is a function of air content. By this procedure, the original definition of n (measure of channel roughness) is retained, and the effect of air entrainment on the expression is given by the variable N.

Of the several methods of modifying Manning's formula for application to aerated flow, Mr. Hall has selected, as the most logical, one in which computed areas and hydraulic radii (assuming no entrained air) are used instead of actual areas and hydraulic radii for the aerated flow. This necessitates the use of hypothetical values of n (n_c, computed by Eq. 26) smaller than those normally used for nonaerated flow, as opposed to results obtained for aerated flow when observed hydraulic radii are used in the Manning formula.

It appears that the most logical procedure for aerated flow would be to compute actual velocities by using actual hydraulic radii and hydraulic gradients, values of n for nonaerated flow, and a variable N instead of 1.486 in Manning's formula. The modified form of the formula then becomes

$$V = \frac{N}{n} R^{\frac{2}{3}} S^{\frac{1}{2}} \qquad \text{(51)}$$

Using Eq. 51, the basic expression, Eq. 17, for flow in steep rectangular chutes becomes

$$\frac{dl}{dV} = \frac{\frac{\alpha\, V}{g} - \frac{q \cos\theta}{V^2}}{\sin\theta - \frac{n^2 V^2}{N^2 R^{4/3}}} \qquad \text{(52)}$$

In Eq. 52, R and N are functions of velocity and air content. The equation can be solved by determining expressions for R and N in terms of V and known quantities.

In developing the expression for R, the data reported in the paper were used to plot percentage of entrained air, u, against $\frac{V^2}{g\,R}$, shown in Fig. 23. This plotting differs from Fig. 10 by the use of u instead of the more complicated ratio $\frac{1-\rho}{\rho}$ and by the use of the hydraulic radius of the aerated flow instead of the computed hydraulic radius of water alone. Mr. Hall erroneously uses $\frac{V^2}{g\,R_c}$ as the Froude number. The Froude number is normally defined as $\frac{V}{\sqrt{g\,D}}$, which probably should become $\frac{V}{\sqrt{g\,\rho\,D}}$ for aerated flow.

The average of all points plotted in Fig. 23 was found to be given approximately by the relation

$$u^2 = 20 \frac{V^2}{g\,R} - 100 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (53)$$

The average curve was drawn to give no air entrainment for values of $\frac{V^2}{g\,R}$ equal to 5 or less. The expression for critical velocity at which air entrainment begins in accelerated high-velocity flow then becomes

$$V = \sqrt{5\,g\,R} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (54)$$

Thus, for a shallow depth in any channel and a value of R equal to 1.0, for example, air entrainment would begin when the velocity is approximately 13 ft per sec. For a greater depth and a value of R equal to 5.0, air entrainment would begin when the velocity is approximately 28 ft per sec. These velocities, in general, are consistent with the meager data available on critical velocity for air entrainment. Additional observations, however, may indicate necessary modifications of Eq. 54.

In Fig. 10, Mr. Hall has drawn all but one of the curves through the origin of coordinates. This procedure results in the computation of some air entrainment for very low velocities. All of the curves should probably be drawn with a finite value of $\frac{V^2}{g\,R_c}$, determined by the critical velocity for air entrainment, when the ratio $\frac{1-\rho}{\rho}$ is zero.

On first inspection, the wide scattering of points in Fig. 23 would seem to cast serious doubt on the application of a single average curve to the several structures. However, by plotting + 10%-error and − 10%-error curves practically all of the plotted points fall within these limits of error. The curves were computed by assuming errors of ± 10% in both the observed depths and velocities of the aerated flow. In view of the difficulties in obtaining accurate depth and velocity measurements, errors of 10% certainly are not excessive.

Because of turbulent entrance conditions the points in group A, Fig. 23, show high percentages of air. In group B, the high percentages of air are caused by flow leaving the chute at a sharp vertical curve. The low air content of the group C points appears to be due to errors in determination of observed velocities.

The observed data listed in Table 9(*a*) were taken from the Bureau of Reclamation's second progress report (unpublished) on studies of the flow of water in open channels with high gradients. In this report the average observed velocity was determined from the oscillograms by taking the time of travel between two sets of electrodes, as given by the distance between centers of gravity of areas under two successive curves representing the passage of a salt cloud past the upper and lower stations. In the first progress report (unpublished) the time distance was obtained by averaging the distances between beginning, maximum, and other characteristic points on the curves. Velocities determined in the first progress report average 24% higher than velocities determined in the second progress report.

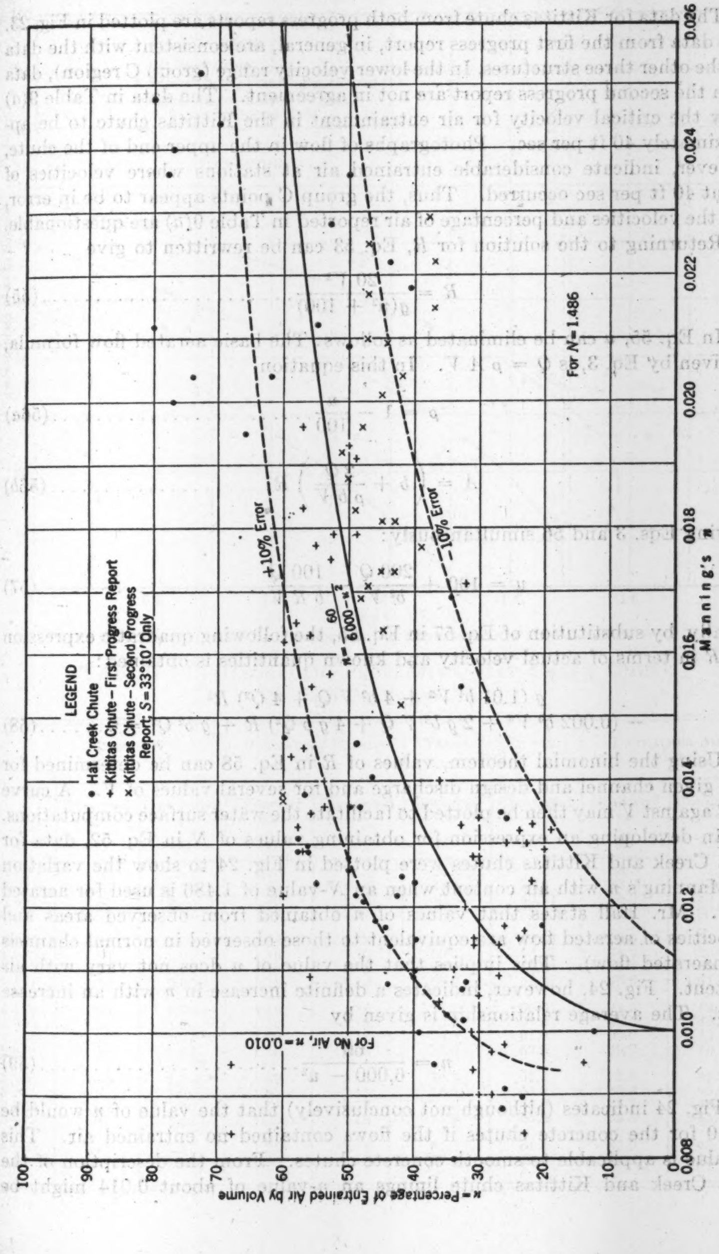

Fig. 24.—Relation of Percentage of Entrained Air to Manning's n

The data for Kittitas chute from both progress reports are plotted in Fig. 23. The data from the first progress report, in general, are consistent with the data for the other three structures. In the lower velocity range (group C region), data from the second progress report are not in agreement. The data in Table 9(*a*) show the critical velocity for air entrainment in the Kittitas chute to be approximately 40 ft per sec. Photographs of flow in the upper end of the chute, however, indicate considerable entrained air at stations where velocities of about 40 ft per sec occurred. Thus, the group C points appear to be in error, and the velocities and percentage of air reported in Table 9(*a*) are questionable.

Returning to the solution for R, Eq. 53 can be rewritten to give

$$R = \frac{20\,V^2}{g(u^2 + 100)} \qquad (55)$$

In Eq. 55, u can be eliminated as follows: The basic aerated flow formula, as given by Eq. 3, is $Q = \rho\,A\,V$. In this equation,

$$\rho = 1 - \frac{u}{100} \qquad (56a)$$

and

$$A = \left(b + \frac{2\,Q}{\rho\,b\,V}\right) R \qquad (56b)$$

Solving Eqs. 3 and 56 simultaneously:

$$u = 100 + \frac{200\,Q}{b^2\,V} - \frac{100\,Q}{b\,R\,V} \qquad (57)$$

Finally, by substitution of Eq. 57 in Eq. 55, the following quadratic expression for R in terms of actual velocity and known quantities is obtained:

$$g\,(1.01\,b^4\,V^2 + 4\,b^2\,V\,Q + 4\,Q^2)\,R^2 - (0.002\,b^4\,V^4 + 2\,g\,b^3\,V\,Q + 4\,g\,b\,Q^2)\,R + g\,b^2\,Q^2 = 0 \qquad (58)$$

Using the binomial theorem, values of R in Eq. 58 can be determined for any given channel and design discharge and for several values of V. A curve of R against V may then be plotted to facilitate the water surface computations.

In developing an expression for obtaining values of N in Eq. 52, data for Hat Creek and Kittitas chutes were plotted in Fig. 24 to show the variation of Manning's n with air content when an N-value of 1.486 is used for aerated flow. Mr. Hall states that values of n obtained from observed areas and velocities of aerated flow are equivalent to those observed in normal channels (nonaerated flow). This implies that the value of n does not vary with air content. Fig. 24, however, indicates a definite increase in n with an increase in u. The average relationship is given by

$$n = \frac{60}{6{,}000 - u^2} \qquad (59)$$

Fig. 24 indicates (although not conclusively) that the value of n would be 0.010 for the concrete chutes if the flows contained no entrained air. This n-value is applicable to smooth concrete chutes. From the description of the Hat Creek and Kittitas chute linings an n-value of about 0.014 might be

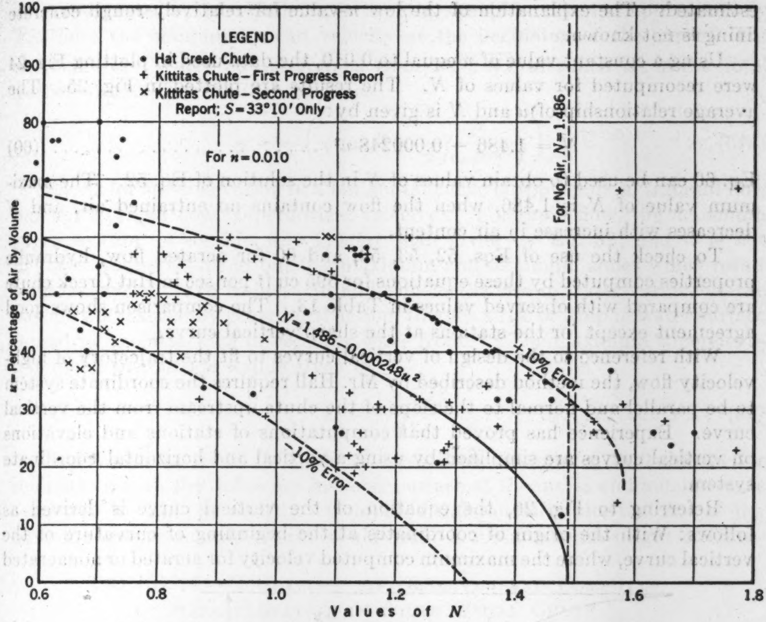

FIG. 25.—RELATION OF PERCENTAGE OF ENTRAINED AIR TO VALUES OF N

TABLE 13.—HYDRAULIC PROPERTIES, HAT CREEK CHUTE

(Q = 395 CU FT PER SEC)

Station	DEPTH (FT)		HYDRAULIC RADIUS (FT)		VELOCITY (FT PER SEC)		ENTRAINED AIR (%)	
	Observed y_o	Computed y	Observed R_o	Computed R	Observed V_o	Computed V	Observed u_o	Computed u
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
0+00	4.00	....	1.74	....	28.5	....	38.5	....
0+25	3.15	3.15	1.57	1.57	33.4	33.4	33.0	33.0
0+50	2.35	2.30	1.36	1.33	40.8	40.2	26.0	25.6
0+75	2.00	2.17	1.23	1.27	48.1	45.9	25.9	30.8
1+00	1.85	2.09	1.18	1.23	54.7	50.0	29.0	34.1
1+25	1.80	2.04	1.16	1.20	60.5	53.3	34.2	36.9
1+50	1.77	2.01	1.15	1.19	65.0	56.0	37.4	39.0
1+75	1.85	2.00	1.18	1.19	68.0	58.2	43.6	41.0
2+00	1.80	1.99	1.16	1.18	71.7	60.0	44.5	42.1
2+25	1.80	1.98	1.16	1.18	73.9	61.0	46.2	43.0
2+50	1.90	1.97	1.20	1.18	75.3	61.7	50.0	43.5
2+75	1.85	1.97	1.18	1.18	75.9	62.0	49.0	43.7
3+00	3.52	1.96	1.59	1.18	76.0	62.3	73.8	44.2
3+25	4.60	1.95	1.83	1.18	75.8	63.9	80.0	45.6
3+50	3.50	1.95	1.64	1.18	74.9	65.2	73.2	46.2
3+75	1.97	1.96	1.23	1.18	72.3	64.2	49.8	45.6
4+00	1.95	1.97	1.21	1.18	61.1	61.8	39.9	43.7
4+09.5	2.17	1.98	1.30	1.18	57.9	60.3	43.4	42.6

estimated. The explanation of the low n-value for relatively rough concrete lining is not known.

Using a constant value of n equal to 0.010, the data used in plotting Fig. 24 were recomputed for values of N. The results are plotted in Fig. 25. The average relationship of u and N is given by:

$$N = 1.486 - 0.000248\, u^2 \qquad (60)$$

Eq. 60 can be used to obtain values of N in the solution of Eq. 52. The maximum value of N is 1.486, when the flow contains no entrained air, and N decreases with increase in air content.

To check the use of Eqs. 52, 53, 58, and 60 for aerated flow, hydraulic properties computed by these equations for 395 cu ft per sec in Hat Creek chute are compared with observed values in Table 13. The comparison shows good agreement except for the stations at the sharp vertical curve.

With reference to the design of vertical curves to fit the trajectory of high-velocity flow, the method described by Mr. Hall requires the coordinate system to be parallel and normal to the slope of the chute upstream from the vertical curve. Experience has proved that computations of stations and elevations on vertical curves are simplified by using a vertical and horizontal coordinate system.

Referring to Fig. 26, the equation of the vertical curve is derived as follows: With the origin of coordinates at the beginning of curvature of the vertical curve, where the maximum computed velocity for aerated or nonaerated flow is V_o, the horizontal distance covered by the flow in time t is given by

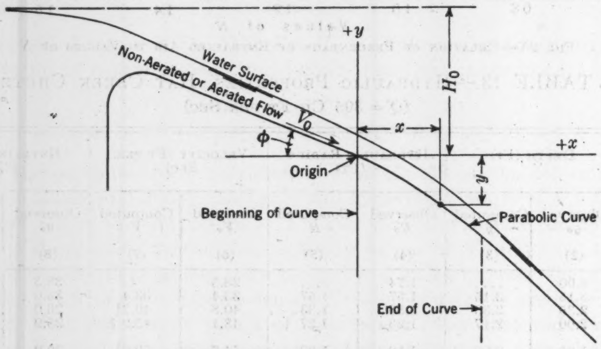

FIG. 26.—VERTICAL INVERT CURVE

$$x = t\, V_o \cos\theta \qquad (61a)$$

and in this same time the flow would drop a vertical distance

$$y = -\left(t\, V_o \sin\theta + \tfrac{1}{2} g\, t^2\right) \qquad (61b)$$

Eliminating t in Eqs. 61a and 61b, the equation for the curve is

$$y = -\left(x \tan\theta + \frac{g\, x^2}{2\, V_o^2 \cos^2\theta}\right) \qquad (61c)$$

As suggested by Mr. Hall, V_o, in Eq. 61*c*, should be taken as approximately 1.2 times the computed mean velocity at the beginning of the curve. For short chutes V_o may be determined from $\sqrt{2\,g\,H_o}$, in which H_o is the total vertical drop. Eq. 61*c* then becomes

$$y = -\left(x \tan\theta + \frac{x^2}{4\,H_o \cos^2\theta}\right) \qquad (61d)$$

Eq. 32 for superelevation in a horizontal curve is applicable to tranquil flow but not to rapid flow (supercritical velocity). For rapid flow, the maximum superelevation is considerably greater than given by Eq. 32; and it is not uniform around the bend but has maximum and minimum zones which result from transverse wave fronts. In addition to the rise in water surface at the outside wall caused by centrifugal action, there is a rise or drop in water surface caused by the attacking or receding wave front.[35] The maximum superelevation along the outside wall is given by

$$e_s = \frac{1.2\,V^2\,b}{g\,r_c} \qquad (62)$$

Observed data listed in Table 11 have been retabulated in Table 14 for comparison with the difference in water surface at the inside and outside walls computed by Eqs. 32 and 62. Computed differences in the last line of the table are maximum values that would occur if the maximum height of transverse wave occurred at each of the stations. The observed values in all cases are smaller than the maximum values. The maximum height of transverse wave probably did not occur exactly at any of the stations listed. The minimum height occurred near some stations. The initial transverse wave at the point of curvature of the 142-ft curve developed by upstream curves is another factor that caused the maximum observed superelevation to be smaller than computed. Transverse waves at the beginning of a curve may either magnify or dampen the new disturbance caused by the curve. From the data it appears that the disturbance was dampened in the 142-ft curve.

TABLE 14.—COMPARISON OF OBSERVED AND COMPUTED SUPERELEVATION IN SOUTH CANAL CHUTE
(Radius of Curve, 142 Ft)

Description	23+00	23+25	23+50	23+75	24+00	24+25	24+50	24+75	25+00	25+25	25+50
Observed velocity, in feet per second	16.6	19.2	21.7	25.2	28.8	29.6	30.5	32.0	33.5	34.3	35.1
Difference in Water Surface at Outside and Inside Walls, in Feet:											
Observed......	0.87	0.25	1.30	0.90	1.20	1.15	1.55	1.30	2.75	2.65	2.00
Computed by—											
Eq. 32.......	0.60	0.69	0.72	0.97	1.27	1.34	1.43	1.57	1.71	1.80	1.88
Eq. 62.......	1.44	1.64	1.71	2.31	3.04	3.20	3.42	3.76	4.11	4.31	4.52

In view of the complex wave pattern that may be developed when a series of closely spaced horizontal curves is located in a chute, the water surface cannot be computed accurately, and, for the sake of efficiency and economy, model studies should be made prior to construction. When a chute has but one horizontal curve, or horizontal curves, spaced some distance apart, the maximum superelevation can be computed by use of Eq. 62. The usual design practice is to provide ample freeboard for the maximum superelevation over the entire length of the curve and for some distance in the downstream tangent although the maximum superelevation will occur at only a few stations. Model studies of structures so designed would probably effect some saving by reduction of wall heights. In some designs superelevating the invert reduces costs. Superelevation in accordance with Eq. 32 will not eliminate the development of transverse waves, but properly designed compound curves, which produce disturbance waves of the same amplitude but of opposite phase from those produced by the main curve, will reduce their height.

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[35] "Testing Theoretical Losses in Open Channel Flow," by J. G. Jobes and J. H. Douma, *Civil Engineering*, November, 1942, pp. 613–615.

The design of hydraulic-jump stilling basins is another problem affected by air entrainment. The commonly used formula for jump height in a rectangular channel is

$$d_2 = -\frac{d_1}{2} + \sqrt{\frac{d^2_1}{4} + \frac{2\, d_1\, V^2_1}{g}} \ldots\ldots\ldots\ldots\ldots\ldots (63)$$

in which d_1 and V_1 are the depth and velocity upstream of the jump and d_2 is the depth after the jump. Although Eq. 63 is not applicable to aerated flow, a formula for aerated flow can be derived by the same method.

Referring to Fig. 27, the formation of a stable hydraulic jump must be accompanied by a change in kinetic momentum from section (1) to section (2)

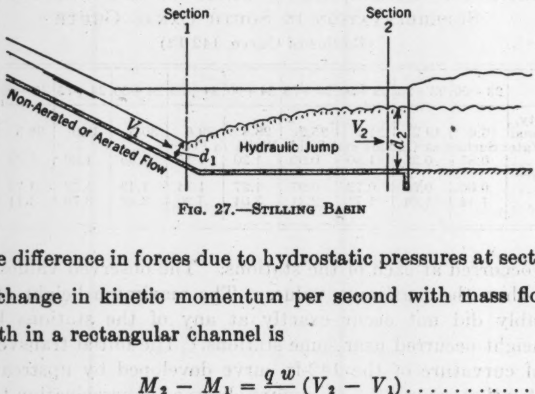

FIG. 27.—STILLING BASIN

equal to the difference in forces due to hydrostatic pressures at sections (1) and (2). The change in kinetic momentum per second with mass flow, $\frac{q\,w}{g}$, per foot of width in a rectangular channel is

$$M_2 - M_1 = \frac{q\,w}{g}(V_2 - V_1) \ldots\ldots\ldots\ldots\ldots\ldots (64a)$$

The resultant force between sections (1) and (2), assuming no entrained air at section (2), is

$$F_1 - F_2 = \frac{w}{2}(\rho\, d^2_1 - d^2_2) \ldots\ldots\ldots\ldots\ldots\ldots (64b)$$

A third equation is

$$q = \rho\, d_1\, V_1 = d_2\, V_2 \ldots\ldots\ldots\ldots\ldots\ldots (64c)$$

Equating Eqs. 64*a* and 64*b* and eliminating V_2 by using Eq. 64*c*, the formula for jump height with aerated flow becomes

$$d_2^2 = \rho\, d_1^2 + \frac{2\,\rho\, d_1\, V_1^2}{g}\left(1 - \rho\frac{d_1}{d_2}\right) \ldots\ldots\ldots\ldots (64d)$$

Eq. 64*d* can be solved for d_2 by trial and error.

The following sample computations are made to illustrate the influence of entrained air on the design of stilling basins: Assume a design discharge of 200 cu ft per sec per ft for a wide spillway channel. Further, assume $V_1 = 100$ ft per sec as calculated by the usual method for nonaerated flow. Then $d_1 = 2.0$ ft and solution of Eq. 63 gives $d_2 = 34.2$ ft. Now, for purposes of comparison, assume $V_1 = 80$ ft per sec as computed by Eq. 52 for aerated flow. Fig. 23 gives an air content of 31% from which $\rho = 0.69$. From Eq. 64*c*, $d_1 = 3.6$ ft; and from Eq. 64*d*, $d_2 = 30.2$ ft, which is 4.0 ft less than computed by the usual method.

Frictional resistance due to entrained air reduces the mean velocity and total kinetic energy at section (1), requiring a smaller d_2 depth for the formation of the hydraulic jump. The result is a significant saving in stilling-basin wall heights, reduced floor thickness because of smaller uplift pressures, less excavation, and a shorter stilling-basin length.

Many additional experimental data and studies by a number of hydraulic engineers are needed to confirm the application of the aerated flow formulas to steep chutes and spillways. In view of usual conservative assumptions of the value of *n*, for purposes of design, and the use of liberal freeboard, further studies should probably be directed toward establishing approximate general formulas before much effort is given to complex refinements.

Joe W. Johnson,[36] **Assoc. M. Am. Soc. C. E.**—Valuable quantitative data have been supplied, by this Symposium, to a field which previously has been discussed only in rather broad generalities. The various data should prove important in future design problems where air entrainment is a factor.

Of interest to the writer was the last part of the paper by Messrs. Kalinske and Robertson. In Fig. 16 experimental data are shown to verify the momentum relationship for a hydraulic jump in a 6-in. pipe. The authors attribute the scatter of the points around a 45° line to the difficulty in measuring values like depth of flow and pressure. The fact, however, that most of the points fall consistently below the 45° line indicates that experimental errors are not entirely the cause of scatter but suggests instead that some other factor may have been in error or was neglected in developing the equation.

A possible explanation of the consistent shift of the points to a position below the 45° line is the fact that the air-water flow ratio, β, may not have had the same value as the air-water ratio of the mixture (hereafter termed Δ). This later ratio refers to the mixture as actually existing between two points in the pipe immediately below the jump where the pipe is completely filled with the flowing mixture. The two ratios β and Δ have equal values only when the air and water are moving at the same velocity. When the air and water are not moving at the same velocity, a "slip" loss occurs, in which case the term β in Eqs. 37 and 38 should be replaced by Δ, the ratio of air to water as actually existing in the pipe.

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[36] Asst. Prof., Mech. Eng., Univ. of California, Berkeley, Calif.

A convincing demonstration of how great may be the difference between these two ratios can be obtained from published data on the flow of gas-liquid mixtures through vertical pipes.[37] In these experiments tests were made on the flow of water, kerosene, and oil through pipes, 1, 2, 2½, 3, and 4 in. in diameter. The flow ratio β was determined by metering the inflows. The volume ratio Δ was obtained by trapping a volume of the flowing mixture between quick-acting gate valves located at two points in the vertical pipe. Then by draining and determining the volume of the trapped liquid and with the volume of the pipe between the gate valves being known, the ratio of the mixture Δ was computed.

A typical example of the observations taken during these experiments is given by run 1 of the tabulated data. Thus, by metering the inflow to the apparatus, 15.7 cu ft per min of gas at mean pipe pressure and 0.86 cu ft per min of water were flowing in the pipe, thereby giving a gas-water flow ratio β of 18.3. The gas-water ratio of the mixture Δ obtained by closing the quick-acting gate valves was found to be only 3.2. The linear velocity of the water was computed to be 10.5 ft per sec, and the velocity of the gas was 56.1 ft per sec.

The wide difference that is possible between the two values of the gas-water ratios is the basis of the foregoing statement that the value of β in Eqs. 37 and 38 perhaps should be replaced by the ratio Δ. In attempting to compare the experiments of T. V. Moore and H. D. Wilde[37] with those of Messrs. Kalinske and Robertson, a comment on the flow conditions in the two experiments is necessary. In the Moore and Wilde experiments the gas expanded as it flowed upward through the pipe and "slipped" past the water. The opposite condition probably occurred in the Kalinske-Robertson experiments—namely, the water was "dragging" air along from a low pressure space to a high pressure space and the velocity of the water slightly exceeded the velocity of the air. Because of this velocity difference, therefore, the mixture ratio Δ was greater than the flow ratio β. Thus, if β in Eqs. 37 and 38 is replaced by Δ, the value of M_2 would be increased and the value of W would be decreased. The net result increases the ordinate of the various experimental points shown in Fig. 16. Closer grouping around the 45° line probably would result. The installation of two quick-closing gate valves in the Kalinske-Robertson apparatus should give interesting information on the relative magnitude of β and Δ. Obviously, their values would not differ as greatly as in the experiments of Messrs. Moore and Wilde where the gas was expanding.

J. C. STEVENS,[38] M. AM. SOC. C. E.—In his part of the Symposium, Mr. Hall has presented a collection of valuable data on air entrainment at high velocities, concerning which there is scanty information available.

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[37] "Experimental Measurement of Slippage in Flow Through Vertical Pipes," by T. V. Moore and H. D. Wilde, *Transactions*, A. I. M. E., Petroleum Div., 1931, pp. 296–313.

[38] Cons. Hydr. Engr. (Stevens & Koon), Portland, Ore.

The writer is interested in devising means of applying the data presented in this paper to the design of chutes, and particularly to the design of stilling basins at the foot of high overflow dams. To design such a basin properly, one should know the velocity, depth, and insufflation characteristics with which the water reaches the basin. To do this, the degree of insufflation must be found in terms of some easily ascertainable quantity. Mr. Hall used $\frac{V^2}{g\,R_c}$ and the ratio of air to water volumes as an index of insufflation. A somewhat better relationship results from expressing the degree of insufflation in terms of the ratio of the kinetic to the potential energy, for brevity termed the "kineticity" of flow. There appears to be no special advantage in using $\frac{1-\rho}{\rho}$ rather than ρ alone to express the degree of insufflation—the straight lines of Fig. 10 are not very convincing.

The observed velocities were "smoothed" through the velocity heads in the energy line. Why not smooth the observed depths also? In the field the only quantities that admit of measurement are the discharge (which can be measured with reasonable accuracy), the depths, and velocities of the water-air mixture for which the observational errors involved are great; hence proper smoothing of both quantities is justified fully.

Alternative Analysis.—An analysis of the observed data that appears to give more effective results is presented herewith. With insufflated flow two conditions are involved: (1) A flow of liquid water measured at the head, a part of which remains liquid but unmeasurable in the field, in the bottom of the channel; and (2) a mixture of air and water riding on top of the liquid water for which depths and velocities may be observed. All other characteristics of this type of flow must be derived from the aforementioned three measured quantities.

A slight departure from the notation of the paper, with some additions, will be made as follows: Let w = a subscript denoting the water (without air); m = a subscript denoting the insufflated mixture; k = kineticity = $\frac{V^2}{2\,g\,y\cos\theta}$; ϕ = angle the friction slope at any cross section makes with the horizontal; S = sine of the friction-slope angle; S_o = sine of the bed-slope angle; e = specific energy head referred to the bed of the channel; and z = elevation above any datum.

There is no question in the writer's mind but that the water in the bottom of the channel is flowing faster than the mixture, but this difference does not admit of observation except in the laboratory. The effect of insufflation is to retard velocities, and the simplest and most direct method of taking account of this retardation is to increase the value of n for the mixture over that for the water alone.

None of the flow formulas in existence are designed to meet the flow conditions under discussion. Since neither the Kutter formula nor the Manning formula is dimensionally correct, it is risky to apply either so far outside of the range of observations on which it is based. However, there is no alternative, and, if the experimental data on chutes are used to determine the factors in these formulas, the same range of factors will render these formulas suitable for design. In this study the Manning formula is used. If strictly applicable, a uniform value of n should be obtained for a given channel unless the rugosity changes throughout its length. However, neither the Manning nor the Kutter formula will produce a constant value of n; hence in design the investigator must adopt a single average value for the water flow and a variable and increasing value for the mixture.

TABLE 15.—FLOW CHARACTERISTICS OF

Explanation: Col. 2, "smoothed" values from Table 3(*d*), Col. 1; Col. 4, from Table 3(*d*), Col. 10; Col. 5 × Col. 10; Col. 15, k = Col. 14 ÷ Col. 13; and Col. 19, n_m = Col. 17 × Col. 18.

Station (1)	Depth y_m (2)	Area A_m (3)	Velocity V_m (4)	Percentage of water ρ (5)	Area A_w (6)	Depth y_w (7)	Hydraulic radius R_w (8)	Z (9)	$(n_w V_m)^2$ (10)	Friction slope S (11)
0+50	2.35	13.05	40.8	74.2	9.68	1.76	1.14	0.380	0.166	0.070
0+75	2.00	11.05	48.1	74.3	8.06	1.48	1.02	0.441	0.231	0.110
1+00	1.85	10.20	54.7	70.9	7.22	1.34	0.94	0.492	0.299	0.163
1+25	1.82	10.02	60.5	65.2	6.54	1.21	0.88	0.537	0.366	0.220
1+50	1.82	10.02	65.0	60.6	6.08	1.13	0.84	0.571	0.422	0.271
1+75	1.82	10.02	68.7	57.4	5.75	1.08	0.81	0.600	0.472	0.314
2+00	1.83	10.07	71.7	54.8	5.51	1.03	0.79	0.620	0.514	0.367
2+25	1.84	10.15	73.9	52.7	5.34	1.01	0.77	0.642	0.546	0.398
2+50	1.87	10.30	75.3	51.0	5.28	0.99	0.76	0.653	0.567	0.421
2+75	1.90	10.47	75.9	49.8	5.21	0.98	0.75	0.665	0.576	0.436

The friction slope of the water undoubtedly differs from that of the mixture but this difference is indeterminate, and one must assume that the same friction slope applies to both in the data presented by Mr. Hall.

Two important factors are sought in this analysis: (1) A relation between kineticity of flow and degree of insufflation, and (2) the increment in the roughness factor due to insufflation.

Hat Creek Chute.—The depths for Hat Creek chute were plotted for each flow and a smooth curve drawn among the points. Only the part of the flume between Stations 0+50 and 2+75 was used. The entrance condition induced insufflation, and some time passed before the normal balance was restored. At the sharp convex vertical curve from Stations 2+75 to 3+00, where the water tends to leave the channel bed, a great slug of air was added to the mixture from which it never recovered. Since the aim was to isolate normal insufflation due to channel roughness and kineticity of flow from other influences, it was necessary to eliminate such extraneous disturbances.

Table 15 gives a sample of the calculations made for this channel for a flow of 395 cu ft per sec. By correspondence, it was learned that there are 8-in. fillets in the corners of this flume which account for observed areas not equaling "width times depth" in Table 3. The formulas for areas (Col. 3) therefore are, for depths greater than 0.67 ft—

$$A = 5.75\,y - 0.45 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (65a)$$

for depths less than 0.67 ft—

$$A = 4.42\,y + y^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (65b)$$

In Col. 5 the percentage of water ρ is computed by

$$\rho = \frac{Q}{A_m\, V_m} \qquad (66)$$

Values in the next column (Col. 6, Table 15) are found from

$$A_w = \rho\, A_m \qquad (67)$$

Hat Creek Chute for 395 Cu Ft per Sec

(see Eq. 66); Col. 6 (see Eq. 67); Col. 9 (see Eq. 68); Col. 10, average value of n_w = 0.010; Col. 11, S = Col. 9

Cos θ (12)	Potential energy $y_w \cos\theta$ (13)	Kinetic energy $\frac{V^2_m}{2g}$ (14)	Kineticity k (15)	Hydraulic radius R_m (16)	$\frac{1.486}{V_m}$ (17)	$(R_m)^{2/3} S^{1/2}$ (18)	Manning n_m (19)	Ratio $\frac{n_m}{n_w}$ (20)	Station (1)
0.896	1.58	25.9	16.4	1.35	0.0365	0.324	0.0118	1.18	0+50
0.896	1.33	36.0	27.1	1.23	0.0310	0.381	0.0118	1.18	0+75
0.896	1.20	46.6	38.8	1.18	0.0272	0.451	0.0123	1.23	1+00
0.896	1.09	57.0	52.3	1.17	0.0246	0.521	0.0127	1.27	1+25
0.886	1.00	65.8	68.5	1.17	0.0229	0.578	0.0132	1.32	1+50
0.886	0.96	73.5	76.6	1.17	0.0217	0.623	0.0135	1.35	1+75
0.886	0.91	80.0	87.9	1.17	0.0208	0.674	0.0140	1.40	2+00
0.917	0.93	85.0	91.4	1.17	0.0202	0.700	0.0141	1.41	2+25
0.917	0.91	88.2	97.0	1.18	0.0198	0.725	0.0144	1.44	2+50
0.917	0.90	89.6	99.6	1.19	0.0196	0.740	0.0148	1.48	2+75

and, in Col. 9,[39]

$$Z = \frac{1}{2.2082\,(R_w)^{4/3}} \qquad (68)$$

Graphs of areas versus depths and of wetted perimeters versus depths were made to facilitate computations.

The average value of $n_w = 0.010$ was obtained by trial. The average value of sin θ for the reach is 0.445. A roughness value had to be selected such that the friction slope for the higher flows would not exceed that value materially, because, if the friction slope tends to exceed the bed slope, the flow will occur in pulsations or waves. This may be expected of the lesser flows, but there is no evidence that the flow was pulsating for the two higher flows. With the observed velocities and an average value of $n_w = 0.010$ the friction slopes obtained were as given in Table 16.

TABLE 16.—Sine of Friction Slope Angles for Hat Creek Chute

Station	Discharge, in Cu Ft Per Sec:			
	395	368	163	115
0+50	0.070	0.073	0.104	0.121
0+75	0.110	0.119	0.199	0.206
1+00	0.163	0.167	0.274	0.283
1+25	0.220	0.221	0.343	0.375
1+50	0.271	0.266	0.400	0.430
1+75	0.314	0.312	0.436	0.490
2+00	0.367	0.354	0.472	0.563
2+25	0.398	0.394	0.493	0.580
2+50	0.421	0.450	0.543	0.619
2+75	0.436	0.456	0.580	0.620

In Table 12 Mr. Hall gives values of n but the slope or slopes on which they were computed is not indicated. Since the flow is nonuniform in all cases, the *n*-value should of course be based on friction slope. The values for Hat Creek chute given in Table 12 and the average value used herein are considerably less than would be expected of a roughened concrete flume with projecting form wires. Nevertheless, the value seems to be correct for the Manning formula, which indicates how cautious one must be in selecting such values for design purposes outside the proved range of this formula.

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[39] "Handbook of Hydraulics," by Horace W. King, McGraw-Hill Book Co., Inc., New York, N. Y., 3d Ed., p. 309, Table 107.

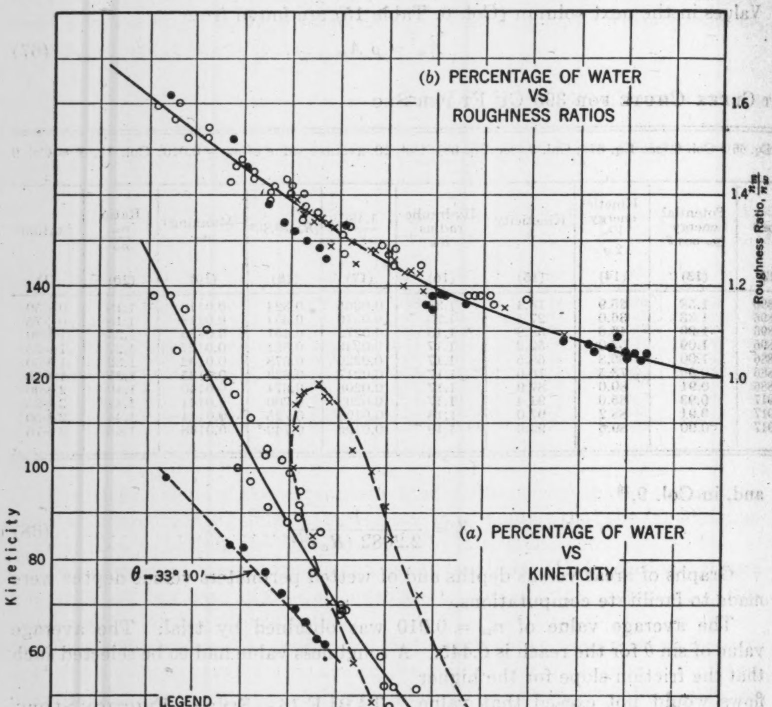

FIG. 28.—RELATION OF PERCENTAGE OF WATER TO ROUGHNESS RATIOS AND KINETICITY

The values in Cols. 5, 15, and 20 of Table 15 are shown in Fig. 28 by the open circles. From this, there appears to be a fairly definite relation between kineticity and the degree of insufflation if no foreign influences are present. The curve must turn sharply at its lower end and intersect the 100% line at some initial value of the kineticity at which appreciable insufflation begins. The two lower points indicated that air was added at the inlet and was ignored in drawing the curve.

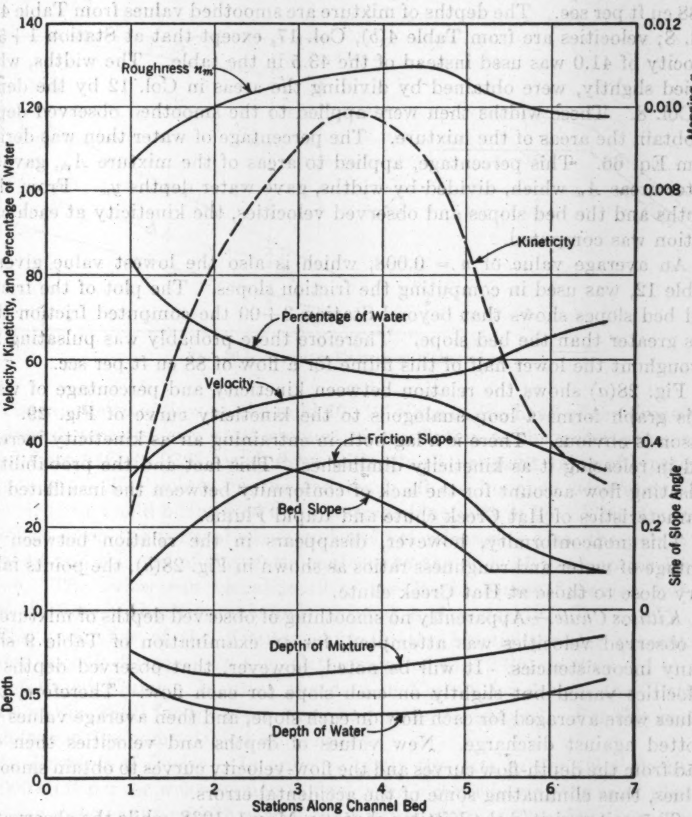

FIG. 29.—INSUFFLATED FLOW CHARACTERISTICS FOR RAPID FLUME WITH A DISCHARGE OF 88 CU FT PER SEC

The conformity of the curve of roughness ratios to the percentage of water in the mixture is quite gratifying and indicates that the effect of insufflation may be accounted for in this manner.

Rapid Flume.—The same method of analyzing the observed data was followed for the Rapid Flume. There were no sharp convex vertical curves to add air, but there was a horizontal curve at Station 2+00 which produced a wave pattern making it difficult to observe depths with any degree of accuracy. On account of air entrained at the inlet, flow in the first 100 ft of flume was not used.

The analysis shows kineticity and insufflation increasing to a maximum as the flow proceeded downstream, followed by reductions as the slope flattened. Fig. 29 shows the insufflated flow characteristics for the flume with a discharge of 88 cu ft per sec. The depths of mixture are smoothed values from Table 4(*b*), Col. 8; velocities are from Table 4(*b*), Col. 17, except that at Station 1+50 a velocity of 41.0 was used instead of the 43.5 in the table. The widths, which varied slightly, were obtained by dividing the areas in Col. 12 by the depths of Col. 8. These widths then were applied to the smoothed observed depths to obtain the areas of the mixture. The percentage of water then was derived from Eq. 66. This percentage, applied to areas of the mixture A_m, gave the water areas A_w which, divided by widths, gave water depths y_w. From these depths and the bed slopes and observed velocities, the kineticity at each cross section was computed.

An average value of $n = 0.008$, which is also the lowest value given in Table 12, was used in computing the friction slopes. The plot of the friction and bed slopes shows that beyond Station 3+00 the computed friction slope was greater than the bed slope. Therefore there probably was pulsating flow throughout the lower half of this flume for a flow of 88 cu ft per sec.

Fig. 28(*a*) shows the relation between kineticity and percentage of water. This graph forms a loop analogous to the kineticity curve of Fig. 29. The reason is obvious. There is a lag both in entraining air as kineticity increases and in releasing it as kineticity diminishes. This fact and the probability of pulsating flow account for the lack of conformity between the insufflated flow characteristics of Hat Creek chute and Rapid Flume.

This nonconformity, however, disappears in the relation between percentage of water and roughness ratios as shown in Fig. 28(*b*), the points falling very close to those at Hat Creek chute.

Kittitas Chute.—Apparently no smoothing of observed depths of mixture and of observed velocities was attempted, for an examination of Table 9 shows many inconsistencies. It will be noted, however, that observed depths and velocities varied but slightly on each slope for each flow. Therefore, these values were averaged for each flow on each slope, and then average values were plotted against discharge. New values of depths and velocities then were read from the depth-flow curves and the flow-velocity curves to obtain smoothed values, thus eliminating some of the accidental errors.

The writer visited the Kittitas chute on May 1, 1938, while the observations for the discharge of 922 cu ft per sec were under way by engineers of the Bureau of Reclamation. He appreciates fully the many difficulties of getting refined and consistent results and believes the smoothing process he has adopted herein to be justified fully.

The values obtained gave ten points on the 10° 12′ slope, nine points on the 33° 10′ slope, and eight points on the practically horizontal extremity of the flume. These points are shown by solid circles in Fig. 28. In computing the friction slopes an average value of $n = 0.014$ was used for Manning's coefficient. The superabundance of air in the mixture for the 33° 10′ slope (resulting from the sharp convex upward vertical curve just above this slope where the water jetted virtually free from the bed) is plainly evident. Fig. 28(*a*) indicates that this curve in the flume increased the volume of air in the mixture by about 10% over the probable volume had the insufflation been the sole result of its kineticity and channel roughness.

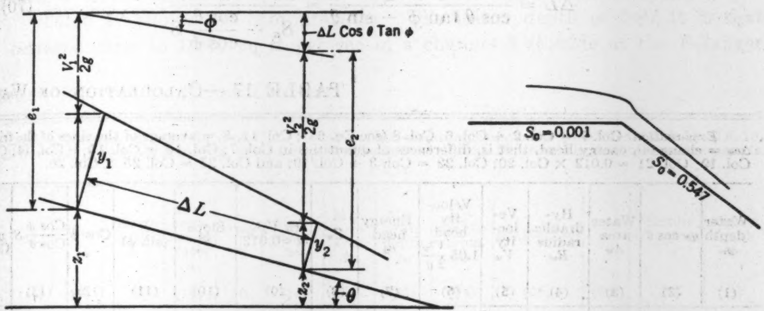

FIG. 30.—ASSUMED ENTRANCE CONDITIONS

The kineticity of flow on the mild 10° 12′ slope at the head of the flume was not enough to cause any great amount of insufflation as the water percentage varied from 84% to 93% for all flows with a fairly constant value of kineticity of about 24. The points for the end reach all are bunched with about 8% more air than would be indexed by its average kineticity of 25. This is surplus air taken in on the vertical curve and not yet released. Also, the points for the head and the end of the chute conform to the ends of the loop for Rapid Flume. The writer can visualize all of the Rapid Flume behavior, but he views the flow of the Kittitas chute "through a glass darkly."

The anomalies of the Kittitas chute in the kineticity-water percentage relation largely disappear in the relationships of Fig. 28(*b*) as they did in the case of Rapid Flume. In using the data of Fig. 28 for design, the Kittitas results in Fig. 28(*a*) for the steep slope had better be ignored. For this reason the details of the computation are omitted.

An Example of Design.—In order to apply the data in Fig. 28 a discharge of 1,000 cu ft per sec was assumed to flow down a chute having the same dimensions and slope as the middle part of the Kittitas chute. The chute is assumed to start at a point where the steep slope ($S_o = 0.547$) joins an upstream part on a mild slope ($S_o = 0.001$) so that critical flow would obtain at the junction and no air entrainment or wave patterns would result at the entrance. The assumed entrance conditions are shown in profile in Fig. 30(*b*).

In any prismatic channel it is not necessary to use the trial-and-error method indicated by Mr. Hall's explanation of Eq. 27. A far simpler method whereby the distance between assumed depths may be computed directly has been outlined by the writer in another connection.[40] The credit for the basis of this method must go to Alva G. Husted.[41] Referring to Fig. 30(a), it is obvious that

$$z_1 + y_1 \cos\theta + C_{e1}\frac{V^2_1}{2\,g} = z_2 + y_2 \cos\theta + C_{e2}\frac{V^2_2}{2\,g} + \Delta L \cos\theta \tan\phi \;. . (69)$$

Obviously, $z_1 - z_2 = \Delta L \sin\theta$; let $e = y \cos\theta + C_e \frac{V^2}{2\,g}$; substitute, and solve for

$$\Delta L = \frac{e_1 - e_2}{\cos\theta \tan\phi - \sin\theta} = \frac{-\,\Delta e}{S_o - \dfrac{\cos\theta}{\cos\phi}S} \;. . . . . (70)$$

TABLE 17.—CALCULATION OF WATER-

Explanation: Col. 7 = Col. 2 + Col. 6; Col. 8 (see Eq. 68); Col. 11, S = average of the sines of the friction- Δe_w = change in energy head, that is, differences of quantities in Col. 7; Col. 16 = Col. 15 ÷ Col. 14; Col. 18 Col. 19; Col. 21 = 0.012 × Col. 20; Col. 22 = Col. 3 + Col. 19; and Col. 27 = Col. 25 × Col. 26.

Water depth y_w (1)	$y_w \cos\theta$ (2)	Water area A_w (3)	Hy-draulic radius R_w (4)	Ve-loc-ity V_w (5)	Veloc-ity head $1.05\frac{V^2_w}{2\,g}$ (6)	Energy head e_w (7)	Z_w (8)	$(n\,V)^2_w$ $n = 0.012$ (9)	Slope S' (10)	S (sin ϕ) (11)	Cos ϕ (12)	$\frac{\text{Cos}\,\phi}{\text{Cos}\,\theta}S$ (13)	S_o minus Col. 13 (14)
7.87	6.60	63.0	2.69	15.87	4.12	10.72	0.121	0.0363	0.00440				
										0.0061	1.00	0.0073	0.540
6.00	5.03	48.0	2.40	20.83	7.09	12.12	0.141	0.0625	0.00882				
										0.0114	1.00	0.0136	0.533
5.00	4.18	40.0	2.22	25.0	10.25	14.43	0.156	0.0901	0.0140				
										0.0196	1.00	0.0234	0.524
4.00	3.35	32.0	2.00	31.2	15.95	19.30	0.180	0.140	0.0252				
										0.0400	0.999	0.0478	0.499
3.00	2.51	24.0	1.72	41.6	28.35	30.86	0.220	0.249	0.0549				
										0.0734	0.997	0.0873	0.460
2.5	2.09	20.0	1.54	50.0	41.0	43.09	0.255	0.360	0.0919				
										0.131	0.991	0.154	0.393
2.0	1.67	16.0	1.33	62.5	63.9	65.57	0.310	0.545	0.169				
										0.202	0.979	0.236	0.311
1.8	1.50	14.4	1.24	69.5	79.0	80.50	0.340	0.694	0.236				
										0.258	0.966	0.298	0.249
1.7	1.42	13.6	1.19	73.6	88.4	89.82	0.359	0.778	0.279				
										0.306	0.952	0.348	0.199
1.6	1.34	12.8	1.14	78.2	100.0	101.34	0.380	0.880	0.334				
										0.348	0.937	0.389	0.158
1.55	1.30	12.4	1.12	80.6	106.0	107.30	0.389	0.935	0.363				
										0.383	0.924	0.422	0.125
1.50	1.26	12.0	1.09	83.4	113.6	114.86	0.404	1.00	0.404				
										0.426	0.905	0.460	0.087
1.45	1.21	11.6	1.06	86.2	121.5	122.71	0.419	1.07	0.449				
										0.472	0.882	0.498	0.049
1.40	1.17	11.20	1.04	89.3	130.4	131.57	0.430	1.15	0.495				
										0.500	0.886	0.518	0.029
1.39	1.17	11.12	1.03	89.9	132.0	133.17	0.435	1.165	0.506				
										0.510	0.860	0.524	0.023
1.38	1.16	11.04	1.03	90.6	134.0	135.16	0.435	1.183	0.514				
										0.522	0.853	0.532	0.015
1.37	1.15	10.96	1.02	91.4	136.5	137.87	0.441	1.203	0.531				

in which S_o = sine of the angle of channel-bed slope; S = sine of the angle of friction slope; e = specific energy head, referred to the bed of the channel; and Δe is the change in the specific energy head of a unit weight of water.

On steep slopes the specific energy head increases downstream and therefore is considered negative in the direction of flow. The coefficient C_e is necessary to convert the kinetic energy of a unit weight of water having the mean velocity of the stream to the mean kinetic energy of a unit weight.[42, 43] For the case in hand, although it probably represents an unnecessary refinement, a coefficient of 1.05 was applied uniformly to the velocity head of the mean velocity to obtain the mean velocity head.

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[40] *Transactions*, Am. Soc. C. E., Vol. 102 (1937), p. 666.

[41] *Engineering News-Record*, April 24, 1925, p. 719.

For this example the channel width is 8 ft; the sides are assumed high enough to contain the mixture; and the roughness coefficient (Manning) was taken to be $n = 0.012$. The sine of the bed slope is 0.547 and the cosine is 0.837.

Table 17 shows the computations. The initial depth of 7.87 ft is that corresponding to 1,000 cu ft per sec in a channel 8 ft wide at the Bélanger critical flow:

$$y_c = \left[\left(\frac{Q}{8} \right)^2 \frac{1}{g} \right]^{1/3} \quad \text{(71)}$$

Additional explanations are given in Table 17. In Fig. 31 the quantities in

Surface Profile in Steep Chute

slope angle, ϕ, between adjacent stations; Col. 14 = sine of bed-slope angle minus the quantities in Col. 13; Col. 15, = Col. 6 + Col. 2; Col. 19, from Fig. 28(a) for kineticities in Col. 18; Col. 20, from Fig. 28(b) for percentages in

Δe_w	ΔL	Station	Kineticity k	Percentage of water ρ	$\frac{n_m}{n_w}$	Manning n_m	Mixture area A_m	Mixture depth y_m	Hydraulic radius R_m	$(R_m)^{2/3}(S')^{1/2}$	$\frac{1.486}{n_m}$	Mixture velocity V_m	Water depth y_w
(15)	(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)	(26)	(27)	(1)
		0+00	0.6	100	1.0	0.0120	63.0	7.87	2.69	0.128	124	15.87	7.87
1.67	3.1												
		0+03	1.4	100	1.0	0.0120	48.0	6.00	2.40	0.168	124	20.83	6.00
2.31	4.3												
		0+07	2.5	100	1.0	0.0120	40.0	5.00	2.22	0.202	124	25.0	5.00
4.87	9.3												
		0+17	4.8	100	1.0	0.0120	32.0	4.00	2.00	0.252	124	31.2	4.00
11.56	23.2												
		0+40	11.3	96	1.02	0.0121	25.0	3.12	1.75	0.340	123	41.8	3.00
12.23	26.6												
		0+57	19.6	83	1.09	0.0131	24.1	3.01	1.72	0.435	113	48.1	2.5
22.48	57.4												
		1+24	38.3	72	1.18	0.0142	22.2	2.77	1.64	0.572	105	60.0	2.0
14.93	48.0												
		1+72	52.6	66	1.24	0.0149	21.9	2.74	1.61	0.667	99.8	66.6	1.8
9.32	37.4												
		2+09	62.2	63	1.28	0.0154	21.6	2.70	1.61	0.726	96.5	70.0	1.7
11.52	58.0												
		2+67	74.6	59	1.33	0.0160	21.7	2.71	1.61	0.794	92.9	73.7	1.6
5.96	37.7												
		3+05	81.7	57	1.35	0.0162	21.8	2.72	1.62	0.831	91.7	76.2	1.55
7.57	60.5												
		3+65	90.2	54	1.40	0.0168	22.2	2.78	1.64	0.884	88.5	78.2	1.50
7.85	90.2												
		4+56	100	52	1.42	0.0170	22.3	2.79	1.64	0.931	87.5	81.4	1.45
8.86	181.0												
		6+37	111	49	1.47	0.0177	22.9	2.86	1.66	0.986	84.0	82.8	1.40
1.60	55.0												
		6+92	112	49	1.47	0.0178	22.8	2.85	1.66	0.995	83.5	83.0	1.39
1.99	87.0												
		7+79	115	48	1.49	0.0179	23.0	2.88	1.67	1.01	83.0	83.8	1.38
2.71	180.0												
		9+59	119	47	1.50	0.0180	23.4	2.92	1.69	1.03	82.5	84.9	1.37

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[42] "Applied Fluid Mechanics," by Morrough P. O'Brien and G. H. Hickox, McGraw-Hill Book Co., Inc., New York, N. Y., 1st Ed., p. 271.

[43] "Velocity Head Correction for Hydraulic Flow," by Morrough P. O'Brien and Joe W. Johnson, *Engineering News-Record*, August 16, 1934, p. 214.

Table 17, Cols. 1, 5, 10, 21, 23, and 27, are plotted against the stations in Col. 17. Insufflation begins at Station 0+25, at which point the water and mixture depths begin to separate. It appears that the terminal velocities of both water and the mixture virtually were reached at Station 9+59. The friction slope is asymptotic to the bed slope, and, as it approaches closer and closer to the bed slope, the flow begins to pulsate. If the chute were indefinitely long, the flow at the end undoubtedly would occur in a series of waves, because there scarcely can be an exact balance between the accelerating and retarding forces, in such cases.

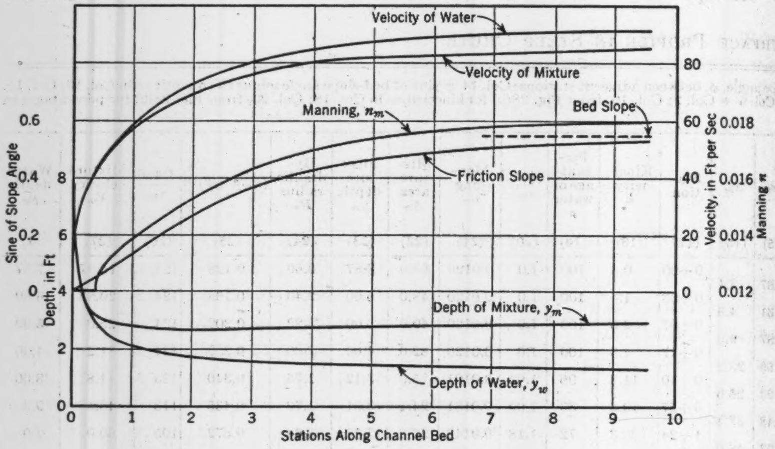

FIG. 31.—FLOW CHARACTERISTICS FOR A STEEP CHUTE

A decided difference in the velocity of the water and that of the mixture can be obtained by calculation in this manner. At Station 9+59 the velocity of the mixture was 7% less than that of the water. Some such difference probably occurs, but this can be demonstrated only in the laboratory.

In applying these data to the design of a stilling basin, one should disregard the air in the mixture and base the design solely on the water depth and its terminal velocity.

This study indicates that the retarding effect of insufflation logically can be accounted for by increasing the roughness factor. The curve of increase is shown in Fig. 31.

D. C. MCCONAUGHY,[44] ESQ.—Mr. Hall's paper treats a phase of hydraulics that is almost unexplored and, it is to be hoped, will stimulate research looking toward the solution of the problems involved.

The Bureau of Reclamation has been interested in this field in recent years in connection with spillways for high concrete dams and for earth dams, in which chutes designed for large volumes of water at high velocities form an important part.

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[44] Senior Engr., Bureau of Reclamation, U. S. Dept. of the Interior, Denver, Colo.

In the absence of any theoretical development of a method of correlating and extending results of experiments on small structures, it is desirable that data be obtained on larger ones. Unfortunately, structures suitable for this purpose are rare. Conditions such as availability of water, a means of measuring the discharge, interference with power or irrigation requirements, etc., must be satisfied, which make it very difficult to conduct a satisfactory experimental program. The data used by the author on the Bureau of Reclamation tests on Kittitas chute were contained in informal memoranda which had not been released for publication because the accuracy of the velocity measurements was considered somewhat questionable and because it was felt that they would add little to existing knowledge on the subject. It was hoped that experiments over a wider range, which could be accepted without reservation, could be obtained before anything was published. However, in view of the publicity which has now been given to the experiments, a more complete description than Mr. Hall's should be given for the benefit of possible future users of the data.

Experiments for perfecting satisfactory apparatus and techniques for making the necessary velocity and depth measurements were started in the hydraulic laboratory about 1937, first, on an existing model in which velocities up to 15 ft per sec could be obtained and, later, on another model specially built for the purpose, giving velocities up to 50 ft per sec. All known methods of measuring velocities were considered, and several were used experimentally. The salt-velocity method was finally selected as offering the best prospect of success, time intervals being measured by means of an oscillograph built in the Denver laboratory of the Bureau. This method was found very successful and arrangements were made to develop the method further by field measurements on the wasteway at Station 1146+30 of the main canal, Kittitas division, Yakima Project. Unfortunately, time did not permit the completion of equipment entirely suitable for field use.

TABLE 18.—ELECTRODE SPACING (SEE TABLE 19)

Slope	Cumulative Distances, in Feet, from the First Electrode											
	0	10	40	50	100	110	200	210	300	310	400	410
	Electrode Numbers											
0° 04′	17	18	19	20	...	...	...	...	...	...	...	...
10° 12′	1	2	...	...	3	4	5	6	7	8	9	10
33° 10′	11	12	...	...	13	14	15	16	...	...	...	...
Interval (ft)	0	10	30	10	50	10	90	10	90	10	90	10

Electrodes were fastened to the walls of the chute, spaced at different intervals; the electrodes were numbered, and these numbers are given in Col. 2, Table 9. Supplementing Mr. Hall's description, electrode No. 1 was 25 ft below the end of the tapered section; No. 10 was about 7 ft above the beginning of the vertical curve between the two steeper slopes; No. 11 was about 64 ft below the end of this curve; Nos. 15 and 16 were equidistant from the beginning of a circular curve of 134.6-ft radius and a central angle of 33° 06′; Nos. 17 and 18 were equidistant from the end of this curve; and No. 20 was at the lower end of the chute. The vertical curve between the two steeper slopes was a parabola with a horizontal length of 84 ft and a vertical drop of 34.87 ft. Distances between electrodes, measured along the slope, are given in Table 18. The salt used was of a fineness to pass a 140-mesh sieve. It was moistened until it could be molded by the hands into balls, which were dropped into the water a short distance upstream from the test reach. The size of the balls necessary was determined by trial and error and varied from 2 to 7 in.

TABLE 19.—VELOCITIES, IN FEET PER SECOND, BETWEEN THE ELECTRODES INDICATED (SEE TABLE 18)

Discharge (cu ft per sec)	(a) Slope, 0° 04′			(b) Slope, 10° 12′									(c) Slope, 33° 10′						
	E[a]	Velocity	E[a]	E[a]	Velocity	E[a]	Velocity	E[a]	Velocity	E[a]	Velocity	E[a]	E[a]	Velocity	E[a]	Velocity	E[a]	Velocity	E[a]
89	..	...	..	1	28.6	3	27.4	5	..	..	..	..	..	...	..	..	..	...	..
	..	...	..	..	...	3	27.4	5	...	..	...	..	..	...	..	...	..	...	..
193	..	...	..	1	30.5	3	29.8	5	31.6	9	...	..	..	...	12	44.5	14	53.3	16
	..	...	..	..	...	..	...	5	30.5	7	...	..	..	...	12	41.5	14	54.0	15
	..	...	..	..	...	..	...	..	...	..	...	..	..	...	12	56.1	14	...	..
362	18	34.0	20	1	37.2	3	40.5	5	41.4	8	49.2	9	11	63.5	14	56.4	15	...	..
	18	35.5	20	1	43.1	3	44.1	5	38.3	8	49.8	9	11	54.7	14	56.7	15	...	..
	18	34.0	20	..	...	..	...	..	...	..	...	..	..	...	12	57.3	14	52.7	16
401	..	...	..	1	37.7	3	42.8	5	...	..	...	..	11	60.1	13	65.3	16	...	..
	..	...	..	1	37.4	3	41.4	5	...	..	...	..	11	60.0	13	63.5	16	...	..
	..	...	..	..	...	..	...	..	...	..	...	..	..	...	13	58.5	16	...	..
491	17	59.5	20	1	42.1	3	39.2	5	43.8	8	...	..	..	...	12	59.0	13	67.5	15
	17	48.1	20	..	...	3	45.1	5	45.5	8	...	..	11	63.8	13	64.3	15	...	..
	18	45.4	20	..	...	3	41.4	5	46.2	9	...	..	..	...	12	62.1	14	61.5	16
587	17	33.7	19	1	44.5	3	44.1	5	46.4	7	50.6	9	11	67.5	13	64.0	15	...	..
	17	45.3	20	..	...	3	47.0	5	49.2	7	53.1	9	11	63.7	13	66.7	15	...	..
	17	60.4	20	..	...	..	...	5	46.2	7	51.6	9	11	60.4	13	67.5	16	...	..
	..	...	..	..	...	..	...	..	...	..	...	..	..	...	13	69.5	15	...	..
719	17	67.6	20	1	46.6	3	46.5	5	41.3	7	56.0	9	11	77.2	13	75.2	16	...	..
	17	60.0	20	1	46.5	3	45.3	5	50.0	7	54.3	9	11	71.1	13	65.1	16	...	..
	17	57.5	19	..	...	..	...	5	56.5	7	51.3	9	..	...	13	66.3	15	...	..
777	17	57.5	19	1	52.1	3	50.1	5	50.9	7	50.8	9	11	65.9	13	94.0	16	...	..
	17	50.6	19	..	...	3	51.7	5	...	7	51.9	9	..	...	13	85.6	15	...	..
922	17	58.6	20	1	58.6	3	60.1	5	53.3	7	54.8	9	..	...	12	68.7	13	75.2	15
	18	56.0	20	1	49.2	3	52.3	5	48.6	7	55.4	9	..	...	12	77.6	13	74.5	15
	..	...	..	..	...	3	49.0	5	...	7	52.2	9	..	...	12	70.5	14	68.6	16
1,005	17	62.3	20	1	58.2	3	51.1	5	58.3	7	54.0	9	..	...	12	91.1	15	...	..
	18	55.4	20	1	50.8	3	57.5	5	55.5	7	51.2	9	..	...	12	75.1	15	...	..
	..	...	..	..	...	..	...	5	53.8	7	54.1	9	..	...	..	...	13	81.6	15
	..	...	..	..	...	..	...	..	...	..	...	..	..	...	..	...	13	82.6	15
	..	...	..	..	...	..	...	..	...	..	...	..	..	...	..	...	14	74.1	16

[a] E denotes "Electrode Number," marking the electrode numbers between which velocities were observed. The position of the electrodes is given in Table 18.

The field trials were not wholly successful, partly because of inadequate apparatus and partly because of imperfect techniques. The separate velocity measurements obtained are given in Table 19. In determining times to obtain mean velocities, the distance between centers of gravity of the oscillograph diagrams should be used. However, the diagrams in many cases were indistinct and in some were not completely recorded. Sixteen of the clearest diagrams were chosen and the distances between the centers of gravity were computed. The ratios of the distances between the beginning points to the distances between the centers of gravity (ratio of maximum to mean velocity) averaged 1.40. The ratios of the distances determined by averaging the beginning points with other characteristic points to the distances between centers of gravity averaged 1.24; when the distances between centers of gravity could not be determined satisfactorily, the distances thus obtained were divided by 1.24 and the result used to determine the velocities given in Table 19. The velocities given by Table 9 are averages of those given in Table 19.

Water-surface profiles at the midpoint of each 10-ft electrode station were measured by point gage at points 1 ft apart. Readings on the floor were made at the same points by the same gage after completion of the tests. The differences were taken as the depths. The water surface was considered to be at the base of the loosely flying spray. Because of the difficulty of visual observation, the vibration of the gage was relied on in making the measurements. There was an easily recognized feel to the gage when it had penetrated this spray and encountered the surface beneath. This surface included many small waves and rollers and the values of depth used were approximate means between crests and valleys. Areas given in Table 9 are averages of the areas in the reach.

Discharges were measured simultaneously with the tests by current meter in the canal above the wasteway. For a few tests, some water passed the inlet for irrigation purposes, and this also was measured by current meter.

The memoranda from which the experimental data were taken contained a mathematical analysis by V. L. Streeter, Assoc. M. Am. Soc. C. E., to which Mr. Hall's is so remarkably similar that it appears one more example may be added to the classic list of cases in which independent investigations arrived at the same result. Mr. Streeter's analysis ended with Eq. 17, and to this point Mr. Hall's equations and illustrations are practically identical with his except that Eqs. 8, 9*a*, and 9*b* have been added and changes made in the two following respects.

(1) Mr. Streeter's form of Eq. 5 was

$$f = K V^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(72)}$$

from which it follows that the normal Chézy formula must be modified if observed quantities are used. Mr. Hall puts the term ρ in the right-hand member of this equation and "proves" exactly the opposite. Evidently Eq. 5 must express a physical fact rather than a mathematical assumption. Eq. 4 expresses the simple fact that, for uniform motion, the force producing and that opposing motion are equal. The former is the component of gravity acting parallel to the slope; the latter is the friction along the wetted perimeter. When an increase in depth is produced by adding air, the force producing motion remains substantially the same, and the area over which the opposing force acts increases. Unless, therefore, the unit force opposing motion is decreased faster by the addition of air than the perimeter is increased, velocities would be reduced.

It has been shown[45] that the mean velocity in an open channel is independent of, or inversely proportional to, the kinematic viscosity. The kinematic viscosity of air, at 60° F, is about thirteen times that of water; hence, there is little reason to expect that velocities would be increased by addition of air. One would expect, rather, a reduced velocity, and, therefore, a higher value of *n*. Mr. Hall seems to recognize this, but, having concluded that the normal coefficient should be used for the aerated section, then proposes to use a smaller value of *n* for water alone, thus assuming that resistance at high velocities is less than at low ones—a strange conclusion.

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[45] "Laws of Turbulent Flow in Open Channels," by Garbis H. Keulegan, *Research Paper RP*1151, National Bureau of Standards.

(2) Mr. Streeter introduced an additional term, $\frac{K V^2}{R^p}$, in the right-hand member of Eq. 14 to allow for air resistance at the contact "surface" of air and water. This is referred to in the text following Eq. 21. That such a resistance exists should be obvious. There is nothing "fictitious" about it; nor is it "* * * based on the area occupied presumably by the water and not by the air." The extent to which empirical formulas allow for this resistance and what further allowance, if any, is required, are other matters. Values of 3.64×10^{-5} for K and 1.46 for p were deduced. However, present data on velocities afford little basis for evaluation of such a correction.

Mr. Hall refers to Eq. 17 as the "* * * basic equation for flow in steep rectangular chutes * * *." Being simply a mathematical combination of the Manning formula and Bernoulli's equation, it will apply to any rectangular channel, steep or otherwise. Aside from its possible mathematical elegance, it seems to possess no advantage over the familiar form

$$l = \frac{\Delta E}{\Delta S} \qquad (73)$$

in which ΔE is the change in energy (normal depth times $\cos \theta$ plus velocity head) and ΔS is the difference between the average friction slope over the length l, determined from the Manning, or other, formula and the bottom slope ($\sin \theta$). For cases to which Eq. 17 applies, Eq. 73 may be solved directly for l by assuming values of the depth. Otherwise, it is Eq. 27 in simple form and is easily solved by trial and error.

At the risk of inviting the obvious rejoinder, the writer confesses to some bewilderment at the multiplicity of formulas and of values of *n*; and as to just how the formulas are to be used. For design purposes, there are no observed values of n, v, y, or R. In his "Conclusions," Mr. Hall recommends use of the "computed area, assuming no air entrainment," which "necessitates the use of a value of *n* smaller than that normally used, and one that depends for its value on the velocity of the water, as indicated by Eq. 26." In Eq. 26 the value of n_e depends on V and on K. How V is to be determined is not clear. Values of K vary from 0.00535 to 0.0104, furnishing quite a range of choice. For methods of determining values of n_e when K_1 is not zero or when the channel is not rectangular with parallel sides, the reader seems to be left to his own devices, or to the use of Eq. 8, which requires that the observed hydraulic radius be known.

There is a velocity below which self-aeration does not take place; from Table 20, its value seems to lie between 10 and 40 ft per sec. In Eq. 24*a*, the constant K_1 should have a value corresponding to this velocity. The plotted points on Mr. Hall's charts do not warrant making the value of K_1 equal to zero. All formulas proposed indicate that, for steady flow, a stable mixture would exist. If this is true, entrance conditions would exercise only a local effect, which could not be included in a general formula.

The experience of the Bureau does not indicate that such confidence could be placed in Mr. Hall's velocity measurements. Due to entrained air the surface of water at high velocities appears very white. The spray above the surface makes it hard to detect the color or the boundaries of the colored mass. The estimation by eye of the position of the center of the colored mass is very uncertain. The reaction time of an individual introduces a time lag in the use of the gun and stop watch, which, for short distances, could result in an appreciable error. For greater distances, this error might be negligible, but the color cloud becomes indistinct and lengthens, making the estimation of the position of its center harder. Because of surface conditions the body of the fluid cannot be seen and the only visible color naturally would be on the surface. Even if errors of measurement could be avoided, the relation between surface and mean velocities is unknown. In short distances the sudden changes in observed velocities, without corresponding depth changes, indicate inaccurate measurements. For example, in Table 3 the decrease in velocity head between Stations 3+75 and 4+00 for a discharge of 395 cu ft per sec is 22 ft, with a decrease in depth of 0.02 ft.

The depth measurement by means of staff gages on the sides of the channel gives the depth more or less to the top of the spray as the gage cannot be seen much below this surface. Just below this surface the percentage of air is much higher than at greater depths. At Kittitas the depth at each side of the chute for a distance of about 1 ft from the walls was appreciably greater than in the center, the difference amounting to as much as 0.8 ft on a practically symmetrical section. The variation of the side depth from the mean amounted in some cases to more than 10%. The same effect has been reported by others.[46] Therefore, the Kittitas data are probably not on the same basis as those of Mr. Hall.

Mr. Hall states in his "Conclusions" that "the retardation factors *n* in the Manning or the Kutter formula, obtained by using observed velocities and areas, agree with those obtained in normal channel flow in similar materials. This relation is demonstrated to be correct both in theory and from actual observations." The theory already has been questioned. Neither do "actual observations" fully support this statement. Data found in literature on the subject are given in Table 20. Inconsistencies therein are evident, particularly in the case of the Ruetz wasteway (Table 20(*b*)). On the assumption that the velocities given represent normal, or terminal, velocities, the writer has computed values of *n* in the Manning formula for the given values of discharge and slope without regard to air entrainment, with the results shown in Table 20. These are comparable to the values of n_c in Table 12.

Many of the values are lower than those which normally might be expected. This could result from too high a velocity or, to a less extent, from too small a

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[46] "Flow of Water in Steep Chutes with Special Reference to Self-Aeration," by R. Ehrenberger, *Osterreichischen Ingenieur und Architecktenvereines*, Nos. 15/16 and 17/18, 1926.

TABLE 20.—Compilation of Earlier Chute Measurements

No.	Description	Width (ft)	Sin θ	n[a]	Q (cu ft per sec)	Depth d (ft)	Hy-draulic radius (ft)	V_1 (ft per sec)	$\frac{Q}{A\,V}$ (%)
	(a) Ehrenberger's Model (Wood) [b]								
1	Chute I	0.82	0.153	0.0071	0.353	0.057	0.050	9.61	79.0
				0.0076	0.706	0.090	0.074	11.94	79.0
				0.0081	1.09	0.123	0.094	13.29	80.7
				0.0086	1.57	0.161	0.116	14.53	80.1
2	Chute II	0.82	0.202	0.0074	0.353	0.054	0.048	10.24	77.0
				0.0080	0.706	0.087	0.072	13.06	75.0
				0.0077	1.09	0.120	0.093	14.70	75.0
				0.0085	1.57	0.156	0.113	16.04	75.8
3	Chute III	0.82	0.305	0.0076	0.353	0.051	0.045	11.48	72.9
				0.0079	0.706	0.082	0.068	14.47	71.6
				0.0084	1.09	0.115	0.090	16.31	70.4
				0.0091	1.57	0.151	0.111	17.55	70.7
4	Chute IV	0.82	0.444	0.0073	0.353	0.048	0.043	13.22	67.5
				0.0075	0.706	0.079	0.066	16.80	64.4
				0.0081	1.09	0.112	0.088	18.73	63.1
				0.0089	1.57	0.148	0.109	20.18	63.5
5	Chute V	0.82	0.606	0.0064	0.353	0.052	0.047	15.78	51.3
				0.0069	0.706	0.085	0.071	19.46	51.2
				0.0074	1.09	0.118	0.092	21.85	51.0
				0.0081	1.57	0.156	0.113	23.62	51.3
	(b) Ruetz Wasteway (Wood)								
6	Trapezoidal[b,c]	8.2[d]	0.606	0.0098	17.7	0.10	0.097	23.6	64
				0.0080	28.3	0.16	0.157	32.2	44
				0.0078	120.0	0.49	0.446	56.1	39
7	Trapezoidal[b]	8.2[d]	0.606	0.0067	27.9	...	0.210	35.5	41
				0.0064	36.0	...	0.272	40.8	38
				0.0063	52.3	...	0.361	47.2	37
				0.0078	129.2	...	0.512	57.0	40
8	Trapezoidal[b]	8.2[d]	0.606	0.0062	32.8	0.26	0.249	39.6	37
				0.0064	56.6	0.38	0.348	48.2	37
				0.0069	62.5	0.39	0.364	47.9	37
				0.0066	70.3	0.45	0.407	51.2	38
				0.0076	151.9	0.74	0.640	62.3	43
9	Trapezoidal[e]	....	....	0.0026	36	0.31	0.29	68.6	20
				0.0036	52	0.36	0.33	65.9	25
				0.0054	56	0.46	0.42	53.5	27
				0.0047	98	0.45	0.41	70.2	39
				0.0058	132	0.84	0.72	70.2	25
				0.0066	159	0.94	0.79	70.2	27
				0.0072	191	...	....	70.2	64
	(c) Dago[e]								
10	....	3.28	0.246	0.0113	49.1	0.64	0.46	33.1	64
				0.0102	26.1	0.43	0.34	28.2	66
	(d) Benkok[e]								
11	....	3.28	0.602	0.0101	211.9	2.20	0.94	77.1	38
				0.0087	102.4	1.21	0.70	67.2	38
	(e) Mallnitz[e] (Trapezoidal)								
12	Trapezoidal	6.56	0.408	0.0066	166.0	1.23	0.92	68.6	....
				0.0067	134.2	1.15	0.88	63.0	....
	(f) Fargo Drop[f] (Wood)								
13	....	6.0	0.124	0.0091	95.0	0.57	0.48	32.3	86

TABLE 20—*Continued*

No.	Description	Width (ft)	Sin θ	n[a]	Q (cu ft per sec)	Depth d (ft)	Hydraulic radius (ft)	V_1 (ft per sec)	$\frac{Q}{A\,V}$ (%)
	(*g*) MORA WASTEWAY[f] (CONCRETE)								
14	....	5.0	0.081	0.0072	27.5	0.32	0.28	22.0	78
	(*h*) VALLEY MOUND CHUTE[f] (CONCRETE)								
15	....	5.0	0.156	0.0066	22.4	0.27	0.22	26.3	63
	(*i*) ARENA CHUTE[f] (CONCRETE)								
16	Section 1	6.0	0.202	0.0104	23.26	0.24	0.22	20.4	79
17	Section 2	6.0	0.151	0.0096	23.26	0.22	0.21	19.6	90
18	....	6.0	0.206	0.0094	50.40	0.32	0.29	29.4	89
	(*j*) LIZARD CHUTE[f] (CONCRETE)								
19	Chute No. 1	3.0	0.082	....	0.44	0.04	0.04	4.97	73
				....	2.04	0.11	0.10	9.33	66
				....	2.80	0.12	0.11	10.9	71
				0.0073	5.72	0.17	0.16	14.3	72
				0.0091	7.97	0.24	0.21	14.2	78
				0.0085	16.42	0.36	0.29	19.2	79
20	Chute No. 2	3.0	0.194	....	0.44	0.05	0.05	6.9	42
				....	2.04	0.10	0.09	7.8	87
				....	2.80	0.10	0.10	11.1	85
				0.0067	5.72	0.16	0.14	19.8	60
				0.0115	7.97	0.21	0.18	16.0	80
				0.0095	16.42	0.31	0.26	23.8	78
	(*k*) KITTITAS WASTEWAY[f] (CONCRETE; MEASURED DOWNSTREAM FROM CHUTE)								
21	....	8.0	0.547	0.0130	231	0.80	0.67	52.2	69
				0.0137	484	1.3	0.98	66.0	70.5
	(*l*) ALBERTA, CANADA[g] (CANADIAN PACIFIC RAILWAY)								
22	Hammerhill Flume (Metal)	5.10	0.057	0.0082	61.0	....	0.573	27.5	85
				0.0102	26.5	....	0.433	18.6	88
23	Dalroy Flume	10.2	0.032	0.0125	59.2	....	0.993	15.2	54
				0.0155	151.4	....	1.311	17.6	76
24	Lateral C-11 Flume	0.9	0.052	....	1.24	....	1.0107	9.82	99
25	Secondary Canal	4.18	0.025	....	6.16	....	0.155	9.25	95
	(*m*) SOUTH CANAL CHUTE,[f] MILEPOST 2, UNCOMPAHGRE PROJECT (CONCRETE; TRAPEZOIDAL)								
26	Section 1; sides ½ : 1	8.84[d]	0.275	0.0175	382	1.15	0.98	40.24	87
				0.0211	465	1.30	1.07	38.17	99
27	Section 2; sides ¼ : 1	8.47[d]	0.070	0.0129	267	1.06	0.87	28.13	103
				0.0131	400	1.39	1.09	31.93	102
				0.0136	463	1.53	1.17	32.58	104

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[a] Value required to give the observed velocity for a given slope and discharge. [b] See footnote 46. [c] Side slopes 1.5 on 1; adjusted values of d and V as given by Ehrenberger. [d] Bottom width of trapezoid. [e] "Kolkabwehr und Stauraumverlandung," by A. Schoklitsch, Julius Springer, 1935. [f] Bureau of Reclamation experiments. [g] "The Flow of Water in Flumes," by Fred C. Scobey, *Bulletin No. 393*, U.S.D.A., 1933.

discharge. In some cases, the method of measuring velocities is unknown. In others it was measured by means of floats. A surface float traveling in the center of the chute, where it naturally would be drawn, very likely might give a velocity higher than the mean. In the writer's opinion, the most dependable measurements of velocities are those on Kittitas wasteway, given in the paper, and those on Ehrenberger's model, given in Table 20(*a*). In both of these cases and in several others the values of *n* are in the normal range. In other cases, in view of the very fair agreement of the Manning formula with the rational formula derived by G. H. Keulegan,[45] there is more reason to suspect the velocity or the discharge measurements than to suppose that higher velocities would result in such radical changes in usual values of *n*, especially when the change is a decrease. As is well known, water at shallow depth on steep slopes tends to travel in waves which overrun each other. Such waves normally would exist in many of the experiments given in Table 20, and it well may be doubted whether a satisfactory determination of mean velocity could be made under such conditions. The data in Table 20 show a tendency toward smaller values of *n* for shallow depths.

Obtaining the velocity, unfortunately, is only a part of the problem, as the depth in some cases is of equal or greater importance. In self-aerated flow the air must enter at the contact between air and water. As a result of the air resistance, this contact is not a definite surface, but a zone in which the percentage of air in the mixture changes from practically 100% to an unknown percentage. Aeration of water not in intimate contact with the air may be caused by air bubbles carried down by turbulence, the air having to pass through this zone into which the turbulence may not extend. Such bubbles must be carried down against their natural buoyancy, which suggests study along the lines of sedimentation studies. It would seem plausible to suppose that the percentage of air might decrease with depth on account of the tendency of the bubbles to rise or of their decrease in size due to increase in pressure. It might be assumed, therefore, that only a limited depth would be affected in deep water; also, that the depth so affected would be a function of time. Such tendencies were noted by Ehrenberger.[46]

It would appear that a formula for aeration should allow for channel roughness. For example, an unlined and a lined channel could have the same Froude number, but whether the air entrainment would be the same remains a matter of speculation. It has been shown that K_1 in Eq. 24*a* should not be zero. If it is not, complications arise; one would conclude from Eq. 24*a*, using constants for the upper Kittitas chute, that no air would be entrained by a velocity of 57 ft per sec if R is 2.5 ft. The case is not unusual; the conclusion doubtful. The not-too-good fit of the plotted data with Mr. Hall's formulas may be due as much to the unfortunate fact that all experiments were made on closely similar channels as to any validity of his use of the Froude number.

It has been suggested that the increase in depth is a function of some power of the velocity, and there seem to be some hypothetical grounds for such an assumption.

The term "depth" should be defined. An accurate method of measuring velocities is badly needed, and, until one is developed, little progress is likely.

A study of the mechanical process by which the air is entrained should be helpful. Possibilities of laboratory research, and perhaps of theoretical investigation, may not have been exhausted. Until reliable information to the contrary can be had, velocities as determined by the usual methods with the ordinary values of n can be regarded as being reasonably satisfactory; but formulas for determination of the quantity of entrained air should be considered as empirical and used with caution. With the data at hand, it should be possible to prepare conservative designs for small structures of the type and size of those used for the experiments. Whether equally satisfactory results can be expected for larger structures, only further information will tell.

T. J. CORWIN, JR.,[47] M. AM. SOC. C. E.—Comparatively little information on water flowing at high velocities in chutes is available to the designing hydraulic engineer. For this reason Mr. Hall's paper is a very valuable contribution.

It is doubtful if the bulking that results from the entrainment of air was anticipated when designs were prepared for the Rapid Flume, South Canal chute, or Hat Creek No. 2 chute twenty-five to forty years ago. Design computations for the Rapid Flume could not be located, but a velocity of 32 ft per sec was used in designing a baffle box which was reconstructed at the end of the chute in 1940. This value substantially checks the observed velocity of 33.3 ft per sec.

For the South Canal chute the computed and observed velocities can be compared in Table 21. In the Hat Creek No. 2 chute (Table 21) the observed velocities are higher than the computed values. The chute was designed for a capacity of 900 cu ft per sec with $C = 100$ or $n =$ about 0.015. No allowance was made for air entrainment. The freeboard allowance was rather limited for such high velocities. The test indicates that, because of the bulking effect of the air, the capacity of the chute is roughly only 50% of the design capacity. The original wooden flume has been replaced with a concrete

TABLE 21.—COMPARISON OF COMPUTED[a] AND OBSERVED VELOCITIES (FEET PER SECOND)

(a) SOUTH CANAL CHUTE								(b) HAT CREEK NO. 2 CHUTE			
$Q = 400$ vs. $Q = 363$				$Q = 400$ vs. $Q = 334$				$Q = 900$ vs. $Q = 395$			
Station	Computed	Observed	Water (%)	Station	Computed	Observed	Water (%)	Station	Computed	Observed	Water (%)
0+50	26.0	28.2	76.1	17+00	14.5	23.2	91.2	0+00	25.0	28.5	61.5
8+00	23.5	27.6	78.3	23+00	14.5	16.6	92.8	1+25	57.5	60.5	65.8
14+00	19.0	26.4	74.9	25+00	21.2	21.7	100.0	2+25	66.3	73.9	53.8
....	....	....	....	26+25	30.7	33.5	70.6	2+75	67.5	75.9	51.0
....	....	....	....	32+00	20.5	25.5	85.6	3+75	74.0	72.3	50.0

[a] In the computed values ($Q = 400$ and 900 cu ft per sec) $n = 0.015$ was used.

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[47] Asst. Engr., Pacific Gas & Elec. Co., San Francisco, Calif.

flume of 600 cu ft per sec capacity. The chute may carry this flow without serious splashing at Station 2+75 (see Table 21).

With regard to "Theory of Flow in Chutes," the three premises stated should be in accord with the desires of all designing engineers. Requirement No. 3 is especially true since the first and most important part of the design is to carry the full flow regardless of any number of reasons as to why the capacity is inadequate.

From actual experience with model tests, one can concur readily with the statement inferred in the "Synopsis" that it is only possible to observe the phenomenon on the prototype.

In designing conduits with horizontal curves and relatively high velocities, ample freeboard should be provided because the actual rise in the water surface on the outside of curves has been observed at about twice the theoretical value. The paper brings out the effect of cavitation on vertical curves when the actual velocity is in excess of the computed velocity and the stream leaves the bottom of the chute. Vertical curves should be designed for velocities computed on the basis of absolute minimum friction losses.

The most difficult problem in connection with the chutes is to develop a suitable transition so the water can enter without entrainment of air or surface waves. The ground topography at spillways, which are placed along the side of bench flumes, makes it necessary for the designer to accept the turbulent condition at the entrance and to provide ample freeboard.

It is fortunate that the wooden Rapid Flume was available for test so that a comparison could be obtained with concrete chutes. Because of the smooth condition of the surfaced timber, the percentage of entrained air at equal velocities was found to be less than for concrete surfaces.

Mr. Hall has been able to draw the conclusion from these tests that the air bulking was reduced with larger cross sections and greater volumes of discharge. This is particularly fortunate; otherwise, many spill chutes at dams might be inadequate for full flood discharge.

In the future one may design chutes with more confidence, knowing what allowance to make for bulking due to air and also that the normal values of *n* are independent of magnitude of the velocities.

L. Standish Hall,[48] M. Am. Soc. C. E.—The discussions of this Symposium have developed many interesting and valuable observations. The writer feels indebted to all those who participated in the discussion of his paper by contributing their time and constructive thoughts to the subject of air entrainment.

The writer hesitated at first to present the data collected on these chutes due to the relatively small range of conditions covered by the tests. However, it was felt that the theoretical approach as presented was sound and that the discussions would bring forth ideas of value. In this respect, the writer has been gratified with the results and trusts that the ideas presented will prove of value to future investigators of air bulking in open channel flow at high velocities.

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[48] Hydr. Engr., East Bay Municipal Utility Dist., Oakland, Calif.

Mr. Corwin presents an interesting comparison of designed and observed velocities on the chutes tested by the writer. At Hat Creek chute, the velocities for the design discharge of 900 cu ft per sec would be appreciably greater than those observed for a flow of 395 cu ft per sec. For example, at Station 2+75 the observed velocity for the larger flow would probably be between 80 and 85 ft per sec and the percentage of water in the mixture would be correspondingly decreased.

From actual observation, the writer has found no cases in which the designed size of a channel was inadequate solely as a result of air bulking, since the generous freeboard usually allowed compensates for deficiencies between design and actual performance. As Mr. Corwin states, the failure of chutes to carry the desired discharge is principally the indirect result of air bulking due to computed velocities falling short of actuality. Excess velocities lead to overtopping of side-walls at: (1) Sharp horizontal curves, (2) sharp vertical curves, and (3) jump basins at chute terminals. In all these cases, the calculation of velocities with provision for air entrainment will yield better conformity between design and results.

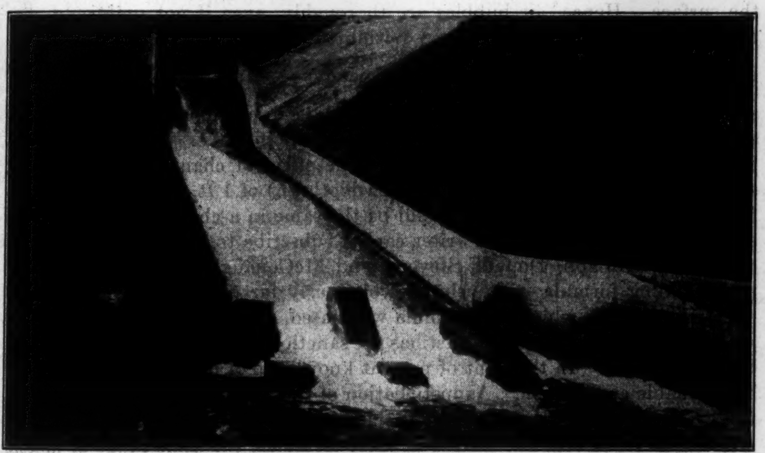

FIG. 32.—LOWER END OF SPILLWAY AT GORDON VALLEY DAM SHOWING A TYPE OF SLUG FLOW AND NONUNIFORM DISTRIBUTION OF WATER

Unsatisfactory entrance conditions may result in excess air entrainment and also in cross waves. Short-radius horizontal curves or an abrupt narrowing of the channel width by too short a transition section at the inlet below the point developing shooting velocities will produce cross waves that result in great variance with the design, and these cross waves may persist throughout the entire length of the chute (see Fig. 32).

Messrs. DeLapp, Douma, and Stevens criticize the use of the parameter $\frac{V^2}{g\,R_e}$, on various grounds. However, its use has certain advantages since, in the design of a new chute involving nonuniform flow, the ratio of air in the water is not known in advance, and the initial use of the computed functions simplifies the calculations.

The Froude number $\frac{V}{\sqrt{g\,D}}$ is a specific case of the general form $\frac{V}{\sqrt{g\,l}}$. In experiments on ship models performed by Froude, a dimension of the model was used, and in the general form l indicates any linear dimension.[49] The form cited by Messrs. DeLapp and Douma is always used in connection with the hydraulic jump.[50] It cannot be questioned that the form $\frac{V^2}{g\,R_c}$ satisfies the requirements of being a dimensionless parameter of the Froude type representing the action of fluid weight.

When the mechanism of air entrainment is studied, the predominant factor not embodied in the Froude number is found to be channel roughness. Insufflation normally starts along the side-walls, although in wide channels air is drawn in across the entire section after a specific velocity is attained. The shearing force along the perimeter produces a torque in the mixture as it slides down the surface. Hence, air bubbles are thoroughly, even if not uniformly, distributed throughout the mixture. Within the range of data presented in the paper, the difference between the parameters $\frac{V^2}{g\,R_c}$ and $\frac{V^2}{g\,y_c \cos\theta}$ is relatively slight. Hence, in the extrapolation of the data to channels of greater dimensions the analysis must be subjective rather than objective. Those advocating the depth as a parameter would ignore the effect of channel width. To be specific, they contend that with a water depth (y_c) of 1 ft and a velocity of 50 ft per sec, the air entrainment would be the same in a channel 1 ft wide as in a channel 100 ft wide. The writer cannot subscribe to this line of reasoning.

Messrs. DeLapp, Douma, Stevens, and McConaughy question the use of the Manning formula with velocities obviously far outside the range of the experiments upon which this formula was based. The use of either the Manning or the Kutter formula at least has the sanction of current practice as stated by Mr. Stevens. In the light of present knowledge of high velocity flow, it is a reasonable assumption. As justification of the use of the Manning formula, the writer concurs with Mr. Stevens that "if the experimental data on chutes are used to determine the factors in these formulas, the same range of factors will render these formulas suitable for design."

Mr. Douma proposes a variation in the Manning formula. Obviously, his Eq. 58 would be more difficult to solve than the method proposed by the writer. Also, in formulating Eq. 60, he has used the values of n from the Hat Creek chute below Station 2+75 where a great slug of air was added to the mixture due to the fact that the trajectory did not follow the profile of the chute. Also, from other tests, he has used values of air entrainment which resulted from unusual disturbances outside of those resulting from channel roughness and water velocity. Obviously, such data cannot be used in determining a general formula.

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[49] "Hydraulic Laboratory Practice," edited by John R. Freeman, A.S.M.E., 1929, pp. 785 and 789; also "Hydraulic Models," *Manual of Engineering Practice No. 25*, Am. Soc. C. E., 1942, p. 14, Eq. 40.

[50] "Fluid Mechanics for Hydraulic Engineers," by Hunter Rouse, McGraw-Hill Book Co., Inc., New York, N. Y., 1938, p. 388.

Mr. DeLapp proposes that a constant K in Eq. 39 (hereafter designated K_L to distinguish it from K in Eq. 24a) can be used to determine the depth of the mixture of air and water after uniform flow conditions are attained. From its derivation, it is obvious that this constant varies with the velocity, and, hence, is influenced by those factors that affect the velocity, namely the slope, the hydraulic radius, and the channel roughness. This relation introduces the use of a retardation factor, and although the validity of either the Manning or Kutter formula for high velocity flows is an assumption, the data approximate the Manning formula as shown by Mr. DeLapp.

The dependence of K_L on the retardation factor can be seen by combining the continuity relation in Eq. 3 with Eq. 41. Then

$$(K_L)^2 = \frac{g\,q_w}{\rho^3\,V^3} \qquad (74a)$$

For uniform flow conditions, the value of V from Eq. 6 can be substituted to determine the value of $(K_L)^{2/3}$ of Eq. 41 as follows:

$$(K_L)^{2/3} = \frac{(g\,q_w)^{1/3}}{\rho\,C\,(R_o\,S_1)^{1/2}} \qquad (74b)$$

Eq. 74b indicates that, for a given discharge and slope, the value of $(K_L)^{2/3}$ varies inversely with the Chézy C and inversely with the observed hydraulic radius. In other words, Mr. DeLapp's constant combines those factors which he criticizes when that constant is segregated into its component parts.

Eq. 41 would apply only to a rectangular chute and indicates that the depth y_o is a proportion of the critical depth. In the design of many chutes this relation will prove of value. In determining the values of $(K_L)^{2/3}$ in Eq. 41 the results from the South Canal were not used by Mr. DeLapp, although there were several stretches in which uniform flow conditions were reached. These data (see Table 22) show a much greater variation in the value of $(K_L)^{2/3}$ than was found in the other chutes. The velocities in the South Canal chute were less than those observed at the other structures. Values of $(K_L)^{2/3}$ appear to increase generally as the velocity decreases, which indicates that the relation proposed by Mr. DeLapp is not as consistent as that presented in Fig. 10. Hence, Eq. 41 should be used with caution.

TABLE 22.—VALUES OF $(K_L)^{2/3}$; SOUTH CANAL CHUTE

Stations	y_c	y_o	V_o	$(K_L)^{2/3}$
2+00 to 5+50	1.16	1.48	32.6	0.420
8+00 to 11+50	1.37	1.67	27.4	0.475
17+00 to 19+50	1.80	2.13	19.0	0.642
21+00 to 23+00	2.02	2.33	16.9	0.695
25+50 to 26+50	1.42	2.02	35.0	0.478
28+00 to 30+50	1.88	2.19	26.3	0.520

In the opinion of Messrs. Douma and McConaughy, the curves in Fig. 10 should not pass through the origin of the coordinates. The writer has realized this to be true for some, although not for all, of the tests, but due to the many uncertainties of design with high velocity flow, feels that no substantial error is introduced and computations are simplified by assuming that the value of

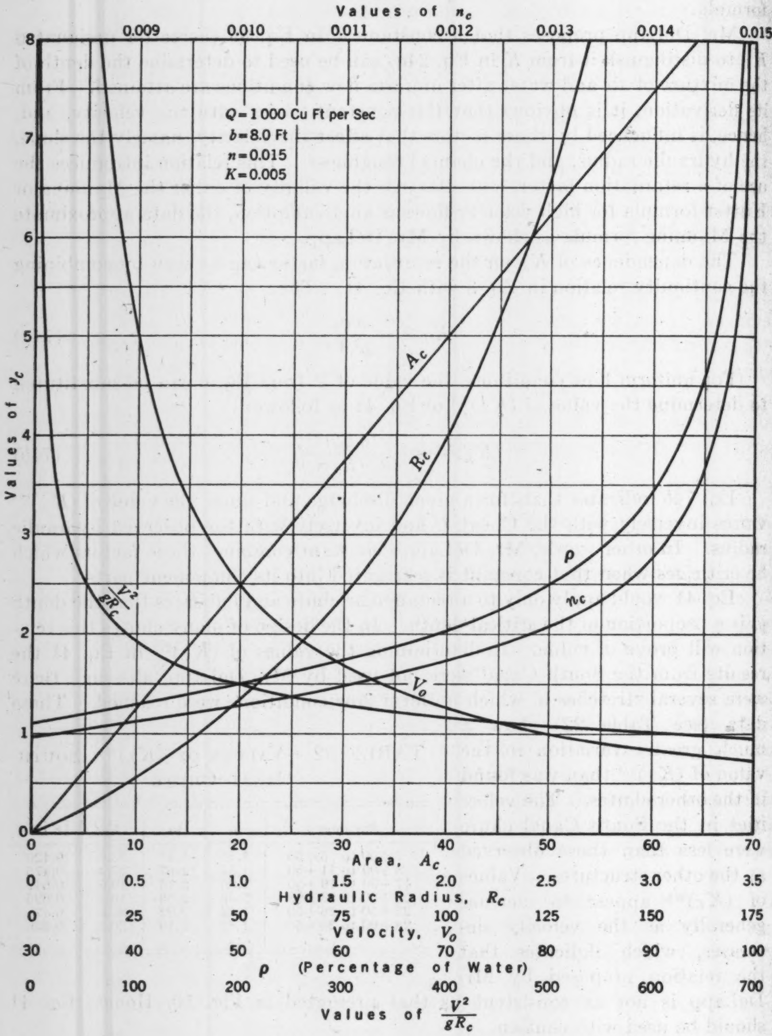

FIG. 33.—CHARACTERISTICS OF FLUME WITH 8-FT RECTANGULAR CROSS SECTION

K_1 is zero in all cases. Although this does result in the computation of air entrainment for low velocities as pointed out by Mr. Douma, the percentage of air as computed is so small that it is practically negligible (see sample computations in Table 23, Col. 11, and Fig. 33). Furthermore, increasing the apparent depth of the water slightly errs on the side of safety. On the other hand, entrance conditions frequently increase the insufflation in a greater degree than indicated by the formula for the relatively low velocities in this zone at the head of a chute.

The relation between the percentage of entrained air and the Froude number shown in Fig. 23 is similar to one of several developed by the writer in his initial studies. However, both of these were discarded in favor of the relationship shown in Fig. 10; but the enveloping curves in Fig. 23 showing the limits of a 10% error in the calculations are very interesting. If such a comparison were applied to Fig. 10, it is believed that in most cases the percentage of error would be substantially less than 10%.

With reference to the velocity measurements on Kittitas chute, Mr. Douma notes that those used by the writer are smaller than the ones originally determined by the Bureau of Reclamation. The observed velocities given in Table 9 are the same as those presented by Mr. McConaughy in Table 19 and are based on the time distances between the centers of gravity of the curves produced on the oscillograms. The velocities first computed, which were 24% larger, were based on time distances between the beginning points and other characteristic points on the curves. It would appear that the second determination was more reliable as indicating the mean velocity. However, Mr. Douma notes that, in some instances, particularly in the 10° 12′ slope, the velocities determined from the center of gravity of the curves appear too small. This may be for the reason cited in the Bureau of Reclamation's first progress report:[51]

> "It may be noted in the table that the observed velocities varied as much as 25% when measured over the same reach with an apparent constant discharge. The average variation is approximately 7%. There are several factors which contribute to this difference in observed velocity. Visual observations and a study of the motion pictures show definite pulsations of velocity in the flow. The apparatus used was evidently sensitive to these pulsations and one record was made of the high velocity mass while another was made at some lower velocity."

Mr. Stevens also refers to the occurrence of pulsating flow. In the foregoing citation and that by Mr. Stevens, the writer believes that reference is made to "slug" flow, which is really surges of small amplitude.[52] This type of flow also is shown in Fig. 32 and apparently occurred in the Kittitas chute, particularly at lower flows. However, it did not occur in any of the chutes tested by the writer. These surges are of a different nature than pulsations which normally occur in turbulent flow, and are noted by many observers[53] in the fluctuations in a pressure gage in a flowing closed conduit or in variations in the rate of revolution in a current meter during the making of stream-flow measurements. The pulsations referred to by Mr. Stevens are possibly of this latter type in an exaggerated form, or a transitional stage between normal pulsating flow and "slug" flow.

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[51] Unpublished report of U. S. Bureau of Reclamation, p. 16.

[52] "Applied Fluid Mechanics," by Morrough P. O'Brien and George H. Hickox, McGraw-Hill Book Co., Inc., New York, N. Y., 1937, p. 38.

[53] "The Mechanics of Turbulent Flow," by Boris A. Bakhmeteff, Princeton Univ. Press, 1936, pp. 11–13.

The writer greatly appreciates Professor Knapp's discussion on the concepts of the flow regimen on horizontal curves. The masterly contribution to this subject by Mr. Ippen and Professor Knapp[54] were not available at the time the paper was prepared. Hence, theory of flow on curves had been confined to the superelevation of horizontal curves in order to eliminate cross waves.

TABLE 23.—CALCULATION OF FLOW

Explanation: $n = 0.015$; $\sin\theta = 0.547$; $\cos\theta = 0.837$; $Q = 1{,}000$ cu ft per sec; $b = 8.0$ ft; $q = 125$ cu ft per entries; these calculations are for a chute of the same dimensions, bottom slope, and discharge as used in Table 17.

Station (1)	Depth y_e (2)	$y_e \times \cos\theta$ (3)	Area A_e (4)	Hydraulic radius R_e (5)	Velocity V (6)	$1.05\,H_v = \frac{1.05\,V^2}{2\,g}$ (7)	Energy head $y_e \times \cos\theta + 1.05\,H_v$ (8)	Kineticity k (9)	Percentage of water ρ (10)	Percentage of air $1-\rho$ (11)
0+00	7.99	6.69	63.9	2.66	15.65	3.99	10.68	1.60	98.6	1.4
0+00.2	7.28	6.10	58.3	2.58	17.16	4.80	10.90	1.79	98.3	1.7
0+00.9	6.63	5.55	53.0	2.49	18.88	5.81	11.36	2.05	97.8	2.2
0+02.0	6.01	5.03	48.1	2.40	20.77	7.03	12.06	2.40	97.4	2.6
0+03.7	5.47	4.58	43.8	2.31	22.85	8.51	13.09	2.86	96.7	3.3
0+06.1	4.97	4.17	39.8	2.22	25.13	10.30	14.47	3.47	95.8	4.2
0+09.2	4.53	3.79	36.2	2.12	27.64	12.46	16.25	4.28	95.0	5.0
0+13.3	4.11	3.69	32.9	2.03	30.40	15.07	18.76	5.08	93.5	6.5
0+18.5	3.75	3.14	30.0	1.94	33.44	18.23	21.37	6.80	92.0	8.0
0+25.1	3.40	2.84	27.2	1.84	36.78	22.04	24.88	8.77	90.0	10.0
0+33.5	3.09	2.59	24.7	1.75	40.46	26.69	29.28	11.30	87.5	12.5
0+44.3	2.81	2.35	22.5	1.65	44.51	32.30	34.65	14.74	84.3	15.7
0+57.9	2.55	2.13	20.4	1.57	48.96	39.08	41.21	19.35	80.8	19.2
0+75.7	2.32	1.94	18.6	1.47	53.86	47.30	49.24	25.38	76.3	23.7
0+98.7	2.11	1.77	16.9	1.39	59.25	57.23	59.00	33.33	71.7	28.3
1+29.0	1.92	1.61	15.4	1.30	65.17	69.24	70.85	44.01	66.2	33.8
1+70.0	1.74	1.46	14.0	1.22	71.69	83.80	85.26	58.40	61.0	39.0
2+27.8	1.59	1.33	12.7	1.14	78.86	101.41	102.74	77.26	54.8	45.2
3+14.8	1.44	1.20	11.6	1.07	86.75	122.69	123.89	103.24	48.0	52.0
4+63.0	1.31	1.10	10.5	0.99	95.43	148.48	149.58	135.98	41.0	59.0
6+96.2	1.19	1.00	9.5	0.92	104.97	179.67	180.67	180.67	35.5	64.5
12+43.9	1.12	0.94	9.0	0.87	111.70	203.43	204.37	217.41	31.0	69.0

In view of Professor Knapp's studies of wave trains on unbanked curved channels, the explanation originally given in connection with Eq. 32 is perhaps somewhat misleading. This equation is to be applied only when the curve is banked and built with proper transition sections at both beginning and end of curve. Then, for equilibrium conditions the cross slope in the portion of uniform curvature will follow Eq. 32 quite closely. The cross slope in the transition sections should be increased linearly with the distance along the transition. Also, the banking should be obtained by lowering the arc on the inside radius, allowing the arc on the outside radius to remain on grade.

Fig. 34 illustrates the flow on a banked curve at the spillway from Upper San Leandro Dam of the East Bay Municipal Utility District near Oakland, Calif. The banking has eliminated cross waves in the straight section below the curve. This curve has a radius of 603.8 ft and a maximum superelevation of 3.6 ft in a width of 25 ft. The channel capacity is 5,000 cu ft per sec, but only 700 cu ft per sec were discharging when the photograph was taken. The same theory of design was used on the San Pablo Spillway and checked by model tests described in the paper.

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[54] "A Study of High Velocity Flow in Curved Sections of Open Channels," March 29, 1936, and "Experimental Investigations of Flow in Curved Channels," July 6, 1938, both by Arthur T. Ippen and Robert T. Knapp; reports prepared at the California Inst. of Technology for the Los Angeles Flood Control Dist.

On unbanked curves and rectangular cross sections, the conditions are as described by Professor Knapp and noted by the writer. The cross slope at maximum points will exceed the equilibrium value as shown in Eq. 49. On the South Canal chute built with horizontal curves having no superelevation, the observations of the depth of water were not taken exactly at the maximum and minimum points, but at 25-ft stations. However, from Table 11, it is seen that at Station 23+50, the actual slope was 1.8 times, and at Station 25+00 it was 1.6 times, the equilibrium slope. At the maximum points, the actual slope was probably twice the equilibrium slope in conformity with Eq. 49.

Characteristics in a Steep Chute

sec; $\alpha = 1.05$; $K = 0.005$; Col. 15 is from Eq. 17; in Col. 16, dV is taken as 10% of V, except for the last two

$\frac{V^2}{g R_e}$ (12)	n_e (13)	Friction slope S (14)	$\frac{dV}{dl}$ (15)	dV (16)	dl (17)	Observed area A_e (18)	Observed depth y_e (19)	Observed hydraulic radius R_e (20)	Station (1)
2.9	0.015	0.0067	0.146	1.56	0.2	64.8	8.10	2.68	0+00
3.5	0.015	0.0084	0.378	1.72	0.7	59.2	7.40	2.60	0+00.2
4.5	0.015	0.0107	0.598	1.89	1.1	54.2	6.78	2.52	0+00.9
5.6	0.015	0.0136	0.818	2.08	1.7	49.4	6.17	2.43	0+02.0
7.0	0.015	0.0181	1.030	2.28	2.4	45.3	5.66	2.34	0+03.7
8.8	0.0145	0.0207	1.238	2.51	3.1	41.5	5.19	2.26	0+06.1
11.2	0.0145	0.0266	1.468	2.76	4.1	38.2	4.77	2.17	0+09.2
14.1	0.0145	0.034	1.713	3.04	5.2	35.4	4.40	2.09	0+13.3
17.9	0.0145	0.044	1.984	3.34	6.6	32.7	4.08	2.02	0+18.5
22.8	0.0145	0.057	2.291	3.68	8.4	30.2	3.78	1.94	0+25.1
29.1	0.0145	0.074	2.658	4.05	10.8	28.3	3.53	1.87	0+33.5
37.3	0.014	0.090	3.063	4.45	13.6	26.7	3.34	1.82	0+44.3
47.6	0.014	0.118	3.625	4.90	17.8	25.3	3.16	1.77	0+57.9
61.4	0.0135	0.143	4.265	5.39	23.0	24.3	3.04	1.73	0+75.7
78.5	0.013	0.175	5.121	5.92	30.3	23.5	2.94	1.70	0+98.7
101.6	0.0125	0.213	6.296	6.52	41.0	23.2	2.90	1.68	1+29.0
130.8	0.012	0.259	8.059	7.17	57.8	22.8	2.85	1.67	1+70.0
169.4	0.0115	0.315	11.026	7.89	87.0	23.2	2.90	1.68	2+27.8
218.0	0.011	0.382	17.079	8.68	148.2	24.0	3.00	1.72	3+14.8
286.0	0.010	0.420	24.449	9.54	233.2	25.5	3.19	1.78	4+63.0
372.0	0.0095	0.505	81.380	6.73	547.7	26.8	3.35	1.82	6+96.2
445.0	0.009	0.547	Infinity	0	Infinity	28.9	3.61	1.90	12+43.9

Mr. Douma also discusses the height of waves on an unbanked horizontal curve. As a result of experiments, he has found that, to allow for secondary waves, the multiplier in Eq. 49 should be increased from 2.0 to 2.4. This is the extreme maximum, because initial transverse waves at the beginning of a curve may dampen rather than intensify the disturbance in the flow path around the curve. However, in the design of new spillways, the maximum height of waves on the outside walls of unbanked horizontal curves probably should be calculated with the use of the factor 2.4 and additional freeboard allowance above the maximum wave height added to this computed depth.

Unusual flow conditions exist at the spillway from the Gordon Valley Dam of the City of Vallejo, Calif. This spillway is 27 ft wide and has an unbanked horizontal curve of 74.43-ft radius near the upper end of the channel. Immediately below the horizontal curve is a vertical curve. Discharge conditions at a relatively moderate flow of about 100 cu ft per sec are shown in Fig. 32. There are two maxima wave points on the outer wall of the horizontal curve, but the inner floor of the curve is dry. Below the vertical curve the wave angle is changed, due to acceleration of velocity, with the result that the flow does not reach a uniform distribution across the bottom before striking the diffuser blocks at the end of the chute. A mathematical analysis of this flow is practically impossible, although the spillway has functioned satisfactorily for many years.

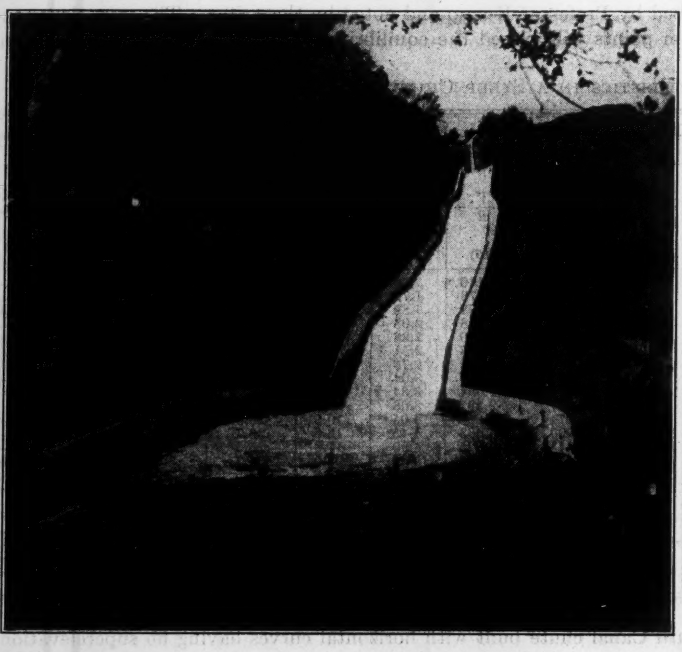

FIG. 34.—SPILLWAY AT UPPER SAN LEANDRO DAM; A FLOW OF 700 CU FT PER SEC ON A SUPERELEVATED HORIZONTAL CURVE OF 603.8-FT RADIUS

Professor Knapp's analysis of the mechanics of air entrainment and his conclusions present many excellent points. In general, there are no conflicts between the theory as originally presented and his conclusions. Some slight exceptions must be recorded.

The analogy between air entrainment and transportation of sediment by Professor Knapp and also by Mr. McConaughy is well taken. Air entrainment is not a static condition, since obviously air is constantly insufflated to replace that escaping. The ratio between air and water expressed in Eq. 24*a*, however, is not necessarily predicated on a uniform vertical distribution of air. The ratio of air to water decreases as the value of R_e increases. A uniform vertical distribution would necessitate smaller bubbles at greater spacing as larger cross sections were approached. Obviously, the most natural distribution in large cross sections would be toward a greater concentration in the upper layers. In the tests made on relatively low depths, the distribution of air, as observed at the end of the chutes, appeared uniform, but this condition would not necessarily apply to larger cross-sectional areas.

With respect to the effect of wind velocity on the phenomenon, this is more theoretical than actual. In all cases of high velocity flow there exists a strong down draft of air parallel with the current. Air drawn into the water naturally continues in the direction of flow after escape. The side-walls of the chute provide protection from cross winds.

The conclusion that air entrainment is not initiated merely by the high velocity of flow is undoubtedly true. With extremely smooth entrance conditions, such as occur on the downstream face of an ogee section overflow dam, the entrainment is delayed. This is illustrated in Fig. 35 by the air entrainment on the face of the Britton Dam on the Pit River System of the Pacific Gas and Electric Company in California. Entrainment occurs immediately at the piers, but, in the intervening space, it does not occur until the water has fallen much farther.

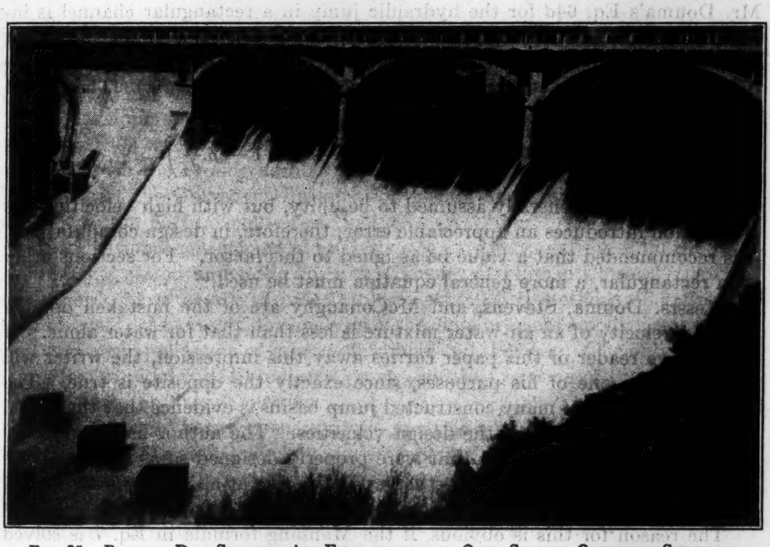

FIG. 35.—BRITTON DAM SHOWING AIR ENTRAINMENT ON OGEE SECTION OVERFLOW SPILLWAY

The hypothesis, presented by both Mr. Stevens and Professor Knapp, that the foam in the upper layers travels at a lower velocity than the lower layers was not verified by observation, although it may occur under certain conditions.

The trajectory of the water surface at the convex vertical curve on Hat Creek chute could only be explained by assuming that the upper layers were traveling at the theoretical velocity generated by the vertical drop between the inlet of the chute and the start of the vertical curve. The ratio of this maximum velocity to the average velocity prompts the conclusion that the layers in contact with the perimeter of the channel were traveling at velocities below the average. The same conclusions have been reached elsewhere.[2] This conclusion is borne out by observations of the retardation of the color cloud along the walls of the chutes. A possible criticism of the color method is that it may yield too great a velocity rather than one too small. Future observations might improve the technique of observing the velocities either by the color or the salt method.

Mr. Douma is mistaken in assuming that Eqs. 30 and 31 require "the coordinate system to be parallel and normal to the slope of the chute upstream from the vertical curve." Eqs. 61*a*, 61*b*, and 61*c* are identical with the writer's except that the origin of the curve has been shifted from the vertex of the parabola to the point of tangency between the upstream slope and the parabola. Mr. Douma's Eq. 64*d* for the hydraulic jump in a rectangular channel is incomplete, since, as a velocity factor, α, was used in Eq. 52, a similar factor, α', for a momentum coefficient should be used.[55] (Also, the first term on the right-hand side of the equation should be $\rho^2 d^2{}_1$ rather than $\rho d^2{}_1$.) The correct equation is:

$$(d_2)^2 = \rho^2 (d_1)^2 + \frac{2\,\alpha'\,\rho\,d_1\,(V_1)^2}{g}\left(1 - \rho\,\frac{d_1}{d_2}\right)$$

The value of α' is generally assumed to be unity, but with high velocities this assumption introduces an appreciable error; therefore, in design computations, it is recommended that a value be assigned to this factor. For sections other than rectangular, a more general equation must be used.[56]

Messrs. Douma, Stevens, and McConaughy are of the mistaken opinion that the velocity of an air-water mixture is less than that for water alone. If the average reader of this paper carries away this impression, the writer will have failed in one of his purposes, since exactly the opposite is true. The inadequate length of many constructed jump basins is evidence that the actual velocities are greater than the design velocities. The author has found very few vertical curves in chutes that were properly designed so as to permit the trajectory of the water to follow the vertical profile at all discharges. Again, this is evidence of insufficient design velocities.

The reason for this is obvious, if the Manning formula in Eq. 7 is solved for V_o,

$$V_o = \frac{1.486}{n} (R_o)^{2/3} S^{1/2} \ldots\ldots\ldots\ldots\ldots\ldots (75)$$

With air entrained in the water the value of n, as far as can be judged from available data, is equal to its value for a channel of similar roughness without insufflation. The entrainment of air increases the cross-sectional area, and hence the value of the hydraulic radius, R_o, of the mixture, more than offsetting a slight reduction in the friction slope, S. As a result, the velocity of the air-water mixture is increased with greater insufflation.

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[55] "Applied Fluid Mechanics," by Morrough P. O'Brien and George H. Hickox, McGraw-Hill Book Co., Inc., New York, N. Y., 1937, pp. 271–272.

[56] *Ibid.*, pp. 283–284.

The statements by Mr. McConaughy of the reason for the Bureau of Reclamation's interest in air entrainment in water at high velocities are identical with those leading to the test and studies described in the paper. The more complete description of the Kittitas chute test contained in his discussion will be welcomed by users of these data.

Reference is made to an analysis by Professor Streeter, which Mr. McConaughy states is "so remarkably similar that it appears one more example may be added to the classic list of cases in which independent investigations arrived at the same result." This statement is not correct as there is only a slight similarity between Professor Streeter's final result and that contained in the paper. To a large extent, this similarity is due to the fact that both writers used, as a starting point, the Chézy and Manning formulas and the Bernoulli equation. In this, neither has shown any startling originality as this mathematical analysis is available in many standard hydraulic textbooks. In fact, the main differences between the writer's analysis and that found in many hydraulic texts for nonuniform flow (without entrained air) are contained in Eqs. 5, 8, 9*a*, and 9*b*, as noted by Mr. McConaughy.

The development of the mathematical analysis presented in this paper was started in 1935 in connection with spillway design, and the correlation of the field data with the Froude number was completed in 1937, presumably about the time that the Bureau of Reclamation began laboratory experiments preparatory to the Kittitas chute tests in 1938.

A conventional mathematical analysis was used purposely to fulfil the first requirements stated under the heading, "Theory of Flow in Chutes," that "it should conform as nearly as practicable to established methods of hydraulic calculations." As a result, the analysis in the main presents merely a rearrangement and new application of existing formulas and theories. The only innovation is Eq. 24*a* and Fig. 10 indicating a straight-line relation of the ratio of air to water with the Froude number.

The differences between the writer's analysis and Professor Streeter's are recognized by Mr. McConaughy as he explains them in detail. His main objection appears to be that the writer's analysis did not agree with that developed by Professor Streeter. Actually, in place of Eq. 14, Professor Streeter used (substituting the nomenclature used in this paper)

$$\frac{dh_E}{dl} = -\frac{n^2 V^2}{2.2082\,(R_c)^{4/3}} - \frac{K_2 V^2}{(R_c)^p} \qquad (76)$$

in which K is given the subscript to differentiate it from K in Eqs. 24*a* and 24*b*.

The first term represented the energy loss due to the water and the second the energy loss due to the air, although the hydraulic radius, in this term, is for the area occupied by the water, despite the denial by Mr. McConaughy.

Substituting Eq. 76 in Eq. 13, Professor Streeter's final equation was:

$$\frac{dl}{dV} = \frac{\alpha \frac{V}{g} - \frac{q \cos \theta}{V^2}}{\sin \theta - \frac{n^2 V^2}{2.2082 (R_c)^{4/3}} - \frac{K^2 V^2}{(R_c)^p}} \ldots\ldots\ldots\ldots\ldots (77)$$

which fails to reconcile fact and theory in several important and fundamental respects. The use of Eq. 72 indicates a disregard of the theory of modern fluid mechanics. Frictional resistance per unit area depends on the velocity of flow, viscosity, and density of the fluid and the size and roughness of the channel. Professor Bakhmeteff, in reference to turbulent resistance, states:[57]

> "With experimental evidence to guide and probe hypothetical reasoning, it was gradually established that frictional resistance as observed in practical cases, was proportional to close the square of the velocity and that friction force depended on the density of the fluid. On the other hand, such resistance was found to be scarcely influenced by temperature, that is, it was not dependent on viscosity."

Professor Bakhmeteff used a formula to represent frictional resistance that is (except for symbols) identical with Eq. 5. Many references may be cited to indicate the correctness of this relation.[58]

With water flowing under normal conditions without entrained air and with usual temperatures, the viscosity and density may be, and customarily are, neglected. However, with air entrained in the water, the density must certainly be taken into account, and possibly even the viscosity. Data are lacking to determine the influence of the latter in an air-water mixture. A point lost sight of by Mr. McConaughy is that, with insufflation, the mixture ceases to be a liquid and becomes a quasi-fluid. Hence, the density of the mixture becomes an important factor.

The use of the relation in Eq. 73 leads to the remarkable conclusion that the flow resistance of the air-water mixture is greater than if the water contained no entrained air. This is the basis of Mr. Stevens' alternative analysis, illustrated by a typical computation given in Table 17. The increase in n in the Manning formula for the mixture is contained in Col. 21 of that table. Having started with the erroneous relation in Eq. 73, the natural conclusion resulting is that n_c is equal to the values of Manning's n obtained in normal channel flow in similar materials. From the relation in Eq. 8, it then follows that the value of n for the mixture must be larger than the normal. Observations of chute flow and laboratory tests prove this to be incorrect, as the water does travel faster once insufflation occurs. The failure of the trajectory of the water to follow the vertical curves on the Hat Creek and Kittitas chutes shows that the velocities were greater than those used in the design of these structures. Many illustrations could be cited of the unsatisfactory operation of jump basins at the end of chutes due to the water velocity being greater than that for which the structure was designed. In tests made in the laboratory, air introduced to a stream discharging under a sluice-gate did not increase the depth of the mixture, but rather caused the mixture to travel at a higher velocity, while maintaining the same depth as without air. This increase in velocity resulted in the movement of the jump downstream.[59]

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[57] "The Mechanics of Turbulent Flow," by Boris A. Bakhmeteff, Princeton Univ. Press, 1936, p. 23.

[58] "Applied Hydro- and Aero-Mechanics," by L. Prandtl and O. G. Tietjens, Engineering Societies Monographs, McGraw-Hill Book Co., Inc., New York, N. Y., 1934, pp. 58–80; "The Mechanics of Turbulent Flow," by Boris A. Bakhmeteff, Princeton Univ. Press, 1936, pp. 1–36; "Applied Fluid Mechanics," by Morrough P. O'Brien and George H. Hickox, McGraw-Hill Book Co., Inc., New York, N. Y., 1937, pp. 98–100, 107, 125, 130–133, and 180; and "Fluid Mechanics for Hydraulic Engineers," by Hunter Rouse, Engineering Societies Monographs, McGraw-Hill Book Co., Inc., New York, N. Y., 1938, pp. 47–54, 192–194, and 274–278.

It is difficult to understand the purpose of Table 20 and the accompanying discussion by Mr. McConaughy in proving that "neither do actual observations" support the statement that "the retardation factors n in the Manning or the Kutter formula, obtained by using observed velocities and areas, agree with those obtained in normal channel flow in similar materials." Certainly, the values of n_c in this table are much less, in most cases, than those normally used for the materials listed. Mr. Scobey has published values of n and n_c for many of the tests listed in Table 20.[60] Aside from the fact that it is stated in this publication that the determination of the velocities was questionable for some of the tests, which may account for some of the inconsistencies, values of n are larger than n_c and, in general, the former do "agree with those obtained in normal channel flow in similar materials." In many older tests, velocities were taken as an average for the entire reach with disregard for acceleration. The use of surface floats is only possible with small proportions of entrained air since, when insufflation is large, the decreased density of the mixture would result in the "float" traveling along the bottom of the chute.

For the design of a new structure n must be assumed on the basis of judgment, and initial values of V, y, and R are computed from the critical section. Subsequent values down the chute can be computed from this starting point. One sample calculation, paralleling Mr. Stevens' example, indicates the procedure (see Table 23). The design of non-rectangular channels was discussed in some detail (see Eq. 27 and supporting text). Eq. 73 is merely Eq. 27 in an oversimplified form. In Eq. 70, Mr. Stevens makes the valuable contribution of an equation to eliminate the use of the trial-and-error method in prismatic channels.

Some of Mr. McConaughy's discussion repeats, in slightly different words, statements that appeared in the paper. These are obvious upon careful reading and require no comment. It would appear from a comparison of the results of the Bureau of Reclamation's test and those of the writer that both were made in a careful and painstaking manner and that the accuracy is as high as might be expected for observations admittedly subject to a relatively high percentage of error. The comparison of the results from Hat Creek and Kittitas concrete chutes, given by the writer in Fig. 10 and by Mr. Stevens in Fig. 17, indicates a remarkable agreement which should lend confidence to the measurements.

Mr. McConaughy, in general, agrees with the writer's hypotheses, his principal criticism being with the form of the equations and the small range of data from which to select coefficients. To the latter, the writer can subscribe heartily, as added knowledge gained from data taken over a wider range of channel sizes, volume of discharge, and with varying degrees of roughness, would correct this deficiency.

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[59] "Effect of Entrained Air on the Hydraulic Jump," by Emery H. Willes, a thesis presented to the College of Civ. Eng., Utah State Agricultural College, Logan, in 1937, in partial fulfilment of the requirements for the degree of Bachelor of Science.

[60] "The Flow of Water in Flumes," by Fred C. Scobey, *Technical Bulletin No. 393*, U. S. Dept. of Agriculture, values of n on pp. 36 and 37, and values of n_c on p. 39.

Mr. Stevens suggests that data of both velocity and depth readings be "smoothed" due to unavoidable observational errors. The need for smoothing velocities was more urgent due to acceleration between observation stations. For example, at Hat Creek chute with a flow of 395 cu ft per sec, the observation showed an elapsed time of 2.16 sec for the color to travel from Station 0+25 to Station 1+25. This time has been distributed to indicate an acceleration from 33.4 ft per sec to 60.5 ft per sec in this distance. There was no obvious reason to indicate a need for "smoothing" depth readings; hence, they were used without correction for observational inaccuracies.

Mr. Stevens has introduced new nomenclature in his alternative analysis, although in most instances the characteristics are not new but were designated by other symbols in the paper. However, his use of subscript *w* to denote water without air and of subscript *m* to denote the insufflated mixture is no doubt a clearer designation than the ones originally used. To avoid confusion the writer will adhere to his nomenclature in this discussion.

Mr. Stevens has attempted to demonstrate a relation between (1) kineticity of flow and degree of insufflation; and (2) the increment in the roughness factor due to insufflation. Discussing the second relation first, Mr. Stevens assumes that the retardation factor for the water without air ($n_c = n_w$) is a constant and that the retardation factor for the mixture ($n = n_m$) increases with the insufflation.

If the analysis given in Eq. 3 to Eq. 8 is correct, and it appears to be correct, it is found that the opposite conclusion results. In other words, n is the constant and n_c diminishes with the insufflation. The ratio n/n_c which Mr. Stevens has computed with considerable effort can be obtained readily from the reciprocal of Eq. 8 using values of R_o and R_c. Computed in this manner, the spread of the observed data is greater in Fig. 36 than the "smoothed" points of Fig. 28(*b*). The relation between ρ and n/n_c is not as simple as indicated by Mr. Stevens by means of the obvious functional relation through A_o and A_c. In Fig. 10 and Eq. 26 the writer endeavored to determine a general relation through the use of the parameter $\frac{V^2}{g\,R_c}$, rather than the more obvious relation. By substituting the value of ρ from Eq. 24*c* in Eq. 26 and replacing B by its equivalent $\frac{2\,q}{b}$, the relation between n_c/n and ρ becomes

$$\frac{n_c}{n} = \left[\rho\left(1 + \frac{2\,K\,V^2}{b\,g}\right)\right]^{2/3} \qquad (78)$$

indicating that the ratio n_c/n depends not only on ρ but also on the velocity and the channel width. The ratio n_c/n is the most convenient form to use in the writer's equations, but the reciprocal has been plotted in Fig. 36 for comparison with Mr. Stevens' diagram (Fig. 28(*b*)). Several curves have been drawn on the graph for values of $\left(1 + \frac{2\,K\,V^2}{b\,g}\right)$ ranging from 1.0 to 1.4,

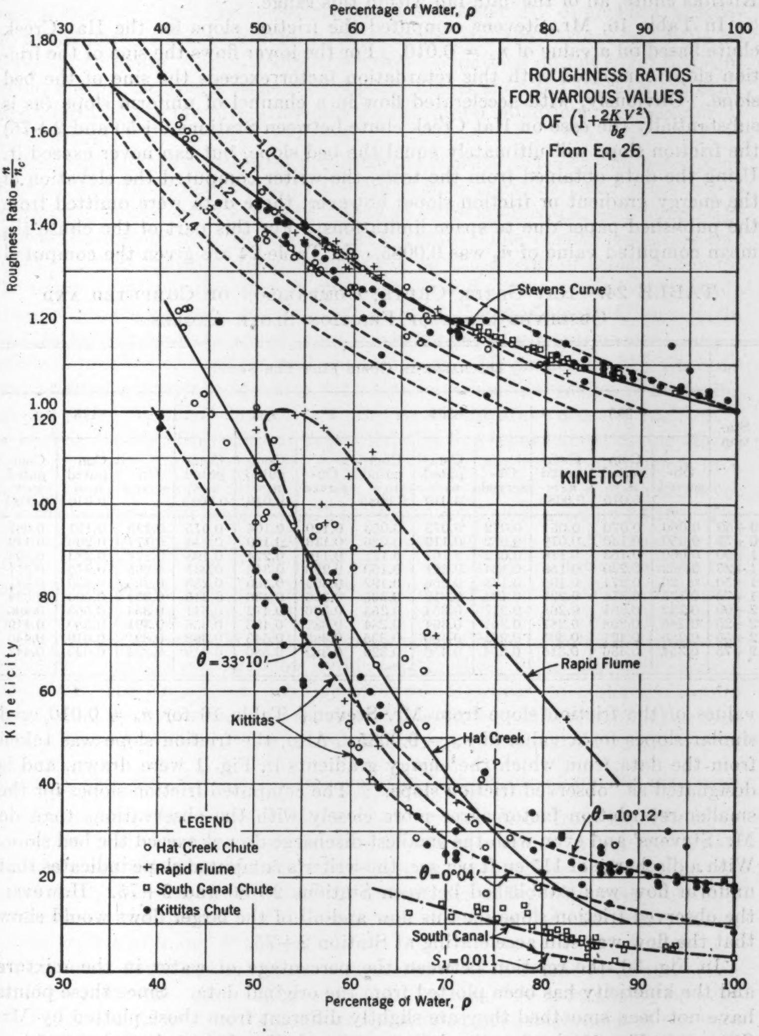

Fig. 36.—Relation of Percentage of Water to Roughness Ratios and Kineticity

which cover the normal range of this factor. Except for a few points on the Kittitas chute, all of the data fall within this range.

In Table 16, Mr. Stevens computed the friction slope for the Hat Creek chute based on a value of $n_c = 0.010$. For the lower flows the sine of the friction slope computed with this retardation factor exceeds the sine of the bed slope. Obviously, with accelerated flow in a channel of uniform slope (as is substantially the case on Hat Creek chute between Stations 0+50 and 2+75) the friction slope will ultimately equal the bed slope, but can never exceed it. Using the data obtained from the tests, the writer computed the elevation of the energy gradient or friction slope; however, these data were omitted from the published paper due to space limitations. For this part of the chute the mean computed value of n_c was 0.0085. In Table 24 are given the computed values of the friction slope from Mr. Stevens' Table 16 for $n_c = 0.010$, and similar slopes for a value of $n_c = 0.0085$. Also, the friction slope was taken from the data from which the energy gradients in Fig. 1 were drawn, and is designated as "observed friction slope." The computed friction slopes for the smaller retardation factor agree more closely with the observations than do Mr. Stevens' and even with the smallest discharge do not exceed the bed slope. With a discharge of 115 cu ft per sec, the writer's computed slope indicates that uniform flow was established between Stations 2+50 and 2+75. However, the observed friction slope for this flow and all of the larger flows would show that the flow was still accelerating at Station 2+75.

TABLE 24.—HAT CREEK CHUTE; COMPARISON OF COMPUTED AND OBSERVED SINES OF FRICTION SLOPE ANGLES

Station	Discharge in Cubic Feet Per Second											
	395			368			163			115		
	Observed	Computed $n_c =$ 0.010	Computed $n_c =$ 0.0085	Observed	Computed $n_c =$ 0.010	Computed $n_c =$ 0.0085	Observed	Computed $n_c =$ 0.010	Computed $n_c =$ 0.0085	Observed	Computed $n_c =$ 0.010	Computed $n_c =$ 0.0085
0+50	0.060	0.070	0.051	0.072	0.073	0.053	0.110	0.104	0.075	0.148	0.121	0.087
0+75	0.060	0.110	0.079	0.072	0.119	0.086	0.110	0.199	0.144	0.177	0.206	0.149
1+00	0.060	0.163	0.118	0.072	0.167	0.121	0.110	0.274	0.198	0.177	0.283	0.204
1+25	0.062	0.220	0.159	0.124	0.221	0.160	0.226	0.343	0.247	0.265	0.375	0.271
1+50	0.120	0.271	0.196	0.175	0.266	0.192	0.309	0.400	0.289	0.265	0.430	0.311
1+75	0.175	0.314	0.227	0.196	0.312	0.225	0.336	0.436	0.315	0.354	0.490	0.354
2+00	0.222	0.367	0.265	0.217	0.354	0.255	0.336	0.472	0.341	0.354	0.563	0.406
2+25	0.286	0.398	0.287	0.261	0.394	0.284	0.382	0.493	0.356	0.391	0.580	0.419
2+50	0.286	0.421	0.304	0.261	0.450	0.325	0.382	0.543	0.392	0.391	0.619	0.446
2+75	0.354	0.436	0.315	0.354	0.456	0.329	0.382	0.580	0.419	0.391	0.620	0.447

In Fig. 36, the relation between the percentage of water in the mixture and the kineticity has been plotted from the original data. Since these points have not been smoothed they are slightly different from those plotted by Mr. Stevens in Fig. 28(*a*); also, the data from South Canal chute have been added. With the latter due to the great number of points, values from adjacent stations having the same kineticity were averaged. The results are of particular interest as the roughness of the channel resulted in insufflation at much lower kineticity than for any of the other channels.

As stated by Messrs. DeLapp and Stevens, Fig. 36 showed a lag in the release of entrained air as velocity decelerates and kineticity diminishes, according to the data on Rapid Flume, Kittitas chute, and the flattest bed slope on South Canal chute. Admittedly, the writer's equations do not cover this condition, but by selecting a value of K slightly higher than the average, the computed depth of the mixture will equal the maximum conditions and a factor of safety provided.

Mr. Stevens has apparently accepted the relation between the percentage of water and the kineticity from the Hat Creek chute as being most representative. In the writer's opinion this is an oversimplification and completely ignores the results from the South Canal chute. The results from this test would apply to chutes of rough concrete or to those built of ashlar masonry. The writer's method, therefore, is more flexible and can be applied over a wide range of conditions once the proper values of the coefficients K and K_1 in Eq. 24*a* have been determined from measurement of air entrainment over a sufficient number of surfaces and channel sizes. Nevertheless, Mr. Stevens' correlation of percentage of water with kineticity has advantages in analyzing data on air entrainment and for this purpose should prove a useful tool.

In Fig. 29 Mr. Stevens has presented the flow characteristics for Rapid Flume in a graphical form that conveys all the essential information in a manner easily grasped. However, the observed value of the roughness factor n for the mixture from Station 1+75 to Station 5+50 was actually 0.010. The friction slope in this stretch of flume was a uniform value of 0.263 (see Fig. 6). Hence, the friction slope at the lower end of the chute did not exceed the bed slope to the degree shown in Fig. 29. This may account for the fact that pulsating (slug) flow actually did not occur in the lower portion of the flume.

For comparison with Mr. Stevens' example of design in Table 17, the writer has made a parallel computation by his own method, using the same channel dimensions, bed slope, and quantity of flow. However, to make the results comparable, the writer has used a value of $n = n_m = 0.015$, or about the mean of Mr. Stevens' values. The characteristics of this channel are plotted in Fig. 33. Values of A_c, R_c, V_o, and $\frac{V^2}{g\,R_c}$ can be computed from the discharge and the channel dimensions. Values of ρ were computed from Eq. 24*b* using $K = 0.005$; and n_c/n was computed from Eq. 26 as modified in Eq. 78. Thus, from the graph it is seen that all of these factors are functions of the depth. It should be noted that the channel characteristics differ with every channel depending on the quantity of discharge, the dimensions of the cross section, and the values of n and K chosen.

The initial depth at the critical section should be increased for α and hence should be 7.99 rather than 7.87 used by Mr. Stevens. (Eq. 29 should be used rather than Eq. 71.)[61] In Table 23 are assembled the calculations of the flow characteristics in this chute. Starting at the critical section the values in Cols. 4 to 13, Table 23, were taken from Fig. 37 either directly or indirectly. The value of n_c was taken only to the nearest 0.005, and decreased from 0.015

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[61] "Applied Fluid Mechanics," by Morrough P. O'Brien and George H. Hickox, McGraw-Hill Book Co., Inc., New York, N. Y., 1937, p. 278.

at the critical section to 0.009 with uniform flow. Cols. 14 and 15 were obtained from Eq. 17. The friction slope is the last term in the denominator on the right side of this equation. The values of dV should be taken in sufficiently small increments to assure accuracy; in this example 10% of V was used. Col. 18 is obtained by dividing Col. 4 by Col. 10, and values of y_o and R_o in Cols. 19 and 20 can be taken from Fig. 33 substituting values A_o, y_o, and R_o for A_c, y_c, and R_c. The maximum velocity for uniform flow would be 111.7 ft per sec and depth of mixture 3.61 ft at Station 12+44, as compared with Mr. Stevens' maximum velocity of 91.4 ft per sec and depth of mixture of 2.92 ft at Station 9+59. Mr. Stevens' computations were not continued until the friction slope equaled the bed slope of 0.547, as was done in Table 23. A comparison of flow characteristics at equal stations has been made in Table 25.

TABLE 25.—COMPARISON OF FLOW CHARACTERISTICS FROM TABLES 17 AND 23

Station	Water Depth, y_c		Velocity, V_o		Kineticity, k		% of Water, ρ		Mixture Depth, y_o		Friction Slope, S	
	Hall	Stevens	Hall	Stevens	Hall	Stevens	Hall	Stevens	Hall	Stevens	Hall	Stevens
1+70	1.74	1.80	71.7	69.5	58.4	52.6	61	66	2.85	2.74	0.259	0.236
3+10	1.44	1.55	86.8	80.6	103.2	81.7	48	57	3.00	2.72	0.382	0.363
4+60	1.31	1.45	95.4	86.2	136.	100.	41	52	3.19	2.79	0.420	0.449
6+95	1.19	1.39	105.0	89.9	181.	112.	36	49	3.35	2.85	0.505	0.506
9+60	1.14	1.37	110.0	91.4	207.	119.	33	47	3.56	2.92	0.522	0.531

The writer's results, particularly with regard to velocity and mixture depth, yield a more conservative design, in addition to being computed by a method that embodies the accepted principles of fluid mechanics.

If the value of the retardation factor in the foregoing calculations had been assumed to be $n = n_m = 0.012$, the velocity for maximum uniform flow would have been 154 ft per sec; the retardation factor for water alone, n_c, would be 0.0055; the water depth, y_c, would be 0.81 ft; the mixture depth, y_o, would be 5.4 ft; and the percentage of water, ρ, would be 15%. The distance required to attain this velocity was not computed. Flow under these conditions would be practically in the form of foam and spray and the velocity would be much greater than in any of the tests available. The flow might even be similar to that characteristic of waterfalls by which "rockets" of water detach themselves from the main mass and are either dissipated into spray or engulfed in the falling water at a lower elevation. Whether the writer's equations would apply at such very high velocities must await verification by future tests.

The computations in Table 23 show that, using a value of $n = 0.015$, a length of channel of 1,244 ft would be required before uniform flow conditions would be attained. The vertical drop would be 680 ft. The slope used in the calculations is the same as the 33° 10′ slope in the Kittitas chute. Since the entire length of that chute was 1,301 ft and the vertical drop 340 ft, it is extremely doubtful that uniform flow conditions had been reached as was assumed in computing the retardation factors by taking the energy slope equal to the bed slope. Disregarding the effect of any possible error in the velocity determination, if the energy slope is actually less than the bed slope the values of n in Table 12 for the Kittitas chute are too large. This may account in part for the larger retardation factors on this chute as compared with Hat Creek and South Canal chutes.

Mr. Stevens has computed a velocity of the air-water mixture in Col. 27 of Table 17 which is less than the velocity of the water in Col. 5. These two velocities should be identical and there is no reason for the water velocity being greater than the mixture velocity. From Table 23 the velocity calculated from the friction slope and the hydraulic radius of the mixture agrees exactly with that computed for the water alone. The small differences that occur resulted from the values of n_c being taken to the nearest 0.005. The reason for the discrepancy in Mr. Stevens' computations has not been checked in detail. However, from Eq. 8, the ratio of $\frac{n}{n_c} = \left(\frac{R_o}{R_c}\right)^{2/3}$. At Station 9+59 in Table 17 the ratio of $\frac{n}{n_c} = \frac{0.0180}{0.150} = 1.20$, whereas the ratio of $\left(\frac{R_o}{R_c}\right)^{2/3} = \left(\frac{1.69}{1.02}\right)^{2/3} = (1.66)^{2/3} = 1.40$. These should be identical, as Eq. 8 is a mathematical relation based on the Manning formula and is independent of the remaining assumptions. The failure of these ratios to coincide accounts for the difference in velocities.

In this closure if the writer appears to be rather critical of those taking part in the discussion, this impression may be created by the fact that points of difference have been covered rather than points of agreement. On the whole, the writer is in accord with the ideas and suggestions brought forth. Many of these were most excellent and indicate the way for future study.

A. A. KALINSKE,[62] ASSOC. M. AM. SOC. C. E., AND J. M. ROBERTSON,[63] JUN. AM. SOC. C. E.—The writers' part of the Symposium was a report on an experimental study of one method of eliminating air pockets from high points in a pipe line. This research is by no means a complete answer to the general problem. The writers did not obtain any experimental data on other possible methods, such as "the trapping of large bubbles of air by surface waves," as mentioned by Mr. Kennison. A little time was spent in trying to establish this particular phenomenon, but without much success. Mr. Kennison shows how the information obtained on the jump in a sloping pipe may be applied to the solution of the problem of the removal of air from high points in pipe lines. Fig. 19, showing the variation of the several flow factors with slope, depth, pipe roughness, etc., for normal flow conditions, indicates how the data may be utilized. Unfortunately, this diagram is a bit involved for easy use of the information presented.

In his discussion of the terms used in evaluating the momentum relation, Mr. Kennison explains how the effect of the entrained air was omitted from Eq. 36. As indicated by Eq. 44 for the special case in which the jump barely fills the pipe, Eq. 36 should be written,

$$F_2 = \left(p_2 - \frac{\gamma D}{2\,(1+\beta)}\cos\theta\right) A_p \ldots\ldots\ldots\ldots\ldots (79)$$

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[62] Asst. Prof., Univ. of Iowa, Iowa Inst. for Hydraulic Research, Iowa City, Iowa.

[63] Asst. Prof., Dept. of Eng. Mechanics, Pennsylvania State College, State College, Pa.

When the jump barely fills the pipe—that is, when $\frac{p_2(1+\beta)}{\gamma}$ is not much greater than the pipe diameter, D—this correction is appreciable. In most of the data for the experiments reported, in which β is large, p_2 was so large that the correction to the data as plotted in Fig. 16 is very small. In general, and especially in the case used by Mr. Kennison, this factor should be included.

The other questions, raised by Mr. Kennison, relating to air removal and the prevention of excessive head loss, are answered in the second phase of the study of this general problem which is now complete (1943) although not yet published. The present paper proposed only to cover the specific item of analyzing the air-removal characteristics of a hydraulic jump occurring in a closed, circular conduit.

The discussions by Messrs. Kindsvater and Johnson, on the various factors causing the measured data on the hydraulic jump (as presented in Fig. 16) to deviate from perfect agreement with the momentum law, are very instructive. It is probably true, as Mr. Kindsvater suggests, that part of the discrepancy is due to the neglect of the wall friction forces acting between the stations at which pressure and momentum are computed. However, as he notes, it is impossible to evaluate this factor correctly. Mr. Johnson suggests that this discrepancy may be due to the fact that Δ, the air-water ratio by volume, is different from β, the air-water flow ratio. The use of Δ rather than β in computing the values shown in Fig. 16 should bring them into closer agreement with theory. In a horizontal pipe these two quantities should be very nearly equal, but, in a sloping pipe, Δ should be larger than β, as the air is being dragged down by the water. In fact, the ratio of Δ to β should increase with pipe slope. Measurement of Δ by means of quick-closing valves, as suggested by Mr. Johnson, is subject to several difficulties. For one thing the valves would have to be very close together in order to cover only the jump itself. Furthermore, such measurements probably would result (due to the ensuing water hammer) in the failure of the "Lucite" pipe line used in these particular tests. Perhaps some form of sampling technique may be developed for this purpose in the future.

The writers agree with Mr. Kindsvater that, in analyzing the jump in sloping conduits, it is essential to know the length of the jump. Considerable data were obtained on the length of the jump, but no good method of correlating them was found. In the light of Mr. Kindsvater's discussion, the data were analyzed, yielding the results shown in Fig. 37. Since the data taken in the horizontal pipe were inadequate, no comparison is possible with the data given in Fig. 22 by Mr. Kindsvater. There is considerable experimental scatter in the data, but it appears that in sloping pipe lines, the jump-length ratio, $\frac{L\gamma}{p_2(1+\beta)}$, is independent of $\frac{y_1}{D}$, in contradistinction to the horizontal pipe. In these measurements, the jump length was taken as the distance from the start of the jump to the point where the water surface left the top of the pipe at the start of the air bubble below the jump (see Figs. 12(*a*), 12(*b*), and 15). A similar point did not exist for the horizontal pipe. The lines representing the average trends of the jump-length ratio for the different slopes have been

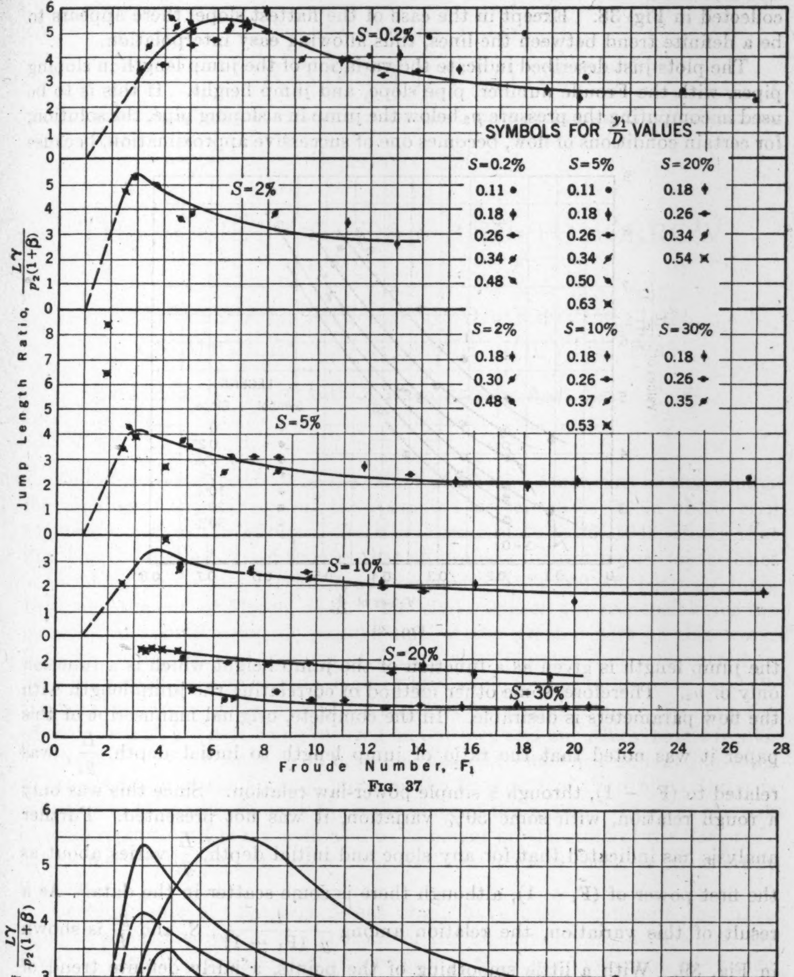
Jump Length Ratio, $\frac{LY}{p_2(1+\beta)}$
Froude Number, F_1
S=0.2%
S=2%
S=5%
S=10%
S=20%
S=30%
SYMBOLS FOR $\frac{y_1}{D}$ VALUES
S=0.2%: 0.11 ●, 0.18, 0.26, 0.34, 0.48
S=5%: 0.11 ●, 0.18, 0.26, 0.34, 0.50, 0.63
S=20%: 0.18, 0.26, 0.34, 0.54
S=2%: 0.18, 0.30, 0.48
S=10%: 0.18, 0.26, 0.37, 0.53
S=30%: 0.18, 0.26, 0.35
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28

FIG. 37

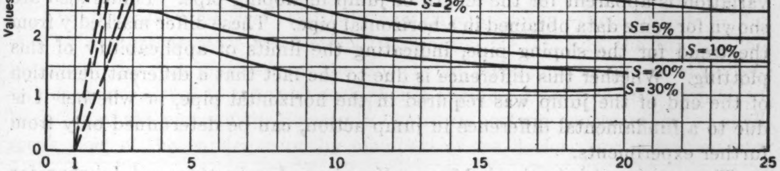
Values of $\frac{LY}{p_2(1+\beta)}$
Froude Number, F_1
S=0.2%
S=2%
S=5%
S=10%
S=20%
S=30%
0 1 5 10 15 20 25
0 1 2 3 4 5 6

FIG. 38

collected in Fig. 38. Except in the case of the flattest slope, there appears to be a definite trend between the lines, thus allowing easy interpolation.

The plots just described indicate the variation of the jump length in sloping pipes, with the Froude number, pipe slope, and jump height. If this is to be used in computing the pressure p_2 below the jump in a sloping pipe, the solution, for certain conditions of flow, becomes one of successive approximation, because the jump length is given as a function of the jump height which is a function only of p_2. Therefore, some other method of correlating the jump length with the flow parameters is desirable. In the complete, original manuscript of this paper it was noted that the ratio of jump length to initial depth, $\frac{L}{y_1}$, was related to $(\mathbf{F}_1 - 1)$, through a simple power-law relation. Since this was only a rough relation, with some 50% variation, it was not presented. Further analysis has indicated that for any slope and initial depth, $\frac{L}{y_1}$ varies about as the first power of $(\mathbf{F}_1 - 1)$, although there is some scatter in the data. As a result of this variation, the relation among $\frac{L}{y_1\,(\mathbf{F}_1 - 1)}$, S, and $\frac{y_1}{D}$ is shown in Fig. 39. With a little smoothing of the points, a fairly definite trend of variation is apparent for the length of jump in sloping pipe. Points also are shown for some data obtained in a horizontal pipe. These differ markedly from the trend for the sloping pipe, indicating the limits of applicability of this plotting. Whether this difference is due to the fact that a different definition of the end of the jump was required in the horizontal pipe, or whether it is due to a fundamental difference in jump action, can be determined only from further experiments.

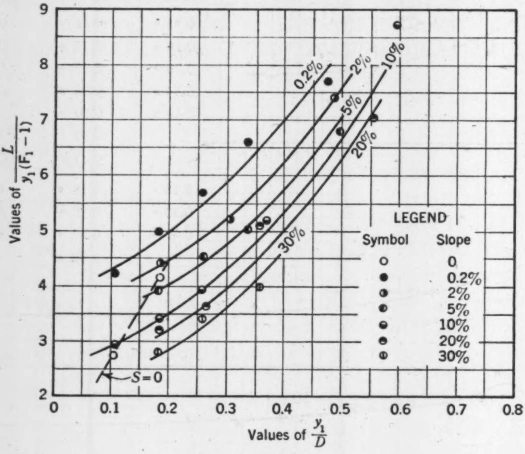

FIG. 39

The writers wish to thank Messrs. Kennison, Kindsvater, and Johnson for their very instructive discussions, which add much to the original paper.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
TRANSACTIONS

Paper No. 2206

THE SOCIETY TODAY AND TOMORROW

ADDRESS AT THE ANNUAL CONVENTION, LOS ANGELES, CALIFORNIA, JULY 28, 1943

BY EZRA B. WHITMAN,[1] PRESIDENT, AM. SOC. C. E.

The objective uppermost in the minds of all true Americans at this time is the winning of the war against the Axis, and we can all be proud of the civil engineer's part in this effort. The *War Survey* of June 26, 1943, states that the War Production Board reports that all projects for war industrial facilities financed by public and private funds reached $19,339,000,000 between June, 1940, and the end of March, 1943. The civil engineer was responsible for the planning and supervision or construction of the greater part of this work. Projects costing tens of millions of dollars were planned and constructed in incredibly short periods of time, making it possible for our industry to produce amazing quantities of the necessities required for waging a successful war, the effects of which are now being realized on our own and our Allies' battle fronts.

Previously, Presidents, upon occasion, have directed their annual addresses to the condition of the Society, much as the President of the United States is called upon to report annually upon the state of the nation. Even at the risk of some repetition it is my wish also to comment on the state of the Society—not so much, however, on where it is and how it came there, as to where it is going.

Developments of relatively recent origin have been significant and have brought about changes in the attitude of the Society, or attitudes which the Board of Direction has adopted with respect to those circumstances, and which it has sought to express on behalf of the Society in new procedures set in process.

I shall speak of several relatively new developments—the addition to the staff of a representative resident in Washington, D. C., and the function he performs; the participation by the Society in resistance to the recently accelerated movement to take the professional engineer employee into "organized labor"; and the plans of the Society in regard to postwar construction. I shall speak, too, of the participation by members of the Society in the tremendous

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[1] (Whitman, Requardt & Smith), Baltimore, Md.

defense and war construction projects; the renegotiation of those contracts; the assistance rendered the Army, the Navy, and other federal agencies in securing the trained personnel they have desired; and of the Society's members in the Armed Services.

First, however, a word on the present condition of the Society: As of July 1, 1943, the membership of the Society was 18,920, representing a net growth of approximately 1,060 over the membership on the same date a year ago. Applications for admission on that date were almost precisely the same in number as those of a year ago. Notwithstanding the cancellation of dues to more than 500 members, as authorized by the Board for those in the lower ranks of the Armed Services, the collection of dues thus far is almost exactly equal to that of last year, a situation which has permitted the Board to authorize certain of those procedures of which I propose to speak further.

The technical activities of the Society, it must be admitted, have suffered from the impact of the war. In large part, the personnel of the technical committees has remained intact, but members have been diverted so generally to the war effort in its many phases that the attention they would have given normally to their respective assigned problems has been necessarily diminished. Also, the reduction in the number of meetings of the Society and of the sessions of the Technical Divisions has led to the production of fewer papers available for the Society's publications from those customary sources. This situation has been intensified by the censorship, imposed by the government on the theory that some material that would interest and inform our members might also inform the enemy. Notwithstanding the consequent inability to publish some highly qualified papers, and the government's requirement of a 10% reduction in the use of paper in 1943, the publications have maintained their usual standard of excellence. The reduction in volume has been felt necessarily in the production of all the technical output of the Society, but it is quantity that has been reduced, not quality.

In general, therefore, it may be said that the Society is in a very healthy condition financially, although in somewhat of a "state-of-being" rather than in full action on technical matters. On the other hand, there has been greatly increased activity in matters relating to the economic condition of its members and, indeed, of the entire civil engineering profession.

Washington Office

What terms historians may use to describe the decade just past and the one in which we are now living is a secret that will not be known until its history is written. Adequate words are available to describe the world's greatest depression and most destructive World War, but new words probably will be coined to give vividness to the description. One fact that historians certainly cannot ignore is that the past decade has seen the greatest concentration of power in Washington since the original thirteen colonies finally agreed to a central government and adopted a Constitution. Engineers could not help but be caught in the vortex centering in Washington. They could hope only to chart a sane course that would prevent their being engulfed by the ever-narrowing swirl. Realizing this trend, the Board of Direction, in

1940, deemed it necessary that it be represented in the Nation's Capital, not only to enable it to better preserve its balance in National affairs, but in order that it might contribute its proper share of professional ability to an ever-increasing technical trend of government.

It was at its July, 1940, meeting, that the Board of Direction decided that a member of the staff should spend as much time as was necessary in Washington, during the development of the National Defense Program, to safeguard the interests of civil engineers in all practical ways. In accordance with this action, Field Secretary Walter E. Jessup, M. Am. Soc. C. E., addressed himself, in July of that year, to matters arising in the Nation's Capital.

With the very generously afforded help of many members, when called upon, he aided in the preparation of contract forms for the professional engineering services desired, and in the determination of fees to be paid to engineers, or engineering firms, to be retained on defense work by the government in the Construction Division of the Quartermaster Corps, later the Corps of Engineers of the Army; the Bureau of Yards and Docks of the Navy; the Federal Works Agency; and the U. S. Housing Authority. Other national societies of engineers and architects collaborated with the Society, in offering their combined services to the government, each in its respective function in a procedure believed to be of advantage both to the government and to the professions.

When Mr. Jessup was called into active service as a Major in the Corps of Engineers, U. S. Army, in the fall of 1941, the Board of Direction concluded that conditions warranted the establishment of a Washington office of the Society and the placing of a full-time representative there. In November, 1941, Hal H. Hale, M. Am. Soc. C. E., was engaged as a member of the staff to take up residence in Washington and was provided with an office and an assistant. It should be emphasized that this action was not taken with the idea of establishing just another pressure point in the Capital, but with the firm conviction that the Society could then keep pace with the rapidly shifting tide of events and be better situated to be of service to the country and to Society members. That these conclusions were well taken has been amply demonstrated by the many calls for service made upon the Washington office, not alone by the Board of Direction, but by various government agencies and elected representatives.

Continuing in the capacity of the Society's representative in that busy city, Mr. Hale is charged to acquaint "himself with those various matters originating in Washington, which may affect, favorably or unfavorably, the welfare of the civil engineer. He has been charged particularly to explore ways and means whereby the Society can be of maximum service in all war measures."

His field of operation is to keep the Board of Direction informed of what is going on in Washington as to matters affecting civil engineers. He is not a lobbyist. Contacts have been established, however, with members of the Congress and with many executives in the various federal agencies, by whom, it is satisfying to record, these contacts have been viewed as helpful. In many instances the function of the civil engineer and his particular capabilities have been made known where they were not formerly understood.

The current of events was moving swiftly in November, 1941, when the Washington office was established. On December 7 of that year, what had been a measure of aid to our friends fighting for their lives in a European war became overnight a fight to the death for our very survival in a world-wide struggle.

One cannot exist in Washington and be unaware that it is a place where laws are made; nor can the Society escape such awareness. Many of the laws so made are of vital concern to engineers. One of the important tasks of the Society's representative is to keep informed about new legislation and be in a position to advise and assist, where necessary, in the review of proposed legislation, as well as to keep the Board constantly informed about such developments.

Recently a law was proposed for the purpose of reorganizing the U. S. Public Health Service. Unfortunately, its original wording would have barred engineers from holding administrative positions in the reorganized agencies, reserving such positions for medical officers only. This inequitable situation was immediately brought to the attention of the Congressman sponsoring the bill, as well as to the Surgeon General of the Public Health Service and other officers of that Service. Numerous conferences were held and, when the bill was brought up for hearings before the appropriate committee of the Congress, our representative testified and placed in the record the proper criticism, with suggested changes to remedy the situation. Strikingly, the testimony given by the Surgeon General on these points was almost identical with that of our Society's representative. Subsequent revisions removed the objectionable restrictions, and it is believed that when the bill is passed engineers in the Public Health Service will be accorded the same rights, privileges, and rewards for efficient service as other officers in the Service would have otherwise alone been granted.

Another experience, typical of the functions performed by Mr. Hale, relates to negotiations with the Federal Public Housing Authority. A new form of contract for professional services was adopted by that agency in September, 1942. Although the new form left much to be desired, particularly from the standpoint of compensation, it was an improvement over the old, and at last placed the engineer on a plane with other professions involved, a status he had not previously been accorded.

It would not be practicable, within the limits of such an address as this, to review in minute detail all of the tasks that the Washington office has been, and is being, called upon to perform. Much of the work can be placed in the category of the general welfare of the engineer. It is proper to mention a few of the agencies with which contact has been made, such as the War Department, Navy Department, Marine Corps, Selective Service System, War Manpower Commission, National Roster of Scientific and Specialized Personnel, Civil Service Commission, and, of course, various committees of the Congress considering legislation of interest to civil engineers.

The Washington office, in a sense, has been a clearing house for information leading to commissions in the Armed Services. It has been called upon on numerous occasions to assist in the clarification and solution of many problems of general professional relations, recent instances being the review of procedures under consideration for the renegotiation of contracts and the important matter of postwar construction, both of which I shall discuss later. More than thirty bills relating to postwar planning, in one form or another, have been introduced into the 78th Congress. Our representative at Washington has collected, analyzed, and commented upon these for the information of the Board of Direction, and, by published articles, for the information of the Society's membership. It is through him that the Board has been enabled to be conversant currently with bills in process of preparation or already introduced, on which hearings have been or are being held, the content of which may or may not be inimical to the interests of our members.

EMPLOYMENT CONDITIONS AND UNIONIZATION OF ENGINEERS

For many years the Society has been engaged in activities aimed at promoting the relative position of the professional civil engineer in his social, financial, and moral relations. These activities aim at increasing the engineer's standing in his community. They aim at placing his financial remuneration in its proper place in comparison with that of the other learned professions, and they aim at promoting proper ethical codes for the engineering profession.

Some of these activities have centered around the Committee on Salaries. In the past, matters relating to salary classifications and schedules have been dealt with actively in many parts of the United States. Under pressure of changing conditions, and because of recent social legislation, these activities have been broadened and another committee has been added—the Committee on Employment Conditions. This Committee has to do with employer-employee relationships, with employment conditions, and with the trend toward pressure grouping and collective bargaining.

It was several years ago that the Board of Direction established a Professional Department consisting of several committees having to do with matters such as the registration of engineers, salaries, and other professional objectives. In January, 1942, however, a Committee on Employment Conditions was authorized, the function of which, although not yet defined in the By-Laws, is to give study to those several matters relating to employees, other than their salaries, such as working hours, vacation and sick leave, overtime pay, tenure of job, etc.

On March 1, 1942, Howard F. Peckworth, M. Am. Soc. C. E., as Assistant to the Secretary, was assigned to devote his full time to these matters. His first step was to make a careful study of federal legislation relating to hours, wages, and conditions of employment. He then made a trip of some 14,000 miles investigating the alleged unionization or pressure groupings of engineers, subsequently reporting his findings to the Board of Direction. In consequence, he was instructed to be of help, wherever practicable, in the settlement of engineer employer-employee difficulties.

His general observation was that engineers do not wish to affiliate with organized labor, but that they do desire some form of group organization to assist them in perfecting salary ranges, normal hours of work, vacation periods, sick leave, tenure of position, and other employment conditions. These, they believe, should be established authoritatively to be recognized as normal conditions of employment. In rendering advice to just such groups in Arizona, California, Florida, Illinois, Kansas, and Louisiana, he traveled another 9,000 miles.

As a result of definite requests from engineer employees, or engineer employers, he has acted as a conciliator, or adviser, in several instances. In this connection he traveled some 7,000 miles (a total of 30,000 miles traveled in 1942) and came into official contact with the highest representatives of the Army, Navy, U. S. Conciliation Service, and National Labor Relations Board.

When the American Society of Civil Engineers was founded in 1852, its membership consisted of only employers. Shortly after, employees were admitted to membership in the Society and, in the present changed economic situation under which we all live, the engineer employer and the engineer employee work together. Because many engineers are employees, they are subject to the recent social legislation which has to do with matters relating to employees. On this subject I shall speak plainly, not as representative of any attitude which the Board of Direction may have, but as an individual, interested in the well-being of the members of the engineering, and particularly the civil engineering, profession. It is a matter about which I am deeply concerned.

This legislation in behalf of employees started in the United States with the Railway Labor Act which gave the employees on railways the right to join together for collective bargaining. This right was further defined, upheld, and extended to practically all employees in succeeding legislation—for instance, in the Norris-LaGuardia Anti-Injunction Act, the National Industrial Recovery Act, and the National Labor Relations Act; collective bargaining is implemented by such boards as the National Labor Relations Board, the U. S. Conciliation Service, and the National War Labor Board.

Efforts on behalf of engineer employees have been put forth by the Board of Direction of the Society in many directions and continuously by the activities of a Committee on Salaries, another on Registration of Engineers, publicity representatives in each Local Section, and a full-time staff member at Headquarters, etc. Later, the Board established a Committee on Unionization, the function of which was subsequently broadened into that of the Committee on Employment Conditions.

The original report of the Committee on Unionization (*Civil Engineering*, March, 1938) gave expression to five specific findings which may be paraphrased somewhat as follows:

1. Membership in a bargaining group should have no more bearing upon a man's qualifications for membership in the Society than has his religious or political affiliations.

2. The National Labor Relations Act has encouraged the formation of new unions and the extension of old unions and is now the law of the land. Subprofessional workers cannot be exempted from its provisions. Pre-professional engineers, mostly young college graduates, cannot be exempted from its provisions. Employee professional engineers also cannot be exempted from its provisions.

3. The Society should support an effort to clarify the distinctions between the subprofessionals, and pre-professional and professional engineers, to prevent the situation of having the latter forced into vertical unions where their identity would be lost and their interests forgotten because of the greater number of subprofessionals.

4. Existing unions are inadequate to represent engineers in collective action. Experience has shown that group action can be taken by them without resort to strikes or other reprehensible tactics. It should be recognized that some engineers feel the need for collective action. Unless the profession sets about to meet this need by some organization of ethical consciousness, it must look forward to the extension of union membership among engineers and the gradual transfer of engineering to the classification of a trade rather than a profession.

5. To minimize the need for collective action by engineers, the Society should adopt and maintain an adequate and reasonably uniform compensation schedule for the several grades of engineering employment. In addition, the Society should be prepared to cite a member for unethical practice when that member pays less to his employees than the established minimum.

These pronouncements, although I have chosen to rephrase them somewhat, were adopted by the Board of Direction at its meeting held in January, 1938, and were printed in the Society's publication, *Civil Engineering*, for March, 1938.

Late in 1941, the Committee on Unionization was disbanded and replaced by a Committee on Employment Conditions. As already stated, early in 1942 Mr. Peckworth was detailed to spend his entire time on the subject of employer-employee relationship. In order to have knowledge of the exact facts in all parts of the Nation about this subject of employment conditions and the reported trend toward the unionization of engineers, Mr. Peckworth, after making his nationwide survey, as previously outlined, was directed to prepare for the Board of Direction a comprehensive report that could be used to guide future actions.

The conclusions reached by him were that the individual citizen seeking improvement, in a democratic form of government, must join with others having the same purpose; that recent legislation in the United States has greatly increased and facilitated the possibilities of group action; that, under these legislative enactments, unions have grown strong, especially those affiliated with organized labor; that, because so many engineers are employees needing group strength, they are aware of this trend toward grouping, collective bargaining, and the forming of unions; and that professional and pre-professional civil engineers, involved in these trends and these conditions, have expressed a definite distaste for affiliation with organized labor groups, and that they have given voice to decided preference for help from this, their national professional society. Thus, they are looking for organizations which, functioning as pressure groups, will protect them from affiliation with organized labor and at the same time will actively promote their economic interests.

The report went on to conclude that one answer has been a reorganized state engineering society. A preferable answer would be a more active interest by the already established and respected national engineering societies in the various conditions of employment of engineers, in the education of their own members in the legal and economic elements of collective bargaining, in recognition of the fact that young engineers and those in the pre-professional group are in need of special consideration, and in providing advice and help where there are acute employment difficulties.

These conclusions were voiced in the report submitted to the Board of Direction of the Society at its meeting held in Minneapolis, Minn., just a year ago this month, at which time the question was asked of the Board of Direction, "What should be done about this matter?" The Board voted unanimously to commission the staff to be of assistance in these matters whenever that assistance was requested and was found to be possible.

In the past twelve months this commission has led to many interesting developments. Individual employee engineers have appealed to the Society for assistance in employment matters, and that assistance has been given to them. Individual employers who were being subjected to the ill-advised actions of subprofessional labor unions appealed to the Society, for help and assistance, and it was given to them. Groups of employees in several different parts of the United States have requested and have been given individual attention. In some cases their interests have been sponsored and promoted in government offices to the very top. In other cases their interests have been carried before war labor panels. In other cases it has been possible to be of assistance to them, at the immediate site of the project, by personal interview and by conciliation between the employer and the employees. Executive engineers at the heads of large technical organizations have been given advice in regard to their methods of dealing with groups of employees, both those affiliated with organized labor and those not so affiliated. Extended efforts have been put forth to improve the contractual conditions under which engineers work when they are employed by the so-called "engineer-architects" and "engineer-architect-managers" under U. S. Engineer Corps contracts for war construction. State engineering societies have been advised of procedures which will protect the professional engineer within their state boundaries. These general citations outline in part the field which has been covered by the commission of the Board of Direction when it directed the staff to be of assistance wherever possible in the subject of employer-employee relationships.

Hardly more than a month ago a situation arose in which the Society was committed to a definite policy, in which, by the way, I heartily concur. Details of the experience will be found in the July, 1943, issue of *Civil Engineering.* Suffice it to say here that, when an American Federation of Labor union of subprofessional employees on a large construction project had demanded that the professional engineer employees on that job, who were greatly outnumbered, should be required to be represented by that union, the Society's staff member, accompanied by local legal counsel retained for the purpose, appeared before a War Labor Board panel in protest.

The average professional engineer, engrossed in the details of his particular professional problems, is often unaware of these modern social trends and the situations existing in other parts of the country regarding the subject of collective bargaining. It has been found in the course of the studies made in the past year, and through the experience gained by giving assistance to engineers actually involved in these matters, that there are inherent implications which affect the entire engineering profession, whether one be a young engineer in pre-professional status just out of school, whether one be a professional engineer working on professional assignments as an employee, or whether one be a professional engineer at the head of a private organization or at the head of a large governmental bureau.

These implications relate to the curtailment of the right of private ownership. Recognition of the right of employees to collective bargaining means, in itself, that the employer must realize that the employee has rights which in the past were seldom accorded him. Another implication involved is the curtailment of freedom of speech. Legislation as it now stands forbids an employer to make certain anti-union statements. Few engineers are involved in the subject to this extent, but it is well for us that cognizance be taken of this fact. Another deep implication is the recognition that contractual relationships nowadays are judged in terms of the social values involved. It is not for us to judge the wisdom of these social changes. It is for us, however, to recognize them and to act accordingly.

A review of the conditions surrounding these Society activities emphasizes the need for renewed professional integration. Our Society is the recognized spokesman for a large and important learned profession and as such it should have among its members a larger percentage of those qualified for membership. In addition, those qualified should be encouraged to apply for state licenses to practice.

Renewed efforts should be made to enroll all qualified technical graduates in the grade of Junior as soon as they graduate. This is necessary in order to give this young group the strength and protection it needs to resist the assumption of its rights by unwarranted subprofessional groups. To facilitate this phase of Society activities, the subprofessional group should be made aware of the newly instituted "engineer-in-training" status and the various states should be encouraged to follow the lead of the Model Law as recently revised in this respect.

The Society has recognized the professional employees' need for group action. Such groups should be properly advised and protected, and this sphere of influence should be widened by committee action in each Local Section and among qualified state professional societies. Forethought in this matter should anticipate the need for action.

Salary classifications and schedules are worthless unless kept up to date, and right at this time one or more such schedules should be adopted and thereafter kept up to date continuously.

These activities are of the utmost importance to the profession. A neglect of action at this time may possibly mean the absorption of the profession by other aggressive groups, the effect of which would be to reduce the profession to the status of a trade, devoid of freedom, life, and the inspiration which in the past has kept the civil engineering profession in a place of leadership in the development of this great country. Action in these matters will mean the continued growth of this great profession. Action will assure the leadership which we must maintain in the stirring days in which we are now living and in the challenging days ahead.

POSTWAR CONSTRUCTION

During preparations for defense, and later for war, the Board of Direction has been reluctant to suggest that postwar construction should be planned for in advance. However, the time has come when men beyond the draft age, and now released from the war construction program, are available, with respect to both capabilities and time, to engage in present planning for postwar construction.

The construction of public works, it is to be admitted, cannot provide for full employment of all those who will be released from war industries or who will be returned from the Armed Services. However, the construction industry is a large industry, employing normally millions of men engaged directly on the site, and, indirectly, in the production of raw materials, their transportation, and their manufacture into articles used in construction. Moreover, there is constantly developing an increasingly large reservoir of useful public works, the repair or construction of which has been forcibly deferred because of attention to the war effort. In addition, many proposed useful public works are greatly needed. I wish particularly to stress the word "useful," and for a reason. The term "public works" came into disrepute during the recent depression. It should not be subjected to that former experience. Public works as constructed then were of two natures:

(*a*) Those that were of sound economic value and useful, as brought into being, in general, in accordance with the policies of the Public Works Administration—carefully designed, subjected to appraisal of their economic soundness, and built, in the customary manner, by skilled contracting organizations.

(*b*) Those public works of another nature were essentially not other than a medium for a social experiment wherein "made" work was devised primarily to provide occupation with pay for those unskilled individuals who, too frequently, were unwilling to work as do those who have a pride in their work, rather than have them "go on the dole." Such were those so-called "public works" which were brought into being, in general, in accordance with the policies of the Work Projects Administration.

It is with the hope that the policies which then prevailed with respect to the projects sponsored by the Public Works Administration shall again prevail that I have stressed the word "useful" as a prerequisite. In this I am supported by the principles formulated by the newly established Society Committee on Post-War Construction. The members of this committee are practical men of wide experience, men who have designed projects that are sound economically and useful in character, and men who have builded well. They have formulated a statement of criteria by which postwar public works should be judged.

RENEGOTIATION OF CONTRACTS

On April 28, 1942, the President of the United States signed the Bill providing for the renegotiation of contracts where it appeared that an excess profit had accrued to the contractor. This Bill related, or course, to those members of the Society who, as individuals, or as partnerships or firms, had had contracts for the construction of Army, Navy, or Maritime Commission projects.

One cannot take exception to the idea of renegotiations, the results of which are to recapture for the government sums which are truly excessive in amount and arise from misconceptions, held at the time when the contracts were awarded, of the nature of the work involved, or which may have accrued through the possibilities of mass production.

The question of what is an "excess profit" with respect to the type of work performed by the so-called architect-engineer, or under the architect-engineer-management contracts, appears to be extremely difficult of definition. In large part, the work of these architect-engineers is that of a mental effort as differentiated from a manual or production procedure and is further complicated by the fact that in general no two problems arising in connection with an architect-engineer contract are alike. The question, therefore, of how excess profit is to be determined with respect to the architect-engineer, or under the architect-engineer-management contract, has given considerable concern to the many members of our Society who have been awarded these contracts on which they have expended time and energy to the utmost.

At the request of a representative of the Corps of Engineers, a committee of Society members consulted with representatives of that branch of the government and, without attempting to formulate precisely a formula by which to calculate excess profit, if any, nonetheless presented those representatives of the government with a memorandum indicating the many experiences which should be taken into consideration when profit, or excess profit, is to be estimated.

WASHINGTON COMMITTEE

Not only was the renegotiation of contracts a matter of vital interest to our many members engaged in construction projects for the various branches of the Armed Services, but another matter of overtime payment by employers for their employees was raised by suits for overtime payment being brought by groups of employees. All salaries paid by architect-engineers, and the payment or nonpayment for overtime, were dictated by the area engineers. Payment of such salaries and for overtime work under the contract terms was reimbursable by the government, but it was the architect-engineer who was being sued for overtime payment not allowed by the area engineers. After many conferences and negotations by members of our Washington Committee, headed by Col. Armour Polk, M. Am. Soc. C. E., and representatives of the architect-engineers, a supplementary contract was devised which protects the employing architect-engineers, relieving them of responsibility for overtime payment not allowed by the area engineers. At the same time the employee who may legally be entitled to further payments can bring his suit and if entitled to further pay can secure it from the federal government.

ARMY, NAVY, AND CIVILIAN PERSONNEL

Ever since 1940, when the thought of the necessity for making preparations for defense began to take form, the Society has been looked to as an organization, the members of which would be capable of rendering assistance to several federal agencies, including those of the armed forces, in their acquisition of suitably trained men to assist in the functions they were called upon to perform. First of all, a consultative committee to the Army and Navy Munitions Board, consisting of five members of the Society and a member of the American Institute of Architects, with the then President of the Society, Col. John P. Hogan, M. Am. Soc. C. E., as Chairman, was appointed by the Secretary of War to advise him with relation to the contracts which would be entered into for the building of the then-called "defense plants," shipyards, etc. This committee, shortly after its appointment, called upon the Society's staff to gather the names, addresses, and significant data of all of those engineers in the United States who designated themselves as consulting individuals, or firms, to whom the government might look for the design and supervision of construction of needed camps, cantonments, defense industries, shipyards, dry docks, etc. The staff, securing the aid of the staffs of other large engineering societies, accumulated a list in excess of 3,100 such individuals or firms with the data requested and made the information thus gathered available to the Quartermaster Corps, Corps of Engineers, and Bureau of Yards and Docks of the Navy.

Continuously since that time, as recounted each year in the Annual Report of the Board of Direction, calls have been made upon the Society to assist in the selection of men for the "Seabees," the Engineer Corps of the Army, the Provost Marshal's Office, and other governmental agencies, in connection with which notices have been printed in *Civil Engineering*, or lists of the Society's membership specializing in certain branches of the civil engineering field, as indicated by membership in the several technical divisions, have been forwarded.

On the other hand, efforts arising at the Headquarters of the Society have been directed toward several of the federal agencies, informing them of the availability of civil engineers and suggesting procedures whereby those inducted into the services might be sought out, appraised, and reallocated to assignments in which their value as trained engineers might be of more significance to the government in its war effort and of more satisfaction to them as participating in that war effort.

MEMBERS IN THE ARMED SERVICES

This has been said to be an "engineers' war" and, although that may apply also to engineers engaged in industrial processes relating to the production of guns, ammunition, and other material, the members of the Society have flocked, naturally, toward service with the Corps of Engineers of the Army and in construction battalions of the Navy, as well as to artillery units and those units engaged in the construction of airfields here and abroad.

It is not known how many members of the Society are in uniform, although it may be safely estimated that probably the number exceeds 4,000 and may even be as great as 4,500. A survey made in the latter part of last year indicated clearly that at that time there were more than 3,000 members of the Society in some one or another of the Armed Service branches, and it is gratifying to know that of those thus known to be in those services, 96% were commissioned officers. That the number has now increased, by perhaps as much as 50% or maybe more, seems probable, particularly in view of the fact that those young men graduating from colleges in the latter part of last year and the early part of this year, although they have joined the Society as Juniors, also stepped from their graduating platforms into uniform. It is a satisfaction to record that these young engineers, although knowingly destined for the Armed Services, were nonetheless desirous of affiliating with the professional society of that branch of engineering in which they have been trained.

From the foregoing I believe that the members can see that the Society has been confronted with a number of problems very different from those which occupied its attention before the war descended upon the vast majority of the peoples of this earth and greatly emphasized still other problems relating to the welfare of engineers. The greater part of the activities of the Society before this time were in connection with technical activities, but since the war started we have been forced to give greater consideration to those problems which may be termed "human relations" in addition to our technical work.

The welfare of the engineers of all grades and the struggle to secure proper recognition of the engineer from a professional point of view are questions which have forced themselves upon us to a far greater degree. That some organization should consider these questions, in my mind, is perfectly manifest, and that our Society is the organization best equipped to undertake this, I think, is self-evident.

The Secretary and his staff of able assistants have carried the burden of this work, and I think they have done and are doing a fine job in these new fields. This paper should not be closed without a word of recognition and hearty commendation for the work which they have been doing.

MEMOIRS OF DECEASED MEMBERS

ARTHUR NEWELL TALBOT, Past-President and Hon. M. Am. Soc. C. E.[1]

DIED APRIL 3, 1942

Arthur Newell Talbot was born on October 21, 1857, at Cortland, Ill. His father, Charles A. Talbot, was born in London, England, and his mother, Harriet Newell Talbot, was born in Brockville, Ontario, Canada. Both had been brought as children to the new settlements in northern Illinois in the tide of immigration, principally from New York and New England, which began after the close of the Blackhawk War. Young Talbot's early life was passed under pioneer conditions among the sturdy people who were developing new homes in a prairie land.

His early education was acquired in the schools in Cortland, a village about 55 miles west of Chicago, Ill., and in the high school at Sycamore, Ill. After completing his high school course he taught a country district school for two years.

In 1877 he entered the University of Illinois (then known as the Illinois Industrial University) at Urbana to study civil engineering. The University at that time had about 300 students of collegiate grade. Arthur Talbot was a brilliant student—indeed, his scholastic average remained the record for many years. However, he did not devote all of his time and energy to study, but was active in extra-curricular activities. He was Secretary, Vice President, and President of Philomathean Literary Society, Associate Editor of the *Illini,* delegate to the Interstate Oratorical Association, class essayist, a leading officer in the student government, and the ranking officer in the Cadet Corps. In addition to these activities, he gave instruction in preparatory mathematics and in his senior year was a student assistant in Physics.

The Philomathean Literary Society was the major student activity during the time that Mr. Talbot was a student. He took a prominent part in this Society, and the titles of his addresses to it covered a wide range of subjects. It cannot be doubted that his interest in student literary activities, and the training he then received, helped to develop the precision and clearness in speech and writing that have been outstanding characteristics.

After graduation in June, 1881, he went west and was for four years engaged in railroad location, construction, and maintenance in Colorado, New Mexico, Kansas, and Idaho. The varied experience gained during those years, and the steadily increasing responsibility as he advanced in railroad work, were such as to broaden his human contacts and develop him in the supervision of construction and in the conduct of business.

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[1] Memoir prepared by a Committee consisting of Daniel W. Mead and Henry E. Riggs, Past-Presidents and Hon. Members, Am. Soc. C. E., F. E. Turneaure, Hon. M. Am. Soc. C. E., and M. L. Enger, M. Am. Soc. C. E.

In September, 1885, he returned to the University of Illinois as assistant professor of engineering and mathematics. He taught a wide range of subjects, which at different times included mathematics, surveying, engineering drawing, contracts and specifications, roads and pavements, railroad engineering, mechanics and materials, hydraulics, tunneling and explosives, water supply, and sewerage.

In 1886 Professor Talbot, together with the late Professor Ira Osborn Baker, M. Am. Soc. C. E., founded the Illinois Society of Engineers at the University of Illinois. Professor Talbot was a faithful member and active worker in this society for fifty-six years, and was the last of the charter members. He was executive secretary from 1886 to 1889, a member of the Executive Board from 1886 to 1891, and President in 1890 and 1891. He also was active as chairman and member of many important committees throughout his long membership.

In 1890 he was appointed professor of municipal and sanitary engineering, in charge of theoretical and applied mechanics. After the era of expansion in engineering schools began, mechanics and engineering materials absorbed his attention even more than sanitary engineering, and without a change in his title the emphasis of his work continued to be placed on mechanics and materials. For more than forty years he moulded and inspired generations of young men and was a leader in the development and advances of this growing engineering school. Always he regarded teaching as the most important part of his life work. He continued in this position until he reached the age limit of the University in September, 1926, when he retired from teaching and administration and was made professor of municipal and sanitary engineering, emeritus. After his retirement he continued until his death to be actively engaged in directing an extensive research program.

On June 7, 1886, Professor Talbot was married to Virginia Mann Hammet in Camargo, Ill. She had been a classmate at the University of Illinois, and was a member of Alethenai Literary Society, a member of the College Senate, and president of the Oratorical Association. Four children were born to them: Kenneth Hammet Talbot, Assoc. M. Am. Soc. C. E., civilian engineer, Brooklyn (N. Y.) Navy Yard; Mildred Virginia, wife of Professor H. J. Gilkey, M. Am. Soc. C. E., of Iowa State College, Ames, Iowa; Rachel Harriet, wife of Dean Harald M. Westergaard, M. Am. Soc. C. E., at the Harvard Graduate School of Engineering, Cambridge, Mass.; and Dorothy Newell, wife of Mr. Warren F. Goodell, farm manager at Urbana. Mrs. Talbot was a charming, cultured woman, devoted to her husband and children. She died on December 4, 1919.

Before the turn of the century Professor Talbot had made contributions to the engineering profession in a number of fields that brought distinction to him and to the College of Engineering. One of the earliest was a formula for areas of waterways for bridges and culverts. Another was a formula for rates of maximum rainfall. Both formulas have been often quoted and widely used. They bear his name today. A small treatise on a very flexible method for laying out easement curves at the ends of circular curves (first described by him in 1891) was published in 1899 as "The Railway Transition Spiral" and has gone through several editions. It has been used by many railroads. His pioneer work in sewage treatment by means of septic tanks later made it possible for municipalities to contest certain patent claims on methods and principles of sewage disposal. During this period his investigations provided methods for the standardization of testing paving brick for strength and abrasion. Before the age of forty, he had made important contributions in hydrology, railway engineering, sanitary engineering, and testing materials.

Prior to 1900 Professor Talbot also developed a background of experience and a ripened judgment which prepared him to take full advantage of the increased facilities that were provided when the Engineering Experiment Station was organized at the University of Illinois. He was active and influential in the formation of the Station. His leadership in formulating policies, ideals, and methods made the Engineering Experiment Station an immediate success. A comprehensive and thorough investigation on reinforced concrete, conducted and directed by him, was started in 1903 and continued for many years on reinforced-concrete beams, slabs, columns, footings, pipes, frames, and buildings. This experimental work became a principal source of the early knowledge on which the properties and requirements for the design of reinforced-concrete structures were based by engineers and engineering organizations and on which principles and methods of practice were formulated. The conception of relations existing between the strength of a concrete mixture and items involving the absolute volume of the cement, the fine and coarse aggregate, and the voids in the mixture, put forth in a paper in 1921, and in a later bulletin, has proved useful to concrete engineers. Tests of stone, brick and concrete, the investigation of steel columns and timber stringers, and a variety of other experimental and analytical work have added to engineering knowledge. Contributions were also made in experimental hydraulics.

A notable piece of research that Professor Talbot directed was the investigation of railroad track, commonly called "Stresses in Railroad Track." This investigation was conducted with a view to obtaining definite and authoritative information on the properties, mode of action, and resistances developed in the various parts of the track structure (rail, ties, ballast, and roadbed) under locomotives and cars moving at various speeds. At the time the work was begun, comparatively little of a scientific nature was known of the stresses in rail and other parts of the track or of the effect on the track of the many variations in action of the rolling stock in its operation. Through twenty-seven years, with the help of a trained staff, a multitude of tests were made with various types of locomotives and cars on track of more than twenty railroads in various parts of the United States, and experimental work was also conducted in the laboratory. Data from all these tests were interpreted and coordinated with analytical treatment to establish principles and findings. In addition to various minor reports of this engineering research, Professor Talbot prepared seven formal reports all of which have been printed in the *Proceedings* of the American Railway Engineering Association and part also in the *Transactions* of the American Society of Civil Engineers. This research project produced reliable knowledge on the interrelation between track and rolling stock and thus aided in putting on a more nearly rational basis the design and construction of the track structure to carry locomotives and cars under modern traffic conditions, as well as giving valuable information applicable to the design of rolling stock. It has been characterized as one of the most significant contributions ever made to the scientific knowledge of railroads.

Professor Talbot's written contributions to the engineering profession are along numerous lines. The reports of the University of Illinois Engineering Experiment Station researches on concrete and reinforced concrete are given in seventeen Station bulletins, with five other bulletins on hydraulics, timber, steel columns, etc. Reports on concrete, reinforced concrete, cast-iron water pipe, methods of testing, and other topics may be found in the *Proceedings* of the American Society for Testing Materials. The report of the first Joint Committee on Standard Specifications for Concrete and Reinforced Concrete, in the preparation of which he participated, was published by several technical societies, including the American Society of Civil Engineers.[2]

Professor Talbot exercised a far-reaching influence on engineering developments through committee activity in engineering societies. Taking a leading part in the work of the first Joint Committee on Concrete and Reinforced Concrete (1904–1916) as a representative of the Society, he was influential in formulating principles and methods of design based on the tests he had made and upon other data and analyses. As chairman of the subcommittee on design, he formulated and advocated many of the views that were adopted by the committee. The report of this committee exercised a marked influence, among engineers and architects, on the ideas and practices in engineering design and on building regulations, in the pioneer period of reinforced-concrete construction. Most of the fundamentals of design then put forth are still accepted. The tests of reinforced concrete made at the Illinois laboratory were widely used by engineering schools and thus the information spread rapidly to engineering offices. In the field of testing materials, he was active in the American Society for Testing Materials from its beginning in 1898 and took a leading part in the work of several of the technical committees that have done constructive work. In sanitary engineering, in railway engineering, and in municipal lines he contributed to technical committee work and in other ways.

In the fifty-four years of his membership in the American Society of Civil Engineers, Professor Talbot rendered a yeoman's service in the technical and educational work. His was hard work rather than honorary assignments, and he performed his duties with his usual ability and thoroughness. In 1918 he was appointed a Member of the John Fritz Medal Board of Award; in 1919 he was appointed a Society representative on the American Engineering Council; in 1920 he served on the Society committee for a questionnaire on a development committee, and in the same year he was chairman of the Committee on Honorary Membership. In 1921 he served as the chairman of the Society committee on technical activities, and in the same year he was made a member of the Research Committee in which capacity he served for about five years. In 1924 he was appointed to confer with the National Bureau of Standards on cement, and was appointed to the Society's special committee on that subject in the same year. In 1936 he served on the committee to consider a memorial for the late William Cowthorne Unwin,[3] Hon. M. Am. Soc. C. E., and in the same year he also served on the committee on specifications for portland cement. In 1927 he was appointed on the Society's committee on soils.

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[2] *Proceedings*, Am. Soc. C. E., October 1924, p. 1153.

Professor Talbot attained high rank among engineering teachers. He was influential in the Society for the Promotion of Engineering Education from its formation in 1893, and held various offices including that of president. He was president of the American Society for Testing Materials from 1913 to 1914. He was a member of a number of other engineering societies, including the Institution of Civil Engineers, American Society of Mechanical Engineers, Western Society of Engineers, American Railway Engineering Association, American Concrete Institute, American Water Works Association, American Public Health Association, and American Association for the Advancement of Science. In all these he has given service in one way or another by writing and direction.

Professor Talbot, by reason of ability, aptitude in teaching, scholarly attainments, and discriminating judgment, was an active participant in the planning and in the development of the College of Engineering of the University of Illinois. He made a noteworthy contribution in the development of laboratories dealing with the testing of materials and structural elements. Starting with a small testing machine in 1888, the facilities have grown to those of the laboratory now housed in the building which bears his name. The development of an hydraulics laboratory whose useful and unique features have been transferred to the new building was another contribution. In both of these early laboratories, and their successors, the variety and completeness of the facilities and the productiveness of the research activities testify to the usefulness of the developments. Few laboratories have produced as great a variety of experimentation and as influential a series of experimental researches.

In his long career as an administrator, Professor Talbot selected and trained many men for teaching and research. He took a great interest in the men on his staff and used various means to promote their progress and development. Those associated with him could not be unaffected by the force of his example, by his high ideals, and by his strong personality. Among those who worked with Professor Talbot are many men in all parts of the world who have become leaders in education and research.

After taking up teaching work, he continued during vacations and at other times to engage in engineering work on railroad construction, pavements, sewerage and water works, and reinforced-concrete design and construction. He acted as consultant to cities and business organizations on various occasions. He served on a board to determine the type of structure for the Galveston (Tex.) causeway and was one of a board in 1927 to make a preliminary report on the location of a bridge over the San Francisco Bay between San Francisco and Oakland, Calif. However, his research and administrative work and his connections with technical committees so occupied his time and energy that he limited the principal contributions of his life to engineering education, engineering research, and the utilization of the fruits of research through engineering society channels.

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[3] For memoir, see *Transactions*, Am. Soc. C. E., Vol. 99 (1934), p. 1391.

In his long career as teacher, investigator, scientist and engineer, Professor Talbot received a great many honors and awards. His fine work was recognized by the granting of honorary memberships by the Institution of Structural Engineers, London, in 1913; by the American Society for Testing Materials in 1923; by the Illinois Society of Engineers in 1924; by the American Society of Civil Engineers in 1925; by the Western Society of Engineers in 1927; by the American Water Works Association in 1930; by the American Concrete Institute in 1932; and by the American Railway Engineering Association in 1933.

In 1932 he was made a national honorary member of Chi Epsilon fraternity. On April 30, 1932, Professor Talbot was made a national honorary member of the Triangle fraternity, a fraternity of engineers which has granted this honor to only two engineers since its organization in 1907. He was granted the degree of Doctor of Science by the University of Pennsylvania at West Philadelphia in 1915; the degree of Doctor of Engineering by the University of Michigan at Ann Arbor in 1916; and the degree of Doctor of Laws by the University of Illinois in 1931.

In 1924 he was granted the Washington Award of the Western Society of Engineers "For pre-eminent services in promoting the public welfare, for his life work as student and teacher, investigator and writer and for his enduring contribution to the science of engineering." In the same year he was granted the George Henderson Medal by the Franklin Institute in Philadelphia, Pa., for "Invention in Railway Engineering," and a tablet was placed on the Urbana and Champaign Sanitary District Building stating, among other things, "On this site, 1897, The Champaign Septic Tank was Built. Designed by Prof. A. N. Talbot. It was among the first of its kind in the country." In 1925 a bronze plaque was awarded Professor Talbot by the American Railway Engineering Association which read "An appreciation to Arthur Newell Talbot, worker in research and scientific advancement," and was accompanied by resolutions reciting "* * * its high appreciation of you as a scientist, and teacher, and investigator and organizer, and, last but not least, as a man." In 1928 Professor Talbot was awarded the Henry C. Turner Medal by the American Concrete Institute "For outstanding contributions to the knowledge of reinforced concrete design and construction." In 1932 the Society for the Promotion of Engineering Education awarded Professor Talbot the Benjamin Garver Lamme Medal for "Achievement in Engineering Education." In 1937 a bronze tablet was given to Professor Talbot by students. It read "Honors the Achievements of Arthur Newell Talbot and His Contributions to Engineering and the Prestige of the College of Engineering, University of Illinois." In the same year Professor Talbot was awarded the

John Fritz Medal by the United Engineering Societies with the citation, "Moulder of men, eminent consultant on engineering projects, leader of research and outstanding educator in civil engineering." In 1938 the Materials Testing Laboratory at the University of Illinois was renamed by the Board of Trustees the Arthur Newell Talbot Laboratory "In honor of Professor Talbot whose achievements in teaching and in research during a period of half a century enriched the lives of students and advanced the science of engineering." In 1942 a bronze plaque was presented to Professor Talbot on his eighty-fourth birthday by The American Railway Engineering Association.

Arthur C. Willard, president of the University of Illinois, in his address at the funeral of Professor Talbot rendered a profound tribute in the course of which he said:

> "His achievements as Professor of Municipal and Sanitary Engineering in the College of Engineering have contributed more to the development and distinction of that College and its Engineering Experiment Station than the work of any other man. His many distinguished colleagues in the faculty both past and present would be the first to approve this statement. * * *
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> "But the influence of this great teacher and investigator extended far beyond the classroom, the laboratory and the campus of his alma mater. For more than 60 years he contributed brilliantly to the scientific knowledge, the standards and the ideals of the profession of Civil Engineering. The list of his scientific and educational writings including reports, bulletins, papers and discussions, embraces more than 325 titles in many different fields of Civil Engineering. It is hardly conceivable that one man could have produced so much even in a long life time of more than fourscore years.
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> "His interests were never superficial as a critical appraisal of any of his writings, even those of fifty years ago, will reveal. Anything Doctor Talbot ever wrote is still sound and authoritative background for our engineering practice of today. * * *
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> "Yet this man was no recluse working alone and apart from the affairs of men. During the most active years of his University career, from his return to the University in 1885 to 1926 when he retired from teaching and administration and became Professor emeritus, he trained and inspired generations of young men. Today these men revere his memory and carry on his work and his ideals through their own teaching, research and professional activities. You will find them everywhere the world around, spreading the fame of a great engineer. Doctor Talbot was no stranger in this modern world of ours. Few men had a broader or more distinguished acquaintance than he had in a half dozen professional fields. He inspired respect and confidence. His judgment and advice were constantly sought by individuals, associations and societies. He traveled far and devoted much time and energy to the affairs of these professional organizations. * * *
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> "Such a record of honors would fully establish the name of Talbot as one of the greatest in the history of the University, but the list is not complete. On the 21st of April, 1938, the Board of Trustees breaking all precedents by using the name of a living person, placed the name of Arthur Newell Talbot on the Materials Testing Laboratory to signalize for future generations of students the enduring contributions of this man to engineering education, research and practice. If the College of Engineering during the 74 years of its existence had graduated only one

> student, and that student had been A. N. Talbot, the State of Illinois would have been well repaid for its entire investment in the College of Engineering.
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> "The career of a most remarkable man has ended; he will long be remembered as a distinguished scholar, a master investigator, a fearless leader and a splendid gentleman."

The deep interest which Professor Talbot took in the professional societies is evidenced by the fact that, in the fifty-four years of his membership in the American Society of Civil Engineers, he failed to attend only three of the 139 national meetings held by the Society after his admission. His friendly attitude, his warm smile, and his genuine modesty made him one of the most widely known and best-beloved members of the Society. During all the years of his membership in the American Railway Engineering Association, he never missed a meeting. He went to Chicago on March 16 to attend the 1942 meeting. On the evening of March 17 he suffered a heart attack at the Palmer House. He was removed to the Passavant Hospital and was apparently showing signs of improvement until Wednesday, April 1, at which time he suffered a stroke from which he failed to rally and passed away on Friday, April 3.

Professor Talbot was elected a Member of the American Society of Civil Engineers on April 4, 1888, and an Honorary Member on November 16, 1925. He served as Director in 1909–1910, as Vice-President in 1911, and as President in 1918.

OTIS ELLIS HOVEY, Hon. M. Am. Soc. C. E.[1]

Died April 15, 1941

Otis Ellis Hovey was born of sturdy Vermont stock in East Hardwick, Vt., on April 9, 1864. His father, Jabez Wadsworth Hovey, who was a farmer and schoolteacher, early imbued Otis with an interest in scientific pursuits and was determined that his son should enjoy all the educational advantages which he himself had been denied. Consequently, after finishing his public school education at East Hardwick, Otis entered the Chandler Scientific School at Dartmouth College, Hanover, N. H., in 1882 and was graduated with the degree of Bachelor of Science in 1885. Throughout his undergraduate work, he attained grades of Phi Beta Kappa rank, but, because of a ruling of the fraternity, he was ineligible for membership. Appropriately enough, he was elected to honorary membership in this fraternity in 1937.

Otis was determined to become a civil engineer from his early boyhood. When a railroad location survey party ran its line through East Hardwick in the early 1870's, young Otis was a constant and fascinated bystander, and it is quite probable that this childhood experience started him on the path to the high engineering achievement which was to be his in later life. After his graduation from Dartmouth, he decided to acquire some practical engineering experience before studying for his professional degree. For one year he worked as engineer of the Hoosac Tunnel and Wilmington Railroad Company in Massachusetts and Vermont. At this time he designed his first bridge, an eighty-foot timber truss, relying on the fundamental background of mathematics and physics which he had acquired in his undergraduate work. Realizing that bridge design was his professional ambition, he spent the following year as draftsman and designer for the Edgemoor Iron Company at Wilmington, Del. The late Onward Bates,[2] Hon. M. and Past-President, Am. Soc. C. E., was then chief engineer of this company, and, when the time came for Mr. Hovey to enter engineering school, Mr. Bates offered him an attractive raise in salary and promotion "to test his determination." As it was throughout his life, Mr. Hovey's determination was firm, however; he declined the tempting offer and entered the Thayer School of Civil Engineering at Dartmouth College. Here he made a fine record, standing at the head of his class, and receiving the degree of Civil Engineer in 1889.

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[1] Memoir prepared by William P. Kimball, Assoc. M. Am. Soc. C. E., and Charles F. Goodrich, M. Am. Soc. C. E.

So outstanding was his record that the late Robert Fletcher,[3] M. Am. Soc. C. E., director of the Thayer School, recommended him for a teaching position at Washington University in St. Louis, Mo., before he had completed his requirements for the engineering degree. Accordingly, Mr. Hovey was engaged as an instructor in civil engineering, and accomplished his final semester's studies by mail. In addition to his "correspondence" course, he taught a heavy schedule of engineering subjects, was in charge of the university's testing laboratory, which was used both by the students and for commercial testing, and did special work with the late Prof. J. B. Johnson,[4] M. Am. Soc. C. E. This experience, although an interlude in his career, was indicative of his entire life, for he was a prodigious worker and possessed a remarkable capacity for both intensive and varied activities. He was an able instructor and an inspiring lecturer, both of which accomplishments meant much to him in later years, for his contacts with younger engineers in his office gave him constant opportunity to teach. Also, he was widely sought as a visiting lecturer in engineering institutions throughout the East. After a year at Washington University, he resigned to enter his real lifework of bridge designing. In 1890 he took a position as designing engineer for the late George S. Morison,[5] Past-President, Am. Soc. C. E., and he never again left this field throughout the following fifty-one years of his life.

During his six years' association with Mr. Morison, Mr. Hovey was responsible for the design of many small bridges and miscellaneous structures and of several major projects. Considering the fact that he was only twenty-six years of age when he went to work for Mr. Morison, amazingly heavy responsibilities were placed on him. A few of the larger structures for the design of which he was entirely responsible were the approaches of the Memphis Bridge at Memphis, Tenn.; the superstructure of the Bellefontaine Bridge across the Mississippi River above St. Louis; the yards and structures for the entrance of the Chicago, Burlington and Quincy Railroad Company into St. Louis; the Alton Bridge across the Mississippi River at Alton, Ill.; and the Leavenworth Bridge across the Missouri River at Leavenworth, Kans. During the winter of 1895 he prepared a complete design and cost estimate for a 3,200-ft suspension bridge across the Hudson River at New York, N. Y. The Brooklyn Bridge was then the longest span ever built, and here was a thirty-year-old engineer actually designing a span more than double the length of that great bridge. It is interesting to note that neither the span length nor the project of bridging the Hudson River, visualized by Mr. Hovey, was accomplished for nearly forty years.

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[2] For memoir, see *Transactions*, Am. Soc. C. E., Vol. 102 (1937), p. 1484.

[3] For memoir, *ibid.*, p. 1538.

[4] For memoir, *ibid.*, Vol. LI (1903), p. 454.

[5] For memoir, *ibid.*, Vol. LIV (1905), p. 513

In 1896 he became engineer for the Union Bridge Company at Athens, Pa., which was later to merge with other firms to form the American Bridge Company. Here he was in complete charge of designs and shop and erection drawings for a wide variety of bridges and buildings, and also was engaged in estimating and contracting for new work for the company. On one occasion, he traveled to London, England, and contracted for a number of bridges which were erected in South Africa. During this decade Mr. Hovey gained much experience, and no better training for his long duties with the American Bridge Company could have been obtained than his years as designing engineer with Mr. Morison and the Union Bridge Company.

His thirty-four years with the American Bridge Company, in an official capacity, constituted by far the largest part of Mr. Hovey's professional career. He entered the employ of this company when it was formed in 1900 by the combination of several small independent bridge companies, one of which was the Union Bridge Company. From 1900 to 1904 he was engineer of design for the American Bridge Company at Pencoyd, Pa.

The United States Steel Corporation, of which the American Bridge Company became a subsidiary, was formed in 1901. During this difficult period Mr. Hovey's organizing and engineering ability was of inestimable value in shaping the course of the engineering department of the latter company. In 1904 his office was transferred to New York, and in 1907 he was appointed assistant chief engineer of the American Bridge Company, which position he held until 1931 when he was appointed consulting engineer and was relieved of many of his administrative duties so that he could devote his entire time to the important engineering problems which arose. In 1934, at the age of seventy, he retired from service with the United States Steel Corporation.

During this thirty-four years of service with the American Bridge Company, Mr. Hovey's duties were confined principally to bridges and especially to movable bridges. The practices in the design and construction of movable bridge machinery in the various plants which formed the company were widely divergent, and it was his no small task to unify them. In so doing Mr. Hovey issued a series of "notes" over a period of several years, on the design of heavy, slow-moving machinery. These "notes" became the text from which the several offices in the company designed movable bridges and their operating machinery. Many of these "notes" were later incorporated, together with other data, in Mr. Hovey's book, in two volumes, entitled "Movable Bridges," which is considered the most authoritative work on the subject ever published. In 1935 he also published an authoritative book on "Steel Dams."

Another one of Mr. Hovey's "specialties" was turntables. All the engineering problems on this subject in the company found their way to his desk, and he designed a series of turntable centers or pivots which were built in considerable number and which he had patented. The different types were named with his initials, the "O-Center," "E-Center," and "H-Center."

In 1904 the company commissioned Mr. Hovey to visit London and Constantinople, Turkey, to investigate structural steel trade conditions, particularly in connection with a proposed bridge over the Golden Horn at Constantinople.

The American Bridge Company had two or three difficult and interesting contracts, the engineering for which fell in Mr. Hovey's province: The construction of the Panama Canal emergency gates in 1912, the strengthening of the Williamsburg Suspension Bridge in New York City in 1914, and the renewal of the four large counterweight sheaves on the vertical-lift span over the Willamette River in Oregon. All of these necessitated intricate designing of special tools and equipment to enable the work to be done without taking the bridges out of commission. Mr. Hovey was justly proud of these jobs, not because they brought him any personal glory, but because they were difficult engineering problems to be solved, not only to the satisfaction of his company and its customers, but more especially to his own satisfaction.

As Mr. Hovey once stated, an engineer for a fabricating company cannot take public credit for his own individual ability. Any glory resulting from his efforts, no matter how arduous, must be subordinated to the welfare of his company. Thus, credit must rest with the engineers employed by the owner of the project. This troubled Mr. Hovey not one whit, as he loved engineering and the effort entailed in the solution of engineering problems. Those who were associated with him remember the many times when, after weeks of hard work, arduous and difficult computations had been solved successfully, and joy, pride, and elation shone from him and infected every one near, not because of any praise that might come to him, but because of the pure pleasure of having done a difficult job to his own satisfaction.

Throughout his thirty-four years with the American Bridge Company, one of his greatest interests lay in helping his younger colleagues along in their chosen profession. His associates had their lives greatly enriched by his character and their professional enthusiasm fired by his example. In the early days of the company and throughout his incumbency as assistant chief engineer he knew practically every engineer and draftsman in the company and lost no opportunity to seek advancement for the deserving. When he visited the various offices and drafting rooms, his stalwart figure and genial and kindly manner drew all to him like a magnet. He was never known to exhibit impatience when he was asked for either professional or personal advice.

A description may well be included here of Mr. Hovey in the office which he occupied for so many years on the fourteenth floor of 71 Broadway, New York City: A tall, dignified gentleman is seated between a large roll-top desk and a table spread with yellow calculation sheets, blueprints, and a slide rule, enveloped in clouds of smoke from a "reverse curve" pipe, in a room lined with huge filing cabinets and bookcases kept in meticulous order and well indexed. A subordinate enters to whom is paid not the slightest attention, except for the one word "Yes?" so that his caller has a minute or two to look at the white-bearded kindly face, which resembled so much that of ex-Chief Justice Hughes. At the end of a certain set of calculations the Doctor removes his nose glasses, lays them carefully on the table, leans back comfortably in his chair with long legs crossed, apparently with all the leisure in the world, and says, "What is it?" The visitor is instantly at ease and knows that he has been silently invited to sit down and ask what he will and that he has the undivided attention of a great man who is the best of listeners and full of sympathy. After the discussion the subordinate goes away happy and absolutely sure he has the right answer and the Doctor replaces his glasses and his pipe and again is immersed in mathematical calculations.

His retirement from service with the American Bridge Company was accepted with a keen sense of loss by all his associates. They missed not only his technical advice, but also his example as a gentleman who always lived up to a very high code of ethics and sense of moral and religious responsibility to his Maker, his family, his friends, his business associates, and himself.

With his characteristic energy and enthusiasm, he opened an office of his own as consulting engineer, also maintaining all his contacts and taking part in committee work for the various engineering societies of which he was a member. In 1930 he was appointed to the Board of the Engineering Foundation, and in 1937 he was elected Director, following the late Alfred D. Flinn,[6] M. Am. Soc. C. E., the only previous Director of this organization. Mr. Hovey's duties necessitated considerable traveling for the purpose of visiting laboratories and shops where experiments were being conducted. His great knowledge of the properties of engineering materials and his experience in testing made his advice almost invaluable in this work. His long term as Treasurer of the American Society of Civil Engineers gave him valuable training in the handling of the funds of the Foundation, and his engineering ability was helpful in the directing of its activities. Because he was a master of the English language far beyond the abilities of most engineers, the Annual Reports which he prepared are examples of conciseness of arrangement and clear detail that are difficult to equal. His dictation was truly remarkable, being very fast, yet accurate. His choice of words and arrangement of phrases, sentences, and paragraphs were well nigh perfect, and he had the peculiar ability to lay emphasis when and where needed. He had a distinct horror of split infinitives.

In addition to his day-in and day-out duties, Mr. Hovey was able, because of his robust health, to engage in many other engineering activities (for which he is perhaps more widely known) such as writing books; lecturing at Yale University, New Haven, Conn., at Princeton University, Princeton, N. J., and at Dartmouth College; acting as treasurer for the American Society of Civil Engineers from 1921 to his death; and serving as a member of the Board of

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[6] For memoir, see *Transactions*, Am. Soc. C. E., Vol. 103 (1938), p. 1787.

Overseers for the Thayer School of Civil Engineering connected with his alma mater, Dartmouth College. The broad scope of his engineering interests is partly indicated by the memberships he maintained in several professional societies. In addition to his membership in the American Society of Civil Engineers, he was also a member of the American Society for Testing Materials, the American Railway Engineering Association, the American Welding Society, the American Institute of Consulting Engineers, the Thayer Society of Engineers, and the Engineers' Club of New York.

Mr. Hovey was also active in many pursuits not associated with engineering. He was an enthusiastic member of the New England Society of New York City. He was interested in amateur photography. Always loyal and devoted to Dartmouth College, he kept in close touch with all his classmates by personal correspondence and regular attendance at class reunions. A deeply religious man in the highest sense of the word, for a long period he was a deacon in the Presbyterian Church. His participation in these varied activities always took the form of service and contribution and might be characterized by his own words when he said, "The secret of contentment and happiness is hard work minus worry and jealousy."

His most cherished avocation was music. Both an enthusiastic and a talented flutist, he was long a member of the New York Flute Club, at one time holding the position of vice-president of this organization. Many members of the Society will remember the performance at the convention in Denver, Colo., during the summer of 1940, in which he participated, accompanied by his daughter, Mrs. Ellen Hovey Davis. A previous occasion was reported[7] as follows:

> "* * * he assisted with J. E. Greiner [M. Am. Soc. C. E.] playing his Stradivarius violin and with Ralph Modjeski [M. Am. Soc. C. E.[8]] at the piano, in an evening of music which is still vividly remembered not only for its artistry but for the celebrity of the performers."

Mr. Hovey's eminence in the engineering profession and his high scholarly ability were recognized in various official citations. As previously mentioned, he was elected an honorary member of Phi Beta Kappa at the 150th anniversary of that organization in 1937. Ten years previous to that, he was awarded the degree of Doctor of Engineering by his alma mater, Dartmouth College. In 1933, he was honored with the degree of Doctor of Science by the Clarkson College of Technology at Potsdam, N. Y. Finally, in 1937, he was elected to Honorary Membership in the American Society of Civil Engineers. The words of John P. Brooks, Assoc. M. Am. Soc. C. E., in presenting Dr. Hovey for Honorary Membership in the Society, give a picture without which no description of Dr. Hovey could be adequate. He said in part:

> "He is a man born to wealth—to a wealth of character as sturdy as his native hills; to a wealth of ancestry that with others made our early history what it is; to wealth of family and local traditions honoring industry and honesty; to a wealth of appreciation of what is fine and beautiful in

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[7] *Civil Engineering*, January, 1938, p. 57.

[8] For memoir, see *Transactions*, Am. Soc. C. E., Vol. 106 (1941), p. 1624.

> art and in nature; and to a wealth of mentality and ingenuity worthy of his pioneer forebears."

In his home life, Dr. Hovey found the peace and happiness which were the natural expression of his serene and loving character. Indicative of the part played in his life by his wife, Martha Owen Hovey, was the dedication of his books on movable bridges which read: "To my wife, who inspired the beginning of its preparation and whose encouragement has made its completion possible, this book is affectionately dedicated." He is survived by his widow and their two children, Ellen Hovey Davis and Otis Wadsworth Hovey, Assoc. M. Am. Soc. C. E.

Dr. Hovey himself can hardly be described more clearly than in his own words expressing the ideals of an engineering career, for to those who knew him he was the embodiment of his own high standards:

> "After having followed the profession of engineering for more than fifty years, one is tempted to look back and try to assess the satisfactions of such a career * * * . A successful engineer must possess and develop a high type of character. He must be meticuluously honest with himself and others. He must be obedient to the laws of nature so far as he can grasp them. He must be logical, thorough, industrious, inventive, practical, firm in well-grounded opinions, yet tolerant of the views of others and able to associate comfortably with them. At the same time, he must see visions and dream dreams, and clearly visualize the embodiment of the dreams, whether in structures, machines, organizations, business or human relationships * * *. While the financial rewards may not be large, the inner satisfactions are great. The engineer feels that he has at least done a little to advance civilization and the enjoyment of life by his friends and the public in general."

Dr. Hovey was elected an Associate Member of the American Society of Civil Engineers on April 4, 1894; a Member on January 3, 1900; and an Honorary Member on October 4, 1937.

JOSEPH BARLOW LIPPINCOTT, Hon. M. Am. Soc. C. E.[1]

Died November 4, 1942

Throughout the southwestern United States, broad areas of green and fertile lands reclaimed from desert waste stand as a memorial to Joseph Barlow Lippincott, a pioneer engineer in water supply and irrigation.

Joseph Barlow Lippincott was born in Scranton, Pa., on October 10, 1864, the son of Joshua Allen and Harriet (Phillips) Lippincott. His paternal forebears were of Quaker stock and first came to Connecticut from England in 1642. Mr. Lippincott attended Dickinson College at Carlisle, Pa., from 1880 to 1882, when the family moved to Kansas. He then entered the University of Kansas at Lawrence where his father was professor of mathematics and later chancellor.

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[1] Memoir prepared by Kenneth Q. Volk, M. Am. Soc. C. E., and Edgar Alan Rowe, Assoc. M. Am. Soc. C. E.

While attending the University of Kansas Mr. Lippincott had a severe case of pneumonia and his health was impaired. To recuperate he went to Texas and worked on a cattle ranch. He often stated that this outdoor work, which consisted mainly of driving cattle over the famous hills from Texas north to the Santa Fe railroad, was the hardest in his life but he regained his strength and enjoyed perfect health for the remainder of his life. Returning to the University of Kansas he was graduated in 1887 with the degree of Bachelor of Science in Civil Engineering. Later, in 1914, he was honored by this institution with the degree of Civil Engineer.

Upon his graduation he worked as track engineer in Missouri for the Santa Fe Railway System, and, in the spring of 1889, was promoted to the position of division engineer, building fifteen miles of line for the Missouri, Kansas and Texas Railway Company. In the fall of the same year he became assistant engineer on the Kansas City Belt Line.

Moving to the Southwest, from 1889 to 1892, Mr. Lippincott was topographer for the U. S. Geological Survey in charge of making topographic maps in New Mexico and California. This work did much to develop Mr. Lippincott's initiative and resourcefulness. At that time this section of the United States was sparsely settled, and the engineers in charge had not only to assemble an organization but also to provide housing and transportation (which was largely by mules) and to finance practically the entire operation. This latter feature was particularly difficult because of the low wages paid and the long delays that sometimes occurred between the submission of expense accounts and the receipt of reimbursing funds. In 1893 he became assistant engineer for the Bear Valley Irrigation Company, on the construction of an early irrigation project on the headwaters of the Santa Ana River in Southern California.

During the time that Mr. Lippincott was with the U. S. Geological Survey he was assigned to make a reconnaissance survey of several hundred thousand acres in the wild and unexplored territory in the Bitterroot Range of the Rocky Mountains on the crest of the Continental Divide on the boundary between Idaho and Montana. There were no roads and few trails. After spending a summer making surveys and photographs of this area for the purpose of establishing National Forest Reserves, Mr. Lippincott was returning with his pack train when the small mule, on which his camera and all of his plates were packed, stopped to gaze over a cliff. The larger mules in the pack train crowded up to enjoy the view and pushed the small mule over the cliff. Mr. Lippincott, with his heart in his mouth, went down into the ravine expecting to find an injured pack animal and all of his photographic efforts during the summer smashed, but he was delighted to find the little mule unhurt, upside down, wedged between two small pine trees. After righting the mule, he inspected the pack and to his amazement found that none of the plates had been broken.

In 1895 Mr. Lippincott accepted the position of resident hydrographer for the State of California, with the Hydrographic Branch of the U. S. Geological

Survey. At this time there were practically no stream-flow measurements available in this area. He established many of the gaging stations on the principal streams and secured local observers who served without pay. He traveled up and down the state in a buckboard visiting stations, many of which were in out-of-the-way places, selecting new stations, maintaining those installed, and collecting records with very meager funds. In an effort to secure private rainfall records within the state he wrote to all the postmasters asking them to put him in touch with local observers in order to make these data available to the public. The rainfall and stream-flow records obtained in this early day were of great value in developing plans for the future water supply of California.

This early hydrographic work gave Mr. Lippincott an intimate knowledge of California which was of great value to him. Recognizing the possibilities and the importance of the development of adequate water supplies, he decided to devote his career to water supply engineering, and at the time of his death he was recognized as one of the outstanding engineers of the country.

In the fall of 1897 and the spring of 1898 Mr. Lippincott served as a member of the board of consulting engineers of the City of Los Angeles, Calif., in connection with litigation in the San Fernando Valley over the pueblo rights of the city. This was a very important case and had much to do with the future growth of Los Angeles. The city maintained that it was a "pure" pueblo founded in compliance with Spanish laws and customs, and, therefore, the use of waters of the Los Angeles River was vested in the city as successor to the pueblo in 1850. In this case the State Supreme Court held that the city had the right to the surface flow of the river or its underground flow at any point from its source or sources to the southern boundary of the city (for domestic and municipal uses) and that this right was not limited to the territory covered by the original pueblo but was attached to any additional territory from time to time brought within the city's limits. Mr. Lippincott served as a member of the board on the appraisal of the water works for the Los Angeles City Water Company, the acquisition of which was the beginning of the present Bureau of Water Works and Supply for the City of Los Angeles.

In the spring of 1899 Mr. Lippincott prepared an exhaustive report for the Geological Survey on the water supply, available reservoir sites, and irrigable areas on the Gila River in southwestern Arizona. After this assignment he made a comprehensive study of the development of the surface waters of the upper Santa Ana River and the San Bernardino artesian basin. The results of both of these studies were published in the *Water Supply Papers* of the U. S. Geological Survey.

With the organization of the U. S. Reclamation Service in 1902 he became one of the group of hydraulic engineers who aided in making its early projects successful. At this time there were few hydrologic records available and much pioneering was necessary. Mr. Lippincott became supervising engineer of all Reclamation Service activities in the Pacific Coast region extending from the Klamath River in Oregon to the lower Colorado River in Arizona and California. His work included the preparation of plans and estimates of cost and construction of the Klamath and the Yuma projects until July, 1905.

The need for additional water for the City of Los Angeles was brought forcibly to the attention of the water commissioners in July of 1904 when, for a ten-day period, the daily consumption exceeded the inflow into the city's reservoirs by nearly four million gallons. In 1905, Mr. Lippincott was appointed with O. K. Parker, M. Am. Soc. C. E., and the late William Mulholland,[2] M. Am. Soc. C. E., to a board of engineers which was to make recommendations in regard to the expansion of the water supply, investigating seven sources. The decision was that the Owens River was the nearest adequate supply and that to augment the city's needs temporarily at lower cost by withdrawals from adjacent artesian areas would result in the curtailment of the development of the surrounding country.

In July, 1906, Mr. Lippincott left the Reclamation Service to become assistant chief engineer of the Owens River Aqueduct. At the time this project was initiated, the City of Los Angeles had a population of 240,000. The planning and construction of an aqueduct, 250 miles in length, to deliver a mountain supply from the Sierra Nevadas to the city was a tremendous undertaking, requiring engineers of vision, energy, and administrative ability to carry it through to completion. When completed, the Owens River Aqueduct had a capacity of 400 cu ft per sec and consisted of open-canal, covered-conduit, steel and concrete siphons, sixty miles of tunnels, five dams, storage reservoirs, and three hydroelectric power plants for construction purposes. One of the first municipal projects of magnitude to be built by day labor under the supervision of its engineering staff, this was a monumental undertaking for such a small city. Mr. Lippincott had much to do with the final location of the aqueduct, was in charge of the preparation of the detail plans and estimates of cost, and initiated and established a new system of construction cost keeping together with the bonus system which developed keen rivalry between different divisions and aided in breaking many world's records for rapid tunnel construction.

From the beginning it was recognized that all the work could be completed within the estimated five-year period with the exception of the 164 tunnels, particularly the Elizabeth Lake Tunnel which was five miles in length. The theory of the bonus system was to pay the tunnel men a fair daily wage and in addition share with them approximately 50% of the estimated savings in case the base footage of progress was exceeded. This sharing of the benefits with the men resulted in improved relations with them; the crews themselves eliminated drones and did not tolerate loafing. Thus, the foremen's and superintendents' duties were confined largely to obtaining necessary supplies and equipment. The higher wages earned by the men under the bonus system also reduced the amout of labor turnover and attracted the best tunnel men in the country.

It was estimated before work started that the cost would not exceed $23,000,000 and that the aqueduct could be completed within five years. Construction work began in 1908, and water was delivered to the city through the aqueduct in 1913, well within the cost and time limit originally set, thereby establishing an amazing record for efficiency and economy.

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[2] For memoir, see *Transactions*, Am. Soc. C. E., Vol. 101 (1936), p. 1604.

Mr. Lippincott was instrumental in the construction by the city of a cement plant at Monolith, Calif., located approximately halfway from the ends of the aqueduct, which eliminated high freight charges for transportation from existing cement mills. In the western end of the Antelope Valley and at Haiwee near the location of the aqueduct there occurred natural deposits of soft volcanic ash called "Tufa." To reduce the length of cement haul from the railroad by long line teams he recommended the construction of grinding mills for blending this natural cement up to 50% with portland cement. This greatly lowered the cost of cement used on the aqueduct. However, the blended cement required more care in handling because it was slower to harden The distinctive feature of this blended cement was its continued growth in strength with age. The J. James R. Croes Medal of the American Society of Civil Engineers was awarded to Mr. Lippincott for his paper entitled "Tufa Cement as Manufactured and Used on the Los Angeles Aqueduct."[3]

Upon the completion of the aqueduct in 1913, Mr. Lippincott entered private practice at Los Angeles, specializing in water supply for irrigation and municipal uses under the firm name of "Engineering Offices of J. B. Lippincott," in which company he was active until the time of his death. During this period he was engaged as consultant on water supply investigations and water works construction for many of the western cities including Los Angeles, San Francisco, Oakland, Alameda, Berkeley, Santa Barbara, San Diego, Long Beach, Riverside, San Luis Obispo, Ventura, Orange, Santa Ana, Whittier, Avalon, and Fullerton, in California; Denver, Colo.; El Paso, Tex.; Phoenix, Ariz.; and others. In addition, he was consultant for many irrigation districts throughout the west including San Fernando Valley Irrigation District, Santa Fe Irrigation District, Vista Irrigation District, Lindsay Strathmore Irrigation District, and numerous others.

Mr. Lippincott made several trips to the Hawaiian Islands where he was employed by the Oahu Sugar Company in connection with the Waiahole water supply and the construction of the Nuuanu reservoir. He also served as consultant on the Wahiawa Water Company's spillway project and later was employed by the City of Honolulu in connection with its domestic system.

His work also included investigations and a report on the diversion of a large water supply from one watershed to another near Fairbanks, Alaska, for a large hydraulic mining operation. For many years he was consulting engineer for the Automobile Club of Southern California and aided in the development and improvements of the early highway system of the State of California. Also, in this connection, he wrote reports on traffic relief and planned improvements for the street system of the City of Los Angeles.

At the outbreak of the World War in 1917 Mr. Lippincott closed his engineering offices and offered his services to the army. As he was beyond the age limit for active service, he aided in the war effort by organizing the William E. Hampton Company. Here he again demonstrated his administrative ability, serving as chief engineer and manager. This company constructed Camp Kearny, California, the flying field at North Island near San Diego for the Bureau of Military Aeronautics, and improvements at Fort MacArthur and Fort Rosecrans in California. Upon the completion of this work, being ambitious to do something further to aid in the war effort, he joined the staff of the U. S. Housing Corporation acting as supervisor on large housing projects at Pittsburgh, Pa., and Bethlehem, Pa., until their abandonment at the end of the war. He then returned to private practice in Los Angeles.

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[3] *Transactions*, Am. Soc. C. E., Vol. LXXVI (1913), p. 520.

With the outbreak of World War II, Mr. Lippincott's offices again became engaged in war construction projects in association with Oliver G. Bowen, M. Am. Soc. C. E. This firm acted as "Architect-Engineer" in drawing up the plans and supervising the construction of Anti-Aircraft Firing Center at Camp Haan, California, the Mojave Desert Firing Range, California, the Basic Air Corps Training Center at Salt Lake City, Utah, and improvements to March Field, California. The plans and layout for the Marysville (Calif.) cantonment, the design of the Modification Center at Daggett, Calif., for the Douglas Aircraft Corporation, and the plans for a navy shipyard for the Pollock-Stockton Shipbuilding Company at Stockton, Calif., also were prepared by this firm. Mr. Lippincott participated actively in this work, and it was while making a tour of inspection of cantonment work at Camp Haan that he passed away suddenly on November 4, 1942.

Mr. Lippincott served on many boards of consulting engineers for the federal government, the State of California, and various municipalities. He was a member of a board of engineers on the preparation of the original flood control plans for Los Angeles County in 1915 and later, for a time, was consultant during the construction of this project. His office also prepared plans for flood control and water conservation for Santa Barbara County and Orange County, California. As a member of the consulting board of the State of California, he aided in the development of a state-wide water plan. At the time of his death he was consulting engineer for the International Boundary Commission between the United States and Mexico, as well as consulting engineer for the U. S. Engineer Office, Los Angeles District, on the design and construction of numerous large flood control dams and appurtenant works. He was also a member of the advisory committee on the U. S. Weather Bureau for the National Sciences Advisory Board.

As a recognized authority on the design and construction of hydraulic fill dams, he was consultant on many of the large structures of this type in California, not only for the owners but also for contractors. He enjoyed the respect of the contracting profession, particularly for his thoroughness, honesty, and fairness in decisions.

Exceedingly interested in public affairs, Mr. Lippincott gave generously of his time to the nation, his state, and his city. In addition to his services on engineering boards, he was state commissioner for the Sixth District Agricultural Association of California from 1911 to 1915. From 1910 to 1917 he was a park commissioner of the City of Los Angeles and had much to do with the development of the splendid park system of that city. One of the Los Angeles park projects for which Mr. Lippincott was largely responsible, and of which he was most proud, was the municipal golf course at Griffith Park—one of the first public courses developed in the United States. He was active in the development and opening up of recreational areas in the Forest Reserves surrounding Los Angeles, making numerous surveys at his own expense for the benefit of the public. From 1918 to 1920 he was civil service commissioner of the City of Los Angeles, and, from 1924 until his death, he was a member of the water and power resources committee of the Los Angeles Chamber of Commerce. This committee has given much time and effort to aid in the adoption of policies for the development of the City of Los Angeles.

A frequent contributor to engineering literature, he prepared numerous papers for the Society's *Transactions* and discussed those of others. While with the U. S. Geological Survey, he wrote several *Water Supply Papers* including "Storage of Flood Waters on the Gila River, Arizona," "Water Supply Problems of Santa Barbara, California," "Underground Water Supplies of the San Bernardino Valley, California," and "California Hydrography." He also contributed many articles to various technical publications.

Mr. Lippincott took an active interest in the affairs of the Society and was willing at any time to advance its interests. By 1914 the City of Los Angeles had started its rapid growth, and numerous engineers were attracted to this region. Mr. Lippincott felt that the Society should have formal representation in the Southwest and, in order to discuss this idea, gave a dinner at his home to the members of the Society residing in this area. This meeting resulted in the formation of the Los Angeles Section, of which he was the first president. He participated actively in the meetings of the Local Section, and his talks not only covered his engineering subjects thoroughly, but also were full of human interest.

In addition to his Section activities he served on numerous committees appointed by the Society, as follows: Chairman of the Committee on a National Reclamation Policy, 1927–1931; member of the Executive Committee of the Irrigation Division, 1928–1931 (chairman in 1909); member of the Committee on Meteorological Data, 1931–1936; member of the Power Division's Committee on Silting, 1927–1934; and member of the Committee on Dams, 1931–1936.

Some of Mr. Lippincott's outstanding traits of character, which in a large measure contributed to his success, were a sense of humor, a gracious smile, an ease of manner which enabled any one to approach him, an ability as an administrator, a desire to help others, an energetic devotion to the immediate task at hand, and a sense of responsibility. An applicant for a position in his organization was always sure of a kind and considerate reception. If he had no work available, he endeavored to the best of his ability to place worthy applicants in positions with others. He was fearless, honest, and fair; his decisions were based on careful thought and preparation. Regardless of the outcome, he did what he thought was right.

Mr. Lippincott had a host of friends not only engineers, but also businessmen, industrialists, civic leaders, and working men. Known affectionately as "J. B.," he always had time for a pleasant word and was particularly helpful when any friend was in trouble and in need of assistance. A very observant man, throughout h[illegible] collected a large amount of valuable engineering data, which he shared [illegible]ly with his fellow engineers. He was beloved not only for his engineering accomplishments and knowledge but also for his ability as a storyteller. Truly a man's man, he greatly enjoyed social meetings with his many friends. An excellent public speaker, he was called upon many times to address gatherings for the discussion of public affairs and proposed civic improvements. To his associates and employees he was more of a father than an employer, always ready with advice and material assistance when needed. He loved hunting and fishing and other out-of-door sports. In his later life he particularly enjoyed his weekly golf games with his friends.

In October, 1929, at the time of the World's Engineering Congress at Tokyo, Japan, he visited the Orient, spending three months traveling throughout Japan and China. He also traveled extensively through Mexico and made trips to Europe, Cuba, the West Indies, and Alaska. In addition to technical subjects, he loved good literature, being particularly interested in history and biography.

His particular hobby, however, was ranching. At the time of his death he was the owner of four citrus groves in the San Fernando Valley in California. In 1913 he planted out his first orange grove in the valley and entered into the study and practice of citriculture with all the thoroughness and energy that marked his engineering activities. He won the admiration and respect of his neighboring farmers whose livelihood depended upon their groves and acted as Director of the Fruit Exchange in the area where his ranches were located. He loved to spend his weekends on his ranches.

Mr. Lippincott was particularly fortunate when he was married in 1890 to Josephine P. Cook of Kansas City, Mo., who enjoyed traveling all over the West with him, frequently living in out-of-the-way places and in tent camps. Mr. Lippincott often told of their first child's crib—a soap box suspended from a tent pole. Mount Josephine in the San Gabriel Forest Reserve was named for Mrs. Lippincott in recognition of her services in preparing mathematical tables for the U. S. Geological Survey. They had two children, a daughter, Rose, who passed away in 1919, and a son, Joseph Reading Lippincott, a lieutenant in the U. S. Naval Reserve, who was on duty with the Navy in the Philippines at the time of their invasion by the Japanese in 1941. No definite word of their son's fate has been received since that date, and, undoubtedly, anxiety over Lieutenant Lippincott's welfare hastened Mr. Lippincott's death. He passed away on November 4, 1942, at Arlington, Calif., at the age of 78.

He was a member of Beta Theta Pi and Tau Beta Pi fraternities, the Los Angeles Country Club, and the California Club.

Mr. Lippincott was elected a Member of the American Society of Civil Engineers on December 6, 1899, and an Honorary Member on October 11, 1936.

FREDERICK ABELS ALLNER, M. Am. Soc. C. E.[1]

DIED JULY 18, 1942

Frederick Abels Allner was born in Skytal, Bohemia (then Austria), on July 19, 1882, the son of Edward Abels Allner and Rosa (Kohn) Allner. He died on July 18, 1942, at the Union Memorial Hospital, in Baltimore, Md.

His accomplishments as an engineer and his participation in the work of various national engineering societies over a long period earned for him recognition as one of the leading engineers in the electric utility field. This recognition was international in scope; Mr. Allner numbered among his friends and acquaintances many European engineers, as well as the notables of his profession in the United States and Canada. Apart from engineering and its related activities, his interests and his services were many. He served his city and his state loyally, and no worthy cause appealed to him in vain.

In 1905, Mr. Allner received the degree of Mechanical Engineer from the Technical College at Prague (then in Austria). Later (1914), he received from the same school the degree of Doctor of Technical Science (Electrical Engineering). He pursued postgraduate courses at the Polytechnic College in Zürich, Switzerland, and at the Electro Technical College in Karlsruhe, Germany. His formal education was not confined to an engineering curriculum. Study of the classics provided a background of interest on which he often drew and to which he continued to add. He never forgot the Greek and Latin he learned in his early years. Besides his native German, he spoke French fluently and had some knowledge of Swedish and Italian. His English vocabulary was exceptionally large, and he used it readily and skilfully.

Mr. Allner began his engineering career with brief periods of employment on design and testing of electrical and mechanical power equipment in Prague and Karlsruhe. At the end of 1907 he came to the United States. Here he hoped to secure a position with the Westinghouse Electric and Manufacturing Company. A depression had decreased the opportunities for engineering employment, however, and forced him to look elsewhere for a job.

After a short interval of odd jobs at drafting in New York, N. Y., he found a position in the shops of the Montreal (Quebec, Canada) Locomotive Works. For six months in 1909 he was employed by the Shawinigan Water and Power Company in the Province of Quebec. Here he entered the field of hydroelectric engineering, and his abilities were soon recognized.

In November, 1909, with several associates from the Shawinigan Company, he went to Lancaster County, Pennsylvania, where a hydroelectric development (later called Holtwood) on the Susquehanna River was nearing completion. The Pennsylvania Water and Power Company was organized in 1910, and Mr. Allner served this company successively as assistant chief engineer, general superintendent, and vice-president (1933 to 1940). He became general superintendent of the Safe Harbor Water Power Corporation at its formation in 1930, and was elected president of this company in 1940.

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[1] Memoir prepared by Edward S. Loane, Assoc. M. Am. Soc. C. E.

Mr. Allner's association with the power developments on the Susquehanna River covered a continuous period of more than thirty-two years. In this time he had either direct or general supervision over the design, construction, and operation of the hydroelectric plant at Holtwood, completed with a capacity of 150,000 hp in 1924; a 30,000 hp steam plant at the same location, completed in 1925; the Safe Harbor project, started in 1930 and now (1942) having a capacity of more than 300,000 hp; and an orderly development of the transmission system reaching to Baltimore, Md., Washington, D. C., the Pennsylvania Railroad at several points, and to York, Lancaster, and Coatesville in Pennsylvania. Mr. Allner contributed much to the economic utilization of these various projects through his negotiation of power contracts with other utilities and through his early recognition of the complementary nature of steam and hydroelectric power supplies and the benefits to be had from their proper coordination.

The coordination of steam and hydroelectric supplies marked a major advance in economic design and operation of electric systems. Mr. Allner was one of the first engineers, or perhaps even the first, to understand clearly the principles involved in such coordinate operation, to apply these principles to a major utility system, and to explain them adequately in the engineering literature.

In the design and operation of the Holtwood and Safe Harbor projects, many other pioneer technical features and methods have been utilized and developed. The advantageous utilization of such new developments can be traced, in many cases, to Mr. Allner's imaginative concept of the problems to be met and to his quick realization of the possible value of untried methods. He fostered a spirit of investigation among the engineers of his organizations, and he was a willing and helpfully critical listener to new ideas advanced by the companies' employees.

In addition to his other direct contributions to the success of the Susquehanna River developments, Mr. Allner had great responsibility in the careful selection and training of engineering and operating personnel. It was characteristic of him, through tact and understanding, to bring out the abilities of the men in his organization to the advantage of both the employee and the company. There existed a spirit of loyalty and mutual respect between him and his associates.

Mr. Allner devoted himself unstintingly to the affairs of the local and national engineering societies. His services to, and his influence on, the Engineers Club of Baltimore were remarkable and far-reaching. He served two years (1930 to 1932) as president. In the depression of the 1930's, it was largely his leadership which enabled the Club not only to weather a critical period but also to expand its facilities and activities. The Club conferred upon him an honorary life membership—an honor shared only with Herbert Hoover, Hon. M. Am. Soc. C. E., and the late John F. Stevens, Past-President and Hon. M. Am. Soc. C. E. He was an active member of other local professional organizations and of three of the Founder Societies—Civil, Electrical, and Mechanical. He was a past-president of the Maryland sections of both the Society and the American Society of Mechanical Engineers. He was a member of the Newcomen Society of England, Engineers Club of Philadelphia, Pa., and Pennsylvania Electric Association.

In 1941, during his term as president of the Maryland Section of the Society, Mr. Allner's was the guiding hand in the arrangements for the Spring Meeting of the Society at Baltimore. To the preparations for this event he gave the same careful and detailed attention, imagination, and enthusiasm that assured success for his undertakings. By his persuasiveness and subtle guidance he obtained from his many assistants a maximum of useful effort. They found themselves accomplishing tasks that to them had seemed almost impossible. As an organizer of engineering meetings, he was without an equal, as this latest of his many activities testified.

In civic affairs he displayed characteristic interest; even where his interests were greatest, he tended to keep himself in the background. He was the civil engineering member of the Maryland State Board of Health; a member of the Smoke Advisory and Appeal Board of Baltimore, the State Aviation Commission, and the Baltimore Association of Commerce; and a member of various commissions and committees engaged in civilian war work. He earlier served as a member of the Baltimore Safety Council and as the American Engineering Council representative on the National Resources Committee.

Busy as he was with professional activities, he never was limited by these labors. He found time for games, extensive travel, family interests, and social intercourse. Chess and color photography were favorite pastimes. He was a member of the Baltimore Country Club, Maryland Club, and University Club of Washington, D. C.

On June 28, 1914, he was married to Adele B. Brock. He is survived by his widow and a son, Frederick Abels Allner, Jr.

Mr. Allner was elected an Associate Member of the American Society of Civil Engineers on December 2, 1914, and a Member on December 9, 1940.

ROBERT ADAMS ALLTON, M. Am. Soc. C. E.[1]

Died June 24, 1942

Robert Adams Allton was born in Nashua, N. H., on March 9, 1892, the son of Norris J. and Della (Haywood) Allton. His early education was obtained in the public schools of Nashua, and he continued his studies at the Massachusetts Institute of Technology (M. I. T.) in Boston, Mass., from which he was graduated in 1913 with the degree of Bachelor of Science in Sanitary Engineering.

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[1] Memoir prepared by James H. Blodgett, Assoc. M. Am. Soc. C. E.

Immediately after graduation, in June, 1913, he was employed as assistant engineer in the New York State Department of Health. Here he received his first practical experience in sanitary engineering by inspecting and reporting on municipal water supplies, sanitary conditions at summer resorts, and other matters pertaining to the protection of public health.

A year later, in June, 1914, he was employed by Langdon Pearse and Samuel A. Greeley, Members, Am. Soc. C. E., sanitary engineers of Chicago, Ill., first as assistant engineer and eventually, by reason of his ability, as principal assistant engineer. His talents as an outstanding engineer began to manifest themselves in his work on the many and sundry problems in design, construction, and operation of water supply, sewerage systems, and sewage treatment and refuse disposal projects (which were handled by the firm of Pearse and Greeley) throughout the Middle West.

During his period of employment with this firm, Mr. Allton served in the armed forces of the United States. He had previously enlisted in the Illinois National Guard, and was called for service on the Mexican border from June to November, 1916, his capacity in this connection being Sergeant of Battery D, 1st Illinois Field Artillery.

He returned to work with the firm of Pearse and Greeley upon the close of the Villa episode and his Mexican border experience, but was again called to active service in May, 1917, just after the entry of the United States into World War I. Not long after his return to service, he was nominated, and successfully met, the requirements for a commission, first as Second Lieutenant, then as First Lieutenant, and later as Captain. He was assigned to Headquarters Company, 331st Field Artillery, at Fort Sill, Okla.

Being of an impatient nature and wishing to take a more intimate and aggressive part in the conflict, Captain Allton applied for more active service, hoping to go overseas with those who were then being sent. However, this was not to be, for his knowledge of the science of engineering and his ability to impart this knowledge to others marked him as too valuable a man for combat duty. He was commissioned a Major and made assistant director, Department of Gunnery, School of Fire, at Fort Sill. As he often afterward said, rather dejectedly, "There's where I fought the war."

Mr. Allton returned to the firm of Pearse and Greeley, after his discharge early in 1919, and continued with it until December of the same year. In January, 1920, he was engaged as assistant to the president of Frank D. Chase, Inc., Industrial Engineers, of Chicago. With this firm he broadened his engineering experience through the design and construction of such special structures, and reports on such special problems, as are to be found in the practice of industrial engineering. In September, 1921, while engaged in this last connection, he was seriously injured in an automobile accident and was forced to retire and return to his home in Nashua for an extended period of recuperation.

Early in 1923 Mr. Allton was engaged by the City of Akron, Ohio, to design a sewage treatment plant suitable to the needs of that fast-growing metropolis. With his usual acumen and boundless energy he made a noteworthy contribution to the science of sanitary engineering in the construction of a 33-mgd plant of the Imhoff tank and trickling filter type. He incorporated several unique but practical devices therein, such as detritus tanks for grease and grit removal and a new vitrified clay pier and grill block type of collector for the trickling filters.

His work at Akron attracted considerable attention in engineering circles, and in January, 1928, he was sought and employed by the City of Columbus, Ohio, to head an organization then being formed to design and construct a new sewage treatment plant for that city. He worked with the late John H. Gregory,[2] M. Am. Soc. C. E., consulting engineer to the city, and the late R. H. Simpson,[3] M. Am. Soc. C. E., chief engineer for the city.

Mr. Allton, as sewage treatment engineer, worked out the basic design and general plan for solving a major problem of pollution of the Scioto River which flows through the City of Columbus. As a part of this plan he recommended, designed, and built, as an adjunct to the sewerage system, the first storm stand-by tanks to be installed in the United States. In spite of a considerable amount of political bickering and unethical practices by mercenary, non-engineer "specialists" in sewage treatment, the recommendations of Mr. Allton and his staff were accepted, and a $4,000,000 activated sludge plant, which he conceived, was constructed. This plant remains as a monument to his genius in the field of sanitary engineering.

His record of accomplishments as a sanitary engineer had by this time gained him national prominence, and he was sought after by various engineers and firms for advice and assistance in their problems. However, "Bob," as he was affectionately known by his many friends and associates, preferred to stand by his first love, the business of sewage treatment. In 1938 he was appointed consultant to the City of Columbus and was placed in charge of operation of the plant he had designed and built.

He acted in this capacity until May, 1941, when he was forced, because of impaired health, to retire in an attempt to improve his physical condition. As he had always been very active, both mentally and physically, he stated that his forced retirement was the hardest work he had ever done. "Bob" Allton died quite suddenly at Hudson, Ohio, where he had settled down to try and find the solution to the toughest problem he had yet encountered—that of regaining his health.

No tribute would be more acceptable to "Bob" Allton than to state that he was a gentleman, that he was always good company, and that he honored his profession by the conscientious application of his keen perceptions and knowledge of engineering to the problems which he encountered. He was a man who respected all men and in turn received their respect and loyalty.

Mr. Allton was awarded the Rudolph Hering Medal by the Society (with Messrs. Gregory and Simpson, and O. Bonney, M. Am. Soc. C. E., who collaborated with him) for a paper entitled "Intercepting Sewers and Storm Stand-By Tanks at Columbus, Ohio."[4]

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[2] For memoir, see *Transactions*, Am. Soc. C. E., Vol. 102 (1937), p. 1552.

[3] For memoir, *ibid.*, Vol. 99 (1934), p. 1507.

[4] *Transactions*, Am. Soc. C. E., Vol. 99 (1934), p. 1295.

He was a member of the American Chemical Society, Ohio Society of Professional Engineers, Engineers Club of Columbus, Optimist Club, and M. I. T. Club of Central Ohio.

In 1924 he was married to Hilda Marie Joyce of Hudson, who survives him. He also leaves a brother, Donald Allton.

Mr. Allton was elected an Associate Member of the American Society of Civil Engineers on April 19, 1920, and a Member on May 13, 1929.

THOMAS CLARK ATWOOD, M. Am. Soc. C. E.[1]

DIED FEBRUARY 2, 1943

Thomas Clark Atwood was born in Provincetown, Mass., on December 5, 1874, the son of John W. and Abbie (Clark) Atwood.

Mr. Atwood's early education was obtained in the public schools of Malden, Mass. He attended Massachusetts Institute of Technology at Cambridge and was graduated in 1897 with the degree of Bachelor of Science in Civil Engineering.

His first work after graduation was as rodman and instrumentman for the Metropolitan Water Board in Boston, Mass., from 1897 to 1900, with time out from June to September, 1898, for service in the U. S. Navy as Gunner's Mate, Second Class, on the *U. S. S. Inca,* during the Spanish-American War. His choice of this branch of the service was perhaps a heritage from his seafaring ancestry—of which he was justly proud.

From August, 1900, to September, 1901, Mr. Atwood was employed as inspector by the Boston District Engineer Office of the U. S. Army Engineer Corps, in charge of the removal of ledges from the main ship channel of Boston harbor. In September, 1901, he became assistant engineer, Bureau of Filtration, Philadelphia, Pa., on design of filter plants, pumping stations, pipe lines, etc., and on the construction of the Torresdale Filtration Plant. This work was completed in 1904. From May, 1904, to January, 1907, Mt. Atwood acted as division engineer, Pittsburgh (Pa.) Bureau of Filtration. In this capacity, he was engaged on the design of the filter plant and supervised the design and construction of a filtered-water reservoir, river crossing, connections to Brilliant Pumping Station, and the installation of station equipment.

In January, 1907, he became assistant designing engineer for the Board of Water Supply, New York, N. Y. In that position he was in charge of studies on filtration plants and of the design of the fourteen steel pipe siphons of the Catskill Aqueduct. In December, 1909, he became division engineer in charge of (1) preparatory field work on the extension of the Catskill Aqueduct within New York City limits and (2) the beginning of construction of the city tunnel. In July, 1911, Mr. Atwood was placed in charge of part of the design of Jerome Park Filter Plant as division engineer for the New York City Department of Water Supply, Gas, and Electricity.

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[1] Memoir prepared by William M. Piatt, M. Am. Soc. C. E., and H. Raymond Weeks, Atwood & Weeks, Inc., Durham, N. C.

In 1914 Mr. Atwood was employed by Yale University, New Haven, Conn., as construction manager of the "Yale Bowl." He then became a consultant on various projects and, in this capacity, was supervising engineer for the Yale Committee of Twenty-One, Inc. His association with Yale University became the foundation for his later splendid work in connection with many educational institutions in the Southern States.

In 1918, Mr. Atwood became supervising engineer for the U. S. Navy at the U. S. Destroyer Plant in Atlantic, Mass. After World War I, he became district plant engineer for the U. S. Shipping Board, Emergency Fleet Corporation, in Baltimore, Md.

In 1920, when American hosiery mills were forced to re-equip their mills for the production of silk stockings instead of the cotton ones of pre-war days, the Durham Hosiery Mills in Durham, N. C. (a chain of fourteen mills and at that time the largest manufacturer of cotton hosiery in the world) began an extensive building program, and Mr. Atwood became chief engineer directly in charge of construction. When this work was suspended during the panic of 1921, Mr. Atwood formed "The T. C. Atwood Organization," which was immediately entrusted with a new $5,000,000 permanent improvement program for the University of North Carolina, at Chapel Hill. In August, 1923, Arthur C. Nash became the architect of the organization which was, at that time, incorporated as Atwood and Nash, Inc. The firm continued to do the architectural and engineering work at the University and designed in all over $9,000,000 worth of buildings and other improvements. When Mr. Nash retired in 1930, Mr. Atwood reformed the organization under the name of Atwood and Weeks, Inc., with H. Raymond Weeks in charge of architectural work. The building program of the University of North Carolina continued to be the chief concern of the organization, and at the time of Mr. Atwood's death it was engaged in considerable work for the Naval Pre-Flight Training Program at that institution. The firm also designed several other projects in North Carolina, among which were the Durham Federal Post Office and Court House; the Raleigh Memorial Auditorium; the new fireproof buildings for the State Hospital at Goldsboro; a large number of public schools throughout the eastern and central part of the state; and new buildings for Peace Junior College (Raleigh), Louisburg College (Louisburg), and Buies Creek Academy (Buies Creek).

During 1932–1933, Mr. Atwood was engineering examiner for the Reconstruction Finance Corporation in Washington, D. C.

Of particular personal interest to Mr. Atwood was the firm's work on such stadiums as the Kenan Stadium at the University of North Carolina and the football stadiums at the University of Alabama at University and the University of Georgia at Athens. Also he was consultant on the Duke University Stadium at Durham.

A paralytic stroke, from which he later fully recovered, forced Mr. Atwood to retire from active work in 1939, but, in 1941 as his contribution to the war effort, he was able to take active charge of the enlargement of the Wright

Automatic Machinery Plant for the Defense Plant Corporation. Although he was apparently in good health, a quite unexpected attack of leucemia caused his death on February 2, 1943.

Mr. Atwood belonged to the First Presbyterian Church and was a member of the Board of Elders.

Unselfish in his devotion to his profession, he gave fully of his time to committee work for the various organizations with which he was actively associated.

Among his professional affiliations were the New England Water Works Association, the Boston Society of Civil Engineers, Connecticut Society of Civil Engineers, the Municipal Engineers of New York, the American Association of Engineers, and the North Carolina Society of Engineers. He was made an Honorary Member of the North Carolina Society of Engineers in January, 1940.

Mr. Atwood was married to Grace M. Winslow on January 1, 1901, in Malden. Two sons were born to them, Theodore W., on October 13, 1907, and Robert Clark, on June 5, 1912. Mrs. Atwood and his sons survive him.

Mr. Atwood was elected an Associate Member of the American Society of Civil Engineers on April 3, 1907, and a Member on October 31, 1911. He became a Life Member in January, 1942.

GEORGE ELLSWORTH BARROWS, M. Am. Soc. C E.[1]

DIED SEPTEMBER 24, 1941

George Ellsworth Barrows was born in Buffalo, N. Y., on July 19, 1879, the son of William A. Barrows and Mary (Ellsworth) Barrows.

Following his education in the public schools of Buffalo and the University of Buffalo, he began his engineering career with the group who were designing and constructing the Pan American Exposition. He always looked back upon this period as a happy introduction to his life's work, for he was working under the inspiring leadership of Newcomb Carlton, who later became president of the Western Union Telegraph Company. This was at the turn of the century, and he remained with the Exposition until the fall of 1901.

During the next ten years, Mr. Barrows had various assignments in the field of railroad engineering, both in the location and construction of lines. It was the period when the railroad network was still being extended, and the old lines were being redesigned to serve the vastly expanded commerce of the United States. From January to August, 1905, he was employed as division engineer in charge of location and preparation of estimates for the New York, Pennsylvania and Southwestern Railroad Company. Upon the termination of this work he went with the Delaware, Lackawanna and Western Railroad Company as assistant engineer. During this time, the Lackawanna's lines into

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[1] Memoir prepared by Norman M. Herthe, Edward P. Lupfer, and Charles J. Pollard, Members, Am. Soc. C. E.

Buffalo were elevated to eliminate grade crossings, and Mr. Barrows had no inconsiderable part in the supervision of this undertaking.

In 1910 he resigned as assistant engineer with the Lackawanna to become a partner of Ellsworth Brothers, Civil Engineers, of Buffalo. This firm had been established by two uncles of Mr. Barrows, the late Alfred B. Ellsworth,[2] Jun. Am. Soc. C. E., and John F. Ellsworth, who were engaged in a general engineering practice. Here he engaged in a considerable consulting practice for municipalities in the design and supervision of sewer and pavement construction and other projects. As an outgrowth of his work for municipal corporations, Mr. Barrows had not been long in private practice before he was called upon to act as expert in a matter of railroad valuation, an assignment for which his previous training had excellently fitted him. It developed that there was a growing need for competent engineers in this department of the profession, and from that time on he devoted the major part of his time to it. In 1916 his firm was employed to undertake the valuation of the buildings and equipment of all the industrial properties in the City of Buffalo for the Department of Finance and Accounts. During the next twenty-five years, Mr. Barrows was engaged in a large number of valuation projects for counties, towns, and cities throughout New York State in connection with public utility and industrial properties, as well as for utilities and other corporations that were setting up their own capital accounts. Until his death, he gave thoughtful and constructive attention to the problems of valuation, including not only the structural aspect, but also the more intricate subject of economic depreciation.

This brief recital covers only the bare outline of some of the activities which characterized his relation to his profession. It has been said that the proper study of mankind is man, and this study Mr. Barrows prosecuted always with keen delight. As he was possessed with that rare quality of entering sympathetically into the problems and difficulties of others, he was always being sought for counsel, and his circle of friends and acquaintances became unusually extensive. In his reflections on religion in the engineering profession, Arthur E. Morgan, M. Am. Soc. C. E., has called marked attention to the narrowing tendencies of modern engineering education, which has been so concerned with stresses and strains in materials that it has not had time to take a philosophical view of those other stresses in the social fabric. The engineer has been so engrossed with his own studies that, of all professional men, those words might be applied best to him, "* * * these ought ye to have done, and not to leave the other undone."[3] Happily there are signs that this is changing; but this restricted outlook never characterized George Barrows, for always he was interested in people, especially young people and their advancement. With material means and with advice and counsel, he aided the work of the Young Men's Christian Association, of which he was a director in Buffalo until his death. He had been a Trustee of Westminster Church, in Buffalo, and in many ways promoted the work of this spiritual center of his native city. The number of young engineers to whom he gave a start or a lift would make an impressive showing. All the men who later became his partners owe him a debt of gratitude for beginning their engineering careers under his inspiring guidance.

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[2] For memoir, see *Proceedings*, Am. Soc. C. E., Vol. XIX, 1893, p. 48.

[3] Matthew 23 : 23 and Luke 11 : 42.

Mr. Barrows was a lover of art, not only as a collector of fine paintings, but as a creator. With palette and brush he found expression for his delight in the visible world of beauty around him.

On June 15, 1907, he was married to Frances Pollard, who survives him. Also surviving is a daughter, Alice Barrows Francisco.

Mr. Barrows was elected an Associate Member of the American Society of Civil Engineers on May 2, 1911, and a Member on October 9, 1917.

GEORGE HENRY BLAKELEY, M. Am. Soc. C. E.[1]

DIED DECEMBER 25, 1942

George Henry Blakeley, the son of Joseph H. and Mary Ann (Gibson) Blakeley, was born at Hanover, N. J., on April 19, 1865. He acquired a technical education at Rutgers University, New Brunswick, N. J., being graduated from the Scientific Department in 1884. Ten years later, this university conferred upon him the honorary degree of Civil Engineer, and, in 1924, further honored him with the degree of Doctor of Science.

After three years of general engineering practice, in 1887, he started on the path that was to lead to a high place in the steel construction industry, by accepting a position as chief draftsman of the Riverside Bridge and Iron Works in Paterson, N. J. In this capacity, he prepared the working drawings for a number of railroad bridges and buildings.

In 1888 Mr. Blakeley entered the employ of the Erie Railroad Company as assistant engineer of bridges. In 1889 he became assistant engineer of the Passaic Rolling Mill Company at Paterson, and, less than a year later, was appointed chief engineer of that organization. During the seventeen years that Mr. Blakeley was with the Passaic Rolling Mill Company, he designed and directed the construction of numerous railroad and highway bridges and also served as an advisory mechanical engineer on the manufacture of structural steel. While with this company, he laid the foundation for his later achievements. He devoted much study to the possibilities of increasing the efficiency of steel as structural material through innovations in rolling practice and became an authority on both the manufacture and uses of structural steel. In 1897 the Passaic Rolling Mill Company published a *Handbook,* written by Mr. Blakeley, which contained much new and original information on structural steel.

In the course of his search for improved methods of rolling structural steel, Mr. Blakeley became acquainted with Henry Grey, who had developed a mill that permitted rolling steel beams with wider flanges, thus providing greater strength for the weight than the older types of beams. Mr. Blakeley became deeply interested in the possibilities of the Grey mill and envisioned a great future for it. When, in 1906, Charles M. Schwab decided to install these mills, in a move to put Bethlehem Steel Company in the structural steel business, he asked Mr. Blakeley to join the Bethlehem organization as manager of structural steel sales. In this capacity, Mr. Blakeley pioneered among builders the type of beam that has since become standard for framing large structures. In 1927 he was appointed a vice-president of Bethlehem Steel Company, and, when, in 1931, the Bethlehem Steel Company acquired the McClintic-Marshall Corporation, he was made president of the new subsidiary.

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[1] Memoir prepared by W. A. Hazard, Assoc. M. Am. Soc. C. E.

Mr. Blakeley was identified actively with the construction of many well-known bridges and buildings, among them the Delaware River Bridge at Philadelphia, Pa.; the Golden Gate Bridge at San Francisco, Calif.; the Peace Bridge at Buffalo, N. Y.; the Merchandise Mart and the Field Museum in Chicago, Ill.; and the Chase National Bank Building in New York, N. Y.

He was a director of the Bethlehem Steel Corporation and a life trustee of Rutgers University. Also, he belonged to the following organizations and societies: American Society of Mechanical Engineers, American Iron and Steel Institute, Engineers' Club of New York, Lawyers Club of New York, University Club of New York, Chi Psi and Phi Beta Kappa fraternities, Bethlehem Club, Saucon Valley Country Club, and New York Yacht Club.

The breadth of Mr. Blakeley's interests is evidenced by the diversity of his hobbies, which included astronomy, radio, clocks, flower gardening, yachting, and college athletics. He was active in the civic life of Bethlehem, Pa., and contributed generously to charities. Very democratic in his way of life, Mr. Blakeley was endowed with a rich fund of humor and was an excellent story-teller. His genial nature and liberal attitude toward associates, fellow employees, and neighbors won him many friends.

In 1893, he was married to Grace Delia Bogart of Passaic, N. J., who died in 1940. He is survived by a son, Lt. G. Bogart Blakeley, U.S.N.R.

Mr. Blakeley was elected a Junior of the American Society of Civil Engineers on December 4, 1889, and a Member on October 2, 1895.

CAROLUS MORTON BROOMALL, M. Am. Soc. C. E.[1]

Died November 1, 1942

Carolus Morton Broomall was born in Upper Providence, Pa., on March 8, 1870, the youngest son of John Martin and Caroline Louise (Larkin) Broomall. His father was the first judge of Delaware County, Pennsylvania. In 1891, Mr. Broomall was married to Henrietta T. Kraker, who, with his only daughter, Carlota Broomall Matienzo, survives him.

Carolus Broomall early became attached to his profession. When the son was about thirteen years old, his father (the Judge) was planning to build a dam on his property. The Judge owned a transit and engaged a surveyor to stake out the work. The boy was much interested in this project and one morning sneaked out of the house with the instrument to do a little surveying of the dam on his own. When the surveyor arrived, the transit could not be found. Finally, someone discovered that young Carolus had it out at the dam site. The Judge said, "All right, if Carolus can handle the transit in fun, he can handle it in earnest." He sent the surveyor away and turned the job over to the young civil engineer.

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[1] Memoir prepared by Scott B. Lilly, M. Am. Soc. C. E.

The Judge believed that every man should know the law and insisted that all of his four sons study for the bar. Carolus, therefore, was admitted to the bar, but, as soon as he had met his father's requirements, he left law and took up civil engineering, the profession he loved. His legal training, however, did two things for him: It improved a natural aptitude for logic, and it made him despise any chicanery that lowers the ethical standards of a profession.

From 1891 until 1920, he acted as Borough Surveyor of Media, Pa. During a part of this period, he was called the Borough Regulator. In 1920, the town had grown so that it needed a regular Borough Engineer, and he filled this position for twenty years. He felt his responsibility as engineer deeply. Four factors in addition to his professional skill contributed to the successful fulfilment of his duties: First, his legal knowledge; second, his familiarity since childhood with local properties and landmarks; third, his continuous study of engineering problems; and fourth, his sense of civic responsibility.

While he had charge of the Media water works, he safeguarded the health of the citizens with intelligence and fidelity. He had the keenest interest in the subject of water purification and studied each new development as it appeared. He spent long hours planning the betterment of the plant. Sometimes these efforts were appreciated by the Borough officials—at other times not; but he always persevered. As an example of his efforts, Mr. Broomall asked permission to install water wheels to cut down the expense of pumping. After a long and unpleasant struggle, he finally succeeded in securing acquiescence to his plan, and the water wheels were installed, with the result that considerable money was saved.

He was appointed Borough Engineer of the adjoining Borough of Swarthmore, Pennsylvania, in 1922, and continued in that position until his death. To show his broad intellectual interests, it is only necessary to point out that, although he went to the University of Pennsylvania, at Philadelphia, for less than a year, he spoke and read Spanish, Italian, and French. He had studied German and, at the age of 70, began the study of Portuguese.

His lack of a college degree may have changed the course of his life. In 1908, an acquaintance in the engineering department of Drexel Institute (Philadelphia) needed an instructor in surveying field work and asked Mr. Broomall if he would give an afternoon a week to the work. The following year, and, for twelve years thereafter, more teaching was handed over to him. During this period, at one time or another—in the day, summer, or night schools—he taught the following subjects: Plane and field surveying, highway engineering, sanitary engineering, steam engineering, structural design, applied physics, hydraulics, mechanics of materials, strength of materials, geology, and Spanish. He also was connected with the mechanical testing laboratory. Finally, Drexel Institute required that all of its teachers have college degrees. At that time Mr. Broomall could afford neither the time nor the money to return to college, so his career as a teacher ended. In any event, it is doubtful whether he would have taken on a full-time teaching job because that would have interfered with his work as Borough Engineer for Media.

Throughout his life, Mr. Broomall was interested in the Delaware County Institute of Science in Media. This Institute was founded in 1833, and his father, Judge Broomall, had been one of its early presidents. In 1938–1939, Mr. Broomall himself was president. This term followed service as treasurer, vice-president, curator, and editor of the Institute's publication, *The Proceedings*. At the Institute, he had fitted up a laboratory, and also had installed cupboards and cases for his many papers and books. He even kept his violin at the Institute and would practice when no one else was in the building.

Evenings at the Institute, he was always ready to talk to the boys who came in. He would make them welcome and would help them to become familiar with the various pieces of apparatus and with the books and collections. Thus he succeeded in interesting many of them in scientific subjects. It was all very informal and very friendly. It is believed that there are many men and women today who owe their love of research and investigation to his quiet but enthusiastic influence.

Mr. Broomall's interests were many. He was constantly studying physics (with special emphasis on wave motion), mathematics, chemistry, mechanics of materials, thermodynamics, mineralogy and geology, anthropology, biology and medicine, Spanish, French, Italian, Portuguese, and German. Notwithstanding this wide range of interests, he never neglected his study of water supply and sewage disposal since he needed such knowledge as Borough Engineer.

Although Mr. Broomall was proud to be a member of the Society of Friends, he did not go to meeting. Conventional worship simply was not to his liking. He possessed many of the characteristics of the old-time Quaker. He was mild-tempered, with occasional emphatic lapses, modest, cheerful, unaggressive, and peace-loving. He disliked gossip, thought conceit was vulgar, and hated cruelty. Despite his gentleness, he was definitely a "fighting Quaker." He had a fine sense of justice, a very great patience, and a good sense of humor.

In the passing of Mr. Broomall, Delaware County lost a conscientious and influential citizen who had added much to the richness of the life of his community.

Mr. Broomall was elected an Associate Member of the American Society of Civil Engineers on November 9, 1920, and a Member on October 5, 1931.

ROBERT FRANCIS BROWN, M. Am. Soc. C. E.[1]

Died October 29, 1941

Robert Francis Brown was born on February 29, 1896, in Osage City, Kans., the son of Robert Lawrence and Lura (Baker) Brown. He received his preparatory education in the schools of Kansas City, Mo., and entered the University of Missouri, at Columbia, in the fall of 1915. Before the completion of the first semester, however, he withdrew from the University and went to work for the engineering firm of Burns and McDonnell. He reentered the University in the fall of 1916 and completed a full year's work. On June 18, 1917, he enlisted in the 110th Engineers, 35th Division, U. S. Army, and served overseas throughout the first World War. He was discharged on July 10, 1919, with the rank of Corporal, and then reentered the University. After one semester, however, he returned to the employ of Burns and McDonnell.

His work in the Burns and McDonnell office was of a diversified character, covering the entire water and sewage field—design, estimates, and supervision of construction, as well as appraisals and preliminary studies. It culminated in his being sent to Los Angeles, Calif., in 1923 to open a branch office for his firm. Mr. Brown was in general charge of this branch until September, 1925, and constructed intercepting and outfall sewers for the City of Glendale, Calif., a sewer system for the City of Blythe, Calif., water works improvements for the City of Needles, Calif., and valuations and preliminary studies for various municipal improvements.

In the meantime, an old friend in the Burns and McDonnell organization, Leon Reynolds, M. Am. Soc. C. E., had become head of the Department of Sanitary Engineering at Leland Stanford Junior University. As a result of this circumstance, Mr. Brown entered the Department of Civil Engineering at Stanford in the fall of 1925. He was graduated with distinction in January, 1927, and received the degree of Bachelor of Arts. At Stanford he was elected to Tau Beta Pi and Sigma Xi. He was also a member of Delta Tau Delta fraternity.

Following his graduation there were two relatively short engagements—first, a year and a half as steel designer with the Western Pipe and Steel Company, and second, a few months on water purification with Thebo Starr and Anderton, Consulting Engineers, of San Francisco, Calif.

In November, 1928, he joined the engineering staff of the California Water Service Company and remained with that company in various capacities until his death. During the early days of this connection he supervised the construction of the Mallard pumping station and pipe line, filter plant, and reservoir. The cost of this work aggregated more than $1,400,000.

The Mallard development involved 5 miles of 33-in. concrete pipe, 2½ miles of 36-in. steel pipe, a 1,000,000,000-gal reservoir formed entirely by gunite-faced earth embankment, a 27-mgd pumping station, and a 4-mgd rapid sand filtration plant. This work was started during the drought of 1928-1929 and, pending its completion, the system was partly supplied by water brought in barges across Suisun Bay from Benicia, Calif., at a cost far in excess of the sale price. Conditions called for maximum possible speed, and the work was completed in record time. Since its completion in 1930 there has been no indication of weakness or faulty construction in any part of this project—the best evidence of the care and skill with which the entire work was done.

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[1] Memoir prepared by J. A. Wade, M. Am. Soc. C. E.

Following this job, no engineering work of any magnitude was scheduled by the company, and Mr. Brown was made operating manager of the Stockton (Calif.) district. Here he remained for eleven years, representing his company in a territory comprising more than 14,000 consumers. In April, 1941, he was transferred to the main office in San Jose, Calif., and was put in charge of all the company's construction work, which amounted to more than $1,000,000 annually.

Mr. Brown's outstanding characteristic was his unfailing geniality. Nobody ever saw him show anger. He was public spirited to a fault, and prodigal in his gifts of time and energy to any worthy civic project. He was active in the American Legion, serving as Commander of the Karl Ross Post of Stockton in 1939. He was also active in the Stockton Rotary Club and was serving as president in 1941 when he was transferred to San Jose. For several years he served as a Lieutenant in Battery C, 143d Field Artillery, National Guard, and saw active duty during the San Francisco general strike in 1934. He was also active in the affairs of the California Section of the American Water Works Association, serving as director for several years, as secretary-treasurer, and, in 1937, as chairman of the Section.

On his graduation from Stanford University, he was united in marriage with Margaret Turner, daughter of C. A. P. Turner, M. Am. Soc. C. E., a leader in the design and construction of reinforced-concrete structures. Mrs. Brown survives her husband.

Mr. Brown was elected an Associate Member of the American Society of Civil Engineers on December 14, 1925, and a Member on September 29, 1930.

ALFRED DICKEY BUTLER, M. Am. Soc. C. E.[1]

Died September 14, 1941

Alfred Dickey Butler was born on May 10, 1878, at the ranch home of his parents on Moran Prairie, Spokane County, Washington. His father, J. N. Butler, and his mother, Jenny (Dickey) Butler were pioneer residents of Washington coming originally from Ohio. Mr. Butler received his early education in the grade school at Moran Prairie and attended the old Spokane High School in Spokane, Wash.

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[1] Memoir prepared by Robert E. Tobin, Jun. Am. Soc. C. E., and Charles E. Davis, M. Am. Soc. C. E.

He entered the State College of Washington, at Pullman, Wash., in September, 1900, and in 1904 he was graduated, with honors, with the degree of Bachelor of Science in Civil Engineering. Because of his brilliant academic record he was elected to membership in the honorary engineering fraternity, Tau Beta Pi. In 1909 the professional degree of Civil Engineer was conferred upon him also by Washington State College.

During his senior year in college Mr. Butler designed and superintended the construction of a $10,000 sewer system at Pullman, which was finished in February, 1905. Following this he served as instrumentman on a survey for a water supply system for the City of Colfax, Wash. Undoubtedly these first two positions formed the basis for the fact that he later selected municipal engineering as his life's work.

Railroad engineering then comprised a considerable part of Mr. Butler's early experience. From February, 1905, until June, 1905, he worked as instrumentman on railroad location in northern Idaho for a private company. From this position he transferred to the employ of the Great Northern Railway Company, where he was engaged in engineering on maintenance of way until October, 1905. Improvements and extensions by the Great Northern Railway Company at this time were responsible for his transfer to instrumentman and field draftsman. Then in May, 1906, he was transferred from a field position to the company's offices. His engineering duties in the office consisted mainly of drafting on location maps and general details. He continued in this capacity until April, 1907, when he accepted a temporary position as transitman with the Oregon-Washington Railroad and Navigation Company until July, 1907. This temporary work included right-of-way location in Washington between Chehalis and Greys Harbor. In addition to his regular work for the different railroad companies, Mr. Butler also engaged in private railway engineering. During the period from July to September, 1907, he acted as engineer in charge of location of a logging railroad for the Weyerhauser Lumber Company near Tolt, Wash. (abandoned logging town close to Seattle, Wash.).

After September, 1907, Mr. Butler's engineering activities seemed to turn back toward municipal work. He was employed in Seattle from September, 1907, until February, 1908, as an instrumentman and engineer on both harbor development work and layout and construction of streets in Interlocken Park.

Mr. Butler's long association with the City of Spokane began on February 29, 1908, and this date perhaps marked the turning point in his career. While he was employed in the city's municipal engineering office many notable achievements in the design and construction of public works were realized. His inception of numerous engineering structures contributed immeasurably to the welfare and growth of Spokane.

His first association with the city engineer's office was as draftsman in the bridge department. Later, as field engineer, he was placed in charge of a $275,000 project on sewer design and construction. This project was built during the period from February, 1908, to July, 1909. Mr. Butler's ability in the successful completion of this work and his general skill along engineering lines won a promotion to principal assistant city engineer in July, 1909. He served in this capacity until November, 1917.

An aggregate amount of $7,540,000 was expended while Mr. Butler was acting principal assistant city engineer in charge of sewers, street grading and paving, and construction projects. In addition, he was also responsible during this period for several notable improvements and additions to Spokane's municipal water supply system. He was placed in charge of the design and construction of a 20,000,000-gal storage reservoir and four reinforced concrete standpipes, each with a capacity of 100,000 gal. These structures form an integral part of the city's water system and are notable for their pleasing architectural qualities.

On November 1, 1917, Mr. Butler was appointed to the position of city engineer but later, on October 7, 1918, he was granted a leave of absence to become an engineer in the construction division of the U. S. Army during the first World War. On December 1, 1918, he returned to the position of city engineer and served continuously in this capacity for twenty-three years—instituting, supervising, and completing many large projects. More than $11,000,000 was expended in municipal work in general, which included street improvements, sewers, paving, bridges, and other engineering structures.

The City of Spokane is frequently referred to as "The City of Bridges," and Mr. Butler deserves much of the credit for this title. He supervised the completion of the Washington Street, the Trent Avenue, and the Post Street bridges. He also supervised the planning and was instrumental in the construction of the Marne, Downriver, Eleventh Avenue, and Freya Street bridges. He was conscientious in seeing to it that the taxpayer received full value for his money and that all construction work met the required standards. At one time there was a difference of opinion between the contractor and the diver employed by the city as to whether bedrock had been reached for the placing of the pier of a bridge. Mr. Butler straightway donned the diving suit and examined the river bottom himself.

The people of Spokane owe much to the efforts of Mr. Butler for the modern city in which they live. He was extremely interested in municipal sanitation and did considerable toward improving public health conditions within the city. Some of the city's larger sewer projects instituted and completed under his supervision are the South Manito and South Manito extension, Union Park, Hillyard, School Section, and Cannon Hill trunk sewers and the down-town intercepting sewers. He also made numerous specialized studies on stream pollution of the Spokane River.

One of Mr. Butler's many assets was his keen interest in all municipal affairs. He did considerable work in behalf of the city's gas tax allowance which resulted in nearly $1,000,000 being allocated from the state fund to Spokane for paving work. Besides being city engineer, he was for a time a member of the Park Board, and was active on the City Plan Commission from its inception.

Mr. Butler was not only an engineer of considerable merit, but he might also be classified as an excellent statesman. He was endowed with an extremely amiable personality which won him a host of lifelong friends. His ability to express his ideas clearly, always in a friendly but forcible manner, did much to gain the esteem of his business associates and fellow citizens.

He was very well liked by all with whom he came in contact and he was never too busy to give his untiring attention to any person or problem presented. By those who knew him best, Mr. Butler was simply called "Fred." Although playing a major rôle in civic affairs, Mr. Butler also found time to be a very active citizen. He served for seventeen years as a member of the Board of Trustees for the Deaconess Hospital to Spokane. He was interested in youth and devoted considerable time to the position of examiner for the Boy Scouts of America and for several years served on the Spokane Boy Scout Council. He took an active part in religious affairs and was a member of the Central Methodist Church. In addition he was a Thirty-Second Degree Mason and also a member of the Scottish Rite. He was a member of the Masonic Temple Association and served on the building committee at the time the Masonic Temple was rebuilt in 1924–1925. He enjoyed music and possessed a fine tenor voice.

Mr. Butler was both a writer and a speaker. He published several technical articles.[2, 3] He proved to be an excellent speaker on engineering as well as other subjects and was frequently called upon to give talks.

His general interest in the advancement of engineering was portrayed in his membership in the Associated Engineers of Spokane and the Northwest Scientific Society. He was both an officer and a charter member of the former organization. He also served as representative of the northwestern states on the committee to revise the constitution of the American Society of Civil Engineers and he performed many services as an officer and member of the Spokane Section of the Society.

On September 4, 1906, Mr. Butler was married to Zella Bisbee, and they became the parents of two sons, Alfred B. and Franklin M. Butler.

In tribute to Mr. Butler, part of a statement by F. G. Sutherlin, Mayor of Spokane, is quoted: "Competent, conservative and even tempered, he inspired confidence and made his work eminently satisfactory." In further tribute, A. B. Colburn, Commissioner for the City of Spokane, stated in part: "Mr. Butler was interested not only in engineering but in everything connected with the city. We consulted him about all civic matters. He was known and honored throughout the state."

Alfred Dickey Butler was elected an Associate Member of the American Society of Civil Engineers on October 5, 1909, and a Member on November 25, 1919.

WILLIAM TATEM COLLINGS, JR., M. Am. Soc. C. E.[1]

DIED NOVEMBER 21, 1941

William Tatem Collings, Jr., the son of William Tatem and Clara Collings, was born on a farm near Beaver City, Nebr., on July 7, 1888. He was grad-

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[2] *Engineering News-Record*, Vol. 84, No. 19, May 6, 1920, p. 899; Vol. 86, No. 12, March 24, 1921, p. 498; and Vol. 89, No. 21, November 23, 1922, p. 890.

[3] *Pacific Builder and Engineer*, Vol. 28, No. 21, May 26, 1922, pp. 7–8.

[1] Memoir prepared by Walter L. Bryant, Jr., Executive Asst., Imperial Irrigation Dist., Imperial, Calif.

uated from the high school in Beaver City in 1906, and, shortly after graduation, he moved to California.

His engineering career began in June, 1906, when he was chainman with the U. S. Indian Service in Los Angeles, Calif. During the next two and one-half years, he was employed by the Indian Service and by private surveyors in and around Los Angeles. For the following seven years, he worked for the City of Los Angeles in various capacities, mainly on the Los Angeles aqueduct and in the harbor district, engaged on conduit, siphon, and tunnel construction, and also on surveys of power sites and transmission lines in Owens River Valley and the distribution system in Los Angeles. During the period, September, 1916, to January, 1917, he was city engineer of Superior, Nebr., supervising a street improvement program involving asphaltic concrete paving.

From February, 1917, to May, 1919, he was with the U. S. Reclamation Service, Rio Grande Project, New Mexico, first on canal location and then on construction. Mr. Collings spent the next two years in private practice at Las Cruces, N. Mex., where he was also town engineer. While in Las Cruces he prepared plans and specifications, estimates, reports, and surveys, and made numerous designs on bridges, etc. In the period from May to August, 1921, he directed surveys for reservoir sites, conduits, and pipe lines for the water supply system of the City of El Paso, Tex. After completing this work, Mr. Collings returned to New Mexico as project engineer for the New Mexico State Highway Department. In July, 1923, he moved to Los Angeles, accepting a position as field engineer with Strong and Dickenson Syndicate, engaged in subdivision and municipal improvement work. He continued in this capacity for the syndicate and other contractors until June, 1927, when he accepted the position of engineer with Imperial Irrigation District. His first work with Imperial Irrigation District was the installation of an irrigation system for 16,000 acres near Niland, Calif. On this project, which was completed in July, 1929, he was in charge of the field work, drew up the plans and specifications, supervised the construction of the distribution system, and was active in studies of the drainage requirements of the district.

Mr. Collings' greatest contribution in the field of engineering came in the decade 1929–1938, in connection with the All-American Canal project. From May, 1929, to May, 1930, he was associated with H. J. Gault, M. Am. Soc. C. E., of the U. S. Bureau of Reclamation, in the field investigations for the canal. Late in 1930, he was sent to Denver, Colo., as the representative of the Imperial Irrigation District at the Bureau of Reclamation offices in that city. For the next seven years, the All-American Canal was his project. A review of his correspondence during this period shows his tremendous capacity for work, his keen understanding of the problems to be met, and his versatility. His reports ranged from hydrographic investigations, cost estimates, and construction method studies, to details in connection with the financing of the project.

In June, 1937, he was recalled from Denver by Imperial Irrigation District to become assistant to the chief engineer at Imperial, Calif. At this time the district had adopted a program which involved the building of the hydro plants on the All-American Canal and construction of the electric transmission and distribution system throughout Imperial Valley to carry All-American Canal power. In February, 1938, Mr. Collings was forced to request sick leave, and from that time until his subsequent passing, he was unable to continue engineering work. In retrospect, it must be remembered that, throughout the strenuous period devoted to the construction of the All-American Canal, Mr. Collings was suffering from the disease which finally caused his death. In view of this, his work is all the more remarkable.

Mr. Collings was married in El Paso, to Serena Perry on April 23, 1919. His widow and one son, William Tatem Collings III, survive him.

Prominent in Masonic circles, Mr. Collings was a member of Santa Fe Lodge, F. and A. M., and the Scottish Rite Consistory of that city. He was a licensed surveyor and registered civil engineer of the State of California and a licensed surveyor and a U. S. mineral surveyor in the State of New Mexico.

Mr. Collings was elected an Associate Member of the American Society of Civil Engineers on March 12, 1923, and a Member on January 20, 1931.

EDWIN FORD DAWSON, M. Am. Soc. C. E.[1]

Died February 25, 1942

Edwin Ford Dawson was born in Philadelphia, Pa., on March 10, 1859, the son of John Collins and Sarah M. (Montgomery) Dawson. He was descended from an old Philadelphia family who for several generations was associated, in executive capacities, with Brown Brothers and Company, one of the oldest banking houses in the East. Although he did not follow in the footsteps of his forebears, Mr. Dawson was always keenly interested in matters relating to economics and finance. He attended the University of Pennsylvania in Philadelphia where he established an outstanding record, and from which he was graduated with honors in 1880 with the degree of Bachelor of Science in Engineering.

During the years 1880 and 1881, he was employed by the U. S. Bureau of Harbor and River Improvements, assigned to development projects in the New York, N. Y., and Philadelphia areas. On leaving this bureau he became associated with the Philadelphia and Reading Railway Company, in Philadelphia (later Reading Company), with which he continued until his retirement on April 1, 1929.

His first assignment of real proportions was the design and construction, in 1890, of a plate-girder and stone-arch-curved double-track bridge over the falls of the Schuylkill River, north of Philadelphia. This project was followed by a number of preliminary surveys, locations, and construction of new lines and coal laterals in the middle anthracite region in the general vicinity of Shamokin, Pa. In later years he served on a joint committee—consisting of engineers representing the Pennsylvania State Highway Department, the

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[1] Memoir prepared by Walter Sebastian, M. Am. Soc. C. E.

Pennsylvania Railroad Company, and the City of Reading, Pa.—that formulated a comprehensive plan for the elimination of all railroad grade crossings within that municipality.

During his term as resident engineer in the office of the chief engineer of the company in Philadelphia, he had charge of the surveys for, and construction of, the numerous and varied works which came within the scope of the chief engineer. His long services in this capacity covered two distinct periods in the later development of railroads: The first was a period of organization adjustment as opposed to earlier railroad expansion—a period during which few new lines were built, but one during which then existing lines were greatly augmented by many additional and improved facilities. More specifically, this was an era of new and enlarged yards, of more modern engine terminals, of new and improved signals and interlocking plants, of piers and wharfs serviced by highly efficient mechanical loading devices, and of additional second, third, and fourth tracks to care properly for the rapidly increasing volumes of business. The second period was characterized by motorized competition in an era of fast, increasing automobile travel, by the introduction of the motor bus and motor truck, by improved electrical protection of highway crossings, by compulsory grade-crossing elimination, and by the more general acceptance of electrification. Thus Mr. Dawson's problems were varied, and his efforts in their solution always were characterized by speed, accuracy, and sound judgment.

Mr. Dawson had a kind and quiet personality; he was decidedly unostentatious in word, deed, and appearance; and he possessed an astute mind. Much of his leisure was devoted to reading, from which he amassed an amazing store of knowledge particularly in the fields of English literature, American history, mineralogy, and economics. He was held in high esteem and affection by his friends and business associates, and it may be truly said of him that he was a gentleman. He died on February 25, 1942, and is survived by his widow, Minnie Shouse Dawson, to whom he was married just fifty-eight years before, and a son, Howard Leslie Dawson.

Mr. Dawson was elected a Member of the American Society of Civil Engineers on March 4, 1908.

CHARLES MEIRS DENISE, M. Am. Soc. C. E.[1]

DIED JUNE 12, 1942

Charles Meirs Denise, the son of Rusha and Louise (Meirs) Denise, was born in Norfolk, Va., on January 24, 1876. After his preparatory schooling, he attended Rutgers College in New Brunswick, N. J., and was graduated in 1895 with the degree of Bachelor of Science. He taught school for two years at North Plainfield Latin School in North Plainfield, N. J., and then entered

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[1] Memoir prepared by Ethan F. Ball, M. Am. Soc. C. E.

Lehigh University in Bethlehem, Pa., from which he was graduated in 1898 with the degree of Civil Engineer. While in college he became interested in sports, especially football, and played on the varsity teams at both institutions. He was a member of the Chi Psi fraternity.

After graduation Mr. Denise spent one year in surveying work for several firms in eastern Pennsylvania. In June, 1899, he began his connection with the structural fabricating industry by entering the employ of the Shiffler Bridge Company, which in 1900 became a part of the American Bridge Company. After six months as a structural draftsman in Philadelphia, Pa., he was transferred to the designing and estimating department in New York, N. Y., and Pencoyd, Pa. In August, 1901, he left the American Bridge Company to become structural engineer for S. V. Huber and Company of Pittsburgh, Pa. While with this firm he designed many plant buildings for various steel companies located near Pittsburgh.

In February, 1903, Mr. Denise accepted a position in the Pittsburgh engineering department of the McClintic-Marshall Company. He remained in Pittsburgh until April, 1904, when he was made contracting engineer in charge of the Chicago (Ill.) Office of the company. He occupied this position until 1919, when he was recalled to Pittsburgh to act as contracting manager in charge of the entire sales department. In August, 1929, he was elected vice-president of the company, with the same duties.

When the McClintic-Marshall Company was acquired by the Bethlehem Steel Company in 1931, and a new subsidiary, the McClintic-Marshall Corporation, was formed, Mr. Denise was appointed vice-president in charge of the sales department and was transferred to Bethlehem. Beginning in 1936, the business of the subsidiary was conducted in the name of the parent company, and Mr. Denise became general manager of sales, fabricated steel construction, Bethlehem Steel Company. He held this position until he resigned, on account of ill health, in October, 1941.

Mr. Denise's friendly manner, great personal charm, and outstanding integrity not only made him invaluable to his company, but won for him the personal confidence and friendship of every one with whom he came in contact. He always retained his early interest in sports and became an enthusiastic and expert golfer.

He was married, in 1910, to Lucy M. Crandall, who, together with two sons, John Robbins and Charles Meirs, Jr., survives him.

Mr. Denise was elected second vice-president of the American Institute of Steel Construction, Inc., in 1936. He was a member of the Western Engineering Society of Chicago. He also was a member of the University Club, the Duquesne Club, and the Oakmont Country Club of Pittsburgh; the University Club and the Engineers' Club of Chicago; and the Saucon Valley Country Club of Bethlehem.

Mr. Denise was elected a Member of the American Society of Civil Engineers on May 3, 1910.

MANUEL VICTOR DOMENECH, M. Am. Soc. C. E.[1]

DIED MARCH 15, 1942

Manuel Victor Domenech was born in the Town of Isabela, Puerto Rico, on March 23, 1869, the son of Juan María Domenech and Nieves (Ferrer) Domenech. He received his early schooling in Isabela and then was sent to the United States to finish his education. Upon completion of his school work he entered Lehigh University, at Bethlehem, Pa., from which he was graduated in 1888 with the degree of Civil Engineer. Being particularly fond of mathematics and an excellent student, he was elected to membership in the honorary fraternity, Tau Beta Pi.

Mr. Domenech began the practice of his profession in 1888 with the Lehigh Valley Railroad Company as draftsman and levelman. Two years later he returned to Puerto Rico, and from 1890 to 1892 he was with La Compañía de Los Ferrocarriles de Puerto Rico, first as designer, and later as construction engineer, in charge of the construction of seventeen miles of track.

From 1892 to 1898 Mr. Domenech was city engineer and municipal architect for the City of Ponce, in charge of construction and maintenance of all streets, municipal roads, water works, public buildings, etc.

In 1899 Mr. Domenech became engineer for the Government of Puerto Rico, in charge of the public roads for the Southern district of the Island. Shortly after this, Mr. Domenech engaged in private practice designing sewerage and water works systems for the cities of Ponce and Mayagüez. He also located, planned, and designed a railroad line for the Ponce and Guayama Railroad Company.

From 1900 to 1906 he was Mayor of the City of Ponce and member of the Island's early Legislature. As a legislator Mr. Domenech helped to shape and adapt laws under the new sovereignty. He was the floor leader of the Republican party during the time that he was a member of the Legislature.

From 1907 to 1914 Mr. Domenech engaged again in private practice of his profession. During this time he was appointed trustee of the University of Puerto Rico, at Rio Piedras.

Following early liberation of the local government under President Woodrow Wilson, Mr. Domenech became Commissioner of Interior of Puerto Rico, holding this high position from 1914 to 1917. He was the first Puerto Rican appointed to this position under the new sovereignty. In this capacity he did splendid work—he encouraged highway development throughout the Island and completed considerable highway and public building construction.

In 1917 Mr. Domenech again engaged in private practice for a short while, later being appointed general manager of the Puerto Rico Railway Light and Power Company, at San Juan, which position he held from 1920 to 1926. His outstanding ability as an engineer and executive was amply manifested as manager of this private concern.

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[1] Memoir prepared by Reinaldo Ramírez, M. Am. Soc. C. E.

In the year 1922 Lehigh University honored Mr. Domenech for his well known public services by conferring upon him the honorary degree of Doctor of Laws.

From 1926 to 1929, Mr. Domenech engaged once more in private practice. In addition to his private work he was engineer in charge of all the property of the Bull Insular Line, Inc., at San Juan. This property consisted of warehouses, piers, tracks, and buildings.

In 1930 the Hon. Theodore Roosevelt, Jr., then Governor of Puerto Rico, appointed Mr. Domenech Treasurer of the Island, a position which he held with exceptional ability until 1935, as it was during this time that the island went through its worst economic crisis. In this position Mr. Domenech served as Acting Governor several times in a very able manner. In July, 1935, he resigned as Treasurer of the Island to become financial adviser to the Governor, which position he held until 1940, when he resigned to become Treasurer of the Island once more, for a short period, after which he resigned to become a member of the newly created Tax Appeals Court. At the time of his death he was serving as a member of this court.

Throughout his life Mr. Domenech was notable as one of those citizens who not only have ability but also the willingness to be useful in civic affairs. It was not strange, therefore, that he was called on repeatedly to serve on various commissions and boards. His unusual executive ability, coupled with sterling honesty and sense of civic duty, made him a leader in the community as witnessed by the following positions which he held: President of Sociedad de Ingenieros de Puerto Rico, President of the Local Chapter of the American Society of Civil Engineers, Chairman of Chamber of Commerce of Puerto Rico, President of the Rotary Club, President of the Union Club, and Trustee of the University of Puerto Rico.

As an executive Mr. Domenech was strict, but always just, with his subordinates.

He occupied a very conspicuous place in public life. He was a founder of the local Republican party, urged early granting of American citizenship to the people of Puerto Rico, and advocated statehood for the Island.

He was, as a general rule, very quiet and unassuming, and never sought publicity for himself or limelight for his actions. He was meticulous in all his undertakings, and any one having dealings with him would soon discover that he was ruggedly honest.

He was married twice and is survived by his widow and ten children, one of whom is Lt. J. P. Domenech of the U. S. Navy.

Mr. Domenech was elected a Member of the American Society of Civil Engineers on July 9, 1906.

EDWARD JAMES DOUGHERTY, M. Am. Soc. C. E.[1]

DIED NOVEMBER 11, 1942

Edward James Dougherty was born at Haverford, Pa., on August 3, 1889, the son of the late James Edward and Mary (Rountree) Dougherty. He received his early education in the public schools at Haverford and was graduated from Villanova College at Villanova, Pa., in 1912, with the degree of Bachelor of Science in Civil Engineering. In 1916 he was awarded the degree of Civil Engineer, and in 1919, the degree of Master of Science by the same college.

Upon graduation he entered the employ of the contracting firm of H. S. Kerbaugh, Inc., advancing to the position of general superintendent. Later, when the company was reorganized as the Empire Construction Company, Mr. Dougherty continued with the firm, becoming president in July, 1931. In February, 1941, he resigned to form his own company, the E. J. Dougherty Construction Company, which he headed at the time of his death.

As general superintendent of the Kerbaugh Company and the Empire Construction Company, he had under his direct supervision the construction on the New York State Barge Canal and many large contracts for the Baltimore and Ohio Railroad Company. Among the more important of the latter projects were: Rebuilding of Fairmount Park Tunnel, Philadelphia, Pa.; elevation of tracks in South Philadelphia; second track improvement of the Metropolitan Branch between Dickerson and Point of Rocks, Md.; Queen Street Underpass, Martinsburg, W. Va.; grade elimination at Hanover Street, Baltimore, Md.; pier improvements at Locust Point, Baltimore; work on the Marley Neck Branch, including the erection of a large bridge across Curtis Creek, Curtis Bay, Md.; bridge across the Potomac River at Harpers Ferry, W. Va.; construction of the "Fair of The Iron Horse" at Halethorpe, Md.; and building the Little Kanawha Railroad.

When he was president of the Empire Construction Company, in addition to the work for the Baltimore and Ohio Railroad Company, a large bridge over South River, a new concrete bridge and approaches on Liberty Road at North Branch, and a large bridge and approaches, involving considerable rock excavation for the cutoff road on Route No. 40 just north of Ellicott City, Md.—all were constructed under his direction for the State Roads Commission of Maryland during the depression years. He also directed the building of a coal dock and dumper foundations for the Chesapeake and Ohio Railway Company at Toledo, Ohio; the foundation for a coal pier for the Norfolk and Western Railway Company at Norfolk, Va.; a floating dry dock for the Bethlehem Shipbuilding Company at Key Highway, Baltimore; bridge, grading, and roadway work for a section of the Pennsylvania Turnpike for the Department of Highways of the State of Pennsylvania, and the change of line and construction of a large bridge for the Pennsylvania Railroad at Elkton, Md.

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[1] Memoir prepared by W. W. Gwathmey, Jr., M. Am. Soc. C. E.

During 1941 and 1942 the company which he formed and headed handled the reconstruction of twenty-three bridges on the West Virginia and Pittsburgh Branch, as well as the grading for a modern engine terminal at Cowen, W. Va., for the Baltimore and Ohio Railroad Company.

Always interested in professional activities, Mr. Dougherty was president of the Maryland Section of the Society in 1939. When he assumed his duties as president, Section affairs were in very bad shape, and largely through his efforts and enthusiasm the organization was restored to its rightful position as one of the leading engineering societies of the state.

Prior to 1939, the State of Maryland had no law governing the registration of professional engineers and land surveyors. During his tenure as president of the Maryland Section, and mainly because of his effort and determination, the Legislature of Maryland passed such an act, which was approved by the Governor on May 27, 1939.

Mr. Dougherty was also a member of the Engineers Club of Baltimore, and served as one of its directors during 1941.

A student of American History, and an acknowledged authority on the history of the Civil War, he was often consulted by writers and historians on this particular phase of our country's history.

Religious, charitable, and educational matters deeply interested him. His philanthropies were performed in such a way as to hide them from every one except those who were recipients or heads of institutions which he helped. He was a man of high ideals and integrity, and his life was filled with active service to his fellowmen. Eddie Dougherty commanded the respect and good wishes of all who had the good fortune to know him.

The life of this man was aptly expressed by Father J. F. Burns who spoke at his funeral, when he said—"Mr. Dougherty was like a great oak, whose branches reached out and enfolded the poor and those who were in distress, and that the Oak had fallen in death, but the place it had made would always be seen."

Mr. Dougherty was elected a Junior of the American Society of Civil Engineers on April 18, 1916; an Associate Member on August 9, 1920; and a Member on September 18, 1939.

ARTHUR SYLVESTER DOUGLASS, M. Am. Soc. C. E.[1]

Died March 6, 1942

Arthur Sylvester Douglass was born on January 16, 1883, in Plymouth, Mass., the son of Roswell Sargent and Frances Josephine (Clark) Douglass. After the completion of his elementary and high school education in Plymouth he spent two years at Wesleyan University, Middletown, Conn. He attended the Massachusetts Institute of Technology at Cambridge, Mass., from which he was graduated in 1908 with the degree of Civil Engineer.

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[1] Memoir prepared by A. C. Marshall, M. Am. Soc. C. E.; Simon Roth, Chf. Draftsman; John S. Nelles, Engr., Drafting and Surveying Bureau; F. D. Campbell, Engr., Electrical System—all of The Detroit Edison Company.

Immediately following graduation, Mr. Douglass entered the employ of the firm of Purdy and Henderson (Corydon T. Purdy, M. Am. Soc. C. E., and the late Lightner Henderson) in Boston, Mass., as a steel detailer. In 1909, he joined the Colorado Power Company where he was promoted from timekeeper to rodman, and finally to assistant resident engineer. Among the jobs on which he was employed were the Bear Trap Dam and Hydro-electric Power Plant and the Barker Meadow Dam. In 1911, he was employed by the Kerbaugh Construction Company and placed in charge of concrete work on the Catskill Aqueduct. Following this, he served as construction superintendent on a power transmission line at Lachine, Quebec, Canada, and later he served as night superintendent on the dam and hydroelectric project at Bonny Eagle, Me.

The spring of 1912 found Mr. Douglass in North Carolina engaged in the construction of power transmission lines in that state. It was here that he became acquainted with Clara Armistead of Rockingham, N.C., and they were married on June 11, 1912. In December of that year Mr. Douglass went to Barcelona, Spain, where he was general foreman in charge of construction of the hydroelectric power plant for the Ebro Power and Irrigation Company. Returning to the United States he became labor foreman for the late Charles M. Jacobs,[2] M. Am. Soc. C. E., and the late John V. Davies,[3] M. Am. Soc. C. E., of New York, N. Y., engaging in subway construction. In 1915 he was appointed Munitions Production Agent of the British Munitions Board for the Northeastern District of the United States and held this position for two years.

In August, 1917, he was commissioned a Captain in the Ordnance Department of the United States Army. In January, 1918, he was promoted to the rank of Major and in March, 1919, he rose to the rank of Lieutenant Colonel. In this period he served as assistant general supervisor, Inspection Section, Ordnance Department; Member of Ordnance Officers Examining Board; Ordnance District Chief, Boston District; and Chairman of the Boston District Claims Board. On April 18, 1919, he received a special letter of commendation from the Chief of Ordnance for services rendered, and on February 15, 1920, he was honorably discharged from the Army.

On March 1, 1920, Mr. Douglass became associated with The Detroit Edison Company with the title of construction engineer which he retained until his death. He had responsible charge of the Construction Bureau and the Drafting and Surveying Bureau as well as the Company's Beecher Shops. It was under his direction that three of the Company's present steam power plants were built: Marysville, Trenton Channel, and Delray Power Plant No. 3. In addition, considerable alteration work was done at the Conners Creek Power Plant. During the same period, many substations and other electric facilities were built as well as several office buildings and warehouses. At one time Mr. Douglass had under his direction a total of 4,370 employees. He pioneered in the use of soil mechanics in foundation problems, correlated with a complete record of subsequent column settlements. He was also quick to take advantage of the new water-cement ratio concrete mix design and field control, and was author of the paper "Three and One-Half Years' Experience of The Detroit Edison Company with Concrete Control."[4] His belief in the future importance of welding as a fabricating method, and his authorization and encouragement of experimental work, led to its development within the Company to the point where, since 1933, it has been accepted as a sound and reliable method of fabricating, even for members subjected to extreme conditions of pressure and temperature.

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[2] For memoir, see *Transactions*, Am. Soc. C. E., Vol. LXXXIII (1919–1920), p. 2236.

[3] For memoir, *ibid.*, Vol. 105 (1940), p. 1826.

Mr. Douglass was a member of the Society of American Military Engineers, the American Welding Society, the Michigan Engineering Society, a director of the American Concrete Institute, and a director and past president of the Engineering Society of Detroit. Other organizations of which he was a member include the Detroit Athletic Club, Detroit Golf Club, Torch Club, Economic Club of Detroit, and the American Legion, Thomas A. Edison Post No. 187.

In addition to his professional activities, Mr. Douglass gave generously of his time and energy to local civic affairs. He was well known for his work in connection with the Detroit Building Congress, the Committee for Trade Recovery, and the Detroit City Plan Commission. He also took an active part in the establishment of Detroit's City Zoning Ordinance.

Although he was first of all a practical construction man, there was another side to Mr. Douglass—a side that one might not expect to find in a man who worked chiefly with concrete and steel: He deeply loved good music. He was familiar with most of the better-known symphonies, and during the past ten years there were few concerts given by the Detroit Symphony Orchestra that did not find Mr. Douglass in his place, an enthusiastic and attentive listener. If he heard a piece that he did not like, he would buy a record of it, and play it over and over until he was convinced that he would never like it or until he had learned to appreciate it. He owned a superb collection of symphony records which he listened to by the hour with a careful and critical ear.

The opera was one of Mr. Douglass' favorite diversions. He was a member of the Metropolitan Opera Guild, and attended many New York performances. He was also a regular visitor to the Detroit Institute of Arts and was especially interested in the annual Michigan Artists Exhibits. He was fond of antiques. Other hobbies included stamp collecting, photography, and fishing.

Limitless enthusiasm, sincere earnestness, an intense personality, and an impressive physique made Arthur Douglass an outstanding figure. He possessed great ability to accomplish what he set out to do; obstacles seemed to fade under pressure of his determination and personal hardships were no challenge to his abundant strength. Yet this zeal detracted in no way from his insistence upon accuracy and quality. No detail was too small to escape his attention and no circumstance was sufficient to justify incompleteness. He was thorough and always followed through on a job from beginning to end.

Perhaps the qualities that were most appreciated by his fellow workers were his profound frankness and fairness. No one ever doubted where he stood on any subject, and no one ever questioned the honesty of his decisions. His firmness of purpose commanded the respect of all who had dealings with him. Personal friendliness, cordiality and hospitality, combined with his outstanding executive qualities, won for him the love and admiration of a great host of friends.

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[4] *Proceedings*, Am. Concrete Inst., Vol. 25, 1929, p. 100.

Mr. Douglass is survived by his wife; three sons, Arthur Sylvester, Jr., William Armistead, and Andrew Dewees; and two sisters, Josephine Douglass and Grace (Mrs. Edgar Schindler).

Mr. Douglass was elected a Member of the American Society of Civil Engineers on March 28, 1932.

WALTER JACOB FUSTON, M. Am. Soc. C. E.[1]

Died December 14, 1942

Walter Jacob Fuston was born at Waxahachie, Tex., on July 20, 1883, the son of Isaac M. and Mary J. (Hendricks) Fuston. His father was a building contractor. His mother's family came to Texas from Illinois in a covered wagon when she was two years old. Mr. Fuston was the second in a family of five children, two of whom survived him.

He attended the public schools of Waxahachie and later completed correspondence courses in electrical, mechanical, and civil engineering. His first practical work in engineering, from July, 1902, to October, 1909, was with the Waxahachie Electric and Gas Company, in the versatile rôle of repairman, lineman, stationary engineer, and meter inspector. From October, 1909, to December, 1915, he was employed as draftsman and purchasing agent by the Texas Traction Company and affiliated companies, which were, at that time, engaged upon a large program of interurban railway construction. When this work was completed, he entered the structural engineering field. From December, 1915, to February, 1920, he was, successively, detailer, checker, and chief draftsman for Mosher Steel Company of Dallas, Tex. In this capacity, he prepared designs and shop drawings for structural steel and reinforced concrete frames of many important buildings in the Southwest.

From February to August, 1920, he was chief engineer for the Munn Construction Company of Dallas, on the preparation of estimates and designs for structures. From August, 1920, until his death, he maintained an office at Dallas as consulting engineer on structural work. His designs were of a high order and covered all kinds of buildings and types of construction.

Mr. Fuston took an active part in civic and professional affairs. He was a member of the Technical Club of Dallas, the American Association for Advancement of Science, and the Church of Christ.

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[1] Memoir prepared by R. O. Jameson, M. Am. Soc. C. E.

On August 25, 1918, in Dallas, he was married to Mrs. Mary Sutton Morris, who survives him. Although Mr. and Mrs. Fuston had no children of their own, they took three homeless children into their home and reared them to maturity.

Mr. Fuston was elected an Associate Member of the American Society of Civil Engineers on October 2, 1922, and a Member on March 5, 1928.

PHILIP DAKIN GILLHAM, M. Am. Soc. C. E.[1]

DIED FEBRUARY 4, 1943

Philip Dakin (P. D.) Gillham was born on a farm in Dover Township, near Princeton, Bureau County, Ill., on May 12, 1883. His father, Julius Gillham, was a farmer. His mother was Elizabeth Dakin. It is interesting to note that through Phoebe McManus Dakin (his grandmother on his mother's side, born in old Fort Louisville, Kentucky), Phil was descended from one of the men who accompanied George Rogers Clark on his history-making expedition into the Northwest Territory.

His mother died when he was only five years old, and he lived on the old home farm with his father until he went away to school. He never outgrew his love of his boyhood home and his intense love of the country.

His early schooling, through the seventh grade, was in the typical midwestern country schoolhouse. He later attended the Princeton Township High School, being graduated in June, 1900. Phil then went to the University of Illinois, at Urbana, and was graduated on June 8, 1904, with the degree of Bachelor of Science in Civil Engineering.

Both in high school and at the university he played football—in Princeton on the high school team and at the university on his class team. He was a charter member of Sigma Rho Chapter of Beta Theta Pi fraternity at the University of Illinois. During his senior year he was president of the Chapter, and, for several years after graduation, he served on the building committee of the Chapter.

From 1904 until January, 1905, P. D. was in charge of land and drainage surveys around Princeton. He then entered the structural engineering field (which became his lifework) as a detailer of steel bridges for the Chicago Bridge and Iron Works in Chicago, Ill. Later, in the same year, he was employed by the Noelke Richards Iron Works, Indianapolis, Ind., as structural detailer on the Grand Leader Building in St. Louis, Mo.

Also, in 1905, P. D. became interested in reinforced concrete construction —then little used. As there were few engineers in the United States familiar with the theory of reinforced concrete design, he felt that there were opportunities for the young engineer in this branch of engineering. In September,

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[1] Memoir prepared by W. O. Snyder and Walter S. Todd, Associate Members, Am. Soc. C. E., and H. R. Creal, Bridge Engr., Kentucky Highway Dept., Frankfort, Ky.

1905, he accepted a position with the Expanded Metal and Corrugated Bar Company, which was engaged in promoting the use of the "corrugated bar"—a deformed section for reinforcing concrete, patented by the company. P. D. soon absorbed the principles of the design and, during the next five years, designed and supervised the construction of many structures of various types. Since contractors were unfamiliar with this new construction material, his work also involved, in most cases, preparing an estimate of the cost of the structure as designed. In addition to this work, in cooperation with other members of the company, he prepared numerous design bulletins for distribution, containing detailed designs for structures.

For four years he was assistant engineer of the firm's St. Louis office, designing, preparing detailed plans, writing specifications, and estimating construction cost for reinforced concrete and structural steel arch and girder bridges, culverts, viaducts, factories, warehouses, office and other buildings, footings, foundations, retaining walls, abutments, reservoirs, sewers, conduits, tanks, grain bins, coal pockets, etc. For a year he was engineer on construction of various reinforced concrete structures, including the eight-story Butler Brothers Building in St. Louis. In 1908, during the period of his service with the Expanded Metal and Corrugated Bar Company, P. D. spent several months in Mexico in connection with the construction of the government penitentiary at Mexico City.

In 1909, as the business of the company expanded, he was transferred to Buffalo, N. Y., for the purpose of opening an office and managing the business of the company in that territory. In 1912, when the company moved its headquarters to Buffalo, he was returned to the St. Louis district as manager.

During the period from October, 1913, to October, 1924, P. D. held a number of assignments on surveying and construction work. From October, 1924, until August, 1932, he was employed by the State Highway Department of Kentucky as bridge designer. At this time the Department of Highways was engaged in a large program of toll bridge construction (financed through the sale of bonds) as well as highway and bridge construction (financed through regular highway funds derived from auto licenses and gasoline taxes). In this program, P. D. had direct charge of a squad of designers and draftsmen, engaged in design and preparation of plans. Some of the larger bridges designed under his direct supervision were the bridge over the Green River at Spottsville, Ky., on U. S. route No. 60; the bridge across Cumberland River at Burnside, Ky., on U. S. route No. 25; and the bridges across Kentucky River at Boonesboro, Ky., and Tyrone, Ky. In addition, he had immediate charge of the preparation of standards for the design and details of steel truss spans.

In 1932 he left the Department of Highways to engage in private engineering practice and to direct Civil Works Administration in Bureau County. He continued in this work until June, 1935, when he was again employed by the State Highway Department of Kentucky in his former capacity as bridge designer.

P. D. was a member of the Illinois Society of Engineers. In Illinois he was licensed as Registered Structural Engineer and in Kentucky as Professional Civil Engineer. He was president of the Frankfort Engineer's Club for the year 1930–1931. He was always interested in the affairs of the Kentucky Section of the Society.

P. D. was quite active in the Masonic Lodge in Frankfort, Ky., and served that organization as Worshipful Master for two terms. He was deacon in the First Presbyterian Church at Frankfort, for a number of years, and, in March, 1937, he was elected an elder in that church. Wherever his work took him throughout his life, he always identified himself with the church of the community in which he resided, serving at various times as trustee and congregational clerk.

Although Mr. Gillham did not have any hobbies in which he was specially interested, in a mild way he collected stamps and accumulated a very interesting collection. A very real lover of nature and the country, he liked birds and studied bird life considerably. He was keenly interested in all new developments in farming and in livestock raising. In addition to engineering magazines, he subscribed to a number of farm magazines and kept informed on all that pertained to the farm.

Mr. Gillham was married to Margaret Ellen Palmer at Princeton, on October 19, 1907. In addition to his widow, he is survived by three daughters, Jane (Mrs. Arthur Buckle), Nancy (Mrs. Ray Hornbaker), and Mary Gillham; a grandson, Howard Philip Buckle; and a half sister, Eva Gillham (Mrs. George Heaton).

P. D. was a home-loving man, happiest when his three girls were with him. His home life was beautiful, and his gentleness and counsel meant much to his family, friends, and engineering associates.

Mr. Gillham was elected a Junior of the American Society of Civil Engineers on October 30, 1906; an Associate Member on April 4, 1911; and a Member on June 26, 1931.

CARLETON GREENE, M. Am. Soc. C. E.[1]

DIED NOVEMBER 14, 1942

Carleton Greene was born in New York, N. Y., on October 24, 1868. He was the son of George Sears Greene, Jr.,[2] M. Am. Soc. C. E., and Moody (Dana) Greene, and grandson of George Sears Greene,[3] Past-President and Hon. M. Am. Soc. C. E., Major General, U. S. Army, who so successfully defended the left flank of the Federal Army at Gettysburg, Pa. Both through his father and mother, he was descended from early seventeenth century settlers of New England.

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[1] Memoir prepared by Chandler Davis, M. Am. Soc. C. E.

[2] For memoir, see *Transactions*, Am. Soc. C. E., Vol. 88 (1925), p. 1392.

[3] For memoir, *ibid.*, Vol. XLIX, December, 1902, p. 335.

He attended the Wilson and Kelly School in New York City and entered Harvard University, Cambridge, Mass., in the fall of 1885, receiving the degree of Bachelor of Arts in 1889. The following two years he studied at Cornell University, Ithaca, N. Y., receiving the degree of Civil Engineer in 1891.

After graduation he served as chainman in the New York State Engineering Department and as special apprentice in the Altoona (Pa.) shops of the Pennsylvania Railroad Company.

In 1893 Mr. Greene joined the Barber Asphalt Company as draftsman and later became superintendent of machinery. In 1897 he was transferred to the company's plant in Trinidad, B. W. I., of which he eventually became general manager.

In 1900 he left the Barber Asphalt Company and revived the firm of Greene and Greene, Consulting Engineers (founded by his grandfather), with his father, specializing in piers, sheds, quay walls, and harbor improvements generally.

Besides his consulting work Carleton Greene was mechanical engineer with Snare and Triest, and consulting engineer with Holbrook, Cabot and Rollins, and the American Shipbuilding Corporation, Hog Island, Pa., during World War I.

In 1911 he was resident engineer, New York State Barge Canal, at Schenectady, N. Y., in charge of the construction of five dams and four docks. Upon completion of this work he was division engineer of the New York State Barge Canal Office in charge of the designing and construction of the New York City terminals. Later he continued his work as a consulting engineer in New York City, specializing in maritime work.

He retired eventually, seeking a well-earned rest.

He was married to Anna Barton Lathrop in Pittsfield, Mass., on June 27, 1901, and was a devoted husband. His wife predeceased him. He is the last of his line.

Carleton Greene joined the 71st Regiment Infantry, National Guard of New York, as a Lieutenant in 1900. He was promoted to Captain, and in 1902 was appointed Regimental Adjutant. He resigned in 1911. In his later years he was interested in, and devoted considerable time to, the Boy Scouts of America.

He published a valuable book on wharfs and piers, particularly stressing conditions such as are encountered in New York Harbor. He was Commissioner of the Oranges, Warden of the Episcopal Church of the Holy Communion in South Orange, N. J., and Vice President of the Borden Mining Company. He was a member of the Century Association of New York City; and a member of the Board of Missions and Church Extension of the Diocese of Newark, N. J.

Carleton Greene was elected an Associate Member of the American Society of Civil Engineers on March 3, 1897, and a Member on March 2, 1915. He served as a Director from 1920 to 1922, and became a Life Member in 1932.

ARTHUR JAMES GRIFFIN, M. Am. Soc. C. E.[1]

DIED JANUARY 20, 1943

Arthur James Griffin was born in Brooklyn, N. Y., on August 23, 1877, the son of John and Margaret (Kelly) Griffin. His parents were born in Ireland and were married there. His father was a foreign correspondent for a newspaper and then studied medicine, becoming a practicing physician. He was Commissioner of Health of the City of Brooklyn and, subsequently, after the creation of Greater New York, Superintendent of Schools for the Borough of Brooklyn. Dr. Griffin performed outstanding work in the control of contagion while Commissioner of Health.

Arthur James Griffin was graduated in 1895 from the Polytechnic Institute of Brooklyn with the degree of Bachelor of Science in Civil Engineering. His earliest engineering activities were in connection with the development of the water supply for Brooklyn and on dock construction for the Bureau of Wallabout Improvement of the Department of City Works. Following this, in 1897 and 1898, Mr. Griffin was with the Twenty-Fourth Improvement Commission on the construction of sewers in what was later the Borough of Bronx, New York, N. Y. He remained with that Borough until 1901, when he was employed by the Borough of Queens, New York, N. Y., on the construction of the Elmhurst Sewage Disposal Plant, St. Nicholas Avenue-Interborough Sewer, and other sewers. In July, 1907, he returned to Brooklyn as engineer of construction for the Bureau of Sewers, remaining until 1918 when he became Chief Engineer of Sewers. He continued in this capacity until 1934, when he was appointed Chief Engineer of Highways and Sewers.

During his connection with the Borough of Brooklyn, Mr. Griffin was in charge of the construction of some of the largest sewers ever built. Under his supervision, many sewage pumping plants and sewage disposal works were constructed, and experiments were conducted in the Twenty-Sixth Ward Disposal Works on all known sewage treatment and disposal methods. These studies resulted in the development of processes later accepted as standard.

The possessor of a strong and engaging personality, Mr. Griffin was the friend of many men prominent in the professional engineering and judicial worlds. Although he was modest to a marked degree, he had a keen and searching mind.

Mr. Griffin was a veteran of the Spanish-American War and a member of Troop C, which served in Puerto Rico.

In 1901 at Brooklyn, he was married to Edith Miller Porter, who died in 1923. He is survived by three daughters: Claire Griffin, Catherine Griffin, and Janice Griffin (Mrs. Joseph James Fitzgerald); and by two grandsons, Arthur Griffin Fitzgerald and Dennis Patrick Fitzgerald.

Mr. Griffin was elected an Associate Member of the American Society of Civil Engineers on December 7, 1904, and a Member on October 21, 1924.

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[1] Memoir prepared by John C. Riedel, M. Am. Soc. C. E.

CLAUDE IRVING GRIMM, M. Am. Soc. C. E.[1]

DIED DECEMBER 1, 1942

A champion tennis player, an accomplished figure skater, and the holder of the highest office open to a civilian in the U. S. Engineer Department, Claude Irving Grimm was gifted with diverse abilities.

It is inevitable that the name Grimm early should acquire the prefix "Peter." So, everywhere from his birthplace in Iowa, to the Ohio River system where he early earned an engineering reputation, to the Pacific Northwest, the site of his crowning achievements, he was "Pete" Grimm (of course not to his subordinates, at least during office hours).

Like many other distinguished sons of the United States, Pete was born in one of its small communities, Clear Lake, Cerro Gordo County, Iowa, on January 12, 1886. His parents were true pioneers. His mother (Henrietta Callanan) had a grandfather who was one of the first white settlers in Cerro Gordo County. The Grimms were among Clear Lake's outstanding citizens. His father, Col. G. W. Grimm, served on the city council for five terms and held the office of Mayor for one term.

Since the dark, glassy surface that northern Iowa lakes put on in winter is too tempting to be ignored, Pete early began to master the intricacies and superb balance required for figure skating—an accomplishment he continually improved and practiced until a few months before his death.

His engineering studies were taken at Iowa State College at Ames, from which institution he received the degree of Bachelor of Science in Civil Engineering in 1908 and, eight years later, the professional degree of Civil Engineer. He was a "Phi Gam" in college, where, doubtless with the applause of his fraternity brothers, he became a runner-up in tennis. He carried his tennis winning technique into the Cincinnati (Ohio) field, winning signal honors. If his profession called him into small communities without tennis facilities, he would organize a tennis club and contrive to get courts built immediately. Soon after his arrival in Portland, Ore., in 1933, he was elected director of the Irvington Tennis Club, an office he held at the time of his death. Furthermore, he won the veterans' (older than 45 years) tennis championship of Portland and of Oregon whenever the matches were played except in one year.

After leaving college and until midsummer of 1912, he designed and built bridges for the American Bridge Company at Ambridge, Pa., and other firms. He was progressively draftsman, inspector for his native county in Iowa, chief draftsman with the Metropolitan Bridge Company of New York, N. Y., and finally assistant chief engineer, Owego Bridge Company, of Owego, N. Y. He then spent six months as engineer of construction on irrigation works at Mission, Tex.; and a year, with the same title, on steel construction for W. W. Kratzer and Company of Pittsburgh, Pa.

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[1] Memoir prepared by J. C. Stevens, M. Am. Soc. C. E.

Ohio River problems of navigation and flood control were calling loudly for a man of his ability. Accordingly, in April, 1914, he entered the employ of the U. S. Engineer Department at Wheeling, W. Va., as designing engineer. He was soon transferred to the Cincinnati office, where he rose from chief designing engineer to principal assistant to the division engineer.

His fifteen years' experience with the problems of the Ohio and Mississippi rivers was the proving ground of his ability and technique. Many of the dams and locks along these rivers, that make that system the longest improved inland waterway in the world today, stand as monuments to this Iowa boy's capacity for sound engineering accomplishments of which his relatives and friends may well be proud.

No man can live on a technical diet all his days. A charming lady appeared on the horizon, romance beckoned, and Pete took "time out" to win the hand and heart of Mary Elizabeth Wealand, to whom he was married in St. Louis, Mo., on April 20, 1930. Although they had no children of their own, they undertook to raise and educate a young girl, Violet Schultz, who was 14 years old at the time of his death.

He then moved to the Pacific Coast where, in February, 1930, he had been appointed principal civil engineer of the Pacific Division Office, U. S. Engineer Department, at San Francisco, Calif. One of his outstanding responsibilities was acting as consultant on federal and state cooperative investigations of the water resources of California, particularly of the Sacramento and San Joaquin rivers (the Central Valley Project).

The so-called "308 Reports"[2] were then being prepared by the U. S. Engineer Department. The water resources, including power, navigation, and irrigation potentialities, of all important streams in the United States were investigated and voluminously reported. In this work Pete had important duties. Since the Pacific Division included the Pacific Northwest, the Columbia River system also came under his jurisdiction.

Following the completion of these projects extensive navigation, power, and flood control projects began to take shape. Soon Pete was transferred to the District Engineer Office in Portland, with the title of head engineer, in charge of the design and construction of Bonneville Dam, which project he saw through to its successful completion in 1935.

Meanwhile the Northwest Pacific Division had been created, and Pete was taken over, title and all. In this capacity he had charge of the design and advisory supervision of construction of the highest earth dam in the world—Mud Mountain on White River, Washington, 425 ft high. He also was largely responsible for planning the Willamette Valley flood control, navigation, and power project, for which two dams, Fern Ridge and Cottage Grove, have been completed.

As extracurricular activities he found time to serve on the Power Committee of the Oregon State Planning Board and was a member of the Regional Drainage Basin Committee of the Pacific Northwest for the National Resources Planning Board.

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[2] Authorized by House Doc. 308, 69th Cong., 1st Session.

His last major assignment, before he became ill, was consultant to the Chief of Engineers on organization and operations of the Massena (N. Y.) Engineer District, which undertook the preparation of plans for improvement of the International Section of the St. Lawrence Waterway. In this connection his work, as usual, was outstanding.

When Maj. Gen. (then Colonel) Thomas M. Robins, M. Am. Soc. C. E., was division engineer for the Pacific Division and later for the Northwest Pacific Division, he often remarked that Pete Grimm could find the answer to an engineering problem quicker than any one he knew.

The author of a number of technical papers published by the Society, president of the Oregon Section at the time of his death, and at one time a candidate for Director from the 12th District (missing election by a very narrow margin), he took an active part in Society affairs. Also, he was a member of the Society of American Military Engineers and contributed frequently to that society's publication, *The Military Engineer.*

Mr. Grimm was elected an Associate Member of the American Society of Civil Engineers on October 10, 1916, and a Member on October 1, 1926.

ALAN GORDON GUTTERIDGE, M. Am. Soc. C. E.[1]

DIED FEBRUARY 21, 1942

Alan Gordon Gutteridge was born at Melbourne, Australia, on February 4, 1892, the son of Dr. W. M. Gutteridge. He was the third of four sons, all of whom had marked careers. He was educated at Melbourne Grammar School and received honors in Physics and Chemistry in his last year of High School. When World War I broke out, he was an engineering student at Calgary, Canada. He enlisted on August 5, 1914, the day after Great Britain declared war on Germany, and embarked on August 27 as Lance-Corporal in the Royal Canadian Regiment.

He was on garrison duty in Bermuda for nearly a year when he was transferred to Europe, arriving in France on November 1, 1915. He took part in the Battle of Ypres and the fighting on the Somme from November 15, 1915, to October 8, 1916, when he was taken prisoner of war at Courcellette and held at Altdamm and Minden camps in Germany. He was released from Germany on December 8, 1918, and returned to Australia.

He at once resumed his studies at the Melbourne University and received honors in electrical engineering, graduating in 1921 with the degree of Bachelor of Civil Engineering.

He then entered the service of the Melbourne Harbour trust as draftsman in March, 1921, and was engaged on civil engineering design for one and one-half years when, as the recipient of a Traveling Fellowship of Rockefeller Foundation, he undertook a special postgraduate course at Harvard University, Cambridge, Mass., attaining the degree of Master of Science in Sanitary and Municipal Engineering. In addition, Mr. Gutteridge obtained practical experience in the design and installation of sewerage works and treatment plants in contact with many leading engineers in the United States of America, Canada, Great Britain, France, Germany, and Italy.

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[1] Memoir prepared by Gerald Haskins, Gutteridge, Haskins & Davey, Sydney, Australia.

On his return to Australia, he was appointed chief engineer of the Commonwealth Department of Health, and during his five years with this Authority, his posts included: Chairman, Royal Commission on Sanitation (1926), Victoria; Royal Commissioner, Brisbane Water Supply (1927); and Chairman, Commonwealth Conference on Public Health Engineering (1927).

In August, 1928, Mr. Gutteridge decided to enter private practice as a chartered consulting civil engineer and subsequently, in 1939, he joined forces with Gerald Haskins and Geoffrey Davey, Assoc. M. Am. Soc. C. E., in an expanded practice covering all States of the Commonwealth of Australia and Tasmania. During this period and until his death, Mr. Gutteridge rendered extremely valuable professional service in the water supply and sewerage field and was identified with the design and construction of a vast number of well-planned and executed works throughout the Commonwealth.

In recognition of his outstanding ability, he was appointed as the Engineering Member of the Defence Panel—an advisory body—aiding the Commonwealth Government in all technical matters relating to the design and construction of the heavy program of defense works undertaken during the course of this war. His death has brought to a close this vital contribution to the security of his land—just another example of his loyalty which also found expression in his service as a soldier during the last war. He was a member of the Institution of Civil Engineers, of London.

Gordon Gutteridge—like all real engineers—was a man of purpose and vision, who subordinated personal interests to the higher development of his profession and to the end that great natural resources should be developed for the betterment of mankind.

Mr. Gutteridge was elected a Member of the American Society of Civil Engineers on May 17, 1937.

ANDREW HARKNESS HARKNESS, M. Am. Soc. C. E.[1]

DIED FEBRUARY 28, 1943

Andrew Harkness Harkness was graduated in Architecture from the University of Toronto, at Toronto, Canada, in 1895 and received the degree of Bachelor of Applied Science in 1897. After three years in an architect's office, he joined the designing staff of the structural department of the Canada

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[1] Memoir prepared by C. R. Young, M. Am. Soc. C. E.

Foundry Company, Ltd., Toronto, remaining with this company for eight years and attaining the position of assistant chief engineer of the structural department.

In 1910 special interest and experience in building prompted him to enter into private practice as a consulting structural engineer in Toronto. This practice he continued until his death—since 1929, in partnership with Maj.-Gen. C. S. L. Hertzberg, under the firm name of Harkness and Hertzberg. During its long existence, this firm was responsible for the structural work on many of the outstanding buildings of Canada. Among these were the Canadian Bank of Commerce, the Canadian Pacific Railway Building, the Canada Life Building, the Dominion Bank Building, the east block of the Parliament Buildings, and the Western Hospital, at Toronto; the Sun Life Building and the Canadian Bank of Commerce Building, at Montreal; the Canadian Bank of Commerce Building and the Confederation Life Building, at Winnipeg; and the Dominion Parliament Buildings and the Civic Hospital, at Ottawa.

In 1935 Mr. Harkness was awarded the Sir John Kennedy Medal by the Engineering Institute of Canada in recognition of his outstanding merit in the engineering profession, and, in 1937, the honorary degree of Doctor of Engineering was conferred upon him by the University of Toronto.

He was a past vice-president of the Engineering Institute of Canada and a past-president of the Ontario Association of Professional Engineers. Always anxious to promote the fortunes of the engineering profession and particularly those of the younger men in it, he gave much time and thought to matters outside the normal responsibilities of his practice. A kindly reception always awaited any one who sought his assistance.

In Toronto he was known in a very wide circle as an enthusiastic gardener, having one of the finest iris gardens in the district. Every year, hundreds of people were welcomed to view his extraordinary display of rare varieties. For more than twenty years he cultivated this interest to his great personal satisfaction and to the delight of his friends.

For a long period before his death at the age of 71, Mr. Harkness had not been in good health. While he retained his interest in engineering matters during this time, since 1941 he had been forced to curtail his professional activities very considerably.

Andrew Harkness Harkness was born at Iroquois, Ont., Canada, on January 31, 1873, the son of Dr. John Harkness and Jane (Graham) Harkness. On September 26, 1900, he was married to Nell Coulter at Iroquois. He is survived by his widow and four daughters, Winnifred Harkness, Mrs. Gordon Duff, Mrs. Max Whitby, and Mrs. Frank Jones.

Mr. Harkness was elected a Member of the American Society of Civil Engineers on June 16, 1919.

JAY RUSSELL HICKOX, M. Am. Soc. C. E.[1]

DIED NOVEMBER 2, 1942

Jay Russell Hickox was born on April 3, 1865, in Southbury, Conn., the elder of two sons of Henry Park and Julia E. (Bradley) Hickox. He attended schools in Southbury and New Haven, Conn., and later Yale University also at New Haven. He was graduated from Sheffield Scientific School at Yale with the degree of Doctor of Philosophy in Civil Engineering in 1886.

He chose railroading as the field in which to apply his engineering training, and, going to Nebraska in July, 1886, entered the service of the Burlington and Missouri River Railroad (later a part of the Chicago, Burlington and Quincy Railroad Company) which was extending its lines in Nebraska, northwestern Kansas, Wyoming, and to Billings, Mont. Beginning as chainman, he advanced rapidly, until, in February, 1887, he became division engineer on construction. However, he soon was transferred to more important railroad construction between Alliance, Nebr., and Newcastle, Wyo., on work involving tunnel and heavy grading over what is known as Pine Ridge, Nebr. Later he was in charge of tracklaying on this line. In February, 1889, he served as chief of one of the parties on the location of the Black Hills line between Edgemont, S. Dak., and Deadwood, S. Dak.

From July, 1891, to September, 1896, he lived in Deadwood, devoting part of his time to private engineering practice and part to the Burlington Lines. Thus, he began a career in engineering which he followed with increasing success throughout his professional life—spent wholly with the Burlington Lines. Even during the years spent at Deadwood, while he was city engineer and deputy U. S. mineral surveyor, he retained his connection with the railroad as its resident engineer, devoting much time to the study of water conditions, as they affected the railroad, and to the development of methods for their improvement.

From 1906 to 1910, having given up his personal practice, he was successively engineer of maintenance of way of the Wyoming District, roadmaster, and trainmaster on the Sterling Division of the Burlington Lines. In this connection, he had further opportunity for study of hydraulic conditions and their effects upon the railroad's properties.

In 1910 he was transferred to Lincoln, Nebr., as principal assistant engineer for that part of the Burlington Lines west of the Missouri River, and assigned to engineering investigations as well as to the development of plans for treatment of natural water supplies requiring chemical preparation for better results in generating steam in locomotives and in stationary boiler plants. In this capacity, he developed plans and installed water-treatment plants at many points on the railroad with such pronounced success that the principles embodied in the plants he designed and the chemicals he used for water treatment were generally adopted by the railroad, and their use was adopted and extended by many other railroads having similar water conditions. Probably no one improvement on railroads has effected greater economies than that of water treatment for steaming purposes.

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[1] Memoir prepared by A. W. Newton, M. Am. Soc. C. E.

Because of these successes he was transferred to Chicago, Ill., in 1924, with the title of System Hydraulic Engineer, which position he filled until his retirement in 1935. During this time, he had full charge of water-treatment plants; and he saw their use expanded over the entire system. In addition, he acted as engineering adviser to the Legal Department of the Burlington Lines on the preparation and presentation of evidence for condemnation proceedings instituted by the War Department in connection with the building of dams on the Upper Mississippi River.

Following his retirement in 1935, he continued in the capacity of consultant on hydraulics for the Burlington Lines, as well as for the Great Northern Railway Company, the Missouri Pacific Railroad Company, and the St. Louis-San Francisco Railway Company.

Those who enjoyed close association with Mr. Hickox were impressed with his sterling character. Without exaggeration, it may be said that he never deliberately did an unfair or unjust thing in his whole life. He was meticulous in the performance of his own duties and obligations and expected a like performance on the part of others. Sometimes this seemed to give him a quality of sternness, but he was so absoluetly fair that he inspired the respect and admiration of all with who he came in contact. He was a congenial, interesting, and loyal companion, ever ready with a good story, a true lover of nature and the sports she provided.

These qualities, combined with a most keen and studious mind and a natural sagacity, especially qualified him for investigation and research—particularly of new problems. He enjoyed intellectual effort and, when he had solved a technical problem to his own satisfaction, he yielded to no one, but maintained his judgment and opinion with courage.

Mr. Hickox was a direct descendant of William Hickock, who came to America in 1640, twenty years after the landing of the Pilgrims, and settled in Farmington, Conn., being one of the original proprietors of the town. Throughout the succeeding years the Hickock (Hickox) family has been characterized by its industry, loyalty, patriotism, and ability to lead. Various members of it have been outstanding as soldiers, clergymen, and scholars.

In 1895 Mr. Hickox was married to Minnie Harding of Deadwood, who survives him.

He was a member of the Congregational Church, and contributed to many worthy charities. Always loyal to his alma mater, he was active in Yale alumni activities until his death. Mr. Hickox belonged to the Masonic Lodge, as well as to such organizations as American Railway Engineering Association, Western Society of Engineers, and Chicago Engineers' Club.

Mr. Hickox was elected a Member of the American Society of Civil Engineers on June 12, 1939.

JOHN MIFFLIN HOOD, JR., M. Am. Soc. C. E.[1]

DIED JUNE 16, 1941

John Mifflin Hood, Jr., was born in Baltimore, Md., on March 18, 1880, the son of John Mifflin and Florence Eloise (Haden) Hood. His first American paternal ancestor sailed from Bridport, England, in 1668 and settled in Maryland on the Patapsco River. He named his large tract of land Hood's Haven and built Hood's Mill. His descendants have served with distinction in all of the wars since his day, besides gaining eminence as doctors, ministers, and civil engineers. They were beloved for their genial manners and their interest in the public welfare. The City of Baltimore erected a statue of John Mifflin Hood, Sr., in honor of his long and faithful service as president of the Western Maryland Railway Company. The city had invested large sums in this company, and he developed it into a successful venture that brought trade and prosperity to the city and the state. The money derived from its sale rebuilt the business section of Baltimore after the conflagration of 1904.

John Mifflin Hood, Jr., was educated in the public schools and City College of Baltimore, attended the Massachusetts Institute of Technology in Boston, Mass., and was graduated as a civil engineer from Princeton University in Princeton, N. J., in 1901. He then entered the service of the Pennsylvania Railroad Company in Philadelphia, Pa., and was engaged chiefly in designing the complicated yard track layouts between the Spring Gardens Bridge and the Schuylkill River.

His next work was on an engineering corps of the Baltimore and Ohio Railroad Company west of Pittsburgh, Pa., in connection with the location of a new line for that company. Then he went to the State of Mississippi and became transitman for an important line west from Hattiesburg to connect the Queen and Crescent Railroad with the Illinois Central Railroad. When this work was completed, he became associated with the Missouri Pacific Railroad Company as division engineer of its new line between Memphis, Tenn., and New Orleans, La. He was in charge of the construction of that part of the extension in the vicinity of Lake Providence, Louisiana.

Upon the termination of this work, he was offered the position of assistant chief engineer of the Atlanta-Birmingham extension of the Seaboard Air Line Railway in Georgia, but chose instead the position of resident engineer in charge of an important section.

On returning to Baltimore he and the late Charles O. Vandevanter,[2] M. Am. Soc. C. E., formed the engineering firm of Vandevanter and Hood. At that time Mr. Hood's father was president of the United Railways and Electric Company of Baltimore, and the son became principal assistant engineer and then engineer of way of the city's street-car system. He was in direct charge of the construction of the North Point extension to Bay Shore Park, and handled the various engineering matters in connection with the building of that summer resort on the shore of Chesapeake Bay. He also supervised construction of the Roland Park and Saint Paul Street Boulevard line and other new extensions of the street-car service of Baltimore and the suburbs, besides having direct supervision over the reconstruction and maintenance of the existing tracks of the entire system. He introduced a new welding process for joining rails, which protected the eyes of the workmen from the white heat. By 1909 he had become chief engineer of the United Railways and Electric Company.

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[1] Memoir prepared by Miss Alice W. Hood, Baltimore, Md.

[2] For memoir, see *Transactions*, Am. Soc. C. E., Vol. 99 (1934), p. 1539.

Turning from his engineering profession, Mr. Hood became an industrialist on entering the service of the Crown Cork and Seal Company which his father-in-law, William Painter, had founded in Baltimore in 1893. He was on the Board of Directors in 1909 and became secretary of the company in 1911. He rose to the position of second vice-president and then first vice-president. In 1914 he was elected president of the company and held that office until he retired from business in 1923.

While in the service of the Crown Cork and Seal Company, he enlarged the main plant on Guilford Avenue in Baltimore, erected the latest style of all-window factory building in Highlandtown (in East Baltimore), and visited the numerous branch factories of the company in the United States, Canada, Mexico, and several European countries. When the World War in 1914 interfered with shipments of cork from Spain, he chartered a special vessel, the *Ruby,* which carried cotton from the Southern States to keep the Spanish cotton mills running and returned with cork to feed the home factories. The Spanish factory supplied cork wool to the American Expeditionary Forces at Coblenz, Germany, while part of the Baltimore factory made munitions for the United States Government.

Mr. Hood was a member of the Presbyterian Church, but was interested in and tolerant of every one else's faith and addressed various groups. He gave large sums to the American Bible Society to send the Bible around the world.

He kept in touch with the latest developments in science and industry, and watched with keen interest the actual polishing of the 200-in. mirror for Mount Palomar in California. He gave interesting talks on his wide travels over the five continents and the seven seas—

> "* * * from Banff to Bali, from Nikko to Karnak, from Peking to Fiji, from Capri to Canberra, from the Isles of the Inland Sea to the Sahara Desert, from the Vale of Kashmir through the Khyber Pass to Afghanistan, from the Taj Mahal to the Pyramids, from the Waiotomo Glowworm Cave of New Zealand to Rio de Janeiro, from the tomb of Lenin to a personal interview with Haile Selassie in Jerusalem."

He spent many winters in Los Angeles, Calif., where he had a host of friends, one of whom thus described him in the publication of a club of which he was a member:

> "Captain of industry, inspiring analyst, and beloved friend John Mifflin Hood Junior, forceful, efficient, creative, is a man who preeminently knows how to live, a gentleman, a man of his word, a man who

always has kept faith according to the highest traditions of his race and creed, of his house and name,—rarely does one have so warm a feeling, so deep a respect."

In 1911 Mr. Hood wrote the story of his father's life, entitled "Memoirs of John Mifflin Hood, 1843–1906. His Life and Works." He himself was a worthy son of a noble father.

He was married in Baltimore on November 5, 1903, to Ethel Gilpin Painter, daughter of the inventor William Painter. They had four children: John Mifflin Hood III, who was drowned at the age of 8; Ethel Painter, a sculptress; Mary Caroline, who married Charles Franklin Mohr and has a son and a daughter; and John, who became his father's constant companion in all his travels. In addition to his three children, Mr. Hood is survived by three sisters, Eloise (wife of Robert A. Cummings, M. Am. Soc. C. E.), Alice W., and Mabel H. (Mrs. F. M. Ridley), and a brother, Richard Haden Hood, M. Am. Soc. C. E. An invalid sister Florence Mifflin Hood was a great favorite of his; she died in July 1942.

Mr. Hood was elected a Junior of the American Society of Civil Engineers on May 31, 1904; an Associate Member on June 6, 1906; and a Member on September 6, 1910.

JERE CHAMBERLAIN HUTCHINS, M. Am. Soc. C. E.[1]

DIED FEBRUARY 24, 1943

Jere Chamberlain Hutchins was born on October 13, 1851, on the family plantation on the Louisiana side of the Mississippi River, not far from Natchez, Miss. His father was Anthony White Hutchins, a planter and native of Natchez. His paternal grandfather, John Hutchins, also had a plantation near Natchez. His mother was born in Bedford Springs, Pa., the daughter of Jeremiah Chamberlain, a Presbyterian minister. The maternal grandfather of Jere Hutchins (a grandson of Col. James Chamberlain, who was an officer under Washington in the Revolutionary War) was graduated from Princeton University at Princeton, N. J.

He was named after his maternal grandfather, Jeremiah, but the last two syllables of his name were omitted, his name being pronounced "Jerry." His father, Anthony Hutchins, whose plantation was located in a district of rich and highly cultivated planters, was a man of culture and considerable wealth. When he was a mere infant, his father chartered a steamboat on which he moved his family and slaves, household effects, horses, and cattle, to Lexington, Mo., in the hope that the cooler climate might benefit Jere's mother's failing health.

He was but ten years old when the Civil War broke out, and he was much intrigued by the glamour of the struggle going on between the Secessionists and Unionists, who were about equally divided in Missouri. After the war, he experienced the agonies of the reconstruction days. Jere attended the public schools and, through the guidance of his mother, was a fair student. The war had ravaged the sources of farm income, and his father became involved in debt, thereby losing his farm. Jere worked during summer vacations on neighboring farms, and contributed his small wages to the family budget.

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[1] Memoir prepared by George B. Russel, Detroit, Mich.

At the age of nineteen, he had determined to go into a lawyer's office and study law. However, he met a railroad contractor, who prevailed upon him to join his organization—then starting construction on a railroad running south from Lexington. Thus he began his career in civil engineering as axman, in the rapidly developing field of railroad construction during the frontier days when the Atlantic and Pacific coasts were being drawn closer together by steam transportation.

His first real job was on road construction for the Great Lakes, Lexington and Mexican Gulf Railroad. Next, he supervised the building of cribs and placing of riprap—in order to protect the piers and approaches of the bridges against abrasion and the high currents of the Arizona and Canadian rivers—along the line of the Missouri, Kansas and Texas Railroad.

Although he would have been able to obtain further construction work on a line connecting Waxahachie, Tex., with the Lexington Central Railroad, he became intrigued by the life of the Texas Rangers and succeeded in acquiring a pony, equipment, and clothes from a cowboy. He then became employed on a range and ranch where it was soon discovered he could ride the range, bust a bronco, rope and brand a maverick, and shoot a gun. Unlike the other Texas Rangers, however, he would drink no whiskey.

Jere soon realized that this occupation was getting him nowhere, so six months later he was off on a long ride to Waco, Tex. Arriving there at the age of twenty three, through the influence of a prominent citizen, he was offered the job of reporter on the *Waco Examiner.* He later became editorial writer and finally editor. Jere learned much in the newspaper business during the five years he was with the *Waco Examiner.* Texas newspaper editors usually resorted to gunfire whenever attacked by rival editors or maligned subscribers on account of critical statements printed in their papers about them. Although Mr. Hutchins, as editor of the *Waco Examiner,* always carried a gun, he averted any such settlement of grievances by rival editors or subscribers by his suavity of manner and his absolute fearlessness in the face of an opponent's threats. These traits of character served him in good stead during the later years of his life when his railroad was fighting politicians and newspapers in Detroit, Mich.

In 1880 Mr. Hutchins was married to Anna Brooks, of Waco. Her younger brother was later to become associated with him on the street railways of Detroit.

After Jere had resigned from the *Waco Examiner,* he was more or less inactive until he was appointed assistant superintendent of roadway on all lines of railroad being built by R. T. Wilson of New York, N. Y., between New

Orleans, La., and Memphis, Tenn., in the interests of the Southern Pacific Railroad Company, then being promoted by the late C. P. Huntington, F. Am. Soc. C. E. For thirteen years he was associated with R. T. Wilson on the construction of railroads in this section of the United States.

In October, 1894, Mr. Wilson telegraphed Mr. Hutchins to meet him in Detroit, where, in partnership with the Hon. Tom L. Johnson, then Congressman from Cleveland, Ohio, he had purchased control of the Detroit Citizens Street Railway, which operated eighty miles of horse-drawn cars in the City of Detroit. Mr. Hutchins was offered the vice-presidency of the road at a salary so much in excess of that he was receiving that he could not refuse the offer. From this beginning he ultimately became the operating head of the largest street railway system in the world. The day he arrived in Detroit, some riotous people of a suburban village tore up some tracks of the railway as a protest against the ten-cent fare that was charged although the regular city fare was only five cents. Mr. Hutchins met the representatives of the Village Council and promised to meet the Council in a spirit of fair adjustment. The tracks were restored and traffic resumed without any further molestation by the public. Thus he started his career in Detroit. Many more difficulties were to arise to make his new career one of the most troublesome that ever beset a street railroad executive in the United States.

The Detroit Citizens Street Railway promised to install a first-class electric system on all its lines, provided the city would grant it a thirty-year franchise with five-cent fares as the minimum. The reply of the Hon. Hazen Pingree, then Mayor of Detroit, was to grant franchise rights to build a rival system in Detroit to a syndicate of Messrs. Albert Pack, Henry A. Everett, and Edward W. Moore, on the basis of a three-cent fare. Detroit was the first city to have a street railroad system operating for less than a five-cent fare. This syndicate was incorporated under the name of The Detroit Electric Company. Mr. Hutchins countered this action by electrifying the horse-car lines and later reducing fares to three cents at all hours of the day or night. The rival company charged four cents after 11 p.m. Finally the two systems were merged in The Detroit United Railway (D.U.R.), and Mr. Hutchins was elected president and general manager at the first annual meeting. This company began to acquire all the roads running out of the city and added more than 200 miles to its operations.

The railway system of Detroit increased from eighty miles of horse-drawn railway in 1899 to 800 miles of electric roads in 1909. In 1899 more than 40,000,000 passengers were handled by the street railway companies; whereas, in 1909, 175,000,000 people were carried by the D.U.R.—yet the opposition of politicians and the press rapidly increased and municipal ownership was strongly agitated.

Mr. Hutchins was offered the presidency of one of the greatest system of street railroads in the world, but before anything was said about the compensation to be given him, he refused the offer, for the reason that he preferred to live in Detroit among his family, friends, and business associates.

Mrs. Hutchins died in January, 1900. For some months, as if by common consent, all criticism of his railroad company was avoided by press and public alike. Although the head of the company, Mr. Hutchins was very highly regarded personally by all who knew him. In fact, the employees chose him as their sole arbitrator in any dispute over wages or discharges from service. There was but one appeal from his decision and that was sustained.

Mr. Hutchins endeared himself to the employees of his railroad company and their morale was excellent. Whenever a conductor was accused of failing to turn in all his collection of fares, Mr. Hutchins personally handled the case. As a result the man was shown that honesty was the best policy and in several instances was retained in service on probation, and in no case was Mr. Hutchins' confidence in the repentance of the accused violated.

There was held in Detroit the annual convention of the American Street Railway Association attended by delegates from nearly all the cities of the United States and Canada, and also some of the larger cities of South America. Mr. Hutchins was elected to the presidency of this Association.

On June 12, 1903, Mr. Hutchins was married to Sarah Russel, daughter of George B. Russel, M. D. She had four brothers and one sister, to whom were born a total of twenty-one children, whom "Uncle Jere" (as he was universally called by friends and family alike) adopted as his own family. Mr. Hutchins was most hospitable and enjoyed entertaining his family frequently. He had a magnetic personality and loved people—especially his newly-acquired relatives. He seldom missed a luncheon at his favorite club, of which, at the time of his death, he was the oldest living member. Since he was a fluent talker and possessed sparkling wit, he was called upon many times to speak at luncheons and dinners given in Detroit in honor of some event or distinguished visitor. He was a professional baseball fanatic, and, during the summer of his ninetieth year, he seldom missed seeing the "Detroit Tigers" play in Detroit against the visiting teams of the American League.

On May 31, 1916, Mr. Hutchins tendered his resignation as president and general manager of the D.U.R.—a great surprise to the directors. With great reluctance they accepted it, appointing him chairman of the board, with a compensation equal to his salary as president. In this way, the company could call on his judgment and recommendations whenever they might be needed.

Municipal ownership of street railways was the platform of all politicians seeking office in the administration of Detroit's civic affairs, from the mayor down to the coroners and, finally after years of opposition to the D.U.R. on the part of politicians and the press, the electorate of Detroit empowered its Mayor, James Couzens, to offer $19,850,000 for the entire assets of the D.U.R., and, on May 15, 1922, the City of Detroit became the owner of the D.U.R.

Mr. Hutchins was a member of the Detroit Historical Society; the Detroit Museum of Art Founders Society; and various clubs, including Yondotega, Grosse Pointe, Detroit, and Detroit Country Club. He belonged to several Masonic Orders in Detroit and Waco. He was a director of several banking institutions and a president of Woodlawn Cemetery Association. He was a member of the Presbyterian Church.

In 1940, a year after the death of his second wife, Mr. Hutchins was induced by some of his family and friends to write a story of his life. Since no children had been born to him in either of his two marriages, it was thought an autobiography would be a source of consolation to him in his loneliness. In the latter part of 1941 it was privately printed and presented to his family and to his host of friends. The book contains an introduction written by the editor of the *Detroit Free Press,* Malcolm W. Bingay, in which he pays Mr. Hutchins the following tribute:

> "The people of Detroit had the craze for municipal ownership. That it was not their inherent hatred for the D.U.R. is best shown by the personality of the author of this book, Mr. Jere C. Hutchins. He was then, as he is now, one of Detroit's best loved citizens. Never did any politician or newspaper say one personal word against Mr. Hutchins.
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> "We of the press, the reporters on the firing line, grew to understand and have a deep affection for the scholarly, courtly gentleman who presided at 'Number 12 Woodward Avenue.' His ready wit and glowing good humor, his keen, quiet perception of life, his gentle philosophy, his personal honor and integrity, endeared him to more than one generation of journalists."

Mr. Hutchins was elected an Associate Member of the American Society of Civil Engineers on March 2, 1898, and a Member on April 4, 1900.

CLARK ELLSWORTH JACOBY, M. Am. Soc. C. E.[1]

Died August 24, 1942

Clark E. Jacoby was born in Iola, Kans., February 11, 1880, son of Miles P. and Elizabeth C. Jacoby. He received his early school training in the public schools of that city, and was graduated from the Iola High School. After attending Lane University at Lecompton, Kans., he entered the University of Kansas, at Lawrence, Kans., receiving the degree of Bachelor of Arts in 1905. He was a member of the Alpha Tau Omega fraternity, and, in addition, he belonged to the National and the Missouri State Societies of the Sons of the American Revolution.

On August 18, 1913, he was married to Colberne Riley, of Carrollton, Mo., a second cousin of Gen. Douglas MacArthur, Commander of the U. S. Armed Forces in the Southern Pacific. Their only child is Maj. Ellsworth R. Jacoby, U. S. Army Air Corps, who was graduated from the U. S. Military Academy, at West Point, N. Y.

Although Mr. Jacoby took little active part in political issues, he was chosen as the Republican candidate for mayor of Kansas City, Mo., in 1934, running a hopeless race against the strong Democratic organization in power at that time.

At different periods during his professional career, Mr. Jacoby was quite active in various technical society groups including—in addition to the American Society of Civil Engineers—the American Society for Testing Materials,

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[1] Memoir prepared by a Committee of the Kansas City Section, composed of E. I. Myers, M. Am. Soc. C. E., and Walter T. Brooks, and Joseph W. Ivy, Assoc. Members, Am. Soc. C. E.

American Society of Agricultural Engineers, American Association of Engineers, The Society of American Military Engineers, Highway Association of Missouri, The Kansas Engineering Society, and the National Drainage Congress. He served the latter as president in 1921.

His engineering career began in March, 1903, as chainman, rodman, and bookkeeper for George H. Griffith, a contractor of Kansas City, and he continued in this capacity until September, 1903. From March to April, 1905, Mr. Jacoby was employed on the engineering staff of the Atchison, Topeka and Santa Fe Railway Company, serving as chainman, rodman, and transitman on field location and maintenance surveys. Later, from April to December, 1909, he was with the Kansas City Southern Railway Company at Kansas City, as draftsman on design of track layout.

Beginning in April, 1908, as assistant engineer, and later, as chief engineer, he prepared plans and supervised construction for District No. 3, Carroll County, Missouri, which included 77,000 acres of farm land, railway right of ways, and other industrial properties.

With Walter T. Brooks, Mr. Jacoby formed the firm of Brooks and Jacoby in July, 1910; this partnership continued until July, 1915. One of the pioneers in applying scientific solutions to drainage and flood control problems, Mr. Jacoby specialized in this work in Missouri and Kansas. In addition, he designed and constructed some of the present brick and concrete pavements in the city of Carrollton, and supervised work on the Broadway Street Viaduct over the Union Station tracks at Joplin, Mo.

In July, 1915, he organized the Clark E. Jacoby Engineering Company and specialized in drainage and flood control work, serving as chief engineer for organized drainage and levee districts, corporate towns, and cities. By this time he was recognized as a national authority on land reclamation problems. Thousands of acres of swamp and overflowed lands in Missouri, Kansas, Illinois, Indiana, Oklahoma, Arkansas, Nebraska, Iowa, and Texas have been transformed into veritable "Gardens of Eden" through drainage and flood control improvements devised by him. Many of these projects will long stand as a living monument to his efforts.

While he was employed by the drainage and levee districts it was necessary that Mr. Jacoby take an active part in the legal proceedings affecting the organization of such districts. Attorneys who were associated with the projects early recognized his thorough knowledge and understanding of the drainage laws under which the districts were incorporated. He freely consulted with attorneys and advised them on the legal technicalities involved in drainage district organization and the legal procedure affecting the financing of districts. As a result of this cooperation, adverse findings by the courts, affecting such organizations, were rare. Legal opinions questioning the legality of bonds, issued by districts organized under the cooperative efforts of Mr. Jacoby with the legal representatives of the district, were practically unanimous in finding no flaws in the proceedings. Attorneys with whom he became associated in the trial of court cases on drainage and flood control matters have often expressed profound respect and admiration for his ability as an expert witness.

In addition to drainage and flood control engineering, Mr. Jacoby was recognized as an authority on river-bank stabilization and land erosion protection. As such he was employed on numerous projects along the Missouri and Arkansas rivers, and along many Kansas streams where bank stabilization problems were involved. The Kansas City Southern Railroad Company recognized his knowledge on such problems and retained him as a consultant on some of their more difficult river-bank erosion problems connected with their railway properties.

Among the many masterpieces of engineering reports upon various types of investigations which he made during his lifetime was an outstanding one written when Mr. Jacoby was industrial engineer for Woods Brothers Corporation, on the industrial development of a large section of land at Houston, Tex. This development involved deep ship canal construction for water transportation along Buffalo Bayou, port, harbor, dock, and warehouse facilities; airport development; and the extension of railway, water supply, and sewage facilities.

For a time, he was retained as consulting engineer under the old Ralph Sewer Act on seven of the large suburban sanitary sewer districts in St. Louis County, Missouri. He also made elaborate investigations and recommendations to the cities of Ottawa, Kans., Fort Smith, Ark., and Wagoner, Okla.

In May, 1935, Mr. Jacoby became associated with the Bureau of Agricultural Engineering, Department of Agriculture, as a consultant and inspector for land drainage work being done by Civilian Conservation Corps (CCC) Camps located in Missouri and Illinois. In 1938 the Department of Agriculture transferred this work from the Bureau of Agricultural Engineering to the Soil Conservation Service, with headquarters in Milwaukee, Wis. Mr. Jacoby then assumed charge of the Soil Conservation Service Drainage Section, operating in Minnesota, Wisconsin, Illinois, Missouri, and Iowa, remaining in this capacity until his death. During this period he not only assisted in developing practices relating to improved land-use measures, but also trained many young and inexperienced engineers in the fundamentals of the best drainage practice.

Mr. Jacoby was outstanding in his ability to make friends with every one and was familiarly known to many of them simply as "Jack." His loyalty to these friends was so pronounced that none ever forgot him. Untiring energy and application to his work contributed much to the success of the projects with which he was associated.

Among his many good qualities were an inexhaustible reservoir of dry humor, a devotion to duty, and an interest in the young engineers—many of whom often felt that Mr. Jacoby was at times too exacting or specific in his training methods. However, many of them have since entered the armed forces of our country and their training under his methods has served them well.

It was his good fortune to be "in harness" until he died. The end came as he was leaving his home for his daily tasks. The suddenness of his death was a great shock to his many friends, but all who knew him best realize that he could look back upon his achievements with pride, because he had filled his niche well and accomplished his tasks with credit. He left the world much better than he found it.

Mr. Jacoby was elected a Member of the American Society of Civil Engineers on April 25, 1921.

DAVID CLAYTON JOHNSON, M. Am. Soc. C. E.[1]

DIED DECEMBER 19, 1942

David Clayton Johnson was born in Brooklyn, N. Y., on August 24, 1885, the son of John Albert and Anna Mary (Jenkins) Johnson. He was of Irish and Welsh descent. One of his ancestors was a Captain in the Continental Army and aide to General Washington during the Revolutionary War.

David Clayton Johnson was educated at Polytechnic Preparatory School in Brooklyn and Stevens Institute of Technology in Hoboken, N. J., from which he was graduated in 1906 with the degree of Mechanical Engineer. After being graduated from college, he was employed first by The Astoria Light, Heat and Power Company in connection with the construction of that company's gas plant. He later became assistant engineer to the late H. de B. Parsons,[2] M. Am. Soc. C. E., consulting engineer of New York. In this connection he gathered much of the wealth of background which was to prove invaluable to him in his later undertakings. He designed and had charge of the erection of power plants and electrical installations, incinerators, and heating plants of important buildings, including fifteen buildings of the New York Zoological Society, the Cathedral of St. John the Divine, New York, and the Metropolitan Club, Washington, D. C. Also, he designed and superintended the construction of a fire boat and a number of ferry boats. As engineer for the firm of Humphreys and Miller he made numerous tests of power plants, boilers, engines, and generators, and made important reports, appraisals, and investigations.

Later, as engineer for Kean, Taylor and Company, bankers, he investigated and reported on industrial companies and railroads in connection with new financing. He continued in this phase of engineering when he became connected, in December, 1917, with The National City Company, an affiliate of The National City Bank of New York. While with this company, the scope of his activities broadened, particularly in the field of investment banking. As manager of the Public Utilities Department, he investigated important public utility companies, not only in all parts of the United States and in Cuba but also in Europe, and negotiated the financing of companies (both foreign and domestic) in amounts aggregating more than a billion dollars.

Several charts for the determination of yields of securities as well as a chart for the comparison of yields of taxable and nontaxable securities (of which several hundred thousand copies have been sold) were devised by Mr. Johnson. Also, he was a co-author of "Yields of Bonds and Stocks," a widely used book on yields.

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[1] Memoir prepared by Angela C. Brennan, Secretary to the late David Clayton Johnson.

[2] For memoir, see *Transactions*, Am. Soc. C. E., Vol. 101 (1936), p. 1617.

In March, 1927, Mr. Johnson was elected a vice-president of New York Steam Corporation, having been a director for several years; and, on July 2, 1928, he was made president of the corporation. His career as engineer, banker, and public utility executive is a brilliant record of outstanding ability and leadership. On May 24, 1937, he was elected a vice-president of Consolidated Edison Company of New York, and, on March 21, 1938, he was made a trustee of that company. While actively managing the affairs of the New York Steam Corporation as its president, Mr. Johnson, as vice-president of Consolidated Edison Company, had jurisdiction over its Treasury Department and was directly in charge of the finances of all the Edison System companies. (More than one and one-quarter billion dollars of the securities of these companies are owned by the investing public.) In addition to handling all financing of these companies, Mr. Johnson was in charge of all securities and funds of the companies, including the proceeds from the sale of bonds and stocks and collections from consumers.

Mr. Johnson was a director of the Long Island Railroad, Fidelity-Phenix Fire Insurance Company, Consolidated Telegraph and Electrical Subway Company, Brooklyn Edison Company, New York and Queens Electric Light and Power Company, and Green Mountain Lake Farms, Inc. He was a trustee and a member of the investment committee of Stevens Institute of Technology, his alma mater, to which he was deeply and actively loyal.

Mr. Johnson is survived by his widow, the former Mary Florence Riley; one son, David Clayton Johnson, Jr.; his sister, Mrs. William H. Dickie, and his brother, Frank Johnson.

His clubs included the Engineers Club, Union League Club, West Side Tennis Club, Manhattan Club, Rockefeller Center Luncheon Club, Stevens Metropolitan Club, University Club of Brooklyn, Bond Club of New York, Westhampton Beach Country Club, Quantuck Beach Club, and the National Golf Club. He was also a member and director of several civic associations, and a member of the National District Heating Association, the American Society of Mechanical Engineers, the Newcomen Society, and the Academy of Political Science.

Mr. Johnson was elected a Junior of the American Society of Civil Engineers on September 1, 1908; an Associate Member on May 7, 1913; and a Member on November 27, 1917.

FREDERICK HENRY KESTER, M. Am. Soc. C. E.[1]

DIED AUGUST 6, 1942

Frederick Henry Kester was born in Richland, Mich., on September 5, 1885, the son of William Henry and Mary (Peake) Kester. He passed his early years on his father's prosperous farm and in the local schools. For about a year he served as apprentice draftsman with The Bucyrus Company in South Milwaukee, Wis., but left this position to attend the University of Wisconsin, in Madison. In May, 1907, he joined the drafting force of The Canadian Bridge Company Limited, of Walkerville, Ontario. Except for a short period from 1908 to 1909, when he left to continue his studies, he remained with this company until June 1, 1941, when he retired because of ill health.

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[1] Memoir prepared by C. M. Goodrich, M. Am. Soc. C. E.

His formative years as an engineer were passed under the influence of Willard Pope and the late F. C. McMath,[2] Members, Am. Soc. C. E., who launched and headed The Canadian Bridge Company Limited, and who cherished the old-fashioned but very salutary beliefs that only after many years of varied experience as a draftsman could an unusually good man become a competent designer, and that such a designer should be trained not only to design, but to price the work, to tender it, and to follow it through the shops and into the field. Such was Mr. Kester's experience, during which he rose steadily through the ranks, becoming president and general manager in 1937. The designs which he made and the contracts which he secured are scattered from coast to coast in Canada, and abroad, as are also the friends he made. One of his trips abroad was in connection with bridges in Jamaica, B. W. I., another with the Ford plant in New Zealand.

In addition to these activities he became, in 1937, general manager of the Canadian Steel Corporation Limited at Ojibway, Ontario, and of the Essex Terminal Railway Company, which serves the industries in the communities across the river from Detroit, Mich. During the past few years the work of The Canadian Bridge Company Limited has been extended to include ship machinery, ships, and a considerable establishment fabricating hulls for universal carriers, or small tanks; and almost all its recent work has been for the Canadian or for the American government, either directly or indirectly.

His interest, both in his work and in his associates, has been keen and unflagging. His own engineering studies never ceased, and after his retirement letters were frequently exchanged between him and the writer, largely on the subjects of the recent advances in the study of columns, of limit design, and of the various developments in the application of elasticity in engineering problems.

During the depression years of the 1930's, Mr. Kester conducted a "refresher" course for the college graduates of the staff, and for such others as could follow the work, which reviewed mathematics through calculus, continued with a study of the methods best used in the design of bridges and buildings, and ended with a study of the more commonly useful elastic methods, notably moment distribution and area moments, including elastic weights. For this work he prepared his own excellent notes, which were blueprinted for the class, and included the only adequate treatment of the shear method for trusses—a method which came to him through Messrs. Pope and McMath from a Mr. Baker, sometime chief engineer of the Detroit Bridge and Iron Works.

Mr. Kester's retirement was a source of deep regret to his associates, both because of his thorough understanding of the problems and the informal and friendly way which characterized all discussions. His services were valued highly by the directors, who insisted that he remain as a consultant. After his retirement he lived in California, but returned for a visit to the companies he had formerly managed. His sudden and unexpected death from heart failure occurred during this visit. Those privileged to be associated with him will not forget his friendliness or his effective and ready helpfulness.

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[2] For memoir, see *Transactions*, Am. Soc. C. E., Vol. 104 (1939), p. 1964.

He was a lifelong member of the Presbyterian Church of Richland, and a member of the Engineering Society of Detroit, the Michigan Engineering Society, the Association of Professional Engineers of the Province of Ontario, the American Iron and Steel Institute, the Engineering Institute of Canada, the New Zealand Institution of Engineers, the Rotary Club of Windsor, Ontario, the Windsor Chamber of Commerce, and the Essex (Ontario) County Golf and Country Club.

Mr. Kester took a lively interest in all the pastimes of the staff, as shown by the ball ground for the ball team, tennis grounds for the tennis club, and a bowling green for the bowling club. He was an active member of the indoor bowling league of the company, and of the golf league as well.

He leaves a wife, two daughters, Elinor M. (Mrs. E. W. Driedger), and Dorothy P. (Mrs. W. R. Stickney), and a son, William H. Kester, an engineer in the employ of the American Bridge Company.

Mr. Kester was elected a Member of the American Society of Civil Engineers on May 19, 1941.

ARTHUR KRAUS, M. Am. Soc. C. E.[1]

Died January 16, 1943

Arthur Kraus was born in New York, N. Y., on May 19, 1887. He was the son of Sigmund and Mary (Breivogel) Kraus.

He studied engineering at Columbia University in New York City, from which he was graduated in 1908, with the degree of Civil Engineer. He was elected to membership in Sigma Xi and Tau Beta Pi fraternities. He also belonged to the Columbia University Club.

His first engineering position, which he accepted immediately after being graduated in June, 1908, was rodman with the Public Service Commission in New York City on track computation and estimates of quantities for subway work. He held this position until February, 1909, when he became engineer with J. Riely Gordon, Architect, of New York. In this capacity he was in charge of engineering work on the structural design, including foundation work, for the Elks Club of Manhattan; Elks Club of the Bronx; Bergen County Court House and Bergen County Jail, Hackensack, N. J.; and other structures, aggregating about $4,000,000 in value. In May, 1910, he resigned from this firm to accept a position with the Hinkle Iron Company of New York, as structural draftsman and designer on detailing, checking, and designing work.

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[1] Memoir prepared by William Buchtenkirk, Assoc. M. Am. Soc. C. E.

In August, 1911, he received an appointment to the Navy Department. He remained with this department until his death, giving many years of valuable service. His career with the Navy Department began as structural draftsman in the Public Works Department of the Brooklyn Navy Yard. In December, 1916, he became supervising draftsman and chief estimator for this department on design and specifications for buildings, piers, trackwork, etc.

In March, 1923, he was appointed assistant shop superintendent in the Hull Division of the Navy Yard, in charge of maintenance and development of the structural, sheet-metal, wood-working, and pipe-fitting shops of the yard. In 1939 he was promoted to senior assistant shop superintendent in the Production Division of the Navy Yard; and, in 1941, he took over his last post, principal assistant shop superintendent, shouldering the responsibility of procuring and maintaining machine tools and heavy equipment.

During the many years that he was connected with the Navy Department, he was very popular with his fellow workers. His conscientiousness, sincerity, and fine personality made him a host of friends both in his professional and in his social life. In loving tribute to him, the following poem was composed by one of the workers in the Machine Shop of the Navy Yard:

Another battle has been lost
But lost without disgrace.
The Lord has reached into our midst
For one we can't replace.

But we'll go on, we'll not complain
We know God has been fair
In granting an eternal rest
For one who's done his share.

He was married to Emma Louise Koehler, in East Orange, N. J., on July 14, 1929. He is survived by his widow; his mother; and two brothers, Frederick Kraus and Walter F. Kraus, Brigadier-General, U. S. Army.

Mr. Kraus was elected a Junior of the American Society of Civil Engineers on March 5, 1912; an Associate Member on September 10, 1918; and a Member on October 1, 1926.

ROBERT LINTON, M. Am. Soc. C. E.[1]

DIED NOVEMBER 12, 1942

Robert Linton was born at Hudson, Ohio, on May 29, 1870, the son of Robert and Caroline (Doolittle) Linton. After being graduated from Washington and Jefferson College at Washington, Pa. (which later awarded him the degrees of Master of Arts and Doctor of Science), he devoted two years to postgraduate studies at the Royal Polytechnic School in Berlin, Germany.

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[1] Memoir prepared by Albert J. Jones, Mining Engr., Los Angeles, Calif.

Mr. Linton was first employed, from 1890 to 1892, by the Pittsburgh (Pa.) civil and mining firm of McKiney and Smith, as an assistant mining engineer. In this position he carried out various underground and construction assignments around the coal mines for the next two years.

In 1896 he joined the R. C. Schmertz Glass Company of Belle Vernon, Pa., as assistant manager. Later when this firm was merged with the American Window Glass Company, he became a member of the latter organization. Advancing from chemist to factory superintendent to chief engineer, he was finally appointed assistant to the chairman of the board of directors. In this capacity he supervised operation and engineering for the company's ten factories in Pennsylvania and Indiana; however, his duties also extended to the mining and preparation of glass sand and the production and transportation of natural gas for the plants. Mr. Linton transferred his interests entirely to mining in 1906 when he became a partner in the mining and engineering firm of Atwater, Linton and Atwater, with offices in New York, N. Y., Duluth, Minn., and Los Angeles, Calif. After the termination of this partnership in 1910, he continued in private consulting practice, maintaining headquarters in New York for the next thirteen years, during which time mine examination and operational work took him to the principal mining districts of Mexico, British Columbia, the Yukon, and the United States. He also held important executive positions, serving as vice-president and general manager of the Sierra Consolidated Mines, Mexico; vice-president and managing director of Consolidated Coppermines Corporation, in Kimberly, Nev.; and from 1918 to 1923, president of the North Butte Mining Company, Butte, Mont.

In 1923 he went to Los Angeles to become vice-president and general manager of the Pacific Clay Products Company. The following twelve years were devoted to building up this company's production of vitrified clay products, firebrick, and other clay derivatives.

After 1935, when Mr. Linton again returned to private practice, he led an active life and was a successful consultant, specializing in the valuation, development, and operation of mines and industrial properties. At the same time, he continued as a director of the Pacific Clay Products Company. In recognition of his accomplishments, as stated, he was given the honorary degree of Doctor of Science by Washington and Jefferson College in 1937, with the following citation:

> "Washington and Jefferson College is proud of your achievements. After your studies here and in Germany and Belgium, you became a pioneer in the glass industry through the introduction of methods of technical control which have since become standard. Turning to mining engineering you again became noted, in this case for your work with cyaniding silver ores and in the standardization of operation in copper mines. More recently you have made engineering contributions to the clay products in-

> dustry which have attracted national attention. Testimony is borne in these accomplishments by the memberships, fellowships and offices which you hold or have held in engineering and technical societies in the United States, England, Germany and South Africa. Your writings have been distinctive in the technical journals of this country as well as Germany, France, Italy and Russia. Most important for us here, you have demonstrated the power of the liberally educated man in the technical field and while maintaining a constant engineering achievement you have enriched your life and thought with the finest culture and with character of both personal and social values."

A voluminous contributor to technical journals, Mr. Linton was also prominently identified with many professional societies. A member of the American Institute of Mining and Metallurgical Engineers since 1900, at the time of his death, he was serving as chairman of the Institute's Rand Foundation Award Committee. He was also a member of the American Society for Testing Materials, American Ceramics Society, the American Association for the Advancement of Science, and the Mining Club of New York. He was a member, and one-time director, of the American Mining Congress, a director of the Los Angeles Chamber of Commerce, and a former president of the Los Angeles Engineering Council.

The following appreciation of Mr. Linton is quoted from *The Explosives Engineer:*[2]

> "If our current efforts to find and produce needed quantities of militarily vital minerals are successful, as there is every reason to hope, much will be owed to the wisdom, experience, and enthusiasm of such members of the mining profession as Robert Linton. His long career as a consultant and executive has taken him to nearly every corner of the continent and has brought him into contact with all phases of mineral exploration and practical mining. * * * As a consulting engineer of outstanding reputation, and in his more immediate capacities as a mining adviser to the Metals Reserve Company and chairman of the Strategic Minerals Sub-Committee of the Los Angeles Chamber of Commerce, he is eminently qualified * * * to outline the avenues of assistance which are open to the prospective producer."

Among the many who mourned his death was former President Herbert Hoover, Hon. M. Am. Soc. C. E., whose condolences referred to "my old and loyal friend, over these many years."

Mr. Linton was married to Margaret McElveen of Pittsburgh at Pittsburgh, on November 4, 1896, and is survived by her and a daughter, Eleanor Linton of New York. Music and art were his hobbies, and golf, his favorite recreation. He was a member of the Los Angeles Country Club.

Mr. Linton was elected a Member of the American Society of Civil Engineers on December 15, 1924.

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[2] *Explosives Engineer,* Vol. 20, No. 10, October, 1942, pp. 285–289.

ALBERT KUALII BRICKWOOD LYMAN, M. Am. Soc. C. E.[1]

DIED AUGUST 13, 1942

Albert K. B. Lyman was born in Hamakua, Hawaii, May 5, 1885, the son of Rufus and Rebecca (Brickwood) Lyman and the descendant of a long line of missionary and distinguished "Alii," or noble families of Hawaii. His grandfather, the Rev. David Beldon Lyman, was one of the Fifth Company, New England Congregational Missionaries, that arrived in Hawaii on April 17, 1832, 172 days out of New Bedford, Mass. The "Kualii" in Albert Lyman's name means "High Chief."

Albert Lyman received his early education in Hawaii, and was one of three brothers to be graduated from the U. S. Military Academy, West Point, N. Y. He was graduated fifteenth in a class of 103, on June 11, 1909, and was commissioned a Second Lieutenant in the Corps of Engineers.

His early service included work with the U. S. Engineer Office, Rock Island, Ill., and with the Isthmian Canal Commission in Panama. Upon his return from Panama, he was engaged on short tours of duty in the Pittsburgh, Pa., and Detroit, Mich., engineer offices. Following these assignments, he attended the Engineer School at Washington Barracks, Washington, D. C., from which he was graduated on January 24, 1912. For a short time he remained at Washington Barracks with the First Battalion of Engineers, and, on October 12, 1912, he was promoted to First Lieutenant, Corps of Engineers. From November, 1913, to July, 1916, he served as First Lieutenant and Captain in the Third Engineers, and as assistant to the department engineer in Hawaii. Following this tour of duty, he was the assistant to the district engineer at Cincinnati, Ohio, until the outbreak of war in April, 1917. From April, 1917, until March, 1918, as Major, Corps of Engineers, he was instructor at engineer officers' training camps at Fort Benjamin Harrison, Indiana, Fort Leavenworth, Kansas, and Camp Lee, Virginia.

In March, 1918, he was promoted to Lieutenant Colonel, Corps of Engineers, U. S. Army, and was assigned as commanding officer of the 602d Engineers, with duty at Camp A. A. Humphreys, Virginia, and in France. He became Colonel, Corps of Engineers, in October, 1918, and after the signing of the Armistice, was on duty both as commanding officer of the 325th Engineers and as instructor at the Engineer School at Camp Humphreys. Colonel Lyman was secretary of the Mississippi River Commission from April, 1920, until February, 1922, when he was appointed military attaché in Havana, Cuba. In February, 1923, he was assigned as chief of the Intelligence Section, Military Division, Office, Chief of Engineers, for a short period prior to his assignment as district engineer in Cincinnati, where he served until July, 1926. From the middle of 1926, until January, 1930, he was chief of the Finance Division, Office, Chief of Engineers. From early 1930 until January, 1932, he was stationed in the Philippine Islands, first as commanding officer of the

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[1] Memoir prepared by James H. Stratton, Assoc. M. Am. Soc. C. E.

14th Engineers and later as division and department engineer. Returning to the United States he was district engineer, U. S. Engineer Office, Duluth, Minn., until August, 1935.

After he was graduated from the Army Industrial College at Washington, D. C., in June, 1936, he was assigned as district engineer, Boston, Mass., where he served until May, 1940. On leaving Boston, he was assigned to the Hawaiian Islands, where he served as commanding officer of the Third Engineers, as department engineer, and as district engineer of the Honolulu, Hawaii, U. S. Engineer District. He was also administrator for the Work Projects Administration for the Territory of Hawaii. On August 10, 1942, he was made Brigadier General, Army of the United States.

General Lyman contributed two valuable papers to Society publications on compaction of soils by explosives.[2,3]

On August 13, General Lyman died unexpectedly at the home of his brother-in-law in Honolulu. General Lyman is survived by his widow, the former Mrs. Gladys Henshaw Gray, to whom he was married on March 19, 1927, in Washington, D. C., and his brother, Col. Charles B. Lyman, U. S. Army.

General Lyman, throughout his long career in the Army, was affectionately known as "Queen," and his loss is keenly felt by his host of friends in and out of the service. He was born a soldier and an engineer of exceptional ability. Always kind to his subordinates and loyal to his superiors, his leadership was outstanding. Never sparing himself in the performance of duty, he died, as he would have wished, serving his country in time of war. It is most fitting that General Lyman was awarded posthumously the Distinguished Service Medal. His service was truly distinguished, as attested by the following editorial:[4]

> "The honor which a grateful country has paid to the late Albert K. B. Lyman, brigadier general of the U. S. Army, is an honor in which Hawaii shares.
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> "Albert Lyman was a son of Hawaii, a product of the elements and forces that make up Hawaii. In his veins flowed the mingled blood of ancient Hawaiians and the blood of those who came to Hawaii's shores as strangers, remained to help build the modern Hawaii."

General Lyman was elected a Member of the American Society of Civil Engineers on December 12, 1938.

WALTER HENRY MEIER, M. Am. Soc. C. E.[1]

DIED NOVEMBER 15, 1941

Walter Henry Meier, the son of Henry Meier and Elizabeth C. (Ruebsam) Meier, was born in Washington, D. C., on October 11, 1886. His father came to

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[2] "Compaction of Soils by Explosives," by A. K. B. Lyman, *Civil Engineering*, April, 1940, p. 205.

[3] "Compaction of Cohesionless Soils by Explosives," by A. K. B. Lyman, *Transactions*, Am. Soc. C. E., Vol. 107 (1942), p. 1330.

[4] *Honolulu Star Bulletin*, November 28, 1942 (editorial).

[1] Memoir prepared by a Committee of the Texas Section consisting of O. H. Koch and L. R. Ferguson, Members, Am. Soc. C. E., and F. M. Smith, Jr., Assoc. M. Am. Soc. C. E.

to the United States from Neilen, Switzerland, to work at the Naval Observatory in Washington, D. C., and later taught at Vanderbilt University (Nashville, Tenn.) and at Centre College (Danville, Ky.) before going to the University of Oklahoma at Norman.

Walter was graduated from the Milton Academy at Taneytown, Md., in 1901. He then volunteered as an Apprentice Seaman in the U. S. Navy, and, during the period of his service, he rose to the positions of Coxswain and Gun Captain. He was stationed first aboard the old *Monongahela,* the last of the sailing ships of the service; and then saw service for three years aboard the *Olympia,* an old veteran of the service. Later, he served aboard the *New Jersey,* Admiral Dewey's flagship. He was honorably discharged in October, 1907, with a rating of 84%.

From September, 1908, until April, 1910, he attended the University of Oklahoma (where his father was teaching) as an engineering student, but, because of poor health resulting from a spell of typhoid fever, he was forced to discontinue his school work. During the next eighteen months, he was employed, in various capacities, on railroad and topographic surveys in Texas and Oklahoma.

In August, 1911, he became draftsman for the Pioneer Telephone and Telegraph Company (later the Southwestern Bell Telephone Company) at Oklahoma City, Okla., and then served as chief draftsman, engineer, and in other positions. He gained considerable experience in the design, construction, and appraisal of aerial and underground local and long-distance telephone plant facilities. In November, 1915, he left this firm to accept a position with the Empire Gas and Fuel Company as an engineer in the geological department, advancing very rapidly to other and greater responsibilities. While with this company, he, in collaboration with Calvin T. Moore, prepared a *Plane Table Manual for Field Geologists.* This manual was used very extensively by geologists and geological schools for a number of years. In 1917, he was sent on a special mission to Kentucky and Tennessee to gather data and report on oil and gas production in those states. This assignment, unfortunately, was cut short in May, 1917, by a call to active duty by the War Department.

As First Lieutenant in the 7th U. S. Engineers, Walter H. Meier arrived in France with the American Expeditionary Forces in March, 1918. During the next twelve months, he saw considerable active service and took part in a number of the major engagements of the World War. He was forced to spend six weeks in the hospital after being gassed at San Mihiel. When he returned to the United States, he was Captain in the 21st Engineers and was honorably discharged by the Army in October, 1919, with a rating of 86%.

In January, 1920, Walter accepted a position as engineer for the Atlantic Oil Producing Company in Dallas, Tex. He was made chief engineer in February, 1927, and held that position continuously until August, 1939, at which time he was promoted to the office of supervisor of gas and gasoline contracts. True to his nature of being intensely interested and loyal to his work, he spent a great deal of his time in helping to develop the oil and gas industry. In a number of instances, the American Petroleum Institute recognized his ability, appointing him to important posts. Among these were—chairman of the Southwestern District, Division of Production; first chairman of the Central Committee on District Activities; first chairman of the Standing Committee on Standardization of Rigs and Derricks; chairman, from its formation, of the Committee on Topical Vocational Training; and a member of the Committee on Measuring, Sampling and Testing. His work on the application of engineering principles and technology in the production of petroleum, and the improvement and standardization of equipment is recognized as most important and a definite contribution toward the development of the oil and gas industry.

Walter Meier was married on May 24, 1912, to Dorotha N. Hazeltine in Norman. He is survived by his widow and eight children: Mary Louise, Alma Carolyn, Walter Henry, Jr., Dorotha Ester, Ruth Enola, Warren Elliott, John Mansfield, and James Valentine.

He was a member of the Trinity Valley Masonic Lodge, Dallas, and a member of the Arlington (Tex.) Presbyterian Church.

A man of outstanding stability of character, he was painstaking and thorough in all of his work. Those who employed him thought highly of him and respected him. He was conscientious and honest in all of his work and dealings, a hard worker, and a man of diversified experience. His good nature, even temper, and careful manner of expressing himself were manifest in all of his undertakings and earned for him the liking and good will of those with whom he came in contact. Always well groomed, his personal appearance, together with his ever-present genuine smile, combined to make him a man one would like and trust instinctively. His ideals of honesty, conscientiousness, and integrity won for him the highest regard of all who knew him.

Walter was vitally interested in his fellowmen. This fact was evidenced many times by his unusual activity in community and civic affairs and by his various technical and engineering society affiliations, as well as by extensive work on committees for organizations with which he was associated. One of the outstanding connections, of which he was justly proud, was his membership on the original committee of the Dallas Technical Club which brought about the establishment of a cooperative plan engineering school at Southern Methodist University in Dallas. His continuing interest and participation on that committee were responsible for the school's subsequent success in training technical men in practical cooperation with the industries of this section of Texas.

He was a member of the Society of American Military Engineers; president and director of the Technical Club of Dallas; and past president of both the Texas Section of the Society and the Dallas Branch. His untiring activities in connection with Society matters are indicated by his active interest in the welfare of the Student Chapter members and the Juniors of the Society.

Walter Meier was elected an Associate Member of the American Society of Civil Engineers on December 4, 1922, and a Member on January 18, 1926.

CLARENCE EUGENE MYERS, M. Am. Soc. C. E.[1]

DIED JANUARY 15, 1943

Clarence Eugene Myers was born in McKeesport, Pa., on January 31, 1888, the son of Harry and Mary L. Myers. His forebears settled in Pennsylvania in 1708, and he was one of the seventh generation of the family.

After being educated in the public schools of McKeesport from which he was graduated in 1902, he entered the Pennsylvania Military College at Chester, Pa., in September, 1905, specializing in military science and civil engineering. He was graduated in June, 1909, with the degree of Bachelor of Science. The College later honored him with degrees of Bachelor of Science in Military Science in 1924, Master in Civil Engineering in 1926, and Doctor of Science in Civil Engineering in 1930. While at college he was elected to membership in Delta Tau Beta fraternity.

Subsequent to his graduation, he was engaged for two years as an instructor in military science and civil engineering by his alma mater. Mr. Myers' early experience was gained from field and office work on surveys and general engineering projects in connection with farms, real estate, coal mines, and electric railways. He began service as a rodman and advanced in grade to chief of party.

Mr. Myers joined the staff of the Highway Department of the Commonwealth of Pennsylvania in August, 1911, as chief of party. Later he was promoted to general superintendent and district engineer, with assignments to supervise work in different parts of the State. He obtained a leave of absence from the department in January, 1918, to enter the U. S. Army.

On his retirement from the Army at the close of World War I, Mr. Myers returned to his duties with the Pennsylvania Department of Highways. Early in 1921, he became engineer of construction for the Bureau of Highways of the City of Philadelphia (Pa.), serving for five years in charge of construction on roads and streets in the city. He was promoted to the position of deputy chief of the Bureau of Highways in February, 1926, and continued in this executive capacity until January, 1928.

The Hon. Harry A. Mackey, then Mayor of the City of Philadelphia, appointed Mr. Myers a member of his cabinet in January, 1928, and placed him in charge of the Department of Transit with the title of director. At this time, the city was engaged in the construction, and equipment for operation, of a comprehensive system of high speed, electrically operated, subway and elevated railways. A substantial volume of the work had been completed by the former administration. A lease for the operation of the system was negotiated with the Street Railway System and the completed parts of the subway were placed in operation. Plans and specifications for the remaining sections of the system were drawn and contracts let for their construction and equipment. Mr. Myers retired from the city service at the termination of Mayor Mackey's term of office in January, 1932.

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[1] Memoir prepared by Charles H. Stevens, Julius Adler, and William E. A. Doherty, Members, Am. Soc. C. E.

In January, 1932, he opened an office in Philadelphia and engaged in business as a consulting engineer until the fall of 1940, when he again entered the armed service of the United States. Mr. Myers held important posts in both the first and second World Wars. In January, 1918, he entered the Army with the rank of Captain in the Corps of Engineers, and six months later was promoted to the rank of Major. He served in this country and overseas, where he was District Purchasing Officer for a time, later becoming Assistant Superintendent of Roads for Base Section No. 2, with headquarters at Bordeaux, France.

Major Myers was mustered out of service in December, 1918. He retained his military standing as a member of the Officers Reserve Corps and subsequently was advanced to the rank of Lieutenant Colonel.

Colonel Myers was recalled to active service in December, 1940, as a Lieutenant Colonel and was promoted to the rank of Colonel in May, 1941. At first, he was assigned to the Construction Division of the Quartermaster Corps where he supervised the construction of a new cantonment and training center for the Corps of Engineers at Fort Belvoir, Virginia. Later, he was charged with supervising the construction of a large ordnance depot in Pennsylvania. This work was well under way when he was relieved of this duty and was sent abroad as Section Engineer of the Western Base Service.

The scope of Colonel Myers' activities abroad and the character of the work performed by him may not be disclosed at this time, but he served the United States faithfully and with distinction (both at home and abroad) as an officer in the Corps of Engineers. He died in the performance of duty on January 15, 1943. He was buried with full military honors in the American plot of Brookwood Cemetery on January 16, 1943, at Woking, Surrey, England. Woking is about 22 miles southeast of London.

The following is a part of the General Order issued by his Commanding General under date, January 16, 1943:

> "1. It is with great regret that the death of Colonel C. E. Myers, C. E., Base Section Engineer, on 15 January 1943 is announced. While Colonel Myers died as the result of an unavoidable automobile accident, yet nevertheless he gave his life for his country in carrying out important duty."

Many letters from overseas have been received from high ranking officers of the armies of Great Britain and the United States, testifying to the character of his service and expressing sympathy for his death. Colonel Myers has been enrolled in the great army of our noble dead—that great multitude of our people who have at different times taken up arms to defend and preserve the precious heritage of liberty that has been handed down to us by our forefathers to cherish and defend.

Always interested in Society affairs, Colonel Myers represented District 4 on the Board of Direction from 1936 to 1938; he was a member of the Executive Committee and a former chairman of the Highway Division. He took an active part in the affairs of the Philadelphia Section and served as Director and President. After the 1926 Annual Convention in Philadelphia, the Section had a considerable fund remaining, and Colonel Myers was instrumental in securing the consent of contributors to the retention of the balance as a permanent fund to finance Section activities.

Colonel Myers was appointed for two terms on the State Registration Board for Professional Engineers by the Governor of the Commonwealth of Pennsylvania. At the time of his death he was president of the Board. He was an honorary member, former vice-president, and president of the City Officials Division of the American Road Builders Association. He also belonged to the American Institute of Consulting Engineers, the Society of American Military Engineers, the Public Works Association, the Engineers Society of Pennsylvania, the American Concrete Institute, the Pennsylvania Society of Professional Engineers, the Engineers Club of Philadelphia, the Union League Club of Philadelphia, and the American Legion.

His chosen field of activity was highway engineering, although he excelled in all the branches of engineering in which he engaged. Colonel Myers possessed outstanding executive ability and, coupled with this, an attractive personality and earnestness of purpose that inspired his associates and staff to render full cooperation in the accomplishment of appointed tasks.

Keenly interested in the encouragement and development of the young men who were on the threshold of the engineering profession, he was tireless in his efforts in their behalf. He was active in the affairs of his alma mater, and, at one time, he was president of its alumni association.

Throughout his professional career, Colonel Myers was steadfast in his convictions and adhered to his conceptions of what he believed to be the right and proper course of procedure. He was loved and esteemed by those who were privileged to be associated with him as co-workers and also by those who were admitted to the closer and more intimate tie of friendship.

In 1925, Colonel Myers was married to Agnes O'Bryan of Philadelphia. He is survived by his wife, a daughter Jane, his mother, and a brother.

Colonel Myers was elected an Associate Member of the American Society of Civil Engineers on January 17, 1921, and a Member on October 1, 1926.

JOHN A. NEWLIN, M. Am. Soc. C. E.[1]

DIED MARCH 27, 1943

John A. Newlin was born at Beech Grove, Marion County, Ind., on February 2, 1872, the son of Charles O. and Sarah (Hayworth) Newlin. His early education was obtained in the rural schools of Marion County. He attended Purdue University at La Fayette, Ind., and was graduated in 1900 with the degree of Bachelor of Science in Civil Engineering.

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[1] Memoir prepared by L. J. Markwardt, Chf., Div. of Timber Mechanics, and G. E. Heck, Engr., U. S. Dept. of Agriculture, Forest Service, Madison, Wis.

Upon the completion of his engineering course, he was employed for four years (1900–1904) as a civil engineer on railroad work by the Chicago, Indianapolis, and Louisville Railway and the Pittsburgh, Cincinnati, Chicago, and St. Louis Railroad.

In 1904 he entered the employ of the U. S. Forest Service as engineer on timber testing. At the time, this work was carried on at La Fayette under the supervision of W. K. Hatt, M. Am. Soc. C. E., and in cooperation with Purdue University. In 1910 he was placed in charge of the Section of Timber Mechanics of the Forest Products Laboratory, U. S. Forest Service, which was established at Madison, Wis., in that year. He held this position until 1938, when he was relieved of administrative duties so that he might give full time to research and consulting activities. Because of the critical need for his knowledge when the Forest Products Laboratory entered upon a greatly expanded program of wartime research for the Army, Navy, and other war agencies, his compulsory retirement in February, 1942, at the age of 70 was deferred a year by special Presidential authorization.

During his connection with the Forest Service Mr. Newlin made many outstanding contributions to the data on wood properties and their uses. Under his guidance a considerable amount of information on strength properties of woods grown in the United States was gathered. He early recognized the importance of systematic procedure in the testing of timber and played a major rôle in the standardization of such methods. In 1927 he prepared a paper for the American Society for Testing Materials on the origin and development of "Tests and Specifications for Timber," which was presented before the International Congress for Testing Materials held at Amsterdam, Holland, in September of that year. The container industry is indebted to his pioneer work in the development of methods of testing shipping containers and in the formulation of specifications for their proper design and construction.

In 1921 he was appointed by Herbert Hoover, Hon. M. Am. Soc. C. E., then Secretary of Commerce, a member of the Building Code Committee of the U. S. Department of Commerce. His membership continued until the disbandment of the committee in 1934. Subsequently he was appointed a member of the American Standards Association's Building Code Correlating Committee, the successor to the Building Code Committee of the Department of Commerce. During and following this period he gave much thought to improving the construction of small dwellings, and, under his direction, a system of prefabricated plywood panels for small house construction was developed and patented. This patent was assigned to the U. S. Department of Agriculture for the free use of the public. The principles of this construction later were applied to the production of thousands of prefabricated houses for war workers.

Mr. Newlin's knowledge of mechanics and the properties of wood was invaluable in connection with aircraft studies during World War I. In cooperation with the National Advisory Committee for Aeronautics he was author of numerous technical reports relating to the design of wooden aircraft. This knowledge and experience enabled him to give important aid to the Forest

Products Laboratory when it began an extensive research program for the nation's air services early in World War II.

He originated several formulas for calculating the strength of wooden members, among the most important of which are the Forest Products Laboratory column formula and a new formula for calculating shear in checked wooden beams. These formulas were adopted by timber engineers as standard for the design of wood structures. Grading rules and working stresses for structural timbers in general use in the United States were developed under Mr. Newlin's supervision. These grading rules and methods of calculating safe working stresses also have served as a guide to engineers in many foreign countries in formulating similar rules and stresses for their structural species. The general adoption of these grading rules and stresses and their incorporation in timber construction specifications were furthered by Mr. Newlin's membership on committees of the American Society for Testing Materials and the American Railway Engineering Association. From time to time he lectured on the mechanical properties of wood and wood products in classes of the University of Wisconsin at Madison.

If Mr. Newlin could be said to have an avocation, it was undoubtedly farming. He not only owned a farm of his own in Indiana but took a keen delight in recounting his life as a boy on his father's farm. Quite appropriately, his experiences always ended with some practical application of mechanics. Many of his stories related to early association and work with his father who was an accomplished woodworker and mechanic as well as a farmer, and most of them referred to mechanics or engineering. This background gave him much of the practical information on wood and its behavior which helped him later in his professional career.

More than most men Mr. Newlin was able to see far into a problem and to visualize the result. With his wide knowledge of wood, this talent enabled him to draw conclusions that others often would not accept until actual substantiating data were presented as proof. The data usually bore out his conclusions. Another related faculty of inestimable value was his ability to avoid being lost in the details of a problem and to eliminate mentally the insignificant factors in a complicated mathematical analysis. His skill in finding short-cut solutions was uncanny. Mr. Newlin was an internationally known authority on the mechanics of wood, and his death is an inestimable loss to the members of his profession, to his friends, and to those who were so fortunate as to be numbered among his associates.

Among the organizations to which Mr. Newlin belonged were International Association for Testing Materials, American Society for Testing Materials, American Railway Engineering Association, and the First Methodist Church of Madison.

On May 1, 1902, at La Fayette, he was married to Anna Ford. His widow and four children, Ruth, Charles Henry, John Dillon, and Benjamin Vestal, survive him.

Mr. Newlin was elected a Member of the American Society of Civil Engineers on May 19, 1924.

WILLIAM WARREN ORCUTT, M. Am. Soc. C. E.[1]

Died April 27, 1942

Matriculating at Leland Stanford Junior University, California, on its opening day with the pioneer class in 1891 to prepare for a career in civil engineering and geology was William Warren Orcutt of Santa Paula, Calif. He was a tall, strongly built, unassuming, earnest, cheerful young man of quiet demeanor and with a good sense of humor. His ruddy complexion, oft-recurring smile, and the bright twinkle in his eyes were accurate indicators of perfect health and of a kindly good nature that fairly teemed with the joy of life and naturally developed most friendly ties with his college mates as well as with other acquaintances.

Born in Dodge Center, Dodge County, Minn., on February 14, 1869, the son of John Hall Orcutt and Adeline Marion (Warren) Orcutt, he was descended through both parents from old Puritan and Virginia stock, some of his ancestors having arrived in America on the *Mayflower*. His father was born in Michigan and his mother in Wisconsin, both of pioneer parents. They were married in Minnesota in 1868, and moved to Nebraska when William was five years old. In 1881 they moved to Ventura County, California, to a tract of land acquired for horticultural purposes. Mr. Orcutt, Sr., fought against the Indians in 1862 with Company M, First Minnesota Mounted Rangers, and was a member of Company B, First Regiment, Minnesota Heavy Artillery, in the Civil War, being honorably discharged at its close. At the time of his death in 1913 he was the oldest Mason in point of service in Ventura County, a most highly respected citizen, a philanthropist, and an authority on horticulture.

From a scholastic viewpoint, William Warren Orcutt obtained his preparatory education in the Santa Paula public schools and then in the Santa Paula Academy (1887–1891). From a personal and a practical standpoint, however, he had other and equally important preparation. His careful home and church training to old New England standards, in a community where like methods prevailed in the families of his companions, early stimulated his innate tendencies toward a dignity, a quiet modesty, a scrupulous honesty, and a tolerance and respect for the rights of others that were gradually to mature, to characterize all of his dealings in later life, and to build up lasting friendships. His love of hunting, fishing, and exploring in the surrounding hills and streams with cherished boyhood friends early brought him in contact with oil seepages, rock exposures, oil pits, tunnels, and oil wells that had been under exploration and production in that neighborhood since 1865. Like all normal lads, there was little that he and his chums did not know in their youthful way regarding the mechanical working of such developments. He naturally acquired a strong desire to gain further knowledge of engineering and geology.

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[1] Memoir prepared by Walter Stalder, Cons. Petroleum Geologist, San Francisco, Calif.

With his professors and instructors at the university he was an industrious and earnest student. That he should play on the football eleven under the coaching of the celebrated Walter Camp and the management of his classmate, Herbert Hoover, Hon. M. Am. Soc. C. E., and earn his "Block S" on the track in 1895, was partly incidental but was natural, due to his fine physique, courageous love of a clean contest, and ability to cooperate well with others. In a physical examination of twelve hundred students he was rated as "Number One and one hundred percent physically perfect."

After receiving the degree of Bachelor of Arts in Civil Engineering in 1895, he and A. C. Hardison, a civil engineering graduate of the University of Maine, in Orono, opened an office in Santa Paula as civil and hydraulic engineers and surveyors, with headquarters in the Santa Paula Hardware Building, which then also housed the main office of the Union Oil Company of California. In 1896 he also became a United States Deputy Surveyor.

On June 9, 1897, William Warren Orcutt and Mary Logan of Santa Paula were united in a happy marriage. Miss Logan had attended the Santa Paula Academy with young Mr. Orcutt and then Pomona College in Claremont, Calif., while he was at Stanford University. The intelligent and broad-minded cooperation that was to characterize their married life served to inspire and round out the many fine accomplishments of his career.

Lyman Stewart, president of the Union Oil Company, had quietly followed the trend of the life of young William Orcutt, first as a boyhood companion of his own son, William L. Stewart, and then in his early engineering work with land surveying, irrigation development, plant and building design, and construction about Santa Paula. He was satisfied that here was an ideal young man to grow with his own organization, and in 1898 Mr. Orcutt became general superintendent of the San Joaquin Valley Division of the Union Oil Company. This new employment renewed close contact with certain boyhood friends then in the same company and was the beginning of a loyal business association which, by the honorable, sound, and intelligent methods for which he had been trained, was to grow to a large, conservative, and successful oil company. In this company he later was successively chief engineer (1900–1917), chief geologist and manager of Land Department (1908–1922), vice-president in charge of Geological and Land departments (1922–1933), and vice-president in charge of Development, Production, Geological, and Land departments (1933–1939). He also was a member of the Executive Committee (1908–1939) and a member of the Board of Directors (1908 to April 27, 1942).

In 1900 he was mapping in the field, surveying for oil pipe lines, and constructing steam plants and other necessary installations at Coalinga, Calif. Shortly afterward, he made a reconnaissance survey for a pipe-line route from the San Joaquin Valley to Port San Luis, Calif., at tidewater on the ocean. This survey was for future use. In 1900 he also organized a geologic department that is credited as being the first such department for an operating oil company.

In 1901, while examining the brea pits on the Hancock ranch in the neighborhood of what later became the Salt Lake oil field in Los Angeles County, California, he realized that there must be an important scientific value to the large accumulation of fossil bones of animals and birds that had become entrapped in the brea. In his quiet and unassuming way he brought this to the attention of John C. Merriam of the University of California, in Berkeley, with the result that the complete skeletons were restored and the knowledge of Pleistocene fauna was greatly enriched. In various museums may be found the skeletons of extinct forms, such as saber-toothed tigers, giant ground sloths, bisons, mastodons, camels, and herons and other birds, from these brea traps. As a token of appreciation, Professor Merriam and other paleontologists have appended the name "Orcutti" to the scientific names of several of these fossils.

In 1901 and 1902 Mr. Orcutt laid out the oil town for the Santa Maria oil field in Santa Barbara County, California, and installed its water system, sewer system, oil pipe lines, gas lines, storage tanks, and equipment for loading oil at railroad sidings. The oil town was named "Orcutt" by his employers in appreciation of their genial, competent, and respected engineer. This recognition, coming as it did so early in his association with the company, indicates the esteem in which he was held and how rapidly he was proving his worth.

In 1903 he laid out and constructed the pipe line from the Santa Maria oil field to Port San Luis. In the Santa Maria oil field his early geological studies and maps again made it possible to select intelligently the best oil properties, and, when highly productive wells "came in," confidence in his geologic, as well as in his engineering, judgment was firmly established. California oil operators in general had not yet been educated to the real value of geology in exploration, and herein he was a pioneer. He was shortly responsible for the selection and development of the Lompoc (Calif.) oil field, for the selection of other extensive holdings in Ventura, Santa Barbara, Los Angeles, Orange, and other California counties, and later for the discovery and development of several fields in the Rocky Mountains, either for oil reserves or for earlier development. His early maps of the Richfield, Santa Fe Springs, and other Los Angeles Basin fields indicate his geologic foresight. In his honor an oil town in Colorado also has been named Orcutt.

Traveling on the job in the earlier days was by buckboard, spring wagon, on horseback, or on foot, and he frequently "camped out." Although slower, walking had the advantage of affording longer, closer, and more friendly contacts with various oil-field operators and with ranch owners with whom he took his meals or stayed overnight. During his work in the San Joaquin Valley, he therefore had an excellent opportunity to build up a mutual respect and confidence among country landowners, small operators, and his company. In the creation of this good will, his own genuine character and affable manner were no small factors. Certain of his company associates also assisted in building this friendly foundation. When, in 1909, approximately 175 small oil-producing companies, which composed the Independent Oil Producers Agency from five San Joaquin Valley fields, desired to improve their marketing condition, the groundwork of mutual confidence and respect was already established for an agreement with his company and for securing right of ways and building the Producers Transportation Company pipe line which followed and for which he had made the early survey. This agreement resulted in the further cementing of loyal friendships and in a cooperation that still exists among the few older operators who survive. This group of pioneers was always like a happy family.

As an engineer, Mr. Orcutt did not publish the technical results of his work. When he reasoned out a new problem, it was all in the day's work for his company and he was very modest about it. As a result, he has not been credited with innovations for which he was primarily responsible. The following is an example: The decline-curve method of oil-land valuation, as introduced to the petroleum industry through the Independent Oil Producers Agency valuation of 1914–1915, was received with enthusiasm. Numerous technical papers have since been written about it. It stemmed from a decline-production method as used by Mr. Orcutt early in 1911. At that time two young engineers from San Francisco, Calif., were struggling with a method for evaluating the Union Oil Company properties. Mr. Orcutt realized their predicament and kindly showed them his method for a valuation that he chanced to be making on a small property. The writer happened to be one of those two young engineers through whom Mr. Orcutt's method was later introduced to the valuation committee and staff of the Independent Oil Producers Agency in 1914, with the aforementioned results. That Mr. Orcutt had any connection with the fundamentals of this method has not been generally known, and he was too generous and broad-minded to be concerned in any way as to its use, other than the satisfaction of realizing that his work had been of some value to others. This is but one of many kindnesses that he was continually extending and that were characteristic of his relations with younger men.

As his executive functions increased, he also became familiar with the financial and all other business phases of the large oil business, in which his analytic and experienced engineering mind was of inestimable value. This ability, combined with his human qualities, respect for the ordinary man, and fair dealing, made business relations pleasant and usually profitable for all concerned.

In 1926, in recognition of his services to the geological profession, the American Association of Petroleum Geologists conferred upon Mr. Orcutt an honorary membership in their organization. His was one of the first of such memberships in that association, and he shared this initial honor with four other outstanding pioneers in the geological profession: Robert T. Hill, George Otis Smith, David White, and I. C. White.

As time went on and one by one his cherished California oil pioneer friends and associates passed on, Mr. Orcutt became the prime mover in, and organizer of, the pioneers Petroleum Society of California and was its only president during his lifetime. The object of the society is to perpetuate the memory of the pioneer oil men through whose vision, courage, and labor the oil industry of California was developed; to establish a museum for the preservation of early California relics and of memorabilia regarding the development of oil in California; and to have a meeting place and recreation grounds for gatherings of oil men. The society purchased thirty-three acres of land near Newhall, Calif., embracing the site of the plant that refined the early oil produced in the Newhall district, where the growing commercial oil business of California was first firmly established. The original oil still was restored, and the building has been completed.

His inherited interest in horticulture and the animal industry expressed itself in the acquisition for a recreational hobby of several ranches, which he and Mrs. Orcutt devoted to citriculture and to the raising of purebred Hereford cattle. He managed his own ranches with the aid of his son and was also a director of the Southern California Livestock Association. An interesting side light is that some of these tracts unexpectedly became important oil producers. As another interest, his boyhood fondness for hunting and fishing was a pleasure that remained with him through life.

Mr. and Mrs. Orcutt were blessed with two children, Gertrude Logan Orcutt and John Logan Orcutt. Gertrude was married to Secondo Guasti, Jr., and has two children, Secondo Guasti III and William Orcutt Guasti. Her husband passed away in 1933 and in 1935 she was married to Jeremiah D. Maguire of New York, N. Y. John was married to Alice Marion Pedersen and has two children, William Logan Orcutt and Marshall Logan Orcutt. During the early 1920's, Mr. and Mrs. Orcutt, Sr., built a beautiful home at Rancho Sombra del Roble, Canoga Park, Calif., where they might enjoy these grandchildren and entertain their friends. This became their permanent residence shortly before his business retirement in 1939.

From 1895 to 1896 Mr. Orcutt was a member of the National Guard of California. In World War I he was in the Reserve Engineering Corps, and in World War II he was chairman of the Advisory Registration Board for District No. 176 from 1940 until his death.

In addition to his direct connection with the Union Oil Company of California, Mr. Orcutt was president of the Canoga Citrus Association, the International Development Company, and La Merced Heights Land and Water Company; vice-president of the Claremont Oil Company, the Far West Oil Company, the Garbutt Oil Company, the Los Angeles Oil Company, the Midway Royal Petroleum Company, the Outer Harbor Wharf and Dock Company, the Union National Petroleum Company, and the Union Steamship Company; and director of the California Coast Oil Company, the Community Mutual Water Company, the Lakeview Oil Company, the Southwestern Ore Company, and the Union Oil Company of Canada. He also served for many years as a director of Security First National Bank of Los Angeles.

Mr. Orcutt was a Presbyterian. He was a member of Canoga Lodge No. 611, F. and A. M.; Southern California Academy of Sciences; Seismological Society of America; Historical Society of Southern California (director); Pacific Geographic Society (member, Board of Regents); Sons of the Revolution; Society, Americans of Royal Descent; Order of the Crown; Descendants of Knights of the Garter; and Society of Mayflower Descendants. He is listed in *Who's Who in America, Who's Who in Engineering, Who's Who in Commerce and Industry, National Cyclopedia of American Biography,* and Burke's *Landed Gentry in America.*

The life of another California oil pioneer has descended below the horizon. In the bright afterglow there is satisfaction in recording that he lived it in a manner true to every inheritance and tradition of his pioneer American ancestry; that Leland Stanford Junior University has reason to be proud of his record as a pioneer graduate in fulfilling the highest ideals for which it was founded; that his successful handling of oil problems was always along lines in which every right-thinking oil man desires to see the industry grow; that, in achieving, he helped others to achieve—always with his characteristic smile and a bright twinkle in his eyes; and that, although world conditions may have grown complicated and his burdens shifted to younger shoulders, the straightforward, clean-cut, and sterling qualities as exemplified in his life afford opportunity for thought on how present and future problems may be most successfully solved.

Mr. Orcutt was elected a Member of the American Society of Civil Engineers on April 18, 1916, and became a Life Member in 1941.

HENRY BLANCHARD PRATT, M. Am. Soc. C. E.[1]

DIED JUNE 4, 1942

Henry Blanchard Pratt was born at Waltham, Mass., on February 1, 1877, a son of Edwin Beecher and Harriet Augusta (Hemmenway) Pratt. His early education was received in the Waltham (Mass.) schools, augmented by a course in structural engineering in the Boston Young Men's Christian Association. His subsequent education was self-acquired by practical experience and home study.

For three years Mr. Pratt was employed by the City of Cambridge, Mass., as rodman and transitman, until the opening of the Spanish-American War, in which he served until April, 1899. During the last three months of his service he worked with a division engineer, making reconnaissance surveys and teaching the use of surveying instruments.

After a few months' work on street railway locations, he became an assistant engineer with the firm of J. R. Worcester and Company, of Boston, Mass., a connection which continued for twenty-three years. During this time he had a varied experience as draftsman; cement tester; and designer of concrete and steel structures, subways and elevated railways, bridges, and submarine tunnels. For eight years he was in charge of the Waltham office of the firm. He also had responsible charge of the execution of much important work.

In 1922 Mr. Pratt moved to Antrim, N. H., where he joined with George H. Caughey in the firm of Caughey and Pratt, engineering contractors. This firm did a general contracting business, specializing in masonry. It designed and built many bridges and dams of small and moderate size, although much of its work consisted of repairs to foundations and buildings. The firm was dissolved in May, 1941, and Mr. Pratt entered the employ of the Public Works Department of the State of New Hampshire. His work for the state consisted of a survey of existing highway bridges, primarily to determine their capacity and to form an accessible record of them. He was still at this work at the time of his sudden death.

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[1] Memoir prepared by Joseph R. Worcester, Hon. M. Am. Soc. C. E. Mr. Worcester died on May 9, 1943.

Mr. Pratt was a useful citizen and had served as a Selectman of Antrim; he was for many years a Trustee of the James A. Tuttle Library, in Antrim, in which he was greatly interested. His greatest pleasure was fine music. He was blessed with a remarkable memory, not only for facts and figures, but for personal experiences which, together with a keen sense of humor, endeared him to all of his large number of friends and associates.

In 1909 he was married to Clara Emerson of Windham, N. H. He is survived by his widow and six children, Henry Blanchard, Jr., Dorothy E. (Mrs. Paul M. Jaquith), Margaret (Mrs. Robert A. Caughey), Ruth L. (Mrs. Paul W. Demarest), Judith D., and Benjamin.

Mr. Pratt was elected an Associate Member of the American Society of Civil Engineers on June 1, 1909, and a Member on January 16, 1917.

CHARLES MILLER REPPERT, M. Am. Soc. C. E.[1]

Died September 16, 1939

Charles Miller Reppert, the son of Henry and Nancy (Miller) Reppert, was born at Verona, Pa., a suburb of Pittsburgh, Pa., on October 10, 1880. His education was obtained in the grade and high schools of Pittsburgh and at Cornell University, Ithaca, N. Y., from which he was graduated in 1904 with the degree of Civil Engineer.

In September of that year, Mr. Reppert began his professional career as draftsman in the Bureau of Filtration of the City of Pittsburgh, under the late Morris Knowles,[2] M. Am. Soc. C. E. For twenty-five years (interrupted for brief periods by World War I and by service with Allegheny County, Pennsylvania) he worked for the city. Early in his career Mr. Reppert showed two outstanding qualities—an unusual ability in design and the capacity to direct large numbers of technical employees effectively. He rose from draftsman to assistant engineer in charge of construction of a 50-in. steel water line more than 5 miles long, including a 36-in. river crossing. In October, 1907, when he was transferred from the Bureau of Filtration to the Bureau of Construction in the Department of Public Works, he was placed in charge of designs, specifications, and contract plans for extensive street and sewer improvements and for miscellaneous structures.

During this period, Mr. Reppert also directed many special investigations and reports upon engineering projects. Among these was a study of retaining wall design. He also urged the adoption of standards for both plain and reinforced-concrete walls under which many large structures were built. The sewerage problems of Pittsburgh particularly interested him, and he was active in developing plans for a comprehensive sewerage system, supervising much of the work. One phase of this investigation was the estimation of the cost of installing a separate sanitary system of sewers for the entire city which, at that time, was served by combined sewers only.

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[1] Memoir prepared by John M. Rice, Nathan Schein, and John D. Stevenson, Members, Am. Soc. C. E. Mr. Schein died on January 8, 1943.

[2] For memoir, see *Transactions*, Am. Soc. C. E., Vol. 98 (1933), p. 1567.

In 1917, Mr. Reppert was assistant to the Chief Engineer of the Bureau of Engineering (formerly the Bureau of Construction). Wishing to take a more active part in the war, he secured a furlough from the city and associated himself with Morris Knowles, as deputy supervising engineer on the construction of Camp Meade, Md. When this work was completed in the fall of 1917, he returned to Pittsburgh, and in the early part of 1918 he became deputy chief engineer and was later made chief engineer.

Leaving city employ, he was appointed assistant general manager of the Housing Division of the Emergency Fleet Corporation, U. S. Shipping Board, and remained in this position until most of the housing developments had been liquidated. He then joined the firm of Morris Knowles, Inc., consulting engineers of Pittsburgh.

Upon the resumption of a public works program in Pittsburgh, Mr. Reppert was induced to return as Chief Engineer of the Bureau of Engineering in 1921. He remained in this position until 1922, when he consented to act as assistant director of the Department of Public Works of Allegheny County, which had just been organized by the late Norman F. Brown,[3] M. Am. Soc. C. E. In this position, he had general supervision of planning and designing major highway and bridge improvements for the county.

Again, in 1926, he returned to city employ as Chief Engineer of the reorganized Department of Public Works, remaining in this position for the next ten years. During this time he had supervision of all public work improvements, including major boulevard and street projects; rehabilitation and improvement of the water works system; bridge construction and reconstruction; design and construction of trunk sewers; and many special engineering problems.

In 1936, he retired from the city's service and entered private practice as a consulting engineer in Pittsburgh. Among the projects which he supervised during this period were Mellon Institute of Industry and Research and a large private low cost housing project in Clairton, Pa.

Mr. Reppert was killed by a fall on September 16, 1939.

Mr. Reppert's professional work was characterized by thoroughness and attention to detail, and its imprint has been left on many of the major projects constructed by the City of Pittsburgh and Allegheny County. He was also interested in city planning and gave a large amount of his time to developing an appreciation of this work. His sincerity of purpose was appreciated by his colleagues as was shown when he was elected first, director, and then, president of the Pittsburgh Section of the Society. For three years he was a member of the Society's licensing committee. At the time of his death, Mr.

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[3] For memoir, see *Transactions*, Am. Soc. C. E., Vol. 105 (1940), p. 1806.

Reppert was Vice-President of the Society. Also, he was an active member and past-president of the Engineers' Society of Western Pennsylvania, and was a former president of the American Society of Municipal Engineers (later American Public Works Association).

Mr. Reppert was married on April 20, 1911, to Blanche Christy Stevenson of Pittsburgh, who, with their son, Charles Miller, Jr., survives him.

Mr. Reppert was elected a Member of the American Society of Civil Engineers on October 1, 1913.

BARZILLAI ALLEN RICH, M. Am. Soc. C. E.[1]

Died January 26, 1943

Barzillai Allen Rich was born in Hingham, Mass., on September 18, 1877, the son of Barzillai Warren and Betsey (Dorethey) Rich. His ancestors were among the early Cape Cod settlers at Truro, Mass.

His early education was received in the public schools at Hingham. He entered Tufts College at Medford, Mass., in 1895, and, although a student there for only one year, he was awarded, in 1924, the degree of Bachelor of Science by Tufts College, designated as a "Degree Extra Ordinem," in recognition of his being the outstanding engineer of his class.

On leaving Tufts College, he obtained employment with the Metropolitan Park Commission of Boston, Mass., which was engaged actively at the time in developing the extensive parkway system of metropolitan Boston. Mr. Rich was employed on this work for about eleven years, successively as rodman, transitman, and chief of party on surveys and road work, and, during most of the time, as assistant engineer on the design and construction of numerous concrete bridges.

In 1907 and 1908, Mr. Rich was engineer and superintendent for Austen Engineering and Construction Company, of Boston, on concrete construction work. During the next two years he was engineer and superintendent for Moore and Company, of Boston, on water works systems built and operated by this company in various New England municipalities. The following year and a half, Mr. Rich was employed by the Metropolitan Water Works, of Boston, on drainage work. From 1912 to 1915, he served as assistant engineer on the Boston Transit Commission, designing subway sections and tunnels.

Mr. Rich resigned from the employ of the Boston Transit Commission to become, in 1915, assistant engineer with the firm of Fay, Spofford and Thorndike, consulting engineers, of Boston. He brought to the firm a broad experience in the design of various types of engineering structures—sewers and drains, bridges, foundations, and buildings. During the time he was with Fay, Spofford and Thorndike, some of his most important work was done, including (1) the design of freight-handling equipment and utilities for the $25,000,000

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[1] Memoir prepared by R. W. Horne, M. Am. Soc. C. E.

Boston Army Supply Base (built in 1918 as an emergency wartime project for the Quartermaster Corps of the U. S. Army), and (2) in 1919, the design of long-span, reinforced concrete arch ribs for the Hampden County Memorial Bridge across the Connecticut River at Springfield, Mass.

In 1920 and 1921, Mr. Rich was engaged as supervising engineer by Lockwood Greene Company of Boston on the design and construction of industrial plants, foundries, and machine shops.

Mr. Rich again became associated with Fay, Spofford and Thorndike in 1922, and, with several other employees of the firm, was made a junior partner. He remained with the firm for somewhat over two years, being engaged largely on investigations and reports on projects involving flood protection, land development, freight handling, sewerage, and drainage. He resigned from his partnership with Fay, Spofford and Thorndike in 1924 to establish the firm of Rich, Bigelow and Tirrell, structural engineers. This firm was dissolved early in 1926. For nearly ten years Mr. Rich served as a structural engineer with Monks and Johnson, of Boston. Then followed two years during which he had miscellaneous engagements on structural engineering work. In 1937, he became connected with Jackson and Moreland of Boston, as a structural engineer, and was employed by this firm at the time of his death on January 26, 1943.

Always interested in technical affairs, Mr. Rich was author of technical papers pertaining to complicated problems in structural design. A keen mathematical mind and an excellent memory account for his unusual ability in matters of engineering design.

He enjoyed the high respect of his numerous friends and those with whom he worked. The possessor of a pleasing personality, he took much pleasure in narrating incidents from his varied and interesting experiences. Relaxation from his technical work was obtained from research on his family genealogy, collecting stamps, and manual labor upon his property.

On October 9, 1900, he was married to Amy Inez Munsey at Lynn, Mass. Five children were born from this marriage. Mr. Rich is survived by his widow; four children, Mrs. John W. Button, Mrs. George Puopolo, Albert Barzillai Rich, and John Warren Rich; and thirteen grandchildren.

Mr. Rich was a member of the Boston Society of Civil Engineers, Boston City Club, and Wollaston Lodge, A. F. and A. M.

Mr. Rich was elected a Member of the American Society of Civil Engineers on January 13, 1919.

WALLACE CARL RIDDICK, M. Am. Soc. C. E.[1]

DIED JUNE 9, 1942

A young Virginian, a descendant of early settlers, left Gates County, North Carolina, some time in the eighteen fifties to go to North Carolina State

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[1] Memoir prepared by Frank T. Miller and George H. Maurice, Members, Am. Soc C. E.

University at Chapel Hill, N. C. When passing through Wake Forest, N. C., he stopped for the night with the result that he remained there, and matriculated at Wake Forest Baptist College. This man, Wiley Goodman Riddick, met his destiny there, was married to Anna Ivey Jones, bought land in Wake County, and settled down on a 600-acre farm adjoining that of his wife's family. When strife between the states developed into warfare, he joined the army of the Confederacy and won a lieutenant's commission. He returned home on a furlough in the late fall of 1863 to harvest his crops, and, on August 5, 1864, a son was born.

Wallace Carl Riddick grew up under the hardships of the reconstruction period. The father's spirit to fight for the cause in which he believed descended to the son, and this spirit was developed by the struggle for existence during the dark days of the seventies and early eighties. All his family had been college graduates, and young Riddick was determined to follow the family tradition, which he achieved by hard work. He entered Wake Forest College in 1881, but, at the end of his junior year, he and all other members of the Kappa Alpha fraternity were expelled, as secret societies were forbidden. He then went to North Carolina University at Chapel Hill, where no credits were allowed from what the university faculty deemed an inferior institution, and examinations on all subjects up to the end of the junior year had to be passed. Mr. Riddick accomplished this, and in June, 1885, was graduated from the university. After teaching school at Mount Airy, N. C., for two years, he went to Lehigh University, Bethlehem, Pa., from which he was graduated with the degree of Civil Engineer in June, 1890.

One of the writers of this memoir, as an undergraduate, remembers Mr. Riddick who, as a senior, was a tall, lean, active man much engaged in the activities of Lehigh University. A rereading of the undergraduate publication *The Epitome* for the years Mr. Riddick was a student demonstrates that, in addition to standing well in his classes (as evidenced by several prizes for excellence in studies and membership in Tau Beta Pi), he was prominent in athletics and class activities. He won four first prizes in the winter sports meet in 1888, breaking the record in the standing high jump, 4 ft 8.5 in., and in the fence vault, 6 ft 8 in. He also won first prizes in heavyweight and middleweight wrestling. In 1888, he was a regular member of the football team, and in 1889, when Lehigh University won the football college championship of Pennsylvania, Mr. Riddick played in almost one half of the games. He was catcher on his 1890 class baseball team.

His other activities at Lehigh included: Vice-president of his class (1890) in his junior year, member of the 1890 *Epitome* board, member of the civil engineering society (its president in his senior year), member of the Chess Club, and member of the Ace Club. During a spring vacation he returned to Wake Forest College and coached the football team to victory over their Southern antagonists. This was one of his pleasantest recollections, for, in those days, football in the South was played as a sport without regard to seasons, although it was discontinued when the weather became too hot. Later, during his early years on the North Carolina State College faculty, Professor Riddick coached the athletic teams, and, when the college stadium was completed, it was named Riddick Stadium for him in recognition of his aid in developing athletics—an essential element of student activity.

His first engineering work after being graduated from Lehigh University was as engineer of construction for the Graystone Granite and Construction Company from 1890 to 1892, on a canal for development of water power, near Weldon, N. C. While on this work he met Lillian Daniel to whom he was married on October 18, 1893.

In 1892 he was appointed professor of civil engineering at North Carolina College of Agriculture and Mechanic Arts (later North Carolina State College), Raleigh, N. C., which was then in its infancy, having been founded in 1889. At this time, he began the work that was to be his principal occupation and ended only with his death. He organized and built up his department to such a recognized position among the best technical schools that he has been acclaimed "the father of professional engineering and engineering education" in North Carolina. For years he was known affectionately by thousands of the men who grew up under his instruction and influence as "Pap Riddick." In 1908 he was elected vice-president of the college, and in 1916, president. During his term as president, the college made remarkable progress—new buildings were built and the faculty and the student enrolment increased nearly 70%. In 1923 he resigned from the presidency to become the first dean of the school of engineering, the development of which had more appeal to him than administrative duties. Until 1937, when he retired from active work, he served in this capacity. However, he insisted on continuing his teaching career and became dean emeritus of engineering and professor of hydraulics. At the time of his death he had rounded out a full fifty years with North Carolina State College.

Aside from his college duties, Dean Riddick was active as a consulting engineer on many and varied projects. Among these were: Design and construction of sewers for Southern Pines, N. C., in 1900; building a system of sewers and additions to water supply for Lumberton, N. C., in 1903; reconstruction of the Raleigh water works in 1912; and work on a water supply and sewerage system for Weldon, in 1912. For years, until 1916, he was consulting expert to the legal department of Seaboard Air Line Railway Company and, from 1936 to his death, consulting hydraulic engineer. As a consulting engineer he was, for many years, in great demand on many private and public utility projects and was outstanding as an expert witness before the courts.

Dean Riddick took an active interest in the Society and in the North Carolina Section, serving on its various committees, and, in 1940, as president of the Local Section. He was a Lieutenant Colonel on the staff of the Hon. Robert B. Glenn, Governor of North Carolina, 1905–1909; a member of the North Carolina State Highway Commission, 1915–1919; a member of the State Board of Vocational Education, 1917–1919; a member of the North Carolina Department of Conservation and Development, 1918; a member of the Board of Visitors, U. S. Naval Academy at Annapolis, Md., 1920–1921; a member of the North Carolina Academy of Science; a member of the North Carolina Teachers Association; and one of the founders of the Raleigh Engineers

Club. In civic life he was also active as a member of the Wake County Board of Elections, as a member of the Wake County Board of Education, and as chairman of the Wake County Selective Draft Board No. 1.

Dean Riddick always had the interest of his young engineers at heart, and, with this end in view, he was active in starting the North Carolina Society of Engineers in 1918 for the purpose of drawing together graduate engineers of all branches before they advanced to membership in the Founder Societies. He was its president in 1919. He was always a fighter and a champion for the cause in which he believed, whether on the athletic field, in the classroom, or on the witness stand. Often, he said that conversation was futile unless argumentative. When the Brookings Institute recommended to the Hon. O. Max Gardner, then Governor of North Carolina, that the engineering school of North Carolina State College be transferred to Chapel Hill, and that North Carolina State College be relegated to the position of a junior college, Dean Riddick rebelled and his influence had much to do with the failure of the proposal.

In 1917 Wake Forest College and Lehigh University both gave Dean Riddick the honorary degree of Doctor of Laws, and, in 1939, North Carolina State College on its fiftieth anniversary honored him with the degree of Doctor of Engineering. He was a Knight Order of Saint Sava (Yugoslavia). He was a member of Capital Club, Rotary Club, and Raleigh Country Club. His farm in Wake County, which he was constantly improving by conservation practices, was his hobby and recreation. It was here that he spent many afternoons, especially in later life, enjoying the *otium cum dignitate* (leisure with dignity).

Dean Riddick was hale and hearty until a year before his death. Mrs. Riddick died on January 28, 1942, and he never fully recovered from the shock. On June 9, 1942, he passed away in a Baltimore (Md.) hospital where he had gone for treatment.

In the words of the head of the North Carolina State College Civil Engineering Department, C. L. Mann, M. Am. Soc. C. E.: "His was a life of service and accomplishments, full of honors and of great and good work exceptionally well done. He was ever a seeker of the truth."

Dean Riddick is survived by five children—Wallace W., and four daughters, Lillian, Narcissa, Anna, and Eugenia.

Dean Riddick was elected a Member of the American Society of Civil Engineers on February 25, 1924.

FRED ELMER RIGHTOR, M. Am. Soc. C. E.[1]

DIED MARCH 1, 1942

Fred Elmer Rightor was born at Rockford, Ill., on August 8, 1882, the son of Elmer Abram and Arabella Victoria (Drummond) Rightor. He attended the University of Illinois at Urbana, and was graduated in 1903 with the degree of Bachelor of Science in Civil Engineering. Later, in 1917, after graduate work on testing bituminous materials, he received a degree from the University of Texas, at Austin. At the University of Illinois he was elected a member of the Delta Upsilon fraternity.

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[1] Memoir prepared by J. H. Briggs, Affiliate, Am. Soc. C. E.

In order to gain practical experience, during the summers of 1900 and 1902, Mr. Rightor worked as a section hand and track apprentice for the Illinois Central Railroad Company. From June, 1903, until June, 1906, he was with the Illinois Central and Yazoo and Mississippi Valley railroad companies, as chainman, rodman, instrumentman, and assistant engineer on maintenance. Leaving railroad work, from June, 1906, to October, 1907, he served as division engineer for the Dering Coal Company of Clinton, Ind., and from October, 1907, until January, 1909, as chief engineer for the Consolidated Indiana Coal Company.

In 1908 Mr. Rightor moved to Texas. A man with an open heart and cordial manner, he fitted perfectly into his new background. In Texas, Mr. Rightor became associated with the Texas Bitulithic Company. He remained with this firm for nearly thirty years, becoming president and treasurer of the Southwest Bitulithic Company of San Antonio, Tex. In September, 1937, he was appointed a member of the First Texas State Board of Registration for Professional Engineers, which had been created, after a hard fight, largely through the efforts of the Texas Section of the Society, of which Mr. Rightor was an active member. Since it was necessary for the new board to proceed with proper caution and regard, Mr. Rightor's appointment was most appropriate. At the first meeting of the Board on October 1, 1937, Mr. Rightor was elected secretary and held this position until his death.

As secretary of the Texas State Board of Registration for Professional Engineers he helped to place it in the front rank of such bodies. Because of his service on a board of such high reputation, Mr. Rightor was soon asked to take an active part in the National Council of State Boards of Engineering Examiners. He was elected Director of the Southern Zone (one of five zones into which the United States is divided) for 1939–1940, and also served on many committees. Through his thoughtfulness for every one, he made a host of friends.

More than most men, he had a sincere love for his fellows. Overpatient with the man seeking employment, he was what is known as an "easy touch." Mr. Rightor was a wonderful mixer and belonged to many organizations. Among his affiliations were the Elks, the Masons, the Texas Society for Professional Engineers, and the Society of American Military Engineers. Throughout the years he attended the Local Section meetings of the Society most regularly.

He has gone on to build engineering structures in other worlds under the Great Engineer who sends the countless spheres whirling through endless space and yet heeds the sparrow's fall. Over Mr. Rightor's grave might be put the

Engineer's Epitaph

Good earth, I've moulded you in many ways,
Together have we been thru all the years
And now my plans are finished, and my days,
Are done. My soul He keep
So tuck me in and lightly cover me,
For I would sleep.

Surviving Mr. Rightor are his wife, Ellen Elizabeth (Nevins) to whom he was married on November 29, 1931; his son, Edward Elmer; one brother, Chester; and his parents.

Mr. Rightor was elected an Associate Member of the American Society of Civil Engineers on May 31, 1910, and a Member on September 10, 1918.

NATHAN SCHEIN, M. Am. Soc. C. E.[1]

Died January 8, 1943

Nathan Schein was born at St. Petersburg (later Leningrad), Russia, on March 22, 1884, the son of the late Joseph D. and Mary Schein. He was graduated from the Birmingham Sub-district School in the old twenty-eighth ward of the City of Pittsburgh, Pa., in June, 1898, and from the Central High School of Pittsburgh in June, 1902. He then matriculated at Cornell University, Ithaca, N. Y., and was graduated in 1906 with the degree of Civil Engineer.

After a short engagement with the Toledo-Massillon Bridge Company, Toledo, Ohio, he entered the City of Pittsburgh Filtration Bureau in January, 1907, as levelman under the late Morris Knowles,[2] M. Am. Soc. C. E., later being promoted to draftsman. In January, 1909, he was transferred to the Bureau of Construction (later the Bureau of Engineering of the Department of Public Works) as draftsman. His term of service with the City of Pittsburgh lasted continuously until January, 1942, with successive promotions to assistant engineer, division engineer, principal assistant engineer, engineer of parks and playgrounds, and senior designing engineer.

For a six months' period in 1917, Mr. Schein was assistant engineer at Camp Meade, Maryland, engaged in the construction of roads, sewers, and buildings. For a short time in 1942 he worked for the Dravo Corporation of Pittsburgh on the expansion program at its Neville Island plant.

His meritorious professional career was eclipsed by his activities in the Society; he was probably the best-known Society man in the Pittsburgh region. In April, 1916, associated with Robert A. Cummings, T. J. Wilkerson, the late C. M. Reppert, and the late Richard Khuen, Jr.,[3] Members, Am. Soc. C. E., and the late George S. Davison, Past-President and Hon. M. Am. Soc.

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[1] Memoir prepared by Louis P. Blum and John D. Stevenson, Members, Am. Soc. C. E.
[2] For memoir, see *Transactions*, Am. Soc. C. E., Vol. 98 (1933), p. 1567.
[3] For memoir, *ibid.*, Vol. 105 (1940), p. 1869.

C. E., he was active in the formation of the Pittsburgh Section. He became secretary-treasurer of the Section about 1919 and served continuously in that office until 1937. No one could have been more earnest and unselfish in service to the Section than Nathan Schein. He knew and was known by practically every member of the Society around Pittsburgh. Earnest and hard working, he was the acknowledged local authority on the workings of the Society and the guide in the work of almost all committees of the Section. After his retirement as secretary-treasurer, he served the Pittsburgh Section as a director.

He is survived by his widow, Sarah J. Schein, to whom he was married on October 24, 1922, in Syracuse, N. Y. Also surviving are a son, Charles Joseph Schein; a daughter, Marian Jean Schein; his mother, Mary Schein; and his brother, Saul Schein.

Mr. Schein was elected an Associate Member of the American Society of Civil Engineers on March 2, 1915, and a Member on October 12, 1925.

ARTHUR MONROE SHAW, M. Am. Soc. C. E.[1]

Died August 2, 1942

Arthur Monroe Shaw was born in Lee Center, Ill., on December 12, 1870, the son of Monroe Shaw and Rebecca (Linn) Shaw.

After his preparatory education in the elementary schools of his home town, he attended Dixon Business College in Dixon, Ill., and Cornell College at Mount Vernon, Iowa.

At the termination of his college career he entered the employ of the Engineering Division of the Illinois Central Railroad Company. From 1891 to 1909 his field of activity embraced railroad, drainage, and irrigation projects of note in the United States, in Mexico, in Honduras, and in Colombia and other sections of South America, both as a private practitioner and as an employee of the United States Government.

From 1909 to 1929, Mr. Shaw continued his work, mainly in these chosen fields, with the exception of that period embracing the World War from 1917 to 1919. With the rank of Major of Engineers, he was assigned to duty with the Construction Division of the Constructing Quartermaster, U. S. Army, and supervised the building of Camp Beauregard in Louisiana, Camp Jesup in Georgia, and Camp Normoyle in Texas. He was later commissioned as Lieutenant Colonel of Engineers, Officers Reserve Corps. His previous military duties had been in the Spanish-American War, where he served as Corporal, and later as Sergeant, in the 2d United States Voluntary Engineers.

Other activities engaged in by Colonel Shaw prior to the year 1929 included the design and supervision of the installation of the National Shipbuilding Corporation at Violet, La.; the Lukens Steel Company in New Orleans, La.; the Harrison County Seawall in Mississippi; and the Todd Engineering and Shipbuilding Company at Algiers, La.

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[1] Memoir prepared by Charles M. Kerr and Ole K. Olsen, Assoc. Members, Am. Soc. C. E., and C. Glenn Cappel and John Riess, Members, Am. Soc. C. E.

From 1929 to 1933 was, perhaps, the most interesting era of Colonel Shaw's professional career. During these years he was retained as chief consultant by the Chinese National and the Chekiang Provincial governments, and supervised the construction of the Hangchow-Kiangshan Railway, which forms part of the National Railway System of China. During his residence in China, Colonel Shaw also made a study of various ancient irrigation works in the eastern section of that country, and worked out revisions of these systems to be used as a basis for their modernization. He also acted as consultant and designing engineer in the reconstruction of the Hangchow Seawall along the Chien Tang River.

Upon the termination of his activities in China, he returned to the United States and entered the employ of the Public Works Administration (PWA) in an administrative capacity, continuing with that organization until 1940.

With the outbreak of World War II he again gave his entire service to his country, but this time he was employed by W. Horace Williams, M. Am. Soc. C. E., as supervising engineer in charge of the construction of Camp Claiborne and Camp Polk, both in Louisiana. He died while on duty in Surinam (Dutch Guiana) in charge of important construction work involving the installation of a project for the Aluminum Company of America.

At various periods of his career Colonel Shaw contributed many valuable technical writings, based upon his experience with, and intimate knowledge of, railway construction, drainage and irrigation developments, and the design and operation of dredging machinery. He was internationally known as an engineer of exceptional ability, and he enjoyed an enviable reputation among the members of his profession. He was recognized as a person of the highest integrity, and his associates knew him as a constant and loyal friend. The death of Colonel Shaw is an inestimable loss to the members of his profession, to a host of friends, and to those who were so fortunate as to be numbered among his associates.

In 1901 he was married to Henriette Otis in Council Bluffs, Iowa. He is survived by his widow and four children, Frances Shaw (Mrs. Eric A. Lansdown), M.D., Phyllis Anne Shaw (Mrs. Fredrick Steinberg), Edwina Shaw (Mrs. James H. Ward), and Richard Monroe Shaw.

He was a member and past vice-president of the Association of Chinese and American Engineers, and a life member and past-president of the Louisiana Engineering Society.

Colonel Shaw was elected an Associate Member of the American Society of Civil Engineers on January 3, 1906, and a Member on September 5, 1911. He became a Life Member in 1941.

FRANK HERBERT SNOW, M. Am. Soc. C. E.[1]

DIED OCTOBER 29, 1942

Frank Herbert Snow was born in Providence, R. I., on November 2, 1865, the son of Levi Franklin and Sarah Ann (Drake) Snow.

In May, 1882, when only seventeen years of age, he was employed by Thomas Keith, surveyor, of Brockton, Mass. Later, he worked for E. L. Brown, engineer, of the same city. From 1887 to January, 1890, he was employed in the city engineer's office in Brockton in succeeding steps: First as rodman, then as chief of party, and finally as assistant engineer. In January, 1890, he became city engineer and surveyor in responsible charge of all city surveying and engineering. In this capacity, he designed and constructed a system of sewerage and sewage treatment for the city and originated the municipal sewer tax, or rental system, which was an innovation of that day. He designed the elimination of, prepared specifications for, and supervised construction of thirteen grade crossings; built the water works, pump station, and standpipe; and made a report on investigation and plans for a new source of water supply. After he left the city employ, his recommendations were adopted, and the works were constructed.

From the spring of 1896 to the summer of 1905, Mr. Snow engaged in private practice with Frank A. Barbour, M. Am. Soc. C. E., with offices in Boston, Mass., and Columbus, Ohio. His firm conducted a general civil and sanitary engineering practice dealing with the design and construction of numerous works. These included the design for sewerage and sewage treatment for the Metropolitan Water Board at Clinton, Mass., the design and construction of water works at Peabody, Attleboro, and Mansfield, Mass., and similar work at St. Johns, N. B., Canada. His company designed a new water supply for Youngstown, Ohio; and designed and constructed sewerage and sewage treatment works at Mansfield and Lakewood, Ohio; Saratoga Springs, N. Y.; Atlantic City, N. J.; Hudson, Peabody, and Bridgewater, Mass.; and numerous other places. While he was in general consulting practice, Mr. Snow was employed first by the City of Boston, in 1897–1898, preparing the rate schedule for the Boston Main Drainage System and later by the City of Taunton, Mass., on a similar project. During this period, he continued to make reports and give expert testimony before various courts.

The years from 1896 to 1905 may be considered as the first period of Mr. Snow's career, ending when he moved from New England to Pennsylvania, where he lived until the time of his death, thirty-seven years later. Early, he displayed the broad vision which characterized his entire life. He was not afraid to try new things, as is shown by the praise given him in the Brockton *Daily Enterprise* (November 3, 1942). The tribute stated:

> "Fifty years ago a young man of promise was City Engineer of Brockton, Massachusetts. At his death in Harrisburg, Pennsylvania, October

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[1] Memoir prepared by Howard E. Moses, M. Am. Soc. C. E.

29, 1942, the promise had been fulfilled. In Brockton, and elsewhere, are monumental works bearing the name of F. Herbert Snow.

"Brockton is credited with being the first city in the country to abolish grade crossings. The engineering on the project was by Mr. Snow, then a young man.

"The City's sewage disposal system, another piece of pioneering, also bears his name. It was a bold step in its time, an experiment. There were those who called it gambling with public funds. But the system proved efficient. As such, it received international attention as the solution of a problem which had beaten the less venturesome.

"He planned the present water supply system, carried out by his successor.

"Later, as a widely known public official and consulting engineer, his impress upon other important public works was as marked.

" 'The City was young,' he once said. 'So was I. We were not afraid to venture beyond the beaten path.' "

The second period in his life began with his appointment as chief engineer of the Pennsylvania Department of Health in 1905. At that time the field of sanitary engineering was not so well developed as at present, and men of outstanding ability and reputation were not plentiful. Samuel G. Dixon, M.D., then organizing the newly formed Pennsylvania Department of Health, is said to have heard a paper presented by Mr. Snow descriptive of work done in Atlantic City and to have been so impressed with his experience, ability, and character that he offered him the position of chief engineer of that department. So he went to Pennsylvania, a state of which he had but little intimate knowledge, and where he was to live the remainder of his life, learning to know it thoroughly by travel, study, and contact with its people. He became a true Pennsylvanian, an outstanding citizen, attaining fame as an engineer, and, at all times, directing his energies toward the advancement of his adopted state.

As chief engineer of the Pennsylvania Department of Health, he went to work with his usual vigor. The first step was to compile data and plans of all public water works and public sewerage systems in Pennsylvania. Comprehensive questionnaires were mailed throughout the commonwealth and the information thus secured formed the basis of the voluminous records now on file in the Department of Health. Always a man of broad vision, he early established the principle of comprehensive planning with an eye toward the future. This was especially true in connection with metropolitan sewerage. In general, he found that sewers had been installed chiefly to meet present needs without much regard to their relation to a comprehensive system for a municipality. He pointed out the poor economy of such a system and urged that further design should take into consideration not only the future development of the community and its environs but also the relationship to the needs of neighbors. Cooperation between communities having mutual problems was stressed on the ground that such problems could best be solved by collaboration rather than individually. In this he naturally encountered the traditional jealousy and enmity between communities, but despite this and other deterrents, he adhered to this principle, and it is still applicable today.

Although, in general, the major supplies in Pennsylvania were secured from the larger surface streams, the field of public water works was unorganized at this time. At the beginning of Mr. Snow's incumbency as chief engineer, few of these supplies were adequately purified. As there were only a few filtration plants, unfiltered water from contaminated rivers was being served in the principal cities of the state. This situation was reflected in the high state typhoid fever rate (as might be expected from present-day knowledge of the relationship between polluted water supplies and water-borne disease). Purification of public water supplies was in its infancy, and, in 1905, chlorine as a purifying agent was just beginning to be considered. In fact, chlorine was first used in Pennsylvania on a public water supply in 1909, four years after Mr. Snow's appointment. Building on Mr. Snow's pioneer work on the purification of Pennsylvania's public water supplies, a vast system of water works has spread throughout the state, supplying water to approximately 8,000,000 people, or 80% of the total population, virtually all of whose supplies are safeguarded by filtration and chlorination, or at least by chlorination.

Perhaps the best measure of the value of this work is found in reduction of typhoid fever and other water-borne diseases. In 1906, the death rate from typhoid fever in Pennsylvania was 54.8 per 100,000 population. This was the rate for the entire state, with the rate for many of the municipalities greatly in excess. Cities were typhoid-ridden and epidemics were expected to occur and they did.

While he was chief engineer, Mr. Snow had responsible charge of the supervision of more than forty water-borne epidemics in Pennsylvania. At the time he left the Department of Health, the typhoid fever rate had been reduced measurably and has since reached rather negligible figures. As part of his duties while chief engineer, he directed design and construction of engineering and surveying for three Tuberculosis Sanatoria at Mont Alto, Cresson, and Hamburg, Pa.; reclamation work at Austin, Pa., after the flood disaster of 1911; and housing and care of Civil War veterans attending the fiftieth anniversary reunion at Gettysburg, Pa., in 1913. For this latter work he was awarded a special medal by the Federal Reunion Commission. At this time he seems to have been given the honorary title "Doctor," by which he was known thereafter—a title which seemed to fit this man who had so much vision and ability and pioneering spirit.

In January, 1914, Mr. Snow was appointed to the position of chief engineer of the Public Service Commission of Pennsylvania and continued in that capacity for eighteen years. As he had done in the Department of Health, he built up the engineering bureau of the commission, making it an important unit in the work of that body. He supervised all engineering work, including such matters as design and construction of bridges, service and facilities of public service companies, plans for elimination of grade crossings, and enforcement of the commission's rules and regulations regarding operation of gas and electric light and power, water, and steam heat companies. He also gave expert testimony before the commission which handles many hundreds of cases in the aforementioned field.

That projects of great magnitude—grade crossings, separation, elimination, and protection—were involved is indicated by the fact that in thirteen years the work done by Mr. Snow for the commission cost $153,000,000. At one time he reported that, for the last few years before his retirement from the commission, the cases tried and passing through the bureau of engineering averaged an annual cost for engineering and allied charges of $500,000 (not including state engineering), which gives a rough measure of the vast volume of engineering coming under the scope of the bureau of engineering.

Age limitation caused Mr. Snow's retirement in 1932, ending the second period of his engineering career, but by no means marking his retirement from active life, a step which would have been unthinkable to a man of his energetic nature. The third period of his life began when he organized the engineering firm of Snow-Weaver Industrial Management, specializing in the rehabilitation of industrial concerns. At this time he was elected secretary of the Pennsylvania Water Works Association, which office he held until his death. He rendered invaluable service to this organization, aiding in the maintenance of its membership and participating in the enlargement of its functions. At its annual meeting in October, 1940, he was elected an honorary life member of the association "in recognition of pioneer work and outstanding accomplishments in matters relating to public water supply."

Since his services as consultant were still in demand, he found time to serve as engineering consultant to the board of surveys of the District of Columbia on the water works system of that District. In the early summer of 1941, he was appointed consulting engineer to the City of Harrisburg. It was while he was at City Hall on October 28, 1942, that he suffered the cerebral hemorrhage which caused his death on the following morning—thus concluding an active life covering a period of sixty years.

Mr. Snow was proud of his profession and labored unceasingly for the welfare of engineers in order that they might be granted their proper place in civil and national life. He had a burning passion to bring about organization among engineers and spent freely both of his time and of his money to accomplish this end. Chiefly responsible for founding the Engineers Society of Pennsylvania, he became its president, and, in recognition of his services to that association, he was made an honorary member in 1937. Also, he was a past-president of the Philadelphia Section of the Society, a member of the American Water Works Association, and a Registered Professional Engineer in Pennsylvania.

A movement for state registration of professional engineers was started in Pennsylvania in 1908 by Mr. Snow, although not much was accomplished for some time. The first real achievement was made when the Hon. John K. Tener, then Governor of Pennsylvania, appointed a committee of five engineers, with Mr. Snow as chairman, to conduct meetings in different parts of the state to formulate a law for the registration of professional engineers. In 1921 such a law was passed licensing professional engineers in Pennsylvania. This was contested and declared unconstitutional by the Pennsylvania Supreme Court. Mr. Snow, still undaunted, continued the battle; a new bill was drafted as changed to meet the unconstitutional parts of the former one; and, after persistent efforts, this law was passed in 1927.

During the course of his public life, Mr. Snow wrote a great many reports, probably numbering several thousand. These are of permanent record in the state departments he served and show not only his facilities as an author, but also his broad grasp of affairs, his keen analytical mind, and his excellent judgment in appraising the conditions pertaining to the case at hand. He wrote numerous technical papers and frequently appeared before municipal, professional, and civic organizations to present papers on current topics. Active in civic affairs, he always could be found consistently advocating measures he deemed beneficial to the general public. He had a wide acquaintanceship and numbered among his friends men prominent in the fields of engineering, business, and municipal life.

Mr. Snow was a member of the Episcopal Church and of both the York Rite and Scottish Rite Masonry. He is survived by his widow, Maude Presby Snow; a sister, Mrs. Charles Sampson; and a brother, William Leonard Snow.

Mr. Snow was elected an Associate Member of the American Society of Civil Engineers on June 4, 1895, and a Member on May 3, 1904.

JOHN RODGERS SPELMAN, M. Am. Soc. C. E.[1]

Died February 27, 1943

John Rodgers Spelman was born at Quincy, Mass., on February 18, 1872. His parents were William DeyErmand and Julia Frances (Rodgers) Spelman. He was educated at the Albany (N. Y.) High School and Pratt Institute, Brooklyn, N. Y., being graduated from the latter in 1891. Participating in athletics during and after his school years, he was captain of the Institute football and baseball teams and later of a club football team called the "Columbians" on whose roster were the names of men later distinguished in business and professional life. His father was a Civil War veteran and, for a time, was supervising architect for the Standard Oil Company. His grandfather, Capt. Benjamin Root Spelman, was a charter member and, from 1850 to 1861, the Commandant of the Albany Burgesses Corps (organized in 1833 as a militia of the Union and later to become the New York National Guard).

Mr. Spelman began his career with George L. Fowler, consulting engineer, New York, N. Y., in February, 1892, on mechanical engineering and construction work. In July, 1893, he left Mr. Fowler to go with Thomas A. Edison, the inventor, as a draftsman and assistant on heavy mill construction. From September, 1894, until July, 1895, he was engaged in private practice making reports to the Brooklyn Building Department in behalf of property owners. From September, 1895, to May, 1898, he was draftsman and designer for the late Henry R. Worthington, M. Am. Soc. C. E., E. W. Bliss

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[1] Memoir prepared by Malcolm S. Spelman, Assoc. M. Am. Soc. C. E.

Company, and the C. and C. Electric Company. From May, 1898, to October, 1902, he was employed by the American Lithographic Company and the Hecla Iron Works as chief draftsman and, for a few months, as acting chief engineer for the latter organization. Becoming particularly interested in steel design, he served as estimator and designing engineer for Milliken Brothers from October, 1902, to April, 1904, and for Post and McCord from April, 1904, to February, 1907.

In February, 1907, Mr. Spelman began his long and varied career as a consulting and supervising engineer, with offices in New York City. Specializing in designs, plans, specifications, and inspection of steel and reinforced concrete construction, he supervised the designs of three new car houses for the Metropolitan Street Railway Company and alterations of fourteen other car houses. He directed the design of sections of subway and elevated structure work and the checking of design calculations for the Public Service Commission on the Second Avenue, Third Avenue, and Ninth Avenue elevated railways, including the design of the Second Avenue and Third Avenue (Harlem River) Bridge and the Putnam Bridge. In association with architects, he designed steel frames for office buildings, hotels, and incidental structures.

From June, 1917, to March, 1918, he was designing and construction engineer at Governors Island, N. Y., under Maj. F. B. Maltby on the construction of fifty-five warehouses, railroad lines, fire protection facilities, ferry slips, transfer bridges, and other work, costing about $1,000,000. From March, 1918, to April, 1919, he was chief engineer for the Foundation Company on a $9,000,000 project at Edgewood Arsenal, Maryland, on a large chlorine plant; railroad, water, and sewer lines; housing; and power plants. Continuing with the Foundation Company, he served as resident engineer until April, 1920, in charge of the design, specifications, field location, and supervision of construction of crusher plants, residences, schools, water and sewer systems, a narrow-gage railroad for ore cars, highway work, and incidental structures for the Midvale Steel Company to procure magnetite iron ore from the Sterling Mines, near Sterlington, N. Y. The total cost of this work was $750,000.

Resuming his private practice in June, 1920, he became consulting and supervising engineer for Nassau County, New York, on the design and construction of the Long Beach Bridge ($1,000,000) and the Bayville Bridge near Oyster Bay, N. Y. ($400,000), and, associated with William J. Beardsley, architect, on the alterations and additions to the Nassau County Court House at Mineola, N. Y. For private interests, he planned and supervised the construction of the Atlantic Beach Bridge ($500,000) and was consultant on the filling, bulkheading, and development of Atlantic Beach into an excellent resort. As consulting engineer for the Town of Southampton, Suffolk County, New York, he designed and supervised the construction of the Westhampton Beach Bridge ($100,000). For private companies and individuals, he furnished plans and specifications for a warehouse and several small apartment buildings. Associated with architects, he was engaged in the structural steel design of schools, theaters, and large residences.

In the fall of 1931 Mr. Spelman gave up his work in New York City and practiced engineering in Rockville Centre, N. Y., where he had lived since

1907. Having previously been a consultant to this growing municipality (with a population of more than 15,000 in 1930), in 1932 he became engaged in the planning and construction of the greatly increased electric power plant owned and operated by the municipality; and finally, in 1935, he was appointed building commissioner. Because of his extensive experience, he not only cared for his own building department work but was engineering consultant to the other municipal departments. For the eight years prior to his death, he carried on investigations and made appraisals for dwellings, apartments, commercial buildings, etc., costing about $9,000,000. At the time of his death he had commenced the details of several postwar projects and was overworked—aiding property owners in heating conversions, alterations and additions, as well as responding to the requests for help from firemen, police, and civilian protection units.

Throughout his life he maintained an active interest in sports and civic matters. He engaged actively in tennis, bowling, hunting, fishing, and boating; and he was an ardent baseball fan. He was an officer for many years of the Rockville Centre Tennis Club. An exempt fireman and a member of the Exempt Firemen's Association, he was a charter member of one of the volunteer fire companies and contributed many years of service. He was a charter member and deacon of the Rockville Centre Presbyterian Church for several years, but he joined Saint Marks Methodist Church in 1926. He was a member of the Massapequa Lodge No. 822, Free and Accepted Masons, and served on its management committee for seventeen years. Also he belonged to the Masonic Club, the Long Island Lodge of Perfection, and the Council of Princes of Jerusalem. He was a Republican and a member of the Rockville Centre Republican Club.

Among the professional organizations of which Mr. Spelman was a member were the American Concrete Institute, the American Society for Testing Materials, the Engineers' Club of New York, the New York State Society of Professional Engineers, and the Society of American Military Engineers. He was licensed in New York State as a professional engineer, an architect, and a land surveyor.

Mr. Spelman possessed great patience and an abiding faith in God and in his fellowmen. His belief in the fundamental values of life and his everlasting service to others endeared him to the many people who knew him. The organizations and landmarks in the community where he lived bear eloquent testimony to his long and useful life. Quiet-mannered, deliberate in thought and action, and responsive to many requests for help from any one, regardless of race, creed, or color, he maintained an outward calm despite many severe trials and personal hardships. Failure to save his strength while helping a friend was a contributing cause to his sudden passing at the age of seventy one.

On January 19, 1899, Mr. Spelman was married to Elsie Rankin Warnock of Marsville (near Toronto), Ont., Canada. They had three sons, Howard Rodgers Spelman, Jun. Am. Soc. C. E., who died in 1932; Malcolm Stewart Spelman, Assoc. M. Am. Soc. C. E.; and Walter DeyErmand Spelman, architect. In addition to his widow and two sons, three grandchildren, Alma, Carol, and Stewart Rodgers Spelman, survive him.

Mr. Spelman was elected an Associate Member of the American Society of Civil Engineers on December 6, 1910, and a Member on January 20, 1922. He became a Life Member in January, 1943.

HERMAN STABLER, M. Am. Soc. C. E.[1]

DIED NOVEMBER 24, 1942

Herman Stabler was born February 3, 1879, in the Quaker community of Sandy Spring, Md. The Stabler family were Quakers of Yorkshire, England. Edward Stabler, Mr. Stabler's great-great-great-grandfather, came to Pennsylvania from England in 1753 and later moved to Petersburg, Va. His great-great-grandfather, also Edward, lived as a boy and young man at Hillsboro, Va., Leesburg, Va., and York, Pa., and in 1792 he opened the Stabler apothecary shop (later the Stabler-Leadbeater Apothecary and Museum) in Alexandria, Va. Harry Stabler, Herman's grandfather, was born in Alexandria and died at Brighton, Md. Herman's father, George Lea Stabler, was born at Brighton in 1844 and was married to Annie Dickinson (Cotton) of Philadelphia, Pa., in 1868. They had three sons, Russell, Lea, and Herman. Herman attended three Quaker schools, a private school in Maryland for five years, Pacific College at Newberg, Ore., for one year, and later he was a student for four years at Earlham College, Richmond, Ind., from which he was graduated in 1899 with a Bachelor of Science degree. At Earlham College he developed an interest in civil engineering as a lifework. In addition to partly working his way through collge, he was active in athletics. One year he won the state championship in pole vaulting, and he used to admit with a smile that "he could outkick any opposing fullback and succeeded in winning one football game with a score of 3 to 0 by a kick from the 45-yd line."

Following a special course of one year in engineering at Columbian (later George Washington University) in Washington, D. C., he was an instructor for about two years in mathematics and engineering at the National Correspondence Institute in Washington, where he learned that "the best way to really learn mathematical and engineering subjects is to try to teach them by the correspondence method."

In 1902 he took the first civil service examination held for the position of irrigation engineer, and, on December 22, 1903, he was appointed a hydrographic aid in the U. S. Geological Survey to make investigations on the economic aspects of pollution of water. His first field work for the Geological Survey was a study of pollution of the Schuylkill River, Pennsylvania. In 1905 he was assigned to a project conducted in cooperation with the State

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[1] Memoir prepared by William G. Hoyt, M. Am. Soc. C. E.

Board of Health of Ohio, the principal object of which was the prevention of stream pollution by sewage and manufacturing wastes. This study, which covered various sewage disposal plants, investigated textile factory wastes, acid-iron waste, and distillery waste. The last named involved a sojourn of several weeks at a distillery where high and low grades of gin and of corn and rye whiskies were made. He there "learned the difference between good and bad liquor and the therapeutic value of corn oil." He there devised a plan, later put into profitable practice, for the conservation of distillery waste, previously discharged into a stream, and its conversion into a concentrated cattle food.

Cooperating with the State Board of Health of Rhode Island in 1906, he investigated various sources of pollution of the Moshassuck and Woonasquatucket rivers, but, as he reported later, "the continued unsanitary condition of these streams offers olfactory evidence that my work there, though interesting and instructive, had little practical effect." He continued his field work on sanitary problems through 1906, being engaged on investigations in Pennsylvania, New York, Iowa, New Mexico, California, and Nevada. Returning to Washington in 1907, he assisted in the nationwide studies of quality of water and denudation, which were then under way, and devised a scheme for the mathematical checking and industrial application of water analyses expressed in ionic form, a system that is still being widely used. Although Mr. Stabler was engaged on problems of sanitary engineering relating to stream pollution for a period of only about five years, his written contributions were outstanding and are still authoritative. During that period he was the author, either alone or with associates, of the following *Water-Supply Papers* relating to stream pollution and erosion:

No. 179—"Prevention of Stream Pollution by Distillery Refuse";
No. 186—"Stream Pollution by Acid-Iron Wastes";
No. 234—Chapter on "Denudation" (with R. B. Dole);
No. 235—"The Purification of Some Textile and Other Factory Wastes" (with G. H. Pratt); and
No. 274—"Some Stream Waters of the Western United States, with Chapters on Sediment Carried by the Rio Grande and the Industrial Application of Water Analyses."

Although he was no longer directly connected with sanitary engineering after about 1908, he never lost interest in quality-of-water problems. For a brief period in 1920, he served as acting chief of the Geological Survey's division of quality of water and was responsible for starting its systematic observations of suspended and dissolved matter in the Colorado River. In 1931 he prepared a section in *Professional Paper No. 163* dealing with the waters of the Salt Creek-Teapot Dome uplift and wrote a chapter on industrial uses of water in "Manual of Industrial Chemistry," by Allen Rogers.

When quality-of-water investigations were temporarily discontinued in the Geological Survey in January, 1908, Mr. Stabler was employed as an irrigation engineer with the U. S. Reclamation Service on design and review of standard structures and the preparation of studies for publication. In August, 1910, he returned to the Geological Survey and for about a year was engaged principally in a study of ground-water resources in southern California. The results of these studies are contained in the following *Water-Supply Papers*: *No. 398,* "Ground Water in San Joaquin Valley, Calif." (with W. C. Mendenhall and R. B. Dole), and *No. 429,* "Ground Water in the San Jacinto and Temecula Basins, Calif." (with G. A. Waring).

His principal duties from 1912 to 1922 involved land-classification studies related to the availability and use of water on the public domain. These necessitated determination of developed and undeveloped water-power resources, classification of land as to irrigability or non-irrigability, feasibility of irrigation projects, and sufficiency of water therefor. During this period he acquired an intimate firsthand knowledge of the West, its resources and its problems. Thus, he was able to take an active part in formulating federal policies concerning conservation and use, and, as early as 1920, he assisted in organizing the work and procedure of the Federal Power Commission and in drafting its early rules, regulations, and orders.

From 1922 to 1925 Herman Stabler was chief of the Land Classification Branch of the Geological Survey, in immediate charge of all matters relating to classification of the public domain (between 20% and 25% of the total area of the United States) with respect to mineral, power, and agricultural values. In this capacity he supervised the taking of an inventory of the national resources of the Western States, planned the surveys of thousands of miles of western rivers, and directed the preparation of several thousand manuscript reports outlining general development programs for the West. He was in direct personal contact with many of the field surveys and joined expeditions making the first complete topographic surveys through Glen Canyon and the Grand Canyon of the Colorado River in Utah and Arizona. As a member of the Grand Canyon Expedition of the Geological Survey in 1923, he had the distinction of being the one person in the party who was not at some time either upset or thrown off a boat in passing through the rapids. This accomplishment was typical of his tenacity in sticking to a problem.

His more important contributions to technical literature on this subject during this period include a mimeographed report on the "History and Purpose of Classifications of Lands by the Geological Survey, with Special Reference to Forest Lands"; "The Rise and Fall of the Public Domain" (an abstract of a more detailed statement presented on July 6, 1932, at the Yellowstone Convention of the American Society of Civil Engineers), published in *Civil Engineering,* September, 1932, pages 541–545; and a mimeographed report entitled "General Discussion of the Open Public Domain and of Its Grazing Resources." For the Colorado River Basin he planned and gave general supervision to surveys, investigations, the preparation of maps, and three major water-utilization reports (*Water-Supply Papers Nos. 556, 617,* and *618*) outlining plans for effective conservation of the water resources of the entire basin. In 1924 he served on a committee that acted in an advisory capacity to the Secretary of the Interior on Colorado River problems.

From 1925 to the time of his death, he was chief of the Conservation Branch of the Geological Survey, having administrative charge of land classification, as in the period 1922 to 1925, and also supervision of mineral leasing operations on public lands under the act of February 25, 1920, on lands included in national petroleum reservations, and on tribal and allotted Indian lands. The operations supervised included about 125 permits for development of power and about 2,000 leases of producing mineral lands with output valued at $125,000,000 to $150,000,000 a year. When Mr. Stabler took over enlarged responsibilities as chief of the Conservation Branch he was already well known as a hydraulic engineer, an expert on quality-of-water problems, and a land-use specialist. While holding this position, he acquired a working knowledge of the many problems, both technical and legal, associated with petroleum production and mining on the public lands but contributed little in technical writing, his activities being largely administrative. In that capacity he continually served as adviser to the Secretary of the Interior and to Congressional committees on problems relating to the use and development of the public domain.

As a Quaker he was peaceful, but only so long as he believed that justice, equity, and honesty prevailed. Aroused by a worthy cause, he fought openly, squarely, and generally successfully with those with whom he disagreed, regardless of whether they were employees, associates, or employers.

Into his duties as an administrator and public servant he carried the high standards that exemplified his private life. He had the ability to obtain and hold the respect and confidence of employees and administrators alike and was the confidant of many in all walks of life with respect to personal problems. In view of his wide knowledge of the several branches of engineering and of the related legal problems, his associates, administrative superiors, and the Congressional leaders sought his advice and valued his helpful suggestions. That his knowledge of these diverse subjects was deeply grounded was confirmed many times by those who contended with him on technical or legal points having to do with the natural resources of the nation. Mr. Mendenhall, Director of the Geological Survey, who was closely associated with Mr. Stabler throughout practically all his professional career, states:

> "Among Stabler's outstanding mental characteristics were first and fundamentally sound scientific and mathematical methods of thought and powers of analysis. To these were added great keenness and delicacy of perception, intuitive and sympathetic insight into human minds and motives, and a rare capacity for lucid expression. These are indeed unusual combinations and when, as in his case, they are united with impeccable personal standards and a high sense of public responsibility, the result is a citizen and an official of the highest type."

All engineers in federal, state, or private employment are indebted to Mr. Stabler for his outstanding efforts in their behalf. Through active work on many committees, he was responsible in part for the Federal Classification Act. He labored hard that the principles of this Act might be extended throughout state and private practice and so raise the basic standards of employment for professional men. With respect to younger or underpaid engineers, he fought unceasingly that their compensation should at all times and in all places reflect their technical and specialized training and that their general economic standing should compare favorably with that of other professions.

Mr. Stabler belonged to many organizations. Among these were: American Institute of Mining and Metallurgical Engineers, American Geographical Society, Washington Academy of Sciences, Washington Society of Engineers, Geological Society of Washington, Cosmos Club and Federal Club (past president) of Washington, and Columbia Country Club, Chevy Chase, Md. He served the American Society of Civil Engineers as Director from 1935 to 1937 and also was on many important committees of the Board of Direction. In politics he was independent. His religious affiliation was with the orthodox branch of the Society of Friends.

By reason of his outstanding qualities as a leader, his record as a public servant, his achievements in advancing professional standards and compensation, his humane interest in the problems of his friends and associates, and his love of life as such, Herman Stabler will be missed and mourned by many.

On November 1, 1905, while on the Ohio assignment, he was married to Bertha R. Buhler of Washington, D. C., who survives him.

Mr. Stabler was elected an Associate Member of the American Society of Civil Engineers on November 1, 1910, and a Member on November 27, 1917.

MILTON GERRY STARRETT, M. Am. Soc. C. E.[1]

Died September 18, 1942

Milton Gerry Starrett was born on January 24, 1861, in Francestown, N. H., son of James H. and Theresa (Morgan) Starrett. He was educated at Francestown Academy, and was graduated from Tufts College, Medford, Mass., as a civil engineer with the degree of Bachelor of Mechanical Arts in 1886. His ability as a mathematician won him the position of Walker instructor of mathematics, a position he filled from 1886 to 1889. During this period his teaching ability was recognized, by both faculty and students, to be of the highest quality.

In 1889 the late F. S. Pearson, M. Am. Soc. C. E.,[2] who was chief engineer of the West End Street Railway Company of Boston, Mass., and his predecessor as Walker instructor, made him assistant chief engineer in charge of construction of two of the company's large power houses, in connection with the changing over of the entire horse railway system to overhead trolley. In this capacity Mr. Starrett designed a high brick stack for a railway power house which was followed in many power houses with much success. An experimental underground trolley of about one mile was built to his design, from which a similar system was installed later in New York, N. Y.

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[1] Memoir prepared by S. A. Spalding, Esq., New Canaan, Conn.

[2] For memoir, see *Transactions*, Am. Soc. C. E., Vol. LXXXVII (1924), p. 1402.

From February, 1891, to February, 1896, he was chief engineer of the Brooklyn Rapid Transit Company in charge of converting the entire street railway system from horse to overhead trolley, including construction of all power houses of the system.

In February, 1896, he joined the Metropolitan Street Railway Company of New York as assistant chief engineer. He had direct charge of the design, construction, and operation of the underground trolley system and cable road in both electrical and mechanical departments, including track, rolling stock, and power stations. In 1901 he was made chief engineer. In 1906 he established a consulting engineering practice in New York City making investigations and reports, appraisals and valuations, for public utilities and bankers in Buffalo, Amsterdam, Cooperstown, N. Y.; Worcester, Mass.; Nashville, Tenn.; Norfolk, Va.; San Francisco, Calif.; Halifax, N. S., Montreal, Que., and Toronto, Ont., Canada; Jamaica, British West Indies; and Sydney, N. S. W., Australia.

From 1915 to 1933, when he retired, he was associated with the late Lewis B. Stillwell, M. Am. Soc. C. E., on various engineering projects, the last of them being the Holland Tunnel and George Washington Bridge construction on which they were consultants to the New York Port Authority.

During the periods mentioned, in which he had direct supervision of work amounting to more than $50,000,000 in the vicinity of New York, he cooperated with city officials in matters affecting service to the public; that is, routing of cars, character of equipment, methods of operation and maintenance of cars, electric lines, and equipment.

As a man and an engineer, Mr. Starrett had the respect and admiration of his associates. He was of an unassuming nature, always thoughtful and considerate and ready to assist in any problem brought to him.

In 1905 Tufts College honored him with a degree of Doctor of Science. He was an Overseer of Tufts College from 1902 to 1907 and an Alumni Trustee from 1907 to 1918.

On December 28, 1893, he was married to Ruth Morrill, who survives him.

Mr. Starrett was elected a Member of the American Society of Civil Engineers on January 3, 1900.

JOHN WILLIAMS STORRS, M. Am. Soc. C. E.[1]

DIED SEPTEMBER 19, 1942

John Williams Storrs was born in Montpelier, Vt., on November 24, 1858, the son of William W. and Elizabeth A. (Roberts) Storrs. In his early childhood the family moved to Concord, N. H., and it was there that he attended the public schools. He began his engineering career by studying and working with the engineering firms of Charles C. Lund and Foss and Merrill.

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[1] Memoir prepared by Frederic E. Everett, M. Am. Soc. C. E.

From 1890 to 1911 Mr. Storrs was connected with the Concord and Montreal and the Boston and Maine railroads in the general field of railroad engineering work. Serving under Frank A. Merrill, chief engineer, he was engineer in charge of construction of the Whitefield and Jefferson Extension in 1891, the Waumbek and New Boston branches in 1892, all of the contract work on the White Mountain and Concord divisions, the Southern Division north of Nashua, N. H., including stations, engine houses, bridges, masonry, and the construction of the Boston and Maine Railroad shops in Concord in 1896.

During 1903 and 1904 he served as State Highway Engineer in charge of construction of highways in the White Mountain region, building the Lafayette Road which runs from Profile Notch, N. H., to Twin Mountain, N. H., and the road from Twin Mountain to Fabyan, N. H.

In 1906 Mr. Storrs established his own engineering firm in Concord, which he conducted until his death. He served as consulting engineer on many important projects, both inside and outside of the state. In 1911 Mr. Storrs was appointed chief engineer of the New Hampshire Public Service Commission, and from 1918 to 1930 was a member of the Commission, being its chairman from 1928 to 1930, inclusive.

Throughout his entire engineering career, Mr. Storrs centered his interest in the construction of bridges and was regarded as an authority in his line of work. In 1918 he wrote and published "A Handbook for the Use of Those Interested in the Construction of Short Span Bridges"—a manual designed to assist a layman in the construction of a bridge up to and including a thirty-foot span.

Mr. Storrs was a Republican but was not active in politics until 1933 when he was elected Mayor of Concord on a nonpartisan ticket. This office he continued to hold by biennial reelection until his death, thus fulfilling his wish of "dying in the harness."

Mr. Storrs was a charter member of the New Hampshire Good Roads League, which was a forerunner of the New Hampshire Good Roads Association. He was also a member of the Boston Society of Civil Engineers, the American Water Works Association, and the Independent Order of Odd Fellows.

He was married to Carrie Etta Dow of Concord on April 29, 1885. He is survived by his widow and one son, Edward Dow Storrs, also a civil engineer.

Mr. Storrs was elected a Member of the American Society of Civil Engineers on February 4, 1914.

WILLIAM JOHN THOMAS, M. Am. Soc. C. E.[1]

DIED JANUARY 1, 1935

William John Thomas, the son of J. W. and Elizabeth (Caswell) Thomas, was born in 1875 in the little coal mining town of Moingona, Iowa, which was made famous in 1881 by Kate Shelly. This fifteen-year-old daughter of a farmer, on a dark and stormy night, crawled on her hands and knees, over stubs of nails, across the Chicago and North Western Railway Company's bridge over the Des Moines River, to give the alarm after a "pusher" and four men had plunged off the damaged bridge into the river. Her heroic efforts resulted in rescuing two of the men and in stopping a limited train before it reached the bridge.

After finishing grade school in Moingona, Mr. Thomas entered Iowa State College at Ames and was graduated with the degree of Bachelor of Civil Engineering in 1895. In 1916, he was awarded the degree of Civil Engineer.

He began his engineering work as a structural steel draftsman with Universal Construction Company in Chicago, Ill., after which he spent three years as structural steel checker and designer for Lewinson and Just, in New York, N. Y. Then he was employed for two years by American Bridge Company in Brooklyn, N. Y. In 1903, he was assistant engineer, Structural Steel Division, for the Lackawanna Bridge Company at Buffalo, N. Y. In 1904, he was with Milliken Brothers, New York City, and, from 1904 to 1908, he designed steel work for Lewinson and Company of New York.

On July 13, 1908, he began an engagement as chief engineer with George B. Post and Sons, Architects, of New York City, continuing in this position for eighteen years. During this period he designed structural work and foundations for many important buildings and also directed the outside superintendents on building projects.

During World War I, as assistant project manager for George B. Post and Sons on work for the United States Housing Bureau, he directed the construction of the town of Craddock, Va., for civilian workers of the Portsmouth (Va.) Navy Yard. After the Armistice he returned to the New York office of his firm.

In 1923, he served with the late George F. Swain,[2] Past-President and Hon. M. Am. Soc. C. E., the late Milo S. Ketchum,[3] Hon. M. Am. Soc. C. E., the late Wilbur J. Watson, M. Am. Soc. C. E., and E. R. Graham on the committee which prepared the Standard Specifications of the American Institute of Steel Construction.

From July, 1927, to February, 1928, he was chief engineer of the Wallin Company, General Contractors, New York City. From February, 1928, to February, 1932, he served as chief engineer for Irwin and Leighton, General

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[1] Memoir prepared by George L. Christy, M. Am. Soc. C. E.

[2] For memoir, see *Transactions*, Am. Soc. C. E., Vol. 98 (1933), p. 1476.

[3] For memoir, *ibid.*, Vol. 102 (1937), p. 1487.

Contractors, of Philadelphia, Pa. During this period he had charge of about twenty-five million dollars' worth of work. The following is a quotation from a letter written in 1933 by Mr. Irwin:

> "We consider Mr. Thomas one of the most experienced and capable structural designers in the country and a very competent engineer for solving any engineering problem presented to him.
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> "Mr. Thomas is a man of the highest integrity and character and while with us showed a keen interest in his work and in the welfare of his employers, together with a splendid spirit of co-operation with all those with whom he came in contact."

Mr. Thomas passed away at West New Brighton, N. Y., on January 1, 1935. His widow, Mrs. Sadie Thomas, survives him.

Mr. Thomas was elected a Member of the American Society of Civil Engineers on February 4, 1913.

ARTHUR CLARENCE TOZZER,[1] M. Am. Soc. C. E.

Died September 9, 1942

Arthur Clarence Tozzer was born on July 18, 1879, in Lynn, Mass., the son of Samuel Clarence Tozzer and Caroline (Marston) Tozzer. He was educated in the public schools of Lynn and entered Dartmouth College at Hanover, N. H., in the fall of 1898. He was graduated with the degree of Bachelor of Science in 1902 and in 1903 received the degree of Civil Engineer from the Thayer School of Engineering at Dartmouth. While at Dartmouth he was a member of the Chi Phi fraternity.

During his college days Mr. Tozzer worked as assistant engineer on construction of Dry Dock No. 2 at the Charlestown, Mass., Navy Yard. From April to December, 1903, he was assistant engineer in both office and field with Henry F. Bryant, Consulting Engineer of Brookline, Mass.

He went to New York, N. Y., in January, 1904, and from then until July, 1904, acted as timekeeper, assistant superintendent, and superintendent for the Foundation Company on pneumatic foundation construction for bridge piers and buildings.

From August, 1904, to January, 1905, Mr. Tozzer served with the Phoenix Construction Company of New York City on engineering design and supervision of construction on an experimental tunnel (freezing method) of the Pennsylvania Railroad Company in connection with the East River tunnels. From February to November, 1905, he was with S. Pearson and Son, Inc., as engineer and superintendent of construction on these tunnels, and was placed in special charge of the installation of the air locks.

In these first three years out of the Thayer School, Arthur Tozzer showed talents which later became outstanding in his life—that is, he demonstrated to his superiors his willingness to accept responsibility and he proved his dependability. He also indicated a penetrating mind and occasionally a bit of inventiveness, but above all he acquired habits of hard work and upright living that were to guide the remainder of his life.

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[1] Memoir prepared by J. P. H. Perry, M. Am. Soc. C. E.

In November, 1905, he entered the employ of the Turner Construction Company as superintendent on the construction of a factory for C. Kenyon and Company in Brooklyn, N. Y. The Turner Construction Company at that time was a small concern engaged in building construction, doing less than half a million dollars of work a year. The contract for the Kenyon Company amounted to $67,000. Reinforced concrete, of which this building was constructed, was in its infancy so far as commercial usage went. There was not much precedent to guide a young superintendent in charge of one of his company's then largest contracts. Personal responsibility was at a premium because things that are done with most other building materials in the laboratory, in the shop, or in the fabricating yard, under routine supervision and control, had to be done in the field on a fast-speed schedule. Integrity of design and integrity of execution were vital. Mr. Tozzer handled his first concrete building job so well and enjoyed his work so much that he remained with the Turner Construction Company until his death thirty-seven years later.

In this more than a third of a century of active practice of his profession, he was personally in charge of the construction of more than 300 buildings. These totaled in cost probably more than $100,000,000 and comprised nearly every known type of building in America, from small factories and warehouses to a great $25,000,000 army base, a 5-mile long bridge, schools, churches, office buildings, laboratories, and residences.

As the Turner Construction Company grew from a modest little concern to one of the great building companies of the United States, doing in the year of Mr. Tozzer's death approximately $90,000,000 of business, Mr. Tozzer grew with it. He was appointed general superintendent in 1913 and manager of the newly opened Boston, Mass., office in 1916. He was recalled to New York to be executive manager of the South Brooklyn Army Supply Base, a $25,000,000 undertaking, in 1918. In 1919 he returned to the Boston office as vice-president and general manager and in 1928 he returned to New York as executive vice-president, which position he retained until two or three years before his death. He was elected a director in 1928 and had been a member of the Executive Committee of the Company for some ten years before he passed away. Throughout this long span of always conscientious endeavor, he gave to the Turner Construction Company the utmost of his broad ability and wide experience.

On June 1, 1916, he was married to Dorothy LaCroix, and they had three daughters, Edith (Mrs. Howard D. Jordan), Caroline (Mrs. James O. Buchanan), and Dorothy. In 1937 Mr. Tozzer was divorced. On September 28, 1937, he was married to Isabel N. Kellers and adopted as his stepson, James McW. Kellers. Mrs. Tozzer and all of the children survive him.

In 1924 his Alma Mater, the Thayer School of Engineering, elected him an Overseer. In 1942 he was further honored by Dartmouth by being elected an honorary member of Phi Beta Kappa. Probably few things in his life gave him more pleasure than the recognition by these institutions at Hanover of his outstanding contribution to his profession.

In 1933, at a time of great national stress and of depression in the building industry, Mr. Tozzer was elected president of the Associated General Contractors of America (A. G. C.), the leading organization of the construction industry. This was at the time of the NRA (National Industrial Recovery Act) codes. Mr. Tozzer's year as president was about as important a one as the A. G. C. has had in its many years of existence. He brought to this important organization all of his energy and power of organization. He did a splendid job in connection with codes. He continued as a member of the so-called "cabinet" of that organization until, in November, 1940, a cardiac disturbance made it obligatory for him to give up all active duties. While he was still a director of the Turner Construction Company and an Overseer of the Thayer School, his days of active work ended with this heart disturbance.

Mr. Tozzer was a member of the American Concrete Institute, The Engineers' Club of New York City, the Architectural League of New York City, the Dartmouth Club of New York City, and the Scarsdale (N. Y.) Golf Club.

One of the outstanding features of Mr. Tozzer's life was his wide experience and great success in handling labor matters for the Turner Construction Company. He had dealt with union building labor practically all his life and for many years had directed labor relations for the Turner Construction Company. His conduct in these vital and important matters brought him into intimate relationship with the leaders of the building trades' movement in the United States. Winning their confidence and respect was one of his outstanding attainments.

He was called to address the United States Army War College in 1938 and 1939 and was a frequent speaker at contractors' banquets all over the United States and at dinners of the Turner Construction Company. He had an easy manner on his feet and always had something important to say to his audiences. The fact that the Army War College asked him to come back two or three times was evidence that they felt that he really contributed something to their studies of how to mobilize the construction industry should war again come to the United States.

Mr. Tozzer was elected a Junior of the American Society of Civil Engineers on April 4, 1905; an Associate Member on February 28, 1911; and a Member on September 12, 1921.

ALVIN DUMOND WILDER, M. Am. Soc. C. E.[1]

DIED SEPTEMBER 18, 1942

Alvin Dumond Wilder was born on August 4, 1885, at Oakland, Calif., the son of Alvin Dumond and Mary E. (Snyder) Wilder. He was graduated from high school in Oakland in 1902 and from the College of Civil Engineering of the University of California, at Berkeley, in 1908.

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[1] Memoir prepared by Austin W. Earl and Clifton E. Hickok, Members, Am. Soc. C. E.

His father was superintendent for the Southern Pacific Railroad Company at Oakland. With this background it was only natural that at the start of his engineering career he should be interested in railroad work, and for the first three years after graduation he worked on railroad location and construction in Washington and Oregon.

From 1911 to 1915 he was designing engineer with the Panama Pacific Exposition Company on the design of wharfs, ferry slips, roads, sewers, electric lighting, and streets. Later he was in charge of structural design in connection with several of the major buildings for the Exposition, including the Varied Industries Building, the Horticulture Building, and the Tower of Progress.

From 1915 to 1917 Mr. Wilder was assistant engineer with the Spring Valley Water Company of San Francisco, Calif., on plans for future development, and was also in charge of construction. He was engaged as a contractor on California state highways during 1917 and 1918. In April, 1918, he went to Canada for the engineering firm of Sidney E. Junkins Company, Ltd., with headquarters in Vancouver, British Columbia. His first assignment was an investigation and report on large undeveloped hydroelectric power sites in British Columbia for the First National Bank of Boston, Mass. He was then in charge of the layout, design, and construction of stockyards for the Canadian Pacific Railway Company at Calgary, Alberta, Canada. After being made vice-president and chief engineer for the Junkins Company in 1920, he was in charge of much work for the Canadian Pacific Railway Company, including the lining of the five-mile Connaught Tunnel; reports on power development and economics of electrifying the railroad through the Selkirk Mountains; a deep-sea pier and passenger terminal at Vancouver; and foundations for a 2,200-ft railway and highway bridge over the Saskatchewan River in Saskatchewan. In addition to this work for the Canadian Pacific Railway Company, he also prepared plans for harbor development for the Vancouver Board of Harbor Commissioners, and made an appraisal of the Parliament Buildings at Ottawa for the Canadian Government. Mr. Wilder returned to the United States in 1930, and was, for a short time, in private practice, but closed his office in 1931 to go to Hawaii as general manager of a pineapple plantation.

In 1933 he was appointed state director for California of the Federal Emergency Administration of Public Works, and held this position until 1937, during which period he administered the construction of five hundred and thirty-one projects of various types, involving a total cost of $171,025,000. In 1937 he became executive secretary of the Housing Authority for the City and County of San Francisco. In 1938 he was appointed director of the Department of Public Works for the City and County of San Francisco, and held this important office until the time of his death.

Mr. Wilder was an engineer and executive of exceptional ability and in his untimely death the City of San Francisco and the engineering profession has suffered a great loss. He was a man of genial disposition, with many friends;

a man respected by all who knew him. In the performance of his duties as administrator of public offices, he exhibited wisdom, impartiality, and tact, which won the liking not only of those who worked with him, but also of all others with whom he came in contact. He was a member of the Delta Kappa Epsilon fraternity and the Family Club of San Francisco.

He was married in 1917 to Emilie Middleton, of San Francisco, who survives him. He is also survived by two sons, Alvin Dumond III and John Francis, and a daughter, Adiel. His son, Alvin Dumond III, a Lieutenant in the Engineer Corps, United States Army, was graduated from the United States Military Academy at West Point, N. Y., in 1942. His son, John Francis, is a midshipman in the United States Navy.

Mr. Wilder was elected a Junior of the American Society of Civil Engineers on October 6, 1908; an Associate Member on December 6, 1915; and a Member on March 16, 1925.

EDWARD GILBERT WILLIAMS, M. Am. Soc. C. E.[1]

Died October 22, 1942

Edward Gilbert Williams, the son of Gilbert and Harriet Gauntlett (Gubb) Williams, was born at Essex, Conn., on May 8, 1865. He was a descendant of William Brewster, one of the *Mayflower* party.

Although he attended the Russell School in New Haven, Conn., and the Lander School in London, England, Mr. Williams prepared for college mainly on his father's ship, trading between China and New England. To his father, Capt. Gilbert Williams, who was a thorough and exacting teacher, the son owed many of the fine qualities that fitted him to become a leader among construction men. In 1887 Mr. Williams was graduated from the Sheffield Scientific School at Yale University in New Haven with the degree of Bachelor of Philosophy.

His engineering career began as rodman on the Burlington and Missouri Railroad in June, 1887. When he left the railroad in August, 1890, he was division engineer of construction in Nebraska, Kansas, South Dakota, and Wyoming.

His next position was in the city engineer's office in New Haven. In 1891 he left to become engineer for the contractors on the extension of the Western Maryland Railroad from Williamsport, Md., to Cherry Run, W. Va.

From November, 1892, to April, 1893, he was in Colombia, South America, reporting on the mining property of the Caribbean Manganese Company, in charge of surveys and the collection of data. After returning home and until January, 1894, he was in charge of the party locating a line from New York, N. Y., to Brewster, N. Y., for the New York, New England and Northern Railroad Company. When this work was completed, Mr. Williams did miscellaneous engineering for A. B. Hill Company of New Haven.

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[1] Memoir prepared by Albert S. Crane, M. Am. Soc. C. E.

In June, 1896, he again went to Colombia as general manager and chief engineer for the Caribbean Manganese Company, remaining there until 1903 when he returned to this country as chief engineer of the Continental Jewel Filtration Company. Until 1909 he worked for this company building the railroad tunnel under the City of Washington, D. C. He then went with J. G. White and Company as construction manager.

While employed by the J. G. White interests, Mr. Williams did his most important work, and when The J. G. White Engineering Corporation was formed in January, 1913, he became vice president in charge of construction. In this capacity he was engaged on the construction of many steam and hydroelectric stations, irrigation works, and industrial plants in the United States; and also dams and irrigation works in Chile and Mexico, railroads in Persia and in Panama, oil refineries in Wyoming and in France, and large oil storage facilities at Pearl Harbor. In 1941, he retired from active work, becoming a consultant to his firm.

When in a reminiscent mood, Mr. Williams liked best to talk about his experiences with the San Blas Indians in Colombia, the building of the Long Island automobile parkway, his connection with the Yale Bowl, the construction of the Nitrate Plant at Sheffield, Ala., the building of Langley Field, and his adventures in Mexico while engaged in irrigation work. He had an excellent memory and the ability to entertain his friends with stories of his adventures.

In addition to being a member of the Mayflower Society, Mr. Williams was a member of the Society of Colonial Wars and of the Sons of the Revolution. Until the close of his active career, he was for many years a member of the Engineers Club and of the Lawyers Club of New York. At the time of his death he was a member of the University and the Yale clubs of New York. He became a member of the American Institute of Mining and Metallurgical Engineers in 1897.

He was married on July 25, 1897, to Alma Heitman, who, with three sons, Gilbert, Norman F., and Edward W., survives him.

Mr. Williams was elected a Member of the American Society of Civil Engineers on February 3, 1897.

JOHN WILLIAM WOERMANN, M. Am. Soc. C. E.[1]

Died December 9, 1942

John William Woermann was born in St. Louis, Mo., on January 18, 1868, the son of Henry William and Catherine M. (Schwiering) Woermann. His father was born in the village of Eggeberg, near the City of Halle, in the province of Westphalia, Germany. At the age of 18 he came to the United States. During the Civil War, when President Lincoln issued a call for volunteers to serve in the Union Army, he enlisted, in St. Louis, as a private in the Fourth Missouri Cavalry. On July 10, 1863, his company was captured, and the men were in the Belle Island Prison until December 27, 1863.

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[1] Memoir prepared by Fred W. Honens, M. Am. Soc. C. E.

John William Woermann received his preliminary education in the Divoll Grade School, and, after completing three years in St. Louis Central High School, he entered the College of Engineering at Washington University, in the same city, where he completed a five-year course in civil engineering, being graduated with the degree of Civil Engineer in 1890. He was elected a member of Tau Beta Pi, the honorary engineering fraternity.

Immediately after leaving college, he entered the services of the U. S. Engineers, U. S. Army, with whom he was employed in various capacities for the greater part of his life. He was first recorder on secondary triangulation, and leveler on surveys of the Missouri River from the headwaters to Fort Benton, Mont.

From January, 1891, to November, 1896, he was first leveler and topographer on location of the Illinois and Mississippi (Hennepin) Canal around the rapids in Rock River at Milan, Ill.; and later resident engineer in charge of construction. At first, this work included building the guard lock, a rock-filled crib dam across Rock River, two miles of earth and rock excavation, seven Tainter gates, about a mile of parallel embankments in Rock River (riprapped on both sides), a highway swing bridge, a railroad bridge, a concrete culvert under the canal; and other miscellaneous projects. The guard lock, known as No. 30, was the first concrete lock ever built in the United States. During his five years as resident engineer, from May, 1897, to September, 1902, he was in charge of construction of miles 20 to 29, inclusive, of the Hennepin Canal, and miles 24 to 29 of the feeder for the same canal. This work included earthwork on the canal trunk (part peat bog), riprapping canal embankments, construction of culverts and foundations for lock No. 22, and building the aqueduct near Mineral, Ill.

For three years, from September, 1900, to October, 1905, he was in immediate charge of the surveys, maps, plans, and estimates for a 14-ft waterway from Chicago, Ill., to St. Louis. The survey included secondary and tertiary triangulation, precise levels, wye levels, topography, hydrography, borings, and discharge measurements. This work covered about 300 miles of the Illinois and Des Plaines rivers, and about 800 sq miles of the adjacent bottom lands. The principal map covered 58 sheets, most of them 12 ft long, with contours 1 ft apart. Later, Mr. Woermann prepared plans and estimates for a lateral canal, and also for a 14-ft waterway from St. Louis to Cairo, Ill.

For thirteen years, from March, 1909, to June, 1922, he was principal assistant to division engineers, first for the Western Division, inspecting projects, plans, specifications, etc., for improvement of thirty-five rivers and canals in the Midwest; and then for the Northwestern Division, in connection with harbors on Lake Michigan and the improvement of the Upper Mississippi River.

In July, 1922, he became principal assistant to the District Engineer at Chicago, and, for ten years, had general supervision of surveys and mapping, construction and dredging inspections, correspondence and reports relating to the Illinois Waterway (the Sanitary District of Chicago), the reclamation of land along the lake front by the South Park Board, the construction of bridges, and other work under War Department permits. He also supervised the preparation of reports on numerous rivers in Illinois, covering navigation, flood control, and water power.

For nine years, from November, 1930, until the time of his retirement, he was senior engineer and principal engineer in charge of the Hydraulic and Report Section, U. S. Engineer Office, Chicago. In this capacity, he prepared miscellaneous letters and reports relating to various subjects, including the diversion of water from Lake Michigan, the preparation of a new 9-ft canalization project for the Lower Illinois River; the opposition to the purchase by the United States of the canals now owned by the Sanitary District of Chicago, and others.

At the recommendation of the Engineer Officer, the time for his retirement was extended one year. Later, he was employed occasionally as expert consultant.

At various times during his career, when government funds were not available, and, between the work on larger projects, he was employed on private work. For eight months, he was inspector in charge of dredging the Chicago River and its branches in Chicago; and, for three years, he was resident engineer on construction of the Illinois Traction Company's bridge across the Illinois River at Peoria, Ill., resident engineer on the hydroelectric plant at the mouth of Des Plaines River, and consulting engineer on drainage and levee projects, reinforced concrete construction, and miscellaneous engineering work.

Painstaking and thorough in all of his work, he was not satisfied until it was complete and clear. His standards were very high. He was a good executive, having the happy faculty of getting the best results from those who served under him, without causing any friction.

His writing was clear, concise, and interesting. As a student, he was editor of *Student Life,* the Washington University paper. He prepared two technical papers: "Concrete Construction on the Illinois and Mississippi Canal" and "Construction of a Low Timber Dam across Rock River," which were presented to the Engineers Club of St. Louis, and published in its *Journal.* Also, he wrote many official reports. He had considerable ability as an amateur photographer, gaining much experience in connection with the survey of the Upper Missouri River (above Fort Benton). He made a complete photographic record of the construction of the Hennepin Canal around the rapids in Rock River at Milan. Several of these photographs were used to illustrate his technical articles. He prepared an article on the Mississippi River for the International Encyclopedia.

He always was deeply interested in the engineering societies of which he was a member, and took an active part in their affairs. He was a past-president of the Engineers Club of St. Louis. He attended many meetings of the Society, and, whenever possible, was present at the meetings of both the Engineers Club of St. Louis and the Engineers Club of Chicago. He was a past-president of the alumni association of Washington University.

Among the organizations to which he belonged were the Western Society of Engineers, Engineers Club of St. Louis, Engineers Club of Chicago, International Association of Navigation Congresses, Presbyterian Church, and A. F. and A. M. of Missouri.

Mr. Woermann was popular with his associates and other employees of the U. S. Engineer Department. At the time of his retirement, January 18, 1939, a testimonial dinner was given in his honor, which was attended by many of them as well as by a number of his other engineering friends. After several lauditory talks had been made by his colleagues, a large number of letters and telegrams of congratulation were read, and he was presented with a beautiful watch, on which had been engraved, "With affection and esteem from your associates in the Chicago Engineering District."

A brief outline of Mr. Woermann's work appears in *Who's Who in Engineering, Who's Who in America,* and the *International Who's Who.*

On October 15, 1891, he was married to Elizabeth Graff Honens at Milan. To them were born two daughters, Elsie H. (Mrs. W. E. Kling), and Lillian H. (Mrs. Raymond Lee Ward).

Besides his widow and daughters, he is survived by four grandchildren (Robert Everett Kling, Everett Woermann Kling, Elizabeth Jane Ward, and Dorothy Lee Ward), two brothers (Frederick C. Woermann, M. Am. Soc. C. E., and Harry A. Woermann), and a sister (Mrs. Marion W. Robards). He is also survived by four nieces, Mrs. F. H. Koetter, Mrs. M. A. Wagner, Mrs. C. W. Eames, and Harryette Woermann.

Mr. Woermann was elected an Associate Member of the American Society of Civil Engineers on May 1, 1895, and a Member on May 7, 1902. He became a Life Member in 1930.

HERBERT SCANDLIN BATTIE, Assoc. M. Am. Soc. C. E.[1]

DIED JUNE 5, 1942

Herbert Scandlin Battie was born in Grafton, Mass., on October 11, 1884, the son of Capt. Charles Wesley Battie and Lizzie (Scandlin) Battie. He was graduated from North Carolina Agricultural and Mechanical College, at Raleigh, in 1907, with the degree of Bachelor of Science in Engineering.

From 1907 to 1909 he served with the McClintic-Marshall Construction Company of Pittsburgh, Pa.; did laboratory work at Cornell University at Ithaca, N. Y.; and was with T. C. Thompson and Brothers, general contractors of Birmingham, Ala. From 1910 to 1912 he was employed by the Pennsylvania Steel Company of Steelton, Pa., in the Erection Department—working out and detailing methods of erecting difficult steel bridges, including the proposed

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[1] Memoir prepared by N. P. Hayes, M. Am. Soc. C. E.

Quebec Bridge; also Scherzer and Strauss spans, usually built in open positions. He also served, during this time, as resident engineer on the Municipal Building, New York, N. Y., and on various highway bridges over the New York State Barge Canal. For part of 1912 he was with Post and McCord of New York City as structural draftsman.

During 1912 and 1913 Mr. Battie worked for the Virginia Bridge and Iron Company of Roanoke, Va., detailing various steel structures. In 1913, continuing in this capacity, he transferred to the Roanoke Bridge Company of Roanoke. For the remainder of 1913 and during 1914, he was employed by the Scofield Engineering Company of Philadelphia, Pa., as resident engineer on a bridge at Plymouth, Pa.

In 1915 and for part of 1916, he was with the Southern Bridge Company of Birmingham, detailing girders and structural steel frames for bridges. Later in 1916 he was appointed assistant to W. B. Spiker, consulting engineer of Atlanta, Ga., on the design of structural steel and reinforced concrete structures. In 1917 he was employed by the American Bridge Company in Chicago, Ill.

Mr. Battie served in the U. S. Army, as First Lieutenant, 212th Regiment of the Engineer Corps, from 1917 to January 21, 1919.

At the close of World War I, he returned to his position with the American Bridge Company in Chicago, remaining until he was called home to Greensboro, N. C., because of his father's illness.

From January to November, 1920, he was with the Carolina Steel and Iron Company of Greensboro, as designer and structural steel detailer. During this time he designed new shops for this company. For the next few years he served as district manager, North and South Carolina, for the Marchant Company on the sale of calculating machines. In 1926 he returned to the Carolina Steel and Iron Company, remaining until 1929. For the next several years he was with the U. S. Geodetic Survey and also served as an inspector with Froehling and Robertson. Again, from 1940 until a few weeks before his death, he was employed by the Carolina Steel and Iron Company.

On March 3, 1928, Mr. Battie was married to Katherine Campbell of Greensboro, by whom he is survived.

Among Mr. Battie's varied interests were the construction of radios, the trends of the stock market, and the study of history. He was a Mason and took a great deal of interest in the building of the Masonic Temple in Greensboro. A member of the Unitarian Church of Grafton (of which his grandfather had been the pastor), upon moving to Greensboro, he joined the First Universalist Church and, for many years, was its treasurer.

Mr. Battie spent many years in Greensboro, serving in many capacities. He was widely known, and his inquiring and well-developed mentality was respected by his many friends and acquaintances.

Mr. Battie was elected a Junior of the American Society of Civil Engineers on December 1, 1908, and an Associate Member on September 12, 1916.

CHESTER ROBERT CHADWICK, Assoc. M. Am. Soc. C. E.[1]

DIED SEPTEMBER 24, 1942

Chester Robert Chadwick was born in Lonsdale, R. I., on June 2, 1883, the son of Robert H. Chadwick and Annie (Tomlinson) Chadwick.

Mr. Chadwick attended the elementary schools of Langley, S. C., and was graduated from the Academy of Richland County, South Carolina. In 1904 he was graduated from Worcester Polytechnic Institute at Worcester, Mass., with the degree of Bachelor of Science in Civil Engineering.

Entering the employment of the American Bridge Company at New York, N. Y., in the same year, he remained with this company until 1909, serving as draftsman and checker, squad chief in charge of mill and office building drawings, and, later, as designer and estimator in the bridge designing department.

From August, 1909, until February, 1910, he was employed by the New York Central and Hudson River Railroad (later New York Central System), New York, as a bridge designer and draftsman in the bridge department of electric zone improvements. In February, 1910, he entered the services of Milliken Brothers, Inc., New York, where he was in charge of the designing and engineering department. His work during this period consisted of the design of several office, loft, and apartment buildings; two complete sugar mills; and a number of other types of mill buildings.

In October, 1912, Mr. Chadwick became affiliated with the National Bridge Company of Canada in Montreal, Canada, as designing engineer in charge of designing, estimating, and laying out a proposed new plant. In January, 1913, he was appointed plant manager in charge of operations. When this company was absorbed by the Dominion Bridge Company of Montreal, Mr. Chadwick remained with the latter company until 1921. He served as assistant to the manager, assistant to the vice-president, and, finally, assistant manager in charge of operations of the Lachine Plant which had a capacity of 6,000 tons per month of fabricated structural steel. In this connection, during World War I, he was instrumental in obtaining contracts for fabricating keels for Hog Island ships and for fabricating shell cases for the British Government.

During 1922 and the early part of 1923 he was associated with the late H. G. Balcom, M. Am. Soc. C. E., of New York, on the design of structural steel for office and hotel buildings.

In March, 1923, Mr. Chadwick entered the employ of Stone and Webster Engineering Corporation, Boston, Mass., remaining with this firm until his death in 1942, except for two short intervals. On leave of absence from Stone and Webster, from May, 1929, to February, 1930, he acted as works manager for the Ames Shovel and Tool Company of North Easton, Mass., in charge of plant operations and plant rehabilitation. In 1932 he left Stone and Webster to become chief engineer of the Reading Iron Company, at Reading, Pa., holding this office for the following three years.

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[1] Memoir prepared by G. R. Strandberg, M. Am. Soc. C. E., and E. E. Bigelow, Stone & Webster Eng. Corp., Boston, Mass.

While affiliated with Stone and Webster Engineering Corporation, Mr. Chadwick served as structural engineer and, later, as industrial engineer. During this time, he was in charge of engineering and design on many large projects, among which were the office building for the Insurance Company of North America in Philadelphia, Pa., the Piney River Plant of the Southern Minerals Products Company, New York, and a large new brass rolling mill for the Bridgeport Brass Company, Bridgeport, Conn. In later years, Mr. Chadwick was engaged mainly in preparing reports on the property and business of many large industrial concerns. Early in 1940, he initiated the engineering work in connection with a new manufacturing plant for the Sperry Gyroscope Company of New York.

In the fall of 1940, he was assigned to work in connection with the construction of the Kankakee (Ill.) Ordnance Works for the U. S. Government. This was one of the first of the new high explosive plants constructed for the defense program in World War II. Later he was associated on similar work for two other large ordnance plants, and was engaged in this work at the time of his death in 1942.

Intensely interested in his work, Mr. Chadwick was loyal to such an extent that he always shouldered more than his share of responsibility, at times to the detriment of his own well-being. He possessed an extremely pleasing personality and was highly respected and loved by all his associates. His personal life was quiet and simple. He enjoyed sailing and was a member of the Wollaston Yacht Club.

Mr. Chadwick was married to Fay A. Billings of Northfield, Mass., on May 20, 1915. He is survived by his widow and two sons, Capt. Robert B. Chadwick and Donald R. Chadwick.

Mr. Chadwick was elected a Junior of the American Society of Civil Engineers on February 5, 1907, and an Associate Member on February 28, 1911.

JOSEPH JAMES CORTI, Assoc. M. Am. Soc. C. E.[1]

Died June 17, 1938

Less than 5% of the Society membership resides outside of the United States, and of this 5% certainly the greater part was born in that country or studied there. In preserving a record of the life of José Santiago Corti (Joseph James Corti) for the annals of the Society, it seems pertinent to call attention to this rather unusual circumstance: He was never in North America, nor in fact in any country where the English language predominates; yet he entered the Society as a young man and was a Life Member at the time of his death, forty-two years later.

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[1] Memoir prepared by Henry Lanark Miller, Assoc. M. Am. Soc. C. E.

These facts accord with certain definite traits in the character of señor Corti: Great constancy, a capacity for, and love of, languages, and a cosmopolitan thirst for knowledge in the lines of his profession. As those who knew him can testify, these traits were combined with a high degree of modesty and a cordial willingness to impart knowledge to others. While still quite a young man, he realized the practical character of engineering in the United States. Throughout his life he advocated such practical engineering in the land of his adoption (Argentina) where, at that time, theoretical studies were pursued, often without regard to their effective application.

José Santiago Corti was born on May 5, 1861, in Videm, Austria, where his maternal grandfather was engaged in forestry. His father, also José, was Italian and came of a family skilled in construction; his mother was Teresa (Marconcic) Corti. The family immigrated to America in 1868, when José Santiago was seven years old, and settled in Buenos Aires, Argentina.

The boy could then speak Italian, German, and Slavic, but, finding no encouragement among his early school companions to retain the latter two, he lost the use of them while learning Spanish. In later life he relearned German enough to use technical books, and also learned, even better, the French and English languages.

At the age of twenty his first employment took him to the western provinces —"Cuyo"—that is, San Luis, Mendoza, and San Juan, where his lifework was destined to be. From May, 1882, to September, 1885, he was assistant engineer in charge of relocation and tracklaying on the Andino Railway, the first to reach the Argentine Andes. It was built as a government enterprise and later was sold to an English company. While employed as division engineer in charge of buildings and bridges on this railway, señor Corti studied for the degree of Civil Engineer, which he received from the University of Buenos Aires in June, 1887, a few years after this degree had been established in this outstanding Argentine university.

After completion of his railroad work with the construction of the station buildings in the City of San Juan, señor Corti settled in that city. He taught geodesy and analytical geometry in the Engineering School, and road making, topography, and descriptive geometry in the School of Mines. During this period (1887-1895) he was also employed by the Province of San Juan on irrigation work and river defense, and in 1895 he headed a national commission charged with planning reconstruction in the City of San Juan after a severe earthquake (1894). For a brief interval (February to August, 1893) he was chief engineer for the City of La Plata (Province of Buenos Aires), but for family reasons preferred to leave that post and return to San Juan where, from 1894 onward, he was employed by the national government.

In 1900 he was transferred to the more important city of Mendoza as head of the 4th Division of Roads and Bridges, comprising the three Cuyo provinces. In this capacity he built the first real macadam roads in this region, between the City of Mendoza and the Town of Guaymallen; planned and built several important bridges; and personally relocated part of the highway across the Andes between Mendoza and the Republic of Chile.

During this period señor Corti taught mathematics and topography in the Mendoza School of Grape and Wineculture (later incorporated in the University of Cuyo). His advice was much sought in private practice. It is a matter of record that he designed the first reinforced-concrete structures in Cuyo, such as the winevats for the del Bono establishment in San Juan (1903) and for D. Ricardo Palencia (1905) in the Department of San Martín (Province of Mendoza).

Such work has attained great importance in this region, and señor Corti might well have achieved financial success. However, he was not commercially inclined, and was quite content to see other men reap practical gain after he had, in a sense, "shown the way." He then would turn to some new pursuit in which, perhaps, there would be no material benefits. Such was his part in establishing in Mendoza the first astronomical observatory.

His interest in the stars had begun when he first used a theodolite. While teaching geodesy in San Juan, he had worked out independently the four-tangent azimuth method, only to learn later that Jacques Babinet had introduced it in France in 1857. The method is taught in the University of Córdoba today under the name Babinet-Corti. In 1910 he was chosen to locate exactly the 35th south latitude as arbitrator in a dispute between the Argentine Great Western and the Buenos Aires Western railways.

In 1911 an intimate friend of señor Corti, D. Juan Carullo, an Italian banker, conceived the idea of building in Mendoza 150 model houses to be sold and raffled on a savings-bank system. The center of interest was to be an observatory in charge of his friend señor Corti. This building was completed, together with some twenty houses, and under señor Corti's direction the instruments were installed. They consisted of one equatorial of 6-in. apochromatic objective, one Bamberg universal, one Bamberg transit instrument, two seismographs, and two electrical pendulums. These were imported from Germany at a cost of about $50,000 (United States currency), and with them señor Corti began observations (1914 to 1919), including a patient determination of latitude with a probable error of $\pm 0''.015$.

It was planned to offer this observatory to the government with the idea that it could serve a double purpose: (1) For use in studying and photographing a certain part of the sky in collaboration with the observatories at La Plata and Córdoba, and (2) for use in teaching practical astronomy to students of the Mendoza high school (Colegio Nacional). The banker, señor Carullo, was himself becoming an enthusiastic amateur under señor Corti's guidance, but their financial plans failed in the general crisis of 1918 and everything went to creditors. That the instruments could be retained in the Province of Mendoza is due to the persistent devotion of señors Corti and Carullo, and to the providential circumstance that the banker in 1930 won the first prize in the Montevideo (Republic of Uruguay) lottery (about a half million dollars in United States currency).

The observatory is now installed in Avenue Emilio Civit with official support, and thus will realize, in part at least, the plans of its founders. To complete the story it must be mentioned, with regret, that señor Carullo's bank failed again and, unwilling to go through trouble a second time, he took his life.

This tragic end to a former associate was not the only cloud that saddened the later years of Joseph James Corti. His oldest son, Julio José Corti, who studied electrical engineering at Cornell University in Ithaca, N. Y. (1907–1910), and mechanical engineering at Brooklyn Polytechnic Institute in Brooklyn, N. Y. (1910–1912), lived for seventeen years in the United States. He contracted sleeping sickness there in 1924 and returned to Mendoza, where he died nine years later after great suffering.

Señor Corti, one of the most gentle and scholarly of men, tasted as well

" * * * the bitter cup
of man's ingratitude * * *."

In 1917 a complete political turnover took place in Argentina, accompanied by much blind hostility to all past administrations. Señor Corti had already earned his pension in government employ and might have retired, as did some of his colleagues; but he "stood by his guns," to proceed or to turn over his post as might be ordered. He had never taken part in politics and, because of his kindly, modest ways, had made no personal enemies. The mere fact of his association with certain people, however, was sufficient excuse for his dismissal from service, and his retirement papers were maliciously sidetracked for several years. The fact that some people in the plots against him were subordinates whom he had befriended and helped in difficulty must have been painful to him. Only after a lapse of years, and by bringing suit against the Argentine nation, did he finally obtain his rights, and his accumulated pension was paid in full.

Señor Corti enjoyed the respect and affection of his fellow citizens during the closing years of his life. In his seventy-first year he was vice-president of the Board of Education, and for a short time director-general of education in the Province of Mendoza. While holding the latter office, he established the first vacation trips for primary students to the mountain valleys of Uspallata.

The recently established University of Cuyo has received as a gift the greater part of señor Corti's valuable technical library. Had the University existed during his lifetime, it no doubt would have had the more valuable benefit of his personal activity, since he was a born teacher. This talent had its chief outlet in the education of his own sons, which he watched with devoted care. It was his custom to leave daily, on each boy's desk, some mathematical or other problem, expecting to find it solved the next day. He would then criticize, correct, or praise the solution, demonstrate a shorter one, or introduce variants.

Señor Corti was a founder member of the Rotary Club established in Mendoza in 1927. He took great interest in its activities, serving as president in 1930–1931, and his discourses and comradeship have made him exceptionally well remembered. On the third anniversary of his death, the organization dedicated a bronze tablet to his memory, and the principal of the Mendoza high school, D. Tomás Silvestre, expressed the feeling of all who knew señor Corti when he said:

"He inspired us with respect by the considerate precision of his convictions. Direct in speech, erudite but simple, he went straight to the fundamentals of every problem as he faced it. He never sought glory or personal advantages: his ambition was to serve others with honesty and intelligence."

Few members of the Society knew señor Corti personally; yet, in future years, many may have occasions to cross the Andes between Argentina and Chile, where the gigantic statue of Christ, erected under Corti's direction, stands on the international border. They may see the observatory or other traces of his activity in Mendoza. It will be a satisfaction to reflect that one of the first and most scholarly of Argentine civil engineers was a lifelong member of the Society.

Señor Corti was married in 1887 to señorita Julia Videla, of San Juan, who died on May 10, 1935. He is survived by one daughter and six sons. Most of his children received professional degrees. Julio José Corti has been mentioned. The others are: Guillermo José Corti, an electrician; Ramón Ceferino Corti, who received the degree of Bachelor of Science in Economics and Finance from New York University in New York, N. Y.; Carlos Emilio Corti Videla, Assoc. M. Am. Soc. C. E., a geographical and cadastral engineer (Córdoba University); Salvador Pedro Corti, a certified accountant; Rafael Benjamin Corti, a cattle expert; Alberto Antonio Corti, a lawyer; and Ernestina Teresa Corti, a professor of languages. Two grandsons, José Guillermo Corti (who received the degree of Doctor of Mathematics from the University of La Plata) and Juan Carlos Corti (who received the degree of Doctor of Medicine from the same University), complete a list that leaves few fields of knowledge unexplored, and may be taken as further proof of señor Corti's devotion to learning.

He was a member of the Sociedad de Ingenieros Civiles; the Centro Nacional de Ingenieros; Sociedad Cientifica Argentina; Mueseo Social Argentino; National Geographic Society (Washington, D. C.); and the Centro Mendocino de Ingenieros. He was a frequent contributor to scientific journals in Argentina.

Señor Corti was elected an Associate Member of the American Society of Civil Engineers on January 1, 1896.

MARIUS SCHOONMAKER DARROW, Assoc. M. Am. Soc. C. E.[1]

DIED APRIL 13, 1942

Marius Schoonmaker Darrow was born on August 11, 1877, at Kingston, N. Y., the son of Henry Dennison and Ella (Schoonmaker) Darrow. On his mother's side he was Holland-Dutch and descended from a family that settled in Esopus (later Kingston), N. Y., in 1659. In 1899, he was graduated from

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[1] Memoir prepared by E. E. Scholer, M. Am. Soc. C. E.

Cornell University, Ithaca, N. Y. He was a member of the Alpha Tau Omega fraternity.

After acting as assistant in the hydraulic laboratory at Cornell University from June to December, 1899, Mr. Darrow went to Chicago, Ill., as recorder for the U. S. Engineers. In 1901, he was employed as assistant engineer at the Stickney Yards in Chicago, but shortly thereafter left to go with the Chicago, Burlington and Quincy Railroad Company, in its Chicago office, as draftsman, designer, and inspector engineer in charge of reinforced-concrete work. Still remaining in Chicago, in 1905 he became superintendent for Thomas Phee and Company, a contracting firm.

In 1906, Mr. Darrow was employed as superintendent for the Columbian Improvement Company of St. Croix Falls, Wis. Again, in 1907, he changed positions—this time to work in the bridge department of the Great Northern Railway Company at St. Paul, Minn. From 1907 to 1911 he was in Utah, Idaho, and Nevada on irrigation projects investigating water rights, surveying, and building dams.

He left the western part of the country in 1911 to go to Maurer, N. J., as superintendent for the Barber Asphalt Corporation. However, in 1916, he returned to the Midwest as manager of the company's Madison, Ill., plant, which specialized in manufacturing asphalt roofings, mastics, paints, asphaltic waterproofing, and insulating compounds. For twenty-three years, until 1939, Mr. Darrow strove to make this plant one of the prominent industries of the Tri-Cities—that group of three towns in Illinois (Venice, Madison, and Granite City) which constitutes the "East Side" heavy industries area of the St. Louis, Mo., manufacturing region. During this period, he greatly expanded the activities of the company, particularly by modernizing the buildings, by constructing new paint and boiler plant units, and by stepping up the roofing mill output per machine fivefold.

At the time of the disastrous Mississippi River flood in 1927, he promptly organized numerous crews of volunteer workers for the repair and reinforcement of the levees at Venice and Madison against the encroachments of the turbulent river, and it was largely due to his generalship and his untiring efforts during those hectic days that the East Side was saved from a devastating flood such as inundated so much other valuable territory farther south.

For many years he was president of the East Side Manufacturers Association. He belonged to the University Club, Engineers Club, and Algonquin Golf Club, all of St. Louis, and Hamilton Club (Chicago). During his lifetime he was a member of the Dutch Reformed Church. His political affiliations were Republican.

On September 27, 1939, Mr. Darrow became ill quite suddenly and was confined to his home. During the following summer, he recovered partly and seemed on the road to normal health again; but he suffered a relapse in the fall, from which he never fully recuperated.

He had a fine presence, a dynamic personality, and commanded the respect of all who knew him and had the loyalty of all the men under him. A great lover of dogs, for many years he had a pair of pedigreed Schnauzers of which he was very proud. Soilless plant cultivation was one of his hobbies.

On August 28, 1924, he was married to Gertrude J. O'Brien, who, with a son, Marius Schoonmaker Darrow, Jr., survives him. He was devoted to his wife and to his son, and was mourned by all his many friends and business associates both in St. Louis and on the East Side.

Mr. Darrow was elected a Junior of the American Society of Civil Engineers on February 6, 1900, and an Associate Member on December 1, 1908.

WILLARD LIVERMORE GORTON, Assoc. M. Am. Soc. C. E.[1]

Died March 25, 1942

Willard Livermore Gorton was born in Eau Claire, Wis., the son of William Eddy and Mary (Livermore) Gorton, on June 13, 1881. His early education was obtained in the grade schools of Warsaw, Ind. From there his parents moved to Oklahoma, where he entered the preparatory school of the University of Oklahoma at Norman. He then took the engineering course at the University of Oklahoma, being graduated with the degree of Bachelor of Arts in Mathematics in June, 1903.

However, while at the University, he was initiated into engineering work as tapeman and rodman on a part of the construction of the Missouri-Kansas and Oklahoma Railroad at Coffeyville, Kans., in the summer of 1902. From October, 1903, to May, 1904, he was employed by Burns and McDonnell, consulting engineers of Kansas City, Mo., as instrumentman on surveys for a water works and sewerage system and on laying out a system of street grades for Chickasha, Indian Territory (later the State of Oklahoma). Also, he directed construction of a part of this work.

In May, 1904, he entered the U. S. Reclamation Service as engineering aide on canal and reservoir surveys for the Milk River Project in Montana. In November, 1904, he was transferred to the North Platte, Interstate Project, being promoted to assistant engineer in January, 1907. Here he displayed distinct ability as chief of party on the surveys of a large part of the second division of the main canal and the corresponding distribution system. He then was appointed resident engineer on the construction (including structures) of this project.

Mr. Gorton showed exceptional executive and engineering ability in building a distribution system on a 40,000-acre tract of land under especially difficult circumstances in the latter part of 1907 and the early part of 1908. This work, followed by equally distinguished service in the initial stages of operation and maintenance, did not fail to attract the attention of the late I. W. McConnell,[2] M. Am. Soc. C. E., who had been supervising engineer for the Central District of the U. S. Reclamation Service and who had just accepted the position of chief irrigation engineer for the J. G. White Company. As a result Mr. McConnell invited Mr. Gorton to accept the position of chief engineer for the Idaho Irrigation Company, and concurrently the position of construction superintendent for the J. G. White Company at Richfield, Idaho.

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[1] Memoir prepared by Andrew Weiss, M. Am. Soc. C. E.

[2] For memoir, see *Transactions*, Am. Soc. C. E., Vol. 101 (1936), p. 1592.

At the close of 1911 Mr. Gorton became acting chief irrigation engineer for the Bureau of Public Works of the Government of the Philippine Islands. On July 1, 1912, he was appointed chief irrigation engineer and, on January 1, 1914, chief designing engineer. While in the Philippine Islands he gave his services without pay to many public activities, serving as assistant on a public committee for placing a valuation of rates with the Manila Electric Company and as designer and supervisor of repairs on the old bridge of Spain in Manila which had been damaged badly by a typhoon.

Concerning his work in the Philippine Islands, Warwich Greene, director of public works, commented as follows:

> "In addition to the regular duties of the above positions, Mr. Gorton discharged many special technical and administrative duties for this Bureau and for the Government."

Mr. Gorton was appreciated both for his personality and for his services, as expressed in the following statement by Mr. Greene:

> "Mr. Gorton discharged the duties of the various positions held by him under this Government in a very satisfactory manner and I take great pleasure in testifying to his unusual capacity as an engineer, to his excellent ability as an administrator, and to his personal character. * * * We consider him one of the ablest engineers and administrators that have ever been employed by this Bureau."

A co-worker and friend, George G. Stroebe, who was with Mr. Gorton in the Philippines, wrote a letter of which the following is a summary:

> "Like all big men, Mr. Gorton was entirely unassuming, patient, painstaking and hard working; kind, generous and unselfish; capable of meeting any obstacle and overcoming it; strong and reliable; courageous and never compromising with what he considered wrong. The qualities which his Philippine friends remember were his gentleness, genuineness, geniality, quiet humor, and thorough sympathy with and readiness to help others, wherever and whenever such help was asked or needed."

Following this period of distinctive service in the Philippines, Mr. Gorton spent some months, February to June, 1915, traveling in China, Japan, and the Hawaiian Islands. On his return to the United States, he accepted the position of chief engineer on maintenance and operation for the Idaho Irrigation Company, in Richfield, serving in this capacity from June, 1915, to September, 1918.

At this time he offered his services to the U. S. Government and was commissioned Captain of Engineers in the U. S. Army, stationed at Camp A. A. Humphreys, Virginia. He served from September to December, 1918, when, with the close of World War I, his services were no longer necessary.

Mr. Gorton then returned to Idaho as consulting engineer on duty of water studies from December, 1918, to April, 1919, for the Idaho Irrigation

Company of Richfield. From April, 1919, to December 31, 1922, Mr. Gorton was chief engineer of the Idaho State Public Utilities Commission. During this time he placed the first valuation on most of the public utilities of the state.

The outstanding quality of his service as chief engineer of the Idaho State Public Utilities Commission is attested in a letter sent the writer by the Hon. Raymond Givens, Justice of the Supreme Court of Idaho, who, as attorney for the Public Utilities Commission at that time, was associated officially with Mr. Gorton during this period. A part of Judge Givens' letter is quoted as follows:

> "Mr. Gorton was not only entirely competent in his own profession, a man of unquestioned integrity, character, and honesty of opinion, but a man of great and broad common sense and understanding, kindly and gentle, possessed of a strong and vigorous character and personality, he was adamant, though in a courteous way, for what he believed to be the right and knew to be the truth. He was a real philosopher, which characteristic he applied to and in the practice of his profession, which made him much sought after for consultations and as an expert witness.
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> "Clear and lucid in his expression of opinion and statements of conclusions, they were always based upon a keen, incisive, thorough, and impartial investigation, search for, and determination of all pertinent, underlying facts.
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> "The civil engineering profession of the state and of the nation can well be proud that Mr. Gorton was for so many years an honored member thereof and devoted his remarkable talents upon such a high plane to the real betterment of mankind."

This engagement was followed by a period of private practice until the end of 1925 as consulting engineer in Boise, Idaho, principally on hydraulic problems and projects. At this time the Republic of Mexico recognized the essential need for irrigation on a much larger scale than had theretofore been feasible by private or corporate enterprise. As a result a national irrigation law was enacted, and the Mexican Government initiated its irrigation enterprises by employing the J. G. White Engineering Corporation to make the necessary studies and plans and to build the projects which appeared to be of first importance in the development of its agricultural resources. This law and the program which followed were modeled largely after the pattern that had been set by the United States when Congress passed the Reclamation Act of June 17, 1902, as a result of which the irrigation and power development of the West has achieved results not elsewhere paralleled to date.

Mr. Gorton was one of the first consultants selected by the White Company in an effort to build a good organization for the large task it had undertaken. His large and notable experience, obtained under widely differing climatic, economic, and social conditions combined with an outstanding personality, provided a most excellent background for the duties at hand. The writer was associated with Mr. Gorton for the last time on this work; in fact, he followed in Mr. Gorton's footsteps in Mexico, when, in October, 1926, Mr. Gorton decided to return to his home and familiar scenes and associates in Idaho. The several weeks spent with Mr. Gorton in Northern Mexico prior to his return to the United States long will be remembered by the writer as it brought about a much needed orientation in a new field and a much appreciated contact with an old and esteemed friend.

Following his return to Boise, he resumed his private practice as consulting engineer, continuing until the Russian Government induced him to accept a position as consulting engineer on a vast irrigation program then being initiated in the vicinity of Tashkent, Uzbekestan, U.S.S.R. He remained in Russia from March 1, 1930, to March 1, 1932, when he returned to his private consulting practice in Boise, after some months of travel in Europe.

Mr. Gorton then accepted the position of chief engineer for both the Boise Water Corporation and the Kellogg Water and Power Company. In this capacity, he rendered notable service until stricken with influenza in October, 1940. This illness was followed by an attack of valvular heart trouble from which he never seemed to recover.

Like most members of his profession Mr. Gorton was averse to being in the limelight, and, as expressed by one of his admirers, Ben Oppenheim, member of the Industrial Accident Board of the State of Idaho:

> "He was an engineer, strictly professional, not exactly secretive but more of a listener than a talker. What he had to say he put down in reports over his signature. After he came back from Russia he changed. His wife persuaded him to talk to her club. He discovered a new field of activity. People, he found, were interested in what he had to say. He became more talkative."

His many influential friends succeeded in inducing him to become a member of the Board of Trustees of the Independent School District of Boise City, on which he served as member from November 13, 1933, to September 14, 1936; then as vice-president until November 11, 1937; and finally as president of the Board until the date of his resignation, February 1, 1938.

In these several positions he rendered most tireless and effective service, giving freely of his time so far as his official duties permitted. As a member of the Board of Trustees he had some very important committee assignments such as:

(*a*) Chairman of the Fire Insurance Committee, initiating notable improvements;

(*b*) Chairman of the Fuel Committee, introducing a system of purchasing coal on a chemical analysis basis, thus enabling the District to obtain coal of a better quality without restricting the field of competition; and

(*c*) Chairman of the Building Committee, during the construction of the Junior High School, the Senior High School gymnasium, and the addition to the Whitney School. (His technical knowledge and engineering experience were very valuable in the building field.)

Regarding his work on this Board of Trustees, from which he resigned due to the pressure of business affairs, J. L. Driscoll, President of the First Security Bank of Boise and member of the Board, writes:

> "Although a very busy man in private life, Mr. Gorton always found time to go to the bottom of all our school problems. He never failed to

keep in mind that the public schools are operated for the benefit of the children and he was anxious that the children receive training of a practical nature which will aid them in later life."

Although severely handicapped as a result of his heart trouble, true to his character and stamina, he carried on as well as he could in his consulting work. On February 27, 1942, he was appointed consulting engineer of the U. S. Reclamation Bureau, a fitting and well-earned tribute from the organization which he served so well during the initial years of his professional career. Mr. Gorton died one month later on March 25, 1942.

Mr. Gorton was a member of the Masonic Lodge in which he ascended to the Scottish Rite and to the Shrine.

Mr. Gorton was married to Catharine Sarchet of Charleston, Ill., on October 30, 1905. Their marriage was a happy one, and to them were born two sons, Willard Livermore Gorton, Jr., and Marquand Sarchet Gorton, Jun. Am. Soc. C. E.

Mr. Gorton was a man of exceptionally fine character and moral worth. With this he combined fine executive ability and a thorough theoretical training which—together with his long and varied professional experience, general knowledge, and wide culture—made him a consultant of the first order in his chosen lifework. The engineering profession has lost one of its outstanding members in his death.

Mr. Gorton was elected an Associate Member of the American Society of Civil Engineers on May 3, 1910.

ALBERT HENRY GREENWOOD, Assoc. M. Am. Soc. C. E.[1]

Died July 11, 1942

Albert Henry Greenwood was born in Lawrence, Mass., on November 8, 1877, the son of James and Mary (Howard) Greenwood, both of whom were of English birth. His early education was received in the schools of Lawrence. After pursuing an engineering course at Dartmouth College, in Hanover, N. H., he was graduated with the degree of Bachelor of Science in 1899. In 1900 he received the degree of Civil Engineer from the Thayer School of Civil Engineering at Dartmouth. Throughout his college course he maintained an exceptionally high scholastic standing. While at Dartmouth he was a member of Alpha Delta Phi fraternity and other college organizations.

During vacations, and for a short time after he was grauated, he did work for the Metropolitan Water Board of Boston, Mass., and for the proprietors of the locks and canals of Lowell, Mass. He also worked as an inspector of bridges for the Boston and Maine Railroad Company.

His first important engineering work was performed as assistant engineer on sea-wall and breakwater construction for the port of Manzanillo on the west coast of Mexico. Upon his return from Mexico he was with the Berlin Construction Company, in Berlin, Conn., for seven years, in charge of the erection of various industrial buildings.

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[1] Memoir prepared by Robert C. Noerr and Robinson D. Buck, Assoc. Members, Am. Soc. C. E.

In 1910, in association with one of the writers, he formed the firm of Greenwood and Noerr, Consulting Engineers, of Hartford, Conn., specializing in factory and general structural design. He maintained this connection until ill health forced him to retire from active practice in 1940. During his connection with the firm he designed and supervised the construction of many buildings and other structures in Hartford and in other parts of the state.

He was a member, and at one time president, of the Hartford Dartmouth Club; a member for many years of the Hartford Civitan Club; and a life member of the Connecticut Society of Civil Engineers.

In 1904, he was married to Mary A. Gage, of Lawrence, who, with two sons, James and John Gage Greenwood, and two grandchildren, survives him.

Mr. Greenwood, though of a naturally quiet and retiring disposition, had a fund of good humor and dry wit, which, with his readiness to help others with their problems, whether professional or personal, made him well liked by all with whom he came in contact. His high character, mental honesty, and practical experience, which enabled him to use his theoretical knowledge to the best advantage, were responsible for the high esteem in which his opinions were held, both by those for whom he did work, and by those who worked under him. With his death the profession has lost an able member, and many have lost a loyal friend.

Mr. Greenwood was elected an Associate Member of the American Society of Civil Engineers on February 1, 1905.

JOSEPH CHURCHILL HILTON, Assoc. M. Am. Soc. C. E.[1]

DIED JANUARY 29, 1942

Joseph Churchill Hilton was born on November 27, 1872, in Yarmouth, Nova Scotia, Canada, the son of Bradford R. and Margery (Churchill) Hilton. He was fifth in direct descent from a line of Nova Scotia sea captains and, from the age of five, spent much of his boyhood at sea with his parents. At the age of seventeen he was serving as able seaman on his father's ship *The Abyssinia*.

Notwithstanding his love of the sea, he attended Acadia University at Wolfville, Nova Scotia, for a time. Later he determined to become a civil engineer and entered Cornell University, Ithaca, N. Y., in 1892, being graduated in 1896, with the degree of Civil Engineer. He was an able and earnest student and was highly regarded by the faculty and members of his class.

At the time he was graduated, the United States was in the midst of a depression, and positions were not to be found, so he returned to Nova Scotia and was employed as an instrumentman on surveys for the Coast Railway Company for several years until 1899. He then set out for larger fields and began a remarkable career which took him to many parts of the United States and South America. For the first two years he was employed by the late Alexander Potter,[3] M. Am. Soc. C. E., a prominent consulting engineer, as engineer on various sewerage systems and on sanitary and water supply investigations.

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[1] Memoir prepared by R. H. Keays, M. Am. Soc. C. E.

In 1901 Mr. Hilton entered the employ of the late Ira A. Shaler, contractor for the important tunnel section of the New York (N. Y.) Rapid Transit Subway on Park Avenue, from 33d Street to 41st Street. This period is well described by Charles E. Fraser, M. Am. Soc. C. E.:

> "Hilton's job was the driving of the tunnels from 33d Street northward. There were two double track tunnels to be built, to be driven in close proximity and beneath an already existing trolley tunnel.
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> "Subway tunneling under the busy streets of New York was an undeveloped art at the time. Misfortunes on the job were the well remembered explosion of a dynamite magazine in front of the Murray Hill Hotel, marks of which are still visible on the polished granite face of the hotel. In July, 1902, Major Shaler was seriously injured by falling rock and died a few minutes later. Hilton remained until the completion of the work, about the end of 1903."

Mr. Hilton was definitely in the contracting business from this time on. In 1903 he was superintendent for the Bergenport Chemical Works at Bayonne, N. J., and, later in the same year, superintendent for James Conway on construction of a sewer system at Millburn, N. J. In 1904, he went to New Orleans, La., as superintendent for T. J. Shea, contractor, on the city sewerage system. In 1904 and 1905, he was superintendent for the Sand Filtration Corporation of America, on construction of a slow sand filtration plant (a large project for those days) at Washington, D. C.

From 1905 to 1907, Mr. Hilton was assistant superintendent for S. Pearson and Son, building the four East River (New York) tunnels for the Pennsylvania Railroad Company. This was a monumental project for that day. Mr. Hilton was in charge of two of the tunnels on the Manhattan side. Great difficulties were encountered in driving these 23-ft diameter tunnels through the sand, rock, and mud underlying the East River. The tunnels were shield-driven and were lined with cast-iron and steel segmental plates. Compressed air was used at all times to support the headings.

Returning to New Orleans, from 1907 to 1909, Mr. Hilton was general superintendent for Black and Laird, contractors for the new Water Purification Plant. In 1909 and 1910 he was a partner in the T. M. McLeod Company and worked on a sewerage project in Hackensack, N. J. From 1910 until 1915 Mr. Hilton was associated with his old friend, the late Francis Mason, in the contracting business, under the name of Mason, Hilton and Company. Many important contracts were undertaken by this company, which was closely associated with the firms of Merrill and Ruckgaber and the McHarg, Barton Company.

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[3] For memoir, see *Transactions*, Am. Soc. C. E., Vol. 106 (1941), p. 1652.

During World War I (1918–1919) Mr. Hilton returned to work with former colleagues as superintendent in charge of excavation and other work for the Turner Construction Company on the Brooklyn Army Base. This engagement was followed by a connection with the Foundation Company of New York, from 1909 to 1921, during which time he had charge of work in Hartford, Conn.; Huntingdon, Pa.; and Bloomfield, N. J. He then went, for a short time, with Ballard, Sprague and Company, contractors on steam plant work; and, from 1922 to 1926, he was with the J. G. White Engineering Corporation on several hydroelectric projects for the Augusta-Aiken Railway and Electric Corporation.

From 1926 to 1929, Mr. Hilton was again with his old associates of the East River tunnels, this time as vice-president of Fraser, Brace and Company, in full charge of construction of La Quiebra Tunnel for the Antioquia Railway, in Colombia, South America. This project included not only building a tunnel 2¼ miles long, but also purchasing and installing equipment suitable for permanent use in the operation of the railway—a high-head Pelton Wheel installation, compressors, air drills, diesel engines, penstock pipe line, and other unusual features. The contract was finished in three years, within the original estimates of construction. For these excellent results, Mr. Hilton received great praise from the officials of the Colombian Government, and from then on he had a great reputation in South America.

From 1930 to 1932, Mr. Hilton was in Chile, with the Ambursen Dam Company as manager of all their work in that country. The only contract actually started, however, was the construction of the Puente Negro Dam, near Santiago. This was a disappointing contract because the plans were changed several times by the government and finally the work was stopped for lack of funds. On their way home from Chile, Mr. Hilton and his wife spent the summer of 1932 traveling in Europe, the first and last planned vacation period he enjoyed during his working years.

From 1933 to 1935 he commuted to Colombia, first representing Gannett, Eastman and Fleming, engineers of Harrisburg, Pa., investigating contract possibilities. (For a short time he returned to the United States as superintendent on the construction of a dam at Reading, Pa.) In 1935, working for the Sanders Engineering Corporation of Portland, Me., he was in charge of the construction of La Regadera Dam, for the City of Bogota, Colombia.

On the occasion of Mr. Hilton's death, a prominent Colombian newspaper in Bogota stated, "We have had the honor to know and to be intimately associated with this very distinguished engineer—natural, unaffected, frank and courteous in all social relations. Mr. Hilton gained the respect of everyone with whom he was associated in one way or another * * *."

From 1936 until his death in 1942, Mr. Hilton was connected with the Public Works Administration, first as resident engineer inspector on the Triborough Bridge of the Port of New York Authority, then as chief resident engineer inspector on several projects, and finally as project engineer on the Circumferential Parkway in Brooklyn and Queens, New York City. His health began to fail in 1940, but, in spite of the painful and crippling nature of his illness, he continued to work until three weeks before he died.

Mr. Hilton became an American citizen in 1907. In 1901 he was married to Margaret L. Williams of Yarmouth, who accompanied him on all his journeys. In addition to his wife, he is survived by two children, Sylvia Caroline, a graduate of Cornell University, and Thomas Bradford, a graduate of McGill University (Montreal, Canada), an aviation cadet in the U. S. Army Air Force.

Mr. Hilton was elected an Associate Member of the American Society of Civil Engineers on January 2, 1907, and became a Life Member in January, 1943.

WILLIAM FREDERICK JENRICK, Assoc. M. Am. Soc. C. E.[1]

Died October 24, 1942

William Frederick Jenrick was born in New York, N. Y., on December 18, 1885, the son of Frederick and Elizabeth (Fengman) Jenrick. He received his early education in the public schools, and was graduated from Cooper Union in New York, in 1907, with the degree of Bachelor of Science. In 1910 he was graduated from Cornell University, Ithaca, N. Y., with the degree of Civil Engineer.

During the years he attended Cooper Union and Cornell, he was with the Foundation Company as first assistant to the chief engineer. His work involved drafting, designing, estimating, and cost keeping on pneumatic foundation work for monumental office buildings, bridge piers, mining shafts, and dams and tunnels; later, he was in the field on construction work. In addition, he inspected and tested air locks, collapsible shafting, and caisson steel at steel mills for use in sinking foundations for the Municipal Building, New York.

Following his graduation from Cornell, he continued in the employ of the Foundation Company until 1915. During this time he designed and made estimates for bidding on heavy foundation work; inspected and reported on conditions for shoring buildings; constructed foundation and steel grillages for heavy structures including buildings, bridges, and other structures; and assisted in the design and sinking of a 200-ft reinforced-concrete mining shaft. When, in May, 1912, he became statistical engineer for the Foundation Company, he investigated and reported on conditions of contracts with regard to cost, efficiency of work, and suggested methods of improvement.

In 1915 Mr. Jenrick became associated with the Stone and Webster Engineering Corporation at Boston, Mass., first as an estimator and later in direct and responsible charge of estimating in the construction division. His connection with Stone and Webster was a permanent one and lasted until his death. The scope of his activities, during this period of twenty-seven years, covered studies and cost estimates of all the various types of construction, which were undertaken by Stone and Webster Engineering Corporation, and included foundations, cofferdams, buildings, harbor and water supply works, hydroelectric projects, steam power plants, transmission lines, and chemical and industrial plants.

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[1] Memoir prepared by J. H. Hood, W. N. Patten, and L. N. Reeve, Members, Am. Soc. C. E.

His work was always outstanding in that he possessed an uncanny ability to size up complicated situations in a very practical manner, and he could substantiate his judgment with the proper estimate of cost, the data for which he had accumulated during many years of study and analysis.

As he functioned largely as a consultant not only on design as affected by cost, but also on construction in working out methods to achieve the maximum of efficiency in building the structures represented by his estimates, his activities were much broader than the title of chief estimator implies.

During his career as an engineer, Mr. Jenrick was particularly interested in the determination of standard methods of measuring quantities for concrete work and, for a number of years, was a member of the American Concrete Institute serving in 1925 and 1926 as secretary of the technical committee dealing with this subject.

From 1923 to 1938, he acted as part-time lecturer at the Massachusetts Institute of Technology, Cambridge, Mass., conducting a course in estimating in the senior year of the curriculum of architectural engineering. For this work, Mr. Jenrick wrote his own text and paired off the students into "partnerships" for the submission of estimates. While teaching, he showed his unusual merit by giving intensive and thorough instruction, and by arousing such interest and enthusiasm that this course was one of the most popular with the students.

Faithful to the duties imposed upon him, he was always ready to cooperate in solving difficult problems even when lack of adequate details, limitation of time, and the like, would have stampeded one having less confidence in his own ability.

He was a man of simple tastes, conservative in all matters, both personal and business, and friendly in a modest manner to all those who claimed his attention. Those who knew him well valued his friendship highly.

On September 10, 1913, he was married at Scotland, Conn., to Mary M. Anthony, who, with their children, Winifred A. and William F., Jr., survives him.

Mr. Jenrick was elected an Associate Member of the American Society of Civil Engineers on December 3, 1913.

HALLETT EDWARD McCLINTOCK, Assoc. M. Am. Soc. C. E.[1]

DIED FEBRUARY 22, 1943

Hallett Edward McClintock was born on October 1, 1882, in Quinnesec, Mich., the son of William Edward McClintock and Katharine (Boucher) McClintock.

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[1] Memoir prepared by Marion McClintock, Continuity Director, KOIL, Omaha, Nebr.

Since he was born and raised in the mining country of northern Michigan, probably it was natural that he would become interested in some form of engineering. Perhaps chance dictated the exact field, because, in June, 1900, at the age of 17, Mr. McClintock began his career as a civil engineer by accepting a job with the Escanaba and Lake Superior Railroad Company in Escanaba, Mich., as a rodman and instrumentman on railroad construction.

After four months in office work, as bookkeeper for E. F. Bradt in Iron Mountain, Mich., Mr. McClintock was employed as surveyor in charge of coal land surveys for the Northern Coal and Coke Company, in Pikeville, Ky. His next position was as instrumentman on mine surveys and heavy stripping for the Oliver Iron Mining Company, Eveleth, Minn.

For a brief time—April, 1903, until September, 1904—Mr. McClintock was in general engineering business for himself in Iron Mountain engaged in surveying and drafting work. He then spent a year as leveler and topographer for the P. A. and W. Railroad in Selma, Ala.

In 1904, when he was well started on his engineering career—although only 22 years old—Mr. McClintock attended the Michigan College of Mining and Technology at Houghton, Mich. While a student, he designed a method for jetting a concrete drop shaft through quicksand. This method was later used to a depth of 200 ft at Negaunee, Mich.

After a year at the Michigan College of Mining and Technology, Mr. McClintock accepted a position with the Rock Island Lines, Chicago, Ill., as transitman and assistant engineer in charge of railroad location. This work involved heavy grade and line revision. While employed by this company, Mr. McClintock was in charge of a fifteen-man field party in Kansas, Oklahoma, and Iowa.

In July, 1907, wishing for more travel and adventure, Mr. McClintock went to Tepic, Mexico, as draftsman and instrumentman on railroad location for the Southern Pacific Railroad Company of Mexico at Guadalajara. He directed all the heavy Sierra Madre mountain location—making reconnaissance surveys and reporting on the 1.5% line which later was built from Tepic 40 miles down to the coast lowlands.

His next venture was as boss in a stamp mill at the Camp Bird Mine, Ouray, Colo., which was followed shortly by his appointment as designer of irrigation structures for the U. S. Indian Service at Fort Duchesne, Utah.

In 1909 Mr. McClintock again went into business for himself in Iron Mountain as engineer on hydraulic studies in connection with available water power on the Menominee Iron Range. His work resulted in the organization of the Peninsula Power Company and in the development of several power sites.

For the next eight years Mr. McClintock was employed by the Union Pacific Railroad Company on work over the entire system. He was first draftsman, then chief draftsman, assistant engineer, and finally location engineer. While with the Union Pacific Railroad Company, Mr. McClintock was engaged on the more difficult office and field problems of construction, location, track work, reports, etc. He represented the company at conferences, was in charge of fifty men in the office, and directed seven field parties on resurveys.

When war was declared in 1917, Mr. McClintock enlisted and was commissioned a Captain in the Engineer Corps of the U. S. Army. He was with the American Expeditionary Force in France and had charge of all material and equipment for the Railway and Dock Section. In this capacity, he built a thousand miles of track, large docks, terminals, etc. Also, he calculated from plans and ordered, allotted, and disposed of all material and equipment. He determined priorities, made reconnaissance surveys, and wrote reports for the Étais Terminal which was being constructed when the Armistice was signed.

On his return to Omaha, Nebr., after war, Mr. McClintock was employed as assistant engineer by the City of Omaha in charge of heavy grading by steam shovel in the business section of the city. Most important in this connection was a special job of grade reduction, removing 300,000 cu yd of streets and property in the western part of the business section. As foreman, he designed and built two crossing fronts and laid out and superintended track laying on a 47° 30′ curve to handle the 2–6–0 standard-gage locomotive. He also designed an interesting two-floor plan for St. Mary Magdalen's Catholic Church, where the cut was 24 ft. Rather than lower the entire building, the altar and the supporting structure were lowered halfway, excavated, and rebuilt, thus adding 60% to seating capacity. After two years with the City of Omaha, Mr. McClintock again went into business for himself as a consulting engineer and contractor on roads, paving, sewers, etc.

In 1925, Mr. McClintock returned to the Union Pacific Railroad Company as locating engineer and assistant engineer on heavy construction and grading and as an expert on valuation. He relocated 20 miles of line south from Torrington, Wyo., saving $100,000 over the prior location, shortening the line. He also secured $3,500,000 extra allowance from the Interstate Commerce Commission on grading valuation.

From March, 1929, until his death, Mr. McClintock was employed as valuation engineer on stockyard work by the Agricultural Marketing Administration of the U. S. Department of Agriculture. This work involved inventory and appraisal of all the structures in stockyards, including large concrete and frame barns and sheds, exchange buildings, water supply and sewerage works, some machinery, and electrical equipment. Although it was not actual construction work, familiarity with advanced construction methods and costs was required.

A talented railroad engineer, probably Mr. McClintock's two most notable characteristics were determination and sincerity. He progressed steadily in his chosen profession until the economic depression, and, even during that time when most construction was at a complete standstill, he was fortunate enough to be employed. When World War II started, he immediately applied for a commission in the Army, but, since he was over the age limit, he was not accepted. He was attempting to secure an engineering position in a vital war industry when he died.

Mr. McClintock was married in July, 1911, to Lillian Bell of Omaha. They had three children, Robert E., who died in 1934 at the age of 18, Claire (Mrs. Bernard McConville), and Marion. On August 6, 1928, he was married to Louise Buford of Memphis, Tenn., who, with his daughters, survives him.

Mr. McClintock was elected an Associate Member of the American Society of Civil Engineers on December 6, 1915, and a Member on April 14, 1919.

WILLIAM GRIFFITH MOLER, Assoc. M. Am. Soc. C .E.[1]

Died September 13, 1942

William Griffith Moler was born in Springfield, Ohio, on October 1, 1858. He was the son of John Douglass and Ella (DuShane) Moler.

Following attendance at public and private schools, Mr. Moler entered Ohio Wesleyan University in Delaware, Ohio, where he received the degree of Bachelor of Science in 1879. Later he received the degree of Master of Arts from the same university. In addition, he studied mathematical and scientific subjects in both England and Germany.

Mr. Moler was a world-wide traveler, a scholar, and an engineer. During his career as a professional engineer, he specialized in the construction of railroads and highways and in the science of hydraulics, attaining prominence in both business and engineering fields.

His first engineering work was in 1879 when he made preliminary railroad surveys from Lancaster, Ohio, to the Ohio Industrial Home at Springfield, Ohio. During the next three years he taught surveying, mathematics, and German at the Portsmouth (Ohio) High School. He was married to Alice Colburn of Portsmouth in 1884 and moved to Greenfield, Ohio, where he was superintendent of schools. Within the next seven years he also served as city engineer of Greenfield and was the chief engineer of the Ohio Southern Railroad Company until 1898.

In 1891 Mr. Moler became associated with his father as a civil and consulting engineer and established the firm of J. D. and William G. Moler in Springfield and Charles Town, W. Va. During this period he served as chief engineer for the Kanawha and Gauley Coal and Coke Company, the Ellijay Land Company's lumber railroad, and the Dayton, Springfield and Urbana Electric Railway Company. He also constructed and landscaped the Ohio Masonic Home at Springfield; built four water works plants; surveyed and designed the landscaping of the Natural Bridge (Virginia); and surveyed and designed several townsites in Ohio, West Virginia, and Georgia.

From 1898 to 1912 his field of activity was transferred to the Republic of Mexico, where he was consulting engineer and special representative for the following: The James Leffel and Company, the Pelton Company, and the Corrugated Bar Company. He also was contractor for the reconstruction of the federal penitentiary in Mexico City; consultant with the late J. A. L. Waddell, Hon. M. Am. Soc. C. E., and the late Ira Grant Hedrick,[2] M. Am. Soc. C. E., of the firm of Waddell and Hedrick, on reinforced concrete foundations in

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[1] Memoir prepared by Eleanor McK. Gibson, Charlottesville, Va.

[2] For memoir, see *Transactions*, Am. Soc. C. E., Vol. 104 (1939), p. 1932.

Mexico; representative of Riter Conley Steel Company; and president of the Santa Fe Mining Company of Mexico City.

In 1905 he was chairman of the Mexican Commission to entertain the Society during its annual meeting in Mexico City. In 1906 he spent a year in Spain studying industrial development and financing of water power and related projects. During 1908 and 1909 he took part in the development of La Sautena—a 1,000,000-acre irrigation project in Northern Mexico. It included a hollow dam on the Conchos River, 180 ft high and 365 ft long, with provisions to use water for power and light, as well as for the irrigation of about 400,000 acres.

While living in Mexico, Mr. Moler became a leader of the American Colony. He was president of the Colony and of the Colony's school board. He also was a director of the Mexico Chamber of Commerce. Mr. Moler numbered the leading Mexicans of that period as his friends, including the late President Porfirio Diaz. It was President Diaz who conferred upon Mr. Moler the Thirty-Third Degree of Masonry, the order's highest honor. His other Masonic degrees included: Clark Lodge, No. 101, Free and Accepted Masons; Springfield Chapter, No. 48, Royal Arch Masons; Springfield Council No. 17, Royal and Select Masters; and Palestine Commandery No. 33, Knights Templar.

In 1912 he returned to the United States and was the New York (N. Y.) representative of the Corrugated Bar Company of St. Louis, Mo., for the eastern territory of the United States in the sale of deformed bars. In 1918 he was director for Mexico of the United States Government War Trade Board in Washington, D. C., in control of exports for the Republic of Mexico.

From 1918 until his retirement in 1939 he was purchasing engineer and fiscal representative in New York City of the Compañía Fundidora de Fierro y Acero de Monterrey, S. A. (Sociedad Anónima), of Mexico City. Mr. Moler spent part of 1925 in Spain studying and developing the power and irrigation project of the Ebro River, and working on its financing, for Stone, Webster and Blodgett. In 1936 Mr. Moler was consulting chief engineer of the Rio Grande Railway and Highway Bridge from Reynosa, Mexico, to Hidalgo, Tex.

Mr. Moler was a member of the Gamma Chapter of Sigma Chi fraternity, member of the American Iron and Steel Institute, and a director of the Mexican Chamber of Commerce of New York. He held many other fraternal and civic honors.

He was an ardent student of the classics and took pride in quoting passages from the Bible in Greek and Hebrew. He had a remarkable memory and an apt quotation from the masters for almost any occasion or situation. He had many friends in many lands, and he will be long remembered for his generosity. One of his last acts was to present a very valuable painting, "Flight into Egypt," to Christ (Episcopal) Church in Springfield, in memory of his mother. He was baptized and confirmed in this church and always remembered this association even though he lived in many other places. After his retirement in 1939, he lived for a short time in Charlottesville, Va., where he had for many years owned and managed a large apple orchard; in Brownsville, Tex., with his brother; and in Bronxville, N. Y., where he had at one time a beautiful home; but he finally returned to the city of his birth in May, 1942.

Mrs. Moler died in 1928 and their two children died in infancy. He is survived by his brother, Harry B. Moler, and three half-sisters, Julia D. (Mrs. J. S. Heaume), Catherine H. (Mrs. Ralph K. Fisher), and Mary I. (Mrs. Frank LeS. Constiner).

Mr. Moler was elected an Associate Member of the American Society of Civil Engineers on January 3, 1900. He became a Life Member in 1929.

HENRY AYLESBURY STRINGFELLOW, Assoc. M. Am. Soc. C. E.[1]

Died November 4, 1942

Henry Aylesbury Stringfellow was born in St. Joseph, Mo., on June 21, 1887. His parents were Ella (Aylesbury) and Horace Stringfellow. After his graduation from the Central High School in St. Joseph, he entered Rensselaer Polytechnic Institute at Troy, N. Y., and was graduated with the degree of Civil Engineer in 1910. For several years following graduation, he was instructor at the Mechanics Institute, Rochester, N. Y., and then, for several years, consulting engineer in Kansas City, Mo., and Detroit, Mich. While in Detroit, he designed the water works for Windsor, Ont., Canada.

Mr. Stringfellow then went to the Far East, where, from 1919 to 1923, he did extensive market research work for American tool manufacturers in Japan, China, and the Philippines.

On his return from China, Mr. Stringfellow was made a partner in the firm of Kiersted and Company of Kansas City. During this time he gained a notable reputation in the engineering field and designed and supervised construction of projects in many parts of the country. Among these were the water works plant at Kansas City, Kans., the water supply for Amarillo, Tex., and the water supply for Barranquilla, Colombia, South America.

Later, while employed by the Southwark Foundry and Machine Company (later Baldwin-Southwark Division, The Baldwin Locomotive Works), Philadelphia, Pa., and by the Elevator Supply Company of Hoboken, N. J., he perfected many inventions. Among them was the improved steel cable hydraulic compacting device patented by him and used on the George Washington Bridge at New York, N. Y. In 1935 he started investigation and research work on back pull drawing from which the present reactive wire drawing method has developed. He held many American, British, and Canadian patents.

In January, 1941, Mr. Stringfellow went to Washington on the engineering staff of the War Production Board (W.P.B.), and, in June of that year, was transferred to the regional office of the W.P.B. in Boston, Mass. There he was placed in charge of the steel and metals division. He was intensely patriotic and took great pride in his work for the war effort. The death of his only son at the University of Michigan at Ann Arbor in 1938 was a sorrow from which he never recovered. This did much to undermine his health, and he died very suddenly on the evening of November 4, 1942, at his home in Boston.

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[1] Memoir prepared by Pauline Smith Stringfellow, St. Thomas, Ont., Canada.

His funeral and burial services, in St. Thomas, Ont., Canada, were conducted by the Rev. Herbert Johnson.

Mr. Stringfellow had many friends. He had a brilliant mind and his character was a model of sincerity, kindliness, and integrity. His colleagues in all branches of his profession will regret deeply the loss of one who was always helpful and reassuring and who overlooked the shortcomings of others in a most charitable way. He was always ready to use his vast store of knowledge and experience to help all who came to him.

In 1915, he was married to Harriet Elizabeth Waklee, of Englewood, N. J. A son, Henry Aylesbury, Jr., and daughter, Margaret Waklee, were born to them. Mrs. Stringfellow died in November, 1919. In June, 1921, he was married to Pauline Smith Bracken, of St. Thomas. He is survived by his widow, his parents, his daughter, Margaret (Mrs. John Broz), and one sister, Grace.

Mr. Stringfellow was a member of the Masonic Lodge at St. Joseph. He belonged to the American Wire Association.

Mr. Stringfellow was elected an Associate Member of the American Society of Civil Engineers on April 19, 1920.

FRANK EDGAR WALTER, Assoc. M. Am. Soc. C. E.[1]

Died August 29, 1942

Frank Edgar Walter was born at Canton, Ohio, on August 3, 1885, the son of Byron and Etha Walter. He received his education in the public schools of Canton and at Ohio Northern University in Ada. Like many other students, he spent his summer vacations working on railroad survey crews.

Continuing in railroad work in the summer of 1903, he served as chainman for the Erie Railroad Company, and, during the summers of 1904 and 1905, he was instrumentman for the same railroad. From October, 1906, to May, 1907, he was employed by the Northern Pacific Railway Company as assistant locating engineer. From May, 1907, to September, 1907, he worked on the relocation of the main line of the Chicago, Milwaukee and Puget Sound Railroad Company through Snoqualmie Pass, Washington. He was assistant locating engineer for the North Coast Railway Company at Spokane, Wash., from September, 1907, to July, 1908. From July to December, 1908, he was employed on bridge work for the Chicago, Milwaukee, St. Paul and Pacific Railroad Company in the Cascade Mountains. From December, 1908, to June, 1911, he served as resident engineer for the North Coast Railway Company, in charge of construction of the line from Benton City to Attalia, Wash.

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[1] Memoir prepared by Julian A. Arntson, M. Am. Soc. C. E.

Leaving railroad work, from September, 1911, to July, 1912, he was employed by the Puget Sound Bridge and Dredge Company on the construction of the bridge crossing the Puyallup River at Tacoma, Wash. However, from July, 1912, to February, 1913, he returned to the employ of the North Coast Railway Company, working on the location of the Naches Pass line from North Yakima to Auburn, Wash. From March, 1913, to December, 1914, he was with Hall and Company, Civil Engineers, in charge of surveys for a proposed power development on the South Fork of the Willapa River, Washington. In January, 1915, he received an appointment as resident engineer in the Highway Department of the State of Washington, remaining until the beginning of World War I.

Entering the U. S. Army as a First Lieutenant of Engineers, he later was promoted to Captain of Company B, 55th Engineers, and served in France.

From October, 1920, to December, 1925, he was employed as resident engineer on the North Hill Viaduct, Akron, Ohio, in charge of maintenance of bridges and highways. Returning to Washington, from May, 1926, to June, 1927, he was locating engineer for the City of Tacoma on a proposed hydroelectric power development. He then served as supervisor of maintenance on roads and bridges for Pierce County, Washington, and as engineer in charge of construction of the Pierce County airport. In September, 1936, Mr. Walter entered the service of the Work Projects Administration as zone engineer, soon was promoted to District Director of Operations, and, one year later, became District Manager, District No. 4 (Pierce, Thurston, Mason, and Grays Harbor counties and a small part of Pacific County). In this capacity, he directed the construction of many school buildings and athletic fields; he had charge of a large part of the extension and expansion of the public utilities at Fort Lewis, McChord Field, Gray Field, the Olympia Airport, and the Hoquiam-Aberdeen Airport, all in Washington. In January, 1942, he was appointed chief engineer of the Inter-County River Improvement District for Pierce and King counties.

Honesty and belief in fair play were his outstanding characteristics. He had a marked ability to handle difficult assignments with exactness and efficiency.

Mr. Walter was a member of the Masonic Order. He attended the Methodist Church. He is survived by his widow and two sons, James and Frank, Jr.

Mr. Walter was elected an Associate Member of the American Society of Civil Engineers on October 7, 1914.

TRANSACTIONS

OF THE

AMERICAN SOCIETY OF CIVIL ENGINEERS

INDEX

VOLUME 108

1943

Subject Index, Page 1685
Author Index, Page 1695

Titles of papers are in quotation marks when given with the author's name.

VOLUME 108

SUBJECT INDEX

ADDRESSES.

"The Society Today and Tomorrow." Address at the Annual Convention, Los Angeles, California, July 28, 1943. Ezra B. Whitman. 1517.

AERATION.

Removal of grease and oil in sewage by aeration. 516.

AGRICULTURE.

"Classification of Irrigable Lands." W. W. Johnston. (With Discussion.) 656.

AIRPORTS.

"Resistance of Soft Fill to Static Wheel Loads." W. Watters Pagon. (With Discussion.) 738.

AIRWAYS.

"Development of Transportation in the United States." J. E. Teal. (With Discussion.) 1.

AMERICAN SOCIETY OF CIVIL ENGINEERS.

"The Society Today and Tomorrow." Address at the Annual Convention, Los Angeles, California, July 28, 1943. Ezra B. Whitman. 1517.

APPARATUS.

Study of air entrainment. 1436.

ARCHES.

See **BRIDGES.**

BEAMS.

"Lateral Stability of Unsymmetrical I-Beams and Trusses in Bending." George Winter. 247. Discussion: Robert K. Schrader, H. N. Hill, and George Winter. 261.

"Numerical Procedure for Computing Deflections, Moments, and Buckling Loads." N. M. Newmark. (With Discussion.) 1161.

BIBLIOGRAPHIES.

Sewage treatment. 522.

Suspended matter in streams. 955.

Water softening. 279.

Weir coefficients. 782.

BRIDGES.

"Design of St. Georges Tied Arch Span." J. M. Garrelts. 543. Discussion: R. W. Abbett, Jacob Karol, Alexander Dodge, Carlos A. Bejarano, A. A. Eremin, A. M. Freudenthal, C. H. Gronquist, Harold E. Langley, and J. M. Garrelts. 555.

CLAY.

"Earth Pressure and Shearing Resistance of Plastic Clay: A Symposium." Karl Terzaghi, Ralph B. Peck, and W. S. Housel. (With Discussion.) 965.

COLUMNS.

"Numerical Procedure for Computing Deflections, Moments, and Buckling Loads." N. M. Newmark. (With Discussion.) 1161.

CONTRACTORS' PLANT.

"Resistance of Soft Fill to Static Wheel Loads." W. Watters Pagon. (With Discussion.) 738.

COSTS OF WORK.

Costs allocated to various systems of Tennessee Valley Authority Projects. 180.

DAMS.

"Allocation of the Tennessee Valley Authority Projects." Theodore B. Parker. (With Discussion.) 174.

"Hydraulic Design of Drop Structures for Gully Control." B. T. Morris and D. C. Johnson. (With Discussion.) 887.

"Model Tests on Structures for Hydroelectric Developments." L. M. Davis. (With Discussion.) 803.

"Permeability of Mud Mountain Dam Core Material." Allen S. Cary, Boyd H. Walter, and Howard T. Harstad. (With Discussion.) 719.

"Stability of Granular Materials." R. E. Glover and F. E. Cornwell. (With Discussion.) 46.

DEPRECIATION.

See **VALUATION.**

DISCHARGE.

Summarized discharge records in the Mississippi River basin. 624.

DRAINAGE.

"Drainage of Leveed Areas in Mountainous Valleys." Gordon R. Williams. 83. Discussion: Merrill Bernard, L. K. Sherman, C. O. Clark, W. W. Horner, Walter T. Wilson, W. B. Langbein, and Gordon R. Williams. 97.

"Hydrodynamics of Model Storm Sewer Inlets Applied to Design." G. S. Tapley. 409. Discussion: Ralph W. Powell, and G. S. Tapley. 444.

EARTH PRESSURES.

Boussinesq formulas for determining pressures. 124.

"Earth Pressure and Shearing Resistance of Plastic Clay: A Symposium." Karl Terzaghi, Ralph B. Peck, and W. S. Housel. 965. Discussion: Ralph H. Burke, L. G. Lenhardt, George E. Shafer, M. E. Chamberlain, A. A. Eremin, A. E. Cummings, D. M. Burmister, Gregory P. Tschebotarioff, D. P. Krynine, L. F. Cooling, Karl Terzaghi, Ralph B. Peck, and W. S. Housel. 1059.

EARTHQUAKES.

"Analytical and Experimental Methods in Engineering Seismology." M. A. Biot. 365. Discussion: George R. Rich, N. J. Hoff, Merit P. White, N. H. Heck, Frank Neumann, Jacob Feld, George W. Housner, Homer M. Hadley, Robert E. Glover, and M. A. Biot. 386.

Limiting slopes for various earthquake intensities. 61.

ECONOMICS.

"Basic Economic Considerations Affecting Single and Multiple Purpose Projects." Otis M. Page. (With Discussion.) 353.

See also **VALUATION.**

EROSION.

"Hydraulic Design of Drop Structures for Gully Control." B. T. Morris and D. C. Johnson. (With Discussion.) 887.

Tests at Safe Harbor. 818.

FILTRATION.

"Reduction of Mineral Content in Water with Organic Zeolites." R. F. Goudey. (With Discussion.) 269.

See also **SEWAGE DISPOSAL.**

FINANCE.

Relationship to public works. 1255.

FLOODS.

"Allocation of the Tennessee Valley Authority Projects." Theodore B. Parker. (With Discussion.) 174.

"Basic Economic Considerations Affecting Single and Multiple Purpose Projects." Otis M. Page. 353. Discussion: G. B. Pillsbury. 363.

"Drainage of Leveed Areas in Mountainous Valleys." Gordon R. Williams. (With Discussion.) 83.

"Statistical Analysis in Hydrology." L. R. Beard. (With Discussion.) 1110.

"Unusual Events and Their Relation to Federal Water Policies." W. G. Hoyt. (With Discussion.) 290.

FLUMES.

"Entrainment of Air in Flowing Water: A Symposium." L. Standish Hall, and A. A. Kalinske and James M. Robertson. (With Discussion.) 1393.

FOUNDATIONS.

"Analytical and Experimental Methods in Engineering Seismology." M. A. Biot. (With Discussion.) 365.

"Timber Friction Pile Foundations." Frank M. Masters. (With Discussion.) 115.

GRAPHICAL CHARTS.

"Distribution Graphs of Suspended-Matter Concentration." Joe W. Johnson. (With Discussion.) 941.

"Statistical Analysis in Hydrology." L. R. Beard. (With Discussion.) 1110.

HIGHWAYS AND ROADS.

"Design of Sign Letter Sizes." Adolphus Mitchell and T. W. Forbes. 233. Discussion: Guy Kelcey, Charles M. Noble, and Adolphus Mitchell and T. W. Forbes. 243.

"Development of Transportation in the United States." J. E. Teal. (With Discussion.) 1.

HYDRAULICS.

"Energy Loss at the Base of a Free Overfall." Walter L. Moore. 1343. Discussion: Merit P. White, Boris A. Bakhmeteff and N. V. Feodoroff, Carl E. Kindsvater, J. E. Christiansen, L. Standish Hall, Hunter Rouse, and Walter L. Moore. 1361.

"Hydraulic Design of Drop Structures for Gully Control." B. T. Morris and D. C. Johnson. (With Discussion.) 887.

"Hydrodynamics of Model Storm Sewer Inlets Applied to Design." G. S. Tapley. (With Discussion.) 409.

"Model Tests on Structures for Hydroelectric Developments." L. M. Davis. 803. Discussion: Emil P. Schuleen, C. M. Allen, Adolf A. Meyer, and L. M. Davis. 824.

"Statistical Analysis in Hydrology." L. R. Beard. (With Discussion.) 1110.

"Viscosity and Surface Tension Effects on V-Notch Weir Coefficients." Arno T. Lenz. (With Discussion.) 759.

See also **WATER, FLOW OF.**

HYDROELECTRIC POWER PLANTS.

See **POWER PLANTS.**

IMHOFF TANKS.

See **SEPTIC TANKS.**

INSTRUMENTS.

Mechanical analyzer for use in connection with earthquake study. 369.

"Viscosity and Surface Tension Effects on V-Notch Weir Coefficients." Arno T. Lenz. (With Discussion.) 759.

IRRIGATION.

"Classification of Irrigable Lands." W. W. Johnston. (With Discussion.) 656.

"Unusual Events and Their Relation to Federal Water Policies." W. G. Hoyt. (With Discussion.) 290.

JOINTS.

"Moment Balance: A Self-Checking Analysis of Rigidly Jointed Frames." R. J. Cornish. (With Discussion.) 677.

LAND.

"Classification of Irrigable Lands." W. W. Johnston. 656. Discussion: R. Earl Storie, Walter W. Weir, W. L. Powers, and W. W. Johnston. 672.

LEVEES.

"Drainage of Leveed Areas in Mountainous Valleys." Gordon R. Williams. (With Discussion.) 83.

MEMOIRS OF DECEASED MEMBERS.

See name of member in Author Index.

MODELS.

"Energy Loss at the Base of a Free Overfall." Walter L. Moore. (With Discussion.) 1343.

"Model Tests on Structures for Hydroelectric Developments." L. M. Davis. (With Discussion.) 803.

MOMENTS.

"Bending Moments in the Walls of Rectangular Tanks." Dana Young. (With Discussion.) 217.

"Moment Balance: A Self-Checking Analysis of Rigidly Jointed Frames." R. J. Cornish. 677. Discussion: D. D. Matthews, William A. Larsen, A. A. Eremin, L. E. Grinter, B. J. Aleck, Frederick S. Merritt, Ralph W. Stewart, G. W. Stokes, Bruce Jameyson, R. E. Bowles, William Morris, John Mason, and R. J. Cornish. 685.

"Numerical Procedure for Computing Deflections, Moments, and Buckling Loads." N. M. Newmark. (With Discussion.) 1161.

NAVIGATION.

"Allocation of the Tennessee Valley Authority Projects." Theodore B. Parker. (With Discussion.) 174.

See also **RIVERS.**

PILES.

"Timber Friction Pile Foundations." Frank M. Masters. 115. Discussion: E. H. Connor, Glenn B. Woodruff, Jacob Feld, G. G. Greulich, G. S. Paxson, Raymond D. Mindlin, John W. Solomon, A. E. Cummings, and Frank M. Masters. 141.

PIPE LINES.

"Development of Transportation in the United States." J. E. Teal. (With Discussion.) 1.

POWER.

"Allocation of the Tennessee Valley Authority Projects." Theodore B. Parker. (With Discussion.) 174.

See also **WATER POWER.**

POWER PLANTS.

"Model Tests on Structures for Hydroelectric Developments." L. M. Davis. (With Discussion.) 803.

PUBLIC WORKS.

"Allocation of the Tennessee Valley Authority Projects." Theodore B. Parker. 174. Discussion: Malcolm Elliott, E. L. Chandler, Eugene L. Grant, Sherman M. Woodward, Albert R. Arledge, C. W. Watchorn and F. A. Allner, and Theodore B. Parker. 188.

"Fundamental Aspects of the Depreciation Problem: A Symposium." Maurice R. Scharff, J. A. Walls, Eugene L. Grant, and Roy W. Crum and Robley Winfrey. (With Discussion.) 1235.

RAILROADS.

"Development of Transportation in the United States." J. E. Teal. (With Discussion.) 1.

See also **SUBWAYS, TUNNELS.**

RAINFALL.

"Drainage of Leveed Areas in Mountainous Valleys." Gordon R. Williams. (With Discussion.) 83.

"Early Contributions to Mississippi River Hydrology." C. S. Jarvis. (With Discussion.) 605.

"Statistical Analysis in Hydrology." L. R. Beard. (With Discussion.) 1110.

REGIONAL PLANNING.

"Unusual Events and Their Relation to Federal Water Policies." W. G. Hoyt. (With Discussion.) 290.

RIVERS.

"Allocation of the Tennessee Valley Authority Projects." Theodore B. Parker. (With Discussion.) 174.

"Distribution Graphs of Suspended-Matter Concentration." Joe W. Johnson. (With Discussion.) 941.

"Early Contributions to Mississippi River Hydrology." C. S. Jarvis. 605. Discussion: Robert Follansbee, R. W. Davenport, Charles Senour, John C. Hoyt, George O. Guesmer, H. T. Cory, and C. S. Jarvis. 629.

RUNOFF.

"Drainage of Leveed Areas in Mountainous Valleys." Gordon R. Williams. (With Discussion.) 83.

"Statistical Analysis in Hydrology." L. R. Beard. 1110. Discussion: L. Standish Hall, H. Alden Foster, Paul V. Hodges, Joe W. Johnson, R. H. Blythe, Jr., R. W. Davenport, Edward J. Bednarski, Ralph W. Powell, E. J. Gumbel, and L. R. Beard. 1122.

See also **RAINFALL.**

SANITARY ENGINEERING.

"The Grease Problem in Sewage Treatment." Almon L. Fales and Samuel A. Greeley. (With Discussion.) 507.

See also **DRAINAGE, FILTRATION, SEWERAGE, WATER.**

SEDIMENTATION.

"Distribution Graphs of Suspended-Matter Concentration." Joe W. Johnson. 941. Discussion: H. A. Einstein, G. C. Dobson, Thomas R. Camp, Frank S. Bailey, and Joe W. Johnson. 957.

SEEPAGE.

"Permeability of Mud Mountain Dam Core Material." Allen S. Cary, Boyd H. Walter, and Howard T. Harstad. (With Discussion.) 719.

SEPTIC TANKS.

Use of Imhoff tanks for grease removal from sewage. 514.

SETTLING BASINS.

Use of sedimentation tanks for grease removal from sewage. 514.

SEWAGE DISPOSAL.

"The Grease Problem in Sewage Treatment." Almon L. Fales and Samuel A. Greeley. 507. Discussion: Arthur D. Weston, Harvey F. Ludwig, W. S. Mahlie, Denis Dickinson, Charles R. Velzy, F. M. Dawson, and Almon L. Fales and Samuel A. Greeley. 525.

SEWAGE TREATMENT.

See **SEWAGE DISPOSAL.**

SEWERAGE.

"Hydrodynamics of Model Storm Sewer Inlets Applied to Design." G. S. Tapley. (With Discussion.) 409.

SLABS.

"Stress Concentrations in Plates Loaded Over Small Areas." H. M. Westergaard. (With Discussion.) 831.

SOILS.

"Classification of Irrigable Lands." W. W. Johnston. (With Discussion.) 656.

"Earth Pressure and Shearing Resistance of Plastic Clay: A Symposium." Karl Terzaghi, Ralph B. Peck, and W. S. Housel. (With Discussion.) 965.

"Permeability of Mud Mountain Dam Core Material." Allen S. Cary, Boyd H. Walter, and Howard T. Harstad. 719. Discussion: F. H. Kellogg, J. E. Christiansen and Milton Fireman, William P. Creager, and Allen S. Cary, Boyd H. Walter, and Howard T. Harstad. 729.

"Resistance of Soft Fill to Static Wheel Loads." W. Watters Pagon. 738. Discussion: Jacob Feld, and W. Watters Pagon. 756.

"Stability of Granular Materials." R. E. Glover and F. E. Cornwell. 46. Discussion: R. G. Hennes, Stanley U. Benscoter, A. Hrennikoff, D. P. Krynine, and R. E. Glover and F. E. Cornwell. 64.

SPILLWAYS.

"Entrainment of Air in Flowing Water: A Symposium." L. Standish Hall, and A. A. Kalinske and James M. Robertson. (With Discussion.) 1393.

STREAM FLOW.

See **WATER, FLOW OF, IN OPEN CHANNELS.**

STREAMS.

See **RIVERS.**

STRESS AND STRAIN.

"Effect of Variation of Elastic Characteristics on Static Unknowns." A. Hrennikoff. (With Discussion.) 317.

"Lateral Stability of Unsymmetrical I-Beams and Trusses in Bending." George Winter. (With Discussion.) 247.

"Mechanics of Creep for Structural Analysis." Joseph Marin. (With Discussion.) 453.

"Stress Concentrations in Plates Loaded Over Small Areas." H. M. Westergaard. (With Discussion.) 831.

STRUCTURES, THEORY OF.

"Effect of Variation of Elastic Characteristics on Static Unknowns." A. Hrennikoff. 317. Discussion: A. H. Finlay, Stefan J. Fraenkel, Joseph A. Wise, and A. Hrennikoff. 340.

"Mechanics of Creep for Structural Analysis." Joseph Marin. 453. Discussion: A. Hrennikoff, C. A. P. Turner, Ralph W. Stewart, and Joseph Marin. 470.

"Moment Balance: A Self-Checking Analysis of Rigidly Jointed Frames." R. J. Cornish. (With Discussion.) 677.

"Numerical Procedure for Computing Deflections, Moments, and Buckling Loads." N. M. Newmark. 1161. Discussion: Bruce Johnston, M. S. Ketchum, Jr., John B. Wilbur, Ralph W. Stewart, Stefan J. Fraenkel, Alfred S. Niles, Camillo Weiss, A. A. Eremin, Myron L. Gossard, Robert A. Williamson, I. Oesterblom, C. W. Dunham, and N. M. Newmark. 1189.

"Stress Concentrations in Plates Loaded Over Small Areas." H. M. Westergaard. 831. Discussion: George R. Rich, Dana Young, A. Nádai, L. A. MacColl, L. W. Teller and Earl C. Sutherland, D. L. Holl, C. A. P. Turner, and H. M. Westergaard. 857.

SUBWAYS.

"Earth Pressure and Shearing Resistance of Plastic Clay: A Symposium." Karl Terzaghi, Ralph B. Peck, and W. S. Houssel. (With Discussion.) 965.

SURVEYING.

"Relative Angular, Linear, and Traverse Accuracies in City Surveys." S. A. Bauer. 577. Discussion: Cleveland B. Coe, W. H. Rayner, Philip Kissam, George D. Whitmore, C. C. Miner, and W. O. Byrd, G. Brooks Earnest, and S. A. Bauer. 591.

TANKS.

"Bending Moments in the Walls of Rectangular Tanks." Dana Young. 217. Discussion: I. K. Silverman, and Dana Young. 231.

TIMBER.

"Timber Friction Pile Foundations." Frank M. Masters. (With Discussion.) 115.

TRAFFIC.

Increase in freight and passenger traffic on railroads. 41.

Passenger-car traffic compared with truck traffic. 10.

See also **TRANSPORTATION.**

TRANSPORTATION.

"Development of Transportation in the United States." J. E. Teal. 1. Discussion: Fred Lavis, William J. Wilgus, Joseph B. Eastman, J. L. Campbell, W. W. Crosby, W. B. Irwin, F. R. Schanck, and J. E. Teal. 24.

TRIANGULATION.

See **SURVEYING.**

TRUSSES.

"Lateral Stability of Unsymmetrical I-Beams and Trusses in Bending." George Winter. (With Discussion.) 247.

TUNNELS.

"Earth Pressure and Shearing Resistance of Plastic Clay: A Symposium." Karl Terzaghi, Ralph B. Peck, and W. S. Housel. (With Discussion.) 965.

VALUATION.

"Fundamental Aspects of the Depreciation Problem: A Symposium." Maurice R. Scharff, J. A. Walls, Eugene L. Grant, and Roy W. Crum and Robley Winfrey. 1235. Discussion: Edwin F. Wendt and L. T. Fleming, Anson Marston, William G. Atwood, George E. Goldthwaite, Nathan B. Jacobs, H. J. Flagg, Nelson Lee Smith, C. Beverley Benson, Paul T. Norton, Jr., H. L. Ripley, Carroll A. Farwell, Thomas R. Agg, Conde B. McCullough, W. V. Burnell, Roger L. Morrison, Wallace B. Carr, W. L. Waters, A. G. Mott, David A. Kosh, E. G. Walker, K. Lee Hyder, James T. Ryan, John C. Page, John S. Worley, John F. Miller, Fred Asa Barnes, Robert F. Legget and Erwin E. Hart, Kernan Robson, J. A. Walls, and Eugene L. Grant. 1262.

Various theories of joint-cost allocations. 177.

VIBRATION.

"Analytical and Experimental Methods in Engineering Seismology." M. A. Biot. (With Discussion.) 365.

WATER.

"Reduction of Mineral Content in Water with Organic Zeolites." R. F. Goudey. 269. Discussion: James M. Montgomery, George L. Davenport, Jr., Robert Spurr Weston, and Harvey F. Ludwig and Russell G. Ludwig. 281.

"Unusual Events and Their Relation to Federal Water Policies." W. G. Hoyt. 290. Discussion: Edward H. Sargent, Dana M. Wood, James S. Sweet, J. L. Campbell, David J. Guy, Albert S. Fry, and W. G. Hoyt. 304.

WATER, FLOW OF, IN OPEN CHANNELS.

"Entrainment of Air in Flowing Water: A Symposium." L. Standish Hall, and A. A. Kalinske and James M. Robertson. 1393. Discussion: Warren DeLapp, Karl R. Kennison, Robert T. Knapp, Carl E. Kindsvater, J. H. Douma, Joe W. Johnson, J. C. Stevens, D. C. McConaughy, T. J. Corwin, Jr., L. Standish Hall, and A. A. Kalinske and J. M. Robertson. 1448.

"Hydraulic Design of Drop Structures for Gully Control." B. T. Morris and D. C. Johnson. 887. Discussion: John Hedberg, L. Standish Hall, J. E. Christiansen, Walter T. Wilson, N. A. Christensen and Dwight Gunder, Boris A. Bakhmeteff and Nicholas V. Feodoroff, G. H. Hickox, and B. T. Morris and D. C. Johnson. 919.

"Profile Curves for Open-Channel Flow." Dwight F. Gunder. 481. Discussion: Carl Rohwer, C. J. Posey, J. C. Stevens, Boris A. Bakhmeteff and Nicholas V. Feodoroff, Hunter Rouse, and Dwight F. Gunder. 489.

WATER, FLOW OF, IN PIPES.

"Entrainment of Air in Flowing Water: A Symposium." L. Standish Hall, and A. A. Kalinske and James M. Robertson. (With Discussion.) 1393.

WATER, FLOW OF, OVER DAMS AND WEIRS.

"Energy Loss at the Base of a Free Overfall." Walter L. Moore. (With Discussion.) 1343.

"Hydrodynamics of Model Storm Sewer Inlets Applied to Design." G. S. Tapley. (With Discussion.) 409.

"Viscosity and Surface Tension Effects on V-Notch Weir Coefficients." Arno T. Lenz. 759. Discussion: A. A. Kalinske, Glen N. Cox and F. J. Germano, Emery H. Willes, Ralph W. Powell, G. H. Hickox, and Arno T. Lenz. 783.

WATER POWER.

"Unusual Events and Their Relation to Federal Water Policies." W. G. Hoyt. (With Discussion.) 290.

WATER SOFTENING.

See **WATER.**

WATERWAYS.

"Development of Transportation in the United States." J. E. Teal. (With Discussion.) 1.

See also **RIVERS.**

WEIRS.

See **WATER, FLOW OF, OVER DAMS AND WEIRS.**

AUTHOR INDEX

ABBETT, R. W.
Tied arch span. 555.

AGG, THOMAS R.
Depreciation. 1287.

ALECK, B. J.
Moment balance. 693.

ALLEN, C. M.
Hydromodels. 828.

ALLNER, FREDERICK ABELS.
Memoir of. 1551.
TVA projects. 210.

ALLTON, ROBERT ADAMS.
Memoir of. 1553.

ARLEDGE, ALBERT R.
TVA projects. 205.

ATWOOD, THOMAS CLARK.
Memoir of. 1556.

ATWOOD, WILLIAM G.
Depreciation. 1268.

BAILEY, FRANK S.
Suspended-matter concentration. 960.

BAKHMETEFF, BORIS A.
Energy loss. 1364.
Gully control. 930.
Open-channel flow. 492.

BARNES, FRED ASA.
Depreciation. 1323.

BARROWS, GEORGE ELLSWORTH.
Memoir of. 1558.

BATTIE, HERBERT SCANDLIN.
Memoir of. 1657.

BAUER, S. A.
"Relative Angular, Linear, and Traverse Accuracies in City Surveys." 577.

BEARD, L. R.
"Statistical Analysis in Hydrology." 1110.

BEDNARSKI, EDWARD J.
Statistical analysis. 1141.

BEJARANO, CARLOS A.
Tied arch span. 566.

BENSCOTER, STANLEY U.
Granular materials. 65.

BENSON, C. BEVERLEY.
Depreciation. 1278.

BERNARD, MERRILL.
Drainage of leveed areas. 97.

BIOT, M. A.
"Analytical and Experimental Methods in Engineering Seismology." 365.

BLAKELEY, GEORGE HENRY.
Memoir of. 1560.

BLYTHE, R. H., JR.
Statistical analysis. 1137.

BOWLES, R. E.
Moment balance. 703.

BROOMALL, CAROLUS MORTON.
Memoir of. 1561.

BROWN, ROBERT FRANCIS.
Memoir of. 1564.

BURKE, RALPH H.
Plastic clay. 1059.

BURMISTER, D. M.
Plastic clay. 1072.

BURNELL, W. V.
Depreciation. 1293.

BUTLER, ALFRED DICKEY.
Memoir of. 1565.

BYRD, W. O.
Surveying accuracies. 598.

CAMP, THOMAS R.
Suspended-matter concentration. 959.

CAMPBELL, J. L.
Federal water policies. 307.
Transportation. 30.

CARR, WALLACE B.
Depreciation. 1297.

CARY, ALLEN S.
"Permeability of Mud Mountain Dam Core Material." 719.

CHADWICK, CHESTER ROBERT.
Memoir of. 1659.

CHAMBERLAIN, M. E.
Plastic clay. 1064.

CHANDLER, E. L.
TVA projects. 189.

CHRISTENSEN, N. A.
Gully control. 927.

CHRISTIANSEN, J. E.
Energy loss. 1375.
Gully control. 921.
Mud Mountain Dam. 733.

CLARK, C. O.
Drainage of leveed areas. 100.

COE, CLEVELAND B.
Surveying accuracies. 591.

COLLINGS, WILLIAM TATEM, JR.
Memoir of. 1568.

CONNER, E. H.
Friction pile foundations. 141.

COOLING, L. F.
Plastic clay. 1086.

CORNISH, R. J.
"Moment Balance: A Self-Checking Analysis of Rigidly Jointed Frames." 677.

CORNWELL, F. E.
"Stability of Granular Materials." 46.

CORTI, JOSEPH JAMES.
Memoir of. 1660.

CORWIN, T. J., JR.
Air entrainment. 1493.

CORY, H. T.
Mississippi River. 638.

COX, GLEN N.
V-notch coefficients. 784.

CREAGER, WILLIAM P.
Mud Mountain Dam. 735.

CROSBY, W. W.
Transportation. 32.

CRUM, ROY W.
"Fundamental Aspects of the Depreciation Problem: A Symposium: Relationship to Public Works and Government Finance." 1255.

CUMMINGS, A. E.
Friction pile foundations. 162.
Plastic clay. 1067.

DARROW, MARIUS SCHOONMAKER.
Memoir of. 1664.

DAVENPORT, GEORGE L., JR.
Organic zeolites. 283.

DAVENPORT, R. W.
Mississippi River. 629.
Statistical analysis. 1139.

DAVIS, L. M.
"Model Tests on Structures for Hydroelectric Developments." 803.

DAWSON, EDWIN FORD.
Memoir of. 1570.

DAWSON, F. M.
Grease in sewage. 539.

DeLAPP, WARREN.
Air entrainment. 1448.

DENISE, CHARLES MEIRS.
Memoir of. 1571.

DICKINSON, DENIS.
Grease in sewage. 537.

DOBSON, G. C.
Suspended-matter concentration. 958.

DODGE, ALEXANDER.
Tied arch span. 558.

DOMENECH, MANUEL VICTOR.
Memoir of. 1573.

DOUGHERTY, EDWARD JAMES.
Memoir of. 1575.

DOUGLASS, ARTHUR SYLVESTER.
Memoir of. 1576.

DOUMA, J. H.
Air entrainment. 1462.

DUNHAM, C. W.
Deflections and moments. 1224.

EARNEST, G. BROOKS.
Surveying accuracies. 600.
EASTMAN, JOSEPH B.
Transportation. 29.
EINSTEIN, H. A.
Suspended-matter concentration. 957.
ELLIOTT, MALCOLM.
TVA projects. 188.
EREMIN, A. A.
Deflections and moments. 1211.
Moment balance. 690.
Plastic clay. 1066.
Tied arch span. 568.
FALES, ALMON L.
"The Grease Problem in Sewage Treatment." 507.
FARWELL, CARROLL A.
Depreciation. 1286.
FELD, JACOB.
Engineering seismology. 397.
Friction pile foundations. 143.
Wheel loads in soft fill. 756.
FEODOROFF, NICHOLAS V.
Energy loss. 1364.
Gully control. 930.
Open-channel flow. 492.
FINLAY, A. H.
Elastic characteristics. 340.
FIREMAN, MILTON.
Mud Mountain Dam. 733.
FLAGG, H. J.
Depreciation. 1274.
FLEMING, L. T.
Depreciation. 1262.
FOLLANSBEE, ROBERT.
Mississippi River. 629.
FORBES, T. W.
"Design of Sign Letter Sizes." 233.
FOSTER, H. ALDEN.
Statistical analysis. 1127.

FRAENKEL, STEFAN J.
Deflections and moments. 1201.
Elastic characteristics. 343.
FREUDENTHAL, A. M.
Tied arch span. 569.
FRY, ALBERT S.
Federal water policies. 312.
FUSTON, WALTER JACOB.
Memoir of. 1579.
GARRELTS, J. M.
"Design of St. Georges Tied Arch Span." 543.
GERMANO, F. J.
V-notch coefficients. 784.
GILLHAM, PHILIP DAKIN.
Memoir of. 1580.
GLOVER, ROBERT E.
Engineering seismology. 405.
"Stability of Granular Materials." 46.
GOLDTHWAITE, GEORGE E.
Depreciation. 1269.
GORTON, WILLARD LIVERMORE.
Memoir of. 1666.
GOSSARD, MYRON L.
Deflections and moments. 1212.
GOUDEY, R. F.
"Reduction of Mineral Content in Water with Organic Zeolites." 269.
GRANT, EUGENE L.
"Fundamental Aspects of the Depreciation Problem: A Symposium: Relationship to Competitive Industry." 1242.
TVA projects. 202.
GREELEY, SAMUEL A.
"The Grease Problem in Sewage Treatment." 507.
GREENE, CARLETON.
Memoir of. 1582.
GREENWOOD, ALBERT HENRY.
Memoir of. 1670.
GREULICH, G. G.
Friction pile foundations. 144.
GRIFFIN, ARTHUR JAMES.
Memoir of. 1584.

GRIMM, CLAUDE IRVING.
Memoir of. 1585.

GRINTER, L. E.
Moment balance. 691.

GRONQUIST, C. H.
Tied arch span. 570.

GUESMER, GEORGE O.
Mississippi River. 638.

GUMBEL, E. J.
Statistical analysis. 1144.

GUNDER, DWIGHT.
Gully control. 927.
"Profile Curves for Open-Channel Flow." 481.

GUTTERIDGE, ALAN GORDON.
Memoir of. 1587.

GUY, DAVID J.
Federal water policies. 309.

HADLEY, HOMER M.
Engineering seismology. 402.

HALL, L. STANDISH.
Energy loss. 1378.
"Entrainment of Air in Flowing Water: A Symposium: Open Channel Flow at High Velocities." 1394.
Gully control. 919.
Statistical analysis. 1122.

HARKNESS, ANDREW HARKNESS.
Memoir of. 1588.

HARSTAD, HOWARD T.
"Permeability of Mud Mountain Dam Core Material." 719.

HART, ERWIN E.
Depreciation. 1326.

HECK, N. H.
Engineering seismology. 394.

HEDBERG, JOHN.
Gully control. 919.

HENNES, R. G.
Granular materials. 64.

HICKOX, G. H.
Gully control. 932.
V-notch coefficients. 796.

HICKOX, JAY RUSSELL.
Memoir of. 1590.
HILL, H. N.
Lateral stability. 264.
HILTON, JOSEPH CHURCHILL.
Memoir of. 1671.
HODGES, PAUL V.
Statistical analysis. 1128.
HOFF, N. J.
Engineering seismology. 388.
HOLL, D. L.
Stress concentrations in plates. 868.
HOOD, JOHN MIFFLIN, JR.
Memoir of. 1592.
HORNER, W. W.
Drainage of leveed areas. 103.
HOUSEL, W. S.
"Earth Pressure and Shearing Resistance of Plastic Clay: A Symposium: Earth Pressure on Tunnels." 1037.
HOUSNER, GEORGE W.
Engineering seismology. 398.
HOVEY, OTIS ELLIS.
Memoir of. 1537.
HOYT, JOHN C.
Mississippi River. 637.
HOYT, W. G.
"Unusual Events and Their Relation to Federal Water Policies." 290.
HRENNIKOFF, A.
"Effect of Variation of Elastic Characteristics on Static Unknowns." 317.
Granular materials. 69.
Mechanics of creep. 470.
HUTCHINS, JERE CHAMBERLAIN.
Memoir of. 1594.
HYDER, K. LEE.
Depreciation. 1311.
IRWIN, W. B.
Transportation. 34.
JACOBS, NATHAN B.
Depreciation. 1272.

JACOBY, CLARK ELLSWORTH.
Memoir of. 1598.

JAMEYSON, BRUCE.
Moment balance. 700.

JARVIS, C. S.
"Early Contributions to Mississippi River Hydrology." 605.

JENRICK, WILLIAM FREDERICK.
Memoir of. 1674.

JOHNSON, D. C.
"Hydraulic Design of Drop Structures for Gully Control." 887.

JOHNSON, DAVID CLAYTON.
Memoir of. 1601.

JOHNSON, JOE W.
Air entrainment. 1473.
"Distribution Graphs of Suspended-Matter Concentration." 941.
Statistical analysis. 1130.

JOHNSTON, BRUCE.
Deflections and moments. 1189.

JOHNSTON, W. W.
"Classification of Irrigable Lands." 656.

KALINSKE, A. A.
"Entrainment of Air in Flowing Water: A Symposium: Closed Conduit Flow." 1435.
V-notch coefficients. 783.

KAROL, JACOB.
Tied arch span. 555.

KELCEY, GUY.
Sign letter sizes. 243.

KELLOGG, F. H.
Mud Mountain Dam. 729.

KENNISON, KARL R.
Air entrainment. 1451.

KESTER, FREDERICK HENRY.
Memoir of. 1602.

KETCHUM, M. S., JR.
Deflections and moments. 1193.

KINDSVATER, CARL E.
Air entrainment. 1459.
Energy loss. 1373.

KISSAM, PHILIP.
  Surveying accuracies. 594.
KNAPP, ROBERT T.
  Air entrainment. 1455.
KOSH, DAVID A.
  Depreciation. 1303.
KRAUS, ARTHUR.
  Memoir of. 1604.
KRYNINE, D. P.
  Granular materials. 72.
  Plastic clay. 1078.
LANGBEIN, W. B.
  Drainage of leveed areas. 109.
LANGLEY, HAROLD E.
  Tied arch span. 572.
LARSEN, WILLIAM A.
  Moment balance. 690.
LAVIS, FRED.
  Transportation. 24.
LEGGET, ROBERT F.
  Depreciation. 1326.
LENHARDT, L. G.
  Plastic clay. 1061.
LENZ, ARNO T.
  "Viscosity and Surface Tension Effects on V-Notch Weir Coefficients." 759.
LINTON, ROBERT.
  Memoir of. 1605.
LIPPINCOTT, JOSEPH BARLOW.
  Memoir of. 1543.
LUDWIG, HARVEY F.
  Grease in sewage. 535.
  Organic zeolites. 286.
LUDWIG, RUSSELL G.
  Organic zeolites. 286.
LYMAN, ALBERT KUALII BRICKWOOD.
  Memoir of. 1608.
MacCOLL, L. A.
  Stress concentrations in plates. 863.
McCLINTOCK, HALLETT EDWARD.
  Memoir of. 1675.

McCONAUGHY, D. C.
Air entrainment. 1484.

McCULLOUGH, CONDE B.
Depreciation. 1289.

MAHLIE, W. S.
Grease in sewage. 537.

MARIN, JOSEPH.
"Mechanics of Creep for Structural Analysis." 453.

MARSTON, ANSON.
Depreciation. 1264.

MASON, JOHN.
Moment balance. 706.

MASTERS, FRANK M.
"Timber Friction Pile Foundations." 115.

MATTHEWS, D. D.
Moment balance. 685.

MEIER, WALTER HENRY.
Memoir of. 1609.

MERRITT, FREDERICK S.
Moment balance. 694.

MEYER, ADOLF A.
Hydromodels. 828.

MILLER, JOHN F.
Depreciation. 1322.

MINDLIN, RAYMOND D.
Friction pile foundations. 147.

MINER, C. C.
Surveying accuracies. 598.

MITCHELL, ADOLPHUS.
"Design of Sign Letter Sizes." 233.

MOLER, WILLIAM GRIFFITH.
Memoir of. 1678.

MONTGOMERY, JAMES M.
Organic zeolites. 281.

MOORE, WALTER L.
"Energy Loss at the Base of a Free Overfall." 1343.

MORRIS, B. T.
"Hydraulic Design of Drop Structures for Gully Control." 887.

MORRIS, WILLIAM.
Moment balance. 704.

MORRISON, ROGER L.
Depreciation. 1296.

MOTT, A. G.
Depreciation. 1301.

MYERS, CLARENCE EUGENE.
Memoir of. 1612.

NÁDAI, A.
Stress concentrations in plates. 860.

NEUMANN, FRANK.
Engineering seismology. 395.

NEWLIN, JOHN A.
Memoir of. 1614.

NEWMARK, N. M.
"Numerical Procedure for Computing Deflections, Moments, and Buckling Loads." 1161.

NILES, ALFRED S.
Deflections and moments. 1206.

NOBLE, CHARLES M.
Sign letter sizes. 244.

NORTON, PAUL T., JR.
Depreciation. 1282.

OESTERBLOM, I.
Deflections and moments. 1224.

ORCUTT, WILLIAM WARREN.
Memoir of. 1617.

PAGE, JOHN C.
Depreciation. 1316.

PAGE, OTIS M.
"Basic Economic Considerations Affecting Single and Multiple Purpose Projects." 353.

PAGON, W. WATTERS.
"Resistance of Soft Fill to Static Wheel Loads." 738.

PARKER, THEODORE B.
"Allocation of the Tennessee Valley Authority Projects." 174.

PAXSON, G. S.
Friction pile foundations. 145.

PECK, RALPH B.
"Earth Pressure and Shearing Resistance of Plastic Clay: A Symposium: Earth-Pressure Measurements in Open Cuts, Chicago (Ill.) Subway." 1008.

PILLSBURY, G. B.
Flood control. 363.

POSEY, C. J.
Open-channel flow. 489.

POWELL, RALPH W.
Statistical analysis. 1144.
Storm sewer inlets. 444.
V-notch coefficients. 792.

POWERS, W. L.
Land classification. 675.

PRATT, HENRY BLANCHARD.
Memoir of. 1622.

RAYNER, W. H.
Surveying accuracies. 592.

REPPERT, CHARLES MILLER.
Memoir of. 1623.

RICH, BARZILLAI ALLEN.
Memoir of. 1625.

RICH, GEORGE R.
Engineering seismology. 386.
Stress concentrations in plates. 857.

RIDDICK, WALLACE CARL.
Memoir of. 1626.

RIGHTOR, FRED ELMER.
Memoir of. 1629.

RIPLEY, H. L.
Depreciation. 1284.

ROBERTSON, JAMES M.
"Entrainment of Air in Flowing Water: A Symposium: Closed Conduit Flow." 1435.

ROBSON, KERNAN.
Depreciation. 1329.

ROHWER, CARL.
Open-channel flow. 489.

ROUSE, HUNTER.
Energy loss. 1383.
Open-channel flow. 502.

RYAN, JAMES T.
Depreciation. 1314.

SARGENT, EDWARD H.
Federal water policies. 304.

SCHANCK, F. R.
Transportation. 36.

SCHARFF, MAURICE R.
"Fundamental Aspects of the Depreciation Problem: A Symposium: Foreword." 1236.

SCHEIN, NATHAN.
Memoir of. 1631.

SCHRADER, ROBERT K.
Lateral stability. 261.

SCHULEEN, EMIL P.
Hydromodels. 824.

SENOUR, CHARLES.
Mississippi River. 633.

SHAFER, GEORGE E.
Plastic clay. 1062.

SHAW, ARTHUR MONROE.
Memoir of. 1632.

SHERMAN, L. K.
Drainage of leveed areas. 98.

SILVERMAN, I. K.
Rectangular tanks. 231.

SMITH, NELSON LEE.
Depreciation. 1276.

SNOW, FRANK HERBERT.
Memoir of. 1634.

SOLOMON, JOHN W.
Friction pile foundations. 149.

SPELMAN, JOHN RODGERS.
Memoir of. 1638.

STABLER, HERMAN.
Memoir of. 1641.

STARRETT, MILTON GERRY.
Memoir of. 1645.

STEVENS, J. C.
Air entrainment. 1474.
Open-channel flow. 492.

STEWART, RALPH W.
Deflections and moments. 1198.
Mechanics of creep. 477.
Moment balance. 696.

STOKES, G. W.
Moment balance. 698.

STORIE, R. EARL.
Land classification. 672.

STORRS, JOHN WILLIAMS.
Memoir of. 1646.

STRINGFELLOW, HENRY AYLESBURY.
Memoir of. 1680.

SUTHERLAND, EARL C.
Stress concentrations in plates. 864.

SWEET, JAMES S.
Federal water policies. 306.

TALBOT, ARTHUR NEWELL.
Memoir of. 1530.

TAPLEY, G. S.
"Hydrodynamics of Model Storm Sewer Inlets Applied to Design." 409.

TEAL, J. E.
"Development of Transportation in the United States." 1.

TELLER, I. W.
Stress concentrations in plates. 864.

TERZAGHI, KARL.
"Earth Pressure and Shearing Resistance of Plastic Clay: A Symposium: Liner-Plate Tunnels on the Chicago (Ill.) Subway." 970.

THOMAS, WILLIAM JOHN.
Memoir of. 1648.

TOZZER, ARTHUR CLARENCE.
Memoir of. 1649.

TSCHEBOTARIOFF, GREGORY P.
Plastic clay. 1075.

TURNER, C. A. P.
Mechanics of creep. 472.
Stress concentrations in plates. 881.

VELZY, CHARLES R.
Grease in sewage. 538.

WALKER, E. G.
Depreciation. 1307.

WALLS, J. A.
"Fundamental Aspects of the Depreciation Problem: A Symposium: Relationship to Regulated Industry." 1237.
WALTER, BOYD H.
"Permeability of Mud Mountain Dam Core Material." 719.
WALTER, FRANK EDGAR.
Memoir of. 1681.
WATCHORN, C. W.
TVA projects. 210.
WATERS, W. L.
Depreciation. 1300.
WEIR, WALTER W.
Land classification. 672.
WEISS, CAMILLO.
Deflections and moments. 1208.
WENDT, EDWIN F.
Depreciation. 1262.
WESTERGAARD, H. M.
"Stress Concentrations in Plates Loaded Over Small Areas." 831.
WESTON, ARTHUR D.
Grease in sewage. 525.
WESTON, ROBERT SPURR.
Organic zeolites. 284.
WHITE, MERIT P.
Energy loss. 1361.
Engineering seismology. 390.
WHITMAN, EZRA B.
"The Society Today and Tomorrow." Address at the Annual Convention, Los Angeles, California, July 28, 1943. 1517.
WHITMORE, GEORGE D.
Surveying accuracies. 598.
WILBUR, JOHN B.
Deflections and moments. 1194.
WILDER, ALVIN DUMOND.
Memoir of. 1651.
WILGUS, WILLIAM J.
Transportation. 25.
WILLES, EMERY H.
V-notch coefficients. 785.

WILLIAMS, EDWARD GILBERT.
Memoir of. 1653.

WILLIAMS, GORDON R.
"Drainage of Leveed Areas in Mountainous Valleys." 83.

WILLIAMSON, ROBERT A.
Deflections and moments. 1220.

WILSON, WALTER T.
Drainage of leveed areas. 108.
Gully control. 926.

WINFREY, ROBLEY.
"Fundamental Aspects of the Depreciation Problem: A Symposium: Relationship to Public Works and Government Finance." 1255.

WINTER, GEORGE.
"Lateral Stability of Unsymmetrical I-Beams and Trusses in Bending." 247.

WISE, JOSEPH A.
Elastic characteristics. 345.

WOERMANN, JOHN WILLIAM.
Memoir of. 1654.

WOOD, DANA M.
Federal water policies. 304.

WOODRUFF, GLENN B.
Friction pile foundations. 142.

WOODWARD, SHERMAN M.
TVA projects. 203.

WORLEY, JOHN S.
Depreciation. 1321.

YOUNG, DANA.
"Bending Moments in the Walls of Rectangular Tanks." 217.
Stress concentrations in plates. 858.